**Title**

Global well-posedness and parametrices for critical Maxwell-Dirac and massive Maxwell-Klein-Gordon equations with small Sobolev data



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**Global well-posedness and parametrices for critical Maxwell-Dirac and massive Maxwell-Klein-Gordon equations with small Sobolev data**

by

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**Abstract**

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In this thesis we prove global well-posedness and modified scattering for the massive Maxwell-Klein-Gordon (MKG) and for the massless Maxwell-Dirac (MD) equations, in the Coulomb gauge on $\mathbb{R}^{1+d}$ ($d \geq 4$), for data with small critical Sobolev norm.

For MKG, this work extends to the general case $m^2 > 0$ the results of Krieger-Sterbenz-Tataru ($d = 4, 5$) and Rodnianski-Tao ($d \geq 6$), who considered the case $m = 0$. We proceed by generalizing the global parametrix construction for the covariant wave operator and the functional framework from the massless case to the Klein-Gordon setting. The equation exhibits a trilinear cancelation structure identified by Machedon-Sterbenz. To treat it one needs sharp $L^2$ null form bounds, which we prove by estimating renormalized solutions in null frames spaces similar to the ones considered by Bejenaru-Herr. To overcome logarithmic divergences we rely on an embedding property of $\Box^{-1}$ in conjunction with endpoint Strichartz estimates in Lorentz spaces.

For MD, the main components of the proof consist of A) uncovering of the null structure of Maxwell-Dirac in the Coulomb gauge, and B) proving solvability of the underlying covariant Dirac equation. A key step for achieving both is to exploit and justify a deep analogy between MD and MKG, which says that the most difficult part of MD takes essentially the same form as parts of the Maxwell-Klein-Gordon structure. As a result, the aforementioned functional framework and parametrix construction become applicable.

# Contents

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# Chapter 1

# Introduction

## 1.1 Introduction and main results

In this thesis we consider the *massive Maxwell-Klein-Gordon* (MKG) equation and the *massless Maxwell-Dirac* (MD) equation on $\mathbb{R}^{d+1}$ for $d \geq 4$.

The Minkowski space $\mathbb{R}^{d+1}$ can be endowed with the metric $g = \mathrm{diag}(1, -1, \ldots, -1)$ or $\eta = \mathrm{diag}(-1, +1, \ldots, +1)$ in the rectilinear coordinates $(x^0, x^1, \ldots, x^d)$. Associated to the Minkowski metric $\eta$ are the *gamma matrices*, which are $N \times N$ complex-valued matrices $\gamma^\mu$ ($\mu = 0, 1, \ldots, d$) satisfying the anti-commutation relations

$$\frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = -\eta^{\mu\nu}\, \mathrm{I}_{4\times 4}, \tag{1.1.1}$$

where $\mathrm{I}_{4\times 4}$ is the $N \times N$ identity matrix, and also the conjugation relations

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0. \tag{1.1.2}$$

On $\mathbb{R}^{1+d}$, the rank of the gamma matrices $\gamma^\mu$ in the standard representation is $N = 2^{\lfloor \frac{d+1}{2} \rfloor}$ [56, Appendix E]. A *spinor field* $\psi$ is a function on $\mathbb{R}^{1+d}$ that takes values in $\mathbb{C}^N$, on which $\gamma^\mu$ acts as multiplication.

Let $\phi : \mathbb{R}^{d+1} \to \mathbb{C}$ be a complex field, while $A_\alpha : \mathbb{R}^{d+1} \to \mathbb{R}$ is a real 1-form with curvature

$$F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha.$$

One defines the *covariant derivatives* and the *covariant Klein-Gordon operator* by

$$D_\alpha \phi := (\partial_\alpha + i A_\alpha)\phi, \qquad \Box_m^A := D^\alpha D_\alpha + m^2$$

Given a real-valued 1-form $A_\mu$ we similarly introduce the *covariant derivative* on spinors $\mathbf{D}_\mu \psi := \partial_\mu \psi + i A_\mu \psi$ which acts componentwisely on $\psi$.

The Maxwell-Klein-Gordon equation arises as the Euler-Lagrange equations for the Lagrangian

$$\mathcal{S}_{MKG}[A_\mu, \phi] = \iint_{\mathbb{R}^{d+1}} \frac{1}{2} D_\alpha \phi \overline{D^\alpha \phi} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{2} m^2 |\phi|^2 \, \mathrm{d}x \, \mathrm{d}t$$

while Maxwell-Dirac arises from the Lagrangian

$$\mathcal{S}_{MD}[A_\mu, \psi] = \iint_{\mathbb{R}^{1+d}} i\langle \gamma^\mu \mathbf{D}_\mu \psi, \gamma^0 \psi\rangle - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - m\langle \psi, \psi\rangle \, \mathrm{d}x\mathrm{d}t.$$

Here $\langle \psi^1, \psi^2\rangle := (\psi^2)^\dagger \psi^1$ is the usual inner product on $\mathbb{C}^N$, where $\psi^\dagger$ denotes the hermitian transpose. Furthermore, we use the standard convention of raising and lowering indices using the Minkowski metric, and the Einstein summation convention of summing repeated upper and lower indices.

A brief computation shows that the Euler-Lagrange equations for $\mathcal{S}_{MKG}[A_\mu, \phi]$ take the form

$$\begin{cases} \partial^\beta F_{\alpha\beta} = \Im(\phi \overline{D_\alpha \phi}), \\ (D^\alpha D_\alpha + m^2)\phi = 0, \end{cases} \tag{1.1.3}$$

while the Euler–Lagrange equations for $\mathcal{S}_{MD}[A_\mu, \psi]$ take the form

$$\begin{cases} \quad \partial^\nu F_{\mu\nu} = - \langle \psi, \alpha_\mu \psi\rangle, \\ i\alpha^\mu \mathbf{D}_\mu \psi = m\beta\psi. \end{cases} \tag{MD}$$

where $\alpha^\mu = \gamma^0\gamma^\mu$ and $\beta = \gamma^0$.

The MKG system (1.1.3) is considered to be the simplest classical field theory enjoying a nontrivial *gauge invariance*. Indeed, for any real valued potential function $\chi$, replacing

$$\phi \mapsto e^{i\chi}\phi, \quad A_\alpha \mapsto A_\alpha - \partial_\alpha \chi, \quad D_\alpha \mapsto e^{i\chi} D_\alpha e^{-i\chi} \tag{1.1.4}$$

one obtains another solution to (1.1.3). The same hold with the transformation $(\tilde{A}, \tilde{\psi}) = (A - \mathrm{d}\chi, e^{i\chi}\psi)$ of $(A, \psi)$ in the case of MD. To remove this gauge ambiguity we will work with the *Coulomb gauge*

$$\mathrm{div}_x A = \partial^j A_j = 0 \tag{1.1.5}$$

where Roman indices are used in sums over the spatial components. Both systems are Lorentz invariant and admit a *conserved energy*, which we will not use here.

When $m = 0$ the equations are invariant under the *scaling*

$$\phi \mapsto \lambda\phi(\lambda t, \lambda x); \qquad A_\alpha \mapsto \lambda A_\alpha(\lambda t, \lambda x)$$

respectively

$$\psi \mapsto \lambda^{\frac{3}{2}}\psi(\lambda t, \lambda x); \qquad A_\alpha \mapsto \lambda A_\alpha(\lambda t, \lambda x)$$

which implies that $\sigma = \frac{d}{2} - 1$ is the *critical regularity* for MKG. We shall refer to $H^\sigma \times H^{\sigma-1} \times \dot{H}^\sigma \times \dot{H}^{\sigma-1}$ as the critical space for $(\phi, A)[0]$ also when $m \neq 0$. In the $(m = 0)$ MD case, the critical space for $(\psi(0), A[0])$ is $\dot{H}^{\sigma - \frac{1}{2}} \times \dot{H}^\sigma \times \dot{H}^{\sigma-1}$.

At this regularity, globally in time, the mass term $m^2\phi$ is not perturbative and must be considered as part of the operator $\Box_m^A$.

In this thesis, we will prove global well-posedness and scattering for the *massive* MKG and for the *massless* (MD) (i.e. $m = 0$) on the Minkowski space $\mathbb{R}^{1+d}$ with $d \geq 4$ under the Coulomb gauge condition (1.1.5), for initial data which are small in the scale-critical Sobolev space.

Under the Coulomb gauge condition (1.1.5), denoting $J_\alpha = -\Im(\phi\overline{D_\alpha \phi})$, the MKG system (1.1.3) becomes

$$\begin{cases} \Box_m^A \phi = 0 \\ \Box A_i = \mathcal{P}_i J_x \\ \Delta A_0 = J_0, \ \Delta \partial_t A_0 = \partial^i J_i \end{cases} \tag{1.1.6}$$

where $\mathcal{P}$ denotes the Leray projection onto divergence-free vector fields

$$\mathcal{P}_j A := \Delta^{-1} \partial^k (\partial_k A_j - \partial_j A_k). \tag{1.1.7}$$

The first result of this thesis consists in extending the results in [31], [47] to the case $m \neq 0$. For a more detailed statement, see Theorem 3.0.1.

**Theorem 1.1.1** ([16] Critical small data global well-posedness and scattering)**.** *Let $d \geq 4$ and $\sigma = \frac{d}{2} - 1$. The MKG equation* (1.1.6) *is globally well-posed for small initial data on* $(\mathbb{R}^{1+d}, g)$ *with $m^2 > 0$, in the following sense: there exists a universal constant $\varepsilon > 0$ such that:*

1. *Let $(\phi[0], A_x[0])$ be a smooth initial data set satisfying the Coulomb condition* (1.1.5) *and the smallness condition*

$$\|\phi[0]\|_{H^\sigma \times H^{\sigma-1}} + \|A_x[0]\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} < \varepsilon. \tag{1.1.8}$$

   *Then there exists a unique global smooth solution $(\phi, A)$ to the system* (1.1.6) *under the Coulomb gauge condition* (1.1.5) *on $\mathbb{R}^{1+d}$ with these data.*

2. *For any $T > 0$, the data-to-solution map $(\phi[0], A_x[0]) \mapsto (\phi, \partial_t \phi, A_x, \partial_t A_x)$ extends continuously to*

$$H^\sigma \times H^{\sigma-1} \times \dot{H}^\sigma \times \dot{H}^{\sigma-1}(\mathbb{R}^d) \cap \{(1.1.8)\} \to C([-T,T]; H^\sigma \times H^{\sigma-1} \times \dot{H}^\sigma \times \dot{H}^{\sigma-1}(\mathbb{R}^d)).$$

3. *The solution $(\phi, A)$ exhibits modified scattering as $t \to \pm\infty$: there exist a solution $(\phi^{\pm\infty}, A_j^{\pm\infty})$ to the linear system*

$$\Box A_j^{\pm\infty} = 0, \qquad \Box_m^{A^{free}} \phi = 0, \qquad \textit{such that}$$

$$\|(\phi - \phi^{\pm\infty})[t]\|_{H^\sigma \times H^{\sigma-1}} + \|(A_j - A_j^{\pm\infty})[t]\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} \to 0 \quad \textit{as } t \to \pm\infty,$$

   *where $A^{free}$ is the free solution of $\Box A^{free} = 0$ with $A_x^{free}[0] = A_x[0]$ and $A_0^{free} = 0$.*

For Maxwell-Dirac we have the following result, which was obtained in collaboration with Sung-Jin Oh [17]:

**Theorem 1.1.2** ([17] Critical small data global well-posedness and scattering). *Consider* (MD) *on* $(\mathbb{R}^{1+d}, \eta)$ *with* $m = 0$ *for* $d \geq 4$ *and let* $\sigma = \frac{d}{2} - 1$. *There exists a universal constant* $\epsilon_* > 0$ *such that the following statements hold.*

1. *Let* $(\psi(0), A_j(0), \partial_t A_j(0))$ *be a smooth initial data set satisfying the Coulomb condition* (1.1.5) *and the smallness condition*

$$\|\psi(0)\|_{\dot{H}^{\sigma-1/2}(\mathbb{R}^d)} + \sup_{j=1,\dots,4} \|(A_j, \partial_t A_j)(0)\|_{\dot{H}^{\sigma} \times \dot{H}^{\sigma-1}(\mathbb{R}^d)} < \epsilon_*. \tag{1.1.9}$$

*Then there exists a unique global smooth solution* $(\psi, A)$ *to the system* (MD) *under the Coulomb gauge condition* (1.1.5) *on* $\mathbb{R}^{1+d}$ *with these data.*

2. *For any* $T > 0$, *the data-to-solution map* $(\psi, A_j, \partial_t A_j)(0) \mapsto (\psi, A_j, \partial_t A_j)$ *extends continuously to*

$$\dot{H}^{\sigma-1/2} \times \dot{H}^{\sigma} \times \dot{H}^{\sigma-1}(\mathbb{R}^d) \cap \{(1.1.9) \text{ holds}\} \to C([0,T]; \dot{H}^{\sigma-1/2} \times \dot{H}^{\sigma} \times \dot{H}^{\sigma-1}(\mathbb{R}^d)).$$

*The same statement holds on the interval* $[-T, 0]$.

3. *For each sign* $\pm$, *the solution* $(\psi, A)$ *exhibits* modified scattering *as* $t \to \pm\infty$, *in the sense that there exist a solution* $(\psi^{\pm\infty}, A_j^{\pm\infty})$ *to the linear system*

$$\begin{cases} \Box A_j^{\pm\infty} = 0, \\ \alpha^{\mu} \boldsymbol{D}_{\mu}^{B} \psi^{\pm\infty} = 0, \end{cases}$$

*such that*

$$\|(\psi - \psi_j^{\pm\infty})(t)\|_{\dot{H}^{\sigma-1/2}(\mathbb{R}^d)} + \|(A_j - A_j^{\pm\infty})[t]\|_{\dot{H}^{\sigma} \times \dot{H}^{\sigma-1}(\mathbb{R}^d)} \to 0 \qquad \text{as } t \to \pm\infty.$$

*Here,* $B_0 = 0$ *and* $B_j$ *can be taken to be either the solution* $A^{free}$ *to* $\Box A^{free} = 0$ *with data* $A_j^{free}[0] = A_j[0]$, *or* $B_j = A_j^{\pm\infty}$.

For both results the case $d = 4$ is the most difficult. When $d \geq 5$ the argument simplifies, in particular the spaces $NE_C^{\pm}$, $PW_C^{\pm}$, $L^2 L^{4,2}$ are not needed. To fix notation, the reader is advised to set $d = 4$, $\sigma = 1$. In fact, for the sake of concreteness, for the MD equation we focus on the case $d = 4$, where we present the proof of Theorem 1.1.2 in detail and then refer to Remarks 1.8.5, 3.0.3, 6.3.8, 6.4.3 and 4.3.1 for the necessary modifications of the argument for $d \geq 5$.

*Remark* 1.1.3. The theorems are stated for Coulomb initial data. However, they can be applied to arbitrary initial data satisfying the smallness conditions by performing a gauge transformation. Given a real 1-form $A_j(0)$ on $\mathbb{R}^d$, one solves the Poisson equation

$$\Delta\chi = \mathrm{div}_x A_j(0), \qquad\qquad \chi \in \dot{H}^{\frac{d}{2}} \cap \dot{W}^{1,d}(\mathbb{R}^d).$$

Then $\tilde{A}(0) = A(0) - \mathrm{d}\chi$ obeys the Coulomb condition (1.1.5). For small $\varepsilon$, the small data condition is preserved up to multiplication by a constant.

In what follows, in the context of MKG we set $m = 1$, noting that by rescaling, the statements for any $m \neq 0$ can be obtained. Notation-wise, we will write $\Box_m$ rather than $\Box_1$. In the case of MD we set $m = 0$, although by examining the proofs of the MKG estimates it becomes clear that one could also obtain the similar result for the massive MD.

In the rest of this chapter we describe the physical motivation and the main ideas ideas of this thesis (null structures, the non-perturbative nonlinearities, the parametrix, adapted function spaces and the the parallelism between MD and MKG). We also review the previous work and set up the notation and definitions. Finally, we present the decompositions of the nonlinearities and state the main estimates as well as the solvability theorems.

In chapter 2 we define the function spaces and present their embeddings properties and the motivation for the choice of norms. With various localizations, we need $X^{s,b}$, Strichartz and $L^1 L^\infty$ spaces, Lorentz spaces and adapted $L^\infty_{t_{\omega,\lambda}} L^2_{x_{\omega,\lambda}}$, $L^2_{t_{\omega,\lambda}} L^\infty_{x_{\omega,\lambda}}$ spaces.

In chapter 3 we give the proofs of the main theorems using the nonlinear estimates and the parametrix theorems stated in this chapter. We discuss existence and uniqueness, frequency envelopes bounds, weak Lipschitz dependence, subcritical well-posedness, persistence of regularity, continuous dependence on the initial data and modified scattering.

In chapter 4 we present the motivation and the construction of the parametrices for covariant Klein-Gordon and Dirac equations. We give the proofs of Theorems 1.6.1, 1.6.2 and Prop. 1.8.11. We discuss the main properties of the phases, decomposable estimates, oscillatory integrals estimates and the conjugation.

Chapter 5 is devoted to a detailed analysis of the core translation-invariant bilinear forms that play a role in our equations. We discuss the classical $\mathcal{N}_0$, $\mathcal{N}_{ij}$ and spinorial null forms, how to adapt $\mathcal{N}_0$ to the Klein-Gordon equation, the geometry of frequency interactions as well as some refinements of Hölder's inequality.

Chapter 6 contains the proofs of the bilinear and trilinear estimates from section 1.8. Using the spaces introduced in chapter 2 these proofs rely on the analysis from chapter 5.

## 1.2 Physical motivation

We begin with a short review of relativistic mechanics and then discuss the historical motivation of our equations following [14]. The units are chosen such that $c = 1$. Consider a free particle on $\mathbb{R}^{3+1}$ with Lagrangian $L = -m\sqrt{1-|\mathbf{v}|^2}$. The conjugate momentum is $\mathbf{p} = \nabla_{\mathbf{v}} L = \frac{m\mathbf{v}}{\sqrt{1-|\mathbf{v}|^2}}$ while the energy equals $E = \mathbf{p}\cdot\mathbf{v} - L = \frac{m}{\sqrt{1-|\mathbf{v}|^2}}$. The energy $E$ and the momentum $\mathbf{p} = (p_x, p_y, p_z)$ combine into the four-momentum vector $p^\mu = (E, \mathbf{p})$. The Minkowski norm of this vector equals the square of the particle's mass: $p^\mu p_\mu = E^2 - |\mathbf{p}|^2 = m^2$.

Now consider a given electromagnetic field with potential $A^\mu = (\phi, \mathbf{A})$. The action functional of a charged particle with mass $m$ and charge $e$ moving in this field is

$$S = \int_{(t_0,x_0)}^{(t_1,x_1)} -m\mathrm{d}s - eA_\mu \mathrm{d}x^\mu.$$

Writing the Lagrangian in coordinates $(t, \mathbf{x})$ we obtain $L = -m\sqrt{1-|\mathbf{v}|^2} + e\mathbf{A}\cdot\mathbf{v} - e\phi$ where $\mathbf{v} = d\mathbf{x}/dt$. We compute the canonical momentum

$$\mathbf{p} = \nabla_{\mathbf{v}} L = \frac{m\mathbf{v}}{\sqrt{1-|\mathbf{v}|^2}} + e\mathbf{A} = \mathbf{p}_k + e\mathbf{A}.$$

The corresponding Hamiltonian

$$H = \mathbf{p}\cdot\mathbf{v} - L = \frac{m}{\sqrt{1-|\mathbf{v}|^2}} + e\phi, \quad \text{and note} \quad H = \sqrt{m^2 + |\mathbf{p} - e\mathbf{A}|^2} + e\phi.$$

In analogy with the free particle we denote by $p^\mu$ the 4-vector consisting of the total energy $H$ and the canonical momentum $\mathbf{p}$. The last equation states that the Lorentz product equals

$$(p - eA)^2 = (p^\mu - eA^\mu)(p_\mu - eA_\mu) = (E - e\phi)^2 - |\mathbf{p} - e\mathbf{A}|^2 = m^2. \tag{1.2.1}$$

## Klein-Gordon and Dirac

The Klein-Gordon and Dirac equations arose out of the desire to obtain relativistic alternatives to the Schrödinger equation

$$i\partial_t \psi = -(1/2m)\Delta\psi + V\psi.$$

In this context the potential $V$ is meant to be replaced by an electromagnetic field, while the energy and momentum are quantized by

$$p^\mu \to i\partial^\mu. \tag{1.2.2}$$

Applying this quantization to the energy-momentum relation $p^2 = m^2$ we obtain the Klein-Gordon equation[1]:

$$(\partial_t^2 - \Delta)\phi + m^2\phi = 0. \tag{1.2.3}$$

[1] Introduced in 1926 by physicists Oskar Klein and Walter Gordon

Because this equation is second order in the time derivative, we must specify initial values for both $\phi$ and $\partial_t\phi$. Thus $\phi$ itself cannot maintain the role of determining the probability density of the position of the particle. To obtain an equation that is of first order in $t$, one could replace (1.2.3) by $i\partial_t\psi = A\psi$ where $A = (-\Delta + m^2)^{1/2}$. This would resemble the Schrödinger equation, but then the resulting equality is no longer a differential equation.

Dirac's idea was that it is possible to obtain a Lorentz-invariant differential equation of type

$$i\partial_t\psi = H\psi \tag{1.2.4}$$

provided that one allows vector-valued wave functions $\psi$. If we impose $H^2 = -\Delta + m^2$ and consider

$$H = \frac{1}{i}(\alpha^1\partial_1 + \alpha^2\partial_2 + \alpha^3\partial_3) + m\beta,$$

by squaring this operator and collecting the terms, we must have

$$\alpha^j\alpha^k + \alpha^k\alpha^j = 2\delta_{jk}I, \qquad \alpha^j\beta + \beta\alpha^j = 0, \qquad \beta^2 = I.$$

Then (1.2.4) can be put in the form of the covariant Dirac equation [2]

$$i\gamma^\mu\partial_\mu\psi = m\psi.$$

One benefit of this model is that it satisfies a conservation of probability law, unlike the Klein-Gordon equation: if $j^\mu = \bar{\psi}\gamma^\mu\psi$, we have $\partial_\mu j^\mu = 0$. Then $j_0 = \psi^\dagger\psi$ is the probability density of the position of the particle.

Varying with respect to $\bar{\psi}$ in the action functional

$$\int i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \,\mathrm{d}x\mathrm{d}t$$

one obtains the Dirac equation (while varying $\psi$ results in the adjoint Dirac equation).

Previously we obtained the energy-momentum vector for a particle in an electromagnetic field from the 4-vector of a free particle by replacing $p^\mu$ by $p^\mu - eA^\mu$. This suggests to replace (1.2.2) by

$$i\partial^\mu \to i\partial^\mu - eA_\mu. \tag{1.2.5}$$

in the action functional, which results in the equation

$$\gamma^\mu(i\partial_\mu - eA_\mu)\psi - m\psi = 0.$$

Most treatments of the Dirac equation consider the electromagnetic field as given and ignore the Dirac current as a source for the Maxwell equations. To couple the Dirac equation with the Maxwell equation we add the term $-1/4F_{\mu\nu}F^{\mu\nu}$ to the Lagrangian, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The resulting system is the Maxwell-Dirac equation. Similarly one couples the Klein-Gordon equation (1.2.3) with Maxwell's equations.

[2] Formulated in 1928 by the British physicist Paul Dirac.

## 1.3 Main ideas

We now provide an outline of the main ideas of this thesis.

### Null structures in the Coulomb gauge.

Null structures arise in equations from mathematical physics which exhibit covariance properties. They manifest through the vanishing of resonant components of the nonlinearities of such equations, and their presence is fundamental in obtaining well-posedness at low regularity.

An important component of the proof is uncovering the null structure of (MD) in the Coulomb gauge, which involves both *classical* (i.e., scalar) and *spinorial* null forms.

A classical null form for scalar inputs refers to a linear combination of

$$\mathcal{N}_{ij}(\phi,\psi) = \partial_i\phi\partial_j\psi - \partial_j\phi\partial_i\psi, \qquad \mathcal{N}_0(\phi,\psi) = \partial_\alpha\phi\cdot\partial^\alpha\psi. \tag{1.3.1}$$

These null forms initially arose in the study of global-in-time behavior of nonlinear wave equations with small, smooth and localized data [24]. Remarkably, in the work [26] of Klainerman and Machedon, it was realized that the same structure is essential for establishing low regularity well-posedness as well.

Among the first applications of this idea was the proof of global well-posedness at energy regularity of the Maxwell–Klein–Gordon equations on $\mathbb{R}^{1+3}$ [25]. A key observation in [25] was that quadratic nonlinearities of Maxwell–Klein–Gordon in the Coulomb gauge consist of null forms of the type $\mathcal{N}_{ij}$. Furthermore, in the proof of essentially optimal local well-posedness of MKG in $\mathbb{R}^{1+3}$ by Machedon and Sterbenz [34], a secondary trilinear null structure involving $\mathcal{N}_0$ was identified in the system after one iteration.

Both of these structures played an important role in [31], and they also do so here. However, special care must be taken since the null form $\mathcal{N}_0$ is adapted to the wave equation while we will also work with Klein-Gordon waves.

Another type of null structures that arise in this work are spinorial null forms. These are bilinear forms with the symbol

$$\Pi_\pm(\xi)\Pi_\mp(\eta), \qquad\qquad \text{where} \quad \Pi_\pm(\xi) := \frac{1}{2}\Big(\mathrm{I}_{4\times4} \mp \frac{\alpha^j\xi_j}{|\xi|}\Big),$$

which were first uncovered by D'Ancona, Foschi, Selberg for the Dirac–Klein–Gordon system in [9]. These authors further investigated the spinorial null forms in the study of the Maxwell–Dirac equation on $\mathbb{R}^{1+3}$ in the Lorenz gauge (in [10]; see also [11]). In the work of Bejenaru–Herr [4, 3] and Bournaveas–Candy [6], these null forms were used in the proof of global well-posedness of the cubic Dirac equation for small critical data.

A more detailed exposition of the null structure of MD-CG is given in Section 5.4. At this point we simply note that the null structure alone is insufficient to close the proofs due to the presence of *non-perturbative* nonlinearity.

## Presence of non-perturbative nonlinearity

As in many previous works on low regularity well-posedness, we take a paradifferential approach in treating the nonlinear terms, exploiting the fact that the high-high to low interactions are weaker and that terms where the derivative falls on low frequencies are weaker as well.

From this point of view, the worst interaction occurs in the frequency-localized components of the scalar part of MKG

$$\sum_k A^{\alpha}_{<k-C} \partial_\alpha \bar{P}_k \phi$$

and of the Dirac part of MD

$$\sum_k \alpha^\mu P_{<k-C} A_\mu P_k \psi.$$

At critical Sobolev regularity these terms are *non-perturbative*, in the sense that even after utilizing all available null structure, they cannot be treated with multilinear estimates for the usual wave, Klein-Gordon and Dirac equations. Instead, following the model set in the work of Rodnianski–Tao [47] and Krieger–Sterbenz–Tataru [31] on MKG-CG, these terms must be viewed as a part of the underlying linear operators, and we must prove their solvability in appropriate function spaces. In fact, in the case of MD we establish solvability of the *covariant Dirac operator* $\alpha^\mu \mathbf{D}_\mu$; see Proposition 1.8.11 below. We note that this is the reason why we have modified scattering, as opposed to scattering to a free field.

The presence of a non-perturbative term is characteristic of geometric wave equations with derivative nonlinearity, whose examples include Wave maps, Maxwell–Klein–Gordon, Yang–Mills.

## Parametrix construction for paradifferential covariant wave equation

The key to addressing the non-perturbative nonlinearity is through a suitable renormalization.

A key breakthrough of Rodnianski and Tao [47] was proving Strichartz estimates for the covariant wave equation by introducing a microlocal parametrix construction, motivated by the gauge covariance of $\Box_A = D^\alpha D_\alpha$ under (1.1.4), i.e., $e^{-i\Psi} \Box_{A'} (e^{i\Psi}\phi) = \Box_A \phi$. The idea was to approximately conjugate (or renormalize) the modified d'Alembertian $\Box + 2iA_{<k-c} \cdot \nabla_x P_k$ to $\Box$ by means of a carefully constructed pseudodifferential gauge transformation

$$\Box^p_A \approx e^{i\Psi_\pm}(t,x,D) \Box e^{-i\Psi_\pm}(D,s,y).$$

These Strichartz estimates were sufficient to prove global regularity of the Maxwell-Klein-Gordon equation at small critical Sobolev data in dimensions $d \geq 6$. We discuss this construction in chapter 4, which also extends to massive case and to the half-wave case.

As explained below, this construction will also provide the basis for the proof of solvability of the covariant Dirac equation. Moreover, a renormalization procedure has been also applied to the Yang-Mills equation at critical regularity [30], [32].

## Function spaces

In [31], Krieger, Sterbenz and Tataru further advanced the parametrix idea in $d = 4$, showing that it interacts well with the function spaces previously introduced for critical wave maps in [54], [52]. In particular, the resulting solution obeys similar bounds as ordinary waves, yielding control of an $X^{s,b}$ norm, null-frame norms and square summed angular-localized Strichartz norms.

Here we will follow their strategy, showing that both the spaces and the renormalization bounds generalize to the Klein-Gordon ($m^2 > 0$) context.

Critical $X^{s,\pm\frac{1}{2}}$ spaces, 'null' energy $L^{\infty}_{t_{\omega,\lambda}} L^2_{x_{\omega,\lambda}}$ and Strichartz $L^2_{t_{\omega,\lambda}} L^{\infty}_{x_{\omega,\lambda}}$ spaces in adapted frames, were already developed for the Klein-Gordon operators by Bejenaru and Herr [4] and we will use some of their ideas. The difficulty here consists in proving the bounds for renormalized solutions

$$\|e^{-i\psi}(t,x,D)\phi\|_{\bar{S}^1_k} \lesssim \|\phi[0]\|_{H^1 \times L^2} + \|\Box_m \phi\|_{\bar{N}_k}.$$

We shall rely on $TT^*$ and stationary phase arguments for both $L^{\infty}_{t_{\omega,\lambda}} L^2_{x_{\omega,\lambda}}$ and $L^2_{t_{\omega,\lambda}} L^{\infty}_{x_{\omega,\lambda}}$ bounds, as well for $P_C L^2 L^{\infty}$, see Corollaries 4.6.9, 4.6.6 and 4.6.4.

However, at low frequency or at high frequencies with very small angle interactions, the adapted frame spaces do not work and we are confronted with logarithmic divergences. To overcome this we rely on Strichartz estimates in Lorentz spaces $L^2 L^{4,2}$ and an embedding property of $\Box^{-1}$ into $L^1 L^{\infty}$.

Here we have been inspired by the paper [50] of Shatah and Struwe. The difference is that instead of inverting $\Delta$ by a type of Sobolev embedding $|D_x|^{-1} : L^{d,1}_x \to L^{\infty}_x$, we have to invert $\Box$ by

$$2^{\frac{1}{2}l} \sum_{k'} P^{\omega}_l Q_{k'+2l} P_{k'} \frac{1}{\Box} : L^1 L^{2,1} \to L^1 L^{\infty}$$

See Proposition 2.2.2 for more details.

## Parallelism between Maxwell–Dirac and Maxwell–Klein–Gordon

In proving solvability of the covariant Dirac operator, as well as uncovering the null structure of MD-CG, we exploit a deep parallelism between the Maxwell–Dirac and the Maxwell–Klein–Gordon equations. On one hand, it provides a clear guiding principle that we hope would be useful in the future study of other Dirac equations. On the other hand, it allows us to borrow some key bounds directly from the massless Maxwell–Klein–Gordon case [31], which simplifies the proof.

Historically, the Dirac equation emerged in an attempt to take the 'square root' of the Klein–Gordon equation in order to obtain an equation that is first order in time. Thus 'squaring' the Dirac component of the system leads to an equation that looks like the Klein–Gordon part of MKG. Unfortunately, as noted in [10], this idea seems to be of limited use, since squaring the Dirac equation destroys most of the spinorial null structure.

An alternative, more fruitful approach was put forth in [10], which we follow in this thesis. The idea is to first take the spatial Fourier transform and diagonalize the Dirac operator $\alpha^\mu \partial_\mu$, decomposing the spinor as $\psi = \psi_+ + \psi_-$ where $\psi_\pm$ obey appropriate half-wave equations. Splitting $\psi$ in the nonlinearity $\alpha^\mu A_\mu \psi$ into $\psi_+ + \psi_-$ as well, we can divide the equation into two parts: the *scalar part*, which consists of contribution of $\psi_\pm$ without multiplication by $\alpha^\mu$, and the remaining *spinorial part*. A similar decomposition can be performed for the nonlinearity of the Maxwell equations.

One of the key observations is that the spinorial part enjoys a more favorable null structure compared to the scalar part. In particular, it is entirely perturbative, and furthermore the secondary null structure à la Machedon–Sterbenz [34] is unnecessary. We refer to Remark 5.4.5 for a more detailed explanation.

For the remaining scalar part, we observe that its structure closely parallels that of MKG; see Remark 1.7.9 for the detailed statement. As a consequence of this parallelism, we show that MD-CG exhibits a nearly identical secondary null structure as the one from MKG (uncovered in [34]); see Section 6.4. Furthermore, the microlocal parametrix construction in [31] can be borrowed as a black box to establish key estimates in the proof of solvability of the covariant Dirac equation, which handles the non-perturbative nonlinearity.

## 1.4 Previous work

The connection between the Maxwell-Klein-Gordon and Maxwell-Dirac equations is that the MKG equation can be considered a scalar counterpart to MD. The paper [31] of Krieger–Sterbenz–Tataru was the main motivation and inspiration for both Theorem 1.1.1 and 1.1.2.

Progress on the Maxwell-Klein-Gordon equation has occurred in conjunction with the Yang–Mills(-Higgs) equations. An early result was obtained by Eardley and Moncrief [12].

On $\mathbb{R}^{2+1}$ and $\mathbb{R}^{3+1}$ the MKG system is energy subcritical. Klainerman-Machedon [25] and Selberg-Tesfahun [49] (in the Lorenz gauge) have proved global regularity as a consequence of local well-posedness at the energy regularity. Further results were obtained by Cuccagna [8]. Machedon-Sterbenz [34] proved an essentially optimal local well-posedness result. In [22] in $\mathbb{R}^{3+1}$, global well-posedness below the energy norm was considered.

On $\mathbb{R}^{4+1}$, an almost optimal local well-posedness result was proved by Klainerman-Tataru [28] for a model problem closely related to MKG and Yang-Mills. This was refined by Selberg [48] for MKG and Sterbenz [51].

At critical regularity all the existing results are for the massless case $m = 0$. Rodnianski-Tao [47] proved global regularity for smooth and small critical Sobolev data in dimensions $6 + 1$ and higher. This result was extended by Krieger-Sterbenz-Tataru [31] to $\mathbb{R}^{4+1}$.

The small data $\mathbb{R}^{4+1}$ energy critical massless result in [31] has been extended to large data global well-posedness by Oh-Tataru ([41],[42],[43]) and independently by Krieger-Lührmann [29]. Proving a similar large data result for the massive case remains an open problem. In contrast, although (MD) is also energy critical on $\mathbb{R}^{1+4}$, the energy for (MD) is *not* coercive; therefore it remains unclear whether Theorem 1.1.2 may be extended to large data.

Now we provide a brief survey of previous results on Maxwell-Dirac. After early work on local well-posedness of (MD) on $\mathbb{R}^{1+3}$ by Gross [20] and Bournaveas [5], D'Ancona–Foschi–Selberg [10] established local well-posedness of (MD) on $\mathbb{R}^{1+3}$ in the Lorenz gauge $\partial^\mu A_\mu = 0$ for data $\psi(0) \in H^\epsilon$, $A_\mu[0] \in H^{1/2+\epsilon} \times H^{-1/2+\epsilon}$, which is almost optimal. In the course of their proof, a deep system null structure of (MD) in the Lorenz gauge was uncovered. Although we work in a different gauge, our work develop upon many ideas from [10]. D'Ancona–Selberg [11] extended this approach to (MD) on $\mathbb{R}^{1+2}$ and proved global well-posedness in the charge class.

Regarding (MD) on $\mathbb{R}^{1+3}$, we also mention [7, 18, 13, 46] on global well-posedness for small, smooth and localized data, [1, 35] on the non-relativistic limit and [36] on unconditional uniqueness at regularity $\psi \in C_t H^{1/2},\ A_x[\cdot] \in C_t(H^1 \times L^2)$ in the Coulomb gauge.

Finally, we note that optimal small data global well-posedness was proved recently for the *cubic Dirac equation* in $\mathbb{R}^{1+2}$ and $\mathbb{R}^{1+3}$ by Bejenaru–Herr [4, 3] (massive) and Bournaveas–Candy [6] (massless). This equation features a spinorial null structure similar to what is considered in this work. Recent works on Yang-Mills include: [30], [32], [40], [39].

## 1.5 Preliminaries

### Notation

We denote

$$\langle \xi \rangle_k = (2^{-2k} + |\xi|^2)^{\frac{1}{2}}, \qquad \langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}.$$

We define $A \prec B$ by $A \leq B - C$, $A \lesssim B$ by $A \leq CB$ and $A = B + O(1)$ by $|A - B| \leq C$, for some absolute constant $C$. We say $A \ll B$ when $A \leq \eta B$ for a small constant $0 < \eta < 1$ and $A \simeq B$ when he have both $A \lesssim B$ and $B \lesssim A$.

Given $\mathcal{C}, \mathcal{C}' \subseteq \mathbb{R}^d$, we use the notation $-\mathcal{C} = \{-\xi : \xi \in \mathcal{C}\}$ and $\mathcal{C} + \mathcal{C}' = \{\xi + \eta : \xi \in \mathcal{C},\ \eta \in \mathcal{C}'\}$. Moreover, we define the *angular distance* between $\mathcal{C}$ and $\mathcal{C}'$ as

$$|\angle(\mathcal{C}, \mathcal{C}')| := \inf\{|\angle(\xi, \eta)| : \xi \in \mathcal{C},\ \eta \in \mathcal{C}'\}.$$

### Frequency projections

Let $\chi$ be a smooth non-negative bump function supported on $[2^{-2}, 2^2]$ which satisfies the partition of unity property

$$\sum_{k' \in \mathbb{Z}} \chi\big(|\xi| / 2^{k'}\big) = 1$$

for $\xi \neq 0$. For $k' \in \mathbb{Z},\ k \geq 0$ we define the Littlewood-Paley operators $P_{k'}, \bar{P}_k$ by

$$\widehat{P_{k'}f}(\xi) = \chi\big(\,|\xi|\,/2^{k'}\big)\hat{f}(\xi), \quad \bar{P}_0 = \sum_{k' \leq 0} P_{k'}, \quad \bar{P}_k = P_k, \text{ for } k \geq 1.$$

The modulation operators $Q_j, Q_j^{\pm},\ \bar{Q}_j, \bar{Q}_j^{\pm}$ are defined by

$$\mathcal{F}(\bar{Q}_j^{\pm} f)(\tau,\xi) = \chi\Big(\frac{|\pm\tau - \langle\xi\rangle|}{2^j}\Big)\mathcal{F}f(\tau,\xi), \quad \mathcal{F}(Q_j^{\pm} f)(\tau,\xi) = \chi\Big(\frac{|\pm\tau - |\xi||}{2^j}\Big)\mathcal{F}f(\tau,\xi).$$

and $Q_j = Q_j^+ + Q_j^-,\ \bar{Q}_j = \bar{Q}_j^+ + \bar{Q}_j^-$ for $j \in \mathbb{Z}$, where $\mathcal{F}$ denotes the space-time Fourier transform.

Given $\ell \leq 0$ we consider a collection of directions $\omega$ on the unit sphere which is maximally $2^{\ell}$-separated. To each $\omega$ we associate a smooth cutoff function $m_\omega$ supported on a cap $\kappa \subset \mathbb{S}^{d-1}$ of radius $\simeq 2^{\ell}$ around $\omega$, with the property that $\sum_\omega m_\omega = 1$. We define $P_\ell^\omega$ (or $P_\kappa$ )to be the spatial Fourier multiplier with symbol $m_\omega(\xi/\,|\xi|)$. In a similar vein, we consider rectangular boxes $\mathcal{C}_{k'}(\ell')$ of dimensions $2^{k'} \times (2^{k'+\ell'})^{d-1}$, where the $2^{k'}$ side lies in the radial direction, which cover $\mathbb{R}^d$ and have finite overlap with each other. We then define $P_{\mathcal{C}_{k'}(\ell')}$ to be the associated smooth spatial frequency localization to $\mathcal{C}_{k'}(\ell')$. For convenience, we choose the blocks so that $P_k P_\ell^\omega = P_{\mathcal{C}_k(\ell)}$.

We will often abbreviate $A_{k'} = P_{k'} f$ or $\phi_k = \bar{P}_k \phi$. We will sometimes use the operators $\tilde{P}_k, \tilde{Q}_{j/<j}, \tilde{P}_\ell^\omega$ with symbols given by bump functions which equal 1 on the support of the multipliers $P_k, Q_{j/<j}$ and $P_\ell^\omega$ respectively and which are adapted to an enlargement of these supports.

Given a sign $s \in \{+,-\}$, define $T_s$ as

$$\widetilde{T_+ f}(\tau,\xi) = 1_{\{\tau>0\}}\tilde{f}(\tau,\xi), \quad \widetilde{T_- f}(\tau,\xi) = 1_{\{\tau\leq 0\}}\tilde{f}(\tau,\xi).$$

For all $j$, we have $Q_{j/<j} T_s = Q^s_{j/<j} T_s$. Moreover, for $j \leq k-3$, we have

$$P_k Q_{j/<j} T_s = P_k Q^s_{j/<j}, \quad P_k Q_{j/<j} = \sum_{s\in\{+,-\}} P_k Q^s_{j/<j}.$$

We call a multiplier disposable when its convolution kernel is a function (or measure) with bounded mass. Minkowski's inequality insures that disposable operators are bounded on translation-invariant normed spaces. Examples include $P_k, P_\ell^\omega, P_{\mathcal{C}}$.

For any $Q^\square_{j/<j} \in \{Q_j^s, Q_{<j}^s, Q_j, Q_{<j}\}$ with $j \in \mathbb{Z}$, the operator $P_k Q^\square_{j/<j}$ is disposable if $j \geq k - C$ [52, Lemma 3]. In general, one has

$$\|P_k Q^\square_{j/<j} f\|_{L^q L^r} \lesssim 2^{d(k-j)_+} \|f\|_{L^q L^r} \qquad (1 \leq q, r \leq \infty) \tag{1.5.1}$$

In the case $r = 2$, we have an unconditional estimate [52, Lemma 4]:

$$\|P_k Q^\square_{j/<j} f\|_{L^q L^2} \lesssim \|f\|_{L^q L^2} \qquad (1 \leq q \leq \infty). \tag{1.5.2}$$

When $j \geq k + 2\ell - C$ the operator $P_k P_\ell^\omega Q_{j/<j}$ is disposable [52, Lemma 6]. Similar considerations apply to $Q_j^{\pm} \bar{Q}_j, \bar{P}_k$ etc.

We also record the following identity.

**Lemma 1.5.1** (Commutator identity)**.** *We can write*

$$P_{<k}(fg) = fP_{<k}g + L(\nabla_x f, 2^{-k}g)$$

*where $L$ is a translation-invariant bilinear operator with integrable kernel.*

*Proof.* See [52, lemma 2]. □

## Sector projections

For $\omega \in \mathbb{S}^{d-1}$ and $0 < \theta \lesssim 1$ we define the sector projections $\Pi^{\omega}_{>\theta}$ by

$$\widehat{\Pi^{\omega}_{>\theta} u}(\xi) = \big(1 - \eta(\angle(\xi,\omega)\theta^{-1})\big)\big(1 - \eta(\angle(\xi,-\omega)\theta^{-1})\big)\hat{u}(\xi) \tag{1.5.3}$$

where $\eta$ is a bump function on the unit scale. We define

$$\Pi^{\omega}_{<\theta} = 1 - \Pi^{\omega}_{>\theta}, \qquad \Pi^{\omega}_{\theta} = \Pi^{\omega}_{>\theta/2} - \Pi^{\omega}_{>\theta}. \tag{1.5.4}$$

## Adapted frames

Following [4], for $\lambda > 0$ and $\omega \in \mathbb{S}^{d-1}$ we define the frame

$$\omega^{\lambda} = \frac{1}{\sqrt{1+\lambda^2}}(\pm\lambda, \omega), \quad \bar{\omega}^{\lambda} = \frac{1}{\sqrt{1+\lambda^2}}(\pm 1, -\lambda\omega), \quad \omega_i^{\perp} \in (0, \omega^{\perp}) \tag{1.5.5}$$

and the coordinates in this frame

$$t_{\omega} = (t,x)\cdot\omega^{\lambda}, \quad x^1_{\omega} = (t,x)\cdot\bar{\omega}^{\lambda}, \quad x'_{\omega,i} = x\cdot\omega_i^{\perp} \tag{1.5.6}$$

When $\lambda = 1$ one obtains the null coordinates as in [54], [52].

For these frames we define the spaces $L^{\infty}_{t_\omega} L^2_{x^1_\omega, x'_\omega}, L^2_{t_\omega} L^{\infty}_{x^1_\omega, x'_\omega}$ in the usual way, which we denote $L^{\infty}_{t_{\omega,\lambda}} L^2_{x_{\omega,\lambda}}, L^2_{t_{\omega,\lambda}} L^{\infty}_{x_{\omega,\lambda}}$ to emphasize the dependence on $\lambda$.

## Pseudodifferential operators

To implement the renormalization we will use pseudodifferential operators. For symbols $a(x,\xi) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ one defines the left quantization $a(x,D)$ by

$$a(x,D)u = \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)\,\mathrm{d}\xi \tag{1.5.7}$$

while the right quantization $a(D,y)$ is defined by

$$a(D,y)u = \iint_{\mathbb{R}^d\times\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(y,\xi)u(y)\,\mathrm{d}y\,\mathrm{d}\xi. \tag{1.5.8}$$

Observe that $a(x,D)^* = \bar{a}(D,y)$. We will only work with symbols which are compactly supported in $\xi$.

## Bilinear forms

We denote by $\mathcal{L}$ a translation-invariant bilinear operator on $\mathbb{R}^d$ whose kernel has bounded mass, i.e.,

$$\mathcal{L}(f,g)(x) = \int K(x-y_1, x-y_2) f(y_1) g(y_2)\, \mathrm{d}y_1 \mathrm{d}y_2$$

where $K$ is a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with bounded mass. As a consequence, $\mathcal{L}(f,g)$ obeys a Hölder-type inequality

$$\|\mathcal{L}(f,g)\|_{L^p} \lesssim \|f\|_{L^{q_1}} \|g\|_{L^{q_2}} \tag{1.5.9}$$

for any exponents $1 \leq p, q_1, q_2 \leq \infty$ such that $p^{-1} = q_1^{-1} + q_2^{-2}$.

We say that the translation-invariant bilinear form $\mathcal{M}(\phi^1, \phi^2)$ has symbol $m(\xi_1, \xi_2)$ if

$$\mathcal{M}(\phi^1, \phi^2)(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{ix\cdot(\xi_1+\xi_2)} m(\xi_1, \xi_2) \hat{\phi}^1(\xi_1) \hat{\phi}^2(\xi_2)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2.$$

We make the analogous definition for functions defined on $\mathbb{R}^{1+d}$ and symbols $m(\Xi^1, \Xi^2)$ where $\Xi^i = (\tau_i, \xi_i)$.

## Stationary/non-stationary phase

We will bound oscillatory integrals using the stationary and non-stationary phase methods. For proofs of these two propositions as stated here see [21].

**Proposition 1.5.2.** *Suppose $K \subset \mathbb{R}^n$ is a compact set, $X$ is an open neighborhood of $K$ and $N \geq 0$. If $u \in C_0^N(K)$, $f \in C^{N+1}(X)$ and $f$ is real valued, then*

$$\left| \int e^{i\lambda f(x)} u(x)\, \mathrm{d}x \right| \leq C \frac{1}{\lambda^N} \sup_{|\alpha| \leq N} |D^\alpha u|\, |f'|^{|\alpha|-2N}, \qquad \lambda > 0 \tag{1.5.10}$$

*where $C$ is bounded when $f$ stays in a bounded set in $C^{N+1}(X)$.*

**Proposition 1.5.3** (Stationary phase)**.** *Suppose $K \subset \mathbb{R}^n$ is a compact set, $X$ is an open neighborhood of $K$ and $k \geq 1$. If $u \in C_0^{2k}(K)$, $f \in C^{3k+1}(X)$ and $f$ is real valued, $f'(x_0) = 0$, $det f''(x_0) \neq 0$, $f \neq 0$ in $K \setminus \{x_0\}$, then for $\lambda > 0$ we have*

$$\left| \int e^{i\lambda f(x)} u(x)\, \mathrm{d}x - e^{i\lambda f(x_0)} \left( \frac{\det(\lambda f''(x_0))}{2\pi i} \right)^{-\frac{1}{2}} \sum_{j<k} \frac{1}{\lambda^j} L_j u \right| \leq C \frac{1}{\lambda^k} \sum_{|\alpha| \leq 2k} \sup |D^\alpha u| \tag{1.5.11}$$

*where $C$ is bounded when $f$ stays in a bounded set in $C^{3k+1}(X)$ and $|x-x_0| / |f'(x)|$ has a uniform bound. $L_j$ are differential operators of order $2j$ acting on $u$ at $x_0$.*

Moreover, one controls derivatives in $\lambda$ (see [38, Lemma 2.35]):

$$\left| \partial_\lambda^j \int e^{i\lambda[f(x)-f(x_0)]} u(x)\, \mathrm{d}x \right| \leq C \frac{1}{\lambda^{\frac{n}{2}+j}}, \qquad j \geq 1. \tag{1.5.12}$$

## $L^p$ estimates

We will frequently use Bernstein's inequality, which states that

$$\|u\|_{L^q_x} \lesssim |V|^{\frac{1}{p}-\frac{1}{q}} \|u\|_{L^p_x}$$

when $\hat{u}$ is supported in a box of volume $V$ and $1 \leq p \leq q \leq \infty$. In particular,

$$\|P_k u\|_{L^q_x} \lesssim 2^{dk(\frac{1}{p}-\frac{1}{q})} \|P_k u\|_{L^p_x}, \qquad \|P_{\mathcal{C}_{k'}(\ell')} u\|_{L^q_x} \lesssim 2^{(dk'+(d-1)\ell')(\frac{1}{p}-\frac{1}{q})} \|P_{\mathcal{C}_{k'}(\ell')} u\|_{L^p_x}$$

For $L^2$ estimates we will rely on

**Lemma 1.5.4** (Schur's test)**.** *Let $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ and the operator $T$ defined by*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \, \mathrm{d}y$$

*which satisfies*

$$\sup_x \int |K(x,y)| \, \mathrm{d}y \leq M, \qquad \sup_y \int |K(x,y)| \, \mathrm{d}x \leq M.$$

*Then*

$$\|T\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq M$$

We also state a simple abstract summation lemma. Roughly speaking, it is the Cauchy-Schwarz inequality for an 'essentially diagonal' sum.

**Lemma 1.5.5.** *Let $\{a_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{b_\beta\}_{\beta \in \mathcal{B}}$ be (countably) indexed sequences of real numbers. Let $\mathcal{J} \subseteq \mathcal{A} \times \mathcal{B}$ be such that for each fixed $\alpha \in \mathcal{A}$, $|\#\{\beta : (\alpha,\beta) \in \mathcal{J}\}| \leq M$, and for each fixed $\beta \in \mathcal{B}$, $|\#\{\alpha : (\alpha,\beta) \in \mathcal{J}\}| \leq M$. Then we have*

$$|\sum_{\alpha,\beta \in \mathcal{J}} a_\alpha b_\beta| \leq M \Big( \sum_{\alpha \in \mathcal{A}} a_\alpha^2 \Big)^{1/2} \Big( \sum_{\beta \in \mathcal{B}} b_\beta^2 \Big)^{1/2}.$$

We omit the straightforward proof.

## Dyadic function spaces

Many function spaces we use will be defined *dyadically*, i.e., the norm of $f$ will be some summation of *dyadic norms* of $P_k f = f_k$. Formally, given a sequence of norms $(X_k)_{k \in \mathbb{Z}}$ or $(X_k)_{k \geq 0}$, $1 \leq p \leq \infty$ and $\sigma \in \mathbb{R}$, we denote by $\ell^p X^\sigma$ the norm

$$\|f\|_{\ell^p X^\sigma} = \Big( \sum_k (2^{\sigma k} \|P_k f\|_{X_k})^p \Big)^{1/p},$$

with the usual modification when $p = \infty$.

## Frequency envelopes

We borrow from [52] the notion of *frequency envelopes*, which is a convenient means to keep track of dyadic frequency profiles. Given $\delta > 0$, we say that a sequence $c = (c_k)_{k\in\mathbb{Z}}$ of positive numbers is a *$\delta$-admissible frequency envelope* if there exists $C_c > 0$ such that for every $k, k' \in \mathbb{Z}$, we have

$$|c_k/c_{k'}| \leq C_c\, 2^{\delta|k-k'|}.$$

Given a sequence $(X_k)_{k\in\mathbb{Z}}$ of dyadic norms, we define the $X_c$-norm as

$$\|f\|_{X_c} = \sup_{k\in\mathbb{Z}} c_k^{-1} \|P_k f\|_{X_k}.$$

Dyadically defined norms are controlled in terms of $c$ and $\|f\|_{X_c}$ in the obvious manner:

$$\|f\|_{\ell^p X^\sigma} \leq \Big( \sum_k (2^{\sigma k} c_k)^p \Big)^{1/p} \|f\|_{X_c}.$$

In the converse direction, we say that $c$ is a frequency envelope for $\|f\|_{\ell^p X^0}$ if

$$\|f\|_{\ell^p X^0} \simeq \Big( \sum_k c_k^p \Big)^{1/p}, \qquad \|P_k f\|_{X_k} \leq c_k.$$

Given any $f \in \ell^p X^0$, we can construct a $\delta$-admissible frequency envelope $c$ for $\|f\|_{\ell^p X^0}$ by defining

$$c_k = \sum_{k'} 2^{-\delta|k-k'|} \|P_{k'} f\|_{X_{k'}}. \tag{1.5.13}$$

By Young's inequality, this frequency envelope inherits any additional $\ell^{p'} X^\sigma$ regularity of $f$ for $1 \leq p' \leq \infty$ and $\sigma \in (-\delta, \delta)$, i.e.,

$$\|2^{\sigma k} c_k\|_{\ell^{p'}} \lesssim \|f\|_{\ell^{p'} X^\sigma}.$$

For the Klein-Gordon equation we will use the analogous notion of frequency envelopes, but indexed by non-negative integer instead. Thus, in this context, given $0 < \delta_1 < 1$, an admissible frequency envelope $(c_k)_{k\geq 0}$ is defined to be a sequence such that $c_p/c_k \leq C 2^{\delta_1|p-k|}$ for any $k, p \geq 0$.

We conclude this subsection with a discussion on simple operations on frequency envelopes. Given a $\delta$-admissible frequency envelope $c \in \ell^p$ $(1 \leq p \leq \infty)$, we may construct a new frequency envelope $\tilde{c}$ by taking $\tilde{c}_k = (\sum_{k'<k} c_{k'}^p)^{1/p}$. For any $\ell \geq 0$, we see (by shifting indices) that

$$|\tilde{c}_{k+\ell}/\tilde{c}_k| \leq C_c 2^{\delta\ell}, \qquad |\tilde{c}_{k-\ell}/\tilde{c}_k| \leq C_c 2^{\delta\ell}.$$

In other words, $\tilde{c}$ is also $\delta$-admissible.

For $\delta'$- and $\delta$-admissible frequency envelopes $b$ and $c$, we denote by $bc = (b_k c_k)_{k\in\mathbb{Z}}$ the product frequency envelope, which is clearly $(\delta+\delta')$-admissible.

By Cauchy-Schwarz inequality, note that the frequency envelopes $(\sum_{k'<k} b_{k'}c_{k'})_{k\in\mathbb{Z}}$ is dominated by $((\sum_{k'<k} b_{k'}^2)^{1/2}(\sum_{k'<k} c_{k'}^2)^{1/2})_{k\in\mathbb{Z}}$, i.e.,

$$\sum_{k'<k} b_{k'}c_{k'} \leq \Big(\sum_{k'<k} b_{k'}^2\Big)^{1/2}\Big(\sum_{k'<k} c_{k'}^2\Big)^{1/2}.$$

In particular, if $b, c \in \ell^2$, then $bc \in \ell^1$.

## 1.6 Parametrices

We now state the renormalization theorems, which are the massive, respectively half-wave, versions of the constructions in [47], [31].

As we mentioned earlier, as parts of the structure of the equations we have the following non-perturbative terms:

$$\sum_k A^\alpha_{<k-C}\partial_\alpha \bar{P}_k\phi$$

and

$$\sum_k \alpha^\mu P_{<k-C}A_\mu P_k\psi.$$

For the purpose of handling these terms, we define the paradifferential covariant Klein-Gordon operator

$$\Box_m^{p,A} = \Box + I - 2i\sum_{k\geq 0} A^j_{<k-C}\partial_j\bar{P}_k \tag{1.6.1}$$

and the paradifferential half-wave operators by

$$(i\partial_t \pm |D|)_A^p = i\partial_t \pm |D| \mp i\sum_{k\in\mathbb{Z}} P_{<k-C}A^j\frac{\partial_j}{|D|}P_k \tag{1.6.2}$$

where $A = A^{free} = (A_1, \dots, A_d, 0)$ is a real-valued 1-form defined on $\mathbb{R}^{1+d}$, assumed to solve the free wave equation and to obey the Coulomb gauge condition

$$\Box A = 0, \qquad \partial^j A_j = 0. \tag{1.6.3}$$

Consider the problem

$$\begin{cases} \Box_m^{p,A}\phi = F \\ \phi[0] = (f, g) \end{cases} \tag{1.6.4}$$

We have the following two solvability results, which we prove in chapter 4. The spaces are defined in chapter 2. We just mention here that $\bar{S}^\sigma$ and $S^{1/2}_\pm$ are the iteration spaces, while $\bar{N}^{\sigma-1}$ and $N^{1/2}_\pm$ are the spaces for the nonlinearity.

**Theorem 1.6.1.** *Let $A$ be a real 1-form obeying* (1.6.3) *on $\mathbb{R}^{d+1}$ for $d \geq 4$. If $\|A[0]\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}}$ is sufficiently small, then for any $F \in \bar{N}^{\sigma-1} \cap L^2 H^{\sigma-\frac{3}{2}}$ and $(f,g) \in H^\sigma \times H^{\sigma-1}$, the solution $\phi$ of* (1.6.4) *exists globally in time and it satisfies*

$$\|\phi\|_{\bar{S}^\sigma} \lesssim \|(f,g)\|_{H^\sigma \times H^{\sigma-1}} + \|F\|_{\bar{N}^{\sigma-1} \cap L^2 H^{\sigma-\frac{3}{2}}} \tag{1.6.5}$$

For the paradifferential half-wave operators operators, we have the following global solvability theorem which we state for $d = 4$. In the general case $d \geq 4$, the theorem holds with substitutions as in Remark 1.8.5.

**Theorem 1.6.2.** *Let $A^{free} = (0, A_1^{free}, \dots, A_4^{free})$ be a real-valued 1-form obeying $\Box A^{free} = 0$ and $\partial^\ell A_\ell^{free} = 0$. Consider the initial value problem*

$$\begin{cases} (i\partial_t + s|D|)^p_{A^{free}} \psi = F, \\ \psi(0) = \psi_0. \end{cases}$$

*If $\|A^{free}[0]\|_{\dot{H}^1 \times L^2}$ is sufficiently small, then for any $F \in N_s^{1/2} \cap L^2 L^2$ and any $\psi_0 \in \dot{H}^{1/2}$ there exists a global (in time) solution $\psi \in S_s^{1/2}$. Moreover, for any admissible frequency envelope $c$, we have*

$$\|\psi\|_{(S_s^{1/2})_c} \lesssim \|\psi_0\|_{\dot{H}_c^{1/2}} + \|F\|_{(N_s^{1/2} \cap L^2 L^2)_c}\ . \tag{1.6.6}$$

*In particular,*

$$\|\psi\|_{S_s^{1/2}} \lesssim \|\psi_0\|_{\dot{H}^{1/2}} + \|F\|_{N_s^{1/2} \cap L^2 L^2}. \tag{1.6.7}$$

Theorem 1.6.2 will be established in Chapter 4 by adapting the parametrix construction for the paradifferential covariant wave equation from [31], which is the massless analogue of Theorem 1.6.1.

## 1.7 Decomposition of the equations

In this section we describe the structure of the nonlinearities of the Maxwell-Klein-Gordon and Maxwell-Dirac equations in the Coulomb gauge.

### Maxwell equations in the Coulomb gauge

We begin by describing the Maxwell equations under the Coulomb condition $\partial^\ell A_\ell = 0$.

Let $J_\mu$ be a 1-form (called the *charge-current 1-form*) on $\mathbb{R}^{1+d}$ such that $\partial^\mu J_\mu = 0$. Consider the Maxwell equations

$$\partial^\mu F_{\nu\mu} = -J_\nu. \tag{1.7.1}$$

Under the Coulomb condition $\partial^\ell A_\ell = 0$, the Maxwell equations (1.7.1) reduce to

$$\Delta A_0 = J_0, \quad \Box A_j = \mathcal{P}_j J_x \tag{1.7.2}$$

where $\Delta := \partial^\ell \partial_\ell$ is the *Laplacian*, $\Box := \partial^\mu \partial_\mu$ is the *d'Alembertian* and $\mathcal{P}$ denotes the Leray projection (1.1.7). Moreover, thanks to $\partial^\mu J_\mu = 0$, we also obtain the following elliptic equation for $\partial_t A_0$:

$$\Delta(\partial_t A_0) = \partial^\ell J_\ell. \tag{1.7.3}$$

## The Maxwell nonlinearity of MKG

Let $(\phi, A)$ be a solution of MKG. The charge-current 1-form $J$ reads

$$J_\alpha = -\Im(\phi \overline{D_\alpha \phi}).$$

*Remark* 1.7.1. When $\phi$ solves a covariant equation $\Box_m^A \phi = 0$ for some real 1-form $A$, denoting the currents $J_\alpha = -\Im(\phi \overline{D_\alpha \phi})$, a simple computation shows $\partial^\alpha J_\alpha = 0$.

By (1.7.2), $A_\mu$ solves the following equations:

$$\begin{aligned} \Delta A_0 &= -\Im(\phi \overline{D_t \phi}), \\ \Box A_j &= -\mathcal{P}_j \Im(\phi \overline{D_x \phi}). \end{aligned}$$

Moreover, thanks to $\partial^\alpha J_\alpha = 0$, which holds by Remark 1.7.1, we have

$$\Delta(\partial_t A_0) = -\partial^i \Im(\phi \overline{D_i \phi}).$$

Momentarily ignoring the cubic terms $\phi \bar{\phi} A_\alpha$ from the products $\phi \overline{D_\alpha \phi}$, we define the main terms

$$\begin{aligned} \mathbf{A}_x(\phi^1, \phi^2) &:= -\Box^{-1} \mathcal{P}_j \Im(\phi^1 \nabla_x \bar{\phi}^2), \\ \mathbf{A}_0(\phi^1, \phi^2) &:= -\Delta^{-1} \Im(\phi^1 \partial_t \bar{\phi}^2). \end{aligned} \tag{1.7.4}$$

where here $\Box^{-1} f$ denotes the solution $\phi$ to the inhomogeneous wave equation $\Box \phi = f$ with $\phi[0] = 0$. Using the formula (1.1.7) for $\mathcal{P}_j$ one identifies the null structure (see (1.3.1))

$$\mathcal{P}_j(\phi^1 \nabla_x \phi^2) = \Delta^{-1} \nabla^i \mathcal{N}_{ij}(\phi^1, \phi^2). \tag{1.7.5}$$

*Remark* 1.7.2. Note that (1.7.5) shows that $\mathcal{P}_j(\phi^1 \nabla_x \phi^2)$ is a skew-symmetric bilinear form.

The main estimates for these nonlinear terms will be given by Proposition 1.8.1.

Moreover, to isolate the more delicate parts of (1.7.4), we define the operators

$$\begin{aligned} \mathcal{H}_{k'} L(\phi, \psi) &= \sum_{j < k' + C_2} P_{k'} Q_j L(\bar{Q}_{<j} \phi, \bar{Q}_{<j} \psi), \\ \mathcal{H} L(\phi, \psi) &= \sum_{\substack{k' < k_2 - C_2 - 10 \\ k' \in \mathbb{Z},\ k_1, k_2 \geq 0}} \mathcal{H}_{k'} L(\bar{P}_{k_1} \phi, \bar{P}_{k_2} \psi), \end{aligned}$$

## The Klein-Gordon nonlinearity

Moving on to the $\phi$ nonlinearity, we expand

$$\Box_m^A \phi = (\Box + 1)\phi + 2iA_\alpha \partial^\alpha \phi + i(\partial^\alpha A_\alpha)\phi - A^\alpha A_\alpha \phi.$$

When $A_x$ is divergence free, we can write $A_j = \mathcal{P}_j A$, which implies

$$A^i \partial_i \phi = \sum \mathcal{N}_{ij}\big(\nabla_i \Delta^{-1} A_j, \phi\big). \tag{1.7.6}$$

As discussed in the introduction, the most difficult interaction occurs when $A_0$ and $A_x$ have frequencies lower than $\phi$. To isolate this part, we introduce the low-high paradifferential operators

$$\pi[A]\phi := \sum_{k \geq 0} P_{<k-C} A_\alpha \, \partial^\alpha \bar{P}_k \phi, \tag{1.7.7}$$

Moreover, we define

$$\begin{aligned} \mathcal{H}^*_{k'} L(A,\phi) &= \sum_{j<k'+C_2} \bar{Q}_{<j} L(P_{k'} Q_j A, \bar{Q}_{<j}\phi), \\ \mathcal{H}^* L(A,\phi) &= \sum_{\substack{k'<k-C_2-10 \\ k' \in \mathbb{Z},\ k,\tilde{k} \geq 0}} \bar{P}_{\tilde{k}} \mathcal{H}^*_{k'} L(A, \phi_k). \end{aligned}$$

The necessary estimates are stated in Prop. 1.8.2 and 1.8.3.

It turns out that the worst part of (1.7.7) occurs for a subpart of $\pi[\mathbf{A}(\phi,\phi)]\phi$, namely $\mathcal{H}^*\pi[\mathcal{H}\mathbf{A}(\phi,\phi)]\phi$. To estimate it, we will need to use the trilinear null structure identified by Machedon-Sterbenz [34] which we present now, following [31].

One can write

$$\mathbf{A}^\alpha(\phi^1,\phi^2)\partial_\alpha \phi = (\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3)(\phi^1,\phi^2,\phi) \tag{1.7.8}$$

where

$$\begin{aligned} \mathcal{Q}_1(\phi^1,\phi^2,\phi) &:= -\Box^{-1}\mathfrak{I}(\phi^1 \partial_\alpha \bar{\phi}^2) \cdot \partial^\alpha \phi, \\ \mathcal{Q}_2(\phi^1,\phi^2,\phi) &:= \Delta^{-1}\Box^{-1}\partial_t \partial_\alpha \mathfrak{I}(\phi^1 \partial_\alpha \bar{\phi}^2) \cdot \partial_t \phi, \\ \mathcal{Q}_3(\phi^1,\phi^2,\phi) &:= \Delta^{-1}\Box^{-1}\partial_\alpha \partial^i \mathfrak{I}(\phi^1 \partial_i \bar{\phi}^2) \cdot \partial^\alpha \phi. \end{aligned} \tag{1.7.9}$$

Indeed, plugging in the Hodge projection $\mathcal{P} = I - \nabla\Delta^{-1}\nabla$ one writes

$$\mathbf{A}^\alpha(\phi^1,\phi^2)\partial_\alpha \phi = \Delta^{-1}\mathfrak{I}(\phi^1 \partial_t \bar{\phi}^2) \cdot \partial_t \phi - \Box^{-1}\mathfrak{I}(\phi^1 \partial_i \bar{\phi}^2) \cdot \partial^i \phi + \frac{\partial^i \partial^j}{\Delta \Box} \mathfrak{I}(\phi^1 \partial_i \bar{\phi}^2) \cdot \partial_j \phi.$$

Now add and subtract $\Box^{-1}\mathfrak{I}(\phi^1 \partial_t \bar{\phi}^2) \cdot \partial_t \phi$ and write

$$\mathbf{A}^\alpha(\phi^1,\phi^2)\partial_\alpha \phi = \mathcal{Q}_1(\phi^1,\phi^2,\phi) + \mathcal{N}.$$

Using $\Delta^{-1} - \Box^{-1} = -\partial_t^2 \Delta^{-1}\Box^{-1}$, $\mathcal{N}$ takes the form

$$\mathcal{N} = -\Delta^{-1}\Box^{-1}\partial_t^2 \Im(\phi^1 \partial_t \bar{\phi^2}) \cdot \partial_t \phi + \Delta^{-1}\Box^{-1}\partial^i\partial^j \Im(\phi^1 \partial_i \bar{\phi^2}) \cdot \partial_j \phi.$$

Adding and subtracting $\Delta^{-1}\Box^{-1}\partial^i\partial_t \Im(\phi^1 \partial_i \bar{\phi^2}) \cdot \partial_t \phi$ we get

$$\mathcal{N} = \mathcal{Q}_2(\phi^1, \phi^2, \phi) + \mathcal{Q}_3(\phi^1, \phi^2, \phi),$$

thus obtaining (1.7.8).

## Diagonalization of the Dirac equation

Our next goal is to rewrite the Dirac operator $\alpha^\mu \partial_\mu$ in a diagonal form. We follow the approach of D'Ancona, Foschi and Selberg [9, 10].

For $\mu = 0, \dots, d$, recall the definition $\alpha^\mu = \gamma^0\gamma^\mu$. Hence $\alpha^0 = \mathrm{I}_{4\times 4}$, whereas $\alpha^j$ are hermitian matrices satisfying

$$\frac{1}{2}(\alpha^j \alpha^k + \alpha^k \alpha^j) = \delta^{jk} \mathrm{I}_{4\times 4}, \tag{1.7.10}$$

thanks to (1.1.1) and (1.1.2). Note that the Dirac operator $\alpha^\mu \partial_\mu$ then takes the form

$$\alpha^\mu \partial_\mu = -i(i\partial_t - \alpha^j D_j).$$

where we use the notation $D_\mu = \frac{1}{i}\partial_\mu$. To diagonalize the operator $\alpha^j D_j$, whose symbol is $\alpha^j \xi_j$, we introduce the multiplier $\Pi(D)$ with symbol

$$\Pi(\xi) := \frac{1}{2}\Big(\mathrm{I}_{4\times 4} - \frac{\alpha^j \xi_j}{|\xi|}\Big).$$

Note that $\Pi(\xi)$ obeys the identities

$$\Pi(\xi)^\dagger = \Pi(\xi), \quad \Pi(\xi)^2 = \Pi(\xi), \quad \Pi(\xi)\Pi(-\xi) = 0.$$

For each sign $s \in \{+, -\}$, we define the multipliers $\Pi_s$ with symbols $\Pi_s(\xi) := \Pi(s\xi)$. By the preceding identities, $\Pi_+$ and $\Pi_-$ form orthogonal projections (i.e., $\Pi_s^\dagger = \Pi_s$, $\Pi_s^2 = \Pi_s$ and $\Pi_+\Pi_- = 0$). Moreover, we have

$$\mathrm{I}_{4\times 4} = \Pi_+(\xi) + \Pi_-(\xi), \quad -\frac{\alpha^j \xi_j}{|\xi|} = \Pi_+(\xi) - \Pi_-(\xi)$$

Thus the Dirac operator can now be written in the form

$$\alpha^\mu \partial_\mu = -i\Big((i\partial_t + |D|)\Pi_+(D) + (i\partial_t - |D|)\Pi_-(D)\Big). \tag{1.7.11}$$

We now present the key identities for revealing the null structure of (MD), which are essentially due to D'Ancona, Foschi and Selberg [9, 10]. We define the self-adjoint operators $\mathcal{R}_\mu$ as

$$\mathcal{R}_\mu := \frac{D_\mu}{|D|} \quad \text{for } \mu = 0, \dots, d.$$

For $\mu = j \in \{1, \dots, d\}$, the operators $\mathcal{R}_j$ are precisely the (self-adjoint) Riesz transforms.

**Lemma 1.7.3.** *For each $\mu = j \in \{1, \dots, d\}$ and sign $s \in \{+, -\}$, we have*

$$\alpha^j \Pi_s = -s\mathcal{R}^j + \Pi_{-s}\alpha^j. \tag{1.7.12}$$

*Proof.* We compute

$$\alpha^j \Pi_s(\xi) - \Pi_{-s}(\xi)\alpha^j = -s\frac{1}{2}\frac{\xi_k}{|\xi|}(\alpha^j\alpha^k + \alpha^k\alpha^j) = -s\frac{\xi^j}{|\xi|}.$$

□

*Remark* 1.7.4. For $\mu = 0$, the analogue of (1.7.12) is

$$\alpha^0 = -s\mathcal{R}^0 + s\frac{i\partial_t + s|D|}{|D|}, \tag{1.7.13}$$

which can be easily justified.

The Riesz transform term $\mathcal{R}^\mu$ is *scalar* in the sense that it does not involve multiplication by $\alpha^j$. Its contribution in (MD) resembles the Maxwell-Klein-Gordon system; Remarkably, the other terms in (1.7.12) and (1.7.13) turn out to contribute parts with more favorable structure. Indeed, in the case of (1.7.13), the presence of the half-wave operator $i\partial_t + s|D|$ (with an appropriate sign $s$) makes this term effectively higher order. In the case of (1.7.12), the following lemma can be used to uncover a null structure.

**Lemma 1.7.5.** *For $z \in \mathbb{C}^N$, $\xi, \eta \in \mathbb{R}^d$ and $\theta := |\angle(\xi, \eta)|$, we have*

$$|\Pi(\xi)\Pi(-\eta)| \leq C\theta. \tag{1.7.14}$$

*Proof.* Using (1.7.10) and the definition of $\Pi(\xi)$, we compute

$$\begin{aligned}\Pi(\xi)\Pi(-\eta) =& \frac{1}{4}\Big(\mathrm{I}_{4\times4} - \frac{\alpha^j\xi_j}{|\xi|}\Big)\Big(\mathrm{I}_{4\times4} + \frac{\alpha^k\eta_k}{|\eta|}\Big) = \frac{1}{4}\Big(\mathrm{I}_{4\times4} - \frac{\alpha^j\xi_j}{|\xi|} + \frac{\alpha^k\eta_k}{|\eta|} - \frac{\alpha^j\alpha^k\xi_j\eta_k}{|\xi||\eta|}\Big)\\ =& -\frac{\alpha^j}{4}\Big(\frac{\xi_j}{|\xi|} - \frac{\eta_j}{|\eta|}\Big) - \frac{\alpha^j\alpha^k}{8}\Big(\frac{\xi_j\eta_k - \xi_k\eta_j}{|\xi||\eta|}\Big) + \frac{\mathrm{I}_{4\times4}}{4}\Big(\frac{|\xi||\eta| - \xi\cdot\eta}{|\xi||\eta|}\Big).\end{aligned}$$

Then the lemma follows. □

We remark that the identity (1.7.13) must be applied judiciously, since $\mathcal{R}^0$ is well-behaved on $\psi_\pm$ only when the modulation does not exceed the spatial frequency.

## Maxwell-Dirac

We are now ready to describe in detail the nonlinearity of the Maxwell-Dirac equation in the Coulomb gauge (MD-CG).

As explained in the introduction, our overall philosophy is that MD-CG can be split into two parts: The scalar part, which does not involve multiplication by the matrix $\alpha^j$, and the spinorial part arising from the spinorial nature of the Dirac equation. The latter part turns out to possess a more favorable null structure; in particular, there is no need to perform a paradifferential renormalization, nor to use a secondary null structure. On the other hand, the former part is deeply related to the massless Maxwell-Klein-Gordon equation in the Coulomb gauge, whose small Sobolev critical global well-posedness was proved in [31]. We refer to Remarks 1.7.9 and 5.4.5 for a further discussion after the nonlinearity of MD-CG is completely described.

## The Maxwell nonlinearity of MD

Let $(A, \psi)$ be a solution to MD-CG. The charge-current 1-form $J$ reads

$$J^\mu = \langle \gamma^\mu \psi, \gamma^0 \psi \rangle = \langle \psi, \alpha^\mu \psi \rangle.$$

where we used (1.1.1), (1.1.2) and the definition of $\alpha^\mu$ in the second identity. By (1.7.2), $A_\mu$ solves the following equations:

$$\Delta A_0 = \langle \psi, \alpha_0 \psi \rangle = -\langle \psi, \alpha^0 \psi \rangle = -\langle \psi, \psi \rangle, \tag{1.7.15}$$

$$\Box A_j = \mathcal{P}_j \langle \psi, \alpha_x \psi \rangle. \tag{1.7.16}$$

Moreover, thanks to $\partial^\mu J_\mu = 0$ (which holds since $\psi$ solves a covariant Dirac equation, see remark 1.7.6), we have

$$\Delta(\partial_t A_0) = \partial^\ell \langle \psi, \alpha_\ell \psi \rangle. \tag{1.7.17}$$

We now introduce bilinear version of the nonlinearities in (1.7.15), (1.7.16) and (1.7.17), in order to set up an iteration scheme for solving MD-CG. Let $\varphi^1, \varphi^2$ be any spinor fields. For (1.7.15), we introduce

$$\mathcal{M}^E(\varphi^1, \varphi^2) := -\langle \varphi^1, \varphi^2 \rangle. \tag{1.7.18}$$

We also define

$$\mathbf{A}_0(\varphi^1, \varphi^2) := \Delta^{-1} \mathcal{M}^E(\varphi^1, \varphi^2), \tag{1.7.19}$$

so that $A_0 = \mathbf{A}_0(\psi, \psi)$ for a solution $(A_\mu, \psi)$ to MD-CG.

For (1.7.16), we use (1.7.12) to decompose the nonlinearity as

$$\mathcal{P}_j \langle \psi, \alpha_x \psi \rangle = \sum_s \mathcal{P}_j \langle \psi, \alpha_x \Pi_s \psi \rangle = \sum_s \Big( - s\mathcal{M}^R_j(\psi, \psi) + \mathcal{M}^S_{j,s}(\psi, \psi) \Big),$$

where

$$\mathcal{M}^R_j(\varphi^1, \varphi^2) := \mathcal{P}_j \langle \varphi^1, \mathcal{R}_x \varphi^2 \rangle, \tag{1.7.20}$$

$$\mathcal{M}^S_{j,s}(\varphi^1, \varphi^2) := \mathcal{P}_j \langle \varphi^1, \Pi_{-s} \alpha_x \varphi^2 \rangle. \tag{1.7.21}$$

We refer to $\mathcal{M}^R_j$ and $\mathcal{M}^S_{j,s}$ as the *scalar* and *spinorial* parts, respectively, of the Maxwell nonlinearity; observe that the scalar part does not involve the matrix $\alpha^j$. We also introduce

$$\mathbf{A}_j(\varphi^1, \varphi^2) := \Box^{-1} \mathcal{P}_j \langle \varphi^1, \alpha_x \varphi^2 \rangle, \tag{1.7.22}$$

$$\mathbf{A}^R_j(\varphi^1, \varphi^2) := \Box^{-1} \mathcal{M}^R_j(\varphi^1, \varphi^2), \tag{1.7.23}$$

$$\mathbf{A}^S_{j,s}(\varphi^1, \varphi^2) := \Box^{-1} \mathcal{M}^S_{j,s}(\varphi^1, \varphi^2) \tag{1.7.24}$$

For a solution $(A_\mu, \psi)$ to MD-CG, we have

$$A_j = A_j^{free} + \mathbf{A}_j(\psi,\psi) = A_j^{free} + \sum_s \Big( - s\mathbf{A}_j^R(\psi, \Pi_s\psi) + \mathbf{A}_{j,s}^S(\psi, \Pi_s\psi)\Big)$$

where $A_j^{free}$ is the free wave with data $A_j^{free}[0] = A_j[0]$.

Finally, corresponding to (1.7.17) we define

$$\partial_t \mathcal{M}^E(\varphi^1, \varphi^2) := \partial^\ell \langle \varphi^1, \alpha_\ell \varphi^2\rangle, \tag{1.7.25}$$

so that $\Delta(\partial_t A_0) = \partial_t \mathcal{M}^E(\psi,\psi)$ for a solution $(A_\mu, \psi)$ to MD-CG.

*Remark* 1.7.6. The notation $\partial_t$ in $\partial_t\mathcal{M}$ is merely formal; the actual $\partial_t$ derivative of $\mathcal{M}^E(\varphi^1,\varphi^2)$ agrees with $\partial_t\mathcal{M}^E(\varphi^1,\varphi^2)$ only if

$$\partial_\mu \langle \varphi^1, \alpha^\mu \varphi^2\rangle = 0.$$

Such an identity holds if, for instance, $\varphi^1$ and $\varphi^2$ obey a (single) covariant Dirac equation $\alpha^\mu(\partial_\mu + i\tilde{A}_\mu)\varphi = 0$ for some connection 1-form $\tilde{A}$, which is not necessarily equal to $A$. We will be careful to ensure that this is the case in our iteration scheme.

## The Dirac nonlinearity

We now turn to the covariant Dirac equation

$$\alpha^\mu \mathbf{D}_\mu \psi = 0. \tag{1.7.26}$$

Expanding $\mathbf{D}_\mu = \partial_\mu + iA_\mu$ and using (1.7.11), we may rewrite the above equation as

$$(i\partial_t + s|D|)\psi_s = \Pi_s(\alpha^\mu A_\mu \psi). \tag{1.7.27}$$

where $s \in \{+,-\}$ and $\psi_s$ is the abbreviation $\psi_s := \Pi_s\psi$. In view of the half-wave decomposition, it is natural to expand $\psi = \psi_+ + \psi_-$ on the RHS of (1.7.27). Using Lemma 1.7.3, as well as the formulae $A_j = \mathcal{P}_j A_x$ and $\psi_s = \Pi_s\psi_s$, we further decompose each of the nonlinearity $\alpha^\mu A_\mu \psi_s$ as

$$\begin{aligned}\alpha^\mu A_\mu \psi_s =& A_0 \Pi_s \psi_s + A_j \alpha^j \Pi_s \psi_s \\ =& \mathcal{N}^E(A_0, \Pi_s\psi_s) - s\mathcal{N}^R(A_x, \psi_s) + \mathcal{N}_s^S(A_x, \psi_s),\end{aligned}$$

where $\mathcal{N}^E$, $\mathcal{N}^R$ and $\mathcal{N}_s^S$ are bilinear forms defined as follows:

$$\mathcal{N}^E(A_0, \varphi) := A_0 \varphi, \tag{1.7.28}$$
$$\mathcal{N}^R(A_x, \varphi) := (\mathcal{P}_j A_x)(\mathcal{R}^j \varphi), \tag{1.7.29}$$
$$\mathcal{N}_s^S(A_x, \varphi) := A_j \Pi_{-s}(\alpha^j \varphi). \tag{1.7.30}$$

We refer to $\mathcal{N}^E, \mathcal{N}^R$ as the *scalar* part of the Dirac nonlinearity, as it does not involve multiplication by $\alpha^\mu$. The remainder $\mathcal{N}_s^S$ is called the *spinorial* part.

We summarize the result of our decomposition so far as follows.

**Lemma 1.7.7.** *Let $\psi$ be a spinor field on $\mathbb{R}^{1+d}$ and $A_\mu$ be a real-valued 1-form obeying $A_j = \mathcal{P}_j A_x$. If $\psi$ is a solution to* (1.7.26)*, then each of $\psi_s = \Pi_s \psi$ $(s \in \{+,-\})$ solves*

$$\begin{aligned}&\Pi_s(i\partial_t + s|D|)\psi_s \\ &= \Pi_s \sum_{s'} \Big( \mathcal{N}^E(A_0, \Pi_{s'}\psi_{s'}) - s'\mathcal{N}^R(A_x, \psi_{s'}) + \mathcal{N}^S_{s'}(A_x, \psi_{s'}) \Big).\end{aligned} \tag{1.7.31}$$

*Conversely, if $(\psi_+, \psi_-)$ is a pair of spinor fields solving* (1.7.31)*, then $\psi := \Pi_+\psi_+ + \Pi_-\psi_-$ is a solution to* (1.7.26)*.*

*Remark* 1.7.8. In the converse statement, $\psi_s$ need not belong to the image of $\Pi_s$, i.e., $\Pi_s\psi_s$ need not equal $\psi_s$ for $s \in \{+,-\}$.

*Proof.* The direct statement has already been proved. To prove the converse statement, we begin by noticing that

$$-s'\mathcal{N}^R(A_x, \psi_{s'}) + \mathcal{N}^S_{s'}(A_x, \psi_{s'}) = A_j \alpha^j \Pi_{s'}\psi_{s'}$$

by Lemma 1.7.3 and $A_j = \mathcal{P}_j A_x$. Therefore, (1.7.31) implies

$$(i\partial_t + s|D|)\Pi_s\psi_s = \Pi_s \Big( A_0\alpha^0(\Pi_+\psi_+ + \Pi_-\psi_-) + A_j\alpha^j(\Pi_+\psi_+ + \Pi_-\psi_-) \Big).$$

Defining $\psi := \Pi_+\psi_+ + \Pi_-\psi_-$, adding up the preceding equation for $s \in \{+,-\}$ and using (1.7.11), the desired statement follows. ☐

As discussed earlier, the most difficult interaction is when $A_0$ and $A_x$ have frequencies lower than $\psi_s$. To isolate this part, we introduce the low-high paradifferential operators

$$\begin{aligned}\pi^E[A_0]\varphi &:= \sum_k \mathcal{N}^E(P_{<k-10}A_0, P_k\varphi) = \sum_k P_{<k-10}A_0\, P_k\varphi, \\ \pi^R[A_x]\varphi &:= \sum_k \mathcal{N}^R(P_{<k-10}A_x, P_k\varphi) = \sum_k \mathcal{P}_j P_{<k-10}A_x\, \mathcal{R}^j P_k\varphi, \\ \pi^S_s[A_x]\varphi &:= \sum_k \mathcal{N}^S_s(P_{<k-10}A_x, P_k\varphi) = \sum_k P_{<k-10}A_j\, \Pi_{\mp}(\alpha^j P_k\varphi).\end{aligned}$$

and the remainders $\widetilde{\mathcal{N}}^E$, $\widetilde{\mathcal{N}}^R$ and $\widetilde{\mathcal{N}}^S_s$ consisting of

$$\begin{aligned}\widetilde{\mathcal{N}}^E(A_0, \varphi) &:= \sum_k \mathcal{N}^E(P_{\geq k-10}A_0, P_k\varphi) = \sum_k P_{\geq k-10}A_0\, P_k\varphi, \\ \widetilde{\mathcal{N}}^R(A_x, \varphi) &:= \sum_k \mathcal{N}^R(P_{\geq k-10}A_x, P_k\varphi) = \sum_k \mathcal{P}_j P_{\geq k-10}A_x\, \mathcal{R}^j P_k\varphi, \\ \widetilde{\mathcal{N}}^S_s(A_x, \varphi) &:= \sum_k \mathcal{N}^S_s(P_{\geq k-10}A_x, P_k\varphi) = \sum_k P_{\geq k-10}A_j\, \Pi_{\mp}(\alpha^j P_k\varphi).\end{aligned}$$

We also recall the paradifferential covariant half-wave operator

$$(i\partial_t + s|D|)^p_{A^{free}} = (i\partial_t + s|D|) + s\sum_k \mathcal{P}_j P_{<k-5} A_x^{free} \mathcal{R}^j P_k. \tag{1.7.32}$$

so that we have

$$(i\partial_t + s|D|)^p_{A^{free}} = (i\partial_t + s|D|) + s\pi^R[A_x^{free}].$$

## Parallelism between MD and MKG

*Remark* 1.7.9. We are now ready to exhibit more concretely the parallelism between MKG in the Coulomb gauge and the scalar part of MD-CG.

We start with MD-CG. Applying (1.7.12), (1.7.13) to the equations for $A_0$ and keeping only the Riesz transform terms, we get

$$\Delta A_0 = -\sum_{s,s'} s' \langle \psi_s, \mathcal{R}_0 \psi_{s'} \rangle + \cdots \tag{1.7.33}$$

Furthermore, consider the equations for $A_x$ and $\psi$ with the spinorial parts $\mathbf{A}^S$ and $\mathcal{N}^S_\pm$ removed. Using also (1.7.13) to the term $A_0\alpha^0\psi$ in the Dirac equation and throwing away the second term in (1.7.13), we arrive at the equations

$$\begin{aligned} \Box A_j &= -\sum_{s,s'} s' \mathcal{P}_j \langle \psi_s, \mathcal{R}_x \psi_{s'} \rangle + \cdots \\ (i\partial_t + s|D|)\psi_s &= -\Pi_s \sum_{s'} s' A_\mu \mathcal{R}^\mu \psi_{s'} + \cdots \end{aligned} \tag{1.7.34}$$

On the other hand, observe that MKG takes the form

$$\begin{cases} \Delta A_0 = -\operatorname{Im}(\phi \overline{\mathbf{D}_0 \phi}) \\ \Box A_j = -\mathcal{P}_j \operatorname{Im}(\phi \overline{\mathbf{D}_x \phi}) \\ \Box \phi = -2iA_\mu \partial^\mu \phi + i\partial_0 A_0 \phi + A_\mu A^\mu \phi \end{cases} \tag{MKG-CG}$$

Using the half-wave decomposition $\phi_s = \frac{1}{2}(\phi + s\frac{\partial_t}{i|D|}\phi)$ $(s \in \{+,-\})$ and keeping only the quadratic nonlinearities (except $\partial_0 A_0 \phi$, which is harmless), we arrive at

$$\begin{aligned} \Delta A_0 &= -\sum_{s,s'} \operatorname{Im}(\phi_s \overline{\partial_0 \phi_{s'}}) + \cdots \\ \Box A_j &= -\sum_{s,s'} \mathcal{P}_j \operatorname{Im}(\phi_s \overline{\partial_x \phi_{s'}}) + \cdots \\ (i\partial_t + s|D|)\phi_s &= \frac{s}{|D|} \sum_{s'} iA_\mu \partial^\mu \phi_{s'} + \cdots \end{aligned} \tag{1.7.35}$$

Modulo constant factors and balance of derivatives, observe the similarity between (1.7.33)–(1.7.34) and (1.7.35). This similarity will be exploited below to prove a crucial trilinear null form estimate (Proposition 1.8.8) and solvability of covariant Dirac equation (Proposition 1.8.11).

## 1.8 Main estimates

In this section we collect the bilinear and trilinear estimates needed to prove Theorems 1.1.1 and 1.1.2. The spaces are defined in chapter 2. We just mention here that $\bar{S}^\sigma$ and $S^{1/2}_\pm$ are the iteration spaces, while $\bar{N}^{\sigma-1}$ and $N^{1/2}_\pm$ are the spaces for the nonlinearity.

For the nonlinearities of the Maxwell parts of MKG we have the following proposition which, in particular, gives control of the terms $(\mathbf{A}_0, \mathbf{A}_x)(\phi^1, \phi^2)$ from (1.7.4).

**Proposition 1.8.1.** *One has the following estimates:*

$$\|\mathcal{P}_j(\phi^1 \nabla_x \phi^2)\|_{\ell^1 N^{\sigma-1}} \lesssim \|\phi^1\|_{\bar{S}^\sigma} \|\phi^2\|_{\bar{S}^\sigma} \tag{1.8.1}$$

$$\|\phi^1 \nabla_{t,x} \phi^2\|_{\ell^1 (L^2 \dot{H}^{\sigma-\frac{3}{2}} \cap L^\infty \dot{H}^{\sigma-2})} \lesssim \|\phi^1\|_{\bar{S}^\sigma} \|\phi^2\|_{\bar{S}^\sigma} \tag{1.8.2}$$

$$\|\phi^1 \phi^2 A\|_{\ell^1 (L^1 \dot{H}^{\sigma-1} \cap L^2 \dot{H}^{\sigma-\frac{3}{2}} \cap L^\infty \dot{H}^{\sigma-2})} \lesssim \|\phi^1\|_{\bar{S}^\sigma} \|\phi^2\|_{\bar{S}^\sigma} \|A\|_{S^\sigma \times Y^\sigma} \tag{1.8.3}$$

Moving on to the Klein-Gordon nonlinearity, we recall definition of $\mathcal{H}^*$ and $\pi[A]\phi$ from (1.7.7). Then we have:

**Proposition 1.8.2.**

1. *For all $\phi$ and $A = (A_x, A_0)$ such that $\partial_j A_j = 0$ one has the null form estimates:*

$$\|A_\alpha \partial^\alpha \phi - \pi[A]\phi\|_{\bar{N}^{\sigma-1}} \lesssim \|A\|_{S^\sigma \times Y^\sigma} \|\phi\|_{\bar{S}^\sigma} \tag{1.8.4}$$

$$\|(I - \mathcal{H}^*)\pi[A]\phi\|_{\bar{N}^{\sigma-1}} \lesssim \|A\|_{\ell^1 S^\sigma \times Y^\sigma} \|\phi\|_{\bar{S}^\sigma} \tag{1.8.5}$$

$$\|\mathcal{H}^* \pi[A]\phi\|_{\bar{N}^{\sigma-1}} \lesssim \|A\|_{Z^\sigma \times Z^\sigma_{ell}} \|\phi\|_{\bar{S}^\sigma} \tag{1.8.6}$$

2. *For all $\phi$ and $A = (A_x, A_0)$ one has*

$$\|A^\alpha \partial_\alpha \phi\|_{L^2 H^{\sigma-\frac{3}{2}}} \lesssim \|A\|_{S^\sigma \times Y^\sigma} \|\phi\|_{\bar{S}^\sigma} \tag{1.8.7}$$

$$\|\partial_t A_0 \phi\|_{\bar{N}^{\sigma-1} \cap L^2 H^{\sigma-\frac{3}{2}}} \lesssim \|A_0\|_{Y^\sigma} \|\phi\|_{\bar{S}^\sigma} \tag{1.8.8}$$

$$\|A^1_\alpha A^2_\alpha \phi\|_{\bar{N}^{\sigma-1} \cap L^2 H^{\sigma-\frac{3}{2}}} \lesssim \|A^1\|_{S^\sigma \times Y^\sigma} \|A^2\|_{S^\sigma \times Y^\sigma} \|\phi\|_{\bar{S}^\sigma}. \tag{1.8.9}$$

The following trilinear bound contains the more delicate estimates occurring in our system. It relies crucially on the cancelation structure given by (1.7.8) and to handle it we will need the norms $L^\infty_{t_{\omega,\lambda}} L^2_{x_{\omega,\lambda}}$, $L^2_{t_{\omega,\lambda}} L^\infty_{x_{\omega,\lambda}}$, the Lorentz norms $L^1 L^{2,1}$, $L^2 L^{4,2}$ as well as the bilinear forms from chapter 5.

**Proposition 1.8.3.** *For $\boldsymbol{A}$ and $\pi$ defined by* (1.7.4) *and* (1.7.7) *one has:*

$$\|\pi[\boldsymbol{A}(\phi^1, \phi^2)]\phi\|_{\bar{N}^{\sigma-1}} \lesssim \|\phi^1\|_{\bar{S}^\sigma} \|\phi^2\|_{\bar{S}^\sigma} \|\phi\|_{\bar{S}^\sigma} \tag{1.8.10}$$

The remaining term $\pi[A^{free}]\phi$ contains the non-perturbative part and to handle it we will use Theorem 1.6.1.

We pause here to give an example of using the estimates above together with Theorem 1.6.1 to solve the Cauchy problem for $\Box_m^A \phi = F$, in the particular case $A = A^{free}$, which will be useful below.

**Proposition 1.8.4.** *Let $A = A^{free}$ be a real 1-form obeying $\Box A = 0,\ \partial^j A_j = 0,\ A_0 = 0$. If $\|A[0]\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}}$ is sufficiently small, then for any $\phi[t_0] \in H^\sigma \times H^{\sigma-1}$ and any $F \in \bar{N}^{\sigma-1} \cap L^2 H^{\sigma-\frac{3}{2}}$, the solution of $\Box_m^A \phi = F$ with data $\phi[t_0]$ satisfies:*

$$\|\phi\|_{\bar{S}^\sigma} \lesssim \|\phi[t_0]\|_{H^\sigma \times H^{\sigma-1}} + \|F\|_{\bar{N}^{\sigma-1} \cap L^2 H^{\sigma-\frac{3}{2}}} \tag{1.8.11}$$

*Proof.* We show that the mapping $\psi \mapsto \phi$ given by $\Box_m^{p,A} \phi = F + \bar{\mathcal{M}}(A,\psi)$ with data $\phi[t_0]$ at $t = t_0$ is a contraction on $\bar{S}^\sigma$, where

$$\bar{\mathcal{M}}(A,\psi) = 2i(A_\alpha \partial^\alpha \psi - \pi[A]\psi) - A^\alpha A_\alpha \psi$$

is chosen so that $\bar{\mathcal{M}}(A,\psi) = \Box_m^{p,A}\psi - \Box_m^A \psi$. Using (1.8.4), (1.8.7), (1.8.9), noting that $\|A\|_{S^\sigma \times Y^\sigma} \lesssim \|A[0]\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} \leq \varepsilon \ll 1$ (since $A_0 = 0$) we obtain

$$\|\bar{\mathcal{M}}(A,\psi)\|_{\bar{N}^{\sigma-1} \cap L^2 H^{\sigma-\frac{3}{2}}} \lesssim \varepsilon \|\psi\|_{\bar{S}^\sigma}$$

which together with Theorem 1.6.1 proves the existence of $\phi$ for $\varepsilon$ small enough. The same estimates imply (1.8.11). $\Box$

Now we collect the ingredients needed to prove Theorem 1.1.2. For the sake of concreteness, we restrict to the case $d = 4$ unless otherwise stated. We use the language of frequency envelopes, which is a convenient way of expressing the weak interaction among different dyadic frequency pieces.

*Remark* 1.8.5. In the case of a general dimension $d \geq 4$, all the estimates below hold with the following substitutions:

$$\begin{aligned} &L^2 \dot{H}^{-3/2} \to L^2 \dot{H}^{\frac{d-7}{2}}, \quad L^2 \dot{H}^{-1/2} \to L^2 \dot{H}^{\frac{d-5}{2}}, \quad L^2 L^2 \to L^2 \dot{H}^{\frac{d-4}{2}}, \\ &N \to N^{\frac{d-4}{2}}, \quad N_s^{1/2} \to N^{\frac{d-3}{2}}, \quad G^{1/2} \to G^{\frac{d-3}{2}}, \\ &S^1 \to S^{\frac{d-2}{2}}, \quad Y^1 \to Y^{\frac{d-2}{2}}, \quad \tilde{S}_s^{1/2} \to \tilde{S}_s^{\frac{d-3}{2}}. \end{aligned}$$

See Remarks 6.3.8, 6.4.3 and 4.3.1.

For the nonlinearity in the $A_0$ and $A_x$ equations, we have the following bilinear estimates.

**Proposition 1.8.6.** *For any admissible frequency envelopes $b, c$ and signs $s, s' \in \{+, -\}$, we have*

$$\|\mathcal{M}^E(\psi, \varphi)\|_{(L^2\dot{H}^{-1/2})_{bc}} + \|\partial_t \mathcal{M}^E(\psi, \varphi)\|_{(L^2\dot{H}^{-3/2})_{bc}} \lesssim \|\psi\|_{(\tilde{S}^{1/2}_s)_b} \|\varphi\|_{(\tilde{S}^{1/2}_{s'})_c}. \tag{1.8.12}$$

$$\|\mathcal{M}^R_j(\psi, \varphi)\|_{(N \cap L^2\dot{H}^{-1/2})_{bc}} \lesssim \|\psi\|_{(\tilde{S}^{1/2}_s)_b} \|\varphi\|_{(\tilde{S}^{1/2}_{s'})_c}, \tag{1.8.13}$$

$$\|\mathcal{M}^S_{j,s'}(\Pi_s\psi, \varphi)\|_{(N \cap L^2\dot{H}^{-1/2})_{bc}} \lesssim \|\psi\|_{(\tilde{S}^{1/2}_s)_b} \|\varphi\|_{(\tilde{S}^{1/2}_{s'})_c}. \tag{1.8.14}$$

For the nonlinearity in the covariant Dirac equation, we first have the following set of bilinear estimates.

**Proposition 1.8.7.** *Let $a$ and $b$ be any admissible frequency envelopes. Then the following statements holds.*

1. *(Remainders $\widetilde{\mathcal{N}}^E$, $\widetilde{\mathcal{N}}^R$ and $\widetilde{\mathcal{N}}^S$) For any signs $s, s'$, we have*

$$\|\widetilde{\mathcal{N}}^E(B, \psi)\|_{(N^{1/2}_{s'})_{ab}} \lesssim \|B\|_{Y^1_a} \|\psi\|_{(\tilde{S}^{1/2}_s)_b}, \tag{1.8.15}$$

$$\|\widetilde{\mathcal{N}}^R(A_x, \psi)\|_{(N^{1/2}_{s'})_{ab}} \lesssim \|A_x\|_{S^1_a} \|\psi\|_{(\tilde{S}^{1/2}_s)_b}, \tag{1.8.16}$$

$$\|\Pi_{s'}\widetilde{\mathcal{N}}^S_s(A_x, \psi)\|_{(N^{1/2}_{s'})_{ab}} \lesssim \|A_x\|_{S^1_a} \|\psi\|_{(\tilde{S}^{1/2}_s)_b}. \tag{1.8.17}$$

2. *(Paradifferential operators $\pi^E$ and $\pi^R$) For opposite signs $s' = -s$, we have*

$$\|\pi^E[B]\psi\|_{(N^{1/2}_{-s})_{ab}} \lesssim \|B\|_{Y^1_a} \|\psi\|_{(\tilde{S}^{1/2}_s)_b}, \tag{1.8.18}$$

$$\|\pi^R[A_x]\psi\|_{(N^{1/2}_{-s})_{ab}} \lesssim \|A_x\|_{S^1_a} \|\psi\|_{(\tilde{S}^{1/2}_s)_b}. \tag{1.8.19}$$

3. *(Paradifferential operator $\pi^S$) For any signs $s, s'$, we have*

$$\|\Pi_{s'}\pi^S_s[A_x]\psi\|_{(N^{1/2}_{s'})_{ab}} \lesssim \|A_x\|_{S^1_a} \|\psi\|_{(\tilde{S}^{1/2}_s)_b}. \tag{1.8.20}$$

4. *(High modulation $L^2L^2$ bounds) For any sign $s$, we have*

$$\|\mathcal{N}^E(B, \psi)\|_{(L^2L^2)_{ab}} \lesssim \|B\|_{Y^1_a} \|\psi\|_{(S^{1/2}_s)_b}, \tag{1.8.21}$$

$$\|\mathcal{N}^R(A_x, \psi)\|_{(L^2L^2)_{ab}} \lesssim \|A_x\|_{S^1_a} \|\psi\|_{(S^{1/2}_s)_b}, \tag{1.8.22}$$

$$\|\mathcal{N}^S_s(A_x, \psi)\|_{(L^2L^2)_{ab}} \lesssim \|A_x\|_{S^1_a} \|\psi\|_{(S^{1/2}_s)_b}. \tag{1.8.23}$$

5. *($\tilde{Z}^{1/2}_s$ bounds) For any sign $s$, we have*

$$\|\mathcal{N}^E(B, \psi)\|_{G^{1/2}_{ab}} \lesssim \|B\|_{Y^1_a} \|\psi\|_{(S^{1/2}_s)_b}, \tag{1.8.24}$$

$$\|\widetilde{\mathcal{N}}^R(A_x, \psi)\|_{G^{1/2}_{ab}} + \|\pi^R[A_x]\psi\|_{G^{1/2}_{ab}} \lesssim \|A_x\|_{S^1_a} \|\psi\|_{(S^{1/2}_s)_b}, \tag{1.8.25}$$

$$\|\mathcal{N}^S_s(A_x, \psi)\|_{G^{1/2}_{ab}} \lesssim \|A_x\|_{S^1_a} \|\psi\|_{(S^{1/2}_s)_b}. \tag{1.8.26}$$

By (1.8.17), (1.8.20), (1.8.23) and (1.8.26), the spinorial nonlinearity $\mathcal{N}^S_{s'}$ can be handled just with bilinear estimates. On the other hand, Proposition 1.8.7 leaves open the treatment of certain parts of $\mathcal{N}^E$ and $\mathcal{N}^R$, namely $\pi^E[A_0]\psi$ and $\pi^R[A_x]\psi$. For a solution to MD-CG, recall the decomposition $A_0 = \mathbf{A}_0(\psi,\psi)$ and $A_x = A_x^{free} + \mathbf{A}_x(\psi,\psi)$. For the terms $\pi^E[\mathbf{A}_0(\psi,\psi)]\psi$ and $\pi^R[\mathbf{A}_x(\psi,\psi)]\psi$, which resemble the MKG-CG nonlinearity (see Remark 1.7.9), we use the following trilinear estimate.

**Proposition 1.8.8.** *For any admissible frequency envelopes $b$, $c$ and $d$, let*

$$f_k = \Big(\sum_{k'<k} c_{k'}^2\Big)^{1/2}\Big(\sum_{k'<k} d_{k'}^2\Big)^{1/2} b_k. \tag{1.8.27}$$

*Then for any signs $s, s_1, s_2 \in \{+,-\}$, we have*

$$\begin{aligned}\|\big(\pi^E[\boldsymbol{A}_0(\Pi_{s_1}\varphi^1,\Pi_{s_2}\varphi^2)] - s\pi^R[\boldsymbol{A}_x(\Pi_{s_1}\varphi^1,\Pi_{s_2}\varphi^2)]\big)\psi\|_{(N_s^{1/2})_f}\\ \lesssim \|\varphi^1\|_{(\tilde{S}_{s_1}^{1/2})_c}\|\varphi^2\|_{(\tilde{S}_{s_2}^{1/2})_d}\|\psi\|_{(\tilde{S}_s^{1/2})_b}.\end{aligned} \tag{1.8.28}$$

*Remark* 1.8.9. In the proof of theorem 1.1.2, the frequency envelopes $a, b, c$ inherit $\ell^2$-summability from the initial data; hence the products $ab$ and $bc$ are $\ell^1$-summable. The bilinear estimates in Propositions 1.8.6 and 1.8.7 therefore imply that certain parts of the solution (in particular, $\mathbf{A}_0$ and $\mathbf{A}_x$) enjoy $\ell^1$-summability of the dyadic norms. As in the case of the massless MKG [31], this fact allows us to cleanly separate $A$ into $\mathbf{A}$ handled by multilinear estimates (Proposition 1.8.8) and $A^{free}$ handled by a parametrix construction (Theorem 1.6.2).

The remaining term $\pi^R[A_x^{free}]\psi$ cannot be treated perturbatively. The optimal estimate, stated in terms of frequency envelopes, is as follows.

**Lemma 1.8.10.** *Let $A^{free} = (0, A_1^{free}, \dots, A_4^{free})$ be a real-valued 1-form obeying $\Box A^{free} = 0$ and $\partial^\ell A_\ell^{free} = 0$. For any admissible frequency envelope $a$ and $b$, let $e_k = (\sum_{k'<k} a_{k'})b_k$. Then for any sign $s \in \{+,-\}$, we have*

$$\|\pi^R[A_x^{free}]\psi\|_{(N_s^{1/2})_e} \lesssim \|A^{free}[0]\|_{(\dot{H}^1\times L^2)_a}\|\psi\|_{(\tilde{S}_s^{1/2})_b} \tag{1.8.29}$$

A sketch of proof of Lemma 1.8.10 will be given in Remark 6.3.7. Instead, $\pi^R[A_x^{free}]$ should be treated as a part of the underlying linear operator $(i\partial_t + s|D|)^p_{A^{free}}$ for which we have Theorem 1.6.2.

Theorem 1.6.2 and the estimates above lead to the following result on solvability of covariant Dirac equations which, in particular, contains the contribution of $A^{free}$. The proof is in Chapter 4.

**Proposition 1.8.11.** *There exists a universal constant $\epsilon_{**} > 0$ such that the following holds. Let $I \subseteq \mathbb{R}$ be a time interval containing $0$. Given spinor fields $\psi_0 \in \dot{H}^{1/2}$ on $\mathbb{R}^4$ and $F$ on*

*$I \times \mathbb{R}^4$ such that $\Pi_s F \in N_s^{1/2} \cap L^2 L^2 \cap G^{1/2}[I]$ $(s \in \{+,-\})$, consider the covariant Dirac equation*

$$\begin{cases} \alpha^\mu \boldsymbol{D}_\mu^A \psi = F & \text{on } I \\ \psi(0) = \psi_0, \end{cases} \tag{1.8.30}$$

*where the potential $A = A_\mu \mathrm{d}x^\mu$ is given by*

$$A_0 = \boldsymbol{A}_0(\psi', \psi'), \quad A_j = A_j^{free} + \boldsymbol{A}_j(\psi', \psi') \quad \text{on } I$$

*for some free wave $A_j^{free} \in C_t \dot{H}^1 \cap \dot{C}_t^1 L^2$ $(j = 1, \ldots, 4)$ and a spinor field $\psi'$ satisfying $\Pi_s \psi' \in \tilde{S}_s^{1/2}[I]$ and $\partial_\mu \langle \psi', \alpha^\mu \psi' \rangle = 0$. If*

$$\sup_{s \in \{+,-\}} \|\Pi_s \psi'\|_{\tilde{S}_s^{1/2}[I]} + \sup_{j \in \{1,\ldots,4\}} \|A_j^{free}[0]\|_{\dot{H}^1 \times L^2} \leq \epsilon_{**}, \tag{1.8.31}$$

*then there exists a unique solution $\psi$ to (1.8.30) on $I \times \mathbb{R}^4$ such that $\Pi_s \psi \in \tilde{S}_s^{1/2}[I]$ for $s \in \{+,-\}$. For any admissible frequency envelope $c$, we have*

$$\|\Pi_s \psi\|_{(\tilde{S}_s^{1/2}[I])_c} \lesssim \|\Pi_s \psi_0\|_{\dot{H}_c^{1/2}} + \|\Pi_s F\|_{(N_s^{1/2} \cap L^2 L^2 \cap G^{1/2}[I])_c}. \tag{1.8.32}$$

*The implicit constants are independent of $I$.*

# Chapter 2

# Function spaces and their embeddings

In this chapter we introduce the main function spaces and their embedding properties that we will use to prove the main Theorems 1.1.1 and 1.1.2. We build on the spaces defined in [31], which in turn build on the prior work [54, 52] on wave maps.

## 2.1 The function spaces

### Strichartz and $X^{s,b}$-type spaces.

We first define the admissible Strichartz norms for the $d+1$ dimensional wave equation. For any $d \geq 4$ and any $k$ we set

$$S_k^{Str,W} = \bigcap_{\frac{2}{q}+\frac{d-1}{r} \leq \frac{d-1}{2}} 2^{(\frac{d}{2}-\frac{1}{q}-\frac{d}{r})k} L^q L^r$$

with norm

$$\|f\|_{S_k^{Str,W}} = \sup_{\frac{2}{q}+\frac{d-1}{r} \leq \frac{d-1}{2}} 2^{-(\frac{d}{2}-\frac{1}{q}-\frac{d}{r})k} \|f\|_{L^q L^r} \tag{2.1.1}$$

Next we define the $X_\infty^{\frac{1}{2}}, X_1^{-\frac{1}{2}}$, the $X_{\pm,\infty}^{\frac{1}{2}}, X_{\pm,1}^{-\frac{1}{2}}$ and the $\bar{X}_\infty^{\frac{1}{2}}, \bar{X}_1^{-\frac{1}{2}}$ spaces, which are logarithmic of refinements of the usual $X^{s,b}$ space. Their dyadic norms are

$$\|F\|_{X_1^{-\frac{1}{2}}} = \sum_{j \in \mathbb{Z}} 2^{-\frac{1}{2}j} \|Q_j F\|_{L^2_{t,x}}, \qquad \|A\|_{X_\infty^{\frac{1}{2}}} = \sup_{j \in \mathbb{Z}} 2^{\frac{1}{2}j} \|Q_j A\|_{L^2_{t,x}}$$

$$\|F\|_{\bar{X}_1^{\pm\frac{1}{2}}} = \sum_{j \in \mathbb{Z}} 2^{\pm\frac{1}{2}j} \|\bar{Q}_j F\|_{L^2_{t,x}}, \qquad \|\phi\|_{\bar{X}_\infty^{\frac{1}{2}}} = \sup_{j \in \mathbb{Z}} 2^{\frac{1}{2}j} \|\bar{Q}_j \phi\|_{L^2_{t,x}}$$

$$\|F\|_{X_{\pm,1}^{-\frac{1}{2}}} = \sum_{j \in \mathbb{Z}} 2^{-\frac{1}{2}j} \|Q_j^{\pm} F\|_{L^2_{t,x}}, \qquad \|\psi\|_{X_{\pm,\infty}^{\frac{1}{2}}} = \sup_{j \in \mathbb{Z}} 2^{\frac{1}{2}j} \|Q_j^{\pm} \psi\|_{L^2_{t,x}}$$

## The spaces for the nonlinearity

For the nonlinearity, we define for $k \geq 0$ and $k' \in \mathbb{Z}$

$$\bar{N}_k = L^1L^2 + \bar{X}_1^{-\frac{1}{2}}, \qquad N_{k'} = L^1L^2 + X_1^{-\frac{1}{2}}, \qquad N_{k'}^{\pm} = L^1L^2 + X_{\pm,1}^{-\frac{1}{2}} \tag{2.1.2}$$

with norms

$$\|F\|_{\bar{N}_k} = \inf_{F=F_1+F_2} \|F_1\|_{L^1L^2} + \|F_2\|_{\bar{X}_1^{-\frac{1}{2}}}, \qquad \|F\|_{N_{k'}} = \inf_{F=F_1+F_2} \|F_1\|_{L^1L^2} + \|F_2\|_{X_1^{-\frac{1}{2}}}$$

$$\|F\|_{N_{k'}^{\pm}} = \inf_{F=F_1+F_2} \|F_1\|_{L^1L^2} + \|F_2\|_{X_{\pm,1}^{-\frac{1}{2}}}$$

By duality we can identify $\bar{N}_k^*$ with $L^\infty L^2 \cap \bar{X}_\infty^{\frac{1}{2}}$. For the scalar and spinorial equation, respectively the $A_i$ equation nonlinearities, for $s \in \mathbb{R}$ we define

$$\|F\|_{\bar{N}^s}^2 = \sum_{k\geq 0} 2^{2sk} \|\bar{P}_k F\|_{\bar{N}_k}^2, \qquad \|F\|_{N_{\pm}^s}^2 = \sum_{k'\in\mathbb{Z}} 2^{2sk'} \|P_{k'}F\|_{N_{k'}^{\pm}}^2$$

$$\|F\|_{\ell^1 N^s} = \sum_{k'\in\mathbb{Z}} 2^{sk'} \|P_{k'}F\|_{N_k'}, \qquad \|F\|_{N^s}^2 = \sum_{k'\in\mathbb{Z}} 2^{2sk'} \|P_{k'}F\|_{N_k'}^2.$$

## The iteration space for $A$

For any $d \geq 4$ and $k' \in \mathbb{Z}$ we define

$$\|A\|_{S_{k'}}^2 = \|A\|_{S_{k'}^{Str,W}}^2 + \|A\|_{X_\infty^{\frac{1}{2}}}^2 + \sup_{\pm} \sup_{l<0} \sum_{\omega} \|P_l^\omega Q_{<k'+2l}^{\pm} A\|_{S_{k'}^\omega(l)}^2$$

where

$$\|A\|_{S_{k'}^\omega(l)}^2 = 2^{-(d-1)k'-(d-3)l} \|A\|_{L^2L^\infty}^2 + \sup_{\substack{k''\in[0,k'],l'\leq 0 \\ k''+l'\leq k'+l}} \sum_{C_{k''}(l')} 2^{-(d-2)k''-(d-3)l'-k'} \|P_{C_{k''}(l')}A\|_{L^2L^\infty}^2.$$

Now we define $\quad \|A\|_{\ell^1 S^\sigma} = \sum_{k'\in\mathbb{Z}} 2^{(\sigma-1)k'} \big( \|\nabla_{t,x} P_{k'} A\|_{S_{k'}} + 2^{-\frac{1}{2}k'} \|\Box P_{k'} A\|_{L^2_{t,x}} \big),$

$$\|A\|_{S^\sigma}^2 = \sum_{k'\in\mathbb{Z}} 2^{2(\sigma-1)k'} \|\nabla_{t,x} P_{k'} A\|_{S_{k'}}^2 + \|\Box A\|_{L^2 \dot{H}^{\sigma-\frac{3}{2}}}^2$$

For the elliptic variable of MKG we set

$$\|A_0\|_{Y^\sigma} = \sum_{k'\in\mathbb{Z}} \|\nabla_{x,t} P_{k'} A_0\|_{L^\infty \dot{H}^{\sigma-1} \cap L^2 \dot{H}^{\sigma-\frac{1}{2}}} \tag{2.1.3}$$

while for the elliptic variable of MD we define $\|A_0\|_{Y^\sigma} = \|\nabla_{x,t} A_0\|_{L^2 \dot{H}^{\sigma-\frac{1}{2}}}$.

## The iteration space for the MKG scalar equation

The solution of the scalar equation will be placed in the space $\bar{S}^{\sigma}$ for $\sigma = \frac{d-2}{2}$ where, for any $s$ we define

$$\|\phi\|_{\bar{S}^s}^2 = \|\bar{P}_0(\phi, \partial_t \phi)\|_{\bar{S}_0}^2 + \sum_{k \geq 1} 2^{2(s-1)k} \|\nabla_{x,t} \bar{P}_k \phi\|_{\bar{S}_k}^2 + \|\Box_m \phi\|_{L^2 H^{s-\frac{3}{2}}}^2$$

where $\bar{S}_k$ are defined below.

When $d = 4$, in addition to (2.1.1), we will also use the Klein-Gordon Strichartz norms below. In general, using these K-G Strichartz norms at high frequencies does not lead to optimal estimates. Therefore, we will only rely on them for low frequencies or when there is enough additional dyadic gain coming from null structures. We set

$$\begin{aligned} &\text{For } d = 4: \qquad \bar{S}_k^{Str} = S_k^{Str,W} \cap 2^{\frac{3}{8}k} L^4 L^{\frac{8}{3}} \cap 2^{\frac{3}{4}k} L^2 L^4 \cap 2^{\frac{3}{4}k} L^2 L^{4,2} \\ &\text{For } d \geq 5: \qquad \bar{S}_k^{Str} = S_k^{Str,W} \end{aligned} \tag{2.1.4}$$

Notice that we incorporate the Lorentz norms $L^{4,2}$. See section 2.2 for more information.

For low frequencies $\{|\xi| \lesssim 1\}$ we define

$$\|\phi\|_{\bar{S}_0} = \|\phi\|_{\bar{S}_0^{Str}} + \|\phi\|_{\bar{X}_\infty^{\frac{1}{2}}} + \sup_{\pm, k'<0} \|\bar{Q}_{<k'}^{\pm} \phi\|_{S_{box(k')}} \qquad (d \geq 4) \tag{2.1.5}$$

where

$$\|\phi\|_{S_{box(k')}}^2 = 2^{-2\sigma k'} \sum_{\mathcal{C} = \mathcal{C}_{k'}} \|P_{\mathcal{C}} \phi\|_{L^2 L^\infty}^2$$

where $(\mathcal{C}_{k'})_{k'}$ is a finitely overlapping collection of cubes of sides $\simeq 2^{k'}$.

For higher frequencies we define as follows. Let $d \geq 4$, $k \geq 1$ and

$$\|\phi\|_{\bar{S}_k}^2 = \|\phi\|_{\bar{S}_k^{Str}}^2 + \|\phi\|_{\bar{X}_\infty^{\frac{1}{2}}}^2 + \sup_{\pm} \sup_{l<0} \sum_{\omega} \|P_l^{\omega} \bar{Q}_{<k+2l}^{\pm} \phi\|_{\bar{S}_k^{\omega \pm}(l)}^2 \tag{2.1.6}$$

where, for $d \geq 5$ we define

$$\|\phi\|_{\bar{S}_k^{\omega \pm}(l)}^2 = \|\phi\|_{S_k^{Str}}^2 + \sup_{\substack{k' \leq k; -k \leq l' \leq 0 \\ k+2l \leq k'+l' \leq k+l}} \sum_{\mathcal{C} = \mathcal{C}_{k'}(l')} 2^{-(d-2)k' - (d-3)l' - k} \|P_{\mathcal{C}} \phi\|_{L^2 L^\infty}^2$$

while for $d = 4$ we set

$$\begin{aligned} \|\phi\|_{\bar{S}_k^{\omega \pm}(l)}^2 = \|\phi\|_{S_k^{Str}}^2 + \sup_{\substack{k' \leq k; -k \leq l' \leq 0 \\ k+2l \leq k'+l' \leq k+l}} \sum_{\mathcal{C} = \mathcal{C}_{k'}(l')} \big( 2^{-2k'-k-l'} \|P_{\mathcal{C}} \phi\|_{L^2 L^\infty}^2 + \\ + 2^{-3(k'+l')} \|P_{\mathcal{C}} \phi\|_{PW_{\mathcal{C}}^{\pm}}^2 + \|P_{\mathcal{C}} \phi\|_{NE_{\mathcal{C}}^{\pm}}^2 \big). \end{aligned}$$

where, for any $\mathcal{C} = \mathcal{C}_{k'}(l')$

$$\|\phi\|_{NE^{\pm}_{\mathcal{C}}} = \sup_{\substack{\bar{\omega},\lambda=\lambda(p) \\ \angle(\bar{\omega},\pm\mathcal{C})\gg 2^{-p},2^{-k},2^{l'+k'-k}}} \angle(\bar{\omega},\pm\mathcal{C})\ \|\phi\|_{L^\infty_{t_{\bar{\omega},\lambda}} L^2_{x_{\bar{\omega},\lambda}}}, \quad \lambda(p) := \frac{1}{\sqrt{1+2^{-2p}}} \tag{2.1.7}$$

$$\|\phi\|_{PW^{\pm}_{\mathcal{C}}} = \inf_{\phi=\sum_i \phi^i} \sum_i \|\phi^i\|_{L^2_{t_{\omega_i,\lambda}} L^\infty_{x_{\omega_i,\lambda}}}, \quad \pm\omega_i \in \mathcal{C},\ \lambda = \frac{|\xi_0|}{\langle \xi_0 \rangle},\ \xi_0 = \text{center}(\mathcal{C}) \tag{2.1.8}$$

The norms $L^\infty_{t_{\bar{\omega},\lambda}} L^2_{x_{\bar{\omega},\lambda}}$ and $L^2_{t_{\omega_i,\lambda}} L^\infty_{x_{\omega_i,\lambda}}$ are taken in the frames (1.5.5), (1.5.6).

In other words, $PW^{\pm}_{\mathcal{C}}$ is an atomic space whose atoms are functions $\phi$ with $\|\phi\|_{L^2_{t_{\omega,\lambda}} L^\infty_{x_{\omega,\lambda}}} \leq 1$ for some $\omega \in \pm\mathcal{C}$, where $\lambda$ depends on the location of $\mathcal{C} = \mathcal{C}_{k'}(l')$.

The purpose of controlling the $NE^{\pm}_{C}$ and $PW^{\pm}_{C}$ norms lies in using the following type of bilinear $L^2_{t,x}$ estimate, which was introduced in [54] for the wave equation (see also [52]). A Klein-Gordon analogue was first developed in [4], which served as inspiration for our implementation.

**Proposition 2.1.1.** *Let $k, k_2 \geq 1$, $k' + C \leq k, k_2$; $l \in [-\min(k,k_2), C]$, and let $\pm_1, \pm_2$ be two signs. Let $\mathcal{C}, \mathcal{C}'$ be boxes of size $2^{k'} \times (2^{k'+l})^3$ located in $\{|\xi| \simeq 2^k\} \subset \mathbb{R}^4$ , resp. $\{|\xi| \simeq 2^{k_2}\} \subset \mathbb{R}^4$ such that*

$$\angle(\pm_1\mathcal{C}, \pm_2\mathcal{C}') \simeq 2^{l'} \gg \max(2^{-\min(k,k_2)}, 2^{l+k'-\min(k,k_2)}) \tag{2.1.9}$$

*Then we have*

$$\|\phi_k \cdot \varphi_{k_2}\|_{L^2_{t,x}(\mathbb{R}^{4+1})} \lesssim 2^{-l'} \|\phi_k\|_{NE^{\pm_1}_{\mathcal{C}}} \|\varphi_{k_2}\|_{PW^{\pm_2}_{\mathcal{C}'}} \tag{2.1.10}$$

*Proof.* The condition (2.1.9) insures that $\pm_1\mathcal{C}$ and $\pm_2\mathcal{C}'$ are angularly separated and the angle between them is well-defined. Since $PW$ is an atomic space, we may assume the second factor is an atom with $\|\varphi_{k_2}\|_{L^2_{t_{\omega,\lambda}} L^\infty_{x_{\omega,\lambda}}} \leq 1$ for some $\omega \in \pm_2\mathcal{C}'$ and $\lambda$ given by (2.1.8). We choose $2^p = |\xi_0| \simeq 2^{k_2}$, so that $\lambda = \lambda(p)$ from (2.1.7) so that together with (2.1.9) we have

$$\|\phi_k\|_{L^\infty_{t_{\omega,\lambda}} L^2_{x_{\omega,\lambda}}} \lesssim 2^{-l'} \|P_{\mathcal{C}} \bar{Q}^{\pm}_{<j} \phi_k\|_{NE^{\pm_1}_{\mathcal{C}}}.$$

Now (2.1.10) follows from Hölder's inequality $L^\infty_{t_{\omega,\lambda}} L^2_{x_{\omega,\lambda}} \times L^2_{t_{\omega,\lambda}} L^\infty_{x_{\omega,\lambda}} \to L^2_{t,x}$. □

*Remark* 2.1.2. When $\Box_m \phi_k = \Box_m \varphi_{k_2} = 0$ and $\phi_k, \varphi_{k_2}$ have Fourier support in $\mathcal{C}$, respectively $\mathcal{C}'$ then one has

$$\|\phi_k \cdot \varphi_{k_2}\|_{L^2_{t,x}(\mathbb{R}^{4+1})} \lesssim 2^{-l'} 2^{\frac{3}{2}(k'+l')} \|\phi_k[0]\|_{L^2 \times H^{-1}} \|\varphi_{k_2}[0]\|_{L^2 \times H^{-1}} \tag{2.1.11}$$

by convolution estimates (see eg. [15], [53]). Thus (2.1.10) is meant as a more general substitute for (2.1.11).

## The iteration space for the MD spinorial equation

Let $d \geq 4$ and $r \in \mathbb{R}$. For the Dirac equation, we need to define analogous spaces adapted to each characteristic cone $\{\tau = \pm|\xi|\}$. Let

$$S_k^{\pm} = S_k^{\mathrm{Str,W}} \cap X_{\pm,\infty}^{0,\frac{1}{2}} \cap Q_{<k-3}^{\pm}\tilde{S}_k.$$

*Remark* 2.1.3. The space $\tilde{S}_k$ is borrowed from [31] (where it is denoted by $S_k$) and is defined below. We have changed the notation because we use $S_k$ for the $A$ equation. We also note that $\tilde{S}_k = S_k^+ + S_k^-$, $N_k = N_k^+ \cap N_k^-$ and $Q_{<k-1}^s N_k^s \subseteq N_k$, $Q_{<k+O(1)}^s \tilde{S}_k \subseteq S_k^s$.

We define

$$\|\psi\|_{S_{\pm}^r}^2 = \sum_{k\in\mathbb{R}} \Big(2^{2rk}\|P_k\psi\|_{S_k^{\pm}}^2 + 2^{(2r-2)k}\|(i\partial_t \pm |D|)P_k\psi\|_{L^2L^2}^2\Big).$$

Let

$$\tilde{S}_k = S_k^{\mathrm{Str,W}} \cap X_{\infty}^{0,\frac{1}{2}} \cap S_k^{\mathrm{ang}}.$$

where $S_k^{\mathrm{ang}}$ is as in [31, Eqs. (6)–(8)]:

$$\|f\|_{S_k^{ang}}^2 = \sup_{l<0} \sum_{\omega} \|P_l^{\omega} Q_{k+2l} f\|_{S_k^{\omega}(l)}^2,$$

where, for $d = 4$:

$$\begin{aligned}\|f\|_{S_k^{\omega}(l)}^2 =& \|f\|_{S_k^{str}}^2 + 2^{-2k}\|f\|_{NE}^2 + 2^{-3k}\sum_{\pm}\|T_{\pm}f\|_{PW_{\omega}^{\mp}(l)}^2 + \\ &+ \sup_{\substack{k'\leq k,\ \ell'\leq 0\\ k+2\ell\leq k'+\ell'\leq k+\ell}} \sum_{\mathcal{C}_{k'}(\ell')} \Big(\|P_{\mathcal{C}_{k'}(\ell')}f\|_{S_k^{str}}^2 + 2^{-2k}\|P_{\mathcal{C}_{k'}(\ell')}f\|_{NE}^2 \\ &+ 2^{-2k'-k}\|P_{\mathcal{C}_{k'}(\ell')}f\|_{L^2L^\infty}^2 + 2^{-3(k'+l')}\sum_{\pm}\|T_{\pm}P_{\mathcal{C}_{k'}(\ell')}f\|_{PW_{\omega}^{\mp}(l)}^2\Big)\end{aligned}$$

while for $d \geq 5$:

$$\|f\|_{S_k^{\omega}(l)}^2 = \|f\|_{S_k^{\mathrm{str}}}^2 + \sup_{\substack{k'\leq k,\ \ell'\leq 0\\ k+2\ell\leq k'+\ell'\leq k+\ell}} \sum_{\mathcal{C}_{k'}(\ell')} 2^{-(d-2)k'}2^{-(d-3)\ell'}2^{-k}\|P_{\mathcal{C}_{k'}(\ell')}f\|_{L^2L^\infty}^2.$$

Here, the $NE$ and $PW_{\omega}^{\mp}(\ell)$ are the *null frame spaces* [54, 52] given by

$$\begin{aligned}\|f\|_{PW_{\omega}^{\mp}(\ell)} &= \inf_{f=\int f^{\omega'}} \int_{|\omega-\omega'|\leq 2^{\ell}} \|f^{\omega'}\|_{L^2_{\pm\omega'}L^\infty_{(\pm\omega')^\perp}}\, \mathrm{d}\omega', \\ \|f\|_{NE} &= \sup_{\omega} \|\,\not\nabla_{\omega}\phi\|_{L^\infty_{\omega}L^2_{\omega^\perp}},\end{aligned}$$

where the $L^q_{\omega}$ norm is with respect to the variable $\ell^{\pm} = t \pm \omega\cdot x$, the $L^r_{\omega^\perp}$ norm is defined on each $\{\ell^{\pm}_{\omega} = const\}$, and $\not\nabla_{\omega}$ denotes derivatives tangent to $\{\ell^{\pm}_{\omega} = const\}$.

*Remark* 2.1.4. We note here that the full exponent of $2^{\ell'}$ in the $L^2L^\infty$ norm of $S_k^{ang}$ was not used in [31], but the extra factor of $2^{-\ell'/2}$ was actually obtained there in Subsection 11.3 for the main parametrix estimate (Theorem 4.3.3). As opposed to the Maxwell-Klein-Gordon case, it turns out that this angular gain is essential here in order to estimate the nonlinear terms of the Maxwell-Dirac system.

The $S^r_\pm$ norm must be augmented with an $L^1L^\infty$ control for high modulations. To this end, consider the dyadic norm

$$\|\psi\|_{\tilde{Z}_k^\pm} = 2^{-2k}\|(i\partial_t \pm |D|)\psi\|_{L^1L^\infty} \tag{2.1.12}$$

and the corresponding $\ell^2$-summed norm, given by

$$\|\psi\|^2_{\tilde{Z}^r_\pm} = \sum_{k\in\mathbb{R}} 2^{2rk}\|\psi_k\|^2_{\tilde{Z}_{\pm,k}}.$$

Define also

$$\|F\|_{G_k} = 2^{-2k}\|F\|_{L^1L^\infty}, \quad \|F\|^2_{G^r} = \sum_k 2^{2rk}\|P_kF\|_{G_k}.$$

For $\psi$ localized at frequency $\{|\xi| \simeq 2^k\}$ and $s \in \{+,-\}$, we have

$$\|\psi\|_{\tilde{Z}_k^\pm} \lesssim \|(i\partial_t \pm |D|)\psi\|_{G_k}.$$

The main iteration space $\tilde{S}^\sigma_s$ for the $s$-components of $\psi$ ($s \in \{+,-\}$) is defined as

$$\|\psi\|^2_{\tilde{S}^r_s} = \|\psi\|^2_{S^r_s} + \|\psi\|^2_{\tilde{Z}^r_s}. \tag{2.1.13}$$

## The $L^1L^\infty$-type norms

We now introduce some auxiliary norms for the Maxwell components $A_0, A_x$ which will be used in the proofs of the trilinear estimates. Let $C_1 > 0$ be a constant and let

$$\|A\|^2_{Z_k} := \sup_{\pm} \sup_{\ell<\frac{1}{2}C_1} 2^{-(d-2)k}2^{\ell}\sum_\omega \|P^\omega_\ell Q^\pm_{k+2\ell}A\|^2_{L^1L^\infty}, \tag{2.1.14}$$

$$\|B\|^2_{Z_{ell,k}} := \sup_{\pm} \sup_{\ell<\frac{1}{2}C_1} 2^{-(d-2)k}2^{-\ell}\sum_\omega \|P^\omega_\ell Q^\pm_{k+2\ell}B\|^2_{L^1L^\infty}. \tag{2.1.15}$$

For $r \in \mathbb{R}$ we set $\|A\|_{Z^r_k} = 2^{kr}\|A\|_{Z_k}$, $\|B\|_{Z^r_{ell,k}} = 2^{kr}\|B\|_{Z_{ell,k}}$ and the $\ell^1$-summed norms

$$\|A\|_{Z^r} = \sum_{k\in\mathbb{R}} \|P_kA\|_{Z^r_k}, \quad \|B\|_{Z^r_{ell}} = \sum_{k\in\mathbb{R}} \|P_kB\|_{Z^r_{ell,k}}.$$

Note that $Z^r_{ell} = \Box^{\frac{1}{2}}\Delta^{-\frac{1}{2}}Z^r$. Moreover, we define the norms $\Box Z^r$ and $\Delta Z^r_{ell}$ so that $\|A\|_{Z^r} = \|\Box A\|_{\Box Z^r}$ and $\|B\|_{Z^r_{ell}} = \|\Delta B\|_{\Delta Z^r_{ell}}$.

By Lemma 2.2.7 we will have

$$\left(\Box^{-1} \times \Delta^{-1}\right) P_{k'} : L^1 L^2 \times L^1 L^2 \to 2^{(\sigma-1)k'} Z^{\sigma}_{k'} \times Z^{\sigma}_{ell,k'} \tag{2.1.16}$$

and

$$\|\Delta^{-1} P_{k'} A_0\|_{Z^{\sigma}_{ell,k'}} \lesssim \sup_{\pm, \ell < C} 2^{\ell} 2^{(\sigma-1)k'} \|Q^{\pm}_{k'+2\ell} P_{k'} A_0\|_{L^1 L^2} \tag{2.1.17}$$

## Time interval localized norms

In a few places we need to consider time interval localization of the function spaces. Given an interval $I \subseteq \mathbb{R}$ and a distribution $f$ on $I \times \mathbb{R}^d$, we define[1]

$$\|f\|_{X[I]} = \inf\{\|\tilde{f}\|_X : \tilde{f} \in X,\ \tilde{f} = f \text{ on } I\},$$

where $X$ may denote any norm, e.g., $S^r$, $N^r$, $\tilde{S}^r_s$ or $N^r_s$.

Let $f \in N^r[I]$. Up to equivalent norms, we may take $\tilde{f}$ above in $N^r$ to be simply the extension by zero outside $I$. Moreover, for $f \in N^r$, we have

$$\lim_{T \to 0} \|f\|_{N^r[0,T]} = 0, \qquad \lim_{T \to \infty} \|f\|_{N^r[T,\infty)} = 0. \tag{2.1.18}$$

Similar properties holds for $N^r_s$. These statements are justified by the following lemma, whose proof can be read off from [41, Proposition 3.3].

**Lemma 2.1.5.** *Let $f \in N^r$ $(r \in \mathbb{R})$. For any interval $I \subseteq \mathbb{R}$, denote by $1_I(t)$ its characteristic function. Then we have $\|1_I(t) f\|_{N^r} \lesssim \|f\|_{N^r}$. Moreover, we have $\lim_{T\to 0+} \|1_{[0,T]}(t) f\|_{N^r} = 0$ and $\lim_{T\to\infty} \|1_{[T,\infty]}(t) f\|_{N^r} = 0$.*

*The same statements hold with $N^r$ replaced by $N^r_s$ $(s \in \{+,-\},\ r \in \mathbb{R})$.*

## Extra derivatives

For $X = S, N, Y, \dot{H}$ and $\bar{X} = \bar{S}, \bar{N}, H$, for any $s, \rho \in \mathbb{R}$ we have

$$\|A\|_{X^{s+\rho}} \simeq \|\nabla^{\rho}_x A\|_{X^s}, \qquad \|f\|_{\bar{X}^{s+\rho}} \simeq \|\langle \nabla_x \rangle^{\rho} f\|_{\bar{X}^s}$$

Similar definitions are made for their dyadic pieces, for instance

$$\|\phi_k\|_{\bar{S}^s_k} \simeq 2^{(s-1)k} \|(\langle D_x \rangle, \partial_t)\phi_k\|_{\bar{S}_k}.$$

[1]We use the convention $\inf \emptyset = \infty$.

## Motivation of the norms

We end this section with a discussion about the choice of norms in the definition of the $S_k, \bar{S}_k, S_k^{\pm}, \tilde{S}_k$ spaces. For solutions $A$ of the free wave equation $\Box A = 0$ we have $\|A\|_{S_k} \simeq \|A[0]\|_{L^2 \times \dot{H}^{-1}}$. The $X_\infty^{0,1/2}$ space provides control of $L^2L^2$ norms that are useful with components of high modulation.

Additionally, one looks for norms that are both useful in proving bilinear estimates and which are controlled for free wave solutions. In fact, by expressing arbitrary functions $A$ as superpositions of free waves, one can obtain boundedness of $\|A\|_{S_k}$ in terms of $\|\Box A\|_{N_k}$. An example of this argument appears in Lemma 2.2.6. The $S_k^{str}$ component corresponds to well-known Strichartz estimates.

Regarding $S_k^{ang}$, the $\ell^2$ summation in $P_l^\omega$ and $P_{\mathcal{C}_{k'}(\ell')}$ is inherited from the initial data. The square summed $L^2L^\infty$ norms play a particularly important role in the estimates. To motivate the choice of dyadic exponents, let us check that these exponents are sharp. We claim that an inequality

$$\|P_{\mathcal{C}_{k'}(\ell')} P_k e^{it|D|} u_0\|_{L^2L^\infty} \lesssim C_{k,k',\ell'} \|u_0\|_{L_x^2} \tag{2.1.19}$$

can be true (uniformly in $k, k', \ell'$) only for $C_{k,k',\ell'}^2 \geq 2^{(d-2)k'} 2^{(d-3)\ell'} 2^k$, and is optimal when the latter is an equality.

We consider the following version of the Knapp example: let $u(t,x)$ be a solution to $\Box u = 0$ with Fourier support in $S = \{\tau = |\xi| \simeq 2^k,\ \xi \in \mathcal{C}_{k'}(\ell')\}$ such that for any $|t| \leq T := \frac{1}{C} 2^k 2^{-2(k'+\ell')}$ one has $|u(t,x)| \simeq 1$ for $x$ in a rectangle of sides $\simeq 2^{-k'} \times (2^{-k'-\ell'})^{d-1}$, dual to $\mathcal{C}_{k'}(\ell')$. The uncertainty principle suggests that $u(t,\cdot)$ becomes dispersed after $|t| \gg T$ because the smallest rectangular box encompassing $S$ has sides $\simeq T^{-1} \times 2^{k'} \times (2^{k'+\ell'})^{d-1}$ (where $T^{-1}$ and $2^{k'}$ are measured in the null directions). In fact, for

$$\mathcal{C}_{k'}(\ell') = \mathcal{C} := \{|\xi_1| \simeq 2^k,\ \left|\xi_1 - \xi_1^0\right| \ll 2^{k'},\ |\xi_i| \ll 2^{k'+\ell'},\ i = 2, d\}$$

one can define

$$u(t,x) = \mathrm{vol}(\mathcal{C})^{-1} \int_{\mathcal{C}} e^{ix\cdot\xi} e^{it|\xi|}\, \mathrm{d}\xi$$

and check that $|u(t,x)| \simeq 1$ for $|t| \lesssim T$, $|x_1 + t| \lesssim 2^{-k'}$, $|x_i| \lesssim 2^{-k'-\ell'}$.

Plugging this example into (2.1.19) gives $T^{\frac{1}{2}} \lesssim C_{k,k',\ell'} \mathrm{vol}(\mathcal{C})^{-\frac{1}{2}}$, which provides the optimal choice of $C_{k,k',\ell'}$ in the definition of $S_k$, $\bar{S}_k$.

Similar arguments apply to the norms $PW$ and $NE$ which are used for $d = 4$. For instance, plugging the same $u(t,x)$ in the inequality

$$\|P_{\mathcal{C}_{k'}(\ell')} e^{it|D|} u_0\|_{L_\omega^2 L_{\omega^\perp}^\infty} \lesssim \tilde{C}_{k',\ell'} \|u_0\|_{L_x^2}, \qquad \text{for } \omega = e_1$$

gives $(2^{-k'})^{\frac{1}{2}} \lesssim \tilde{C}_{k',\ell'} \mathrm{vol}(\mathcal{C})^{-\frac{1}{2}}$, thus $\tilde{C}_{k',\ell'} = 2^{\frac{3}{2}(k'+\ell')}$ is the optimal choice for $PW$.

## 2.2 The embeddings

### Lorentz spaces and $\Box^{-1}$ embeddings

For functions $f$ in the Lorentz space $L^{p,q}$, by decomposing

$$f = \sum f_m, \quad \text{where} \quad f_m(x) := f(x) 1_{\{|f(x)| \in [2^m, 2^{m+1}]\}}$$

we have the following equivalent norm (see [19])

$$\|f\|_{L^{p,q}} \simeq \| \|f_m\|_{L^p(\mathbb{R}^d)} \|_{\ell^q_m(\mathbb{Z})}. \tag{2.2.1}$$

The Lorentz spaces also enjoy a Hölder-type inequality which is due to O'Neil [44]. We will need the following case

$$\|\phi\psi\|_{L^{2,1}} \lesssim \|\phi\|_{L^{4,2}} \|\psi\|_{L^{4,2}} \tag{2.2.2}$$

For $M \in \mathbb{Z}$ and $l \leq 0$ let

$$T^\omega_l = \sum_{k' \leq M} P^\omega_l Q^\pm_{k'+2l} P_{k'} \frac{1}{\Box} \tag{2.2.3}$$

*Remark* 2.2.1. We will use the $T^\omega_l$ operators on $\mathbb{R}^{4+1}$ to estimate parts of the potential $A$ in $L^1 L^\infty$, using the embedding (2.2.5) together with Lorentz space Strichartz estimates $L^2 L^{4,2}$ for $\phi$ and (2.2.2). We have been motivated by [50], where $A \approx \frac{\partial}{\Delta}(du)^2$, and where essentially a Sobolev-type emdedding $\frac{1}{|D_x|} : L^{d,1}_x \to L^\infty_x(\mathbb{R}^d)$ is used.

When $l = 0$ the symbol of the operator $T^\omega_l$ makes it resemble $\Delta^{-1}_x$.

The main point here will be that it is crucial to keep the $k'$ summation inside the norm in order to overcome logarithmic divergences in (6.2.6).

**Proposition 2.2.2.** *On $\mathbb{R}^{4+1}$ the following embeddings hold uniformly in $l \leq 0$ and $M$:*

$$2^{\frac{1}{2}l} T^\omega_l : L^2 L^{\frac{4}{3}} \to L^2 L^4, \tag{2.2.4}$$

$$2^{\frac{1}{2}l} T^\omega_l : L^1 L^{2,1} \to L^1 L^\infty. \tag{2.2.5}$$

*Proof.* **S**tep 1.Proof of (2.2.4). Apply an angular projection such that $\tilde{P}^\omega_l P^\omega_l = P^\omega_l$. Now (2.2.4) follows by composing the following embeddings

$$2^{-\frac{3}{4}l} P^\omega_l |D_x|^{-1} : L^2 L^{\frac{4}{3}} \to L^2_{t,x}, \quad 2^{2l} \frac{|D_x|^2}{\Box} \sum_{k' \leq M} Q^\pm_{k'+2l} P_{k'} : L^2_{t,x} \to L^2_{t,x} \tag{2.2.6}$$

$$2^{-\frac{3}{4}l} \tilde{P}^\omega_l |D_x|^{-1} : L^2_{t,x} \to L^2 L^4. \tag{2.2.7}$$

When $l = 0$, the first and third mappings follow from Sobolev embedding. For smaller $l$ we make a change of variable that maps an angular cap of angle $\simeq 2^l$ into one of angle $\simeq 2^0$, which reduces the bound to the case $l = 0$.

The second mapping holds because the operator has a bounded multiplier.

**S**tep 2. Proof of (2.2.5). Let $k(t,x)$ be the kernel of $2^{\frac{1}{2}l}T_l^\omega$. It suffices to show

$$2^{\frac{1}{2}l}T_l^\omega[\delta_0(t)\otimes\cdot\,] : L_x^{2,1}\to L^1L^\infty,\ \text{i.e.}\ \|\int f(y)k(t,x-y)\,\mathrm{d}y\|_{L_t^1L_x^\infty}\lesssim\|f\|_{L_x^{2,1}} \tag{2.2.8}$$

Indeed, assuming (2.2.8), denoting $\phi_s(\cdot)=\phi(s,\cdot)$, we have

$$\|2^{\frac{1}{2}l}T_l^\omega\phi\|_{L^1L^\infty}\le\int\|\int\phi(s,y)k(t-s,x-y)\,\mathrm{d}y\|_{L_t^1L_x^\infty}\,\mathrm{d}s\lesssim\int\|\phi_s\|_{L^{2,1}}\,\mathrm{d}s$$

using the time translation-invariance in (2.2.8).

To prove (2.2.8), since $q=1$, by (2.2.1) we may assume that $f=f_m$, i.e. $|f(x)|\simeq 2^m$ for $x\in E$ and $f(x)=0$ for $x\notin E$. We normalize $\|f\|_{L^{2,1}}\simeq\|f\|_{L_x^2}=1$, which implies $|E|\simeq 2^{-2m}$. We have

$$\|\int f(x-y)k(t,y)\,\mathrm{d}y\|_{L_x^\infty}\lesssim 2^m\sup_{|F|\simeq 2^{-2m}}\int_F|k(t,y)|\,\mathrm{d}y \tag{2.2.9}$$

For $x_\omega=x\cdot\omega,\ x'_{\omega,i}=x\cdot\omega_i^\perp$, we will show

$$|k(t,x)|\lesssim 2^{\frac{1}{2}l}\frac{2^{3l}}{(2^{2l}|t|+|x_\omega|+2^l|x'_\omega|)^3}. \tag{2.2.10}$$

Assuming this, we integrate it on $F$ and since the fraction is decreasing in $|x_\omega|,|x'_\omega|$,

$$\begin{aligned}\text{RHS (2.2.9)}&\lesssim 2^m2^{\frac{1}{2}l}\int_{[-R,R]\times(2^{-l}[-R,R])^3}\frac{2^{3l}}{(2^{2l}|t|+|x_\omega|+2^l|x'_\omega|)^3}\,\mathrm{d}x_\omega\,\mathrm{d}x'_\omega\\&\lesssim 2^m2^{\frac{1}{2}l}\int_{[-R,R]^4}\frac{1}{(2^{2l}|t|+|(x_\omega,x'_\omega)|)^3}\,\mathrm{d}x_\omega\,\mathrm{d}x'_\omega\lesssim 2^m2^{\frac{1}{2}l}\frac{R^4}{(2^{2l}|t|)^3+R^3}\end{aligned}$$

for $R^4\simeq 2^{3l}2^{-2m}$. Integrating this bound in $t$ we obtain (2.2.8).

**S**tep 3. Proof of (2.2.10). Let $k_0(t,x)$ be the kernel of $P_0Q_{2l}^\pm P_l^\omega\frac{1}{\Box}$. Then

$$k(t,x)=2^{\frac{1}{2}l}\sum_{k'\le M}2^{3k'}k_0\big(2^{k'}(t,x)\big). \tag{2.2.11}$$

Let $(t_\omega,x_\omega^1,x'_\omega)$ be the coordinates in the frame (1.5.5), (1.5.6) for $\lambda=1$. Then

$$2^{-3l}k_0(t_\omega,2^{-2l}x_\omega^1,2^{-l}x'_\omega)$$

is a Schwartz function, being the Fourier transform of a bump function. Thus,

$$|k_0(t,x)|\lesssim\frac{2^{3l}}{\langle|t_\omega|+2^{2l}|x_\omega^1|+2^l|x'_\omega|\rangle^N}\lesssim\frac{2^{3l}}{\langle 2^{2l}|t|+|x_\omega|+2^l|x'_\omega|\rangle^N}.$$

Using this and (2.2.11), denoting $S=2^{2l}|t|+|x_\omega|+2^l|x'_\omega|$, we have

$$|k(t,x)|\lesssim 2^{\frac{1}{2}l}2^{3l}\big(\sum_{2^{k'}\le S^{-1}}2^{3k'}+\sum_{S^{-1}<2^{k'}}2^{-(N-3)k'}S^{-N}\big)\lesssim 2^{\frac{1}{2}l}2^{3l}S^{-3}$$

obtaining (2.2.10). □

## Further properties

For iterating Maxwell's equation we will use the following proposition.

**Proposition 2.2.3.** *For any $A$ such that $A[0]=0$ one has*

$$\|A\|_{\ell^1 S^\sigma} \lesssim \|\Box A\|_{\ell^1(N^{\sigma-1}\cap L^2\dot{H}^{\sigma-\frac{3}{2}})} \tag{2.2.12}$$

*For any free solution $A^{free}$, i.e. $\Box A^{free}=0$, one has $\|A^{free}\|_{S^\sigma} \simeq \|A[0]\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}$. Thus, for any $A$,*

$$\|A\|_{S^\sigma} \lesssim \|A[0]\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} + \|\Box A\|_{N^{\sigma-1}\cap L^2\dot{H}^{\sigma-\frac{3}{2}}} \tag{2.2.13}$$

*In addition, for any $A_0$ one has*

$$\|A_0\|_{Y^\sigma} \lesssim \|\Delta A_0\|_{\ell^1(L^\infty\dot{H}^{\sigma-2}\cap L^2\dot{H}^{\sigma-\frac{3}{2}})} + \|\Delta\partial_t A_0\|_{\ell^1(L^\infty\dot{H}^{\sigma-3}\cap L^2\dot{H}^{\sigma-\frac{5}{2}})} \tag{2.2.14}$$

*Proof.* The $A_0$ bound follows easily from the definition of $Y^\sigma$. The $A$ bounds are reduced to

$$\|\nabla_{t,x}A_{k'}\|_{S_{k'}} \lesssim \|A_{k'}[0]\|_{\dot{H}^1\times L^2} + \|\Box A_{k'}\|_{N_{k'}}$$

The $X_\infty^{\frac{1}{2}}$ part follows easily from Lemma 2.2.4. Using the argument of Lemma 4.10.2 (with $\psi=0$), we reduce to showing

$$e^{\pm it|D|}P_{k'}: L_x^2 \to S^{Str,W}, \qquad e^{\pm it|D|}P_{k'}P_l^\omega : L_x^2 \to S_{k'}^\omega(l) \tag{2.2.15}$$

The first mapping represents well-known Strichartz estimates. By orthogonality, the second one follows from

$$2^{-\frac{d-1}{2}k'-\frac{d-3}{2}l}e^{\pm it|D|}P_{k'}P_l^\omega : L_x^2 \to L^2L^\infty,$$
$$2^{-\frac{d-2}{2}k''-\frac{1}{2}k'-\frac{d-3}{2}l'}e^{\pm it|D|}P_{k'}P_{C_{k''}(l')} : L_x^2 \to L^2L^\infty$$

By a $TT^*$ argument, these are reduced to the dispersive estimate (4.6.9), like in Cor. 4.6.4 (with $\psi=0$ and $|D|$ instead of $\langle D\rangle$, which does not affect the proof). □

The following Sobolev-type embedding holds.

**Lemma 2.2.4.** *Let $p\geq q$. For any sign $\pm$ we have*

$$\|\bar{Q}_j^\pm u\|_{L^pL^2} \lesssim 2^{(\frac{1}{q}-\frac{1}{p})j}\|\bar{Q}_j^\pm u\|_{L^qL^2} \lesssim 2^{(\frac{1}{q}-\frac{1}{p})j}\|u\|_{L^qL^2}.$$

*The same statements holds for $Q_j^\pm$.*

*Proof.* We conjugate by the operator $U$ defined by

$$\mathcal{F}(Uu)(\tau,\xi) = \mathcal{F}u(\tau\pm\langle\xi\rangle,\xi),$$

which acts at each $t$ as the unitary multiplier $e^{\mp it\langle D\rangle}$. Thus we have

$$Q_j^{\pm}u = U^{-1}\chi(\frac{D_t}{2^j})Uu.$$

This clearly implies the second inequality. For the first one we write

$$\|Q_j^{\pm}u\|_{L^pL^2} \lesssim \|\chi(\frac{D_t}{2^j})Uu\|_{L^pL^2} \lesssim 2^{(\frac{1}{q}-\frac{1}{p})j}\|\chi(\frac{D_t}{2^j})Uf\|_{L^qL^2} \lesssim 2^{(\frac{1}{q}-\frac{1}{p})j}\|\bar{Q}_j^{\pm}u\|_{L^qL^2}.$$

The same argument works for $Q_j^{\pm}$, conjugating by $e^{\mp it|D|}$ instead. □

Next we prove the embedding $\bar{X}_1^{\frac{1}{2}} \subset \bar{S}_k$.

**Proposition 2.2.5.** *For $k \geq 0$, $k' \in \mathbb{Z}$ and $\phi, \psi$ with Fourier support in $\{\langle\xi\rangle \simeq 2^k\}$, resp. $\{|\xi| \simeq 2^{k'}\}$, we have*

$$\|\phi\|_{\bar{S}_k} \lesssim \|\phi\|_{\bar{X}_1^{\frac{1}{2}}}, \qquad \|\psi\|_{\bar{S}_{k'}} \lesssim \|\psi\|_{X_1^{\frac{1}{2}}}.$$

*Proof.* We consider the first inequality, since the proof of the second one is analogous. We may assume that $\phi$ has Fourier support in $\{|\tau - \langle\xi\rangle| \simeq 2^j,\ \tau \geq 0\}$. The bound clearly holds for the $\bar{X}_\infty^{\frac{1}{2}}$ component of $\bar{S}_k$. For the other norms we claim $\|e^{it\langle D\rangle}u\|_{\bar{S}_k} \lesssim \|u\|_{L_x^2}$. Assuming this, we write $\tau = \rho + \langle\xi\rangle$ in the inversion formula

$$\phi(t) = \int e^{it\tau + ix\xi}\mathcal{F}\phi(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau = \int_{|\rho|\simeq 2^j} e^{it\rho}e^{it\langle D\rangle}\phi_\rho\,\mathrm{d}\rho$$

for $\hat{\phi}_\rho(\xi) = \mathcal{F}\phi(\rho + \langle\xi\rangle, \xi)$. Then by Minkowski and Cauchy-Schwarz inequalities

$$\|\phi\|_{\bar{S}_k} \lesssim \int_{|\rho|\simeq 2^j} \|e^{it\langle D\rangle}\phi_\rho\|_{\bar{S}_k}\,\mathrm{d}\rho \lesssim \int_{|\rho|\simeq 2^j} \|\phi_\rho\|_{L_x^2}\,\mathrm{d}\rho \lesssim 2^{\frac{j}{2}}\|\phi\|_{L_{t,x}^2} \simeq \|\phi\|_{\bar{X}_1^{\frac{1}{2}}}.$$

By an orthogonality argument, for any $l < 0$ it remains to establish

$$e^{it\langle D\rangle}\bar{P}_k : L_x^2 \to \bar{S}_k^{Str}, \qquad e^{it\langle D\rangle}\bar{P}_k P_l^{\omega} : L_x^2 \to \bar{S}_k^{\omega\pm}(l)$$

The first mapping follows by taking $\psi_{k,\pm} = 0$ in (4.10.5). The second one follows similarly, by orthogonality and (4.10.8) for $L^2L^\infty$, (4.10.9) for $PW_C^{\pm}$ and Corollary 4.6.10 for $NE_C^{\pm}$. For $k = 0$, the $S_{box(k')}$ component follows similarly. □

Similarly, we have

**Lemma 2.2.6.** *Suppose $f$ is localized at frequency $\{|\xi| \simeq 2^k\}$ and $s \in \{+,-\}$.*

*1. If $f$ is localized at $Q^s$-modulation $\lesssim 2^k$ then*

$$\|f\|_{L^2L^2} \lesssim 2^{\frac{k}{2}}\|f\|_{N_k^s}. \tag{2.2.16}$$

*2. If $f$ is localized at $Q^s$-modulation $\gtrsim 2^k$ and $u$ is defined by*

$$\mathcal{F}u(\tau,\xi) = \frac{1}{\tau - s\,|\xi|}\mathcal{F}f(\tau,\xi) \tag{2.2.17}$$

*then*

$$\|u\|_{S_k^s} \lesssim \|f\|_{\dot H^{-1/2}} \tag{2.2.18}$$
$$\|u\|_{L^\infty L^2} \lesssim \|f\|_{N_k^s} \tag{2.2.19}$$

*Proof.* In view of the low modulation, (2.2.16) follows by duality from the embedding in Prop. 2.2.5. Similarly, (2.2.18) follows from the inequalities

$$\|u\|_{S_k^s} \lesssim \|u\|_{X_{s,1}^{0,\frac{1}{2}}} \lesssim \|f\|_{X_{s,1}^{0,-\frac{1}{2}}} \lesssim \|f\|_{\dot H^{-1/2}}.$$

Now we prove (2.2.19). Since $N_k^s$ is an atomic space we consider two cases. First, if $f$ is an $X_{s,1}^{0,-1/2}$-atom then we write

$$u(t) = \int e^{it\rho}e^{ist|D|}\phi_\rho\,\mathrm{d}\rho$$

where $\phi_\rho$ satisfies

$$\widehat{\phi_\rho}(\xi) = \mathcal{F}u(\rho + s\,|\xi|\,,\xi), \qquad \int \|\phi_\rho\|_{L^2}\,\mathrm{d}\rho \lesssim \|f\|_{X_{s,1}^{0,-1/2}}.$$

If $f$ is an $L^1L^2$-atom we write $u$ as a superposition of truncated homogenous waves

$$u(t) = \int e^{i(t-t')s|D|}f(t)1_{t>t'}\,\mathrm{d}t'.$$

In both cases (2.2.19) follows from the basic inequality for free waves

$$\|e^{ist|D|}\phi\|_{L^\infty L^2} \lesssim \|\phi\|_{L^2}.$$ □

The following lemma concerns the $Z$ spaces.

**Lemma 2.2.7.** *For $F$ with frequency support in $\{|\xi| \simeq 2^k\}$, we have*

$$\|F\|_{\Box Z^{\frac{d-2}{2}}} \lesssim \sup_{\ell<\frac{1}{2}C_1} 2^{-2k}2^{-\frac{3}{2}\ell}\Big(\sum_\omega \|P_{\ell_-}^\omega Q_{k+2\ell}P_kF\|_{L^1L^\infty}^2\Big)^{\frac{1}{2}}, \tag{2.2.20}$$
$$\|F\|_{\Box Z^{\frac{d-2}{2}}} \lesssim \|Q_{<k+C_1}P_kF\|_{L^1\dot H^{\frac{d-4}{2}}}, \tag{2.2.21}$$
$$\|F\|_{\Delta Z_{ell}^{\frac{d-2}{2}}} \lesssim \sup_{\ell<\frac{1}{2}C_1} 2^{-2k}2^{-\frac{1}{2}\ell}\Big(\sum_\omega \|P_{\ell_-}^\omega Q_{k+2\ell}P_kF\|_{L^1L^\infty}^2\Big)^{\frac{1}{2}}, \tag{2.2.22}$$
$$\|F\|_{\Delta Z_{ell}^{\frac{d-2}{2}}} \lesssim \|Q_{<k+C_1}P_kF\|_{L^1\dot H^{\frac{d-4}{2}}}, \tag{2.2.23}$$
$$\|F\|_{\Delta Z_{ell}^{\frac{d-2}{2}}} \lesssim \sup_{\ell<\frac{1}{2}C_1} 2^{\ell}\|Q_{k+2\ell}P_kF\|_{L^1\dot H^{\frac{d-4}{2}}}. \tag{2.2.24}$$

*Proof.* To prove (2.2.20), note that the symbol of the operator $(2^{2k+2\ell}/\Box)\tilde{P}^{\omega}_{\ell\_}\tilde{Q}_{k+2\ell}\tilde{P}_k$ obeys the same bump function estimates as the symbol of $P^{\omega}_{\ell\_}Q_{k+2\ell}P_k$ on the rectangular region of size $(2^{k+\ell})^{d-1}\times 2^{k+2\ell}\times 2^k$ where it is supported. Thus, this operator is disposable. Similarly, the operator $(2^{2k}/\Delta)\tilde{P}_k$ is disposable, which implies (2.2.22). The bound (2.2.21) [resp. (2.2.23), (2.2.24)] follows from (2.2.20) [resp. (2.2.22)] by applying Bernstein's inequality and using the orthogonality property of the sectors associated to $(P^{\omega}_{\ell\_})_{\omega}$. We note that the proof of (2.2.21), (2.2.23) are sharp only in $d = 4$. □

*Remark* 2.2.8. Notice the following simple inequalities:

$$\|P_kF\|_{N^s_k} \lesssim \|P_kF\|_{N_s}. \tag{2.2.25}$$

If the functions $f_{k'}$ have Fourier support in the regions $\{|\xi| \simeq 2^{k'}\}$ and $f = \sum_{k'} f_{k'}$ then

$$\|P_kf\|_{N^0_s} \lesssim \sum_{k'=k+O(1)} \|f_{k'}\|_{N^s_{k'}} \tag{2.2.26}$$

$$\|P_kf\|_{S^\sigma_s} \lesssim 2^{\sigma k} \sum_{k'=k+O(1)} \left(\|f_{k'}\|_{S^s_{k'}} + \|(i\partial_t + s\,|D|)f_{k'}\|_{L^2\dot{H}^{-1/2}}\right). \tag{2.2.27}$$

Finally, we have

**Proposition 2.2.9.** *Let $k \geq 0$ and $\mathcal{C}_{k'}(l')$ be a finitely overlapping collection of boxes. We have*

$$\sum_{\mathcal{C}_{k'}(l')} \|P_{\mathcal{C}_{k'}(l')}F\|^2_{\bar{N}_k} \lesssim \|F\|^2_{\bar{N}_k}$$

*Proof.* Since $\bar{N}_k$ is an atomic space the property reduces to the corresponding inequalities for $L^1L^2$ and $L^2_{t,x}$, which are standard inequalities. □

# Chapter 3

# Proofs of the main well-posedness theorems

Assuming the estimates in sections 1.6 and 1.8 we prove Theorems 1.1.1 and 1.1.2. The MD-CG system takes the form

$$\begin{cases} \alpha^{\mu}\mathbf{D}_{\mu}\psi = 0 \\ \Box A_j = \mathcal{P}_j\langle\psi, \alpha_x\psi\rangle \\ \Delta A_0 = -\langle\psi,\psi\rangle \end{cases} \tag{MD-CG}$$

while, for $J_\alpha = -\Im(\phi\overline{D_\alpha\phi})$, the MKG system is written as

$$\begin{cases} \Box_m^A\phi = 0 \\ \Box A_i = \mathcal{P}_i J_x \\ \Delta A_0 = J_0 \end{cases} \tag{MKG}$$

We begin with a more detailed formulation of the main parts of Theorems 1.1.1 and 1.1.2. After proving these we proceed to the proofs of statements (2) and (3) of the main theorems.

**Theorem 3.0.1.** *There exists a universal constant $\varepsilon > 0$ such that*

1. *For any initial data $\phi[0] \in H^\sigma \times H^{\sigma-1}$, $A_x[0] \in \dot{H}^\sigma \times \dot{H}^{\sigma-1}$ for* MKG *satisfying the smallness condition* (1.1.8) *and* (1.1.5)*, there exists a unique global solution $(\phi, A_x, A_0) \in \bar{S}^\sigma \times S^\sigma \times Y^\sigma$ to* MKG *with this data.*

2. *For any admissible frequency envelope $(c_k)_{k\geq 0}$ such that $\|\bar{P}_k\phi[0]\|_{H^\sigma\times H^{\sigma-1}} \leq c_k$, we have*

$$\|\bar{P}_k\phi\|_{\bar{S}^\sigma} \lesssim c_k, \quad \|P_{k'}[A_x - A_x^{free}]\|_{S^\sigma} + \|P_{k'}A_0\|_{Y^\sigma} \lesssim \begin{cases} c_{k'}^2, & k' \geq 0 \\ 2^{\frac{k'}{2}}c_0^2, & k' \leq 0 \end{cases}. \tag{3.0.1}$$

3. *(Weak Lipschitz dependence) Let* $(\phi', A') \in \bar{S}^\sigma \times S^\sigma \times Y^\sigma$ *be another solution to* MKG *with small initial data. Then, for* [1] $\delta \in (0, \delta_1)$ *we have*

$$\|\phi - \phi'\|_{\bar{S}^{\sigma-\delta}} + \|A - A'\|_{S^{\sigma-\delta} \times Y^{\sigma-\delta}} \lesssim \|(\phi - \phi')[0]\|_{H^{\sigma-\delta} \times H^{\sigma-\delta-1}} + \|(A_x - A'_x)[0]\|_{\dot{H}^{\sigma-\delta} \times \dot{H}^{\sigma-\delta-1}} \tag{3.0.2}$$

4. *(Persistence of regularity) If* $\phi[0] \in H^N \times H^{N-1}$, $A_x[0] \in \dot{H}^N \times \dot{H}^{N-1}$ $(N \geq \sigma)$, *then* $(\phi, \partial_t \phi) \in C_t(\mathbb{R}; H^N \times H^{N-1})$, $\nabla_{t,x} A_x \in C_t(\mathbb{R}; \dot{H}^{N-1})$. *In particular, if the data* $(\phi[0], A_x[0])$ *are smooth, then so is the solution* $(\phi, A)$.

Now let $(\psi(0), A_x[0])$ be an initial data set for MD-CG. We say that $c = (c_k)_{k \in \mathbb{Z}}$ is a *frequency envelope for* $(\psi(0), A_x[0])$ if

$$\|P_k \psi(0)\|_{\dot{H}^{1/2}} + \|P_k A_x[0]\|_{\dot{H}^1 \times L^2} \leq c_k.$$

**Theorem 3.0.2.** *There exists a universal constant* $\epsilon_* > 0$ *such that the following statements hold.*

1. *For any initial data* $\psi(0) \in \dot{H}^{1/2}$, $A_x[0] \in \dot{H}^1 \times L^2$ *for MD-CG satisfying the smallness condition* (1.1.9)*, there exists a unique global solution* $(A, \psi)$ *to MD-CG with these data in the space* $\Pi_s \psi \in \tilde{S}_s^{1/2}$, $A_0 \in Y^1$, $A_j \in S^1$. *Given any admissible frequency envelope* $c$ *for* $(\psi(0), A_x[0])$*, we have*

$$\sup_{s \in \{+,-\}} \|\Pi_s \psi\|_{(\tilde{S}_s^{1/2})_c} + \|A_x - A_x^{free}\|_{(S^1)_{c^2}} + \|A_0\|_{Y^1_{c^2}} \lesssim 1. \tag{3.0.3}$$

2. *Let* $(A', \psi')$ *be another solution to MD-CG such that* $\Pi_s \psi' \in \tilde{S}_s^{1/2}$, $A'_0 \in Y^1$, $A_j \in S^1$ *and the data* $\psi'(0), A'_x[0]$ *satisfies* (1.1.9). *Assume also that* $(\psi - \psi')(0) \in \dot{H}^{1/2-\delta_2}$ *and* $(A_x - A'_x)[0] \in \dot{H}^{1-\delta_2} \times \dot{H}^{-\delta_2}$ *for some* $\delta_2 \in (0, \delta_1)$. *Then we have*

$$\begin{aligned} \sup_{s \in \{+,-\}} &\|\Pi_s(\psi - \psi')\|_{\tilde{S}^{1/2-\delta_2}} + \|A_x - A'_x\|_{S^{1-\delta_2}} + \|A_0 - A'_0\|_{Y^{1-\delta_2}} \\ &\lesssim \|(\psi - \psi')(0)\|_{\dot{H}^{1/2-\delta_2}} + \|(A_x - A'_x)[0]\|_{\dot{H}^{1-\delta_2} \times \dot{H}^{-\delta_2}}. \end{aligned} \tag{3.0.4}$$

3. *If* $\psi(0) \in \dot{H}^{1/2+N}$, $A_x[0] \in \dot{H}^{1+N} \times \dot{H}^N$ $(N \geq 0)$*, then* $\psi \in C_t(\mathbb{R}; \dot{H}^{1/2+N})$, $\partial_{t,x} A_x \in C_t(\mathbb{R}; \dot{H}^N)$. *In particular, if the data* $(\psi(0), A_x[0])$ *are smooth, then so is the solution* $(A, \psi)$.

Theorems 3.0.1 and 3.0.2 are proved by a Picard-type iteration argument as in [31]. The presence of a non-perturbative interaction with $A^{free}$ precludes both the usual iteration procedure based on inverting $\Box$ or the free Dirac operator and the possibility of proving Lipschitz dependence in the full space $\bar{S}^\sigma \times S^\sigma \times Y^\sigma$. Instead, we will rely on Theorem 1.6.1 which provides linear estimates for $\Box_m^{p,A^{free}}$ and on the solvability result for the covariant Dirac equation given by Prop. 1.8.11.

[1] $\delta_1$ is the admissible frequency envelope constant.

*Remark* 3.0.3. The Maxwell-Dirac statements and proofs are presented for $d = 4$. In a general dimension $d \geq 4$, all arguments in this chapter apply with substitutions as in Remark 1.8.5.

## 3.1 Existence and frequency envelope bound for MD. Uniqueness.

We first prove Statement (1) of Theorem 3.0.2 except uniqueness, which is proved in the next step. We proceed by a Picard-type iteration, where the iterates are constructed recursively as follows. For the zeroth iterate, we take the trivial pair $(A^0, \psi^0) = 0$. Then for any $n \geq 0$, we first define

$$A_0^{n+1} = \mathbf{A}_0(\psi^n, \psi^n), \quad A_j^{n+1} = A_j^{free} + \mathbf{A}_j(\psi^n, \psi^n),$$

where $\mathbf{A}_0, \mathbf{A}_j$ are given by (1.7.19), (1.7.22) and $A_j^{free}$ denotes the free wave development of $A_j[0] = (A_j, \partial_t A_j)(0)$. Next, we define $\psi^{n+1}$ by solving the covariant Dirac equation

$$\alpha^\mu \mathbf{D}_\mu^{A^{n+1}} \psi^{n+1} = 0, \quad \psi^{n+1}(0) = \psi(0).$$

In order to construct $\psi^{n+1}$, we wish to apply Proposition 1.8.11 with $A = A^{n+1}$, or equivalently, $\psi' = \psi^n$ and $A_j^{free}[0] = A_j[0]$. When $n = 0$ we have $\psi^0 = 0$, so the hypothesis of Proposition 1.8.11 is verified simply by recalling (1.1.9) and taking $\epsilon_* \leq \epsilon_{**}$. For $n \geq 1$, we make the induction hypothesis

$$\sup_{s \in \{+,-\}} \|\Pi_s(\psi^m - \psi^{m-1})\|_{\tilde{S}_s^{1/2}} \leq (C_* \epsilon_*)^m \quad \text{for all } 1 \leq m \leq n. \tag{3.1.1}$$

for some universal constant $C_* > 0$. Recalling (1.1.9), summing up (3.1.1) for $1 \leq m \leq n$ and taking $\epsilon_*$ sufficiently small compared to $\epsilon_{**}$ (independent of $n$), we may ensure that the hypothesis (1.8.31) of Proposition 1.8.11 holds. Moreover, since $\psi^n$ obeys a covariant Dirac equation, the condition $\partial_\mu \langle \psi^n, \alpha^\mu \psi^n \rangle = 0$ is satisfied by Remark 1.7.6.

With an appropriate choice of $C_*$ and $\epsilon_*$, we claim that the $(n+1)$-th iterate $(A^{n+1}, \psi^{n+1})$ has the following properties:

$$\sup_{s \in \{+,-\}} \|\Pi_s \psi^{n+1}\|_{(\tilde{S}_s^{1/2})_c} + \|A_x^{n+1} - A_x^{free}\|_{(S^1)_{c^2}} + \|A_0^{n+1}\|_{Y^1_{c^2}} \lesssim 1, \tag{3.1.2}$$

$$\sup_{s \in \{+,-\}} \|\Pi_s(\psi^{n+1} - \psi^n)\|_{\tilde{S}_s^{1/2}} + \|A_x^{n+1} - A_x^n\|_{S^1} + \|A_0^{n+1} - A_0^n\|_{Y^1} \leq (C_* \epsilon_*)^{n+1}. \tag{3.1.3}$$

Assuming these, the proof of existence and (3.0.3) may be concluded as follows. Note that (3.1.3) ensures that the induction hypothesis (3.1.1) remains valid up to $m = n+1$. Moreover, these estimates immediately imply convergence of $(A^n, \psi^n)$ in the topology $\Pi_s \psi^n \in \tilde{S}_s^{1/2}$, $A_j \in S^1$ and $A_0 \in Y^1$ to a solution $(A, \psi)$ to MD-CG; furthermore, the solution obeys the frequency envelope bound (3.0.3).

It only remains to establish (3.1.2) and (3.1.3); we start with (3.1.2). Decomposing $\Delta A_0^{n+1}$, $\Delta \partial_t A_0^{n+1}$ and $\Box A_x^{n+1}$ as in Section 1.7 and applying Proposition 1.8.6, the proof of (3.1.2) is reduced to establishing

$$\sup_{s\in\{+,-\}} \|\Pi_s \psi^m\|_{(\tilde{S}^{1/2}_s)_c} \lesssim 1 \quad \text{for } m = 1, \dots, n+1. \tag{3.1.4}$$

Choosing $\epsilon_*$ sufficiently small and summing up the induction hypothesis (3.1.1), we obtain

$$\sup_{s\in\{+,-\}} \|\Pi_s \psi^m\|_{\tilde{S}^{1/2}_s} + \|A_x[0]\|_{\dot{H}^1\times L^2} \leq C\epsilon_* \leq \epsilon_{**} \quad \text{for } m = 0, \dots, n. \tag{3.1.5}$$

This bound allows us to apply Proposition 1.8.11, which implies (3.1.4) as desired.

Next, we turn to (3.1.3). For any $\mu \in \{0, 1, \dots, 4\}$, we may write

$$A^{n+1}_\mu - A^n_\mu = \mathbf{A}_\mu(\delta\psi^n, \psi^n) + \mathbf{A}_\mu(\psi^{n-1}, \delta\psi^n),$$

where we have used the shorthand $\delta\psi^n = \psi^n - \psi^{n-1}$. Decomposing $\Delta\mathbf{A}_0 = \mathcal{M}_0$, $\Delta\partial_t\mathbf{A}_0 = \partial_t\mathcal{M}_0$ and $\Box\mathbf{A}_x = \mathcal{M}_x$ as in Section 1.7 and applying[2] Proposition 1.8.6, we obtain

$$\begin{aligned}
&\|A^{n+1}_0 - A^n_0\|_{L^2\dot{H}^{3/2}} + \|\partial_t A^{n+1}_0 - \partial_t A^n_0\|_{L^2\dot{H}^{1/2}} + \|A^{n+1}_x - A^n_x\|_{S^1} \\
&\lesssim \sup_{s,s'\in\{+,-\}} \Big(\|\Pi_s\psi^n\|_{\tilde{S}^{1/2}_s} + \|\Pi_s\psi^{n-1}\|_{\tilde{S}^{1/2}_s}\Big)\|\Pi_{s'}\delta\psi^n\|_{\tilde{S}^{1/2}_s}
\end{aligned}$$

By (3.1.1) and (3.1.2) for $\psi^n$ and $\psi^{n-1}$, it follows that

$$\|A^{n+1}_0 - A^n_0\|_{Y^1} + \|A^{n+1}_x - A^n_x\|_{S^1} \lesssim \epsilon_*(C_*\epsilon_*)^n$$

which is acceptable by choosing $C_*$ larger than the implicit (universal) constant.

We now estimate the $\tilde{S}^{1/2}_s$ norm of $\delta\psi^{n+1} = \psi^{n+1} - \psi^n$. We begin by computing

$$\begin{aligned}
\alpha^\mu \mathbf{D}^{A^n}_\mu \delta\psi^{n+1} &= -i\alpha^\mu(A^{n+1}_\mu - A^n_\mu)\psi^{n+1} \\
&= -i\alpha^\mu\Big(\mathbf{A}_\mu(\delta\psi^n, \psi^n) + \mathbf{A}_\mu(\psi^{n-1}, \delta\psi^n)\Big)\psi^{n+1}.
\end{aligned}$$

By symmetry, it suffices to consider only the contribution of $\mathbf{A}_\mu(\delta\psi^{n-1}, \psi^n)$. Using the shorthand $\psi^{n+1}_s = \Pi_s\psi^{n+1}$, we expand

$$\Pi_s\Big(\alpha^\mu\mathbf{A}_\mu(\delta\psi^n, \psi^n)\psi^{n+1}\Big)$$

$$= \Pi_s\Big(\pi^E[\mathbf{A}_0(\delta\psi^n, \psi^n)]\psi^{n+1}_s - s\pi^R[\mathbf{A}_x(\delta\psi^n, \psi^n)]\psi^{n+1}_s\Big) \tag{3.1.6}$$

$$+ \Pi_s\widetilde{\mathcal{N}}^E(\mathbf{A}_0(\delta\psi^n, \psi^n), \Pi_s\psi^{n+1}_s) - s\Pi_s\widetilde{\mathcal{N}}^R(\mathbf{A}_x(\delta\psi^n, \psi^n), \psi^{n+1}_s) \tag{3.1.7}$$

$$+ \Pi_s\mathcal{N}^E(\mathbf{A}_0(\delta\psi^n, \psi^n), \Pi_{-s}\psi^{n+1}_{-s}) + s\Pi_s\mathcal{N}^R(\mathbf{A}_x(\delta\psi^n, \psi^n), \psi^{n+1}_{-s}) \tag{3.1.8}$$

$$+ \Pi_s\mathcal{N}^S_s(\mathbf{A}_x(\delta\psi^n, \psi^n), \psi^{n+1}_s) + \Pi_s\mathcal{N}^S_{-s}(\mathbf{A}_x(\delta\psi^n, \psi^n), \psi^{n+1}_{-s}) \tag{3.1.9}$$

[2] Proposition 1.8.6 is stated in terms of admissible frequency envelopes. Constructing frequency envelopes as in (1.5.13), Proposition 1.8.6 easily implies the non-frequency envelope version, which we use here. The same remark applies to the application of estimates in Propositions 1.8.7 and 1.8.8 below.

We wish to estimate the $N_s^{1/2} \cap L^2 L^2 \cap G^{1/2}$ norm of the RHS using Proposition 1.8.7 and 1.8.8. More precisely, For the $N_s^{1/2}$ norm, we apply (1.8.28) for (3.1.6); (1.8.15)–(1.8.16) for (3.1.7); (1.8.15)–(1.8.16), (1.8.18)–(1.8.19) for (3.1.8) and (1.8.17), (1.8.20) for (3.1.9). For the $L^2 L^2 \cap G^{1/2}$ norm, we simply use (1.8.21)–(1.8.23) and (1.8.24)–(1.8.26). Then we obtain

$$\|\Pi_s\Big(\alpha^\mu \mathbf{A}_\mu(\delta\psi^n, \psi^n)\psi^{n+1}\Big)\|_{N_s^{1/2}\cap L^2L^2\cap G^{1/2}} \lesssim \sup_{s_1,s_2,s_3} \|\delta\psi^n_{s_1}\|_{\tilde{S}^{1/2}_{s_1}} \|\psi^n_{s_2}\|_{\tilde{S}^{1/2}_{s_2}} \|\psi^{n+1}_{s_3}\|_{\tilde{S}^{1/2}_{s_3}}$$

Hence by Proposition 1.8.11, (3.1.1) and (3.1.2) for $\psi^{n+1}$ and $\psi^n$, we arrive at

$$\sup_{s\in\{+,-\}} \|\Pi_s \delta\psi^{n+1}\|_{\tilde{S}^{1/2}_s} \lesssim \epsilon_*^2 (C_* \epsilon_*)^n, \tag{3.1.10}$$

which is acceptable.

## Uniqueness

To finish the proof of Statement (1) of Theorem 3.0.2, we need to show that the solution $(A, \psi)$ is unique in the iteration space. Let $(A', \psi')$ be another solution to MD-CG with the same data, which obeys $\Pi_s \psi' \in \tilde{S}^{1/2}_s$, $A'_x \in S^1$ and $A'_0 \in Y^1$. To prove the desired uniqueness, by a simple continuity argument, it is enough show that $(A, \psi) = (A', \psi')$ on $[0, T]$ for some $T = T(\psi(0), A_x[0]) > 0$. Moreover, it is clear from MD-CG that $A'_0 = \mathbf{A}_0(\psi', \psi')$ and $A'_x = A^{free}_x + \mathbf{A}_x(\psi', \psi')$; hence it suffices to establish

$$\psi(t) = \psi'(t) \quad \text{ for } t \in [0, T]. \tag{3.1.11}$$

Define $\delta\psi = \psi' - \psi$. By Proposition 1.8.11, we have

$$\sup_{s\in\{+,-\}} \|\Pi_s \delta\psi\|_{\tilde{S}^{1/2}_s[0,T]} \lesssim \sup_{s\in\{+,-\}} \|\Pi_s \alpha^\mu \mathbf{D}^A_\mu \delta\psi\|_{N_s^{1/2}\cap L^2L^2\cap G^{1/2}[0,T]} \tag{3.1.12}$$

Moreover, writing out the equations for $\alpha^\mu \mathbf{D}^A_\mu \delta\psi$ and $\alpha^\mu \partial_\mu \delta\psi$ in terms of $\psi$, $\delta\psi$ and analyzing it as in the proof of (3.1.3), we arrive at

$$\text{RHS of (3.1.12)} \lesssim \Big(\epsilon_* + \sup_{s\in\{+,-\}} \|\Pi_s \delta\psi\|_{\tilde{S}^{1/2}_s[0,T]}\Big)^2 \sup_{s\in\{+,-\}} \|\Pi_s \delta\psi\|_{\tilde{S}^{1/2}_s[0,T]}$$

In particular, the RHS of (3.1.12) is finite; hence the LHS of (3.1.12) can be made as small as we want by choosing $T$ sufficiently small (we use (2.1.18) for $N_s^{1/2}$). Combining (3.1.12) with the preceding estimate, and taking $\epsilon_*$ smaller if necessary, we may conclude that $\delta\psi = 0$ on $[0, T]$ as desired.

## 3.2 Existence and uniqueness for MKG. Frequency envelopes bounds.

We now prove Statement (1) of Theorem 3.0.1.

**S**tep 1. We set up a Picard iteration. For the zeroth iterate, we take $(\phi^0, A_j^0, A_0^0) = (0, A_j^{free}, 0)$ and for any $n \geq 0$ define $J_\alpha^n = -\Im(\phi^n \overline{D_\alpha^{A_n} \phi^n})$ and, recursively,

$$\begin{aligned} \Box_m^{A^n} \phi^{n+1} &= 0 && (3.2.1)\\ \Box A_j^{n+1} &= \mathcal{P}_j J_x^n && (3.2.2)\\ \Delta A_0^{n+1} &= J_0^n && (3.2.3) \end{aligned}$$

with initial data $(\phi[0], A_x[0])$. Differentiating (3.2.3) and using Remark 1.7.1, we get

$$\Delta \partial_t A_0^{n+1} = \partial^i J_i^n. \tag{3.2.4}$$

Note that $A_0^1 = 0$. We claim that

$$\|A_x^1\|_{S^\sigma} = \|A_x^{free}\|_{S^\sigma} \leq C_0 \|A_x[0]\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} \leq C_0 \varepsilon, \qquad \|\phi^1\|_{\bar{S}^\sigma} \leq C_0 \varepsilon \tag{3.2.5}$$

where $A_j^{free}$ denotes the free wave development of $A_j[0] = (A_j, \partial_t A_j)(0)$.

For $n \geq 1$, denoting $A^m = (A_x^m, A_0^m)$ we make the induction hypothesis

$$\|\phi^m - \phi^{m-1}\|_{\bar{S}^\sigma} + \|A^m - A^{m-1}\|_{\ell^1 S^\sigma \times Y^\sigma} \leq (C_* \varepsilon)^m \qquad m = 2, n. \tag{3.2.6}$$

for a universal constant $C_* > 0$. By summing this up and adding (3.2.5) we get

$$\|\phi^m\|_{\bar{S}^\sigma} + \|A_x^m - A_x^{free}\|_{\ell^1 S^\sigma} + \|A_x^m\|_{S^\sigma} + \|A_0^m\|_{Y^\sigma} \leq 2C_0 \varepsilon \quad m = 1, n. \tag{3.2.7}$$

These estimates imply convergence of $(\phi^n, A_x^n, A_0^n)$ in the topology of $\bar{S}^\sigma \times S^\sigma \times Y^\sigma$ to a solution of MKG.

**S**tep 2. Notice that we can decompose

$$\begin{aligned} A_0^{n+1} &= \mathbf{A}_0(\phi^n, \phi^n) + A_0^{R,n+1}, \qquad && A_0^{R,n+1} := -\Delta^{-1}(|\phi^n|^2 A_0^n)\\ A_j^{n+1} &= A_j^{free} + \mathbf{A}_j(\phi^n, \phi^n) + A_j^{R,n+1}, \qquad && A_j^{R,n+1} := -\Box^{-1}\mathcal{P}_j(|\phi^n|^2 A_x^n) \end{aligned}$$

for $\mathbf{A} = (\mathbf{A}_0, \mathbf{A}_j)$ defined in (1.7.4), and set $A^{R,n} = (A_x^{R,n}, A_0^{R,n})$. To estimate $A^{n+1} - A^n$ we write

$$\begin{aligned} A^{n+1} - A^n &= \mathbf{A}(\phi^n - \phi^{n-1}, \phi^n) + \mathbf{A}(\phi^{n-1}, \phi^n - \phi^{n-1}) + \big(A^{R,n+1} - A^{R,n}\big)\\ \partial_t A_0^{n+1} - \partial_t A_0^n &= \Delta^{-1} \nabla_x \Im\big(\phi^{n-1} \nabla_x \overline{\phi^{n-1}} - \phi^n \nabla_x \bar{\phi}^n + i\left|\phi^{n-1}\right|^2 A_x^{n-1} - i\left|\phi^n\right|^2 A_x^n\big) \end{aligned} \tag{3.2.8}$$

The difference $A^{n+1} - A^n$ is estimated in $\ell^1 S^\sigma \times Y^\sigma$ using Proposition 2.2.3 and (1.8.1)-(1.8.3), together with (3.2.6), (3.2.7). With an appropriate choice of $C_*$ and $\varepsilon$, this insures the induction hypothesis (3.2.6) for $A$ remains valid with $m = n + 1$.

Moreover, using (2.1.16) and (1.8.3) with (3.2.6), (3.2.7) we obtain

$$\|A^{R,n}\|_{(Z^\sigma\cap\ell^1 S^\sigma)\times(Z^\sigma_{ell}\cap Y^\sigma)} \lesssim \varepsilon, \quad \|A^{R,n}-A^{R,n-1}\|_{(Z^\sigma\cap\ell^1 S^\sigma)\times(Z^\sigma_{ell}\cap Y^\sigma)} \lesssim (C_*\varepsilon)^{n+1} \tag{3.2.9}$$

**S**tep 3. In order to solve (3.2.1), we rewrite it as

$$\Box_m^{p,A^{free}}\phi^{n+1} = \mathcal{M}(A^n,\phi^{n+1})$$

where

$$\begin{aligned}(2i)^{-1}\mathcal{M}(A^n,\phi) = &\big(A^n_\alpha\cdot\partial^\alpha\phi - \pi[A^n]\phi\big) + \pi[A^{R,n}]\phi \\ &+\pi[\mathbf{A}(\phi^{n-1},\phi^{n-1})]\phi - (2i)^{-1}\big(\partial_t A^n_0\phi + A^{n,\alpha}A^n_\alpha\phi\big)\end{aligned}$$

We prove that the map $\phi\mapsto\psi$ defined by $\Box_m^{p,A^{free}}\psi = \mathcal{M}(A^n,\phi)$ is a contraction on $\bar{S}^\sigma$. This follows from Theorem 1.6.1 together with

$$\|\mathcal{M}(A^n,\phi)\|_{\bar{N}^{\sigma-1}\cap L^2H^{\sigma-\frac{3}{2}}} \lesssim \varepsilon\|\phi\|_{\bar{S}^\sigma}. \tag{3.2.10}$$

which holds due to (1.8.4)-(1.8.9), (1.8.10) since we have (3.2.7) and (3.2.9).

Moreover, this argument also establishes (3.2.5) for $\phi^1$ since we are assuming $A^{R,0} = \mathbf{A}(\phi^{-1},\phi^{-1}) = 0$.

**S**tep 4. To estimate $\phi^{n+1}-\phi^n$ using Theorem 1.6.1 in addition to applying (3.2.10) with $\phi=\phi^{n+1}-\phi^n$ we also need

$$\|\mathcal{M}(A^n,\phi^n)-\mathcal{M}(A^{n-1},\phi^n)\|_{\bar{N}^{\sigma-1}\cap L^2H^{\sigma-\frac{3}{2}}} \lesssim (C_*\varepsilon)^n\|\phi^n\|_{\bar{S}^\sigma}$$

This follows by applying (1.8.4), (1.8.8), (1.8.9) with $A=A^n-A^{n-1}$, then (1.8.5), (1.8.6) with $A=A^{R,n}-A^{R,n-1}$, and finally (1.8.10) with $\mathbf{A}(\phi^{n-1},\phi^{n-1}-\phi^{n-2})$ and $\mathbf{A}(\phi^{n-1}-\phi^{n-2},\phi^{n-2})$. We use these together with (3.2.6) and (3.2.9). We conclude that, with appropriate $C_*$ and $\varepsilon$, the induction hypothesis (3.2.6) remains valid with $m=n+1$ for $\phi$ as well.

**S**tep 5. To prove uniqueness, assume that $(\phi,A)$ and $(\phi',A')$ are two solutions with the same initial data. Then the same $A^{free}$ is used in $\Box_m^{p,A^{free}}$ for both $\phi,\phi'$ and using the same estimates as above one obtains

$$\|A-A'\|_{\ell^1S^\sigma\times Y^\sigma}+\|\phi-\phi'\|_{\bar{S}^\sigma} \lesssim \varepsilon\big(\|A-A'\|_{\ell^1S^\sigma\times Y^\sigma}+\|\phi-\phi'\|_{\bar{S}^\sigma}\big).$$

Choosing $\varepsilon$ small enough the uniqueness statement follows.

## The frequency envelope bounds (3.0.1)

The main observation here is that all estimates used in the proof of existence have a frequency envelope version. Using Remark 4.2.3 and $\Box_m^{p,A^{free}}\phi=\mathcal{M}(A,\phi)$ we have

$$\|\phi\|_{\bar{S}^\sigma_c} \lesssim \|\phi[0]\|_{(H^\sigma\times H^{\sigma-1})_c} + \|\mathcal{M}(A,\phi)\|_{(\bar{N}^{\sigma-1}\cap L^2H^{\sigma-\frac{3}{2}})_c} \tag{3.2.11}$$

By (6.1.26), (6.1.27), (6.1.31), (6.1.32), (6.1.34), (6.2.2) , (6.2.13), (6.2.15), Lemma 6.1.1 and the proof of (1.8.7)-(1.8.9) we have

$$\|\mathcal{M}(A,\phi)\|_{(\bar{N}^{\sigma-1}\cap L^2H^{\sigma-\frac{3}{2}})_c} \lesssim \left(\|A\|_{S^\sigma\times Y^\sigma} + \|A^R\|_{(Z^\sigma\cap\ell^1S^\sigma)\times(Z^\sigma_{ell}\cap Y^\sigma)} + \|\phi\|^2_{\bar{S}^\sigma}\right)\|\phi\|_{\bar{S}^\sigma_c} \quad (3.2.12)$$

The term in the bracket is $\lesssim \varepsilon$, thus from (3.2.11) we obtain $\|\phi\|_{\bar{S}^\sigma_c} \lesssim \|\phi[0]\|_{(H^\sigma\times H^{\sigma-1})_c}$ which implies $\|\bar{P}_k\phi\|_{\bar{S}^\sigma} \lesssim c_k$.

Now we turn to $A$. We define $\tilde{c}_{k'} = c^2_{k'}$ for $k' \geq 0$ and $\tilde{c}_{k'} = 2^{\frac{k'}{2}} c_0^2$ for $k' \leq 0$. One has

$$\|A_x - A_x^{free}\|_{S^\sigma_{\tilde{c}}} + \|A_0\|_{Y^\sigma_{\tilde{c}}} \lesssim \|\Box A_x\|_{N^{\sigma-1}_{\tilde{c}}\cap L^2\dot{H}^{\sigma-\frac{3}{2}}_{\tilde{c}}} + \|\Delta A_0\|_{\Delta Y^\sigma_{\tilde{c}}} \lesssim \|\phi\|^2_{\bar{S}^\sigma_c} \lesssim 1$$

using (6.1.18) and the proofs of (1.8.2), (1.8.3). This concludes the proof of (3.0.1).

*Remark* 3.2.1. A consequence of (3.0.1) is that if we additionally assume $(\phi[0], A_x[0]) \in H^s \times H^{s-1} \times \dot{H}^s \times \dot{H}^{s-1}$ for $s \in (\sigma, \sigma+\delta_1)$ then we can deduce

$$\|\phi\|_{L^\infty(H^s\times H^{s-1})} + \|A\|_{L^\infty(\dot{H}^s\times\dot{H}^{s-1})} \lesssim \|\phi[0]\|_{H^s\times H^{s-1}} + \|A_x[0]\|_{\dot{H}^s\times\dot{H}^{s-1}} \quad (3.2.13)$$

Indeed, choosing the frequency envelope

$$c_k = \sum_{k_1\geq 0} 2^{-\delta_1|k-k_1|}\|\bar{P}_{k_1}\phi[0]\|_{H^\sigma\times H^{\sigma-1}}, \qquad \|c_k\|_{\ell^2(\mathbb{Z}_+)} \simeq \|\phi[0]\|_{H^\sigma\times H^{\sigma-1}} \quad (3.2.14)$$

from (3.0.1) we obtain

$$\|\phi\|_{L^\infty(H^s\times H^{s-1})} \lesssim \| \langle D\rangle^{s-\sigma} \phi\|_{\bar{S}^\sigma} \lesssim \|2^{k(s-\sigma)}c_k\|_{\ell^2(\mathbb{Z}_+)} \lesssim \|\phi[0]\|_{H^s\times H^{s-1}}$$

and similarly with $(A_x - A_x^{free}, A_0)$; meanwhile $\|A_x^{free}\|_{L^\infty(\dot{H}^s\times\dot{H}^{s-1})} \lesssim \|A_x[0]\|_{\dot{H}^s\times\dot{H}^{s-1}}$.

## 3.3 Weak Lipschitz dependence

### The MD case

Here we outline the proof of Statement (2) of Theorem 3.0.2. Let $\delta\psi = \psi - \psi'$ and $\delta A = A - A'$. It is clear from MD-CG that $A'_0 = \mathbf{A}_0(\psi',\psi')$ and $A'_x = (A'_x)^{free} + \mathbf{A}_x(\psi',\psi')$, where $(A'_x)^{free}$ is the free wave development of $A'_x[0]$. Applying Proposition 1.8.6 with appropriate frequency envelopes, we see that establishing (3.0.4) reduces to showing

$$\sup_{s\in\{+,-\}} \|\Pi_s\delta\psi\|_{\tilde{S}^{1/2-\delta_2}_s} \lesssim \|\delta\psi(0)\|_{\dot{H}^{1/2-\delta_2}} + \|\delta A_x[0]\|_{\dot{H}^{1-\delta_2}\times\dot{H}^{-\delta_2}}. \quad (3.3.1)$$

For simplicity of exposition, we will assume that $\Pi_s\delta\psi \in \tilde{S}^{1/2-\delta_2}_s$ and prove (3.3.1). This assumption can be bypassed by establishing (3.3.1) for the difference $\delta\psi = \psi^n - (\psi')^n$ of Picard iterates in Step 1; we omit the details.

The difference $\delta\psi$ obeys the covariant equation

$$\alpha^\mu \mathbf{D}^A_\mu \delta\psi = -i\alpha^\mu\Big(\mathbf{A}_\mu(\delta\psi,\psi) + \mathbf{A}_\mu(\psi',\delta\psi)\Big)\psi' - i\alpha^\ell \delta A^{free}_\ell \psi' =: \delta I_1 + \delta I_2.$$

We claim that

$$\sup_{s'\in\{+,-\}} \|\Pi_{s'}\delta I_1\|_{N^{1/2-\delta_2}_{s'}\cap L^2\dot{H}^{-\delta_2}\cap G^{1/2-\delta_2}} \lesssim \epsilon_*^2 \sup_{s\in\{+,-\}} \|\Pi_s \delta\psi\|_{\tilde{S}^{1/2-\delta_2}_s}, \tag{3.3.2}$$

$$\sup_{s'\in\{+,-\}} \|\Pi_{s'}\delta I_2\|_{N^{1/2-\delta_2}_{s'}\cap L^2\dot{H}^{-\delta_2}\cap G^{1/2-\delta_2}} \lesssim \epsilon_* \|\delta A_x[0]\|_{\dot{H}^{1-\delta_2}\times\dot{H}^{-\delta_2}}. \tag{3.3.3}$$

Assuming that (3.3.2)–(3.3.3) hold, we may finish the proof as follows. Applying Proposition 1.8.11 with an appropriate frequency envelope, we obtain

$$\sup_{s\in\{+,-\}} \|\Pi_s\delta\psi\|_{S^{1/2-\delta_2}_s} \lesssim \|\delta\psi(0)\|_{\dot{H}^{1/2-\delta_2}} + \sup_{s\in\{+,-\}} \|\Pi_s(\alpha^\mu \mathbf{D}^A_\mu \delta\psi)\|_{N^{1/2-\delta_2}_s\cap L^2\dot{H}^{-\delta_2}\cap G^{1/2-\delta_2}}$$

The last terms can be estimated using (3.3.2)–(3.3.3). Taking $\epsilon_*$ sufficiently small to absorb the contribution of $\|\Pi_s\delta\psi\|_{\tilde{S}^{1/2-\delta_2}_s}$ (which is finite by assumption) into the LHS, the desired inequality (3.3.1) follows in a straightforward manner.

It only remains to establish (3.3.2)–(3.3.3). The proof of (3.3.2) is very similar to that of (3.1.3) in Step 1; we omit the details. To prove (3.3.3), we start by writing

$$\Pi_{s'}\delta I_2 = -i\sum_s \Pi_{s'}(\alpha^\ell \delta A^{free}_\ell)\Pi_s\psi_s = i\sum_s \Big(s\Pi_{s'}\mathcal{N}^R(\delta A^{free}_x,\psi_s) - \mathcal{N}^S_s(\delta A^{free}_x,\psi_s)\Big)$$

where $\psi_s = \Pi_s\psi$. The $L^2\dot{H}^{-\delta_2}\cap G^{1/2-\delta_2}$ norm of both terms can be handled by applying (1.8.21)–(1.8.23) and (1.8.24)–(1.8.26) with appropriate frequency envelopes. Henceforth, we focus on the $N^{1/2-\delta_2}_s$ norm. The term $\mathcal{N}^S_s(\delta A^{free}_x,\psi_s)$ can be treated using (1.8.17) and (1.8.20). For the term $\mathcal{N}^R(\delta A^{free}_x,\psi_s)$, application of (1.8.16) and (1.8.19) leaves us only with the term $s'\Pi_{s'}(\pi^R[\delta A^{free}_x]\psi_{s'})$. For this term, we apply (1.8.29) with frequency envelopes $a$ and $b$ for $\|\delta A_x[0]\|_{\dot{H}^1\times L^2}$ and $\|\psi\|_{\tilde{S}^{1/2}_{s'}}$, respectively. Observe that $\sum_{k'<k} a_{k'} \lesssim 2^{\delta_2}\|\delta A_x[0]\|_{\dot{H}^{1-\delta_2}\times\dot{H}^{-\delta_2}}$, so

$$\|\pi^R[A^{free}_x]\psi_{s'}\|_{N^{1/2-\delta_2}_{s'}} \lesssim \|\delta A_x[0]\|_{\dot{H}^{1-\delta_2}\times\dot{H}^{-\delta_2}}\|\psi_{s'}\|_{\tilde{S}^{1/2}_{s'}}$$

which is exactly what we need (it is this point where $\delta_2 > 0$ is used).

## The MKG case (3.0.2)

Let $\delta\phi = \phi - \phi'$ and $\delta A = A - A'$. Similarly to the equations in (3.2.8) we write

$$\delta A = \mathbf{A}(\delta\phi,\phi) + \mathbf{A}(\phi',\delta\phi) + \big(A^R - A'^R\big)$$

and similarly for $\delta\partial_t A_0$. Applying (6.1.18) and the estimates in the proofs of (1.8.2), (1.8.3) we get

$$\|\delta A\|_{S^{\sigma-\delta}\times Y^{\sigma-\delta}} \lesssim \|\delta A_x[0]\|_{\dot{H}^{\sigma-\delta}\times\dot{H}^{\sigma-\delta-1}} + \varepsilon\|\delta\phi\|_{\bar{S}^{\sigma-\delta}} + \varepsilon\|\delta A\|_{S^{\sigma-\delta}\times Y^{\sigma-\delta}}.$$

By Remark 4.2.3 we have

$$\|\delta\phi\|_{\bar{S}^{\sigma-\delta}} \lesssim \|\delta\phi[0]\|_{H^{\sigma-\delta}\times H^{\sigma-\delta-1}} + \|\Box_m^{p,A^{free}}\delta\phi\|_{\bar{N}^{\sigma-\delta-1}}.$$

The equation for $\delta\phi$ is

$$\Box_m^{p,A^{free}}\delta\phi = \mathcal{M}(A,\delta\phi) + \big(\mathcal{M}(A,\phi') - \mathcal{M}(A',\phi')\big) + 2i\sum_{k\geq 0}\delta A^{free}_{<k-C}\cdot\nabla_x\phi'_k$$

By applying (3.2.12) with an appropriate frequency envelope $c$ we get

$$\|\mathcal{M}(A,\delta\phi)\|_{\bar{N}^{\sigma-\delta-1}\cap L^2H^{\sigma-\frac{3}{2}-\delta}} \lesssim \varepsilon\|\delta\phi\|_{\bar{S}^{\sigma-\delta}}$$

Similarly we obtain

$$\|\mathcal{M}(A,\phi') - \mathcal{M}(A',\phi')\|_{\bar{N}^{\sigma-\delta-1}\cap L^2H^{\sigma-\frac{3}{2}-\delta}} \lesssim \varepsilon\big(\|\delta A\|_{S^{\sigma-\delta}\times Y^{\sigma-\delta}} + \|\delta\phi\|_{\bar{S}^{\sigma-\delta}}\big)$$

Using (6.1.32) (note that the $\mathcal{H}^*$ term is 0 for $A^{free}$) we get

$$\|\sum_{k\geq 0}\delta A^{free}_{<k-C}\cdot\nabla_x\phi'_k\|_{\bar{N}^{\sigma-\delta-1}\cap L^2H^{\sigma-\frac{3}{2}-\delta}} \lesssim \|\delta A^{free}\|_{S^{\sigma-\delta}}\|\phi'\|_{\bar{S}^{\sigma}} \lesssim \varepsilon\|\delta A_x[0]\|_{\dot{H}^{\sigma-\delta}\times\dot{H}^{\sigma-\delta-1}}$$

At this point is where $\delta > 0$ was used, to do the $k' < k$ $\ell^2$-summation of $\delta A^{free}$. Putting the above together we obtain

$$\begin{aligned}\|\delta\phi\|_{\bar{S}^{\sigma-\delta}} + \|\delta A\|_{S^{\sigma-\delta}\times Y^{\sigma-\delta}} \lesssim\ & \|\delta\phi[0]\|_{H^{\sigma-\delta}\times H^{\sigma-\delta-1}} + \|\delta A_x[0]\|_{\dot{H}^{\sigma-\delta}\times\dot{H}^{\sigma-\delta-1}} \\ & + \varepsilon\big(\|\delta\phi\|_{\bar{S}^{\sigma-\delta}} + \|\delta A\|_{S^{\sigma-\delta}\times Y^{\sigma-\delta}}\big).\end{aligned}$$

For $\varepsilon$ small enough we obtain (3.0.2).

## 3.4 Subcritical local well-posedness

Here we review some local wellposedness facts that will be used in the proofs below. We assume $\partial^\ell A_\ell(0) = \partial^\ell\partial_t A_\ell(0) = 0$. Given $s, N \in \mathbb{R}$, we introduce shorthands $H^{s,N} = \dot{H}^s \cap \dot{H}^N$ and $\mathcal{H}^{s,N} = (\dot{H}^s \times \dot{H}^{s-1}) \cap (\dot{H}^N \times \dot{H}^{N-1})$. Note that for $s > \frac{d}{2} + 1$, $H^{s-1}$ becomes a Banach Algebra of functions on $\mathbb{R}^d$.

**Proposition 3.4.1.** *For any initial data $\psi(0) \in H^{1/2,5/2}$ and $A_x[0] \in \mathcal{H}^{1,3}$, there exists a unique local solution $(A,\psi)$ to MD-CG with these data in the space $\psi \in C_t([0,T]; H^{1/2,5/2})$ and $\partial_{t,x}A_x \in C_t([0,T]; \mathcal{H}^{0,2})$, where $T > 0$ depends only on $\|\psi(0)\|_{H^{1/2,5/2}}$ and $\|A_x[0]\|_{\mathcal{H}^{1,3}}$. The data-to-solution map in these spaces is Lipschitz continuous. Moreover, if $\psi(0) \in \dot{H}^{1/2+N}$, $A_x[0] \in \dot{H}^{1+N}\times\dot{H}^N$ for $N \geq 2$, then $\psi \in C_t([0,T]; \dot{H}^{1/2+N})$, $\partial_{t,x}A_x \in C_t([0,T]; \dot{H}^N)$.*

For MKG we have:

**Proposition 3.4.2.** *Let $s > \frac{d}{2} + 1$. For any initial data $\phi[0] \in H^s \times H^{s-1}$ and $A_x[0] \in \mathcal{H}^{\sigma,s}$ there exists a unique local solution $(\phi, A)$ to* MKG *with these data in the space $(\phi, \partial_t \phi; A, \partial_t A) \in C_t([0,T], H^s \times H^{s-1}; \mathcal{H}^{\sigma,s})$ where $T > 0$ depends continuously on $\|\phi[0]\|_{H^s \times H^{s-1}}$ and $\|A_x[0]\|_{\mathcal{H}^{\sigma,s}}$. The data-to-solution map in these spaces is Lipschitz continuous. Moreover, additional Sobolev regularity of the initial data is preserved by the solution.*

We omit the proofs, which proceeds by usual Picard iteration (based on the d'Alembertian $\Box$ and the free Dirac operator $\alpha^\mu \partial_\mu$) and the algebra and multiplication properties of the spaces above. Here, the massive term $\phi$ can be treated perturbatively.

We remark that a stronger subcritical result - almost optimal local well-posedness (i.e. initial data in $H^{1+\varepsilon}(\mathbb{R}^4)$) was proved in [48].

## 3.5 Persistence of regularity

Now we sketch the proof of Statement (3) of Theorem 3.0.2 and Statement (4) of Theorem 3.0.1. In view of Propositions 3.4.1 and 3.4.2, it suffices to show that

$$\sup_{s \in \{+,-\}} \|\Pi_s \psi\|_{\tilde{S}_s^{1/2+N}} + \|A_x\|_{S^{1+N}} \lesssim \|\psi(0)\|_{\dot{H}^{1/2+N}} + \|A_x[0]\|_{\dot{H}^{1+N} \times \dot{H}^N} \tag{3.5.1}$$

and

$$\|\nabla^N \phi\|_{\bar{S}^\sigma} + \|\nabla^N (A - A_x^{free})\|_{\ell^1 S^\sigma \times Y^\sigma} \lesssim \|\phi[0]\|_{H^{\sigma+N} \times H^{\sigma+N-1}} + \|A_x[0]\|_{\dot{H}^{\sigma+N} \times \dot{H}^{\sigma+N-1}}. \tag{3.5.2}$$

for $N = 1, 2$, whenever the RHS is finite. Henceforth, we only consider the case $N = 1$; the case $N = 2$ can be handled similarly. Moreover, for simplicity, we will already assume that $\Pi_s \psi \in \tilde{S}_s^{1/2+N}$ and $\nabla \phi \in \bar{S}^\sigma$, $\nabla A \in S^\sigma \times Y^\sigma$ . As before, this assumption may be bypassed by repeating the proof of (3.5.1), (3.5.2) for each iterate.

### The case of MD

By Proposition 1.8.6 (for $\Box \nabla \mathbf{A}_x$), it suffices to bound only the contribution of $\psi$ in (3.5.1). Observe that $\nabla \psi$ obeys

$$\alpha^\mu \mathbf{D}_\mu^A \nabla \psi = -i \alpha^\mu \Big( \mathbf{A}_\mu(\nabla \psi, \psi)\psi + \mathbf{A}_\mu(\psi, \nabla \psi) \Big) \psi - i \alpha^\ell \nabla A_\ell^{free} \psi =: I_1 + I_2.$$

We claim that

$$\sup_{s' \in \{+,-\}} \|\Pi_{s'} I_1\|_{N_{s'}^{1/2} \cap L^2 L^2 \cap G^{1/2}} \lesssim \epsilon_*^2 \sup_{s \in \{+,-\}} \|\Pi_s \psi\|_{\tilde{S}_s^{3/2}}, \tag{3.5.3}$$

$$\sup_{s' \in \{+,-\}} \|\Pi_{s'} I_2\|_{N_{s'}^{1/2} \cap L^2 L^2 \cap G^{1/2}} \lesssim \epsilon_* ( \sup_{s \in \{+,-\}} \|\Pi_s \psi\|_{\tilde{S}_s^{3/2}} + \|A_x[0]\|_{\dot{H}^2 \times \dot{H}^1}), \tag{3.5.4}$$

Then by Proposition 1.8.11 and (3.5.3)–(3.5.4), we would have

$$\sup_{s\in\{+,-\}} \|\Pi_s \psi\|_{\tilde{S}_s^{3/2}} \lesssim \|\psi(0)\|_{\dot{H}^{3/2}} + \epsilon_* \|A_x[0]\|_{\dot{H}^2\times\dot{H}^1} + \epsilon_* \sup_{s\in\{+,-\}} \|\Pi_s \psi\|_{\tilde{S}_s^{3/2}}.$$

Taking $\epsilon_*$ smaller if necessary, we may absorb the last term into the LHS, which would prove (3.5.1).

It remains to justify (3.5.3)–(3.5.4); below we only discuss (3.5.4), as the other bounds can be proved in a similar fashion to Step 1 (in parallel with Step 3). By (1.8.16)–(1.8.17), (1.8.19)–(1.8.20), (1.8.22)–(1.8.23) and (1.8.25)–(1.8.26), it is straightforward to show that

$$\|\Pi_{s'}(I_2 + is'\pi^R[\nabla A_x^{free}]\psi_{s'})\|_{N_{s'}^{1/2}\cap L^2L^2\cap G^{1/2}} \lesssim \epsilon_* \|A_x[0]\|_{\dot{H}^2\times\dot{H}^1}.$$

Moreover, the $L^2L^2 \cap G^{1/2}$ norm of $s'\Pi_{s'}(\pi^R[\nabla A_x^{free}]\psi_{s'})$ can be bounded by the same RHS using (1.8.22) and (1.8.25). To handle its $N_{s'}^{1/2}$ norm, we apply (1.8.29) with frequency envelopes $a$ and $b$ for $\|\nabla A_x[0]\|_{\dot{H}^1\times L^2}$, $\|\psi\|_{\tilde{S}_{s'}^{1/2}}$, respectively. For any $0 < \delta < \delta_1$, we have

$$\|P_k \Pi_{s'}(\pi^R[\nabla A_x^{free}]\psi_{s'})\|_{N_{s'}^{1/2}} \leq (\sum_{k'<k} a_{k'})b_k \leq 2^{\delta k}\|\nabla A_x[0]\|_{\dot{H}^{1-\delta}\times\dot{H}^{-\delta}} b_k$$

Square summing over $k$, we see that the $N_{s'}^{1/2}$ norm of $\Pi_{s'}(\pi^R[\nabla A_x^{free}]\psi_{s'})$ is bounded by $\|\nabla A_x[0]\|_{\dot{H}^{1-\delta}\times\dot{H}^{-\delta}}\|\psi_{s'}\|_{\tilde{S}_{s'}^{1/2+\delta}}$. By a simple interpolation, the desired bound (3.5.4) follows.

## The case of MKG

We write

$$\nabla(A_x - A_x^{free}) = \mathbf{A}_x(\nabla\phi, \phi) + \mathbf{A}_x(\phi, \nabla\phi) + \nabla A_x^R, \quad A_x^R = -\Box^{-1}\mathcal{P}_x(|\phi|^2 A_x)$$

Using the product rule we distribute the derivative on the terms inside $A_x^R$. We also write the similar formula for $\nabla A_0$. From Prop. 1.8.1 we get

$$\|\nabla(A - A_x^{free})\|_{\ell^1 S^\sigma\times Y^\sigma} \lesssim \varepsilon(\|\nabla\phi\|_{\bar{S}^\sigma} + \|\nabla A\|_{S^\sigma\times Y^\sigma}) \tag{3.5.5}$$

The equation for $\nabla\phi$ is

$$\Box_m^{p,A^{free}} \nabla\phi = \nabla\mathcal{M}(A,\phi) + 2i\sum_{k\geq 0} \nabla A_{<k-C}^{free}\cdot\nabla_x\phi_k$$

Using the product rule on $\nabla\mathcal{M}(A,\phi)$ and Prop. 1.8.1, 1.8.2, 1.8.3 we obtain

$$\|\nabla\mathcal{M}(A,\phi)\|_{\bar{N}^{\sigma-1}\cap L^2H^{\sigma-\frac{3}{2}}} \lesssim \varepsilon(\|\nabla\phi\|_{\bar{S}^\sigma} + \|\nabla A\|_{S^\sigma\times Y^\sigma})$$

Using (6.1.32) (note that the $\mathcal{H}^*$ term is 0 for $A^{free}$) we get

$$\|\sum_{k\geq 0} \nabla A_{<k-C}^{free}\cdot\nabla_x\phi_k\|_{\bar{N}^{\sigma-1}\cap L^2H^{\sigma-\frac{3}{2}}} \lesssim \varepsilon\|\nabla A_x[0]\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}$$

We bound $\nabla\phi$ using Theorem 1.6.1 so that together with (3.5.5) we have

$$\|\nabla\phi\|_{\bar{S}^\sigma} + \|\nabla(A - A_x^{free})\|_{\ell^1 S^\sigma \times Y^\sigma} \lesssim \|\nabla\phi[0]\|_{H^\sigma \times H^{\sigma-1}} + \varepsilon\|\nabla A_x[0]\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} + \varepsilon(\|\nabla\phi\|_{\bar{S}^\sigma} + \|\nabla A\|_{S^\sigma \times Y^\sigma})$$

Choosing $\varepsilon$ small enough gives (3.5.2).

*Remark* 3.5.1. An alternative approach would be to use (3.2.13) for $s \in (\sigma, \sigma + \delta_1)$ together with the almost optimal local well-posedness result in [48] and its higher dimensional analogue.

## 3.6 Proof of continuous dependence on data

Here we prove Statement (2) of Theorem 1.1.2. The same argument proves statement (2) of Theorem 1.1.1, we omit the repetitive details.

Along the way, we also show that every solution obtained by Theorem 3.0.2 arises as an approximation by smooth solutions.

Let $\psi(0) \in \dot{H}^{1/2}$, $A_x[0] \in \dot{H}^1 \times L^2$ be an initial data set for MD-CG. Given $m \in \mathbb{Z}$, let $\psi^{(m)}(0), A_x^{(m)}[0]$ be the regularization $\psi^{(m)}(0) = P_{\leq m}\psi(0)$, $A_x^{(m)}[0] = P_{\leq m}A_x[0]$. Denote by $(A, \psi)$ [resp. $(A^{(m)}, \psi^{(m)})$] the solution with the data $\psi(0), A_x[0]$ [resp. $\psi^{(m)}(0), A_x^{(m)}[0]$ ] given by Theorem 3.0.2.

**Lemma 3.6.1** (Approximation by smooth solutions). *Let $c$ be an admissible frequency envelope for $\psi(0)$, $A_x[0]$. In the above setting, we have*

$$\sup_{s\in\{+,-\}} \|\Pi_s(\psi - \psi^{(m)})\|_{\tilde{S}_s^{1/2}} + \|A_x - A_x^{(m)}\|_{S^1} + \|A_0 - A_0^{(m)}\|_{Y^1} \lesssim \Big(\sum_{k>m} c_k^2\Big)^{1/2}.$$

*Proof.* Let $c$ be an admissible frequency envelope for $(\psi(0), A_x[0])$; observe that it is also a frequency envelope for $(\psi^{(m)}(0), A_x^{(m)}[0])$. Applying the frequency envelope bound (3.1.2) to $(A, \psi)$ and $(A^{(m)}, \psi^{(m)})$ separately, the above estimate follows for $P_{>m}(\psi - \psi^{(m)})$ and $P_{>m}(A - A^{(m)})$. On the other hand, for $P_{\leq m}(\psi - \psi^{(m)})$ and $P_{\leq m}(A - A^{(m)})$ we use weak Lipschitz continuity (3.0.4). Observe that

$$\begin{aligned}\|P_{\leq m}\Pi_s(\psi - \psi^{(m)})\|_{\tilde{S}_s^{1/2}} &\lesssim 2^{\delta_2 m}\|P_{\leq m}\Pi_s(\psi - \psi^{(m)})\|_{\tilde{S}_s^{1/2-\delta_2}} \\ &\lesssim 2^{\delta_2 m}(\|P_{>m}\psi(0)\|_{\dot{H}^{1/2-\delta_2}} + \|P_{>m}A_x[0]\|_{\dot{H}^1 \times L^2}),\end{aligned}$$

where the last line is bounded by $(\sum_{k>m} c_k^2)^{1/2}$. Combined with similar observations for $A_x - A_x^{(m)}$ in $S^1$ and $A_0 - A_0^{(m)}$ in $Y^1$, the lemma follows. □

We are now ready to prove Statement (2) of Theorem 1.1.2. Let $\psi^n(0), A_x^n[0]$ be a sequence of initial data sets for MD-CG such that $\psi^n(0) \to \psi(0)$ in $\dot{H}^{1/2}$ and $A_x^n[0] \to A_x[0]$

in $\dot{H}^1 \times L^2$. Denote by $(A^n, \psi^n)$ the corresponding solution to MD-CG, which exists for large $n$ by Theorem 3.0.2. For any $\epsilon > 0$, we claim that

$$\sup_{s\in\{+,-\}} \|\Pi_s(\psi^n - \psi)\|_{\tilde{S}_s^{1/2}} + \|A_x^n - A_x\|_{S^1} < \epsilon \tag{3.6.1}$$

for sufficiently large $n$. The desired continuity statement is equivalent to this claim.

Let $c$ be an admissible frequency envelope for $(\psi(0), A_x[0])$. Applying Lemma 3.6.1, we may find $m \in \mathbb{Z}$ such that for sufficiently large $n$,

$$\begin{aligned} \sup_{s\in\{+,-\}} \|\Pi_s(\psi - \psi^{(m)})\|_{\tilde{S}_s^{1/2}} + \|A_x - A_x^{(m)}\|_{S^1} <& \frac{1}{4}\epsilon, \\ \sup_{s\in\{+,-\}} \|\Pi_s(\psi^n - \psi^{n(m)})\|_{\tilde{S}_s^{1/2}} + \|A_x^n - A_x^{n(m)}\|_{S^1} <& \frac{1}{4}\epsilon, \end{aligned} \tag{3.6.2}$$

where $(A^{n(m)}, \psi^{n(m)})$ is defined in the obvious manner. By persistence of regularity and Proposition 3.4.1, we have (as $n \to \infty$)

$$\|(\psi^{n(m)} - \psi^{(m)})(t)\|_{C_t([0,T];H^{1/2,5/2})} + \|(A_x^{n(m)} - A_x^{(m)})[t]\|_{C_t([0,T];\mathcal{H}^{1,3})} \to 0.$$

Reiterating the preceding bound in MD-CG, we also obtain (as $n \to \infty$)

$$\|\alpha^\mu \partial_\mu(\psi^{n(m)} - \psi^{(m)})\|_{C_t([0,T];H^{1/2,5/2})} + \|\Box(A_x^{n(m)} - A_x^{(m)})\|_{C_t([0,T];H^{0,2})} \to 0.$$

In a straightforward manner, the preceding two statements imply

$$\sup_{s\in\{+,-\}} \|\Pi_s(\psi^{n(m)} - \psi^{(m)})\|_{\tilde{S}_s^{1/2}[0,T]} + \|A_x^{n(m)} - A_x^{(m)}\|_{S^1[0,T]} < \frac{1}{2}\epsilon$$

for sufficiently large $n$. Combined with (3.6.2), the desired conclusion (3.6.1) follows.

## 3.7 Proof of modified scattering

Here we conclude the proof of Theorems 1.1.1 and 1.1.2 by sketching the proof of Statement (3). Without loss of generality, we fix $\pm = +$.

### The MD case

Let $(A, \psi)$ be a solution to MD-CG with data $(\psi(0), A_x[0])$ given by Theorem 3.0.2, and let $A_x^{free}$ denote the free wave development of $A_x[0]$. To prove modified scattering for $\psi$, we first decompose the covariant Dirac equation into

$$\alpha^\mu \mathbf{D}_\mu^{A^{free}} \psi = -i\alpha^\mu \mathbf{A}_\mu(\psi, \psi)\psi.$$

For any $t < t'$, Proposition 1.8.11 implies that

$$\|\psi(t') - S^{A^{free}}(t', t)\psi(t)\|_{\dot{H}^{1/2}} \lesssim \sup_{s\in\{+,-\}} \|\Pi_s(\alpha^\mu \mathbf{A}_\mu(\psi,\psi)\psi)\|_{(N_s^{1/2}\cap L^2L^2\cap G^{1/2})[t,\infty)},$$

where $S^{A^{free}}(t', t)$ denotes the propagator from time $t$ to $t'$ for the covariant Dirac equation $\alpha^\mu \mathbf{D}_\mu^{A^{free}}\varphi = 0$. An analysis as in Section 4.4 using Propositions 1.8.6 and 1.8.7 shows that the RHS is finite for (say) $t = 0$; by (2.1.18), it follows that the RHS vanishes as $t \to \infty$. Using the uniform boundedness of $S^{A^{free}}(0, t')$ on $\dot{H}^{1/2}$ (again by Proposition 1.8.11), as well as the formula $S^{A^{free}}(t'', t) = S^{A^{free}}(t'', t')S^{A^{free}}(t', t)$, it follows that (as $t \to \infty$)

$$\|S^{A^{free}}(0, t')\psi(t') - S^{A^{free}}(0, t)\psi(t)\|_{\dot{H}^{1/2}} \lesssim \|\psi(t') - S^{A^{free}}(t', t)\psi(t)\|_{\dot{H}^{1/2}} \to 0.$$

Hence $\lim_{t\to\infty} S^{A^{free}}(0, t)\psi(t)$ tends to some limit $\psi^\infty(0)$ in $\dot{H}^{1/2}$, which is precisely the data for $\psi^\infty$ in Theorem 1.1.2.

The proof of scattering for $A_x$ is more standard and straightforward. In fact, since $\|\mathcal{M}_x(\psi,\psi)\|_{\ell^1(N\cap L^2\dot{H}^{-1/2})[0,\infty)} < \infty$ by Proposition 1.8.6, $\lim_{t\to\infty} S[0, t]A_x[t]$ tends to a limit $A_x^\infty[0]$ in $\ell^1(\dot{H}^1 \times L^2)$; here $S[t', t]$ denotes the propagator for the free wave equation. In particular, we have $A_x[0] - A_x^\infty[0] \in \ell^1(\dot{H}^1 \times L^2)$; this fact allows us to replace $A^{free}$ by $A^\infty$ as claimed in Theorem 1.1.2. We leave the details to the reader.

## The MKG case

Let $(\phi, A)$ be the solutions with initial data $(\phi[0], A_x[0])$ given by Theorem 3.0.1 and let $A^{free}$ be the free wave development of $A_x[0]$. We denote by $S^{A^{free}}(t', t)$ the propagator from time $t$ to $t'$ for the covariant equation $\Box_m^{A^{free}}\phi = 0$, given by Prop. 1.8.4, which implies, for any $t < t'$

$$\|\phi[t'] - S^{A^{free}}(t', t)\phi[t]\|_{H^\sigma\times H^{\sigma-1}} \lesssim \|\Box_m^{A^{free}}\phi\|_{(\bar{N}^{\sigma-1}\cap L^2H^{\sigma-\frac{3}{2}})[t,\infty)}$$

the last one being the time interval localized norm. Using the estimates from Prop. 1.8.2 like in the proof of existence shows that the RHS is finite for, say $t = 0$, and the RHS vanishes as $t \to \infty$. By the uniform boundedness of $S^{A^{free}}(0, t)$ on $H^\sigma \times H^{\sigma-1}$ (Prop. 1.8.4) and the formula $S^{A^{free}}(t'', t) = S^{A^{free}}(t'', t')S^{A^{free}}(t', t)$ it follows that, as $t \to \infty$

$$\|S^{A^{free}}(0, t')\phi[t'] - S^{A^{free}}(0, t)\phi[t]\|_{H^\sigma\times H^{\sigma-1}} \lesssim \|\phi[t'] - S^{A^{free}}(t', t)\phi[t]\|_{H^\sigma\times H^{\sigma-1}} \to 0$$

Therefore the limit $\lim_{t\to\infty} S^{A^{free}}(0, t)\phi[t] =: \phi^\infty[0]$ exists in $H^\sigma \times H^{\sigma-1}$ and $\phi^\infty[0]$ is taken as the initial data for $\phi^\infty$ in Theorem 1.1.1.

The proof of scattering for $A_x$ is similar, we omit the details.

# Chapter 4

# The parametrices for Klein-Gordon and Dirac equations

This chapter is dedicated to the proofs of Theorems 1.6.1, 1.6.2 and Prop. 1.8.11. We will present the motivation and the construction of the parametrices for covariant Klein-Gordon and Dirac equations. We then discuss the main properties of the phases, decomposable estimates, oscillatory integrals estimates, the conjugation and the mapping properties.

## 4.1 Motivation

We begin by recalling some heuristic considerations motivating the construction in [47], which also extends to the massive case and to the half-wave case (which will lead to the solvability of the covariant Dirac equation).

Suppose that one is interested in solving the equation

$$\Box_m^A \phi = 0, \qquad \Box_m^A := D^\alpha D_\alpha + I \tag{4.1.1}$$

where $D_\alpha \phi = (\partial_\alpha + i A_\alpha)\phi$ and $\Box A = 0$. After solving (4.1.1), one can also obtain solutions to the inhomogeneous equation $\Box_m^A \phi = F$ by Duhamel's formula. The equation (4.1.1) enjoys the following gauge invariance. For any real function $\psi$, replacing

$$\phi \mapsto e^{i\psi}\phi, \quad A_\alpha \mapsto A_\alpha - \partial_\alpha \psi, \quad D_\alpha \mapsto e^{i\psi} D_\alpha e^{-i\psi}$$

we obtain another solution. To make use of this, one expects that by choosing $\psi$ appropriately ($\nabla\psi \approx A$) one could reduce closer to the free wave equation $\Box\phi \approx 0$.

However, this is not in general possible since $A$ is not a conservative vector field. Instead, one makes the construction microlocally and for each dyadic frequency separately. Taking $e^{ix\cdot\xi}$ as initial data, considering $\phi = e^{-i\psi_\pm(t,x)} e^{\pm it\langle\xi\rangle + ix\cdot\xi}$ we compute

$$\Box_m^A \phi = 2\left(\pm \langle\xi\rangle \partial_t \psi_\pm - \xi \cdot \nabla\psi_\pm + A\cdot\xi\right)\phi + \big( - i\Box\psi_\pm + (\partial_t\psi_\pm)^2 - |\nabla\psi_\pm|^2 - A\cdot\nabla\psi_\pm\big)\phi$$

The second bracket is expected to be an error term, while for the first, one wants to define $\psi_\pm$ so as to get as much cancelation as possible, while also avoiding to make $\psi_\pm$ too singular. Defining

$$L_\pm = \pm\partial_t + \frac{\xi}{\langle\xi\rangle}\cdot\nabla_x, \qquad \text{one has}$$

$$-L_+L_- = \Box + \Delta_{\omega^\perp} + \frac{1}{\langle\xi\rangle^2}(\omega\cdot\nabla_x)^2, \quad \omega = \frac{\xi}{|\xi|}. \tag{4.1.2}$$

We would like to have $L_\mp\psi_\pm = A\cdot\xi/\langle\xi\rangle$ thus applying $L_\pm$ and neglecting $\Box$ in (4.1.2) (since $\Box A = 0$) one obtains, for fixed $\xi$:

$$\psi_\pm(t,x) = \frac{-1}{\Delta_{\omega^\perp} + \frac{1}{\langle\xi\rangle^2}(\omega\cdot\nabla_x)^2} L_\pm\left(A(t,x)\cdot\frac{\xi}{\langle\xi\rangle}\right). \tag{4.1.3}$$

Taking general initial data $\int e^{ix\cdot\xi}h_\pm(\xi)\,\mathrm{d}\xi$, using linearity, one obtains the approximate solutions

$$\phi_\pm(t,x) = \int e^{-i\psi_\pm(t,x,\xi)}e^{\pm it\langle\xi\rangle}e^{ix\cdot\xi}h_\pm(\xi)\,\mathrm{d}\xi.$$

Thus, the renormalization is done through the pseudodifferential operators $e^{-i\psi_\pm}(t,x,D)$.

In what follows, $\xi$ will be restricted to dyadic frequencies $|\xi|\simeq 2^k$ or $|\xi|\lesssim 1$, while $A(t,x)$ (and thus $\psi$ too) will be localized to strictly lower frequencies $\ll 2^k$. When $|\xi|\lesssim 1$, the denominator in (4.1.3) is essentially $\Delta_x$. If $\xi$ is a high frequency then the dominant term is $\Delta_{\omega^\perp}^{-1}$ and the construction needs to be refined to remove the singularity; see the next subsection for precise definitions.

For more details motivating the construction see [47, sec. 7,8].

The construction in [31] slightly differs from the one in [47] in that they further localize the exponentials in the $(t,x)$-frequencies $\left(e^{-i\psi_\pm(t,x,\xi)}\right)_{<k-c}$. By Taylor expansion one can see that these constructions are essentially equivalent. Indeed, since

$$e^{i\psi_{<k-c}(t,x,\xi)} = 1 + i\psi_{<k-c}(t,x,\xi) + O\big(\psi^2_{<k-c}(t,x,\xi)\big)$$

we see that they differ only by higher order terms, which are negligible due to the smallness assumption on $A$. Here, following [31], it will be technically convenient to do this localization. We denote by

$$e^{\pm i\psi^k_\pm}_{<h}(t,x,D), \qquad e^{\pm i\psi^k_\pm}_{<h}(D,s,y)$$

the left and right quantizations of the symbol $e^{\pm i\psi^k_\pm}_{<h}(t,x,\xi)$ where the $<h$ subscript denotes $(t,x)$-frequency localization to frequencies $\leq h-C$, pointwise in $\xi$. Thus

$$e^{\pm i\psi^k_\pm}_{<h}(t,x,\xi) = \int_{\mathbb{R}^{d+1}} e^{\pm iT_{(s,y)}\psi^k_\pm(t,x,\xi)}m_h(s,y)\,\mathrm{d}s\,\mathrm{d}y \tag{4.1.4}$$

where $T_{(s,y)}\psi(t,x,\xi) = \psi(t+s,x+y,\xi)$ and $m_h = 2^{(d+1)h}m(2^h\cdot)$ for a bump function $m(s,y)$. By averaging arguments such as Lemmas 4.5.11, 4.10.1, estimates for $e^{-i\psi_\pm}(t,x,D)$ will automatically transfer to $e^{-i\psi_\pm}_{<k}(t,x,D)$.

## 4.2 The parametrix for covariant Klein-Gordon operators

We consider the paradifferential covariant Klein-Gordon operator

$$\Box_m^{p,A} = \Box + I - 2i \sum_{k \geq 0} A^j_{<k-C} \partial_j \bar{P}_k \tag{4.2.1}$$

where $A = A^{free} = (A_1, \dots, A_d, 0)$ is a real-valued 1-form defined on $\mathbb{R}^{1+d}$, assumed to solve the free wave equation and to obey the Coulomb gauge condition

$$\Box A = 0, \qquad \partial^j A_j = 0. \tag{4.2.2}$$

By the argument in Prop. 2.2.3 one may show

$$\|\phi\|_{\bar{S}^\sigma} \lesssim \|\phi[0]\|_{H^\sigma \times H^{\sigma-1}} + \|\Box_m \phi\|_{\bar{N}^{\sigma-1}}$$

Following [31], the goal is to generalize this inequality, showing that $\Box_m$ can be replaced by $\Box_m^{p,A}$.

We consider the problem

$$\begin{cases} \Box_m^{p,A} \phi = F \\ \phi[0] = (f, g) \end{cases} \tag{4.2.3}$$

which is the object of Theorem 1.6.1. The proof of this theorem will reduce to its frequency localized approximate version:

**Theorem 4.2.1.** *Let $A$ be a real 1-form obeying (1.6.3) on $\mathbb{R}^{d+1}$ for $d \geq 4$ and let $k \geq 0$. If $\|A[0]\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}}$ is sufficiently small, then for any $(f_k, g_k)$ with Fourier support in $\{\langle \xi \rangle \simeq 2^k\}$ and any $F_k$ with Fourier support in $\{\langle \xi \rangle \simeq 2^k,\ ||\tau| - \langle \xi \rangle| \ll 2^k\}$ there exists a function $\phi_k$ with Fourier support in $\{\langle \xi \rangle \simeq 2^k,\ ||\tau| - \langle \xi \rangle| \ll 2^k\}$ such that*

$$\|(\langle D_x \rangle, \partial_t)\phi_k\|_{\bar{S}_k} \lesssim \|(f_k, g_k)\|_{H^1 \times L^2} + \|F_k\|_{\bar{N}_k} =: M_k \tag{4.2.4}$$

$$\|(\Box_m - 2i A^j_{<k-c} \partial_j)\phi_k - F_k\|_{\bar{N}_k} \lesssim \varepsilon^{\frac{1}{2}} M_k \tag{4.2.5}$$

$$\|(\phi_k(0) - f_k, \partial_t \phi_k(0) - g_k)\|_{H^1 \times L^2} \lesssim \varepsilon^{\frac{1}{2}} M_k. \tag{4.2.6}$$

The approximate solution will be defined by $2\phi_k = T^+ + T^- + S^+ + S^-$ where

$$\begin{aligned} T^\pm &:= e^{-i\psi^k_\pm}_{<k}(t,x,D) \frac{e^{\pm it\langle D \rangle}}{i\langle D \rangle} e^{i\psi^k_\pm}_{<k}(D,y,0)(i\langle D \rangle f_k \pm g_k) \\ S^\pm &:= \pm e^{-i\psi^k_\pm}_{<k}(t,x,D) \frac{K^\pm}{i\langle D \rangle} e^{i\psi^k_\pm}_{<k}(D,y,s) F_k, \end{aligned} \tag{4.2.7}$$

The phase $\psi_\pm^k(t,x,\xi)$ is defined in Section 4.5 and $K^\pm F$ are the Duhamel terms

$$K^\pm F(t) = u(t) = \int_0^t e^{\pm i(t-s)\langle D\rangle} F(s)\,\mathrm{d}s, \qquad (\partial_t \mp i\langle D\rangle)u = F, \quad u(0) = 0.$$

To implement this one needs estimates for the operators $e_{<k}^{-i\psi_\pm^k}(t,x,D)$ and their adjoints, adapted to the function spaces used in the iteration.

**Theorem 4.2.2.** *For any $k \geq 0$, the frequency localized renormalization operators have the following properties for any $X \in \{\bar{N}_k, L_x^2, \bar{N}_k^*\}$:*

$$e_{<k}^{\pm' i\psi_\pm^k}(t,x,D) : X \to X \tag{4.2.8}$$

$$2^{-k}\partial_{t,x} e_{<k}^{\pm' i\psi_\pm^k}(t,x,D) : X \to \varepsilon X \tag{4.2.9}$$

$$e_{<k}^{-i\psi_\pm^k}(t,x,D) e_{<k}^{i\psi_\pm^k}(D,y,s) - I : X \to \varepsilon^{\frac{1}{2}} X \tag{4.2.10}$$

*as well as*

$$2^k \| e_{<k}^{-i\psi_\pm^k}(t,x,D) u_k \|_{\bar{S}_k} \lesssim \|u_k\|_{L^\infty(H^1\times L^2)} + \|\Box_m u_k\|_{\bar{N}_k} \tag{4.2.11}$$

$$\begin{aligned} \| e_{<k}^{-i\psi_\pm^k}(t,x,D)\Box_m u_k - \Box_m^{A_{<k}} e_{<k}^{-i\psi_\pm^k}(t,x,D) u_k \|_{\bar{N}_k} &\lesssim \\ \varepsilon \|u_k\|_{L^\infty H^1} + \varepsilon 2^k \|(i\partial_t \pm \langle D\rangle) u_k\|_{\bar{N}_k}& \end{aligned} \tag{4.2.12}$$

Moreover, by (4.2.10) and (4.5.25) one obtains

$$e_{<k}^{-i\psi_\pm^k}(t,x,D) \frac{1}{\langle D\rangle} e_{<k}^{i\psi_\pm^k}(D,y,s) - \frac{1}{\langle D\rangle} : X \to \varepsilon^{\frac{1}{2}} 2^{-k} X \tag{4.2.13}$$

The proof of Theorem 4.2.2 is given later in this chapter. Now we show how these mappings imply Theorems 1.6.1, 4.2.1.

*Proof of Theorem 1.6.1.* **S**tep 1. We first look to define an approximate solution $\phi^a = \phi^a[f,g,F]$ satisfying, for some $\delta \in (0,1)$:

$$\|\Box_m^{p,A}\phi^a - F\|_{\bar{N}^{\sigma-1}\cap L^2 H^{\sigma-\frac{3}{2}}} + \|\phi^a[0] - (f,g)\|_{H^\sigma\times H^{\sigma-1}} \leq \delta\big[\|F\|_{\bar{N}^{\sigma-1}\cap L^2 H^{\sigma-\frac{3}{2}}} + \|(f,g)\|_{H^\sigma\times H^{\sigma-1}}\big] \tag{4.2.14}$$

and

$$\|\phi^a\|_{\bar{S}^\sigma} \lesssim \|F\|_{\bar{N}^{\sigma-1}\cap L^2 H^{\sigma-\frac{3}{2}}} + \|(f,g)\|_{H^\sigma\times H^{\sigma-1}}. \tag{4.2.15}$$

We define $\phi^a$ from its frequency-localized versions

$$\phi^a := \sum_{k\geq 0} \phi_k^a, \qquad \phi_k^a = \phi_k^1 + \phi_k^2$$

which remain to be defined. We decompose $\bar{P}_k F = \bar{Q}_{<k-6}\bar{P}_k F + \bar{Q}_{>k-6}\bar{P}_k F$ and first define $\phi_k^2$ by

$$\mathcal{F}\phi_k^2(\tau,\xi) := \frac{1}{-\tau^2 + |\xi|^2 + 1}\mathcal{F}(\bar{Q}_{>k-6}\bar{P}_k F)(\tau,\xi)$$

so that $\Box_m \phi_k^2 = \bar{Q}_{>k-6}\bar{P}_k F$. We have

$$\|(\langle D_x\rangle, \partial_t)\phi_k^2\|_{\bar{S}_k} \lesssim \|\phi_k^2\|_{L^\infty(H^1\times L^2)} + \|\bar{Q}_{>k-6}\bar{P}_k F\|_{\bar{N}_k} \lesssim \|\bar{P}_k F\|_{\bar{N}_k}.$$

Then we apply Theorem 4.2.1 to $\bar{Q}_{<k-6}\bar{P}_k F$ and $\bar{P}_k(f,g) - \phi_k^2[0]$ which defines the function $\phi_k^1$. We are left with estimating

$$\|A^j_{<k-C}\partial_j \phi_k^2\|_{L^1L^2\cap L^2 H^{-\frac{1}{2}}} \lesssim \|A^j_{<k-C}\|_{L^2L^\infty}\|\nabla\phi_k^2\|_{L^2_{t,x}\cap L^\infty H^{-\frac{1}{2}}} \lesssim \varepsilon\|\bar{P}_k F\|_{\bar{N}_k}$$

and similarly, using also Lemma 2.2.4,

$$\begin{aligned} 2^{-\frac{1}{2}k}\|\Box_m\phi_k^1\|_{L^2_{t,x}} &\lesssim \|\Box_m^{A_{<k}}\phi_k^1 - \bar{Q}_{<k-6}\bar{P}_k F\|_{\bar{N}_k} + \|\bar{Q}_{<k-6}\bar{P}_k F\|_{\bar{N}_k} + \|A^j_{<k-C}\partial_j\phi_k^1\|_{L^2H^{-\frac{1}{2}}} \\ &\lesssim \|\bar{P}_k F\|_{\bar{N}_k} + \|\bar{P}_k(f,g)\|_{H^1\times L^2} \end{aligned}$$

The following error term, for $k', k'' = k \pm O(1)$, follows from (6.1.27), (1.8.7)

$$\|A^j_{k'}\partial_j\bar{P}_{k''}\phi_k^a\|_{\bar{N}_k\cap L^2H^{-\frac{1}{2}}} \lesssim \varepsilon\|\phi_k^a\|_{\bar{S}_k^1}$$

**S**tep 2. Now we iterate the approximate solutions from Step 1 to construct an exact solution. We define $\phi := \lim \phi^{\leq n}$ where

$$\phi^{\leq n} := \phi^1 + \cdots + \phi^n$$

and the $\phi^n$ are defined inductively by $\phi^1 := \phi^a[f,g,F]$ and

$$\phi^n := \phi^a[(f,g) - \phi^{\leq n-1}[0], F - \Box_m^{p,A}\phi^{\leq n-1}]$$

Normalizing $\|F\|_{\bar{N}^{\sigma-1}\cap L^2H^{\sigma-\frac{3}{2}}} + \|(f,g)\|_{H^\sigma\times H^{\sigma-1}} = 1$ it follows by induction using (4.2.14), (4.2.15) that

$$\|\Box_m^{p,A}\phi^{\leq n} - F\|_{\bar{N}^{\sigma-1}\cap L^2H^{\sigma-\frac{3}{2}}} + \|\phi^{\leq n}[0] - (f,g)\|_{H^\sigma\times H^{\sigma-1}} \leq \delta^n \tag{4.2.16}$$

and

$$\|\phi^n\|_{\bar{S}^\sigma} \lesssim \delta^{n-1}. \tag{4.2.17}$$

Thus $\phi^{\leq n}$ is a Cauchy sequence in $\bar{S}^\sigma$ and $\phi$ is well-defined, satisfying (1.6.5). Passing to the limit in (4.2.16) we see that $\phi$ solves (1.6.4). □

*Remark* 4.2.3. The argument above also implies a frequency envelope version of (1.6.5), which will be useful in proving continuous dependence on the initial data :

$$\|\phi\|_{\bar{S}^\sigma_c} \lesssim \|(f,g)\|_{H^\sigma_c\times H^{\sigma-1}_c} + \|F\|_{(\bar{N}^{\sigma-1}\cap L^2H^{\sigma-\frac{3}{2}})_c} \tag{4.2.18}$$

*Proof of Theorem 4.2.1.* We define $\phi_k$ by

$$\phi_k = \frac{1}{2}\big(T^+ + T^- + S^+ + S^-\big)$$

where $T^\pm, S^\pm$ are defined by (4.2.7).

The bound (4.2.4) follows from (4.2.11) and (4.2.8), where for $\partial_t\phi_k$ we use the low modulation support of $\phi_k$. We turn to (4.2.6) and write

$$\phi_k(0) - f_k = \frac{1}{2i}\sum_\pm [e^{-i\psi^k_\pm}_{<k}(0,x,D)\frac{1}{\langle D\rangle}e^{i\psi^k_\pm}_{<k}(D,y,0) - \frac{1}{\langle D\rangle}](i\langle D\rangle f_k \pm g_k)$$

$$\begin{aligned}\partial_t\phi_k(0) - g_k = \frac{1}{2}\sum_\pm \Big[&[e^{-i\psi^k_\pm}_{<k}(0,x,D)e^{i\psi^k_\pm}_{<k}(D,y,0) - I](\pm i\langle D\rangle f_k + g_k)\\ &+ [\partial_t e^{-i\psi^k_\pm}_{<k}](0,x,D)\frac{1}{i\langle D\rangle}e^{i\psi^k_\pm}_{<k}(D,y,0)(i\langle D\rangle f_k \pm g_k)\\ &\pm [e^{-i\psi^k_\pm}_{<k}(0,x,D)\frac{1}{i\langle D\rangle}e^{i\psi^k_\pm}_{<k}(D,y,0) - \frac{1}{i\langle D\rangle}]F_k(0)\Big]\end{aligned}$$

These are estimated using (4.2.13), (4.2.10), (4.2.8), respectively (4.2.13), together with

$$\|F_k(0)\|_{L^2_x} \lesssim \|F_k\|_{L^\infty L^2} \lesssim 2^k\|F_k\|_{\bar{N}_k}$$

which follows from Lemma 2.2.4 considering the modulation assumption on $F_k$.

Now we prove (4.2.5). We write

$$\Box^{A_{<k}}_m\phi_k - F_k = \sum_\pm \big[[\Box^{A_{<k}}_m e^{-i\psi^k_\pm}_{<k}(t,x,D) - e^{-i\psi^k_\pm}_{<k}(t,x,D)\Box_m]\phi_\pm \tag{4.2.19}$$

$$\pm \frac{1}{2}e^{-i\psi^k_\pm}_{<k}(t,x,D)\frac{\partial_t \pm i\langle D\rangle}{i\langle D\rangle}e^{i\psi^k_\pm}_{<k}(D,y,s)F_k\big] - F_k. \tag{4.2.20}$$

where

$$\phi_\pm := \frac{1}{2i\langle D\rangle}\big[e^{\pm it\langle D\rangle}e^{i\psi^k_\pm}_{<k}(D,y,0)(i\langle D\rangle f_k \pm g_k) \pm K^\pm e^{i\psi^k_\pm}_{<k}(D,y,s)F_k\big]$$

Using (4.2.12) we estimate

$$\|(4.2.19)\|_{\bar{N}_k} \lesssim \sum_\pm \varepsilon[\|e^{i\psi^k_\pm}_{<k}(D,y,0)(i\langle D\rangle f_k \pm g_k)\|_{L^2} + \|e^{i\psi^k_\pm}_{<k}(D,y,s)F_k\|_{\bar{N}_k}]$$

and then we use (4.2.8). Now we turn to (4.2.20) and write

$$(4.2.20) = \sum_\pm \frac{1}{2}\Big[[e^{-i\psi^k_\pm}_{<k}(t,x,D)e^{i\psi^k_\pm}_{<k}(D,y,s) - I]F_k \tag{4.2.21}$$

$$\pm i^{-1}[e^{-i\psi^k_\pm}_{<k}(t,x,D)\frac{1}{\langle D\rangle}e^{i\psi^k_\pm}_{<k}(D,y,s) - \frac{1}{\langle D\rangle}]\partial_t F_k \tag{4.2.22}$$

$$\pm e^{-i\psi^k_\pm}_{<k}(t,x,D)\frac{1}{i\langle D\rangle}[\partial_t e^{i\psi^k_\pm}_{<k}](D,y,s)F_k\Big]. \tag{4.2.23}$$

For (4.2.21) we use (4.2.10), for (4.2.22) we use (4.2.13), and for (4.2.23) we use (4.2.8), (4.2.9), all with $X = \bar{N}_k$. $\square$

## 4.3 The parametrix for half-wave operators

The goal of this section is to prove Theorem 1.6.2.

Suppose $\Box A^{free} = 0$ with $\|A^{free}\|_{\dot{H}^1 \times L^2} \leq \varepsilon$ together with the Coulomb condition $\partial^\ell A^{free}_\ell = 0$. Without loss of generality we assume the sign $s = +$. Define the paradifferential half-wave operators by

$$(i\partial_t + |D|)^p_A = i\partial_t + |D| - i \sum_{k \in \mathbb{Z}} P_{<k-C} A^{free,j} \frac{\partial_j}{|D|} P_k \tag{4.3.1}$$

where

$$(i\partial_t + |D|)^p_{A_{<k}} = i\partial_t + |D| - iP_{<k-C} A^{free,j} \frac{\partial_j}{|D|} P_k \tag{4.3.2}$$

and the paradifferential covariant (massless) wave $\Box$ operator by

$$\Box^p_{A_{<k}} = \Box - 2iP_{<k-C} A^{free,j} P_k \partial_j, \tag{4.3.3}$$

Consider the problem

$$\begin{cases} (i\partial_t + |D|)^p_A \psi = F \\ \psi(0) = f. \end{cases} \tag{4.3.4}$$

*Remark* 4.3.1. In this section we set $d = 4$. The construction in [31, Sections 6–11] (for $m = 0$) may be generalized to $\mathbb{R}^{1+d}$ with $d \geq 5$ without much difficulty, in essentially the same way as we do below in this chapter for $m^2 > 0$. Then, for $d \geq 5$, the argument in this section goes through with the substitutions as in Remark 1.8.5.

The proof of Theorem 1.6.2 reduces to the following proposition (whose proof is later in this section) by the same way argument as Theorem 1.6.1 reduces to Theorem 4.2.1.

**Proposition 4.3.2.** *For any $F \in N^{1/2}_+ \cap L^2L^2$ and any $f \in \dot{H}^{1/2}$ there exists $\psi^a \in S^{1/2}_+$ such that for any admissible frequency envelope c, we have*

$$\begin{aligned} \|\psi^a(0) - f\|_{\dot{H}^{1/2}_c} + \|(i\partial_t + |D|)^p_A \psi^a - F\|_{(N^{1/2}_+ \cap L^2L^2)_c} \\ \leq \delta \left( \|f\|_{\dot{H}^{1/2}_c} + \|F\|_{(N^{1/2}_+ \cap L^2L^2)_c} \right), \end{aligned} \tag{4.3.5}$$

$$\|\psi^a\|_{(S^{1/2}_+)_c} \lesssim \|f\|_{\dot{H}^{1/2}_c} + \|F\|_{(N^{1/2}_+ \cap L^2L^2)_c} \ . \tag{4.3.6}$$

**Construction of the parametrix**

The parametrix constructed in [31] for the (massless) equation

$$\begin{cases} \Box^p_{A_{<0}}\phi = F \\ \phi[0] = (g,h) \end{cases} \tag{4.3.7}$$

takes the form

$$\phi_{app} = \frac{1}{2}\left(T^+ + T^- + S^+ + S^-\right) \tag{4.3.8}$$

where[1]

$$T^{\pm} = e^{-i\Psi_\pm}_{<0}(t,x,D)\frac{1}{|D|}e^{\pm it|D|}e^{i\Psi_\pm}_{<0}(D,y,0)(|D|\,g \pm i^{-1}h) \tag{4.3.9}$$

$$S^{\pm} = \mp e^{-i\Psi_\pm}_{<0}(t,x,D)\frac{1}{|D|}K^{\pm}e^{i\Psi_\pm}_{<0}(D,y,s)i^{-1}F \tag{4.3.10}$$

where $K^{\pm}F$ denotes the solution $u$ of the equation

$$(\partial_t \mp i\,|D|)u = F, \qquad u(0) = 0 \tag{4.3.11}$$

given by the Duhamel formula

$$K^{\pm}F(t) = \int_0^t e^{\pm i(t-s)|D|}F(s)\,\mathrm{d}s.$$

More precisely, the result in [31] states

**Theorem 4.3.3.** *Assume that $F, g, h$ are localized at frequency $1$, and also that $F$ is localized at modulation $\lesssim 1$. Then $\phi_{app}$ is an approximate solution for (4.3.7), in the sense that*

$$\|\phi_{app}\|_{\tilde{S}_0} \lesssim \|g\|_{L^2} + \|h\|_{L^2} + \|F\|_{N_0} \tag{4.3.12}$$

*and*

$$\|\phi_{app}[0] - (g,h)\|_{L^2} + \|\Box^p_{A_{<0}}\phi_{app} - F\|_{N_0} \leq \delta(\|g\|_{L^2} + \|h\|_{L^2} + \|F\|_{N_0}) \tag{4.3.13}$$

The spaces $\tilde{S}_0$ and $N_0$ are defined in chapter 2.

[1] Note that if the $e^{\pm i\Psi_\pm}_{<0}$ terms are removed one obtains the solution of the ordinary wave equation $\Box\phi = F,\ \phi[0] = (g,h)$.

## Renormalization for $(i\partial_t + |D|)^p_A$

Now our goal is to similarly obtain a parametrix (or approximate solution) for (4.3.4) in order to prove Proposition 4.3.2.

Suppose $F, g, h$ are localized at frequency 1, and consider $S^{\pm}, T^{\pm}$ defined by (4.3.9), (4.3.10). If $F$ has small $Q^+$ -modulation, then so do $S^+$ and $T^+$. This also applies to $S^-$, except for a part with Fourier support in the lower characteristic cone. Therefore we decompose

$$S^- = Q^+_{\leq -2} S^- + S^-_0, \qquad S^-_0 := e^{-i\Psi_-}_{<0}(t,x,D) Q^-_{\leq -1} \left( \frac{1}{|D|} u \right), \tag{4.3.14}$$

according to the following definitions

$$u := \frac{1}{i} K^- \tilde{F}, \qquad \tilde{F} := e^{i\Psi_{\pm}}_{<0}(D,y,s) F, \tag{4.3.15}$$

so that $(i\partial_t - |D|)u = \tilde{F},\ u(0) = 0$. Let us define the function $v$ such that

$$\mathcal{F}v(\tau,\xi) := \frac{-1}{\tau + |\xi|} \mathcal{F}(\tilde{F})(\tau,\xi), \text{ so } \quad (i\partial_t - |D|)v = \tilde{F}. \tag{4.3.16}$$

The term $S^-_0$ can be controlled by $\|F\|_{N^0}$ as follows.

**Lemma 4.3.4.** *Suppose $F$ is localized at frequency 1 and at $Q^+$-modulation $\leq 1$. Then for $S^-_0$ and $v$ defined by (4.3.14) and (4.3.16) we have:*

$$\|v(0)\|_{L^2} \lesssim \|F\|_{N_0} \tag{4.3.17}$$

$$S^-_0 = -e^{-i\Psi_-}_{<0}(t,x,D) \frac{1}{|D|} e^{-it|D|}(v(0)) \tag{4.3.18}$$

$$\|(i\partial_t - |D|) S^-_0(0)\|_{L^2} \lesssim \varepsilon \|F\|_{N_0}. \tag{4.3.19}$$

*Proof.* The proof is divided into three steps.

**S**tep 1: Proof of (4.3.17). Since $F$ and $\tilde{F}$ are localized at $Q^-$-modulation $\gtrsim 1$ from (2.2.19) and (4.2.8) we have

$$\|v(0)\|_{L^2} \lesssim \|v\|_{L^\infty L^2} \lesssim \|\tilde{F}\|_{N^-_0} \lesssim \|\tilde{F}\|_{N_0} \lesssim \|F\|_{N_0}. \tag{4.3.20}$$

**S**tep 2: Proof of (4.3.18). Subtracting $v$ from $u$ we get

$$(i\partial_t - |D|)(u - v) = 0, \qquad (u - v)(0) = -v(0). \tag{4.3.21}$$

Thus $Q^-_{\leq -1} u = Q^-_{\leq -1}(u - v) = e^{-it|D|}(-v(0))$ from which (4.3.18) follows.

Step 3: Proof of (4.3.19). Using (4.3.17), it suffices to show $\|(i\partial_t - |D|)S_0^-(0)\|_{L^2} \lesssim \varepsilon \|v(0)\|_{L^2}$.

$$(i\partial_t - |D|)S_0^-(0) = i[\partial_t e_{<0}^{-i\Psi_-}](0,x,D)\left(\frac{v(0)}{|D|}\right) + \left[e_{<0}^{-i\Psi_-}, |D|\right]\left(\frac{v(0)}{|D|}\right). \tag{4.3.22}$$

The first term is estimated by (4.2.9). For the second, we use the dual of Lemma 4.5.10 and (4.2.9) to obtain

$$\| \, |D| \, e_{<0}^{-i\Psi_-}(0,x,D) - e_{<0}^{-i\Psi_-}(0,x,D) \, |D| \, \|_{L^2 \to L^2} \lesssim \|\partial_x e_{<0}^{-i\Psi_-}(0,x,D)\|_{L^2\to L^2} \lesssim \varepsilon$$

□

The following proposition is essentially a restatement of Theorem 4.3.3 in a convenient form for our application.

**Proposition 4.3.5.** *Suppose $F$ and $f$ are localized at frequency $\{|\xi| \in [2^{-2}, 2^2]\}$ and $F$ is also localized at $Q^+$-modulation $\{|\tau - |\xi|| \leq 2^{-4}\}$. Then there exists $\phi$ localized at $\{|\xi| \in [2^{-3}, 2^{+3}], \ |\tau - |\xi|| \leq 2^{-3}\}$ such that*

$$\|(i\partial_t - |D|)\phi(0) - f\|_{L^2} + \|\Box^p_{A_{<0}}\phi - F\|_{N_0} \leq \delta\left(\|f\|_{L^2} + \|F\|_{N_0}\right) \tag{4.3.23}$$

$$\|\phi\|_{\tilde{S}_0} \lesssim \|f\|_{L^2} + \|F\|_{N_0}. \tag{4.3.24}$$

*Proof.* Let us choose $g$ and $h$ such that

$$ih + |D|\,g = 0, \qquad ih - |D|\,g = f \tag{4.3.25}$$

and apply Theorem 4.3.3 to $(F, g, h)$. Then $T^- = 0$ in the definition of $\phi_{app}$ from (4.3.8)–(4.3.10). From Theorem 4.3.3 we have

$$\|\phi_{app}[0] - (g,h)\|_{L^2} + \|\Box^p_{A_{<0}}\phi_{app} - F\|_{N_0} \ll B, \quad \|\phi_{app}\|_{\tilde{S}_0} \lesssim B. \tag{4.3.26}$$

where $B = \|g\|_{L^2} + \|h\|_{L^2} + \|F\|_{N_0}$. Observe that it suffices to bound the LHS of (4.3.23) by $\delta B$. We define

$$\phi := \frac{1}{2}\left(T^+ + S^+ + Q^+_{\leq -2}S^-\right) \tag{4.3.27}$$

and observe that $\phi$ has the stated $Q^+$-modulation. Furthermore,

$$\phi_{app} = \phi + \frac{1}{2}S_0^-$$

where $S_0^-$ is given by (4.3.14), (4.3.18). We write

$$\Box^p_{A_{<0}}\phi - F = \left(\Box^p_{A_{<0}}\phi_{app} - F\right) + \left(\Box^p_{A_{<0}} e_{<0}^{-i\Psi_-}(t,x,D) - e_{<0}^{-i\Psi_-}(t,x,D)\Box\right)\frac{e^{\pm it|D|}}{|D|}(v(0))$$

The first term is estimated by (4.3.26), while for the second use (4.2.12) and (4.3.17). Moreover,

$$\begin{aligned}(i\partial_t - |D|)\phi(0) - f =& [(i\partial_t - |D|)\phi_{app} - (ih - |D|\, g)] \\ &+ [(ih - |D|\, g) - f] - \frac{1}{2}(i\partial_t - |D|)S_0^-(0)\end{aligned} \tag{4.3.28}$$

The first term is estimated by (4.3.26), the second term is zero, and the third term follows from (4.3.19). This proves (4.3.23).

The bound (4.3.24) follows from (4.3.26), (4.3.18), (4.3.17) and the bound $e_{<0}^{-i\Psi_\pm}(t,x,D) : S_0^\# \to \tilde{S}_0$ from [31]. □

We are now ready to construct the key part of our parametrix for (4.3.4).

**Proposition 4.3.6.** *Suppose $F$ and $f$ are localized at frequency $\{|\xi| \in [2^{k-2}, 2^{k+2}]\}$ and $F$ is also localized at $Q^+$ -modulation $\{|\tau - |\xi|| \leq 2^{k-4}\}$. Then there exists $\psi_k^1$ localized at $\{|\xi| \in [2^{k-3}, 2^{k+3}],\ |\tau - |\xi|| \leq 2^{k-3}\}$ such that*

$$\|\psi_k^1(0) - f\|_{L^2} + \|(i\partial_t + |D|)^p_{A_{<k}}\psi_k^1 - F\|_{N_k^+} \leq \delta\left(\|f\|_{L^2} + \|F\|_{N_k^+}\right) \tag{4.3.29}$$

$$\|\psi_k^1\|_{S_k^+} \lesssim \|f\|_{L^2} + \|F\|_{N_k^+}. \tag{4.3.30}$$

*Proof.* By scaling invariance, we may assume $k = 0$. Define

$$\psi_0^1 := (i\partial_t - |D|)\phi$$

where $\phi$ is obtained by applying Proposition 4.3.5 to $F, f$ and $-A^{free}$. At this low $Q^+$-modulation, the norms of $N_0$ and $N_0^+$ coincide. Observe that on that space-time frequency region, the symbol of $(i\partial_t - |D|)$ is $\sim 1$ and behaves as a bump function. Moreover,

$$\|\psi_0^1\|_{S_0^+} \lesssim \|\psi_0^1\|_{\tilde{S}_0} \lesssim \|\phi\|_{\tilde{S}_0}$$

which implies (4.3.30). We write

$$\begin{aligned}(i\partial_t + |D|)^p_{A_{<0}}\psi_0^1 &= \Box\phi - iA_{<-C}^{free,\ell}\frac{\partial_\ell}{|D|}P_0(i\partial_t + |D| - 2\,|D|)\phi \\ &= \Box^p_{-A_{<0}}\phi - iA_{<-C}^{free,\ell}\frac{\partial_\ell}{|D|}(i\partial_t + |D|)P_0\phi.\end{aligned} \tag{4.3.31}$$

Since $\|A_{<-C}^{free}\|_{L^2L^\infty} \lesssim \varepsilon$, we estimate

$$\begin{aligned}\|A_{<-C}^{free,\ell}\frac{\partial_\ell}{|D|}(i\partial_t + |D|)P_0\phi\|_{L^1L^2} &\lesssim \varepsilon\sum_{j\leq 0} 2^j\|Q_j^+P_0\phi\|_{L^2L^2} \lesssim \varepsilon\|\phi\|_{X_\infty^{0,\frac{1}{2}}} \\ &\lesssim \varepsilon\|\phi\|_{\tilde{S}_0} \lesssim \varepsilon(\|f\|_{L^2} + \|F\|_{N_0})\end{aligned} \tag{4.3.32}$$

where the last inequality comes from Proposition 4.3.5, which completes the proof. □

## Proof of Proposition 4.3.2

We are now ready to prove Proposition 4.3.2.

### The approximate solution $\psi^a$.

We define $\psi^a := \sum_k \psi_k^a$ from its frequency-localized versions

$$\psi_k^a := \psi_k^1 + \psi_k^2$$

which remain to be defined.

We decompose $F = \sum_k P_k F$ and $P_k F = Q^+_{<k-6} P_k F + Q^+_{>k-6} P_k F$. We first define $\psi_k^2$ by

$$\mathcal{F}\psi_k^2(\tau,\xi) := \frac{1}{-\tau+|\xi|}\mathcal{F}(Q^+_{>k-6}P_k F)(\tau,\xi) \tag{4.3.33}$$

so that $(i\partial_t + |D|)\psi_k^2 = Q^+_{>k-6} P_k F$.

Then we apply Proposition 4.3.6 to $Q^+_{<k-6} P_k F$ and $P_k f - \psi_k^2(0)$ which defines the function $\psi_k^1$.

### Reduction to the frequency-localized case.

By redefining $\delta$ (taking $\varepsilon$ smaller), it suffices to show

$$\begin{aligned} &\|P_k[\psi^a(0) - f]\|_{\dot H^{1/2}} + \|P_k[(i\partial_t + |D|)^p_A \psi^a - F]\|_{N^{1/2}_+ \cap L^2L^2} \\ &\qquad \lesssim \delta \sum_{k'=k+O(1)} \left( \|P_{k'} f\|_{\dot H^{1/2}} + \|P_{k'} F\|_{N^{1/2}_+ \cap L^2L^2} \right), \end{aligned} \tag{4.3.34}$$

$$\|P_k \psi^a\|_{S^{1/2}_+} \lesssim \sum_{k'=k+O(1)} \|P_{k'} f\|_{\dot H^{1/2}} + \|P_{k'} F\|_{N^{1/2}_+ \cap L^2L^2}. \tag{4.3.35}$$

Notice that

$$P_k[(i\partial_t + |D|)^p_A \psi^a - F] = \sum_{k'=k+O(1)} P_k[(i\partial_t + |D|)^p_A \psi^a_{k'} - P_{k'} F], \tag{4.3.36}$$

and the analogous summation for $P_k\psi^a$ and $P_k[\psi^a(0) - f]$. By disposing of $P_k$ it suffices to show the following estimates:

$$\begin{aligned} &\|(i\partial_t + |D|)^p_{A_{<k'}} \psi^a_{k'} - P_{k'} F\|_{N^+_{k'}} + \|\psi^a_{k'}(0) - P_{k'} f\|_{L^2} \\ &\qquad \lesssim \delta \left( \|P_{k'} f\|_{L^2} + \|P_{k'} F\|_{N^+_{k'} \cap L^2 \dot H^{-1/2}} \right) \end{aligned} \tag{4.3.37}$$

$$2^{-k'/2}\|(i\partial_t + |D|)^p_{A_{<k'}} \psi^a_{k'} - P_{k'} F\|_{L^2L^2} \lesssim \delta \left( \|P_{k'} f\|_{L^2} + \|P_{k'} F\|_{N^+_{k'} \cap L^2 \dot H^{-1/2}} \right) \tag{4.3.38}$$

$$\|\psi^a_{k'}\|_{S^+_{k'}} \lesssim \|P_{k'}f\|_{L^2} + \|P_{k'}F\|_{N^+_{k'}\cap L^2\dot{H}^{-1/2}} \tag{4.3.39}$$

$$2^{-\frac{k}{2}}\|(i\partial_t + |D|)\psi^a_{k'}\|_{L^2L^2} \lesssim \|P_{k'}f\|_{L^2} + \|P_{k'}F\|_{N^+_{k'}\cap L^2\dot{H}^{-1/2}} \tag{4.3.40}$$

and the following error term, where $k'', k''' = k' \pm O(1)$:

$$\|A^{free,j}_{k''-c}\frac{\partial_j}{|D|}P_{k'''}\psi^a_{k'}\|_{N^+_{k'}\cap L^2\dot{H}^{-1/2}} \lesssim \varepsilon\|\psi^a_{k'}\|_{S^+_{k'}} \tag{4.3.41}$$

**Proof of claims** (4.3.37)–(4.3.41)

It only remains to prove (4.3.37)–(4.3.41).

**S**tep 1: Proof of (4.3.39). For $\psi^2_k$ we have, by Lemma 2.2.6

$$\|\psi^2_k\|_{S^+_k} \lesssim \|Q^+_{>k-6}P_kF\|_{N^+_k\cap L^2\dot{H}^{-1/2}}. \tag{4.3.42}$$

For the function $\psi^1_k$, by Proposition 4.3.6, we have

$$\|\psi^1_k\|_{S^+_k} \lesssim \|P_kf - \psi^2_k(0)\|_{L^2} + \|Q^+_{<k-6}P_kF\|_{N^+_k} \lesssim \|P_kf\|_{L^2} + \|P_kF\|_{N^+_k\cap L^2\dot{H}^{-1/2}}. \tag{4.3.43}$$

We have used (4.3.42) to bound $\|\psi^2_k(0)\|_{L^2}$.

**S**tep 2: Proof of (4.3.37). By Proposition 4.3.6, we have

$$\begin{aligned}
&\|\psi^1_k(0) - [P_kf - \psi^2_k(0)]\|_{L^2} + \|(i\partial_t + |D|)^p_{A_{<k}}\psi^1_k - Q^+_{<k-6}P_kF\|_{N^+_k}\\
\le&\delta\left(\|P_kf - \psi^2_k(0)\|_{L^2} + \|Q^+_{<k-6}P_kF\|_{N^+_k}\right)\\
\lesssim&\delta(\|P_kf\|_{L^2} + \|P_kF\|_{N^+_k\cap L^2\dot{H}^{-1/2}})
\end{aligned} \tag{4.3.44}$$

It remains to estimate

$$\begin{aligned}
&\|(i\partial_t + |D|)^p_{A_{<k}}\psi^2_k - Q^+_{>k-6}P_kF\|_{N^+_k} \le \|A^{free,j}_{<k-C}\frac{\partial_j}{|D|}P_k\psi^2_k\|_{N^+_k}\\
\lesssim&\|A^{free}_{<k-C}\|_{L^2L^\infty}\|\psi^2_k\|_{L^2L^2} \lesssim (\varepsilon 2^{k/2})2^{-k/2}\|Q^+_{>k-6}P_kF\|_{N^+_k\cap L^2\dot{H}^{-1/2}}\\
\lesssim&\varepsilon\|P_kF\|_{N^+_k\cap L^2\dot{H}^{-1/2}}
\end{aligned}$$

The first inequality follows from the definition (4.3.2). The third inequality follows from (4.3.42).

**S**tep 3: Proof of (4.3.38). We estimate

$$2^{-\frac{k}{2}}\|(i\partial_t + |D|)^p_{A_{<k}}\psi^1_k - Q^+_{<k-6}P_kF\|_{L^2L^2} \lesssim \delta(\|P_kf\|_{L^2} + \|P_kF\|_{N^+_k\cap L^2\dot{H}^{-1/2}}) \tag{4.3.45}$$

using (2.2.16) and (4.3.44). For $(i\partial_t + |D|)^p_{A_{<k}}\psi^2_k - Q^+_{>k-6}P_kF$, using (4.3.42) we estimate

$$\|A^{free,j}_{<k-C}\frac{\partial_j}{|D|}P_k\psi^2_k\|_{L^2L^2} \lesssim \|A^{free}_{<k-C}\|_{L^2L^\infty}\|\psi^2_k\|_{L^\infty L^2} \lesssim 2^{k/2}\varepsilon\|Q^+_{>k-6}P_kF\|_{N^+_k\cap L^2\dot{H}^{-1/2}}.$$

**S**tep 4: Proof of (4.3.40). We write

$$(i\partial_t + |D|)\psi_k^a = Q^+_{>k-6} P_k F + Q^+_{<k-6} P_k F + \\ + ((i\partial_t + |D|)^p_{A_{<k}} \psi^1_k - Q^+_{<k-6} P_k F) + A^{free,j}_{<k-C} \frac{\partial_j}{|D|} P_k \psi^1_k.$$

We use (4.3.45) and it remains to estimate

$$2^{-k/2} \| A^{free,j}_{<k-C} \frac{\partial_j}{|D|} P_k \psi^1_k \|_{L^2 L^2} \lesssim 2^{-k/2} \| A^{free}_{<k-C} \|_{L^2 L^\infty} \| \psi^1_k \|_{L^\infty L^2} \tag{4.3.46}$$

$$\lesssim \varepsilon (\| P_k f \|_{L^2} + \| P_k F \|_{N^+_k \cap L^2 \dot{H}^{-1/2}}). \tag{4.3.47}$$

**S**tep 5: Proof of (4.3.41). The $N^+_{k'}$ bound follows from (6.3.20), while the $L^2 \dot{H}^{-1/2}$ bound follows from the estimate (4.3.46) with $k$ replaced by $k', k'', k'''$.

## 4.4 Solvability of the covariant Dirac equation

We now prove Proposition 1.8.11 by using the construction of the previous section and employing the estimates stated in section 1.8.

Recall the equation (1.8.30):

$$\begin{cases} \alpha^\mu \mathbf{D}^A_\mu \psi = F & \text{on } I \\ \psi(0) = \psi_0, \end{cases} \tag{4.4.1}$$

where the potential $A = A_\mu \mathrm{d}x^\mu$ is given by

$$A_0 = \mathbf{A}_0(\psi', \psi'), \quad A_j = A^{free}_j + \mathbf{A}_j(\psi', \psi') \quad \text{on } I$$

where $(A_0, A_j)$ are defined by (1.7.22), (1.7.19)

### Proof of Proposition 1.8.11

To solve (1.8.30), we introduce an auxiliary equation (see (4.4.2) below), which on one hand reduces to (1.8.30) after suitable manipulation, and on the other hand possess appropriate structure so that it could be solved via an iteration argument. More precisely, we look for a pair $(\varphi_+, \varphi_-)$ of spinor fields which obeys

$$(i\partial_t + s|D|)\varphi_s = \mathcal{N}^E(A_0, \Pi_+ \varphi_+) + \mathcal{N}^E(A_0, \Pi_- \varphi_-) + \Pi_{-s}(\pi^E[A_0]\varphi_s) \\ - \mathcal{N}^R(A_x, \varphi_+) + \mathcal{N}^R(A_x, \varphi_-) \\ + \Pi_s \mathcal{N}^S_+(A_x, \varphi_+) + \Pi_s \mathcal{N}^S_-(A_x, \varphi_-) + i\Pi_s F. \tag{4.4.2}$$

with $\varphi_s(0) = \Pi_s \psi(0)$ for $s \in \{+, -\}$.

Taking $\Pi_s$ of both sides, a computation similar to Lemma 1.7.7 shows that $\psi = \Pi_+\varphi_+ + \Pi_-\varphi_-$ solves the desired covariant Dirac equation; a key observation here is that the last term on the first line vanishes. Therefore, in order to establish the existence statement in Proposition 1.8.11, it suffices to show that, under the hypotheses of Proposition 1.8.11, there exists a solution $(\varphi_+, \varphi_-)$ to (4.4.2) obeying

$$\|\varphi_s\|_{(\tilde{S}^{1/2}[I])_c} \lesssim \|\Pi_s\psi_0\|_{\dot{H}^{1/2}_c} + \|\Pi_s F\|_{(N_s^{1/2}\cap L^2L^2\cap G^{1/2}[I])_c}. \tag{4.4.3}$$

Our goal in the remainder of this subsection is to prove the preceding statement. The remaining uniqueness statement in Proposition 1.8.11 follows by a similar argument applied to $\Pi_s(4.4.2)$; we omit the repetitive details.

Before analyzing (4.4.2), we begin with some simple remarks. First, extending $\Pi_s F$ by zero outside of $I$ results in an equivalent $N_s^{1/2} \cap L^2L^2 \cap G^{1/2}$ norm (see Lemma 2.1.5 and the preceding discussion); therefore, it suffices to focus on the case $I = \mathbb{R}$. Next, by Proposition 1.8.6 (note that $\partial_t A_0 = \partial_t \mathcal{M}^E(\psi', \psi')$ thanks to the hypothesis $\partial_\mu\langle\psi', \alpha^\mu\psi'\rangle = 0$), $A$ obeys the following bound: Given an admissible frequency envelope $b$ with $\sup_{s\in\{+,-\}} \|\Pi_s\psi'\|_{(\tilde{S}_s^{1/2})_b} \leq 1$, we have

$$\|A_0\|_{Y^1_{b^2}} + \|A_x - A_x^{free}\|_{S^1_{b^2}} \lesssim 1. \tag{4.4.4}$$

Constructing $b$ appropriately, we have $\|b^2\|_{\ell^1} \leq \|b\|_{\ell^2}^2 \lesssim \epsilon_{**}^2$ by hypothesis.

We are now ready to begin the analysis of (4.4.2). Using the decomposition in Section 1.7 and the identity

$$\pi^E[A_0]\Pi_s\varphi_s + \Pi_{-s}\pi^E[A_0]\varphi_s = \pi^E[A_0](1 - \Pi_{-s})\varphi_s + \Pi_{-s}\pi^E[A_0]\varphi_s,$$

the system (4.4.2) can be rewritten as $(i\partial_t + s|D|)^p_{A^{free}}\varphi_s = \mathcal{E}_s\varphi + i\Pi_s F$, where

$$\begin{aligned}
\mathcal{E}_s\varphi = \mathcal{E}_s[A^{free}, \psi']\varphi = &\pi^E[\mathbf{A}_0(\psi', \psi')]\varphi_s - s\pi^R[\mathbf{A}_x(\psi', \psi')]\varphi_s && (4.4.5)\\
&+ \widetilde{\mathcal{N}}^E(A_0, \Pi_s\varphi_s) - s\widetilde{\mathcal{N}}^R(A_x, \varphi_s) && (4.4.6)\\
&+ \mathcal{N}^E(A_0, \Pi_{-s}\varphi_{-s}) + s\mathcal{N}^R(A_x, \varphi_{-s}) && (4.4.7)\\
&+ \Pi_s\mathcal{N}^S_+(A_x, \varphi_+) + \Pi_s\mathcal{N}^S_-(A_x, \varphi_-) && (4.4.8)\\
&+ [\Pi_{-s}, \pi^E[A_0]]\varphi_s. && (4.4.9)
\end{aligned}$$

For any admissible frequency envelope $c$ and $\varphi' = (\varphi'_+, \varphi'_-) \in (\tilde{S}_+^{1/2} \times \tilde{S}_-^{1/2})_c$, we claim that

$$\|\mathcal{E}_s\varphi'\|_{(N_s^{1/2}\cap L^2L^2\cap G^{1/2})_c} \lesssim \epsilon_{**} \sup_{s\in\{+,-\}} \|\varphi'_s\|_{(\tilde{S}_s^{1/2})_c}. \tag{4.4.10}$$

For the moment, we assume the claim and complete the proof. Let $\varphi' = (\varphi'_+, \varphi'_-) \in (\tilde{S}_+^{1/2} \times \tilde{S}_-^{1/2})_c$, and consider a solution $\varphi$ to

$$(i\partial_t + s|D|)^p_{A^{free}}\varphi_s = \mathcal{E}_s\varphi' + i\Pi_s F$$

given by Theorem 1.6.2. By the same theorem and (4.4.10), we have

$$\|\varphi_s\|_{S_c^{1/2}} \lesssim \epsilon_{**} \sup_{s\in\{+,-\}} \|\varphi'_s\|_{(\tilde{S}_s^{1/2})_c} + \|\varphi(0)\|_{\dot{H}^{1/2}} + \|\Pi_s F\|_{(N_s^{1/2}\cap L^2L^2)_c}.$$

Combined with the inequality

$$\|\varphi\|_{(\tilde{Z}_s^{1/2})_c} = \|(i\partial_t + s|D|)\varphi\|_{G_c^{1/2}} \leq \|(i\partial_t + s|D|)^p_{A^{free}}\varphi\|_{G_c^{1/2}} + \|\pi^R[A_x^{free}]\varphi\|_{G_c^{1/2}}$$

and (1.8.25) (which only involves the $S_s^{1/2}$ norm on the RHS), we have

$$\|\varphi_s\|_{\tilde{S}_c^{1/2}} \lesssim \epsilon_{**} \sup_{s\in\{+,-\}} \|\varphi'_s\|_{(\tilde{S}_s^{1/2})_c} + \|\varphi(0)\|_{\dot{H}^{1/2}} + \|\Pi_s F\|_{(N_s^{1/2}\cap L^2L^2\cap G^{1/2})_c}.$$

Taking $\epsilon_{**} > 0$ sufficiently small, we may ensure that the map $\varphi' \mapsto \varphi$ is a contraction in $(\tilde{S}_+^{1/2} \times \tilde{S}_-^{1/2})_c$. By iteration (or Banach fixed point theorem), we may then obtain the desired solution $\varphi$ to (4.4.2).

Now it only remains to prove (4.4.10). For (4.4.5), we use Proposition 1.8.8 with appropriate frequency envelopes. For (4.4.6)–(4.4.8), we apply Proposition 1.8.7 and (4.4.4). Finally, (4.4.9) is handled using (4.4.4) and the following lemma.

**Lemma 4.4.1.** *Let $a$, $b$ be any admissible frequency envelopes, and $s \in \{+,-\}$. Then we have*

$$\|[\Pi_{-s}, \pi^E[A_0]]\psi\|_{(N_s^{1/2}\cap L^2L^2\cap G^{1/2})_{ab}} \lesssim \|A_0\|_{Y_a^1}\|\psi\|_{(\tilde{S}_s^{1/2})_b} \tag{4.4.11}$$

*Proof.* By (1.8.21) and (1.8.24), (4.4.11) holds for the $L^2L^2 \cap G^{1/2}$ norm on the LHS even without the commutator structure; hence it remains to show

$$\|[\Pi_{-s}, \pi^E[A_0]]\psi\|_{(N_s^{1/2})_{ab}} \lesssim \|A_0\|_{Y_a^1}\|\psi\|_{(\tilde{S}_s^{1/2})_b} \tag{4.4.12}$$

Write $A_k = P_k A_0$, $\psi_k = P_k\psi$ and $\tilde{P}_k := \Pi_{-s_0}P_k$, so that

$$[\Pi_{-s_0}, \pi^E[A_0]]\psi = \sum_{k',k_1,k:k_1<k-5} [\tilde{P}_{k'}, A_{k_1}]\psi_k.$$

Observe that the summand vanishes unless $k' = k+O(1)$. Moreover, we have the well-known commutator identity

$$[\tilde{P}_{k'}, A_{k_1}]f = 2^{-k'}\mathcal{L}(\nabla A_{k_1}, f)$$

where $\mathcal{L}$ is a translation-invariant bilinear operator with bounded mass kernel (see [52, Lemma 2]). Applying Lemma 5.6.1, we have

$$\begin{aligned}\|[\tilde{P}_{k'}, A_{k_1}]\psi_k\|_{N_{s_0}^{1/2}} \lesssim& 2^{-k}\|\mathcal{L}(\nabla A_{k_1}, \psi_k)\|_{N_{s_0}^{1/2}} \\ \lesssim& 2^{-\frac{1}{2}k}\|\nabla A_{k_1}\|_{L^2L^2}\Big(\sum_{\mathcal{C}_{k_1}(0)} \|P_{\mathcal{C}_{k_1}(0)}\psi_k\|^2_{L^2L^\infty}\Big)^{1/2} \\ \lesssim& 2^{\frac{1}{2}(k_1-k)}\|A_{k_1}\|_{Y^1}\|\psi_k\|_{S_{s_0}^{1/2}}.\end{aligned}$$

Thanks to the gain $2^{\frac{1}{2}(k_1-k)}$, the frequency envelope bound (4.4.12) follows. □

## 4.5 The definition and properties of the phase

Now we return to the massive MKG equation and prepare the preliminaries to proving Theorem 4.2.2.

### The construction of the phase

We recall that $A$ is real-valued, it solves the free wave equation $\Box A = 0$ and satisfies the Coulomb gauge condition $\nabla_x \cdot A = 0$.

For $k = 0$ we define

$$\begin{aligned} \psi^0_{\pm}(t,x,\xi) &:= \sum_{j<-C} \psi^0_{j,\pm}(t,x,\xi), \qquad \text{where} \\ \psi^0_{j,\pm}(t,x,\xi) &:= \frac{-L_{\pm}}{\Delta_{\omega^\perp} + \frac{1}{\langle \xi \rangle^2}(\omega \cdot \nabla_x)^2} \left( P_j A(t,x) \cdot \frac{\xi}{\langle \xi \rangle} \right) \end{aligned} \tag{4.5.1}$$

For $k \geq 1$ we define

$$\psi^k_{\pm}(t,x,\xi) := \frac{-1}{\Delta_{\omega^\perp} + \frac{1}{\langle \xi \rangle^2}(\omega \cdot \nabla_x)^2} L_{\pm} \sum_{k_1 < k-c} \left( \Pi^{\omega}_{>\delta(k_1-k)} P_{k_1} A \cdot \frac{\xi}{\langle \xi \rangle} \right) \tag{4.5.2}$$

It will be convenient to rescale the angular pieces that define $\psi^k_{\pm}$ to $|\xi| \simeq 1$:

$$\psi^k_{j,\theta,\pm}(t,x,2^k\xi) := \frac{-L_{\pm,k}}{\Delta_{\omega^\perp} + 2^{-2k}\frac{1}{\langle \xi \rangle_k^2}(\omega \cdot \nabla_x)^2} \left( \Pi^{\omega}_{\theta} P_j A \cdot \omega \frac{|\xi|}{\langle \xi \rangle_k} \right) \tag{4.5.3}$$

for $2^{\delta(j-k)} < \theta < c$ and $j < k - c$, where

$$L_{\pm,k} = \pm \partial_t + \frac{|\xi|}{\langle \xi \rangle_k} \omega \cdot \nabla_x, \qquad \omega = \frac{\xi}{|\xi|}, \qquad \langle \xi \rangle_k = \sqrt{2^{-2k} + |\xi|^2}.$$

Note that $\Pi^{\omega}_{\theta}, \Pi^{>\omega}_{\theta}$ defined in (1.5.3), (1.5.4) behave like Littlewood-Paley projections in the space $\omega^\perp$.

*Remark* 4.5.1. It will be important to keep in mind that $\psi^k_{\pm}$ is real-valued, since it is defined by applying real and even Fourier multipliers to the real function $A$.

*Remark* 4.5.2. Due to the Coulomb condition $\nabla_x \cdot A = 0$ the expression in (4.5.2) acts like a null form, leading to an angular gain. Indeed, a simple computation shows

$$\left| \widehat{\Pi^{\omega}_{\theta} A}(\eta) \cdot \omega \right| \lesssim \theta \left| \widehat{\Pi^{\omega}_{\theta} A}(\eta) \right|,$$

which implies $\| \Pi^{\omega}_{\theta} A \cdot \omega \|_{L^2_x} \lesssim \theta \| \Pi^{\omega}_{\theta} A \|_{L^2_x}$.

We denote by

$$\varphi_\xi(\eta) = |\eta|^2_{\omega^\perp} + 2^{-2k} \langle \xi \rangle_k^{-2} (\omega \cdot \eta)^2$$

the Fourier multiplier of the operator $\Delta_{\omega^\perp} + 2^{-2k} \frac{1}{\langle \xi \rangle_k^2} (\omega \cdot \nabla_x)^2$. We have the following bounds on $\varphi_\xi(\eta)$:

**Lemma 4.5.3.** *Let $k \geq 1$. For any $\eta$ and $\xi = |\xi|\,\omega$ such that $\angle(\xi, \eta) \simeq \theta$ and $|\eta| \simeq 2^j$ we have*

$$\left| (\theta \nabla_\omega)^\alpha \frac{1}{\varphi_\xi(\eta)} \right| \leq \frac{C_\alpha}{(2^j \theta)^2 + 2^{2j-2k}} \tag{4.5.4}$$

$$\left| \partial^l_{|\xi|} (\theta \nabla_\omega)^\alpha \frac{1}{\varphi_\xi(\eta)} \right| \leq \frac{C_{\alpha,l}}{(2^j \theta)^2 + 2^{2j-2k}} \cdot \frac{2^{-2k}}{\theta^2 + 2^{-2k}}, \qquad l \geq 1. \tag{4.5.5}$$

*Remark* 4.5.4. Suppose we want to estimate $\partial^l_{|\xi|} (\theta \partial_\omega)^\alpha \psi^k_{j,\theta,\pm}(t_0, \cdot, 2^k \xi)$ in $L^2_x$. By lemma 4.5.3 and the Coulomb condition (remark 4.5.2), the following multiplier applied to $A(t_0)$

$$\partial^{\alpha_1}_{|\xi|} (\theta \partial_\omega)^\alpha \frac{-L_{\pm,k}}{\Delta_{\omega^\perp} + 2^{-2k} \frac{1}{\langle \xi \rangle_k^2} (\omega \cdot \nabla_x)^2} \left( \Pi^\omega_\theta P_j(\;) \cdot \omega \frac{|\xi|}{\langle \xi \rangle_k} \right)$$

may be replaced by

$$\frac{2^{-j}\theta}{\theta^2 + 2^{-2k}} \Pi^{\omega,\alpha}_\theta \tilde{P}_j \quad (\text{if } l = 0), \qquad \frac{2^{-2k} 2^{-j} \theta}{(\theta^2 + 2^{-2k})^2} \Pi^{\omega,\alpha,l}_\theta \tilde{P}_j \quad (\text{if } l \geq 1),$$

for the purpose of obtaining an upper bound for the $L^2_x$ norm, where $\Pi^{\omega,\alpha,l}_\theta$ and $\tilde{P}_j$ obey the same type of localization properties and symbol estimates as $\Pi^\omega_\theta$ and $P_j$.

*Proof.* For $\alpha = 0,\ l = 0$ the bound is clear since

$$\varphi_\xi(\eta) \simeq (2^j \theta)^2 + 2^{2j-2k}. \tag{4.5.6}$$

For $N \geq 1$ we prove the lemma by induction on $N = l + |\alpha|$. We focus on the case $l \geq 1$ since the proof of (4.5.4) is entirely similar. Suppose the claim holds for all $l', \alpha'$ such that $0 \leq l' + |\alpha'| \leq N - 1$. Applying the product rule to $1 = \varphi_\xi(\eta) \frac{1}{\varphi_\xi(\eta)}$ we obtain

$$\varphi_\xi(\eta) \cdot \partial^l_{|\xi|} (\theta \nabla_\omega)^\alpha \frac{1}{\varphi_\xi(\eta)} = \sum C^{\alpha',\beta'}_{\alpha'',\beta''} \cdot \partial^{l'}_{|\xi|} (\theta \nabla_\omega)^{\alpha'} \varphi_\xi(\eta) \cdot \partial^{l''}_{|\xi|} (\theta \nabla_\omega)^{\alpha''} \frac{1}{\varphi_\xi(\eta)}$$

where we sum over $l' + l'' = l,\ \alpha' + \alpha'' = \alpha,\ l'' + |\alpha''| \leq N - 1$. Given the induction hypothesis and (4.5.6), for the terms in the sum it suffices to show

$$\left| \partial^{l'}_{|\xi|} (\theta \nabla_\omega)^{\alpha'} \varphi_\xi(\eta) \right| \lesssim 2^{2j-2k} \qquad \text{for } l' \geq 1, \tag{4.5.7}$$

$$\left| \partial^{l'}_{|\xi|} (\theta \nabla_\omega)^{\alpha'} \varphi_\xi(\eta) \right| \lesssim (2^j \theta)^2 \qquad \text{for } \ l' = 0,\ |\alpha'| \geq 1 \ (l'' = l \geq 1) \tag{4.5.8}$$

We write

$$\varphi_\xi(\eta) = C_\eta - (\omega\cdot\eta)^2(1 - 2^{-2k}\langle\xi\rangle_k^{-2}).$$

We have $|\omega\cdot\eta| \lesssim 2^j$ and thus for $l' \geq 1$ we obtain (4.5.7).

Now suppose $l' = 0$ and thus $|\alpha'| \geq 1$. Observe that $\partial_\omega(\omega\cdot\eta)^2 \simeq 2^{2j}\theta$ and thus for all $|\alpha'| \geq 1$ we have $(\theta\nabla_\omega)^{\alpha'}(\omega\cdot\eta)^2 \lesssim 2^{2j}\theta^2$, which implies (4.5.8). □

The following proposition will be used in stationary phase arguments.

**Proposition 4.5.5.** *For $k \geq 0$, $|\xi| \simeq 1$, denoting $T = |t-s| + |x-y|$ we have:*

$$\left|\psi^k_\pm(t,x,2^k\xi) - \psi^k_\pm(s,y,2^k\xi)\right| \lesssim \varepsilon\log(1+2^kT) \tag{4.5.9}$$

$$\left|\partial^\alpha_\omega\left(\psi^k_\pm(t,x,2^k\xi) - \psi^k_\pm(s,y,2^k\xi)\right)\right| \lesssim \varepsilon(1+2^kT)^{(|\alpha|-\frac{1}{2})\delta}, \quad 1 \leq |\alpha| \leq \delta^{-1} \tag{4.5.10}$$

$$\left|\partial^l_{|\xi|}\partial^\alpha_\omega\left(\psi^k_\pm(t,x,2^k\xi) - \psi^k_\pm(s,y,2^k\xi)\right)\right| \lesssim \varepsilon 2^{-2k}(1+2^kT)^{(|\alpha|+\frac{3}{2})\delta}, \quad l \geq 1, (|\alpha| + \frac{3}{2})\delta < 1 \tag{4.5.11}$$

*Proof.* Using $\| |\nabla|^\sigma A\|_{L^\infty L^2} \lesssim \varepsilon$, Bernstein's inequality $P_j\Pi^\omega_\theta L^2_x \to (2^{dj}\theta^{d-1})^{\frac{1}{2}}L^\infty_x$ and the null form (Remark 4.5.2), for $k \geq 1$ we obtain

$$\left|\psi^k_{j,\theta,\pm}(t,x,2^k\xi)\right| \lesssim \varepsilon(2^{dj}\theta^{d-1})^{\frac{1}{2}}\theta\frac{2^j2^{-\sigma j}}{(2^j\theta)^2 + 2^{2j-2k}} \lesssim \varepsilon\theta^{\frac{1}{2}}$$

Thus, for both $k = 0$ and $k \geq 1$, one has

$$\left|\psi^k_{j,\pm}(t,x,2^k\xi)\right| \lesssim \varepsilon, \qquad \left|\nabla_{x,t}\psi^k_{j,\pm}(t,x,2^k\xi)\right| \lesssim 2^j\varepsilon$$

We sum the last bound for $j \leq j_0$ and the previous one for $j_0 < j \leq k - c$:

$$\left|\psi^k_\pm(t,x,2^k\xi) - \psi^k_\pm(s,y,2^k\xi)\right| \lesssim \varepsilon\left(2^{j_0}T + (k - j_0)\right)$$

Choosing $k - j_0 = \log_2(2^kT) - O(1)$ we obtain (4.5.9).

For the proof of (4.5.10) and (4.5.11) we use Remark 4.5.4. Since their proofs are similar we only write the details for (4.5.11). First suppose $k \geq 1$.

From Bernstein's inequality and Remark 4.5.4 we obtain

$$\left|\partial^l_{|\xi|}\partial^\alpha_\omega\psi^k_{j,\theta,\pm}(t,x,\xi)\right| \lesssim (2^{dj}\theta^{d-1})^{\frac{1}{2}}\theta\frac{1}{(2^j\theta)^2 + 2^{2j-2k}}\frac{2^{-2k}}{\theta^2}\theta^{-|\alpha|} \lesssim 2^{-2k}\varepsilon\theta^{-\frac{3}{2}-|\alpha|}$$

$$\left|\nabla_{x,t}\partial^l_{|\xi|}\partial^\alpha_\omega\psi^k_{j,\theta,\pm}(t,x,\xi)\right| \lesssim 2^j2^{-2k}\varepsilon\theta^{-\frac{3}{2}-|\alpha|}.$$

We sum after $2^{\delta(j-k)} < \theta < c$. Summing one bound for $j \leq j_0$ and the other one for $j_0 < j \leq k - c$ we obtain

$$\left|\partial^l_{|\xi|}\partial^\alpha_\omega\left(\psi^k_\pm(t,x,2^k\xi) - \psi^k_\pm(s,y,2^k\xi)\right)\right| \lesssim \varepsilon 2^{-2k}\left(2^{j_0}T2^{-(\frac{3}{2}+|\alpha|)\delta(j_0-k)} + 2^{-(\frac{3}{2}+|\alpha|)\delta(j_0-k)}\right)$$

Choosing $2^{-j_0} \sim T$ , we obtain (4.5.11). When $k = 0$ the same numerology, but without the $\theta$ factors, implies (4.5.11). □

## Decomposable estimates

The decomposable calculus was introduced in [47]. The formulation that we use here is similar to [31], which we have modified to allow for non-homogeneous symbols.

For $k = 0$ we define

$$\|F\|_{D_0(L^qL^r)} = \sum_{\alpha \leq 10d} \sup_{|\xi| \leq C} \|\partial_\xi^\alpha F(\cdot, \cdot, \xi)\|_{L^qL^r}.$$

For $k \geq 1$ we define

$$\|F^\theta\|^2_{D_k^\theta(L^qL^r)} = \sum_\phi \sum_{\alpha_1, |\alpha| \leq 10d} \sup_{|\xi| \sim 2^k} \|\chi_\phi^\theta(\xi)(2^k\partial_{|\xi|})^{\alpha_1}(2^k\theta\nabla_\omega)^\alpha F^\theta(\cdot, \cdot, \xi)\|^2_{L^qL^r}. \tag{4.5.12}$$

where $\chi_\phi^\theta(\xi)$ denote cutoff functions to sectors centered at $\phi$ of angle $\lesssim \theta$ and $\phi$ is summed over a finitely overlapping collection of such sectors.

The symbol $F(t, x, \xi)$ is in $D_k(L^qL^r)$ if we can decompose $F = \sum F^\theta$ such that

$$\sum_\theta \|F^\theta\|_{D_k^\theta(L^qL^r)} < \infty$$

and we define $\|F\|_{D_k(L^qL^r)}$ to be the infimum of such sums.

**Lemma 4.5.6.** *Suppose that for $i = 1, N$ the symbols $a(t, x, \xi), F_i(t, x, \xi)$ satisfy $\|F_i\|_{D_k(L^{q_i}L^{r_i})} \lesssim 1$ and*

$$\sup_{t \in \mathbb{R}} \|A(t, x, D)\|_{L^2_x \to L^2_x} \lesssim 1.$$

*The symbol of $T$ is defined to be*

$$a(t, x, \xi) \prod_{i=1}^N F_i(t, x, \xi).$$

*Then, whenever $q, \tilde{q}, r, r_i \in [1, \infty]$, $q_i \geq 2$ are such that*

$$\frac{1}{q} = \frac{1}{\tilde{q}} + \sum_{i=1}^N \frac{1}{q_i}, \qquad \frac{1}{r} = \frac{1}{2} + \sum_{i=1}^N \frac{1}{r_i},$$

*we have*

$$T(t, x, D)\bar{P}_k : L^{\tilde{q}}L^2 \to L^qL^r,$$

*By duality, when $r = 2$, $r_i = \infty$, the same mapping holds for $T(D, s, y)$.*

*Proof for $k = 0$.* For each $i = 1, N$ we decompose into Fourier series

$$F_i(t,x,\xi) = \sum_{j\in\mathbb{Z}^d} d_{i,j}(t,x)e_j(\xi), \qquad e_j(\xi) = e^{ij\cdot\xi}$$

on a box $[-C/2, C/2]^d$. From the Fourier inversion formula and integration by parts we obtain

$$\langle j\rangle^M \|d_{i,j}\|_{L^{q_i}L^{r_i}} \lesssim \|F_i\|_{D_0(L^{q_i}L^{r_i})} \lesssim 1$$

for some large $M$. Then

$$Tu(t,x) = \sum_{j_1,\dots j_N\in\mathbb{Z}^d} \prod_{i=1}^N d_{i,j_i}(t,x) \int e^{ix\cdot\xi} e^{i(j_1+\dots+j_N)\cdot\xi} a(t,x,\xi)\hat{u}(t,\xi)\,\mathrm{d}\xi$$

From Hölder's inequality we obtain

$$\begin{aligned}\|Tu\|_{L^qL^r} &\le \sum_{j_1,\dots j_N\in\mathbb{Z}^d} \prod_{i=1}^N \|d_{i,j_i}\|_{L^{q_i}L^{r_i}} \|A(t,x,D)e^{i(j_1+\dots+j_N)D}u\|_{L^{\tilde{q}}L^2} \lesssim \\ &\lesssim \sum_{j_1,\dots j_N\in\mathbb{Z}^d} \prod_{i=1}^N \langle j_i\rangle^{-M} \|u\|_{L^{\tilde{q}}L^2} \lesssim \|u\|_{L^{\tilde{q}}L^2},\end{aligned}$$

which proves the claim. □

*Proof for $k \ge 1$.* We present the proof for the case $N = 2$. It is straightforward to observe that the following works for any $N$. From the decompositions $F_i = \sum_{\theta_i} F_i^{\theta_i}$ and definition of $D_k(L^qL^r)$ we see that it suffices to restrict attention to the operator $T$ with symbol

$$a(t,x,\xi)F_1(t,x,\xi)F_2(t,x,\xi)$$

in the case $F_i = F_i^{\theta_i},\ i = 1,2$ and to prove

$$\|T(t,x,D)\tilde{P}_k\|_{L^{\tilde{q}}L^2\to L^qL^r} \lesssim \|F_1\|_{D_k^{\theta_1}(L^{q_1}L^{r_1})}\|F_2\|_{D_k^{\theta_2}(L^{q_2}L^{r_2})}. \tag{4.5.13}$$

For $i = 1,2$ we decompose

$$F_i = \sum_{T_i} F_i^{T_i}, \qquad F_i^{T_i}(t,x,\xi) := \varphi_{\theta_i}^{T_i}(\xi)F_i(t,x,\xi)$$

where $\varphi_{\theta_i}^{T_i}(\xi)$ are cutoff functions to sectors $T_i$ of angle $\theta_i$, where the $T_i$ is summed over a finitely overlapping collection of such sectors. We also consider the bump functions $\chi_{T_i}(\xi)$ which equal 1 on the supports of $\varphi_{\theta_i}^{T_i}(\xi)$ and are adapted to some enlargements of the sectors $T_i$. We expand each component as a Fourier series

$$F_i^{T_i}(t,x,\xi) = \sum_{j\in\mathbb{Z}^d} d_{i,j}^{T_i}(t,x)e_{\theta_i,j}^{T_i}(\xi), \qquad e_{\theta_i,j}^{T_i}(\xi) = \exp(i2^{-k}j\cdot(|\xi|, \tilde{\omega}\theta_i^{-1})/C)$$

on the tube $T_i = \{|\xi| \sim 2^k, \angle(\xi, \phi_i) \lesssim \theta_i\}$ where $\xi = |\xi|\,\omega$ in polar coordinates so that $\omega$ is parametrized by $\tilde{\omega} \in \mathbb{R}^{d-1}$ such that $|\tilde{\omega}| \lesssim \theta_i$. Integrating by parts in the Fourier inversion formula for $d_{i,j}^{T_i}(t,x)$ we obtain

$$\langle j\rangle^M \|d_{i,j}^{T_i}(t)\|_{L^{r_i}} \lesssim \sum_{\alpha_1,|\alpha|\le 10d} \sup_{|\xi|\sim 2^k} \|(2^k\partial_{|\xi|})^{\alpha_1}(2^k\theta_i\nabla_\omega)^\alpha F_i^{T_i}(t,\cdot,\xi)\|_{L^{r_i}}$$

and since $q_i \ge 2$ we have

$$\|d_{i,j}^{T_i}\|_{L_t^{q_i} l_{T_i}^2 L^{r_i}} \lesssim \langle j\rangle^{-M} \|F_i\|_{D_k^{\theta_i}(L^{q_i}L^{r_i})} \tag{4.5.14}$$

Since for $\xi \in T_i$ we have $F_i^{T_i} = F_i^{T_i}\chi_{T_i}(\xi)$, we can write

$$Tu(t,x) = \sum_{T_1,T_2}\sum_{j_1,j_2} d_{1,j_1}^{T_1}(t,x) d_{2,j_2}^{T_2}(t,x) \int e^{ix\xi} a(t,x,\xi) e_{\theta_1,j_1}^{T_1} e_{\theta_2,j_2}^{T_i} \chi_{T_1}\chi_{T_2}\tilde{\chi}(\xi/2^k)\hat{u}(t,\xi)\,\mathrm{d}\xi.$$

Thus

$$\begin{aligned}
\|Tu(t)\|_{L_x^r} &\lesssim \sum_{j_1,j_2\in\mathbb{Z}^d}\sum_{T_1,T_2} \|d_{1,j_1}^{T_1}(t)\|_{L_x^{r_1}} \|d_{2,j_2}^{T_2}(t)\|_{L_x^{r_2}} \|\chi_{T_1}\chi_{T_2}\hat{u}(t)\|_{L^2} \\
&\lesssim \sum_{j_1,j_2\in\mathbb{Z}^d} \|d_{1,j_1}^{T_1}(t)\|_{l_{T_1}^2 L_x^{r_1}} \sum_{T_2} \|d_{2,j_2}^{T_2}(t)\|_{L_x^{r_2}} \|\chi_{T_2}\hat{u}(t)\|_{L^2} \\
&\lesssim \sum_{j_1,j_2\in\mathbb{Z}^d} \|d_{1,j_1}^{T_1}(t)\|_{l_{T_1}^2 L_x^{r_1}} \|d_{2,j_2}^{T_2}(t)\|_{l_{T_2}^2 L_x^{r_2}} \|\hat{u}(t)\|_{L^2}
\end{aligned}$$

Applying Hölder's inequality and (4.5.14) we obtain

$$\|Tu\|_{L^qL^r} \lesssim \|u\|_{L^{\tilde{q}}L^2} \sum_{j_1,j_2\in\mathbb{Z}^d} \langle j_1\rangle^{-M}\langle j_2\rangle^{-M} \|F_1\|_{D_k^{\theta_1}(L^{q_1}L^{r_1})}\|F_2\|_{D_k^{\theta_2}(L^{q_2}L^{r_2})}$$

which sums up to (4.5.13). □

## Decomposable estimates for the phase

Now we apply the decomposable calculus to the phases $\psi_\pm^k(t,x,\xi)$.

**Lemma 4.5.7.** *Let $q \ge 2$, $\frac{2}{q} + \frac{d-1}{r} \le \frac{d-1}{2}$. For $k \ge 1$ we have*

$$\|(\psi_{j,\theta,\pm}^k, 2^{-j}\nabla_{t,x}\psi_{j,\theta,\pm}^k)\|_{D_k^\theta(L^qL^r)} \lesssim \varepsilon 2^{-(\frac{1}{q}+\frac{d}{r})j} \frac{\theta^{\frac{d+1}{2}-(\frac{2}{q}+\frac{d-1}{r})}}{\theta^2 + 2^{-2k}}. \tag{4.5.15}$$

$$\|P_\theta^\omega A_{k_1}(t,x)\cdot\omega\|_{D_k^\theta(L^2L^\infty)} \lesssim \theta^{\frac{3}{2}} 2^{\frac{k_1}{2}}\varepsilon. \tag{4.5.16}$$

*For $k = 0$ we have*

$$\|(\psi_{j,\pm}^0, 2^{-j}\nabla_{t,x}\psi_{j,\pm}^0)\|_{D_0(L^qL^r)} \lesssim \varepsilon 2^{-(\frac{1}{q}+\frac{d}{r})j}. \tag{4.5.17}$$

*Proof.* Suppose $k \geq 1$. Without loss of generality, we will focus on $\psi^k_{j,\theta,\pm}$, since exactly the same estimates hold for $2^{-j}\nabla_{t,x}\psi^k_{j,\theta,\pm}$. In light of the definition (4.5.3), for any $\xi = |\xi|\,\omega$ and any $\alpha_1, |\alpha| \leq 10d$, the derivatives $\partial^{\alpha_1}_{|\xi|}(\theta\partial_\omega)^\alpha \psi^k_{j,\theta,\pm}$ are localized to a sector of angle $O(\theta)$ in the $(t,x)$-frequencies and they solve the free wave equation

$$\Box_{t,x}\partial^{\alpha_1}_{|\xi|}(\theta\partial_\omega)^\alpha \psi^k_{j,\theta,\pm}(t,x,2^k\xi) = 0$$

Let $r_0$ be defined by $\frac{2}{q} + \frac{d-1}{r_0} = \frac{d-1}{2}$. The Bernstein and Strichartz inequalities imply

$$\begin{aligned}\|\partial^{\alpha_1}_{|\xi|}(\theta\partial_\omega)^\alpha \psi^k_{j,\theta,\pm}(\cdot,2^k\xi)\|_{L^qL^r} &\lesssim \theta^{(d-1)(\frac{1}{r_0}-\frac{1}{r})} 2^{d(\frac{1}{r_0}-\frac{1}{r})j} \|\partial^{\alpha_1}_{|\xi|}(\theta\partial_\omega)^\alpha \psi^k_{j,\theta,\pm}(\cdot,2^k\xi)\|_{L^qL^{r_0}} \\ &\lesssim 2^{(1-\frac{1}{q}-\frac{d}{r})j}\theta^{(d-1)(\frac{1}{r_0}-\frac{1}{r})} \|\partial^{\alpha_1}_{|\xi|}(\theta\partial_\omega)^\alpha \psi^k_{j,\theta,\pm}(2^k\xi)[0]\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} \qquad (4.5.18)\end{aligned}$$

By Remark 4.5.4 (which uses the null form) we deduce

$$(4.5.18) \lesssim 2^{-(\frac{1}{q}+\frac{d}{r})j} \frac{\theta^{\frac{d+1}{2}-(\frac{2}{q}+\frac{d-1}{r})}}{\theta^2 + 2^{-2k}} \|\Pi^{\omega,\alpha}_\theta \tilde{P}_j A[0]\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}.$$

By putting together this estimate, definition 4.5.12, the finite overlap of the sectors and the orthogonality property, we obtain

$$\|\psi^k_{j,\theta,\pm}\|_{D^\theta_k(L^qL^r)} \lesssim 2^{-(\frac{1}{q}+\frac{d}{r})j} \frac{\theta^{\frac{d+1}{2}-(\frac{2}{q}+\frac{d-1}{r})}}{\theta^2 + 2^{-2k}} \|\tilde{P}_j A[0]\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}},$$

which proves the claim, since $\|\tilde{P}_j A[0]\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} \lesssim \varepsilon$.

The same argument applies to (4.5.16). The only difference is that one uses the angular-localized Strichartz inequality $\|P^\omega_\theta A_j\|_{L^2L^\infty} \lesssim \theta^{\frac{d-3}{2}} 2^{\frac{j}{2}} \|A_j\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}$, which holds for free waves, in addition to the null form which gives an extra $\theta$.

When $k = 0$ the same argument goes through without angular projections and with no factors of $\theta$ in (4.5.18). □

*Remark* 4.5.8. As a consequence of the above we also obtain

$$\|\nabla_\xi(\Pi^\omega_{\leq\delta(k_1-k)} A_{k_1}(t,x)\cdot\xi)\|_{D^1_k(L^2L^\infty)} \lesssim 2^{-10d\delta(k_1-k)} 2^{\frac{k_1}{2}}\varepsilon. \qquad (4.5.19)$$

**Corollary 4.5.9.** *For $k \geq 0$ we have*

$$\|(\psi^k_{j,\pm}, 2^{-j}\nabla_{t,x}\psi^k_{j,\pm})\|_{D_k(L^qL^\infty)} \lesssim 2^{-\frac{j}{q}}\varepsilon, \qquad q > 4 \qquad (4.5.20)$$

$$\|\nabla_{t,x}\psi^k_\pm\|_{D_k(L^2L^\infty)} \lesssim 2^{\frac{k}{2}}\varepsilon \qquad (4.5.21)$$

$$\|\nabla_{t,x}\psi^k_\pm\|_{D_k(L^\infty L^\infty)} \lesssim 2^k\varepsilon. \qquad (4.5.22)$$

*Proof.* The bound (4.5.22) follows by summing over (4.5.20). For $k = 0$, (4.5.20) and (4.5.21) follow from (4.5.17).

Now assume $k \geq 1$. The condition $q > 4$ makes the power of $\theta$ positive in (4.5.15) for any $d \geq 4$. Thus (4.5.20) follows by summing in $\theta$. For (4.5.21), summing in $\theta$ gives the factor $2^{\frac{\delta}{2}(k-j)}$, which is overcome by the extra factor of $2^j$ when summing in $j < k$. □

## Further properties

**Lemma 4.5.10.** *Let $a(x,\xi)$ and $b(x,\xi)$ be smooth symbols. Then one has*

$$\|a^r b^r - (ab)^r\|_{L^r(L^2)\to L^q(L^2)} \lesssim \|(\nabla_x a)^r\|_{L^r(L^2)\to L^{p_1}(L^2)} \|\nabla_\xi b\|_{D_k^1 L^{p_2}(L^\infty)} \tag{4.5.23}$$

$$\|a^l b^l - (ab)^l\|_{L^r(L^2)\to L^q(L^2)} \lesssim \|\nabla_\xi a\|_{D_k^1 L^{p_2}(L^\infty)} \|(\nabla_x b)^l\|_{L^r(L^2)\to L^{p_1}(L^2)} \tag{4.5.24}$$

*where $q^{-1} = \sum p_i^{-1}$. Furthermore, if $b = b(\xi)$ is a smooth multiplier supported on $\{\langle\xi\rangle \simeq 2^k\}$, then for any two translation invariant spaces $X, Y$ one has:*

$$\|a^r b^r - (ab)^r\|_{X\to Y} \lesssim 2^{-k} \|(\nabla_x a)^r\|_{X\to Y}\ . \tag{4.5.25}$$

*Proof.* See [31, Lemma 7.2]. □

**Lemma 4.5.11.** *Let $X, Y$ be translation-invariant spaces of functions on $\mathbb{R}^{n+1}$ and consider the symbol $a(t,x,\xi)$ such that*

$$a(t,x,D) : X \to Y.$$

*Then the $(t,x)$-frequency localized symbol $a_{<h}(t,x,\xi)$ also satisfies*

$$a_{<h}(t,x,D) : X \to Y.$$

*Proof.* We write

$$a_{<h}(t,x,D)u = \int m(s,y) T_{(s,y)} a(t,x,D) T_{-(s,y)} u \, \mathrm{d}s \, \mathrm{d}y$$

where $m(s,y)$ is a bump function and $T_{(s,y)}$ denotes translation by $(s,y)$. Now the claim follows from Minkowski's inequality and the $T_{\pm(s,y)}$-invariance of $X, Y$. □

# 4.6 Oscillatory integrals estimates

In this section we prove estimates for oscillatory integrals that arise as kernels of $TT^*$ operators used in proofs of the mapping (4.2.11) and (4.2.8), (4.2.9), (4.2.10). These bounds are based on stationay and non-stationary phase arguments (see Prop. 1.5.2 and 1.5.3).

## Rapid decay away from the cone

We consider

$$K_k^a(t,x,s,y) = \int e^{-i\psi_\pm^k(t,x,\xi)} a\Big(\frac{\xi}{2^k}\Big) e^{\pm i(t-s)\langle\xi\rangle + i(x-y)\xi} e^{+i\psi_\pm^k(s,y,\xi)} \, \mathrm{d}\xi \tag{4.6.1}$$

where $a(\xi)$ is a bump function supported on $\{|\xi| \simeq 1\}$ for $k \geq 1$ and on $\{|\xi| \lesssim 1\}$ for $k = 0$.

**Proposition 4.6.1.** *For $k \geq 0$ and any $N \geq 0$, we have*

$$|K_k^a(t,x,s,y)| \lesssim 2^{dk} \frac{1}{\langle 2^k(|t-s|-|x-y|)\rangle^N} \tag{4.6.2}$$

*whenever* $2^k\,||t-s|-|x-y|| \gg 2^{-k}\,|t-s|$.

*Moreover, the implicit constant is bounded when $a(\xi)$ is bounded in $C^N(|\xi| \lesssim 1)$.*

*Proof.* We first assume $k \geq 1$. Suppose without loss of generality that $t-s \geq 0$, and $\pm = +$. Denoting $\lambda = ||t-s|-|x-y||$ it suffices to consider $2^k\lambda \gg 1$. By a change of variables we write

$$K_k^a = 2^{dk} \int_{|\xi|\simeq 1} e^{i2^k(\phi_k+\varphi_k)(t,x,s,y,\xi)} a(\xi)\,\mathrm{d}\xi$$

where

$$\phi_k(t,x,s,y,\xi) = (t-s)\langle\xi\rangle_k + (x-y)\cdot\xi$$
$$\varphi_k(t,x,s,y,\xi) = -(\psi_\pm^k(t,x,2^k\xi) - \psi_\pm^k(s,y,2^k\xi))/2^k.$$

By Prop 4.5.5 and noting that $T = |t-s|+|x-y| \lesssim 2^{2k}\lambda$ we have

$$|\nabla\varphi_k| \lesssim \varepsilon(2^kT)^{3\delta}/2^k \lesssim \varepsilon\lambda$$

Furthermore,

$$\nabla\phi_k = (t-s)\frac{\xi}{\langle\xi\rangle_k} + (x-y)$$

If $|x-y| \geq 2\,|t-s|$ or $|t-s| \geq 2\,|x-y|$, by non-stationary phase, we easily estimate $|K_k^a| \lesssim 2^{dk}\left\langle 2^kT\right\rangle^{-N}$.

Now we assume $|t-s| \simeq |x-y| \gg 2^{-k}$. On the region $\angle(-\xi, x-y) > 10^{-3}$ we have $|\nabla\phi_k| \gtrsim |t-s|$, thus by a smooth cutoff and non-stationary phase, that component of the integral is $\lesssim 2^{dk}\left\langle 2^kT\right\rangle^{-N}$. Now we can assume $a(\xi)$ is supported on the region $\angle(-\xi, x-y) \leq 10^{-2}$.

If $|\nabla\phi_k| \geq 1/4\lambda$ on that region, we get the bound $2^{dk}\left\langle 2^k\lambda\right\rangle^{-N}$. We claim this is always the case. Suppose the contrary, that there exists $\xi$ such that $|\nabla\phi_k| \leq 1/4\lambda$. Then $(t-s)\frac{|\xi'|}{\langle\xi\rangle_k} \leq 1/4\lambda$ writing in coordinates $\xi = (\xi_1, \xi')$ where $\xi_1$ is in the direction $x-y$ while $\xi'$ is orthogonal to it.

$$\nabla\phi_k \cdot \frac{x-y}{|x-y|} = (t-s)\frac{\xi_1}{\langle\xi\rangle_k} + |x-y| = \pm\lambda + (t-s)(1+\frac{\xi_1}{\langle\xi\rangle_k})$$

Thus $\xi_1 \leq 0$ and using that $\lambda \gg 2^{-2k}(t-s)$ we have

$$(t-s)(1+\frac{\xi_1}{\langle\xi\rangle_k}) = \frac{t-s}{\langle\xi\rangle_k}\frac{2^{-2k}+|\xi'|^2}{\langle\xi\rangle_k+|\xi_1|} < 2^{-2k}(t-s) + \frac{1}{4}\lambda \leq \frac{1}{2}\lambda$$

which implies $|\nabla\phi_k| \geq 1/2\lambda$, a contradiction. This concludes the case $k \geq 1$.

When $k = 0$ we have $|x-y| \gg |t-s|$. For the corresponding phase we have $|\nabla\phi_0| \geq \frac{1}{2}|x-y|$ and thus we get the factor $\langle x-y\rangle^{-N}$. □

## Dispersive estimates

Dispersive estimates for the Klein-Gordon equation are treated in places like [38, Section 2.5], [4]. The situation here is slightly complicated by the presence of the $e^{\pm i\psi}$ transformations. To account for this we use Prop. 4.5.5. Let

$$K^k := \int e^{-i\psi^k_{\pm}(t/2^k,x/2^k,2^k\xi)+i\psi^k_{\pm}(s/2^k,y/2^k,2^k\xi)}e^{\pm it(t-s)\langle\xi\rangle_k}e^{i(x-y)\xi}a(\xi)\,\mathrm{d}\xi$$

where $a(\xi)$ is a bump function supported on $\{|\xi| \simeq 1\}$ for $k \geq 1$ and on $\{\langle\xi\rangle \simeq 1\}$ for $k = 0$.

**Proposition 4.6.2.** *For any $k \geq 0$ one has the inequalities*

$$\left|K^k(t,x;s,y)\right| \lesssim \begin{cases} \dfrac{1}{\langle t-s\rangle^{\frac{d-1}{2}}} & (4.6.3) \\ \dfrac{2^k}{\langle t-s\rangle^{d/2}} & (4.6.4) \end{cases}$$

*Proof.* **S**tep 1. We first prove (4.6.3) for $k \geq 1$. We assume $|t-s| \simeq |x-y| \gg 1$ and that $a(\xi)$ is supported on the region $\angle(\mp\xi, x-y) \leq 10^{-2}$, since in the other cases the phase is non-stationary and we obtain the bound $\langle t-s\rangle^{-N}$ from the proof of Prop. 4.6.1. We denote

$$\varphi(t,x,s,y,\xi) = -\psi^k_{\pm}(t/2^k, x/2^k, 2^k\xi) + \psi^k_{\pm}(s/2^k, y/2^k, 2^k\xi)$$

and write

$$(x-y)\cdot\xi \pm |x-y|\,|\xi| = \pm 2\,|x-y|\,|\xi|\sin^2(\theta/2)$$

where $\theta = \angle(\mp\xi, x-y)$. We write $\xi = (\xi_1, \xi')$ in polar coordinates, where $\xi_1 = |\xi|$ is the radial component. Then

$$K^k = \int_{\xi_1\simeq 1} \xi_1^3 e^{\pm i(t-s)\langle\xi_1\rangle_k \mp i|x-y|\xi_1}\Omega(\xi_1)\,\mathrm{d}\xi_1 \tag{4.6.5}$$

$$\text{where}\quad \Omega(\xi_1) = \int e^{\pm i|x-y|2\xi_1\sin^2(\theta/2)}a(\xi_1,\xi')e^{i\varphi}\,\mathrm{d}S(\xi')$$

For each $\xi_1$ we bound

$$|\Omega(\xi_1)| \lesssim |x-y|^{-\frac{d-1}{2}} \tag{4.6.6}$$

as a stationary-phase estimate (see Prop. 1.5.3). When derivatives fall on $e^{i\varphi}$ we get factors of $\varepsilon\,|x-y|^{\delta}$ by (4.5.10); however, these are compensated by the extra factors $|x-y|^{-1}$ from the expansion (1.5.11). Integrating in $\xi_1$ we obtain (4.6.3).

Furthermore, using (1.5.12) we obtain

$$\left|\partial^j_{\xi_1}\Omega(\xi_1)\right| \lesssim |x-y|^{-\frac{d-1}{2}}\,\langle 2^{-2k}\,|x-y|^{4\delta}\rangle \qquad j = 1,2. \tag{4.6.7}$$

The term $\langle 2^{-2k}|x-y|^{4\delta}\rangle$ occurs by (4.5.11) when $\partial_{\xi_1}$ derivatives fall on $e^{i\varphi}$.

**S**tep 2. Now we prove (4.6.4).

First we consider $k=0$ and $|t-s|\gg 1$. When $|t-s|\leq c|x-y|$ the phase is non-stationary and we obtain $\langle t-s\rangle^{-N}$. Otherwise, we consider the phase $\langle\xi\rangle+\frac{x-y}{|t-s|}\cdot\xi$ and get the bound $\langle t-s\rangle^{-d/2}$ as a stationary-phase estimate using Prop. 4.5.5.

Now we take $k\geq 1$ under the assumptions from Step 1. We may also assume $|t-s|\gg 2^{2k}$ (otherwise (4.6.4) follows from (4.6.3)).

In (4.6.5) we have the phase $2^{-2k}|t-s|f(\xi_1)$ where

$$f(\xi_1)=2^{2k}\Big(\langle\xi_1\rangle_k-\frac{|x-y|}{|t-s|}\xi_1\Big),\quad f'(\xi_1)=2^{2k}\Big(\frac{\xi_1}{\langle\xi_1\rangle_k}-\frac{|x-y|}{|t-s|}\Big),\quad |f''(\xi_1)|\simeq 1,$$

and $\left|f^{(m)}(\xi_1)\right|\lesssim 1$ for $m\geq 3$. Using stationary phase in $\xi_1$ (Prop. 1.5.3/1.5.2) one has

$$\left|K^k\right|\lesssim\frac{1}{\left|2^{-2k}|t-s|\right|^{\frac{1}{2}}}\sup|\Omega|+\frac{1}{2^{-2k}|t-s|}\sup_{j\leq 2}\left|\partial_{\xi^1}^j\Omega\right|,$$

which, together with (4.6.6), (4.6.7), implies (4.6.4). □

Now we consider more localized estimates.

Let $\mathcal{C}$ be a box of size $\simeq 2^{k'}\times(2^{k'+l'})^{d-1}$ located in an annulus $\{\langle\xi\rangle\simeq 2^k\}$ for $k\geq 0$. Suppose $a_{\mathcal{C}}$ is a bump function adapted to $\mathcal{C}$ and define

$$K^{k',l'}(t,x;s,y):=\int e^{-i\psi_\pm^k(t,x,\xi)+i\psi_\pm^k(s,y,\xi)}e^{\pm it(t-s)\langle\xi\rangle}e^{i(x-y)\xi}a_{\mathcal{C}}(\xi)\,\mathrm{d}\xi. \tag{4.6.8}$$

**Proposition 4.6.3.** *Let $k\geq 0$, $k'\leq k$ and $-k\leq l'\leq 0$. Then, we have*

$$\left|K^{k',l'}(t,x;s,y)\right|\lesssim 2^{dk'+(d-1)l'}\frac{1}{\langle 2^{2(k'+l')-k}(t-s)\rangle^{\frac{d-1}{2}}} \tag{4.6.9}$$

*Proof.* We assume $2^{2(k'+l')-k}|t-s|\gg 1$ (otherwise we bound the integrand by absolute values on $\mathcal{C}$) and assume $|t-s|\simeq|x-y|$ (otherwise the phase is non-stationary). Let $k\geq 1$. By a change of variable we rescale to $|\xi|\simeq 1$ and write $K^{k',l'}$ as $2^{dk}\times$ (4.6.5)- applied to $2^k(t,x;s,y)$, with $a(\cdot)$ supported on a box $2^{k'-k}\times(2^{k'+l'-k})^{d-1}$. Like before, for each $\xi_1$ we bound the inner integral $\Omega(\xi_1)$ by $(2^k|t-s|)^{-\frac{d-1}{2}}$ by stationary-phase. Integrating in $\xi_1$ on a radius of size $2^{k'-k}$ we get $2^{dk}2^{k'-k}(2^k|t-s|)^{-\frac{d-1}{2}}$ which gives (4.6.9). When $k=0$, $l'=O(1)$ the estimate is straightforward. □

**Corollary 4.6.4.** *Let $k\geq 0$, $k'\leq k$ and $-k\leq l'\leq 0$. Then*

$$e^{-i\psi_\pm^k}(t,x,D)e^{\pm it\langle D\rangle}P_{C_{k'}(l')}:L_x^2\to 2^{\frac{k}{2}+\frac{d-2}{2}k'+\frac{d-3}{2}l'}L^2L^\infty \tag{4.6.10}$$

*Proof.* By a $TT^*$ argument this follows from

$$2^{-k-(d-2)k'-(d-3)l'} e^{-i\psi^k_\pm}(t,x,D) e^{\pm it(t-s)\langle D\rangle} P^2_{C_{k'}(l')} e^{i\psi^k_\pm}(D,s,y) : L^2L^1 \to L^2L^\infty$$

We use (4.6.9) to bound the kernel of this operator, and the mapping follows since $2^{2k'+2l'-k}\left\langle 2^{2k'+2l'-k}|r|\right\rangle^{-(d}$ has $L^1_rL^\infty_x$ norm $\lesssim 1$. □

## The PW decay bound, $d=4$

Let $\mathcal{C}$ be a box of size $\simeq 2^{k'} \times (2^{k'-k})^3$ with center $\xi_0$ located in an annulus $\{|\xi| \sim 2^k\} \subset \mathbb{R}^4$. We consider the decay of the integral $K^{k',-k}$ defined in (4.6.8), in the frame (1.5.5), (1.5.6), where $\omega$ is the direction of $\xi_0$ and $\lambda = \frac{|\xi_0|}{\langle \xi_0\rangle}$.

This type of bound is similar to the one used by Bejenaru and Herr [4, Prop. 2.3] to establish null-frame $L^2_{t_{\omega,\lambda}}L^\infty_{x_{\omega,\lambda}}$- Strichartz estimates, an idea we will also follow in this paper.

**Proposition 4.6.5.** *When $|t_\omega - s_\omega| \gg 2^{k'-3k}|t-s|$, we have*

$$\left|K^{k',-k}(t,x;s,y)\right| \lesssim 2^{4k'-3k}\frac{1}{\langle 2^{k'}(t_\omega - s_\omega)\rangle^2} \tag{4.6.11}$$

*Proof.* Denoting $T = |t-s| + |x-y|$, we clearly have $|t_\omega - s_\omega| \leq T$. In the cases when $|t-s| \geq 2|x-y|$ or $|x-y| \geq 2|t-s|$ from integrating by parts radially we obtain the decay $\left\langle 2^{k'}T\right\rangle^{-N} 2^{4k'-3k}$. Now suppose $|t-s| \simeq |x-y|$, $\pm = +$ and let

$$\phi(\xi) = (t-s)\langle\xi\rangle + (x-y)\cdot\xi, \qquad \nabla\phi = (t-s)\frac{\xi}{\langle\xi\rangle} + x - y.$$

For $\xi \in \mathcal{C}$ we have $\frac{|\xi|}{\langle\xi\rangle} = \lambda + O(2^{k'-3k})$ and

$$\frac{\xi}{|\xi|} = \omega + \sum_i O(2^{k'-2k})\omega_i + O(2^{2(k'-2k)}), \qquad \omega_i \in \omega^\perp.$$

Therefore

$$\omega\cdot\nabla\phi = (t_\omega - s_\omega)\sqrt{1+\lambda^2} + O(2^{k'-3k}|t-s|).$$

Due to the assumption, the phase is non-stationary $|\omega\cdot\nabla\phi| \gtrsim |t_\omega - s_\omega|$ and we obtain (4.6.11) by integrating by parts with $\partial_\omega = \omega\cdot\nabla$.

When derivatives fall on $e^{-i\psi^k_\pm(t,x,\xi)+\psi^k_\pm(s,y,\xi)}$ we get extra factors of $2^{k'-k}(2^kT)^\delta$ from Prop 4.5.5. However, we compensate this factors by writing the integral in polar coordinates similarly to (4.6.5) and using stationary-phase for the inner integral like in the proof of (4.6.9), (4.6.3), giving an extra $(2^{2k'-3k}T)^{-3/2}$, which suffices. □

**Corollary 4.6.6.** *For $k \geq 1$ let $\xi_0$ be the center of the box $C_{k'}(-k)$, $\lambda = \frac{|\xi_0|}{\langle \xi_0 \rangle}$ and $\omega = \frac{\xi_0}{|\xi_0|}$. Then*

$$2^{-\frac{3}{2}(k'-k)} e^{-i\psi_{\pm}^{k}}(t,x,D) e^{it\langle D \rangle} P_k P_{C_{k'}(-k)} : L_x^2 \to L^2_{t_{\omega,\lambda}} L^\infty_{x_{\omega,\lambda}}$$

*Proof.* By a $TT^*$ argument this follows from the mapping

$$2^{-3(k'-k)} e^{-i\psi_{\pm}^{k}}(t,x,D) e^{it(t-s)\langle D \rangle} P_k^2 a_{\mathcal{C}}(D) e^{i\psi_{\pm}^{k}}(D,s,y) : L^2_{t_{\omega,\lambda}} L^1_{x_{\omega,\lambda}} \to L^2_{t_{\omega,\lambda}} L^\infty_{x_{\omega,\lambda}}$$

which holds since the kernel of this operator is bounded by $2^{k'} \left\langle 2^{k'}(t_\omega - s_\omega) \right\rangle^{-3/2} \in L^1_{t_\omega - s_\omega} L^\infty$. When $|t_\omega - s_\omega| \gg 2^{k'-3k} |t-s|$ this follows from (4.6.11), while for $|t_\omega - s_\omega| \lesssim 2^{k'-3k} |t-s|$ it follows from (4.6.9) with $l' = -k$. □

## The null frame decay bound, $d = 4$

Let $\bar{\omega} \in \mathbb{S}^3$ and let $\kappa_l$ be a spherical cap of angle $2^l$ such that $\angle(\kappa_l, \pm\bar{\omega}) \simeq 2^l$. Let $\lambda = \frac{1}{\sqrt{1+2^{-2p}}}$, which together with $\bar{\omega}$ defines the frame (1.5.5) and the coordinates in this frame

$$t_{\bar{\omega}} = (t,x) \cdot \bar{\omega}^\lambda, \quad x^1_{\bar{\omega}} = (t,x) \cdot \bar{\bar{\omega}}^\lambda, \quad x'_{\bar{\omega},i} = x \cdot \bar{\omega}_i^\perp$$

Suppose $a_l(\xi)$ is a smooth function adapted to $\{|\xi| \simeq 2^k, \ \frac{\xi}{|\xi|} \in \kappa_l\}$ and consider

$$K_l^{a_l}(t,x;s,y) := \int e^{-i\psi_{\pm}^{k}(t,x,\xi) + \psi_{\pm}^{k}(s,y,\xi)} e^{\pm it(t-s)\langle \xi \rangle} e^{i(x-y)\xi} a_l(\xi) \, \mathrm{d}\xi.$$

**Proposition 4.6.7.** *Suppose* $\max(2^{-p}, 2^{-k}) \ll 2^l \simeq \angle(\kappa_l, \pm\bar{\omega})$. *Then, we have*

$$|K_l^{a_l}(t,x;s,y)| \lesssim 2^{4k+3l} \frac{1}{\left\langle 2^{k+2l} |t-s| \right\rangle^N} \frac{1}{\left\langle 2^{k+l} |x'_{\bar{\omega}} - y'_{\bar{\omega}}| \right\rangle^N} \left\langle 2^k |t_{\bar{\omega}} - s_{\bar{\omega}}| \right\rangle^{2N} \tag{4.6.12}$$

*Moreover, the implicit constant depends on only $2N+1$ derivatives of $a_l$.*

*Proof.* We prove that the phase is non-stationary due to the angular separation. Suppose $\pm = +$ and let

$$\phi(\xi) = (t-s)\langle \xi \rangle + (x-y) \cdot \xi, \qquad \nabla\phi = (t-s)\frac{\xi}{\langle \xi \rangle} + x - y.$$

Choosing the right $\bar{\omega}_i^\perp$ we obtain

$$\nabla\phi \cdot \bar{\omega}_i^\perp \simeq 2^l(t-s) + |x'_{\bar{\omega}} - y'_{\bar{\omega}}|.$$

When $2^l |t-s| \ll |x'_{\bar\omega} - y'_{\bar\omega}|$ we obtain $|K_l^{a_l}| \lesssim 2^{4k+3l} \left\langle 2^{k+l} |x'_{\bar\omega} - y'_{\bar\omega}| \right\rangle^{-2N}$, which implies (4.6.12). Similarly when $|x'_{\bar\omega} - y'_{\bar\omega}| \ll 2^l |t-s|$ we get $|K_l^{a_l}| \lesssim 2^{4k+3l} \left\langle 2^{k+l} 2^l |t-s| \right\rangle^{-2N}$ which also suffices.

We use Prop 4.5.5 to control the contribution of $\psi_\pm^k(t,x,\xi) - \psi_\pm^k(s,y,\xi)$.

Now assume $2^l |t-s| \simeq |x'_{\bar\omega} - y'_{\bar\omega}|$. When $(2^{2l} |t-s| \simeq)\ 2^l |x'_{\bar\omega} - y'_{\bar\omega}| \lesssim |t_{\bar\omega} - s_{\bar\omega}|$ estimating the integrand by absolute values we get $|K_l^{a_l}| \lesssim 2^{4k+3l}$, which suffices in this case.

Now we assume $|t_{\bar\omega} - s_{\bar\omega}| \ll 2^{2l} |t-s| \simeq 2^l |x'_{\bar\omega} - y'_{\bar\omega}|$.

Since $(x-y)\cdot\bar\omega = -\lambda(t-s) + (t_{\bar\omega} - s_{\bar\omega})\sqrt{1+\lambda^2}$, we have

$$\nabla\phi \cdot \bar\omega = (t-s)\left(\frac{|\xi|}{\langle\xi\rangle}\frac{\xi}{|\xi|}\cdot\bar\omega - \lambda\right) + (t_{\bar\omega} - s_{\bar\omega})\sqrt{1+\lambda^2}$$

We estimate

$$\frac{|\xi|}{\langle\xi\rangle} - 1 \simeq -2^{-2k}, \quad \frac{\xi}{|\xi|}\cdot\bar\omega - 1 \simeq -2^{2l}, \quad \lambda - 1 \simeq -2^{-2p}$$

From the hypothesis on $2^l$ we conclude that this term dominates so

$$|\nabla\phi \cdot \bar\omega| \gtrsim 2^{2l} |t-s|,$$

which implies (4.6.12) as a non-stationary phase estimate. □

Now we consider frequency localized symbols and look at the $TT^*$ operator

$$e_{<k}^{-i\psi_\pm^k}(t,x,D) e^{\pm it(t-s)\langle D\rangle} a_l(D) e_{<k}^{i\psi_\pm^k}(D,s,y) \tag{4.6.13}$$

from $L^2(\Sigma) \to L^2(\Sigma)$, where $\Sigma = (\bar\omega^\lambda)^\perp$ with kernel

$$K_{<k}^l(t,x;s,y) := \int e_{<k}^{-i\psi_\pm^k(t,x,\xi)} e^{\pm it(t-s)\langle\xi\rangle} e^{i(x-y)\xi} a_l(\xi) e_{<k}^{i\psi_\pm^k(s,y,\xi)} \, d\xi.$$

for $(t,x;s,y) \in \Sigma\times\Sigma$, i.e. $t_{\bar\omega} = s_{\bar\omega} = 0$.

**Proposition 4.6.8.** *Suppose* $\max(2^{-p}, 2^{-k}) \ll 2^l \simeq \angle(\kappa_l, \pm\bar\omega)$. *Then,*

$$\left|K_{<k}^l(t,x;s,y)\right| \lesssim 2^{4k+3l} \frac{1}{\left\langle 2^{k+2l} |x_{\bar\omega}^1 - y_{\bar\omega}^1| \right\rangle^N} \frac{1}{\left\langle 2^{k+l} |x'_{\bar\omega} - y'_{\bar\omega}| \right\rangle^N} \tag{4.6.14}$$

*holds when* $\lambda(t-s) + (x-y)\cdot\bar\omega = 0$.

**Corollary 4.6.9.** *Suppose* $\max(2^{-p}, 2^{-k}) \ll 2^l \simeq \angle(\kappa_l, \pm\bar\omega)$. *Then*

$$2^l e_{<k}^{-i\psi_\pm^k}(t,x,D) e^{\pm it\langle D\rangle} P_k P_{\kappa_l} : L_x^2 \to L_{t_{\bar\omega,\lambda}}^\infty L_{x_{\bar\omega,\lambda}}^2. \tag{4.6.15}$$

**Corollary 4.6.10.** *Let* $\mathcal{C} = \mathcal{C}_{k'}(l')$. *Then*

$$e_{<k}^{-i\psi_\pm^k}(t,x,D) e^{\pm it\langle D\rangle} P_k P_{\mathcal{C}} : L_x^2 \to NE_{\mathcal{C}}^\pm. \tag{4.6.16}$$

*Proof of Prop. 4.6.8.* We average using (4.1.4) to write $K^l_{<k}(t,x;s,y)$ as

$$\iint e^{-iT_z\psi^k_\pm(t,x,\xi)}e^{+iT_w\psi^k_\pm(s,y,\xi)}a_l(\xi)e^{\pm i(t-s)\langle\xi\rangle}e^{i(x-y)\xi}\,\mathrm{d}\xi m_k(z)m_k(w)\,\mathrm{d}z\,\mathrm{d}w =$$

$$= \int T_zT_wK_l^{a(z,w)}(t,x;s,y)m_k(z)m_k(w)\,\mathrm{d}z\,\mathrm{d}w \tag{4.6.17}$$

where $a(z,w)(\xi) = e^{-i(z-w)\cdot(\pm\langle\xi\rangle,\xi)}a_l(\xi)$. Since $t_{\bar\omega} = s_{\bar\omega} = 0$ using (4.6.12) we obtain

$$\left|T_zT_wK_l^{a(z,w)}(t,x;s,y)\right| \lesssim \left\langle 2^k(|z|+|w|)\right\rangle^{2N+1} 2^{4k+3l}\times$$
$$\times\left\langle 2^{k+2l}\,|t-s+z_1-w_1|\right\rangle^{-N}\left\langle 2^{k+l}\,|x'_{\bar\omega}-y'_{\bar\omega}+z'_{\bar\omega}-w'_{\bar\omega}|\right\rangle^{-N}\left\langle 2^k\,|z-w|\right\rangle^{2N}$$

We obtain (4.6.14) from the integral (4.6.17) using the rapid decay

$$\left\langle 2^k(|z|+|w|)\right\rangle^{2N+1}\left\langle 2^k\,|z-w|\right\rangle^{2N} m_k(z)m_k(w) \lesssim \left\langle 2^k\,|z|\right\rangle^{-N_2}\left\langle 2^k\,|w|\right\rangle^{-N_2}.$$

for any $N_2$, and by repeatedly applying

$$\int_{\mathbb{R}}\frac{1}{\langle\alpha\,|a-r|\rangle^N}\frac{2^k}{\langle 2^k\,|r|\rangle^{N_2}}\,\mathrm{d}r \lesssim \frac{1}{\langle\alpha\,|a|\rangle^N}$$

for $\alpha \leq 2^k$ and $N_2$ large enough. Note that here $|t-s| \simeq |x^1_{\bar\omega} - y^1_{\bar\omega}|$. □

*Proof of Corollary 4.6.9.* By translation invariance, it suffices to prove that the operator is bounded from $L^2_x \to L^2(\Sigma)$. By a $TT^*$ argument this follows if we prove $2^{2l}\times$ (4.6.13) $: L^2(\Sigma) \to L^2(\Sigma)$, for which we use Schur's test. Indeed, the kernel of this operator is $2^{2l}K^l_{<k}(t,x;s,y)$ on $\Sigma\times\Sigma$, which is integrable on $\Sigma$ by (4.6.14). □

*Proof of Corollary 4.6.10.* Recall definition (2.1.7). For any $\bar\omega$, $\lambda = \lambda(p)$ such that $\angle(\bar\omega, \pm\mathcal{C}) \gg \max(2^{-p}, 2^{-k}, 2^{l'+k'-k})$ we may define $2^l \simeq \angle(\bar\omega, \pm\mathcal{C})$ and $\kappa_l \supset \mathcal{C}$ so that Corollary 4.6.9 applies. □

## 4.7 Proof of the fixed time $L^2_x$ estimates (4.2.8), (4.2.9), (4.2.10)

The remainder of this chapter is devoted to the proof of Theorem 4.2.2.

The following proposition establishes the $L^2_x$ part of (4.2.8), (4.2.9).

**Proposition 4.7.1.** *For any $k \geq 0$, the mappings*

$$e^{\pm i\psi_\pm^k}(t_0, x, D)\bar{P}_k : L_x^2 \to L_x^2 \tag{4.7.1}$$

$$e_{<h}^{\pm i\psi_\pm^k}(t_0, x, D)\bar{P}_k : L_x^2 \to L_x^2 \tag{4.7.2}$$

$$\nabla_{t,x} e_{<h}^{\pm i\psi_\pm^k}(t_0, x, D)\bar{P}_k : L_x^2 \to \varepsilon 2^k L_x^2 \tag{4.7.3}$$

*hold for any fixed $t_0$, uniformly in $h, t_0 \in \mathbb{R}$. By duality, the same mappings hold for right quantizations.*

*Proof.* **S**tep 1. First we prove (4.7.1) by considering the $TT^*$ operator

$$e^{\pm i\psi_\pm^k}(t_0, x, D)\bar{P}_k^2 e^{\mp i\psi_\pm^k}(D, y, t_0)$$

with kernel $K_k^a(t_0, x, t_0, y)$ defined by (4.6.1).

Due to the $(x, y)$ symmetry and Schur's test it suffices to show

$$\sup_x \int |K_k^a(t_0, x, t_0, y)| \, \mathrm{d}y \lesssim 1.$$

This follows from Prop. 4.6.1.

**S**tep 2. Now we prove (4.7.2) using (4.1.4) and (4.7.1). For $u \in L_x^2$ we write

$$e_{<h}^{\pm i\psi_\pm^k}(t_0, x, D)\bar{P}_k u = \int_{\mathbb{R}^{d+1}} m_h(s, y) e^{\pm i\psi_\pm^k}(t_0 + s, x + y, D)[\bar{P}_k u_y] \, \mathrm{d}s \, \mathrm{d}y$$

where $\hat{u}_y(\xi) = e^{-iy\xi}\hat{u}_0(\xi)$. By Minkowski's inequality, (4.7.1) for $t_0 + s$, translation invariance of $L_x^2$, and the bound $\|u_y\|_{L_x^2} \leq \|u\|_{L_x^2}$ we obtain

$$\begin{aligned}
\|e_{<h}^{\pm i\psi_\pm^k}(t_0, x, D)\bar{P}_k u\|_{L_x^2} &\lesssim \int_{\mathbb{R}^{d+1}} m_h(s, y) \|e^{\pm i\psi_\pm^k}(t_0 + s, \cdot, D)[\bar{P}_k u_y]\|_{L_x^2} \, \mathrm{d}s \, \mathrm{d}y \\
&\lesssim \int_{\mathbb{R}^{d+1}} m_h(s, y) \|\bar{P}_k u_y\|_{L_x^2} \, \mathrm{d}s \, \mathrm{d}y \lesssim \|u\|_{L_x^2}.
\end{aligned}$$

**S**tep 3. Since we have

$$\nabla_{t,x} e^{\pm i\psi_\pm^k(t,x,\xi)} = \pm i \nabla_{t,x}\psi_\pm^k(t, x, \xi) e^{\pm i\psi_\pm^k(t,x,\xi)}$$

using (4.5.22), (4.7.1) and Lemma 4.5.6 we obtain

$$\|\nabla_{t,x} e^{\pm i\psi_\pm^k}(t, x, D)P_k\|_{L^1L^2 \to L^1L^2} \lesssim \varepsilon 2^k$$

Applying this to $\phi(t, x) = \delta_{t_0}(t) \otimes u(x)$ (or rather with an approximate to the identity $\eta_\varepsilon$ converging to $\delta_{t_0}$ in $t$) we obtain

$$\nabla_{t,x} e^{\pm i\psi_\pm^k}(t_0, x, D)P_k : L_x^2 \to \varepsilon 2^k L_x^2$$

for any $t_0$. By averaging this estimate as in Step 2 we obtain (4.7.3). □

*Remark* 4.7.2. The same argument also shows

$$e^{\pm i\psi^k_{<k'',\pm}}(t_0,x,D)P_k : L^2_x \to L^2_x \tag{4.7.4}$$

Now we turn to the proof of (4.2.10).

**Proposition 4.7.3.** *Let $k \geq 0$. For any $t_0$ we have*

$$e^{-i\psi^k_\pm}_{<k}(t_0,x,D)e^{i\psi^k_\pm}_{<k}(D,t_0,y) - I : \bar{P}_k L^2_x \to \varepsilon^{\frac{1}{2}} L^2_x$$

*Proof.* **S**tep 1. First, let us note that

$$e^{-i\psi^k_\pm}_{<k}(t_0,x,D)[e^{i\psi^k_\pm}_{<k}(D,t_0,y)\bar{P}_k - \bar{P}_k e^{i\psi^k_\pm}_{<k}(D,t_0,y)] : L^2_x \to \varepsilon L^2_x$$

This follows from (4.7.2) and from (4.5.25), (4.7.3).

The kernel of $e^{-i\psi^k_\pm}_{<k}(t_0,x,D)\bar{P}_k e^{i\psi^k_\pm}_{<k}(D,t_0,y)$ is

$$K_{<k}(x,y) = \int e^{-i\psi^k_\pm}_{<k}(t_0,x,\xi)a(\xi/2^k)e^{i(x-y)\xi}e^{+i\psi^k_\pm}_{<k}(t_0,y,\xi)\,\mathrm{d}\xi$$

while the kernel of $\bar{P}_k$ is $2^{dk}\check{a}(2^k(x-y))$. Thus, by Schur's test it remains to prove

$$\sup_x \int \left|K_{<k}(x,y) - 2^{dk}\check{a}(2^k(x-y))\right| \mathrm{d}y \lesssim \varepsilon^{\frac{1}{2}}. \tag{4.7.5}$$

**S**tep 2. For large $|x-y|$ we will use

$$2^{dk}\left|\check{a}(2^k(x-y))\right|,\ |K_{<k}(x,y)| \lesssim \frac{2^{dk}}{(1+2^k|x-y|)^{2d+1}}. \tag{4.7.6}$$

The bound for $\check{a}$ is obvious. Recalling (4.1.4) we write $K_{<k}(x,y)$ as

$$\iint e^{-iT_z\psi^k_\pm(t_0,x,\xi)}e^{+iT_w\psi^k_\pm(t_0,y,\xi)}a(\xi/2^k)e^{i(x-y)\xi}\,\mathrm{d}\xi m_k(z)m_k(w)\,\mathrm{d}z\,\mathrm{d}w =$$

$$= \int T_z T_w K^{a(z,w)}_k(t_0,x,t_0,y)m_k(z)m_k(w)\,\mathrm{d}z\,\mathrm{d}w \tag{4.7.7}$$

where $z=(t,z'),\ w=(s,w'),\ a(z,w)(\xi) = e^{-i2^k(z-w)\cdot(\pm\langle\xi\rangle_k,\xi)}a(\xi)$ and

$$K^a_k(t,x,s,y) = \int e^{-i\psi^k_\pm}(t,x,\xi)a(\xi/2^k)e^{\pm i(t-s)\langle\xi\rangle + i(x-y)\xi}e^{+i\psi^k_\pm}(s,y,\xi)\,\mathrm{d}\xi$$

From Prop. 4.6.1, on the region $||t-s|-|x-y+z'-w'|| \gg 2^{-2k}|t-s|$ we have

$$\left|T_z T_w K_k^{a(z,w)}(t_0,x,t_0,y)\right| \lesssim \left\langle 2^k(|z|+|w|)\right\rangle^N \frac{2^{dk}}{\langle 2^k(|t-s|-|x-y+z'-w'|)\rangle^N}$$

Over this region, the integral (4.7.7) obeys the upper bound in (4.7.6). This can be seen by repeatedly applying

$$\int_{\mathbb{R}} \frac{1}{(1+2^k|r-a|)^N} \frac{2^k}{(1+2^k r)^{N_1}} \,\mathrm{d}r \lesssim \frac{1}{(1+2^k a)^{N-1}}, \quad N_1 \geq 2N$$

and for any $N_2$

$$\left\langle 2^k(|z|+|w|)\right\rangle^N m_k(z) m_k(w) \lesssim \left\langle 2^k|z|\right\rangle^{-N_2} \left\langle 2^k|w|\right\rangle^{-N_2}.$$

On the region $||t-s|-|x-y+z'-w'|| \lesssim 2^{-2k}|t-s|$, we use the term $\left\langle 2^k(t-s)\right\rangle^{-N}$ from the rapid decay of $m_k(z), m_k(w)$ and bound

$$\frac{1}{\langle 2^k(t-s)\rangle^N} \lesssim \frac{1}{\langle 2^k(|t-s|-|x-y+z'-w'|)\rangle^N}, \quad \left|T_z T_w K_k^{a(z,w)}\right| \lesssim 2^{dk}$$

which imply the upper bound in (4.7.6) as before.

**S**tep 3. The kernel of $e_{<k}^{-i\psi_\pm^k}(t_0,x,D)\bar{P}_k e_{<k}^{i\psi_\pm^k}(D,t_0,y) - \bar{P}_k$ obeys the bound

$$\left|K_{<k}(x,y) - 2^{dk}\check{a}(2^k(x-y))\right| \lesssim \varepsilon 2^{dk}(3+2^k|x-y|). \tag{4.7.8}$$

Indeed, we write $K_{<k}(x,y) - 2^{dk}\check{a}(2^k(x-y))$ as

$$2^{dk} \iint (e^{-iT_z\psi_\pm^k(t_0,x,2^k\xi)+iT_w\psi_\pm^k(t_0,y,2^k\xi)} - 1) a(\xi) e^{i2^k(x-y)\xi} \,\mathrm{d}\xi\, m_k(z) m_k(w) \,\mathrm{d}z \,\mathrm{d}w$$

and by (4.5.9), we bound

$$\begin{aligned}\left|e^{-iT_z\psi_\pm^k(t_0,x,2^k\xi)+iT_w\psi_\pm^k(t_0,y,2^k\xi)} - 1\right| &\lesssim \varepsilon \log(1+2^k(|x-y|+|z|+|w|)) \\ &\lesssim \varepsilon[1+2^k(|x-y|+|z|+|w|)].\end{aligned}$$

Bounding by absolute values and integrating in $z$ and $w$ we obtain (4.7.8).

**S**tep 4. Now we prove (4.7.5). We integrate (4.7.8) on $\{y \mid |x-y| \leq R\}$ and integrate (4.7.6) on the complement of this set, for $(2^k R)^{d+1} \simeq \varepsilon^{-\frac{1}{2}}$. We obtain

$$\text{LHS (4.7.5)} \lesssim \varepsilon (2^k R)^{d+1} + \frac{1}{(2^k R)^{d+1}} \lesssim \varepsilon^{\frac{1}{2}}.$$

□

## 4.8 Proof of the $\bar{N}_k$, $\bar{N}_k^*$ estimates (4.2.8), (4.2.9), (4.2.10)

In the proof we will need the following lemma.

**Lemma 4.8.1.** *For $k \geq 0$, $k \geq k' \geq j - O(1)$ and for both quantizations, we have:*

$$2^{j/2} \|\bar{Q}_j e_{k'}^{\pm i\psi_\pm^k} \bar{P}_k G\|_{L^2_{t,x}} \lesssim \varepsilon 2^{\delta(j-k')} \|G\|_{\bar{N}_k^*}, \tag{4.8.1}$$

*and thus, by duality*

$$2^{\frac{j}{2}} \|\bar{P}_k e_{k'}^{\pm i\psi_\pm^k} \bar{Q}_j F_k\|_{\bar{N}_k} \lesssim \varepsilon 2^{\delta(j-k')} \|F_k\|_{L^2_{t,x}} \tag{4.8.2}$$

**Corollary 4.8.2.** *For $k \geq 0,\ l \leq 0$ we have*

$$\bar{Q}_{<k+2l}(e_{<k}^{-i\psi_\pm^k} - e_{<k+2l}^{-i\psi_\pm^k})(t,x,D)\bar{P}_k : \bar{N}_k^* \to \bar{X}_1^{1/2}$$

*Proof.* This follows by summing over (4.8.1). □

The proof of this lemma is a bit long and is defered to the end of this section. The following proposition completes the proofs of (4.2.8), (4.2.9), (4.2.10).

**Proposition 4.8.3.** *For any $k \geq 0$, denoting $\psi = \psi_\pm^k$, one has:*

$$e_{<k}^{\pm i\psi}(t,x,D), e_{<k}^{\pm i\psi}(D,s,y) : \bar{N}_k \to \bar{N}_k \tag{4.8.3}$$

$$\partial_{t,x} e_{<k}^{\pm i\psi}(t,x,D), \partial_{t,x} e_{<k}^{\pm i\psi}(D,s,y) : \bar{N}_k \to \varepsilon 2^k \bar{N}_k \tag{4.8.4}$$

$$e_{<k}^{-i\psi}(t,x,D) e_{<k}^{i\psi}(D,s,y) - I : \bar{N}_k \to \varepsilon^{\frac{1}{2}} \bar{N}_k \tag{4.8.5}$$

*By duality, the same mappings hold for $\bar{N}_k^*$ in place of $\bar{N}_k$.*

*Proof.* **S**tep 1. Since $\bar{N}_k$ is defined as an atomic space, it suffices to prove (4.8.3), (4.8.4) applied to $F$ when $F$ is an $L^1 L^2$-atom ($\|F\|_{L^1 L^2} \leq 1$) or to $\bar{Q}_j F$ an $\bar{X}_1^{-\frac{1}{2}}$-atom ( $2^{-\frac{j}{2}} \|\bar{Q}_j F\|_{L^2_{t,x}} \leq 1$). The first case follows from integrating the pointwise in $t$ bounds (4.7.2), (4.7.3)

$$\|(e_{<k}^{\pm i\psi}, \varepsilon^{-1} 2^{-k} \nabla e_{<k}^{\pm i\psi}) F_k(t)\|_{L^2_x} \lesssim \|F_k(t)\|_{L^2_x}$$

for both the left and right quantizations.

Now consider the second case. We split

$$e_{<k}^{\pm i\psi} = e_{<\min(j,k)}^{\pm i\psi} + (e_{<k}^{\pm i\psi} - e_{<\min(j,k)}^{\pm i\psi}).$$

Note that $e^{\pm i\psi}_{<\min(j,k)}\bar{Q}_j F = \tilde{\bar{Q}}_j e^{\pm i\psi}_{<\min(j,k)}\bar{Q}_j F$ and thus the bound

$$\|e^{\pm i\psi}_{<\min(j,k)}\bar{Q}_j F\|_{\bar{N}_k} \lesssim 2^{-\frac{j}{2}}\|e^{\pm i\psi}_{<\min(j,k)}\bar{Q}_j F\|_{L^2_{t,x}} \lesssim 2^{-\frac{j}{2}}\|\bar{Q}_j F\|_{L^2_{t,x}}$$

follows from integrating (4.7.2). The same argument applies to $\nabla e^{\pm i\psi}_{<\min(j,k)}$ using (4.7.3). The remaining estimate, for $j \leq k$

$$\|(e^{\pm i\psi}_{<k} - e^{\pm i\psi}_{<j})\bar{Q}_j F_k\|_{\bar{N}_k} \lesssim \varepsilon 2^{-\frac{j}{2}}\|\bar{Q}_j F_k\|_{L^2_{t,x}} \tag{4.8.6}$$

follows by summing (4.8.2) in $k'$. Note that (4.8.6) remains true with $e^{\pm i\psi}$ replaced by $2^{-k}\nabla e^{\pm i\psi}$, because (4.8.2) remains true, which concludes (4.8.4). To see this, one writes $2^{-k'}\nabla e^{\pm i\psi}_{k'} = L e^{\pm i\psi}_{k'}$ where $L$ is disposable and use translation invariance and (4.8.2).

**S**tep 2. To prove (4.8.5), since that operator is self-adjoint, we prove that it is bounded from $\bar{N}^*_k \to \varepsilon^{\frac{1}{2}}\bar{N}^*_k$, where $\bar{N}^*_k \simeq L^\infty L^2 \cap \bar{X}^{\frac{1}{2}}_\infty$. The $L^\infty L^2$ mapping follows from Prop. 4.7.3, so it remains to prove

$$2^{\frac{j}{2}}\|\bar{Q}_j\tilde{P}_k[e^{-i\psi}_{<k}(t,x,D)e^{i\psi}_{<k}(D,s,y) - I]F_k\|_{L^2_{t,x}} \lesssim \varepsilon^{\frac{1}{2}}\|F_k\|_{L^\infty L^2 \cap \bar{X}^{\frac{1}{2}}_\infty}$$

For $\bar{Q}_{>j-c}F_k$ we can discard $\bar{Q}_j\tilde{P}_k$ and since $\|\bar{Q}_{>j-c}F_k\|_{L^2_{t,x}} \lesssim 2^{-j/2}\|F_k\|_{\bar{X}^{\frac{1}{2}}_\infty}$, the bound follows for this component from Prop. 4.7.3 by integration .

For $\bar{Q}_{\leq j-c}F_k$ the claim follows by adding the following

$$2^{\frac{j}{2}}\bar{Q}_j\tilde{P}_k[e^{-i\psi}_{<k} - e^{-i\psi}_{<j}](t,x,D)e^{i\psi}_{<k}(D,s,y)\bar{Q}_{\leq j-c} : \bar{N}^*_k \to \varepsilon L^2_{t,x}$$

$$2^{\frac{j}{2}}\bar{Q}_j\tilde{P}_k e^{-i\psi}_{<j}(t,x,D)[e^{i\psi}_{<k} - e^{i\psi}_{<j}](D,s,y)\bar{Q}_{\leq j-c} : \bar{N}^*_k \to \varepsilon L^2_{t,x}$$

since

$$\bar{Q}_j\tilde{P}_k I\bar{Q}_{\leq j-c} = 0, \qquad \bar{Q}_j\tilde{P}_k e^{-i\psi}_{<j}e^{i\psi}_{<j}\bar{Q}_{\leq j-c} = 0.$$

These mappings follow from (4.8.1), (4.8.3) for $\bar{N}^*_k$ and Prop. 4.7.3 and writing $\bar{Q}_j\tilde{P}_k e^{-i\psi}_{<j} = \bar{Q}_j\tilde{P}_k e^{-i\psi}_{<j}\bar{Q}_{[j-5,j+5]}$. □

## Proof of Lemma 4.8.1

We follow the method from [31] based on Moser-type estimates. The more difficult case is $d = 4$ and the argument can be simplified for $d \geq 5$. In the proof we will need the following lemmas.

**Lemma 4.8.4.** *Let $1 \le q \le p \le \infty$ and $k \ge 0$. Then for all $j - O(1) \le k' \le k$ we have*

$$\|e_{k'}^{\pm i\psi^k_{<j,\pm}}(t,x,D)\bar{P}_k\|_{L^pL^2 \to L^qL^2} \lesssim \varepsilon 2^{(\frac{1}{p}-\frac{1}{q})j} 2^{2(j-k')} \tag{4.8.7}$$

$$\|e_{k'}^{\pm i\psi^k_{\pm}}(t,x,D)\bar{P}_k\|_{L^2L^2 \to L^{\frac{10}{7}}L^2} \lesssim \varepsilon 2^{-\frac{1}{5}k'} \tag{4.8.8}$$

*By duality, the same bounds holds for the right quantization.*

*Remark* 4.8.5. To motivate the proof, we note that applying the $k'(>j)$ localization in the power series

$$e^{i\psi_{<j}(t,x,\xi)} = 1 + i\psi_{<j}(t,x,\xi) + O\big(\psi^2_{<j}(t,x,\xi)\big)$$

makes the linear term $1 + i\psi$ vanish. For the higher order terms Hölder inequality applies (in the form of decomposable lemma 4.5.6).

*Proof.* To prove (4.8.7), let $L_{k'}$ be a disposable multiplier in the $(t,x)$-frequencies such that

$$e_{k'}^{\pm i\psi^k_{<j,\pm}} = 2^{-2k'} L_{k'} \Delta_{t,x} e^{\pm i\psi^k_{<j,\pm}} = 2^{-2k'} L_{k'}(-\left|\partial_{t,x}\psi^k_{<j,\pm}\right|^2 \pm i\Delta_{t,x}\psi^k_{<j,\pm}) e^{\pm i\psi^k_{<j,\pm}}.$$

We may dispose of $L_{k'}$ by translation invariance. Then (4.8.7) follows from (4.5.20).

To prove (4.8.8) we write

$$e_{k'}^{\pm i\psi^k_{\pm}} = e_{k'}^{\pm i\psi^k_{<k'-C,\pm}} \pm i \int_{[k'-C,k-C]} \left(\psi^k_{\pm,l} e^{\pm i\psi^k_{<l,\pm}}\right)_{k'} \mathrm{d}l$$

For the first term we use (4.8.7). For the second term, we have

$$\|\psi^k_{\pm,l} e^{\pm i\psi^k_{<l,\pm}}(t,x,D)\|_{L^2L^2 \to L^{\frac{10}{7}}L^2} \lesssim \varepsilon 2^{-\frac{l}{5}}$$

by Lemma 4.5.6, (4.5.20) and (4.7.4), from which (4.8.8) follows. □

**Lemma 4.8.6.** *For $k \ge 0$, $k \ge k' \ge j - O(1)$, $j - C \le l' \le l \le k - C$ and for both quantizations, we have:*

$$\|\bar{Q}_j[(\psi^k_{k'} e_{<j}^{\pm i\psi^k_{<j,\pm}})_{k'} \bar{Q}_{\prec j} G_k]\|_{L^2_{t,x}} 2^{\frac{j}{2}} \lesssim \varepsilon 2^{\frac{1}{4}(j-k')} \|G\|_{L^\infty L^2} \tag{4.8.9}$$

$$\|\bar{Q}_j[(\psi^k_l \psi^k_{l'} e_{<j}^{\pm i\psi^k_{<j,\pm}})_{k'} \bar{Q}_{\prec j} G_k]\|_{L^2_{t,x}} 2^{\frac{j}{2}} \lesssim \varepsilon^2 2^{\frac{1}{12}(j-l)} 2^{\frac{1}{6}(j-l')} \|G\|_{L^\infty L^2}. \tag{4.8.10}$$

*Proof.* **S**tep 1. By translation invariance we may discard the outer $k'$ localization. By Lemma 5.5.2 we deduce that in (4.8.9) the contribution of $\psi^{k,\pm}_{k',\theta}$ (which define $\psi^k_{k',\pm}$ in (4.5.3)) is zero unless $\theta \gtrsim 2^{\frac{1}{2}(j-k')}$ and $j \ge k' - 2k - C$. For these terms, from (4.5.15) we get

$$2^{\frac{j}{2}} \sum_{\theta \gtrsim 2^{\frac{1}{2}(j-k')}} \|\psi^{k,\pm}_{k',\theta}\|_{D^\theta_k(L^2L^\infty)} \lesssim \varepsilon 2^{\frac{1}{4}(j-k')}$$

from which (4.8.9) follows by Lemma 4.5.6. When $k = 0$ no angular localization are needed.

**S**tep 2. Now we prove (4.8.10). First we consider the case $l' + c \le l = k' + O(1)$ and define $\theta_0 := 2^{\frac{1}{2}(j-l)}$. By appropriately applying Lemma 5.5.2 we deduce that the terms that contribute to (4.8.10) are $\psi_l^k \psi_{l',\theta'}^k$ for $\theta' \gtrsim \theta_0$ and $\psi_{l,\theta}^k \psi_{l',\theta'}^k$ for $\theta' \ll \theta_0$, $\theta \gtrsim \theta_0$. We use (4.5.15) with $q = 3$ for the large angle terms and with $q = 6$ for the other term, obtaining

$$\sum_{\theta' \gtrsim \theta_0} \| \psi_l^k \psi_{l',\theta'}^k e^{i\psi} \|_{L^\infty L^2 \to L^2_{t,x}} + \sum_{\substack{\theta' \ll \theta_0 \\ \theta \gtrsim \theta_0}} \| \psi_{l,\theta}^k \psi_{l',\theta'}^k e^{i\psi} \|_{L^\infty L^2 \to L^2_{t,x}} \lesssim \varepsilon^2 2^{-\frac{j}{2}} 2^{\frac{1}{12}(j-l)} 2^{\frac{1}{6}(j-l')} \tag{4.8.11}$$

In the high-high case $l = l' + O(1) \ge k'$ the same argument applies with $\theta_0 := 2^{\frac{1}{2}(j-k')} 2^{k'-l}$ and (4.8.11) also follows in this case. □

*Proof of Lemma 4.8.1.* For brevity, we supress the $k$ superscript and write $\psi$ to denote $\psi_\pm^k$.

**S**tep 1. [The contribution of $\bar{Q}_{>j-c} G_k$]

We use Lemma 2.2.4 and (4.8.8)

$$\| \bar{Q}_j e_{k'}^{\pm i\psi} \bar{Q}_{>j-c} G_k \|_{L^2_{t,x}} \lesssim 2^{\frac{j}{5}} \| e_{k'}^{\pm i\psi} \bar{Q}_{>j-c} G_k \|_{L^{\frac{10}{7}} L^2} \lesssim \varepsilon 2^{\frac{j-k'}{5}} \| \bar{Q}_{>j-c} G_k \|_{L^2_{t,x}}.$$

and the last norm is $\lesssim 2^{-j/2} \| G \|_{\bar{N}_k^*}$.

**S**tep 2. [The contribution of $\bar{Q}_{\prec j} G_k$] Motivated by remark 4.8.5, by iterating the fundamental theorem of calculus, we decompose the symbol

$$e^{\pm i\psi}(t,x,\xi) = \mathcal{T}_0 \pm i \mathcal{T}_1 - \mathcal{T}_2 \mp i \mathcal{T}_3$$

where $\mathcal{T}_0 = e^{i\psi_{<j}}$ and

$$\mathcal{T}_1 = \int_{[j-C,k-C]} \psi_l e^{i\psi_{<j}} \, \mathrm{d}l, \qquad \mathcal{T}_2 = \iint_{j-C \le l' \le l \le k-C} \psi_l \psi_{l'} e^{i\psi_{<j}} \, \mathrm{d}l \, \mathrm{d}l'$$

$$\mathcal{T}_3 = \iiint_{j-C \le l'' \le l' \le l \le k-C} \psi_l \psi_{l'} \psi_{l''} e^{i\psi_{<l''}} \, \mathrm{d}l \, \mathrm{d}l' \, \mathrm{d}l''$$

The term $\mathcal{T}_0$ is estimated by (4.8.7):

$$\| (\mathcal{T}_0)_{k'}(t,x,D) \bar{Q}_{\le k} G_k \|_{L^2_{t,x}} \lesssim \varepsilon 2^{\frac{-j}{2}} 2^{2(j-k')} \| \bar{Q}_{\le k} G_k \|_{L^\infty L^2}$$

Next, we split $\mathcal{T}_1 = \mathcal{T}_1^1 + \mathcal{T}_1^2$ where

$$\mathcal{T}_1^1 = \int_{[j-C,k-C]} \psi_l e_{<j}^{i\psi_{<j}} \, \mathrm{d}l, \qquad \mathcal{T}_1^2 = \int_{[j-C,k-C]} \psi_l e_{>j-c/2}^{i\psi_{<j}} \, \mathrm{d}l$$

By appying the $k'$ localization, the integral defining $\mathcal{T}_1^1$ vanishes unless $l = k' + O(1)$ for which we may apply (4.8.9). To estimate $\mathcal{T}_1^2$ we use Lemma 4.5.6 with (4.5.20) for $q = 6$ and (4.8.7) with $L^\infty L^2 \to L^3 L^2$.

We turn to $\mathcal{T}_2$ and separate $e^{i\psi_{<j}} = e^{i\psi_{<j}}_{<j} + e^{i\psi_{<j}}_{>j-c/2}$ as before. For the first component we use (4.8.10). For the second, we use (4.5.20) with $q = 6$ for $\psi_l, \psi_{l'}$ and (4.8.7) with $L^\infty L^2 \to L^6 L^2$ obtaining:

$$2^{\frac{1}{2}j} \|\psi_l \psi_{l'} e^{i\psi_{<j}}_{>j-c/2}\|_{L^\infty L^2 \to L^2 L^2} \lesssim \varepsilon 2^{\frac{1}{6}(j-l)} 2^{\frac{1}{6}(j-l')} \qquad \text{for } l > k' - c$$
$$2^{\frac{1}{2}j} \|\psi_l \psi_{l'} e^{i\psi_{<j}}_{k'}\|_{L^\infty L^2 \to L^2 L^2} \lesssim \varepsilon 2^{\frac{1}{6}(j-l)} 2^{\frac{1}{6}(j-l')} 2^{2(j-k')} \qquad \text{for } l < k' - c.$$

For $\mathcal{T}_3$ we use (4.5.20) for $q = 6$. When $l < k' - C$ we use (4.8.7) with $p = q = \infty$ and it remains to integrate

$$2^{\frac{1}{2}j} \|\psi_l \psi_{l'} \psi_{l''} e^{i\psi_{<l''}}_{k'}\|_{L^\infty L^2 \to L^2 L^2} \lesssim \varepsilon 2^{\frac{1}{2}j} 2^{-\frac{1}{6}l} 2^{-\frac{1}{6}l'} 2^{-\frac{1}{6}l''} 2^{2(l''-k')}.$$

On $l \geq k' - C$ it suffices to integrate

$$2^{\frac{1}{2}j} \|\psi_l \psi_{l'} \psi_{l''} e^{i\psi_{<l''}}\|_{L^\infty L^2 \to L^2 L^2} \lesssim \varepsilon 2^{\frac{1}{2}j} 2^{-\frac{1}{6}l} 2^{-\frac{1}{6}l'} 2^{-\frac{1}{6}l''}.$$

□

## 4.9 Proof of the conjugation bound (4.2.12)

In general, for pseudodifferential operators one has the composition property $a(x, D)b(x, D) = c(x, D)$ where, in an asymptotic sense

$$c(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi).$$

In the present case this formula will be exact, as seen by differentiating under the integral in (1.5.7).

By definition (1.5.7), the symbol of $e^{-i\psi^k_\pm}_{<k}(t, x, D)\Box_m$ is

$$e^{-i\psi^k_\pm(t,x,\xi)}_{<k} (\partial_t^2 + |\xi|^2 + 1). \tag{4.9.1}$$

By differentiating (1.5.7), we see that the symbol of $\Box_m e^{-i\psi^k_\pm}_{<k}(t, x, D)$ is

$$e^{-i\psi^k_\pm}_{<k} (\partial_t^2 + |\xi|^2 + 1) + \Box e^{-i\psi^k_\pm}_{<k} + 2 \left( \partial_t e^{-i\psi^k_\pm}_{<k} \partial_t - i(\nabla e^{-i\psi^k_\pm}_{<k}) \cdot \xi \right) \tag{4.9.2}$$

while the symbol of the operator $2i(A_{<k} \cdot \nabla) e^{-i\psi^k_\pm}_{<k}(t, x, D)$ is

$$-2e^{-i\psi^k_\pm(t,x,\xi)}_{<k} A_{<k}(t, x) \cdot \xi + 2i \nabla e^{-i\psi^k_\pm(t,x,\xi)}_{<k} \cdot A_{<k}(t, x) \tag{4.9.3}$$

Now, the inequality (4.2.12) follows from the following proposition.

**Proposition 4.9.1.** *Denoting $\psi = \psi_{\pm}^{k}$, we can decompose*

$$e_{<k}^{-i\psi_{\pm}^{k}}(t,x,D)\Box_m - \Box_m^{A_{<k}} e_{<k}^{-i\psi_{\pm}^{k}}(t,x,D) = \sum_{i=0}^{5} F_i(t,x,D)$$

*where*

$$\begin{aligned}
F_0(t,x,\xi) :=& 2\left[((\pm\langle\xi\rangle\partial_t - \xi\cdot\nabla)\psi(t,x,\xi) + A_{<k}(t,x)\cdot\xi)e^{-i\psi(t,x,\xi)}\right]_{<k} \\
F_1(t,x,\xi) :=& -\Box e_{<k}^{-i\psi(t,x,\xi)} \\
F_2(t,x,\xi) :=& 2i\nabla e_{<k}^{-i\psi(t,x,\xi)}\cdot A_{<k}(t,x) \\
F_3(t,x,\xi) :=& 2i^{-1}\partial_t e_{<k}^{-i\psi(t,x,\xi)}(i\partial_t \pm \langle\xi\rangle) \\
F_4(t,x,\xi) :=& 2\left[\left(A_{<k}(t,x)e^{-i\psi(t,x,\xi)}\right)_{<k} - A_{<k}(t,x)e_{<k}^{-i\psi(t,x,\xi)}\right]\cdot\xi
\end{aligned}$$

*and for all $i = 0, 4$ we have*

$$\|F_i(t,x,D)u_k\|_{\bar{N}_k} \lesssim \varepsilon\|u_k\|_{L^\infty H^1} + \varepsilon 2^k\|(i\partial_t \pm \langle D\rangle)u_k\|_{\bar{N}_k} \tag{4.9.4}$$

*Proof.* The decomposition follows from (4.9.1)-(4.9.3) and basic manipulations. We proceed to the proof of (4.9.4). We will make use of the bound

$$\|u_k\|_{\bar{N}_k^*} \lesssim \|u_k\|_{L^\infty L^2} + \|(i\partial_t \pm \langle D\rangle)u_k\|_{\bar{N}_k},$$

for which we refer to the proof of Lemma 4.10.2. Recall that we identify $\bar{N}_k^* \simeq L^\infty L^2 \cap \bar{X}_\infty^{\frac{1}{2}}$.

**S**tep 1.[The main term $F_0$] Recall the identity (4.1.2) and the definitions (4.5.1), (4.5.2). For $k=0$ the term $F_0(t,x,\xi)$ vanishes. Now assume $k \geq 1$ and write

$$F_0(t,x,\xi) = 2((\sum_{k_1<k-c}\Pi^{\omega}_{\leq\delta(k_1-k)}A_{k_1}\cdot\xi)e^{-i\psi(t,x,\xi)})_{<k} = 2F'(t,x,\xi)_{<k}$$

where

$$F'(t,x,\xi) = a(t,x,\xi)e_{<k}^{-i\psi(t,x,\xi)}, \qquad a(t,x,\xi) := \sum_{k_1<k-c}\Pi^{\omega}_{\leq\delta(k_1-k)}A_{k_1}(t,x)\cdot\xi$$

By (4.5.24) we have

$$\|F'(t,x,D) - a(t,x,D)e_{<k}^{-i\psi}(t,x,D)\|_{L^\infty L^2\to L^1L^2} \lesssim \|\nabla_\xi a\|_{D_k^1 L^2(L^\infty)}\|\nabla_x e_{<k}^{-i\psi}(t,x,D)\|_{L^\infty(L^2)\to L^2(L^2)}$$

By lemma 4.5.6, (4.5.21) and lemma 4.5.11 we have

$$\|(\nabla_x\psi e^{-i\psi})_{<k}(t,x,D)\|_{L^\infty(L^2)\to L^2(L^2)} \lesssim \|\nabla_x\psi_{\pm}^{<k}\|_{D_k(L^2L^\infty)} \lesssim 2^{\frac{k}{2}}\varepsilon$$

Summing over (4.5.19), we get $\|\nabla_\xi a\|_{D_k^1 L^2(L^\infty)} \lesssim 2^{\frac{k}{2}}\varepsilon$. Thus

$$F'(t,x,D) - a(t,x,D)e^{-i\psi}_{<k}(t,x,D) : \bar{N}_k^* \to 2^k \varepsilon \bar{N}_k$$

and it remains to prove

$$a(t,x,D)e^{-i\psi}_{<k}(t,x,D) : \bar{N}_k^* \to 2^k \varepsilon \bar{N}_k$$

By Proposition 4.8.3, $e^{-i\psi}_{<k}(t,x,D)$ is bounded on $\bar{N}_k^*$.

Assume $-k \leq \delta(k_1 - k)$ (the case $\delta(k_1-k) \leq -k$ is analogous). We decompose

$$a(t,x,\xi) = \sum_{k_1 < k-c} \sum_{\theta \in [2^{-k}, 2^{\delta(k_1-k)}]} a^\theta_{k_1}(t,x,\xi),$$

$$a^{2^{-k}}_{k_1}(t,x,\xi) := \Pi^\omega_{\leq 2^{-k}} A_{k_1}(t,x)\cdot\xi, \qquad a^\theta_{k_1}(t,x,\xi) := \Pi^\omega_\theta A_{k_1}(t,x)\cdot \xi \quad (\theta > 2^{-k}),$$

and it remains to prove

$$\|a^\theta_{k_1}(t,x,D)v_k\|_{\bar{N}_k} \lesssim \theta^{\frac{1}{2}}\varepsilon 2^k \|v_k\|_{\bar{N}_k^*}.$$

for all $\theta = 2^l,\ l \geq -k$. First, using (4.5.16) we have

$$\|a^\theta_{k_1}(t,x,D)\bar{Q}_{>k_1+2l-c}v_k\|_{L^1L^2} \lesssim \|a^\theta_{k_1}\|_{D_kL^2L^\infty}\|\bar{Q}_{>k_1+2l-c}v_k\|_{L^2_{t,x}} \lesssim \theta^{\frac{1}{2}}\varepsilon 2^k \|v_k\|_{\bar{N}_k^*}$$

Then, denoting $f(t,x) = a^\theta_{k_1}(t,x,D)\bar{Q}_{<k_1+2l-c}v_k$ we have

$$\|f\|_{L^2_{t,x}} \lesssim \|a^\theta_{k_1}\|_{D_kL^2L^\infty}\|\bar{Q}_{<k_1+2l-c}v_k\|_{L^\infty L^2} \lesssim 2^{\frac{3}{2}l}2^{\frac{k_1}{2}}\varepsilon 2^k \|v_k\|_{\bar{N}_k^*}$$

For each $\xi$, the term $\bar{Q}_j[\Pi^\omega_\theta A_{k_1}(t,x)\xi e^{ix\xi}\bar{Q}_{<k_1+2l-c}\hat{v}_k(t,\xi)]$ is non-zero only for $j = k_1 + 2l + O(1)$ (by Remark 5.5.3 of Lemma 5.5.2). Thus,

$$\|f\|_{\bar{N}_k} \leq \|f\|_{\bar{X}_1^{-1/2}} \lesssim \sum_{j=k_1+2l+O(1)} \|\bar{Q}_j f\|_{L^2_{t,x}} 2^{-\frac{j}{2}} \lesssim \theta^{\frac{1}{2}}\varepsilon 2^k \|v_k\|_{\bar{N}_k^*}$$

**S**tep 2. [The terms $F_1$ and $F_2$] Since $\Box_{t,x}\psi(t,x,\xi) = 0$ we have

$$\begin{aligned} F_1(t,x,\xi) &= \left[(|\partial_t\psi(t,x,\xi)|^2 - |\nabla\psi(t,x,\xi)|^2)e^{-i\psi(t,x,\xi)}\right]_{<k}, \\ F_2(t,x,\xi) &= 2iA^j_{<k}(t,x)\left(\partial_j\psi(t,x,\xi)e^{-i\psi(t,x,\xi)}\right)_{<k} \end{aligned}$$

By lemma 4.5.6 and (4.5.21) we have

$$\begin{aligned} (\partial_j\psi e^{-i\psi})(t,x,D) &: L^\infty L^2 \to \varepsilon 2^{\frac{k}{2}} L^2L^2 \\ (|\partial_\alpha\psi|^2 e^{-i\psi})(t,x,D) &: L^\infty L^2 \to \varepsilon^2 2^k L^1L^2 \end{aligned}$$

By lemma 4.5.11 the same mappings hold for the $< k$ localized symbols, which proves (4.9.4) for $F_1$, while for $F_2$ we further apply Hölder's inequality together with $\|A_{<k}\|_{L^2L^\infty} \lesssim 2^{k/2}\varepsilon$.

**S**tep 3. [The term $F_3$] The bound follows by using (4.8.4) to dispose of

$$2^{-k}\varepsilon^{-1}\partial_t e^{-i\psi}_{<k}(t,x,D).$$

**S**tep 4.[The term $F_4$] Using Lemma 1.5.1 we write

$$F_4(t,x,\xi) = 2^{-k}\xi_j L(\nabla_{t,x}A^j_{<k}(t,x), e^{-i\psi(t,x,\xi)})$$

As in lemma 4.5.11, by translation-invariance it suffices to prove

$$2^{-k}\|\nabla_{t,x}A^j_{<k}e^{-i\psi}(t,x,D)\partial_j u_k\|_{\bar{N}_k} \lesssim \varepsilon 2^k \|e^{-i\psi}(t,x,D)u_k\|_{\bar{N}^*_k} \lesssim \varepsilon 2^k \|u_k\|_{\bar{N}^*_k}$$

which follows from (6.1.32) (observe that the $\mathcal{H}^*_{k'}$ term is zero when $\Box A^j = 0$ and in this case the $\bar{N}^*_{k'}$ norm of $\phi$ suffices. One uses the derivative on $A^j$ to do the $k'$ summation). □

# 4.10 Proof of the $\bar{S}_k$ bound (4.2.11)

We begin by stating a simple lemma that provides bounds for localized symbols.

**Lemma 4.10.1.** *Let $X$ be a translation-invariant space of functions defined on $\mathbb{R}^{d+1}$. Let $P$ be a bounded Fourier multiplier. Suppose we have the bounded map*

$$e^{-i\psi}(t,x,D)e^{\pm it\langle D\rangle}P : L^2_x \to X. \tag{4.10.1}$$

*Then, uniformly in $h$, we also have the bounded map for localized symbols:*

$$e^{-i\psi}_{<h}(t,x,D)e^{\pm it\langle D\rangle}P : L^2_x \to X. \tag{4.10.2}$$

*Proof.* Recalling (4.1.4), for $u_0 \in L^2_x$ we write

$$e^{-i\psi}_{<h}(t,x,D)e^{\pm it\langle D\rangle}Pu_0 = \int_{\mathbb{R}^{d+1}} m_h(s,y)e^{-i\psi}(t+s,x+y,D)e^{\pm i(t+s)\langle D\rangle}Pu_{s,y}\,\mathrm{d}s\,\mathrm{d}y$$

where $\hat{u}_{s,y}(\xi) = e^{\mp is\langle\xi\rangle}e^{-iy\xi}\hat{u}_0(\xi)$. By Minkowski's inequality, translation invariance of $X$, (4.10.1) and the bound $\|u_{s,y}\|_{L^2_x} \le \|u_0\|_{L^2_x}$ we obtain (4.10.2). □

We will apply this lemma for $X$ taking the various norms that define $\bar{S}_k$.

The next lemma will be used to reduce estimates to the case of free waves.

**Lemma 4.10.2.** *Let $k \geq 0$ and $X$ be a space of functions on $\mathbb{R}^{1+n}$ with Fourier support in $\{\langle\xi\rangle \simeq 2^k\}$ (or a subset of it, such as a $2^{k'} \times (2^{k'+l'})^3$ box) such that*

$$\|e^{it\sigma} f\|_X \lesssim \|f\|_X, \quad \forall \sigma \in \mathbb{R}$$

$$\|1_{t>s} f\|_X \lesssim \|f\|_X, \quad \forall s \in \mathbb{R}$$

$$\|e^{-i\psi}_{<h}(t,x,D) e^{\pm it\langle D\rangle} u_0\|_X \lesssim C_1 \|u_0\|_{L^2}. \tag{4.10.3}$$

*hold for all $f, u_0$ and both signs $\pm$. Then, we have*

$$2^k \|e^{-i\psi}_{<h}(t,x,D) u\|_X \lesssim C_1 (\|u[0]\|_{H^1 \times L^2} + \|\Box_m u\|_{\bar{N}_k}) \tag{4.10.4}$$

*If we only assume that* (4.10.3) *holds for one of the signs $\pm$, then* (4.10.4) *still holds for functions $u$ with Fourier support in $\{\pm\tau \geq 0\}$.*

*Proof.* We decompose $\Box_m u = F^1 + F^2$ such that $\|\Box_m u\|_{\bar{N}_k} \simeq \|F^1\|_{L^1 L^2} + \|F^2\|_{\bar{X}_1^{-\frac{1}{2}}}$. By (4.10.3) we can subtract free solutions from $u$ and so we may assume that $u[0] = (0,0)$. We may also assume that $F^2$ is modulation-localized to $|\tau - \langle\xi\rangle| \simeq 2^j$, $\tau \geq 0$. We define $v = \frac{1}{\Box_m} F^2$ and write $u = u^1 + u^2$ where $u^1$ is the Duhamel term

$$u^1(t) = \int_{\mathbb{R}} \frac{\sin((t-s)\langle D\rangle)}{\langle D\rangle} 1_{t>s} F^1(s)\,\mathrm{d}s - \sum_{\pm} \pm e^{\pm it\langle D\rangle} \int_{-\infty}^0 e^{\mp is\langle D\rangle} \frac{F^1(s)}{2i\langle D\rangle}\,\mathrm{d}s$$

$$\text{and} \qquad u^2 = v - e^{it\langle D\rangle} w^1 - e^{-it\langle D\rangle} w^2$$

so that $\Box_m u^2 = 0$ and $w^1, w^2$ are chosen such that $u^2[0] = (0,0)$.

For the second part of $u^1$ we use (4.10.3) together with

$$\|\int_{-\infty}^0 e^{\mp is\langle D\rangle} \frac{F^1(s)}{2i\langle D\rangle}\,\mathrm{d}s\|_{L^2} \leq \int_{-\infty}^0 \|e^{\mp is\langle D\rangle} \frac{F^1(s)}{2i\langle D\rangle}\|_{L^2}\,\mathrm{d}s \lesssim 2^{-k} \|F^1(s)\|_{L^1 L^2}.$$

For the first part of $u^1$ we again write $\sin((t-s)\langle D\rangle)$ in terms of $e^{\pm i(t-s)\langle D\rangle}$, and

$$\|e^{-i\psi_{k,\pm}}_{<h} \int_{\mathbb{R}} \frac{e^{\pm i(t-s)\langle D\rangle}}{\langle D\rangle} 1_{t>s} F^1(s)\,\mathrm{d}s\|_X \leq \int_{\mathbb{R}} \|1_{t>s} e^{-i\psi_{k,\pm}}_{<h} e^{\pm i(t-s)\langle D\rangle} \frac{F^1(s)}{\langle D\rangle}\|_X\,\mathrm{d}s$$

$$\lesssim 2^{-k} C_1 \int_{\mathbb{R}} \|e^{\mp is\langle D\rangle} F^1(s)\|_{L^2}\,\mathrm{d}s \leq 2^{-k} C_1 \|F^1(s)\|_{L^1 L^2}.$$

Now we turn to $u^2$. For $w^1, w^2$ we use (4.10.3) and, using Lemma (2.2.4)

$$\|w^i\|_{L^2} \lesssim \|(v, \frac{\partial_t}{\langle D\rangle} v)\|_{L^\infty L^2} \lesssim 2^{\frac{j}{2}} \|(\frac{1}{\Box_m}, \frac{i\partial_t - \langle D\rangle}{\langle D\rangle \Box_m}) F^2\|_{L^2_{t,x}} \lesssim 2^{\frac{-j}{2}} 2^{-k} \|F^2\|_{L^2_{t,x}}$$

Next, we write $\tau = \rho + \langle \xi \rangle$ in the Fourier inversion formula

$$v(t) = \int e^{it\tau + ix\xi} \mathcal{F}v(\tau, \xi) \,\mathrm{d}\xi \,\mathrm{d}\tau = \int_{|\rho| \simeq 2^j} e^{it\rho} e^{it\langle D \rangle} \phi_\rho \,\mathrm{d}\rho$$

for $\hat{\phi}_\rho(\xi) = \mathcal{F}v(\rho + \langle \xi \rangle, \xi)$. Then

$$\|e^{-i\psi_{k,\pm}}_{<h} v\|_X \lesssim \int_{|\rho| \simeq 2^j} \|e^{it\rho} e^{-i\psi_{k,\pm}}_{<h}(t,x,D) e^{it\langle D \rangle} \phi_\rho\|_X \,\mathrm{d}\rho \lesssim C_1 \int_{|\rho| \simeq 2^j} \|\phi_\rho\|_{L^2_x} \,\mathrm{d}\rho$$

By Cauchy-Schwarz we bound this by $2^{\frac{j}{2}} C_1 \|v\|_{L^2_{t,x}} \lesssim 2^{\frac{-j}{2}} 2^{-k} C_1 \|F^2\|_{L^2_{t,x}}$.

If we only assume that (4.10.3) holds for one of the signs $\pm$, then we have the following variant

$$\|e^{-i\psi}_{<h}(t,x,D)u\|_X \lesssim C_1(\|u(0)\|_{L^2} + \|(i\partial_t \pm \langle D \rangle)u\|_{\bar{N}_k})$$

Now the Duhamel term is expressed in terms of one of the $e^{\pm it\langle D \rangle}$. For functions with Fourier support in $\{\pm\tau \geq 0\}$ we have $\|(i\partial_t \pm \langle D \rangle)u\|_{N_k} \simeq 2^{-k} \|\Box_m u\|_{N_k}$. □

Now we are ready to begin the proof of (4.2.11). We will implicitly use Prop. 2.2.9.

For brevity, we drop the $k$ and $\pm$ subscripts and denote $\psi = \psi^k_\pm$.

## The Strichartz norms

By Lemma 4.10.2, the bound for $\bar{S}^{Str}_k$ reduces to

**Lemma 4.10.3.** *For all $k \geq 0$ we have*

$$\|e^{-i\psi}_{<k}(t,x,D) e^{\pm it\langle D \rangle} v_k\|_{\bar{S}^{Str}_k} \lesssim \|v_k\|_{L^2_x}$$

*Proof.* Using Lemma 4.10.1 this bound follows from

$$e^{-i\psi}(t,x,D) e^{\pm it\langle D \rangle} : \bar{P}_k L^2_x \to \bar{S}^{Str}_k. \tag{4.10.5}$$

We use the result of Keel-Tao on Strichartz estimates from [23]. As noticed in that paper (see sec. 6 and the end of sec. 5; see also [50, sec. 5], the $L^2 L^r$ estimate also holds with $L^r$ replaced by the Lorentz space $L^{r,2}$. We need this only when $d = 4$ for the $L^2 L^{4,2}$ norm in (2.1.4).

By change of variable, we rescale at frequency $2^0$:

$$U(t) := e^{-i\psi(\cdot/2^k, \cdot/2^k, 2^k \cdot)}(t,x,D) e^{\pm it\langle D \rangle_k}$$

The $L^2_x \to L^2_x$ bound follows from Prop. 4.7.1. The $L^1 \to L^\infty$ bound for $U(t)U(s)^*$ follows from (4.6.3) for $S^{Str,W}_k$ and from (4.6.4) for the other $\bar{S}^{Str}_k$ norms in (2.1.4) when $d = 4$. □

### The $\bar{X}^{\frac{1}{2}}_{\infty}$ norms.

For any $j \in \mathbb{Z}$ we show

$$2^{\frac{1}{2}j}\|\bar{Q}_j e^{-i\psi}_{<k}(t,x,D)\phi_k\|_{L^2_{t,x}} \lesssim \|\phi_k\|_{L^\infty(H^1\times L^2)} + \|\Box_m \phi_k\|_{\bar{N}_k}. \tag{4.10.6}$$

We separate

$$e^{-i\psi}_{<k} = e^{-i\psi}_{<\min(j,k)} + \sum_{k'+C\in[j,k]} e^{-i\psi}_{k'}$$

For the first term we write

$$\bar{Q}_j e^{-i\psi}_{<\min(j,k)}\phi_k = \bar{Q}_j e^{-i\psi}_{<\min(j,k)} \bar{Q}_{[j-1,j+1]}\phi_k$$

Then we discard $\bar{Q}_j e^{-i\psi}_{<\min(j,k)}$ and the estimate becomes trivial. The second term follows by summing over (4.8.1).

### The $S_{box(k')}$ norms in (2.1.5), $k = 0$

For $k' < 0$ we prove

$$2^{-\sigma k'}(\sum_{C_{k'}} \|\bar{Q}^{\pm}_{<k'} P_{C_{k'}} e^{-i\psi^0_{\pm}}_{<0}(t,x,D)\phi\|^2_{L^2L^\infty})^{1/2} \lesssim \|(\phi,\partial_t\phi)(0)\|_{L^2_x} + \|\Box_m\phi\|_{\bar{N}_0}$$

We may assume $\phi$ is Fourier supported in $\pm\tau \geq 0$. We split

$$e^{-i\psi^0_{\pm}}_{<0} = (e^{-i\psi^0_{\pm}}_{<0} - e^{-i\psi^0_{\pm}}_{<k'}) + e^{-i\psi^0_{\pm}}_{<k'}$$

The estimate for the first term follows from Prop. 2.2.5 and Cor. 4.8.2. For the second term we write

$$P_{C_{k'}} e^{-i\psi^0_{\pm}}_{<k'} = P_{C_{k'}} e^{-i\psi^0_{\pm}}_{<k'} \tilde{P}_{C_{k'}}.$$

Then we can discard $\bar{Q}^{\pm}_{<k'} P_{C_{k'}}$ and prove

$$2^{-\sigma k'}\|e^{-i\psi^0_{\pm}}_{<k'}(t,x,D)\tilde{P}_{C_{k'}}\phi\|_{L^2L^\infty} \lesssim \|\tilde{P}_{C_{k'}}(\phi,\partial_t\phi)(0)\|_{L^2_x} + \|\tilde{P}_{C_{k'}}\Box_m\phi\|_{\bar{N}_0}$$

By Lemma 4.10.2, this reduces to

$$2^{-\sigma k'} e^{-i\psi^0_{\pm}}_{<k'}(t,x,D) e^{\pm it\langle D\rangle} \tilde{P}_{C_{k'}} \bar{P}_0 : L^2_x \to L^2 L^\infty$$

which follows from Corollary 4.6.4 using Lemma 4.10.1.

## The square summed $\bar{S}_k^{\omega\pm}(l)$ norms, $k \geq 1$, first part

For any fixed $l < 0$ we split

$$e^{-i\psi}_{<k} = (e^{-i\psi}_{<k} - e^{-i\psi}_{<k+2l}) + e^{-i\psi}_{<k+2l}.$$

Here we treat the first term, while the second one is considered below. The bound

$$2^k (\sum_{\omega} \| P_l^\omega \bar{Q}^{\pm}_{<k+2l} (e^{-i\psi}_{<k} - e^{-i\psi}_{<k+2l}) \phi \|^2_{\bar{S}_k^{\omega,\pm}(l)})^{\frac{1}{2}} \lesssim \| \nabla_{t,x} \phi(0) \|_{L^2_x} + \| \Box_m \phi \|_{\bar{N}_k}$$

follows from Prop. 2.2.5 and Cor. 4.8.2.

## The square-summed $L^2 L^\infty$ and $\bar{S}_k^{Str}$ norms, $k \geq 1$

Let $l < 0$. It remains to consider $e^{-i\psi}_{<k+2l}$. We fix $\omega$ and the estimate we need boils down to square-summing the following over $\omega$, after taking supremum over $k' \leq k$, $l' < 0$, for $k + 2l \leq k' + l' \leq k + l$

$$2^{-\frac{k}{2} - \frac{d-2}{2} k' - \frac{d-3}{2} l'} (\sum_{\mathcal{C} = \mathcal{C}_{k'}(l')} \| P_{\mathcal{C}} P_l^\omega \bar{Q}^{\pm}_{<k+2l} e^{-i\psi}_{<k+2l} \phi \|^2_{L^2 L^\infty})^{\frac{1}{2}} \lesssim \| \tilde{P}_l^\omega \nabla_{t,x} \phi(0) \|_{L^2_x} + \| \tilde{P}_l^\omega \Box_m \phi \|_{\bar{N}_k}$$

Fix $\mathcal{C} = \mathcal{C}_{k'}(l')$. Since $k + 2l \leq k' + l'$, one can write

$$P_{\mathcal{C}} e^{-i\psi}_{<k+2l}(t,x,D) \phi = P_{\mathcal{C}} e^{-i\psi}_{<k+2l}(t,x,D) \tilde{P}_{\mathcal{C}} \phi. \tag{4.10.7}$$

Then one can can discard $P_{\mathcal{C}} P_l^\omega \bar{Q}_{<k+2l}$ and prove

$$2^{-\frac{k}{2} - \frac{d-2}{2} k' - \frac{d-3}{2} l'} \| e^{-i\psi}_{<k+2l}(t,x,D) \tilde{P}_{\mathcal{C}} \phi \|_{L^2 L^\infty} \lesssim \| \tilde{P}_{\mathcal{C}} \nabla_{t,x} \phi(0) \|_{L^2_x} + \| \tilde{P}_{\mathcal{C}} \Box_m \phi \|_{\bar{N}_k}$$

By Lemma 4.10.2, this reduces to

$$e^{-i\psi}_{<k+2l}(t,x,D) e^{\pm it\langle D \rangle} \tilde{P}_{\mathcal{C}} : L^2_x \to 2^{\frac{k}{2} + \frac{d-2}{2} k' + \frac{d-3}{2} l'} L^2 L^\infty \tag{4.10.8}$$

which follows by Lemma 4.10.1 from Corollary 4.6.4.

The same argument applies to $\bar{S}_k^{Str}$ except that one uses (4.10.5) and Lemma 4.10.1 instead of (4.10.8).

## The PW norms $(d = 4, k \geq 1)$

We fix $l, -k \leq l', k', \omega, \mathcal{C} = \mathcal{C}_{k'}(l')$ as before and use (4.10.7). We discard $P_{\mathcal{C}} P_l^\omega \bar{Q}^{\pm}_{<k+2l}$ and prove

$$2^{-\frac{3}{2}(k'+l')+k} \| e^{-i\psi}_{<k+2l}(t,x,D) \bar{Q}^{\pm}_{<k+2l} \tilde{P}_{\mathcal{C}} \phi \|_{PW^{\pm}_{\mathcal{C}}} \lesssim \| \tilde{P}_{\mathcal{C}} \nabla_{t,x} \phi(0) \|_{L^2_x} + \| \tilde{P}_{\mathcal{C}} \Box_m \phi \|_{\bar{N}_k}$$

Let's assume $\pm = +$. By Lemma 4.10.2, we reduce to

$$\|e^{-i\psi}_{<k+2l}(t,x,D)e^{it\langle D\rangle}\tilde{P}_{\mathcal{C}}u_k\|_{PW^{\pm}_{\mathcal{C}}} \lesssim 2^{\frac{3}{2}(k'+l')}\|\tilde{P}_{\mathcal{C}}u_k\|_{L^2_x} \tag{4.10.9}$$

From Corollary 4.6.6 and Lemma 4.10.1 we deduce that

$$2^{-\frac{3}{2}(k'-k)}e^{-i\psi}_{<k+2l}(t,x,D)e^{it\langle D\rangle}P_k P_{C^i_{k'}(-k)} : L^2_x \to L^2_{t_{\omega_i,\lambda}}L^\infty_{x_{\omega_i,\lambda}}$$

holds for $C^i_{k'}(-k) \subset \mathcal{C}$ where $\omega_i$ is the direction of the center of $C^i_{k'}(-k)$.

We can cover $\mathcal{C} = \mathcal{C}_{k'}(l')$ by roughly $2^{3(l'+k)}$ boxes of size $2^{k'} \times (2^{k'-k})^3$:

$$\mathcal{C} = \cup_{i=1}^{O(2^{3(l'+k)})} C^i_{k'}(-k).$$

Notice that $\lambda$ can be chosen the same for all $i$. By the definition of $PW^{\pm}_{\mathcal{C}}$ (2.1.8)

$$\text{LHS (4.10.9)} \leq \sum_i \|e^{-i\psi}_{<k+2l}(t,x,D)e^{it\langle D\rangle}P_{C^i_{k'}(-k)}u_k\|_{L^2_{t_{\omega_i,\lambda}}L^\infty_{x_{\omega_i,\lambda}}} \lesssim$$

$$\lesssim 2^{\frac{3}{2}(k'-k)}\sum_i \|\tilde{P}_{C^i_{k'}(-k)}u_k\|_{L^2_x} \lesssim 2^{\frac{3}{2}(k'-k)}2^{\frac{3}{2}(l'+k)}(\sum_i \|\tilde{P}_{C^i_{k'}(-k)}u_k\|^2_{L^2_x})^{\frac{1}{2}} \lesssim 2^{\frac{3}{2}(k'+l')}\|\tilde{P}_{\mathcal{C}}u_k\|_{L^2_x}$$

where we have used Cauchy-Schwarz and orthogonality. This proves (4.10.9).

## The NE norms ($d = 4, k \geq 1$)

We fix $l, -k \leq l', k', \mathcal{C} = \mathcal{C}_{k'}(l')$ as before and use (4.10.7). We prove

$$2^k\|P_{\mathcal{C}}P^{\omega}_l\bar{Q}^{\pm}_{<k+2l}e^{-i\psi}_{<k+2l}\bar{Q}^{\pm}_{<k+2l}\tilde{P}_{\mathcal{C}}\phi\|_{NE^{\pm}_{\mathcal{C}}} \lesssim \|\tilde{P}_{\mathcal{C}}\nabla_{t,x}\phi(0)\|_{L^2_x} + \|\tilde{P}_{\mathcal{C}}\Box_m\phi\|_{\bar{N}_k}$$

Now we split again $e^{-i\psi}_{<k+2l} = (e^{-i\psi}_{<k+2l} - e^{-i\psi}_{<k}) + e^{-i\psi}_{<k}$. The first term

$$P_{\mathcal{C}}P^{\omega}_l\bar{Q}^{\pm}_{<k+2l}(e^{-i\psi}_{<k+2l} - e^{-i\psi}_{<k})\bar{Q}^{\pm}_{<k+2l}\tilde{P}_{\mathcal{C}}\phi$$

is estimated by appropriately applying Prop. 2.2.5 and Cor. 4.8.2.

For the second term we may discard $P_{\mathcal{C}}P^{\omega}_l\bar{Q}^{\pm}_{<k+2l}$ and prove

$$2^k\|e^{-i\psi}_{<k}\bar{Q}^{\pm}_{<k+2l}\tilde{P}_{\mathcal{C}}\phi\|_{NE^{\pm}_{\mathcal{C}}} \lesssim \|\tilde{P}_{\mathcal{C}}\nabla_{t,x}\phi(0)\|_{L^2_x} + \|\tilde{P}_{\mathcal{C}}\Box_m\phi\|_{\bar{N}_k}.$$

This is reduced by Lemma 4.10.2 to

$$e^{-i\psi}_{<k}(t,x,D)e^{\pm it\langle D\rangle}\tilde{P}_{\mathcal{C}} : L^2_x \to NE^{\pm}_{\mathcal{C}},$$

which follows from Corollary 4.6.10.

# Chapter 5

# The core null and bilinear forms

The rest of the thesis is concerned with the proofs of the estimates from Section 1.8. This chapter is devoted to preparing some preliminaries regarding translation-invariant bilinear forms that play a role in those proofs. We discuss the classical $\mathcal{N}_0$, $\mathcal{N}_{ij}$ and spinorial null forms, how to adapt $\mathcal{N}_0$ to the Klein-Gordon equation, the geometry of frequency interactions as well as some refinements of Hölder's inequality.

Let $L$ be a bilinear operator on $\mathbb{R}^d$ or $\mathbb{R}^{1+d}$ with symbol $m(\xi_1, \xi_2)$, respectively $m(\Xi^1, \Xi^2)$ (which is possibly a distribution), i.e.,

$$L(f_1, f_2)(x) = \int e^{ix\cdot(\xi_1+\xi_2)} m(\xi_1, \xi_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \frac{\mathrm{d}\xi_1 \, \mathrm{d}\xi_2}{(2\pi)^{2d}}.$$

The translation-invariant operator

$$\mathcal{L}(f, g)(x) = \int K(x - y_1, x - y_2) f(y_1) g(y_2) \, \mathrm{d}y_1 \mathrm{d}y_2$$

can be written in this form by defining

$$m(\xi_1, \xi_2) = \hat{K}(\xi_1, \xi_2).$$

Conversely, $L$ can be written in the form (1.5), if we ensure that $K \in L^1$ or that it has bounded mass. Some important examples will be provided below.

To understand $L(f_1, f_2)$, we may consider the 'dualized' expression

$$\iint f_0 L(f_1, f_2) \, \mathrm{d}t\mathrm{d}x = \int_{\{\Xi^0+\Xi^1+\Xi^2=0\}} m(\Xi^1, \Xi^2) \hat{f}_0(\Xi^0) \hat{f}_1(\Xi^1) \hat{f}_2(\Xi^2) \frac{\mathrm{d}\Xi^1 \, \mathrm{d}\Xi^2}{(2\pi)^{2(d+1)}}. \tag{5.0.1}$$

## 5.1 The $\mathcal{M}$ and $\mathcal{M}_0$ forms

### The $\mathcal{M}$ form

During the proof of the trilinear estimate, we will need to consider terms like

$$P_{k'}Q_j\mathcal{M}(\bar{Q}_{<j}\phi^1_{k_1}, \bar{Q}_{<j}\phi^2_{k_2})$$

where

$$\mathcal{M}(\phi^1, \phi^2) := \partial_\alpha(\phi^1 \cdot \partial^\alpha \phi^2) \tag{5.1.1}$$

is a null-form adapted to the wave equation, while $\phi^1_{k_1}, \phi^2_{k_2}$ are assumed to be high-frequency Klein-Gordon waves of low $\bar{Q}$-modulation, with low frequency output.

To obtain effective bounds, we need to split

$$\mathcal{M} = \mathcal{R}^{\pm}_0 + \mathcal{M}_0 - \mathcal{N}_0 \tag{5.1.2}$$

where, denoting $\Xi^i = (\tau_i, \xi_i)$, the symbols of $\mathcal{M}, \mathcal{R}^{\pm}_0, \mathcal{M}_0, \mathcal{N}_0$ are

$$m(\Xi^1, \Xi^2) = (\tau_1 + \tau_2)\tau_2 - (\xi_1 + \xi_2) \cdot \xi_2, \tag{5.1.3}$$

and, respectively,

$$r^{\pm}_0(\Xi^1, \Xi^2) := \tau_1(\tau_2 \pm \langle \xi_2 \rangle) + (\langle \xi_1 \rangle \mp \tau_1) \langle \xi_2 \rangle + (\tau_2^2 - \langle \xi_2 \rangle^2), \tag{5.1.4}$$

$$m_0(\Xi^1, \Xi^2) := 1 + |\xi_1| |\xi_2| - \langle \xi_1 \rangle \langle \xi_2 \rangle, \tag{5.1.5}$$

$$n_0(\Xi^1, \Xi^2) := |\xi_1| |\xi_2| + \xi_1 \cdot \xi_2. \tag{5.1.6}$$

### The $\mathcal{M}_0$ form

Let $\mathcal{M}_0(\phi^1, \phi^2)$ be the bilinear form with symbol

$$m_0(\xi_1, \xi_2) = 1 + |\xi_1| |\xi_2| - \langle \xi_1 \rangle \langle \xi_2 \rangle .$$

Notice that this multiplier is a radial function in $\xi_1$ and $\xi_2$.

The following two statements are aimed at obtaining an exponential gain for $\mathcal{M}_0$ in the high $\times$ high $\to$ low frequency interactions.

**Lemma 5.1.1.** *The following bounds hold:*

$$\begin{aligned}
|m_0(\xi_1, \xi_2)| &\leq \frac{|\xi_1 + \xi_2|^2}{\langle \xi_1 \rangle \langle \xi_2 \rangle} \\
|\partial_{\xi_i} m_0(\xi_1, \xi_2)| &\leq \frac{|\xi_1 + \xi_2|}{\langle \xi_i \rangle} \Big( \frac{1}{\langle \xi_1 \rangle} + \frac{1}{\langle \xi_2 \rangle} \Big), \qquad i = 1, 2 \\
\left| \partial^\alpha_{\xi_i} m_0(\xi_1, \xi_2) \right| &\lesssim \frac{\langle \xi_1 \rangle \langle \xi_2 \rangle}{\langle \xi_i \rangle^{|\alpha| + 2}}, \qquad |\alpha| \geq 2, \ i = 1, 2.
\end{aligned}$$

We return to the proof of this lemma after the following proposition which provides an exponential gain needed for estimate (6.2.15).

**Proposition 5.1.2.** *Let $k \geq 0$, $k' \leq k - C$ and $1 \leq p, q_1, q_2 \leq \infty$ with $p^{-1} = q_1^{-1} + q_2^{-1}$. Let $\mathcal{C}_1, \mathcal{C}_2$ be boxes of size $\simeq (2^{k'})^d$ located to $\mathcal{C}_i \subset \{\langle \xi_i \rangle \simeq 2^k\}$ so that*

$$\mathcal{C}_1 + \mathcal{C}_2 \subset \{|\xi| \leq 2^{k'+2}\}$$

*Then, for all functions $\phi_1, \phi_2$ with Fourier support in $\mathcal{C}_1, \mathcal{C}_2$ we have*

$$\|\mathcal{M}_0(\phi^1, \phi^2)\|_{L^p} \lesssim 2^{2(k'-k)} \|\phi_1\|_{L^{q_1}} \|\phi_2\|_{L^{q_2}}. \tag{5.1.7}$$

*Proof.* We expand $m_0(\xi_1, \xi_2)$ as a rapidly decreasing sum of tensor products

$$m_0(\xi_1, \xi_2) = \sum_{\mathbf{j},\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{j},\mathbf{k}} \, a^1_{\mathbf{j}}(\xi_1) a^2_{\mathbf{k}}(\xi_2) \quad \text{for } (\xi_1, \xi_2) \in \mathcal{C}_1 \times \mathcal{C}_2 \tag{5.1.8}$$

where, denoting $\mu = 2^{2(k'-k)}$, for any $n \geq 0$, $c_{\mathbf{j},\mathbf{k}}$ obeys

$$|c_{\mathbf{j},\mathbf{k}}| \lesssim_n \mu (1 + |\mathbf{j}| + |\mathbf{k}|)^{-n}, \tag{5.1.9}$$

and for some universal constant $n_0 > 0$, the $a^i_{\mathbf{j}}$ satisfy

$$\|a^i_{\mathbf{j}}(D)\|_{L^q \to L^q} \lesssim (1 + |\mathbf{j}|)^{n_0}, \qquad i = 1, 2. \tag{5.1.10}$$

Assuming (5.1.8)–(5.1.10), the desired estimate (5.1.7) follows immediately. Indeed, (5.1.8) implies that

$$\mathcal{M}_0(\phi_1, \phi_2) = \sum_{\mathbf{j},\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{j},\mathbf{k}} \cdot a^1_{\mathbf{j}}(D)\phi_1 \cdot a^2_{\mathbf{k}}(D)\phi_2,$$

so (5.1.7) follows by applying Hölder's inequality and (5.1.10), then using (5.1.9) to sum up in $\mathbf{j}, \mathbf{k} \in \mathbb{Z}^d$.

Let the boxes $\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2$ be enlargements of $\mathcal{C}_1, \mathcal{C}_2$ of size $\simeq (2^{k'})^d$ and let $\chi_1, \chi_2$ be bump functions adapted to these sets which are equal to 1 on $\mathcal{C}_1$, respectively $\mathcal{C}_2$.

Then for $(\xi_1, \xi_2) \in \mathcal{C}_1 \times \mathcal{C}_2$, we have $m_0(\xi_1, \xi_2) = m_0(\xi_1, \xi_2)\chi_1(\xi_1)\chi_2(\xi_2)$. Performing a Fourier series expansion of $m_0(\xi_1, \xi_2)\chi_1(\xi_1)\chi_2(\xi_2)$ by viewing $\tilde{\mathcal{C}}_1 \times \tilde{\mathcal{C}}_2$ as a torus, we may write

$$m_0(\xi_1, \xi_2) = \sum_{\mathbf{j},\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{j},\mathbf{k}} \, e^{2\pi i \mathbf{j} \cdot \xi_1'/2^{k'+c}} e^{2\pi i \mathbf{k} \cdot \xi_2'/2^{k'+c}} \quad \text{for } (\xi_1, \xi_2) \in \mathcal{C}_1 \times \mathcal{C}_2. \tag{5.1.11}$$

for $\xi_i' = \xi_i - \xi_i^0$ where $\xi_i^0$ is the center of $\mathcal{C}_i$. Defining

$$a^i_{\mathbf{j}}(\xi) = \chi_i(\xi_i) e^{2\pi i \mathbf{j} \cdot \xi_1'/2^{k'+c}}, \qquad i = 1, 2,$$

we obtain the desired decomposition (5.1.8) from (5.1.11).

To prove (5.1.9), we use the Fourier inversion formula

$$c_{\mathbf{j},\mathbf{k}} = \frac{1}{\mathrm{Vol}\ (\tilde{\mathcal{C}}_1 \times \tilde{\mathcal{C}}_2)} \int_{\tilde{\mathcal{C}}_1 \times \tilde{\mathcal{C}}_2} m_0(\xi_1^0 + \xi_1', \xi_2^0 + \xi_2') \chi_1 \chi_2 e^{-2\pi i (\mathbf{j}\cdot\xi_1' + \mathbf{k}\cdot\xi_2')/2^{k'+c}} \mathrm{d}\xi_1' \, \mathrm{d}\xi_2'.$$

By Lemma 5.1.1, for $(\xi_1, \xi_2) \in \mathcal{C}_1 \times \mathcal{C}_2$, since $|\xi_1 + \xi_2| \lesssim 2^{k'}$, for any $|\alpha| \geq 0$ we have

$$\left| (2^{k'} \partial_{\xi_i})^\alpha m_0(\xi_1, \xi_2) \right| \lesssim \mu, \qquad i = 1, 2$$

Thus, integrating by parts in $\xi_1'$ [resp. in $\xi_2'$], we obtain

$$|c_{\mathbf{j},\mathbf{k}}| \lesssim_n \mu (1 + |\mathbf{j}|)^{-n}, \qquad |c_{\mathbf{j},\mathbf{k}}| \lesssim_n \mu (1 + |\mathbf{k}|)^{-n}, \quad n \geq 0.$$

These bounds imply (5.1.9). Next, we have

$$\left| (2^{k'} \partial_{\xi_i})^\alpha a^i_{\mathbf{j}}(\xi_i) \right| \lesssim (1 + |\mathbf{j}|)^{|\alpha|}, \qquad |\alpha| \geq 0, \ i = 1, 2$$

This implies that the convolution kernel of $a^i_{\mathbf{j}}(D_i)$ satisfies $\|\check{a}^i_{\mathbf{j}}\|_{L^1} \lesssim (1 + |\mathbf{j}|)^{n_0}$ for $n_0 = d + 1$, which gives (5.1.10) □

*Proof of Lemma 5.1.1.* The bounds follow from elementary computations. Indeed,

$$-m_0(\xi_1, \xi_2) = \frac{(|\xi_1| - |\xi_2|)^2}{1 + |\xi_1| \, |\xi_2| + \langle \xi_1 \rangle \langle \xi_2 \rangle} \leq \frac{|\xi_1 + \xi_2|^2}{\langle \xi_1 \rangle \langle \xi_2 \rangle}.$$

Next, wlog assume $i = 1$. Since $m_0$ is radial in $\xi_1$ it suffices to compute

$$\partial_{|\xi_1|} m_0(\xi_1, \xi_2) = \frac{1}{\langle \xi_1 \rangle} \big( \langle \xi_1 \rangle |\xi_2| - \langle \xi_2 \rangle |\xi_1| \big) = \frac{1}{\langle \xi_1 \rangle} \frac{|\xi_2|^2 - |\xi_1|^2}{\langle \xi_1 \rangle |\xi_2| + \langle \xi_2 \rangle |\xi_1|}$$

which gives the desires bound.

Finally, the estimate for higher derivatives follows from $|\partial_r^n \langle r \rangle| \lesssim \langle r \rangle^{-n-1}$ for $n \geq 2$, which is straightforward to prove by induction. □

## 5.2 The $\mathcal{N}_0$ and $\tilde{\mathcal{N}}_0$ forms

We consider the bilinear forms $\tilde{\mathcal{N}}_0(\phi^1, \phi^2)$ on $\mathbb{R}^{d+1}$ with symbol

$$\tilde{n}(\Xi^1, \Xi^2) = \frac{1}{|(\tau_1, \xi_1)|} \frac{1}{|(\tau_2, \xi_2)|} (\tau_1 \tau_2 - \xi_1 \cdot \xi_2) \tag{5.2.1}$$

and $\mathcal{N}_0(\phi^1, \phi^2)$ on $\mathbb{R}^d$ with symbol

$$n_0(\xi_1, \xi_2) = |\xi_1| \, |\xi_2| + \xi_1 \cdot \xi_2. \tag{5.2.2}$$

**Proposition 5.2.1.** *Let $k_1, k_2 \in \mathbb{Z}$, $l' \leq 0$, and signs $\pm_1, \pm_2$. Let $\kappa_1, \kappa_2$ be spherical caps of angle $\simeq 2^{l'}$ centered at $\omega_1, \omega_2$ such that $\angle(\pm_1\omega_1, \pm_2\omega_2) \lesssim 2^{l'}$. Let $X_1, X_2$ be translation-invariant spaces and $L$ be a translation-invariant bilinear operator. Suppose that*

$$\|L(\phi^1, \phi^2)\|_X \lesssim C_{S_1,S_2} \|\phi^1\|_{X_1} \|\phi^2\|_{X_2}$$

*holds for all $\phi_1, \phi_2$ which are Fourier-supported, respectively, in some subsets*

$$S_i \subset E_i := \{|\xi_i| \simeq 2^{k_i},\ |\tau_i \mp_i |\xi_i|| \lesssim 2^{k_i+2l'},\ \frac{\xi_i}{|\xi_i|} \in \kappa_i\}, \qquad i = 1, 2.$$

*Then one also has*

$$\|L(\partial_\alpha \phi^1, \partial^\alpha \phi^2)\|_X \lesssim 2^{2l'} C_{S_1,S_2} \|\nabla_{t,x}\phi^1\|_{X_1} \|\nabla_{t,x}\phi^2\|_{X_2} \tag{5.2.3}$$

*for all such $\phi_1, \phi_2$ .*

**Corollary 5.2.2.** *Under the conditions from Proposition 2.1.1, for $j \leq \min(k, k_2) + 2l' - C$ one has*

$$\|\partial^\alpha P_{\mathcal{C}} \bar{Q}^{\pm_1}_{<j} \phi_k \cdot \partial_\alpha P_{\mathcal{C}'} \bar{Q}^{\pm_2}_{<j} \varphi_{k_2}\|_{L^2_{t,x}} \lesssim 2^{l'} \|P_{\mathcal{C}} \bar{Q}^{\pm_1}_{<j} \nabla \phi_k\|_{NE^{\pm_1}_{\mathcal{C}}} \|P_{\mathcal{C}'} \bar{Q}^{\pm_2}_{<j} \nabla \varphi_{k_2}\|_{PW^{\pm_2}_{\mathcal{C}'}}$$

*Remark* 5.2.3. One may of course formulate analogues of Prop. 5.2.1 also for multilinear forms, such as the trilinear expressions $L(\phi^1, \partial_\alpha \phi^2, \partial^\alpha \phi^3)$ that occur in the proofs of (6.2.5), (6.2.6), (6.2.7). Checking that the same argument applies for them is straightforward and is left to the reader.

*Proof of Prop. 5.2.1.* **S**tep 1. Let $\ell(\Xi^1, \Xi^2)$ be the multiplier symbol of $L$. In (5.2.3) we have the operator with symbol $\ell(\Xi^1, \Xi^2)\tilde{n}(\Xi^1, \Xi^2)$ applied to $|D_{t,x}|\,\phi^1, |D_{t,x}|\,\phi^2$.

The idea is to perform a separation of variables in the form

$$\tilde{n}(\Xi^1, \Xi^2) = \sum_{\mathbf{j},\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{j},\mathbf{k}}\, a_{\mathbf{j}}(\Xi^1) b_{\mathbf{k}}(\Xi^2) \quad \text{ for } (\Xi^1, \Xi^2) \in E_1 \times E_2 \tag{5.2.4}$$

where for each $n \geq 0$ the coefficients obey

$$|c_{\mathbf{j},\mathbf{k}}| \lesssim_n 2^{2l'} (1 + |\mathbf{j}| + |\mathbf{k}|)^{-n}, \tag{5.2.5}$$

and for some universal constant $n_0 > 0$, the operators $a_{\mathbf{j}}$ and $b_{\mathbf{k}}$ satisfy

$$\|a_{\mathbf{j}}(D_{t,x})\|_{X_1 \to X_1} \lesssim (1 + |\mathbf{j}|)^{n_0}, \quad \|b_{\mathbf{k}}(D_{t,x})\|_{X_2 \to X_2} \lesssim (1 + |\mathbf{k}|)^{n_0}, \tag{5.2.6}$$

From these, (5.2.3) follows immediately.

We do a change of variables such that $\tau_i^\omega$ is the (essentially null vector) radial coordinate, $\tau_i^{\omega^\perp}$ is orthogonal to it, and $\xi_i'$ are angular type coordinates in the $\xi$ hyperplane, so that $|\xi_i'| \simeq 2^{k_i}\theta_i$ where $\theta_i$ are the angles between $\xi_i$ and the center of $\kappa_i$. We denote $\tilde{\Xi}_i = (\tau_i^\omega, \tau_i^{\omega^\perp}, \xi_i')$.

Denote by $\tilde{E}_i$ an enlargement of $E_i$, chosen be a rectangular region of size $\simeq 2^{k_i} \times 2^{k_i+2l'} \times (2^{k_i+l'})^{d-1}$ (consistently with the coordinates $(\tau_i^\omega, \tau_i^{\omega^\perp}, \xi_i')$). Let $\chi_i$ be a bump function adapted to $\tilde{E}_i$, which is equal to 1 on $E_i$.

**S**tep 2. We claim the following bounds for $(\Xi^1, \Xi^2) \in E_1 \times E_2$:

$$\left|\tilde{n}(\Xi^1, \Xi^2)\right| \lesssim 2^{2l'} \tag{5.2.7}$$

$$\left|\partial_{\xi_i'}\tilde{n}(\Xi^1, \Xi^2)\right| \lesssim 2^{-k_i}2^{l'}, \qquad i = 1, 2; \tag{5.2.8}$$

$$\left|\partial_{\Xi_i}^\alpha\tilde{n}(\Xi^1, \Xi^2)\right| \lesssim \left|\Xi^i\right|^{-|\alpha|}, \qquad i = 1, 2. \tag{5.2.9}$$

Recall (5.2.1). We write

$$\tau_1\tau_2 - \xi_1 \cdot \xi_2 = (\tau_1 \mp_1 |\xi_1|)\tau_2 \pm_1 |\xi_1| (\tau_2 \mp_2 |\xi_2|) \pm_1 \pm_2 |\xi_1| |\xi_2| \left(1 - \cos\angle(\pm_1\xi_1, \pm_2\xi_2)\right)$$

which clearly implies (5.2.7). It is easy to see that

$$\left|\partial_{\xi_i'}\tilde{n}(\Xi^1, \Xi^2)\right| \lesssim 2^{-k_i}\sin\angle(\xi_1, \xi_2)$$

which implies (5.2.8), while (5.2.9) follows from the fact that $\tilde{n}$ is homogeneous in both $\Xi^1, \Xi^2$.

**S**tep 3. Performing a Fourier series expansion of $\tilde{n}(\tilde{\Xi_1}, \tilde{\Xi_2})\chi_1(\tilde{\Xi_1})\chi_2(\tilde{\Xi_2})$ by viewing $\tilde{E}_1 \times \tilde{E}_2$ as a torus, we may write

$$\tilde{n}(\tilde{\Xi_1}, \tilde{\Xi_2}) = \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{j},\mathbf{k}}\, e^{2\pi i\mathbf{j}\cdot D_1\tilde{\Xi_1}}e^{2\pi i\mathbf{k}\cdot D_2\tilde{\Xi_2}} \quad \text{for } (\tilde{\Xi_1}, \tilde{\Xi_2}) \in E_1 \times E_2, \tag{5.2.10}$$

where $D_1, D_2$ are diagonal matrices of the form

$$D_i = \operatorname{diag}\left(O(2^{-k_i}), O(2^{-k_i-2l'}), O(2^{-k_i-l'}), \ldots, O(2^{-k_i-l'})\right). \tag{5.2.11}$$

Defining

$$a_{\mathbf{j}}(\Xi_1) = (\chi_1(\tilde{\Xi_2})e^{2\pi i\mathbf{j}\cdot D_1\tilde{\Xi_1}})(\Xi_1), \quad b_{\mathbf{k}}(\Xi_2) = (\chi_2(\tilde{\Xi_2})e^{2\pi i\mathbf{k}\cdot D_2\tilde{\Xi_2}})(\Xi_2),$$

we obtain the desired decomposition (5.2.4) from (5.2.10).

To prove (5.2.5), by the Fourier inversion formula

$$c_{\mathbf{j},\mathbf{k}} = \frac{1}{\operatorname{Vol}(\tilde{E}_1 \times \tilde{E}_2)}\int_{\tilde{E}_1\times\tilde{E}_2} n(\tilde{\Xi_1}, \tilde{\Xi_2})\chi_1(\tilde{\Xi_1})\chi_2(\tilde{\Xi_2})e^{-2\pi i\mathbf{j}\cdot D_1\tilde{\Xi_1}}e^{-2\pi i\mathbf{k}\cdot D_2\tilde{\Xi_2}}\, d\tilde{\Xi_1}\, d\tilde{\Xi_2}.$$

Integrating by parts w.r.t. to $\tau_i^\omega$ by the homogeneity of $\tilde{n}$ and (5.2.7) we obtain

$$|c_{\mathbf{j},\mathbf{k}}| \lesssim_n 2^{2l'}(1 + |\mathbf{j}_1|)^{-n} \quad [\text{resp. } |c_{\mathbf{j},\mathbf{k}}| \lesssim_n 2^{2l'}(1 + |\mathbf{k}_1|)^{-n}],$$

for any $n \geq 0$. On the other hand, for any $j = 2, \dots, d+1$, integration by parts in $\tau_i^{\omega^\perp}$ or in $\xi_i'$ and using (5.2.7)-(5.2.9) yields

$$|c_{\mathbf{j},\mathbf{k}}| \lesssim_n 2^{2l'} |\mathbf{j}_j|^{-n} \quad \text{[resp. } |c_{\mathbf{j},\mathbf{k}}| \lesssim_n 2^{2l'} |\mathbf{k}_j|^{-n}\text{]}.$$

The preceding bounds imply (5.2.5) as desired.

Finally, we need to establish (5.2.6). We will describe the case of $a_{\mathbf{j}}(D)$. Consider the differential operators

$$D_{\omega_1} = (2^{k_1} \partial_{\tau_i^\omega}, 2^{k_1+2l'} \partial_{\tau_i^{\omega^\perp}}, 2^{k_1+l'} \partial_{\xi_1'})$$

For any multi-index $\alpha$, observe that

$$|D_{\omega_1}^\alpha (\chi_1(\tilde{\Xi}_1) e^{2\pi i \mathbf{j} \cdot D_1 \tilde{\Xi}_1})| \lesssim_\alpha (1 + |\mathbf{j}|)^{|\alpha|}.$$

From this bound, it is straightforward to check that the convolution kernel of $a_{\mathbf{j}}(D)$ obeys $\|\check{a}_{\mathbf{j}}\|_{L^1} \lesssim (1 + |\mathbf{j}|)^{n_0}$ for some universal constant $n_0$, which implies the bound (5.2.6) for $a_{\mathbf{j}}(D)$. □

*Proof of Corollary 5.2.2.* The corollary follows from Prop. 5.2.1 and Prop. 2.1.1. Indeed, with $k = k_1, k_2 = k_2$ we take $C_{S_1,S_2} = 2^{-l'}$ with

$$S_1 = \{(\tau_1, \xi_1) \mid \xi_1 \in \mathcal{C}, \ |\xi_1| \simeq 2^k, \ |\tau_1 \mp_1 \langle \xi_1 \rangle| \lesssim 2^j\}$$

and $S_2$ defined analogously. We check that $S_i \subset E_i$. The condition (2.1.9) insures that we can define $\kappa_1, \kappa_2$ appropriately. It remains to verify

$$|\tau_i \mp_i |\xi_i|| \leq |\tau_i \mp_i \langle \xi_i \rangle| + \langle \xi_i \rangle - |\xi_i| \lesssim 2^j + 2^{-k_i} \lesssim 2^{k_i+2l'}$$

by the condition on $j$ and (2.1.9). □

If we replace $\tau_i$ by $\pm |\xi_i|$ in (5.2.1) we remove the time dependence in Prop. 5.2.1 and may formulate a spatial analogue for the bilinear form defined by $|\xi_1| |\xi_2| \pm \xi_1 \cdot \xi_2$. We consider the + case for $\mathcal{N}_0(\phi_1, \phi_2)$ in (5.2.2), which will be useful for high × high → low frequency interactions.

**Proposition 5.2.4.** *Let $k \in \mathbb{Z}$, $l \leq 0$ and $1 \leq p, q_1, q_2 \leq \infty$ with $p^{-1} = q_1^{-1} + q_2^{-1}$. Let $\kappa_1, \kappa_2$ be spherical caps of angle $\simeq 2^l$ such that $\angle(\kappa_1, -\kappa_2) \lesssim 2^l$.*

*Then, for all functions $\phi_1, \phi_2$ with Fourier support, respectively, in $\{|\xi_i| \simeq 2^k, \ \xi_i / |\xi_i| \in \kappa_i\}$, $i = 1, 2$, we have*

$$\|\mathcal{N}_0(\phi^1, \phi^2)\|_{L^p} \lesssim 2^{2l+2k} \|\phi_1\|_{L^{q_1}} \|\phi_2\|_{L^{q_2}}.$$

*Proof.* The proof is very similar to the proof of Prop. 5.2.1 and is omitted. The basic difference is that here one performs the Fourier series expansion on a $\left(2^k \times (2^{k+l})^{d-1}\right)^2$-sized region in $\mathbb{R}^d_\xi \times \mathbb{R}^d_\xi$ instead of $\mathbb{R}^{d+1}_{\tau,\xi} \times \mathbb{R}^{d+1}_{\tau,\xi}$. □

## 5.3 Abstract null forms

To unify the treatment of various null forms that arise in MKG and MD, we consider the following proposition.

**Proposition 5.3.1.** *Let $N$ be a bilinear form with symbol $m(\xi,\eta)$ assumed to be homogeneous of degree 0 in $\xi,\eta$ and to obey*

$$|m(\xi,\eta)| \leq A\, |\angle(\xi,\eta)|\,.$$

*Let $\omega_1,\omega_2 \subset \mathbb{S}^{d-1}$ be angular caps of radius $|r_i| \leq 2^{-10}$, $i=1,2$ and define $\theta := \max\{\angle(|\omega_1,\omega_2)|\,, r_1, r_2\}$. Let $1 \leq p, q_1, q_2 \leq \infty$ be such that $p^{-1} = q_1^{-1} + q_2^{-1}$. Let the functions $f_1, f_2$ be defined on $\mathbb{R}^d$ with Fourier support in*

$$\{|\xi| \simeq 2^{k_i},\ \frac{\xi}{|\xi|} \in \omega_i\}, \qquad i=1,2.$$

*Then we have*

$$\|N(f_1,f_2)\|_{L^p} \lesssim \theta \|f_1\|_{L^{q_1}} \|f_2\|_{L^{q_2}}. \tag{5.3.1}$$

*Remark* 5.3.2. Under the assumptions of Prop. 5.3.1 we have the following bounds:

$$|S_\xi^{n_1} S_\eta^{n_2} m(\xi,\eta)| \leq A_{n_1,n_2} |\angle(\xi,\eta)| \tag{5.3.2}$$

$$|\partial_\xi^{\alpha_1} \partial_\eta^{\alpha_2} m(\xi,\eta)| \leq A_{\alpha_1,\alpha_2} |\xi|^{-|\alpha_1|} |\eta|^{-|\alpha_2|} \tag{5.3.3}$$

where $S_\xi = \xi \cdot \partial_\xi$ and $S_\eta = \eta \cdot \partial_\eta$. Under these conditions, we shall call $N$ an *abstract null form*.

*Proof.* The idea of the proof is to perform a separation of variables to write the symbol $m(\xi,\eta)$ of $N$ in the form

$$m(\xi,\eta) = \sum_{\mathbf{j},\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{j},\mathbf{k}}\, a_{\mathbf{j}}(\xi) b_{\mathbf{k}}(\eta) \quad \text{ for } (\xi,\eta) \in E_1 \times E_2 \tag{5.3.4}$$

where for each integer $n \geq 0$ the coefficient $c_{\mathbf{j},\mathbf{k}}$ obeys

$$|c_{\mathbf{j},\mathbf{k}}| \lesssim_n \theta (1 + |\mathbf{j}| + |\mathbf{k}|)^{-n}, \tag{5.3.5}$$

and for some universal constant $n_0 > 0$, the quantizations of the symbols $a_{\mathbf{j}}$ and $b_{\mathbf{k}}$ satisfy

$$\|a_{\mathbf{j}}(D)\|_{L^q \to L^q} \lesssim (1+|\mathbf{j}|)^{n_0}, \quad \|b_{\mathbf{k}}(D)\|_{L^q \to L^q} \lesssim (1+|\mathbf{k}|)^{n_0}, \tag{5.3.6}$$

for every $1 \leq q \leq \infty$.

Assuming (5.3.4)–(5.3.6), the desired estimate (5.3.1) follows immediately. Indeed, (5.3.4) implies that

$$N(f_1,f_2) = \sum_{\mathbf{j},\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{j},\mathbf{k}} \cdot a_{\mathbf{j}}(D) f_1 \cdot b_{\mathbf{k}}(D) f_2,$$

so (5.3.1) follows by applying Hölder's inequality and (5.3.6), then using (5.3.5) to sum up in $\mathbf{j}, \mathbf{k} \in \mathbb{Z}^d$.

Without loss of generality, we may assume that $k_1 \geq k_2$ and that $\omega_1, \omega_2$ are angular caps of an equal diameter, denoted by $r$. Moreover, in view of the scaling invariance of the bounds (5.3.2) and (5.3.3), we may set $k_1 = 0$. Let $\hat{E}_j$ be an enlargement of $E_j$ $(j = 1, 2)$ with a *fixed* angular dimension and let $a(\xi), b(\eta)$ be bump function adapted to these sets, which are equal to 1 on $E_1$, respectively $E_2$, so that $\hat{f}_1 = a\hat{f}_1$ and $\hat{f}_2 = b\hat{f}_2$. Then

$$N(f_1, f_2) = N^{m'}(f_1, f_2),$$

where $N^{m'}$ is the bilinear operator with symbol $m'(\xi, \eta) = a(\xi)b(\eta)m(\xi, \eta)$.

The first step is to make an invertible change of variables $\xi \mapsto \tilde{\xi} = \tilde{\xi}(\xi)$, so that $S_\xi = \tilde{\xi}_1 \partial_{\tilde{\xi}_1}$ and the Jacobian and its derivatives obey appropriate bounds of all order for $\xi \in \hat{E}_1$. We also need to perform a similar change of variables $\eta \mapsto \tilde{\eta}(\eta)$ for $\eta \in \hat{E}_2$. Essentially, what we need is a polar coordinate system with the radial variable as the first component.

One concrete way to proceed is as follows. Denote the center of the angular cap $\omega_1$ by $\mathbf{p}_1 \in \mathbb{S}^{d-1}$. Let $(\zeta_2, \dots, \zeta_d) \in \mathbb{R}^{d-1}$ be a smooth positively oriented coordinate system on the hemisphere $\mathbb{S}^{d-1} \cap \{\xi : \mathbf{p}_1 \cdot \xi > 0\}$, such that $(\zeta_2, \dots, \zeta_d) = (0, \dots, 0)$ corresponds to $\mathbf{p}_1$. Define

$$\tilde{\xi}(\xi) = \Big(|\xi|, |\xi|\zeta_2\Big(\frac{\xi}{|\xi|}\Big), \dots, |\xi|\zeta_d\Big(\frac{\xi}{|\xi|}\Big)\Big) \qquad \text{for } \xi \in \{\xi : \mathbf{p}_1 \cdot \xi > 0\}.$$

We define $\tilde{\eta}(\eta)$ for $\eta \in \{\eta : \mathbf{p}_2 \cdot \eta > 0\}$ similarly, with the point $\mathbf{p}_1$ replaced by the center $\mathbf{p}_2$ of the cap $\omega_2$. Observe that $(\tilde{\xi}, \tilde{\eta})$ are well-defined and invertible on $\tilde{E}_1 \times \tilde{E}_2$, in which $m'$ is supported. Abusing the notation a bit, we write $m(\tilde{\xi}, \tilde{\eta}) = m(\xi(\tilde{\xi}), \eta(\tilde{\eta}))$ and simply $E_j$ for the region $\{\tilde{\xi} : \xi(\tilde{\xi}) \in E_j\}$ etc.

With such definitions, it is clear that $\tilde{\xi}_1 \partial_{\tilde{\xi}_1} = S_\xi$ and $\tilde{\eta}_1 \partial_{\tilde{\eta}_1} = S_\eta$. Hence (5.3.2) translates to

$$|\partial_{\tilde{\xi}_1}^{n_1} \partial_{\tilde{\eta}_1}^{n_2} m(\tilde{\xi}, \tilde{\eta})| \lesssim_{A, n_1, n_2} \theta |\tilde{\xi}_1|^{-n_1} |\tilde{\eta}_1|^{-n_2}. \tag{5.3.7}$$

Moreover, since each component of $\tilde{\xi} \in \mathbb{R}^d$ [resp. $\tilde{\eta}$] is homogeneous of degree 1 in $\xi$ [resp. in $\eta$], we immediately have the bounds

$$|\partial_\xi^\alpha \tilde{\xi}(\xi)| \lesssim_\alpha |\xi|^{1-|\alpha|} \quad [\text{resp. } |\partial_\eta^\alpha \tilde{\eta}(\eta)| \lesssim_\alpha |\eta|^{1-|\alpha|}] \quad \text{for any multi-index } \alpha. \tag{5.3.8}$$

Observe that we have $|\tilde{\xi}(\xi)| \simeq |\xi|$ for $\xi \in \hat{E}_1$ [resp. $|\tilde{\eta}(\eta)| \simeq |\eta|$ on $\eta \in \hat{E}_2$]. Further straightforward computations using (5.3.8) show that

$$|\partial_{\tilde{\xi}}^\alpha \xi(\tilde{\xi})| \lesssim_\alpha |\tilde{\xi}|^{1-|\alpha|} \quad [\text{resp. } |\partial_{\tilde{\eta}}^\alpha \eta(\tilde{\eta})| \lesssim_\alpha |\tilde{\eta}|^{1-|\alpha|}] \quad \text{for any multi-index } \alpha, \tag{5.3.9}$$

for $(\xi(\tilde{\xi}), \eta(\tilde{\eta})) \in \hat{E}_1 \times \hat{E}_2$. Combined with (5.3.3) and the support property of $m$, we have

$$|\partial_{\tilde{\xi}}^{\alpha_1} \partial_{\tilde{\eta}}^{\alpha_2} m(\tilde{\xi}, \tilde{\eta})| \lesssim_{A, \alpha_1, \alpha_2} |\tilde{\xi}|^{-|\alpha_1|} |\tilde{\eta}|^{-|\alpha_2|}. \tag{5.3.10}$$

We now introduce rectangular boxes $R_1$ and $R_2$, which are defined as

$$R_1 = \{\tilde{\xi} : \tilde{\xi}_1 \simeq 1, \sup_{j=2,\dots,d} |\tilde{\xi}_j| \lesssim r\}, \quad R_2 = \{\tilde{\eta} : \tilde{\eta}_1 \simeq 2^{k_2}, \sup_{j=2,\dots,d} |\tilde{\eta}_j| \lesssim 2^{k_2} r\},$$

where the implicit constants are chosen so that $E_1 \subseteq R_1$ and $E_2 \subseteq R_2$. Let $\tilde{a}(\tilde{\xi})$ and $\tilde{b}(\tilde{\eta})$ be the bump functions adapted to the boxes $R_1$ and $R_2$, respectively such that $\tilde{a}$ and $\tilde{b}$ are equal to 1 on $E_1$ and $E_2$, respectively.

Thus we have the following bounds for $j = 2, \dots d$:

$$|(r\partial_{\tilde{\xi}_j})^n m(\tilde{\xi}, \tilde{\eta})\tilde{a}(\tilde{\xi})\tilde{b}(\tilde{\eta})| \lesssim_{A,n} \theta, \quad |(2^{k_2} r\partial_{\tilde{\eta}_j})^n m(\tilde{\xi}, \tilde{\eta})\tilde{a}(\tilde{\xi})\tilde{b}(\tilde{\eta})| \lesssim_{A,n} \theta \tag{5.3.11}$$

Performing a Fourier series expansion of $m(\tilde{\xi}, \tilde{\eta})\tilde{a}(\tilde{\xi})\tilde{b}(\tilde{\eta})$ in the variables $(\tilde{\xi}, \tilde{\eta})$ by viewing $R_1 \times R_2$ as a torus, we may write

$$m(\tilde{\xi}, \tilde{\eta}) = \sum_{\mathbf{j},\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{j},\mathbf{k}}\, e^{2\pi i \mathbf{j}\cdot D_1 \tilde{\xi}} e^{2\pi i \mathbf{k}\cdot D_2 \tilde{\eta}} \quad \text{for } (\tilde{\xi}, \tilde{\eta}) \in E_1 \times E_2, \tag{5.3.12}$$

where $D_1, D_2$ are diagonal matrices of the form

$$\begin{aligned} D_1 =& \operatorname{diag}(O(1), O(r^{-1}), \dots, O(r^{-1})), \\ D_2 =& \operatorname{diag}(O(2^{-k_2}), O(2^{-k_2} r^{-1}), \dots, O(2^{-k_2} r^{-1})). \end{aligned}$$

Defining

$$a_{\mathbf{j}}(\xi) = (\tilde{a}(\tilde{\xi}) e^{2\pi i \mathbf{j}\cdot D_1 \tilde{\xi}})(\xi), \quad b_{\mathbf{k}}(\eta) = (\tilde{b}(\tilde{\eta}) e^{2\pi i \mathbf{k}\cdot D_2 \tilde{\eta}})(\eta),$$

we obtain the desired decomposition (5.3.4) from (5.3.12).

To prove (5.3.5), we begin with the following formula for the Fourier coefficient $c_{\mathbf{j},\mathbf{k}}$:

$$c_{\mathbf{j},\mathbf{k}} = \frac{1}{\operatorname{Vol}(R_1 \times R_2)} \int_{R_1 \times R_2} m(\tilde{\xi}, \tilde{\eta})\tilde{a}(\tilde{\xi})\tilde{b}(\tilde{\eta}) e^{-2\pi i \mathbf{j}\cdot D_1 \tilde{\xi}} e^{-2\pi i \mathbf{k}\cdot D_2 \tilde{\eta}}\, \mathrm{d}\tilde{\xi}\, \mathrm{d}\tilde{\eta}.$$

Integrating by parts in $\tilde{\xi}_1$ [resp. in $\tilde{\eta}_1$] and using (5.3.7), we obtain

$$|c_{\mathbf{j},\mathbf{k}}| \lesssim_n \theta(1 + |\mathbf{j}_1|)^{-n} \quad [\text{resp. } |c_{\mathbf{j},\mathbf{k}}| \lesssim_n \theta(1 + |\mathbf{k}_1|)^{-n}],$$

for each integer $n \geq 0$. On the other hand, for any $j = 2, \dots, d$, integration by parts in $\tilde{\xi}_j$ [resp. in $\tilde{\eta}_j$] and using (5.3.11) yields

$$|c_{\mathbf{j},\mathbf{k}}| \lesssim_n \theta|\mathbf{j}_j|^{-n} \quad [\text{resp. } |c_{\mathbf{j},\mathbf{k}}| \lesssim_n \theta|\mathbf{k}_j|^{-n}],$$

The preceding bounds imply (5.3.5) as desired.

Finally, we need to establish (5.3.6). We will describe the case of $a_{\mathbf{j}}(D)$ in detail, and leave the similar proof for $b_{\mathbf{k}}(\eta)$ to the reader. For any multi-index $\alpha$, observe that

$$|\partial_{\tilde{\xi}}^{\alpha}(\tilde{a}(\tilde{\xi}) e^{2\pi i \mathbf{j}\cdot D_1 \tilde{\xi}})| \lesssim_\alpha (1 + |\mathbf{j}|)^{|\alpha|} r^{-(\alpha_2 + \dots + \alpha_d)}. \tag{5.3.13}$$

By rotation, we may assume that the center of $\omega_1$ is aligned with the $\xi_1$-axis, i.e., $\mathbf{p}_1 = (1, 0, \dots, 0)$. Then we claim that

$$|\partial_\xi^\alpha a_{\mathbf{j}}(\xi)| \lesssim_\alpha (1 + |\mathbf{j}|)^{|\alpha|} r^{-(\alpha_2 + \cdots + \alpha_d)}. \tag{5.3.14}$$

From such a bound, it is straightforward to check that the convolution kernel (i.e., inverse Fourier transform) $\check{a}_{\mathbf{j}}(x)$ of $a_{\mathbf{j}}(D)$ obeys $\|\check{a}_{\mathbf{j}}\|_{L^1} \lesssim (1 + |\mathbf{j}|)^{n_0}$ for some universal constant $n_0$ (in fact, $n_0 = d$ would work), which implies the desired $L^q$ bounds (5.3.6) for $a_{\mathbf{j}}(D)$.

In order to verify (5.3.14), the key is to ensure that each $\partial_{\xi_1}$ derivative does not lose a factor of $r^{-1}$. Recall that $\tilde{\xi}_j = |\xi| \zeta_j(\xi/|\xi|)$ for $j = 2, \dots, d$. Observe that $\partial_{\xi_1}^n \zeta_j(\xi/|\xi|) \restriction_{\xi = \mathbf{p}_1} = 0$ for every $n \geq 0$ (in fact, $\zeta_j$ can be chosen to be independent of the first coordinate $\xi_1$ everywhere on $\mathbb{S}^{d-1} \cap \{\xi_1 > 0\} \subseteq \mathbb{R}^d$). Therefore, we have

$$\Big| \frac{\partial^n \tilde{\xi}_j}{\partial \xi_1^n} \Big| \lesssim \sum_{i=0}^n \Big| \partial_{\xi_1}^i \zeta_j \Big( \frac{\xi}{|\xi|} \Big) \Big| \lesssim_n \mathrm{dist} \Big( \frac{\xi}{|\xi|}, \mathbf{p}_1 \Big) \lesssim r \quad \text{for every } n \geq 0, \ \xi \in \operatorname{supp} a_{\mathbf{j}}.$$

Let $c(\xi)$ be any smooth function. By an iteration of the chain rule $\partial_{\xi_1} = (\partial_{\xi_1} \tilde{\xi}_1) \partial_{\tilde{\xi}_1} + \sum_{j=2}^d (\partial_{\xi_1} \tilde{\xi}_j) \partial_{\tilde{\xi}_j}$, it follows that

$$|\partial_{\xi_1}^{\alpha_1} c(\xi)| \lesssim_{\alpha_1} \sum_{|\beta| \leq \alpha_1} r^{\beta_2 + \cdots + \beta_d} |(\partial_{\tilde{\xi}}^\beta c)(\xi)| \quad \text{for every } \alpha_1 \geq 0, \ \xi \in \operatorname{supp} a_{\mathbf{j}}.$$

Substituting $c(\xi) = \partial_{\xi_2}^{\alpha_2} \cdots \partial_{\xi_d}^{\alpha_d} a_{\mathbf{j}}(\xi)$ and using (5.3.8), (5.3.13), the desired bound (5.3.14) follows after a straightforward computation.

□

## 5.4 Null structures of MD and MKG in the Coulomb gauge

We begin with the null forms

$$\mathcal{N}_{ij}(\phi, \varphi) = \partial_i \phi \partial_j \varphi - \partial_j \phi \partial_i \varphi. \tag{5.4.1}$$

which arise in the MKG equation by writing

$$\mathcal{P}_j(\phi^1 \nabla_x \phi^2) = \Delta^{-1} \nabla^i \mathcal{N}_{ij}(\phi^1, \phi^2). \tag{5.4.2}$$

and, whenever $A_x$ is divergence free, since $A_j = \mathcal{P}_j A$ we can write

$$A^i \partial_i \phi = \sum \mathcal{N}_{ij} \big( \nabla_i \Delta^{-1} A_j, \phi \big). \tag{5.4.3}$$

To exploit these identities, we have the following corollary of Prop. 5.3.1.

**Corollary 5.4.1.** *Under the conditions on the support of $f_1, f_2$ from Prop. 5.3.1 we have*

$$\|\mathcal{N}_{ij}(f_1,f_2)\|_{L^p} \lesssim \theta \|\nabla_x f_1\|_{L^{q_1}} \|\nabla_x f_2\|_{L^{q_2}}. \tag{5.4.4}$$

*Proof.* This follows by writing $\mathcal{N}_{ij}(f_1,f_2) = N(|D|\,f_1, |D|\,f_2)$ for $N$ with symbol $n(\xi,\eta) = \frac{\xi_i}{|\xi|}\frac{\eta_j}{|\eta|} - \frac{\xi_j}{|\xi|}\frac{\eta_i}{|\eta|}$ and applying (5.3.1). □

We now recast the null structure of MD-CG in terms of abstract null forms. It is at this point that we may fully explain an important point discussed in the introduction, namely, how the spinorial nonlinearities $\mathcal{M}^S$ and $\mathcal{N}^S$ exhibit *more favorable* null structure compared to the Riesz transform parts $\mathcal{M}^R$ and $\mathcal{N}^R$. (see Remark 5.4.5).

We begin with some schematic definitions.

**Definition 5.4.2** (Symbols $\mathcal{N}$ and $\mathcal{N}_\pm$)**.** We denote by $\mathcal{N}_+$ an abstract null form (Remark 5.3.2 and Prop. 5.3.1), and by $\mathcal{N}_-$ a bilinear operator such that $(f,g) \mapsto \mathcal{N}_-(f,\overline{g})$ is an abstract null form. We call $\mathcal{N}_s$ an *abstract null form* of type $s \in \{+,-\}$. Denoting the symbol of $\mathcal{N}_s$ by $m_s$, note that it satisfies

$$|S_\xi^{k_1} S_\eta^{k_2} m_s(\xi,\eta)| \leq A_{s,k_1,k_2} |\angle(\xi, s\eta)|.$$

We write $\mathcal{N}$ for a bilinear operator which is an abstract null form of both types; in short, $\mathcal{N} = \mathcal{N}_+$ and $\mathcal{N}_-$.

**Definition 5.4.3** (Symbols $\mathcal{N}^*$ and $\mathcal{N}^*_\pm$)**.** For $s \in \{+,-\}$, we denote by $\mathcal{N}^*_s$ (called a *dual abstract null form* of type $s$) a bilinear operator such that

$$\int h\,\mathcal{N}^*_s(f,g)\,\mathrm{d}x = \int f\,\mathcal{N}_{-s}(h,g)\,\mathrm{d}x \tag{5.4.5}$$

for some abstract null form $\mathcal{N}_{-s}$ of type $-s$. We denote by $\mathcal{N}^*$ a bilinear operator which is a dual abstract null form of both types, i.e., $\mathcal{N}^* = \mathcal{N}^*_+$ and $\mathcal{N}^*_-$. (Note that the second input $g$ plays a special role in $\mathcal{N}^*$ and $\mathcal{N}^*_s$.)

We are now ready to describe the (bilinear) null structure of MD-CG in terms of abstract null forms.

**Proposition 5.4.4.** *The Maxwell nonlinearities $\mathcal{M}^S_s$, $\mathcal{M}^R$ have the null structure*

$$\mathcal{M}^S_{s_2}(\Pi_{s_1}\psi,\varphi) = \mathcal{P}_j\langle \Pi_{s_1}\psi, \Pi_{-s_2}\alpha_x\varphi\rangle = \mathcal{N}_{s_1 s_2}(\psi,\varphi), \tag{5.4.6}$$

$$\mathcal{M}^R(\psi,\varphi) = \mathcal{P}_j\langle \psi, \mathcal{R}_x\varphi\rangle = \mathcal{N}^*(\psi,\varphi). \tag{5.4.7}$$

*The Dirac nonlinearities $\mathcal{N}^S_s$, $\mathcal{N}^R$ have the null structure*

$$\Pi_{s_0}\mathcal{N}^S_{s_2}(A_x,f) = \Pi_{s_0}(a_j\Pi_{-s_2}(\alpha^j f)) = \mathcal{N}^*_{s_0 s_2}(A_x,f), \tag{5.4.8}$$

$$\mathcal{N}^R(A_x,\psi) = \mathcal{P}_j A_x \mathcal{R}^j\psi = \mathcal{N}(A_x,\psi). \tag{5.4.9}$$

*Proof.* Statements (5.4.6) and (5.4.8) follow from Lemma 1.7.5. To prove the remaining statements, we use (1.1.7) to compute

$$\begin{aligned}\mathcal{P}_j A_x \mathcal{R}^j \psi =&(\delta^{k\ell}\delta^{ji} - \delta^{j\ell}\delta^{ik})\mathcal{R}_k\mathcal{R}_\ell A_i \mathcal{R}_j \psi \\ =&\delta^{k\ell}\delta^{ij}(\mathcal{R}_k\mathcal{R}_\ell A_i\mathcal{R}_j\psi - \mathcal{R}_j\mathcal{R}_\ell A_i \mathcal{R}_k\psi) \\ =&\delta^{k\ell}\delta^{ij}\mathcal{N}_{kj}(\mathcal{R}_\ell A_i, \psi)\end{aligned}$$

where $\mathcal{N}_{kj}$ is a bilinear operator with symbol $|\xi|^{-1}|\eta|^{-1}(\xi_k\eta_j - \eta_j\xi_k)$. It is clear that each $\mathcal{N}_{kj}$ is an abstract null form of the form $\mathcal{N}$, which proves (5.4.7) and (5.4.9) (the former follows by duality). □

*Remark* 5.4.5. A crucial observation here is that the spinorial nonlinearities have a more favorable null structure compared to the Riesz transform counterparts. To see this, consider the Dirac nonlinearities $\mathcal{N}^R$ and $\mathcal{N}_s^S$ in the low-high interaction case, which is the worst frequency balance scenario:

$$\pi^R[A_x]\psi = \sum_k \sum_{k'<k-10} \mathcal{N}^R(P_{k'}A_x, P_k\psi), \quad \pi_s^S[A_x]\psi = \sum_k \sum_{k'<k-10} \mathcal{N}_s^S(P_{k'}A_x, P_k\psi).$$

Proposition 5.4.4 shows that $\mathcal{N}^R$ gains in the angle $\theta$ between (the Fourier variables of) $A_x$ and $\psi$, whereas $\mathcal{N}_s^S$ gains in the angle $\theta^*$ between $\psi$ and the output. In this frequency balance scenario, observe that $\theta^*$ is smaller than $\theta$. Indeed, for each fixed $k, k'$, the law of sines implies that $\theta^* \simeq 2^{k'-k}\theta$. This extra exponential high-low gain leads to the improved estimate (1.8.20) for $\pi_s^S[A_x]$, which fails for $\pi^R[A_x]$. Similarly, $\mathcal{M}_s^S$ exhibits an extra exponential off-diagonal gain compared to $\mathcal{M}^R$ in the worst frequency balance scenario (high-high, in this case), which leads to the improved $Z^1$ norm bound (6.4.13) below.

Heuristically, the preceding observation leaves us with only the contribution of the scalar part $\mathcal{M}^E, \mathcal{M}^R, \mathcal{N}^E, \mathcal{N}^R$ to be handled; this is the main point of Proposition 1.8.8 and Theorem 1.6.2. The redeeming feature of this scalar remainder is that it closely resembles the massless MKG; see Remark 1.7.9. In particular, exploiting this similarity, we are able to borrow a trilinear null form estimate (Proposition 6.4.1) and parametrix construction (Theorem 4.3.3) from the massless MKG case [31] at key steps in the proof below.

## 5.5 The geometry of frequency interactions

### An orthogonality property

In view of performing summation arguments later on, we present below various 'orthogonality' statements concerning the vanishing property of the expression (5.0.1) based on the Fourier supports of $f_i$ ($i = 0, 1, 2$).

Given a triple $k_0, k_1, k_2 \in \mathbb{R}$, we denote by $k_{\min}$, $k_{\text{med}}$ and $k_{\max}$ the minimum, median and maximum of $k_0, k_1, k_2$. Similarly we consider $j_{\min}, j_{\text{med}}, j_{\max}$ for $j_0, j_1, j_2$ If $f_i = P_{k_i} f_i$, then (5.0.1) vanishes unless the maximum and the median of $k_0, k_1, k_2$ (i.e., the two largest numbers) are apart by at most (say) 5; this is the standard Littlewood-Paley trichotomy. We furthermore have the following refinement, which is useful when $k_{\min}$ is very small compared to $k_{\max}$:

**Lemma 5.5.1.** *Let $k_0, k_1, k_2 \in \mathbb{Z}$ be such that $|k_{\mathrm{med}} - k_{\max}| \leq 5$. For $i = 0, 1, 2$, let $\mathcal{C}^i$ be a cube of the form $\mathcal{C}_{k_{\min}}(0)$ (i.e., of dimension $2^{k_{\min}} \times \cdots \times 2^{k_{\min}}$) situated in $\{|\xi| \simeq 2^{k_i}\}$.*

1. *Then the expression*
$$\iint P_{\mathcal{C}^0} h_{k_0} \, L(P_{\mathcal{C}^1} f_{k_1}, P_{\mathcal{C}^2} g_{k_2}) \, \mathrm{d}t\mathrm{d}x \tag{5.5.1}$$
*vanishes unless $\mathcal{C}^0 + \mathcal{C}^1 + \mathcal{C}^2 \ni 0$.*

2. *If $\mathcal{C}^0 + \mathcal{C}^1 + \mathcal{C}^2 \ni 0$, then the cubes situated in the non-minimal frequency annuli are almost diametrically opposite. More precisely, we have*
$$|\angle(\mathcal{C}^i, -\mathcal{C}^{i'})| \lesssim 2^{k_{\min} - k_{\max}},$$
*where $k_i, k_{i'}$ $(i \neq i')$ are the median and maximal frequencies.*

3. *Without loss of generality, assume that $k_0$ is non-minimal, i.e., $k_0 = k_{\mathrm{med}}$ or $k_{\max}$. For any fixed cube $\mathcal{C}^0$ of the form $\mathcal{C}_{k_{\min}}(0)$ situated in $\{|\xi| \simeq 2^{k_0}\}$, there are only (uniformly) bounded number of cubes $\mathcal{C}^1, \mathcal{C}^2$ of the form $\mathcal{C}_{k_{\min}}(0)$ in $\{|\xi| \simeq 2^{k_1}\}, \{|\xi| \simeq 2^{k_2}\}$ such that $\mathcal{C}^0 + \mathcal{C}^1 + \mathcal{C}^2 \ni 0$.*

*Proof.* Statement (1) is obvious from the Fourier space representation of (5.5.1). For the proof of Statements (2) and (3), we assume without loss of generality that $k_2 = k_{\min}$. Since $\mathcal{C}^0 + \mathcal{C}^1 + \mathcal{C}^2 \ni 0$, there exists $\xi^i \in \mathcal{C}^i$ $(i = 0, 1, 2)$ forming a triangle, i.e., $\sum_i \xi^i = 0$. By the law of cosines,
$$|\xi^0|^2 + |\xi^1|^2 - 2|\xi^0||\xi^1| \cos\angle(\xi^0, -\xi^1) = |\xi^2|^2.$$
Rearranging terms, we see that
$$2|\xi^0||\xi^1|(1 - \cos\angle(\xi^0, -\xi^1)) = |\xi^2|^2 - (|\xi^0| - |\xi^1|)^2.$$
The LHS is comparable to $2^{2k_{\max}}|\angle(\xi^0, -\xi^1)|$, whereas the RHS is bounded from above by $\lesssim 2^{2k_{\min}}$. Statement (2) now follows.

It remains to establish Statement (3). Since there are only bounded number of cubes $\mathcal{C}_{k_{\min}}(0)$ in $\{|\xi| \simeq 2^{k_{\min}}\}$, the desired statement for $\mathcal{C}^2$ follows. Observing that $\mathcal{C}^0 + \mathcal{C}^2$ is contained in a cube of dimension $\lesssim 2^{k_{\min}}$, we see that there are only bounded number of cubes $\mathcal{C}^1 = \mathcal{C}_{k_{\min}}(0)$ such that $\mathcal{C}^0 + \mathcal{C}^2 \cap (-\mathcal{C}^1) \neq \emptyset$, or equivalently, $\mathcal{C}^0 + \mathcal{C}^1 + \mathcal{C}^2 \ni 0$. □

## The geometry of frequency interactions

Before we state the core bilinear estimates that will be used to estimate the nonlinearities, we analyze the geometry of the frequencies of two hyperboloids interacting with a cone at low modulations. The method of doing this is well-known, see [52, sec. 13], [4, Lemma 6.5].

Note that the analogue of the Littlewood-Paley trichotomy does not hold for modulations; However, modulation localization forces certain angular conditions among the spatial Fourier supports of the functions.

**Lemma 5.5.2.** *Let* $(k_0, k_1, k_2) \in \mathbb{Z} \times \mathbb{Z}_+ \times \mathbb{Z}_+$, $j_i \in \mathbb{Z}$ *for* $i = 0, 1, 2$ *and let* $\omega_i \subset \mathbb{S}^{d-1}$ *be angular caps of radius* $r_i \ll 1$. *Let* $\phi^1, \phi^2$ *have Fourier support, respectively, in*

$$S_i = \{\langle\xi\rangle \simeq 2^{k_i},\ \frac{\xi}{|\xi|} \in \omega_i,\ |\tau - s_i \langle\xi\rangle| \simeq 2^{j_i}\}, \qquad i = 1, 2$$

*and let* $A$ *have Fourier support in*

$$S_0 = \{|\xi| \simeq 2^{k_0},\ \frac{\xi}{|\xi|} \in \omega_0,\ |\tau - s_0 |\xi|| \simeq 2^{j_0}\},$$

*for some signs* $s_0, s_1, s_2$. *Let* $L$ *be translation-invariant and consider*

$$\int A \cdot L(\phi^1, \phi^2)\, \mathrm{d}x\, \mathrm{d}t. \tag{5.5.2}$$

1. *Suppose* $j_{\max} \le k_{\min} + C_0$. *Then* (5.5.2) *vanishes unless*

$$j_{\max} \ge k_{\min} - 2\min(k_1, k_2) - C.$$

2. *Suppose* $j_{\max} \le k_{\min} + C_0$ *and define* $\ell := \frac{1}{2}(j_{\max} - k_{\min})_-$.
*Then* (5.5.2) *vanishes unless* $2^\ell \gtrsim 2^{-\min(k_1,k_2)}$ *and*

$$\angle(s_i\omega_i, s_{i'}\omega_{i'}) \lesssim 2^\ell 2^{k_{\min} - \min(k_i, k_{i'})} + \max(r_i, r_{i'}) \tag{5.5.3}$$

*for every pair* $i, i' \in \{0, 1, 2\}$.

3. *If in addition we assume* $j_{\text{med}} \le j_{\max} - 5$, *then in* (5.5.3) *we have* $\simeq$ *instead of* $\lesssim$.

4. *If* $j_{\text{med}} \le j_{\max} - 5$ *then* (5.5.2) *vanishes unless either* $j_{\max} = k_{\max} + O(1)$ *or* $j_{\max} \le k_{\min} + \frac{1}{2}C_0$.

*Proof.* If (5.5.2) does not vanish, there exist $(\tau^i, \xi^i) \in S_i$, $(i = 0, 1, 2)$ such that $\sum_i (\tau^i, \xi^i) = 0$. Consider

$$H := s_0 |\xi_0| + s_1 \langle\xi_1\rangle + s_2 \langle\xi_2\rangle .$$

Using $\sum_i \tau^i = 0$, note that

$$|H| = \left|(s_0 |\xi_0| - \tau^0) + (s_1 \langle \xi_1 \rangle - \tau^1) + (s_2 \langle \xi_2 \rangle - \tau^2)\right| \lesssim 2^{j_{\max}}. \tag{5.5.4}$$

When the signs $s_i$ of the two highest frequencies are the same, we have $|H| \simeq 2^{k_{\max}}$. This implies $j_{\max} \geq k_{\max} - C$ and with the assumption $j_{\max} \leq k_{\min} + C_0$ we deduce $|k_{\max} - k_{\min}| \leq C$ and $\ell = O(1)$, in which case the statements are obvious.

Now suppose the high frequencies have opposite signs. By conjugation symmetry we may assume $s_0 = +$. By swapping $\phi^1$ with $\phi^2$ if needed, we may assume $s_2 = -$ and that $k_2 \neq k_{\min}$. We write

$$H = |\xi_0| + s_1 \langle \xi_1 \rangle - \langle \xi_2 \rangle = \frac{(|\xi_0| + s_1 \langle \xi_1 \rangle)^2 - (1 + |\xi_0 + \xi_1|^2)}{|\xi_0| + s_1 \langle \xi_1 \rangle + \langle \xi_2 \rangle} =$$
$$= \frac{2s_1 |\xi_0| \langle \xi_1 \rangle - 2\xi_0 \cdot \xi_1}{|\xi_0| + s_1 \langle \xi_1 \rangle + \langle \xi_2 \rangle} = \frac{2s_1 |\xi_0| |\xi_1| - 2\xi_0 \cdot \xi_1}{|\xi_0| + s_1 \langle \xi_1 \rangle + \langle \xi_2 \rangle} + \frac{2s_1 |\xi_0|}{\langle \xi_1 \rangle + |\xi_1|} \frac{1}{|\xi_0| + s_1 \langle \xi_1 \rangle + \langle \xi_2 \rangle}.$$

where we have used $\langle \xi_1 \rangle - |\xi_1| = (\langle \xi_1 \rangle + |\xi_1|)^{-1}$.

If $k_0 = k_{\min}$ we are in the case $(s_0, s_1, s_2) = (+, +, -)$. If $k_0 = k_{\max} + O(1)$, we are in the case $k_1 = k_{\min}$. Either way, we deduce

$$|H| \simeq 2^{k_{\min}} \angle(\xi^0, s_1 \xi^1)^2 + 2^{k_0 - k_1 - k_2}.$$

This and (5.5.4) proves Statement (1) and (2) for $(i, i') = (0, 1)$. The other pairs $(i, i')$ are reduced to this case. Indeed, denote by $\xi^l$ and $\xi^h$ the low and high frequencies among $\xi_0, \xi_1$. By the law of sines we have

$$\sin \angle(\xi^h, -\xi_2) = \frac{|\xi^l|}{|\xi_2|} \sin \angle(\xi^l, \xi^h) \lesssim 2^{\ell} 2^{k_{\min} - k_2}$$

which implies (5.5.3) in the high-high case. The remaining low-high case now follows from the previous two cases and the triangle inequality.

Statement (3) follows by noting that in the case $j_{\text{med}} \leq j_{\max} - 5$ we have $|H| \simeq 2^{j_{\max}}$. Similarly, for statement (4), since either $|H| \simeq 2^{k_{\max}}$ or $|H| \lesssim 2^{k_{\min}}$, the statement follows by choosing $C_0$ large enough. □

*Remark* 5.5.3. In the case $k_{\min} \in \{k_i, k_{i'}\}$, Statement (3) can be rephrased as follows. Denoting $2^{\ell_0} = \angle(s_i \omega_i, s_{i'} \omega_{i'})$ and choosing $r_i, r_{i'} \ll 2^{\ell_0}$, then (5.5.2) vanishes unless

$$j_{\max} = k_{\min} + 2\ell_0 + O(1).$$

In the case when all three functions are localized by the wave equation modulations, which correspond to the $Q_j^{\pm}$ operators, we have the following version. The proof is very similar to the above and is omitted.

**Lemma 5.5.4** (Geometry of the cone)**.** *Let $k_0$, $k_1$, $k_2$, $j_0, j_1, j_2 \in \mathbb{Z}$ be such that $|k_{\mathrm{med}} - k_{\max}| \leq 5$. For $i = 0, 1, 2$, let $\omega_i \subseteq \mathbb{S}^{d-1}$ be an angular cap of radius $0 < r_i < 2^{-5}$ and let $f_i$ have Fourier support in the region $\{|\xi| \simeq 2^{k_i}, \frac{\xi}{|\xi|} \in \omega_i, |\tau - s_i|\xi|| \simeq 2^{j_i}\}$. Then there exists a constant $C_0 > 0$ such that the following statements hold:*

1. *Suppose that $j_{\max} \leq k_{\min} + C_0$. Define $\ell := \frac{1}{2}(j_{\max} - k_{\min})_-$. Then the expression $\iint f_0 L(f_1, f_2) \,\mathrm{d}t\mathrm{d}x$ vanishes unless*

$$|\angle(s_i\omega_i, s_{i'}\omega_{i'})| \lesssim 2^{k_{\min} - \min\{k_i, k_{i'}\}} 2^{\ell} + \max\{r_i, r_{i'}\} \tag{5.5.5}$$

*for every pair $i, i' \in \{0, 1, 2\}$ $(i \neq i')$.*

2. *Suppose that $j_{\mathrm{med}} \leq j_{\max} - 5$. Then the expression $\iint f_0 L(f_1, f_2) \,\mathrm{d}t\mathrm{d}x$ vanishes unless either $j_{\max} = k_{\max} + O(1)$ or $j_{\max} \leq k_{\min} + \frac{1}{2}C_0$.*

From Lemma 5.5.4, we immediately obtain the following refinement of Lemma 5.5.1.

**Lemma 5.5.5.** *Let $k_0, k_1, k_2, j_0, j_1, j_2 \in \mathbb{Z}$ be such that $|k_{\mathrm{med}} - k_{\max}| \leq 5$ and $j_{\max} \leq k_{\min} + C_0$. Define $\ell := \frac{1}{2}(j_{\max} - k_{\min})_-$. For $i = 0, 1, 2$, let $\mathcal{C}^i$ be a rectangular box of the form $\mathcal{C}_{k_{\min}}(\ell)$ (i.e., of dimension $2^{k_{\min}} \times 2^{k_{\min}+\ell} \times \cdots \times 2^{k_{\min}+\ell}$, with the longest side aligned in the radial direction) situated in $\{|\xi| \simeq 2^{k_i}\}$.*

1. *Then the expression*

$$\iint P_{\mathcal{C}^0} Q^{s_0}_{j_0} h_{k_0} \, L(P_{\mathcal{C}^1} Q^{s_1}_{j_1} f_{k_1}, P_{\mathcal{C}^2} Q^{s_2}_{j_2} g_{k_2}) \,\mathrm{d}t\mathrm{d}x \tag{5.5.6}$$

*vanishes unless*

$$\mathcal{C}^0 + \mathcal{C}^1 + \mathcal{C}^2 \ni 0 \quad \text{and} \quad |\angle(s_i\mathcal{C}^i, s_{i'}\mathcal{C}^{i'})| \lesssim 2^{\ell} 2^{k_{\min} - \min\{k_i, k_{i'}\}} \tag{5.5.7}$$

*for every $i, i' \in \{0, 1, 2\}$ $(i \neq i')$.*

2. *Let $k_i = k_{\mathrm{med}}$ or $k_{\max}$; without loss of generality, assume that $i = 0$. Then for any fixed rectangular box $\mathcal{C}^0$ of the form $\mathcal{C}_{k_{\min}}(\ell)$ situated in $\{|\xi| \simeq 2^{k_0}\}$, there are only (uniformly) bounded number of boxes $\mathcal{C}^1, \mathcal{C}^2$ in $\{|\xi| \simeq 2^{k_1}\}, \{|\xi| \simeq 2^{k_2}\}$ such that (5.5.7) holds.*

*Proof.* Statement (1) follows immediately from Lemma 5.5.4. Statement (2) can be proved in a similar fashion as Lemma 5.5.1. We first assume without loss of generality that $k_2 = k_{\min}$. It is clear that there are only bounded number of $\mathcal{C}^2 = \mathcal{C}_{k_{\min}}(\ell)$ in $\{|\xi| \simeq 2^{k_{\min}}\}$ such that $|\angle(s_0\mathcal{C}^0, s_2\mathcal{C}^2)| \lesssim 2^{\ell}$. Moreover, observe that $\mathcal{C}^0 + \mathcal{C}^2$ is contained in a cube of sidelength $\lesssim 2^{k_{\min}}$. Combined with the angular restriction $|\angle(s_0\mathcal{C}^0, s_1\mathcal{C}^1)| \lesssim 2^{k_{\min} - k_{\max}} 2^{\ell}$, it follows that there are only bounded number of $\mathcal{C}^1$ such that (5.5.7) holds. □

## 5.6 Core bilinear estimates

We begin with a Hölder-type estimate which will be useful for dealing with the high modulation contribution, as well as the elliptic equations.

**Lemma 5.6.1.** *Let $k_0, k_1, k_2 \in \mathbb{Z}$ be such that $|k_{\text{med}} - k_{\max}| \leq 5$. Let $\mathcal{L}$ be a translation invariant bilinear operator on $\mathbb{R}^d$ with bounded mass kernel. Then we have*

$$\|P_{k_0}\mathcal{L}(f_{k_1}, g_{k_2})\|_{L^2L^2} \lesssim \|f_{k_1}\|_{L^\infty L^2}\Big(\sum_{\mathcal{C}_{k_{\min}}} \|P_{\mathcal{C}_{k_{\min}}} g_{k_2}\|^2_{L^2L^\infty}\Big)^{1/2}, \tag{5.6.1}$$

$$\|P_{k_0}\mathcal{L}(f_{k_1}, g_{k_2})\|_{L^1L^2} \lesssim \|f_{k_1}\|_{L^2L^2}\Big(\sum_{\mathcal{C}_{k_{\min}}} \|P_{\mathcal{C}_{k_{\min}}} g_{k_2}\|^2_{L^2L^\infty}\Big)^{1/2}. \tag{5.6.2}$$

*The same estimates hold in the cases $(k_0, k_1, k_2) \in \mathbb{Z}\times\mathbb{Z}_+\times\mathbb{Z}_+$, $(k_0, k_1, k_2) \in \mathbb{Z}_+\times\mathbb{Z}\times\mathbb{Z}_+$ or $(k_0, k_1, k_2) \in \mathbb{Z}_+\times\mathbb{Z}_+\times\mathbb{Z}$ when we replace the LHS by $P_{k_0}\mathcal{L}(\bar{P}_{k_1}f, \bar{P}_{k_2}g)$, $\bar{P}_{k_0}\mathcal{L}(P_{k_1}f, \bar{P}_{k_2}g)$, respectively $\bar{P}_{k_0}\mathcal{L}(\bar{P}_{k_1}f, P_{k_2}g)$.*

We now state the main bilinear estimates for $\mathcal{L}$, $\mathcal{N}_s$, $\mathcal{N}_s^*$ and $\mathcal{N}_{ij}$ when the inputs and the output have low modulation (i.e. less than the minimum frequency).

**Proposition 5.6.2.** *Let $k_0, k_1, k_2, j \in \mathbb{Z}$ be such that $|k_{\max} - k_{\text{med}}| \leq 5$ and $j \leq k_{\min} + C_0$. Define $\ell := \frac{1}{2}(j - k_{\min})_-$ and let $\mathcal{L}$ be translation invariant with bounded mass. Then for any signs $s_0, s_1, s_2 \in \{+, -\}$, the following estimates hold:*

$$\|P_{k_0}Q_j^{s_0}\mathcal{L}(Q_{<j}^{s_1}f_{k_1}, Q_{<j}^{s_2}g_{k_2})\|_{L^2L^2} \lesssim \|f_{k_1}\|_{L^\infty L^2}\Big(\sum_{\mathcal{C}_{k_{\min}}(\ell)} \|P_{\mathcal{C}_{k_{\min}}(\ell)}Q_{<j}^{s_2}g_{k_2}\|^2_{L^2L^\infty}\Big)^{1/2} \tag{5.6.3}$$

$$\|P_{k_0}Q_{<j}^{s_0}\mathcal{L}(Q_j^{s_1}f_{k_1}, Q_{<j}^{s_2}g_{k_2})\|_{L^1L^2} \lesssim \|Q_j^{s_1}f_{k_1}\|_{L^2L^2}\Big(\sum_{\mathcal{C}_{k_{\min}}(\ell)} \|P_{\mathcal{C}_{k_{\min}}(\ell)}Q_{<j}^{s_2}g_{k_2}\|^2_{L^2L^\infty}\Big)^{1/2} \tag{5.6.4}$$

*The same statement holds when we consider $(k_0, k_1, k_2) \in \mathbb{Z}_+\times\mathbb{Z}_+\times\mathbb{Z}$ and we replace $(Q_j^{s_0}, Q_{<j}^{s_1}, Q_{<j}^{s_2})$ by $(\bar{Q}_j^{s_0}, \bar{Q}_{<j}^{s_1}, Q_{<j}^{s_2})$ and all the similar variations.*

**Proposition 5.6.3** (Core estimates for $\mathcal{N}_s$)**.** *Let $k_0, k_1, k_2, j \in \mathbb{Z}$ be such that $|k_{\max} - k_{\text{med}}| \leq 5$ and $j \leq k_{\min} + C_0$. Define $\ell := \frac{1}{2}(j - k_{\min})_-$ and let $\mathcal{N}_s$ be an abstract null form as in Definition 5.4.2. Then, for any signs $s_0, s_1, s_2 \in \{+, -\}$, the following estimates hold:*

$$\begin{aligned}&\|P_{k_0}Q_j^{s_0}\mathcal{N}_{s_1s_2}(Q_{<j}^{s_1}f_{k_1}, Q_{<j}^{s_2}g_{k_2})\|_{L^2L^2}\\ &\quad\lesssim 2^\ell 2^{k_{\min}-\min\{k_1,k_2\}}\|f_{k_1}\|_{L^\infty L^2}\Big(\sum_{\mathcal{C}_{k_{\min}}(\ell)} \|P_{\mathcal{C}_{k_{\min}}(\ell)}Q_{<j}^{s_2}g_{k_2}\|^2_{L^2L^\infty}\Big)^{1/2}\end{aligned} \tag{5.6.5}$$

$$\begin{aligned}&\|P_{k_0}Q_{<j}^{s_0}\mathcal{N}_{s_1s_2}(Q_j^{s_1}f_{k_1}, Q_{<j}^{s_2}g_{k_2})\|_{L^1L^2}\\ &\quad\lesssim 2^\ell 2^{k_{\min}-\min\{k_1,k_2\}}\|Q_j^{s_1}f_{k_1}\|_{L^2L^2}\Big(\sum_{\mathcal{C}_{k_{\min}}(\ell)} \|P_{\mathcal{C}_{k_{\min}}(\ell)}Q_{<j}^{s_2}g_{k_2}\|^2_{L^2L^\infty}\Big)^{1/2}\end{aligned} \tag{5.6.6}$$

**Proposition 5.6.4** (Core estimates for $\mathcal{N}_s^*$)**.** *Let $k_0, k_1, k_2, j \in \mathbb{Z}$ be such that $|k_{\max} - k_{\mathrm{med}}| \leq 5$ and $j \leq k_{\min} + C_0$. Define $\ell := \frac{1}{2}(j - k_{\min})_-$ and let $\mathcal{N}_s^*$ be an abstract null form as in Definition 5.4.3. Then, for any signs $s_0, s_1, s_2 \in \{+,-\}$, the following estimates hold:*

$$\begin{aligned}\|P_{k_0} Q_j^{s_0} \mathcal{N}^*_{s_0 s_2}(Q^{s_1}_{<j} f_{k_1}, Q^{s_2}_{<j} g_{k_2})\|_{L^2 L^2} \\ \lesssim 2^{\ell} 2^{k_{\min} - \min\{k_0, k_2\}} \|f_{k_1}\|_{L^\infty L^2} \Big( \sum_{\mathcal{C}_{k_{\min}}(\ell)} \|P_{\mathcal{C}_{k_{\min}}(\ell)} Q^{s_2}_{<j} g_{k_2}\|^2_{L^2 L^\infty} \Big)^{1/2}\end{aligned} \tag{5.6.7}$$

$$\begin{aligned}\|P_{k_0} Q_{<j}^{s_0} \mathcal{N}^*_{s_0 s_2}(Q^{s_1}_{j} f_{k_1}, Q^{s_2}_{<j} g_{k_2})\|_{L^1 L^2} \\ \lesssim 2^{\ell} 2^{k_{\min} - \min\{k_0, k_2\}} \|Q^{s_1}_j f_{k_1}\|_{L^2 L^2} \Big( \sum_{\mathcal{C}_{k_{\min}}(\ell)} \|P_{\mathcal{C}_{k_{\min}}(\ell)} Q^{s_2}_{<j} g_{k_2}\|^2_{L^2 L^\infty} \Big)^{1/2}\end{aligned} \tag{5.6.8}$$

$$\begin{aligned}\|P_{k_0} Q_{<j}^{s_0} \mathcal{N}^*_{s_0 s_2}(Q^{s_1}_{<j} f_{k_1}, Q^{s_2}_{j} g_{k_2})\|_{L^1 L^2} \\ \lesssim 2^{\ell} 2^{k_{\min} - \min\{k_0, k_2\}} \Big( \sum_{\mathcal{C}_{k_{\min}(\ell)}} \|P_{\mathcal{C}_{k_{\min}(\ell)}} Q^{s_1}_{<j} f_{k_1}\|^2_{L^2 L^\infty} \Big)^{1/2} \|Q^{s_2}_j g_{k_2}\|_{L^2 L^2}\end{aligned} \tag{5.6.9}$$

*Remark* 5.6.5. It is clear from the proof that each of the inequalities holds (with an adjusted constant) when we replace any of the multipliers $Q^{s_i}_{<j}$ by $Q^{s_i}_{\leq j}$ or $Q^{s_i}_{<j-C}$ for any fixed $C \geq 0$.

**Proposition 5.6.6.** *Let $k_0 \in \mathbb{Z},\ k_1, k_2 \geq 0,\ j \in \mathbb{Z}$ be such that $|k_{\max} - k_{\mathrm{med}}| \leq 5$ and $j \leq k_{\min} + C_0$. Define $\ell := \frac{1}{2}(j - k_{\min})_-$ and let $\mathcal{N}$ be any of the null forms $\mathcal{N}_{ij}$. Then, the following estimates hold:*

$$\begin{aligned}\|P_{k_0} Q_j \mathcal{N}(\bar{Q}_{<j} f_{k_1}, \bar{Q}_{<j} g_{k_2})\|_{L^2 L^2} \\ \lesssim 2^{\ell} 2^{k_{\min} + k_{\max}} \|f_{k_1}\|_{L^\infty L^2} \Big( \sup_{\pm} \sum_{\mathcal{C}_{k_{\min}}(\ell)} \|P_{\mathcal{C}_{k_{\min}}(\ell)} \bar{Q}^{\pm}_{<j} g_{k_2}\|^2_{L^2 L^\infty} \Big)^{1/2}\end{aligned} \tag{5.6.10}$$

$$\begin{aligned}\|P_{k_0} Q_{<j} \mathcal{N}(\bar{Q}_{j} f_{k_1}, \bar{Q}_{<j} g_{k_2})\|_{L^1 L^2} \\ \lesssim 2^{\ell} 2^{k_{\min} + k_{\max}} \|\bar{Q}_j f_{k_1}\|_{L^2 L^2} \Big( \sup_{\pm} \sum_{\mathcal{C}_{k_{\min}}(\ell)} \|P_{\mathcal{C}_{k_{\min}}(\ell)} \bar{Q}^{\pm}_{<j} g_{k_2}\|^2_{L^2 L^\infty} \Big)^{1/2}\end{aligned} \tag{5.6.11}$$

*The same statement holds in the case $(k_0, k_1, k_2) \in \mathbb{Z}_+ \times \mathbb{Z} \times \mathbb{Z}_+$ when we replace the LHS of (5.6.5), (5.6.6) by $\bar{P}_{k_0} \bar{Q}_j \mathcal{N}(Q_{<j} f_{k_1}, \bar{Q}_{<j} g_{k_2})$ and $\bar{P}_{k_0} \bar{Q}_{<j} \mathcal{N}(Q_j f_{k_1}, \bar{Q}_{<j} g_{k_2})$ respectively; or in the case $(k_0, k_1, k_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}$ when we replace the LHS of (5.6.5), (5.6.6) by $\bar{P}_{k_0} \bar{Q}_j \mathcal{N}(\bar{Q}_{<j} f_{k_1}, Q_{<j} g_{k_2})$ and $\bar{P}_{k_0} \bar{Q}_{<j} \mathcal{N}(\bar{Q}_j f_{k_1}, Q_{<j} g_{k_2})$ respectively.*

Although there are numerous cases, all the estimates may be proved in an identical fashion, which combines Lemma 5.5.5 with either (1.5.9) or the following estimate:

**Lemma 5.6.7.** *Let $k_0, k_1, k_2, j, \ell$ be as in Propositions 5.6.3 and 5.6.4. For $i = 0, 1, 2$, let $s_i \in \{+, -\}$ and $\mathcal{C}^i$ be a rectangular box of the form $\mathcal{C}_{k_{\min}}(\ell)$ situated in $\{|\xi| \sim 2^{k_i}\}$ such that (5.5.7) holds. Then for any $1 \le q_0, q_1, q_2 \le \infty$ such that $q_0^{-1} + q_1^{-1} + q_2^{-1} = 1$, we have*

$$
\begin{aligned}
&|\int P_{\mathcal{C}^0} h_{k_0} \mathcal{N}_{s_1 s_1}(P_{\mathcal{C}^1} f_{k_1}, P_{\mathcal{C}^2} g_{k_2})\, \mathrm{d}x| \\
&\quad \lesssim 2^{\ell} 2^{k_{\min} - \min\{k_1, k_2\}} \|P_{\mathcal{C}^0} h_{k_0}\|_{L^{q_0}} \|P_{\mathcal{C}^1} f_{k_1}\|_{L^{q_1}} \|P_{\mathcal{C}^2} g_{k_2}\|_{L^{q_2}}
\end{aligned}
$$

*Proof.* Upon verifying that the inputs obey the hypothesis of Proposition 5.3.1, the lemma follows immediately. □

*Proof of Propositions 5.6.1, 5.6.2, 5.6.3, 5.6.4 and 5.6.6.* We present the details in the case of Propositions 5.6.3 and 5.6.4; Proposition 5.6.6 follows from the same proof since in this case we have Corollary 5.4.1 and Lemma 5.5.2; Proposition 5.6.2 follows from the same proof with Lemma 5.6.7 replaced by (1.5.9), which removes $2^{\ell} 2^{k_{\min} - \min\{k_1, k_2\}}$ in (5.6.13). The proof of Lemma 5.6.1 is similar and simpler, since it uses Lemma 5.5.1 instead of Lemma 5.5.5.

For $t \in \mathbb{R}$ and rectangular boxes $\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2$ of the form $\mathcal{C}_{k_{\min}}(\ell)$, we introduce the expression

$$
I_{\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2}(t) = \int P_{\mathcal{C}^0} Q^{s_0}_{j/<j} h_{k_0}\, \mathcal{N}_{s_1 s_2}(P_{\mathcal{C}^1} Q^{s_1}_{j/<j} f_{k_1}, P_{\mathcal{C}^2} Q^{s_2}_{j/<j} g_{k_2})(t)\, \mathrm{d}x
$$

where $Q^{s_i}_{j/<j}$ stands for either $Q^{s_i}_{j}$ or $Q^{s_i}_{<j}$. Note that

$$
\iint Q^{s_0}_{j/<j} h_{k_0}\, \mathcal{N}_{s_1 s_2}(Q^{s_1}_{j/<j} f_{k_1}, Q^{s_2}_{j/<j} g_{k_2})\, \mathrm{d}t \mathrm{d}x = \sum_{\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2} \int I_{\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2}(t)\, \mathrm{d}t. \tag{5.6.12}
$$

By Lemma 5.5.5, the summand on the RHS vanishes unless $\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2$ satisfy (5.5.7). Using the shorthand $\tilde{h} = Q^{s_0}_{j/<j} h_{k_0}$, $\tilde{f} = Q^{s_1}_{j/<j} f_{k_1}$ and $\tilde{g} = Q^{s_2}_{j/<j} h_{k_2}$, Lemma 5.6.7 implies

$$
|I_{\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2}(t)| \lesssim 2^{\ell} 2^{k_{\min} - \min\{k_1, k_2\}} \|P_{\mathcal{C}^0} \tilde{h}(t)\|_{L^{q_0}} \|P_{\mathcal{C}^1} \tilde{f}(t)\|_{L^{q_1}} \|P_{\mathcal{C}^2} \tilde{g}(t)\|_{L^{q_2}} \tag{5.6.13}
$$

for any $1 \le q_0, q_1, q_2 \le \infty$ such that $q_0^{-1} + q_1^{-1} + q_2^{-1} = 1$. We now sum up the RHS of (5.6.13) in $(\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2)$ for which (5.5.7) holds. As in the proof of Lemma 5.6.1, we first sum up the boxes in $\{|\xi| \simeq 2^{k_{\min}}\}$ (for which there are only bounded many summands) and then apply Lemma 1.5.5 to the remaining (essentially diagonal) summation. We then obtain

$$
\begin{aligned}
\sum_{\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2 : (5.5.7)} |I_{\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2}(t)| \lesssim & 2^{\ell} 2^{k_{\min} - \min\{k_1, k_2\}} \Big( \sum_{\mathcal{C}^0} \|P_{\mathcal{C}^0} \tilde{h}(t)\|_{L^{q_0}}^2 \Big)^{1/2} \\
& \times \Big( \sum_{\mathcal{C}^1} \|P_{\mathcal{C}^1} \tilde{f}(t)\|_{L^{q_1}}^2 \Big)^{1/2} \Big( \sum_{\mathcal{C}^2} \|P_{\mathcal{C}^2} \tilde{g}(t)\|_{L^{q_2}}^2 \Big)^{1/2}.
\end{aligned}
$$

We are ready to complete the proof in a few strokes. To prove estimates (5.6.5) and (5.6.6), take $(q_0, q_1, q_2) = (2, 2, \infty)$. By orthogonality in $L^2$, factors involving $\tilde{h}$ and $\tilde{f}$ can be bounded by $\|\tilde{h}(t)\|_{L^2}$ and $\|\tilde{f}(t)\|_{L^2}$, respectively. Integrating and applying Hölder's inequality in $t$, estimates (5.6.5) and (5.6.6) follow by duality as in the proof of Lemma 5.6.1. Next, by the definition of $\mathcal{N}_s^*$ in (5.4.5), (5.6.7) and (5.6.8) follow from the same method as well (we note that, since we use the pairing $\int fg$, the transpose of $Q^{s_0}_{j/<j}$ is $Q^{-s_0}_{j/<j}$). Finally, (5.6.9) is proved by taking $(q_0, q_1, q_2) = (\infty, 2, 2)$ and proceeding analogously. $\square$

# Chapter 6

# Bilinear and trilinear estimates

The proofs in this chapter are based on the Littlewood-Paley trichotomy which states that $P_{k_0}(P_{k_1}f_1 P_{k_2}f_2)$ vanishes unless $|k_{\text{med}} - k_{\max}| \leq 5$, where $k_{\text{med}}, k_{\max}$ are the the median and the maximum of $\{k_0, k_1, k_2\}$.

## 6.1 Bilinear estimates for MKG

Most of the arguments in this section originate in [31]. However, we have tried to give a thorough exposition in order to justify that the arguments carry over when two of the inputs/output correspond to Klein-Gordon waves.

### Additional bilinear estimates

Before we begin the proofs we state some additional bilinear estimates that will be used in the proof of the trilinear estimate in the next section.

We separate the high-high and low-high parts of $\mathbf{A}_0$ from (1.7.4)

$$\mathbf{A}_0(\phi^1, \phi^2) = \mathbf{A}_0^{LH}(\phi^1, \phi^2) + \mathbf{A}_0^{HH}(\phi^1, \phi^2)$$

$$\text{where} \qquad \mathbf{A}_0^{HH}(\phi^1, \phi^2) = \sum_{\substack{k_0, k_1, k_2 \\ k_0 < k_2 - C_2 - 5}} P_{k_0} \mathbf{A}_0(\bar{P}_{k_1}\phi^1, \bar{P}_{k_2}\phi^2). \tag{6.1.1}$$

**Lemma 6.1.1.** *With the decomposition above, one has:*

$$\|\pi[(0, A_0)]\phi\|_{\bar{N}^{\sigma-1}} \lesssim \|A_0\|_{\ell^1 L^1 L^\infty} \|\phi\|_{\bar{S}^\sigma}. \tag{6.1.2}$$

$$\|\boldsymbol{A}_0^{LH}(\phi^1, \phi^2)\|_{\ell^1 L^1 L^\infty} \lesssim \|\phi^1\|_{\bar{S}^\sigma} \|\phi^2\|_{\bar{S}^\sigma} \tag{6.1.3}$$

$$\|(\boldsymbol{A}_x, \boldsymbol{A}_0^{HH})(\phi^1, \phi^2)\|_{\ell^1 S^\sigma \times Y^\sigma} \lesssim \|\phi^1\|_{\bar{S}^\sigma} \|\phi^2\|_{\bar{S}^\sigma} \tag{6.1.4}$$

$$\|(I - \mathcal{H})(\boldsymbol{A}_x, \boldsymbol{A}_0^{HH})(\phi^1, \phi^2)\|_{Z^\sigma \times Z^\sigma_{ell}} \lesssim \|\phi^1\|_{\bar{S}^\sigma} \|\phi^2\|_{\bar{S}^\sigma}. \tag{6.1.5}$$

*For $d \geq 5$ one also has:*

$$\|(\boldsymbol{A}_x, \boldsymbol{A}_0^{HH})(\phi^1, \phi^2)\|_{Z^\sigma \times Z^\sigma_{ell}} \lesssim \|\phi^1\|_{\bar{S}^\sigma} \|\phi^2\|_{\bar{S}^\sigma} \tag{6.1.6}$$

*Proof.* By doing dyadic decompositions, (6.1.2) follows trivially from Hölder's inequality $L^1L^\infty \times L^\infty L^2 \to L^1L^2$. The bound (6.1.3) follows from

$$\|P_{k'}(\phi^1_{k_1}\partial_t\phi^2_{k_2})\|_{L^1L^\infty} \lesssim \|\phi^1_{k_1}\|_{L^2L^\infty}\|\partial_t\phi^2_{k_2}\|_{L^2L^\infty}.$$

The bound (6.1.4) follows from Prop. 1.8.1 and from the proof of (1.8.2).

The proofs of estimates (6.1.5), (6.1.6) are longer and are deferred to the end of this section. □

## Dyadic norms

For easy referencing in the arguments below, here we collect the norms that we control. Recall that we denote

$$\|A_x\|_{S^s_{k'}} = 2^{(s-1)k'}\|\nabla_{t,x}A\|_{S_{k'}}, \qquad \|\phi_k\|_{\bar{S}^s_k} = 2^{(s-1)k}\|(\langle D_x\rangle, \partial_t)\phi_k\|_{\bar{S}_k}$$

For $k' \in \mathbb{Z}$ and $k \geq 0$ we have:

$$\|\nabla_{t,x}P_{k'}A_x\|_{L^\infty L^2} \lesssim \|P_{k'}A_x\|_{S^1_{k'}}, \qquad \|(\langle D_x\rangle, \partial_t)\phi_k\|_{L^\infty L^2} \lesssim \|\phi_k\|_{\bar{S}^1_k} \tag{6.1.7}$$

$$\|Q_jP_{k'}A_x\|_{L^2_{t,x}} \lesssim 2^{-\frac{1}{2}j}\|P_{k'}A_x\|_{S_{k'}}, \qquad \|\bar{Q}_j\phi_k\|_{L^2_{t,x}} \lesssim 2^{-\frac{1}{2}j}\|\phi_k\|_{\bar{S}_k} \tag{6.1.8}$$

$$\|P_{k'}A_x\|_{L^2L^\infty} \lesssim 2^{\frac{1}{2}k'}\|P_{k'}A_x\|_{S^\sigma_{k'}}, \qquad \|\phi_k\|_{L^2L^\infty} \lesssim 2^{\frac{1}{2}k}\|\phi_k\|_{\bar{S}^\sigma_k} \tag{6.1.9}$$

For any $k' \leq k$ and $l' \in [-k, C]$, $j = k' + 2l'$ and any $\pm$:

$$\begin{aligned}\Big(\sum_{\mathcal{C}=\mathcal{C}_{k'}(l')} \|P_{\mathcal{C}}\bar{Q}^\pm_{<j}\phi_k\|^2_{L^2L^\infty}\Big)^{1/2} &\lesssim 2^{\frac{1}{2}l'}2^{\sigma(k'-k)}2^{\frac{1}{2}k}\|\phi_k\|_{\bar{S}^\sigma_k},\\ \Big(\sum_{\mathcal{C}=\mathcal{C}_{k'}(0)} \|P_{\mathcal{C}}\phi_k\|^2_{L^2L^\infty}\Big)^{1/2} &\lesssim 2^{\sigma(k'-k)}2^{\frac{1}{2}k}\|\phi_k\|_{\bar{S}^\sigma_k}.\end{aligned} \tag{6.1.10}$$

The former follows by choosing $k + 2l = k' + 2l'$ in (2.1.6). When $k = 0$ it suffices to consider $l' = 0$. The latter inequality holds for $\bar{Q}_{<k'}\phi_k$, while for $\bar{Q}_{\geq k'}\phi_k$ it follows from (6.1.8), orthogonality and Bernstein's inequality (with $l' = 0$)

$$P_{C_{k'}(l')}L^2_x \subset 2^{\frac{d}{2}k' + \frac{d-1}{2}l'}L^\infty_x \tag{6.1.11}$$

Using (6.1.11) we also obtain, when $d = 4$, $\sigma = 1$,

$$\begin{aligned}\big(\sum_{C_{k'}(l')} \|P_{C_{k'}(l')}(\partial_t \mp i\langle D\rangle)\bar{Q}^\pm_{<j}\phi_k\|^2_{L^2L^\infty}\big)^{\frac{1}{2}} &\lesssim 2^{\frac{3}{2}l'}2^{2k'}2^{\frac{1}{2}j}\|\phi_k\|_{\bar{X}^{\frac{1}{2}}_\infty}\\ &\lesssim 2^{\frac{3}{2}l'}2^{2k'}2^{\frac{1}{2}j}2^{-k}\|\phi_k\|_{\bar{S}^1_k}.\end{aligned} \tag{6.1.12}$$

For any $k' \leq k''$ and $l' \leq 0$, $j = k' + 2l'$ and any $\pm$ we have

$$\begin{aligned} \Big( \sum_{\mathcal{C}=\mathcal{C}_{k'}(l')} \|P_{\mathcal{C}} Q^{\pm}_{<j} A_{k''}\|^2_{L^2 L^\infty} \Big)^{1/2} &\lesssim 2^{\frac{1}{2}l'} 2^{\sigma(k'-k'')} 2^{\frac{1}{2}k''} \|P_{k''} A_x\|_{S^\sigma_{k''}}, \\ \Big( \sum_{\mathcal{C}=\mathcal{C}_{k'}(0)} \|P_{\mathcal{C}} A_{k''}\|^2_{L^2 L^\infty} \Big)^{1/2} &\lesssim 2^{\sigma(k'-k'')} 2^{\frac{1}{2}k''} \|P_{k''} A_x\|_{S^\sigma_{k''}}. \end{aligned} \tag{6.1.13}$$

For $A_0$ we have the following bounds

$$\|\nabla_{t,x} P_{k'} A_0\|_{L^\infty L^2} \lesssim \|P_{k'} A_0\|_{Y^1}. \tag{6.1.14}$$

Since we control $\partial_t A_0$, for $j \geq k'$ we have both

$$\|P_{k'} A_0\|_{L^2_{t,x}} \lesssim 2^{-(\sigma+\frac{1}{2})k'} \|P_{k'} A_0\|_{Y^\sigma}, \qquad \|Q_j P_{k'} A_0\|_{L^2_{t,x}} \lesssim 2^{-j} 2^{-(\sigma-\frac{1}{2})k'} \|P_{k'} A_0\|_{Y^\sigma} \tag{6.1.15}$$

and for $j = k' + 2l'$, using (6.1.11) and orthogonality, we have

$$\Big( \sum_{\mathcal{C}=\mathcal{C}_{k'}(l')} \|P_{\mathcal{C}} (Q^{\pm}_{<j}) A^0_{k'}\|^2_{L^2 L^\infty} \Big)^{1/2} \lesssim 2^{\frac{d}{2}k' + \frac{d-1}{2}l'} \|A^0_{k'}\|_{L^2_{t,x}} \lesssim 2^{\frac{1}{2}k' + \frac{3}{2}l'} \|P_{k'} A_0\|_{Y^\sigma} \tag{6.1.16}$$

In particular,
$$\|P_{k'} A_0\|_{L^2 L^\infty} \lesssim 2^{\frac{1}{2}k'} \|P_{k'} A_0\|_{Y^\sigma}. \tag{6.1.17}$$

Now we turn to the proofs of Prop. 1.8.1, 1.8.2.

## Proof of (1.8.1)

This follows from proving, for $k' \in \mathbb{Z}$, $k_1, k_2 \geq 0$:

$$\|P_{k'} \mathcal{P}_j (\phi^1_{k_1} \nabla_x \phi^2_{k_2})\|_{N^{\sigma-1}_{k'}} \lesssim 2^{\frac{1}{2}(k_{\min}-k_{\max})} \|\phi^1_{k_1}\|_{\bar{S}^\sigma_{k_1}} \|\phi^2_{k_2}\|_{\bar{S}^\sigma_{k_2}}. \tag{6.1.18}$$

Note that the factor $2^{\frac{1}{2}(k_{\min}-k_{\max})}$ provides the $\ell^1$ summation in (1.8.1). Here $k_{\min}, k_{\max}$ are taken from the set $\{k', k_1, k_2\}$.

We first treat the high modulation contribution. Since $\mathcal{P}_j(\phi^1 \nabla_x \phi^2)$ is skew adjoint (see Remark 1.7.2), in the low-high case ($2^{k'} \simeq 2^{k_{max}}$) we may assume $k_2 = k_{\min}$ (i.e. the derivative falls on the lower frequency). By Lemma 5.6.1 we have

$$\|P_{k'} \mathcal{P}_j (\bar{Q}_{\geq k_{\min}} \phi^1_{k_1} \nabla_x \phi^2_{k_2})\|_{L^1 L^2} \lesssim \|\bar{Q}_{\geq k_{\min}} \phi^1_{k_1}\|_{L^2_{t,x}} \Big( \sum_{\mathcal{C}_{k_{\min}}} \|P_{\mathcal{C}_{k_{\min}}} \nabla_x \phi^2_{k_2}\|^2_{L^2 L^\infty} \Big)^{\frac{1}{2}}, \tag{6.1.19}$$

$$\|P_{k'} \mathcal{P}_j (\bar{Q}_{<k_{\min}} \phi^1_{k_1} \nabla_x \bar{Q}_{\geq k_{\min}} \phi^2_{k_2})\|_{L^1 L^2} \lesssim \Big( \sum_{\mathcal{C}_{k_{\min}}} \|P_{\mathcal{C}_{k_{\min}}} \bar{Q}_{<k_{\min}} \phi^1_{k_1}\|^2_{L^2 L^\infty} \Big)^{\frac{1}{2}} \|\bar{Q}_{\geq k_{\min}} \nabla_x \phi^2_{k_2}\|_{L^2_{t,x}} \tag{6.1.20}$$

$$\|P_{k'} Q_{\geq k_{\min}} \mathcal{P}_j (\bar{Q}_{<k_{\min}} \phi^1_{k_1} \nabla_x \bar{Q}_{<k_{\min}} \phi^2_{k_2})\|_{L^2_{t,x}} \lesssim \Big( \sum_{\mathcal{C}_{k_{min}}} \|P_{\mathcal{C}_{k_{min}}} \bar{Q}_{<k_{\min}} \phi^1_{k_1}\|^2_{L^2 L^\infty} \Big)^{\frac{1}{2}} \|\bar{Q}_{<k_{\min}} \nabla_x \phi^2_{k_2}\|_{L^\infty L^2}. \tag{6.1.21}$$

Using (6.1.8), (6.1.10) for (6.1.19), using (6.1.10), (6.1.8) for (6.1.20), and using (6.1.10), (6.1.7) and the $X_1^{-1/2}$ norm for (6.1.21), we see that these terms are acceptable.

We continue with the low modulation term

$$P_{k'} Q_{<k_{\min}} \mathcal{P}_j (\bar{Q}_{<k_{\min}} \phi^1_{k_1} \nabla_x \bar{Q}_{<k_{\min}} \phi^2_{k_2}),$$

which, summing according to the highest modulation, using (1.7.5), we decompose into sums of

$$I_0 = \sum_{j<k_{\min}} P_{k'} Q_j \Delta^{-1} \nabla^l \mathcal{N}_{lm} (\bar{Q}_{<j} \phi^1_{k_1}, \bar{Q}_{<j} \phi^2_{k_2}), \tag{6.1.22}$$

$$I_1 = \sum_{j<k_{\min}} P_{k'} Q_{\leq j} \Delta^{-1} \nabla^l \mathcal{N}_{lm} (\bar{Q}_j \phi^1_{k_1}, \bar{Q}_{<j} \phi^2_{k_2}), \tag{6.1.23}$$

$$I_2 = \sum_{j<k_{\min}} P_{k'} Q_{\leq j} \Delta^{-1} \nabla^l \mathcal{N}_{lm} (\bar{Q}_{\leq j} \phi^1_{k_1}, \bar{Q}_j \phi^2_{k_2}). \tag{6.1.24}$$

for which we have

$$\| \, |D|^{\sigma-1} I_0 \|_{X_1^{-1/2}} + \|I_1\|_{L^1 \dot{H}^{\sigma-1}} + \|I_2\|_{L^1 \dot{H}^{\sigma-1}} \lesssim 2^{\frac{1}{2}(k_{\min}-k_{\max})} \|\phi^1_{k_1}\|_{\bar{S}^\sigma_{k_1}} \|\phi^2_{k_2}\|_{\bar{S}^\sigma_{k_2}}. \tag{6.1.25}$$

These are estimated by Proposition 5.6.6 and (6.1.7), (6.1.8), (6.1.10), which concludes the proof of (6.1.18).

## Proof of (1.8.4).

We separate $A_0 \partial_t \phi$ and $A^j \partial_j \phi$. Since we subtract $\pi[A]\phi$, this effectively eliminates low-high interactions in the Littlewood-Paley trichotomy. Thus for $k, k_0 \geq 0$, $k' \geq k - C$ it suffices to prove

$$\|\bar{P}_{k_0} \big( A^0_{k'} \partial_t \phi_k \big) \|_{L^1 H^{\sigma-1}} \lesssim 2^{k_{\min}-k_{\max}} \|A^0_{k'}\|_{L^2 \dot{H}^{\sigma+\frac{1}{2}}} \|\partial_t \phi_k\|_{\bar{S}^{\sigma-1}_k}, \tag{6.1.26}$$

$$\|\bar{P}_{k_0} \big( A^j_{k'} \partial_j \phi_k \big) \|_{\bar{N}^{\sigma-1}_{k_0}} \lesssim 2^{\frac{1}{2}(k_{\min}-k_{\max})} \|A_{k'}\|_{S^\sigma_{k'}} \|\phi_k\|_{\bar{S}^\sigma_k}. \tag{6.1.27}$$

The bound (6.1.26) follows immediately from (5.6.2). Now we turn to (6.1.27).

We first treat the high modulation contribution. By Lemma 5.6.1 we have

$$\|\bar{P}_{k_0} \bar{Q}_{\geq k_{\min}} \big( A^j_{k'} \partial_j \phi_k \big) \|_{L^2_{t,x}} \lesssim \|A_{k'}\|_{L^\infty L^2} \big( \sum_{\mathcal{C}_{k_{\min}}} \|P_{\mathcal{C}_{k_{\min}}} \nabla_x \phi_k\|^2_{L^2 L^\infty} \big)^{\frac{1}{2}},$$

$$\|\bar{P}_{k_0} \bar{Q}_{<k_{\min}} (Q_{\geq k_{\min}} A^j_{k'} \partial_j \phi_k) \|_{L^1 L^2} \lesssim \|Q_{\geq k_{\min}} A_{k'}\|_{L^2_{t,x}} \big( \sum_{\mathcal{C}_{k_{\min}}} \|P_{\mathcal{C}_{k_{\min}}} \nabla_x \phi_k\|^2_{L^2 L^\infty} \big)^{\frac{1}{2}},$$

$$\|\bar{P}_{k_0} \bar{Q}_{<k_{\min}} (Q_{<k_{\min}} A^j_{k'} \partial_j \bar{Q}_{\geq k_{\min}} \phi_k) \|_{L^1 L^2} \lesssim \big( \sum_{\mathcal{C}_{k_{\min}}} \|P_{\mathcal{C}_{k_{\min}}} Q_{<k_{\min}} A_{k'}\|^2_{L^2 L^\infty} \big)^{\frac{1}{2}} \|\bar{Q}_{\geq k_{\min}} \nabla_x \phi_k\|_{L^2_{t,x}}.$$

Using (6.1.7), (6.1.10) and the $\bar{X}_1^{-1/2}$ norm for the first term, (6.1.8), (6.1.9) for the second, and (6.1.13), (6.1.8) for the third, we see that these terms are acceptable.

We continue with the low modulation term

$$\bar{P}_{k_0}\bar{Q}_{<k_{\min}}\big(Q_{<k_{\min}}A^j_{k'}\partial_j\bar{Q}_{<k_{\min}}\phi_k\big)$$

which, summing according to the highest modulation, using (1.7.6), we decompose into sums of

$$I_0 = \sum_{j<k_{\min}} \bar{P}_{k_0}\bar{Q}_j\mathcal{N}_{lm}(\Delta^{-1}\nabla^l Q_{<j}A^m_{k'}, \bar{Q}_{<j}\phi_k), \tag{6.1.28}$$

$$I_1 = \sum_{j<k_{\min}} \bar{P}_{k_0}\bar{Q}_{\leq j}\mathcal{N}_{lm}(\Delta^{-1}\nabla^l Q_j A^m_{k'}, \bar{Q}_{<j}\phi_k), \tag{6.1.29}$$

$$I_2 = \sum_{j<k_{\min}} \bar{P}_{k_0}\bar{Q}_{\leq j}\mathcal{N}_{lm}(\Delta^{-1}\nabla^l Q_{\leq j}A^m_{k'}, \bar{Q}_j\phi_k). \tag{6.1.30}$$

These are estimated using Proposition 5.6.6. We use (5.6.10) with (6.1.7) and (6.1.10) to estimate $I_0$ in $\bar{X}_1^{-1/2}$. For $I_1$ we use (5.6.11) with (6.1.8) and (6.1.10), while for $I_2$ we use (5.6.11) with (6.1.13) and (6.1.8). This concludes the proof of (1.8.4).

## Proof of (1.8.5).

We separate $A_0\partial_t\phi$ and $A^j\partial_j\phi$. This case corresponds to low-high interactions in the Littlewood-Paley trichotomy. Thus for $k, k_0 \geq 0$, $k' \leq k - C$ (and $|k - k_0| \leq 5$) it suffices to prove

$$\|\bar{P}_{k_0}\big(A^0_{k'}\partial_t\phi_k\big) - \bar{P}_{k_0}\mathcal{H}^*_{k'}\big(A_0\partial_t\phi_k\big)\|_{\bar{N}_{k_0}} \lesssim \|P_{k'}A_0\|_{Y^\sigma}\|\phi_k\|_{\bar{S}^1_k} \tag{6.1.31}$$

$$\|\bar{P}_{k_0}\big(A^j_{k'}\partial_j\phi_k\big) - \bar{P}_{k_0}\mathcal{H}^*_{k'}\big(A^j\partial_j\phi_k\big)\|_{\bar{N}_{k_0}} \lesssim \|P_{k'}A_x\|_{S^\sigma_{k'}}\|\phi_k\|_{\bar{S}^1_k}. \tag{6.1.32}$$

Notice that the lack of an exponential gain of type $2^{\frac{1}{2}(k_{\min}-k_{\max})}$ (as in (6.1.26), (6.1.27)) is responsible for the need of $\ell^1$ summation in the norm on the RHS of (1.8.5).

We first treat the high modulation contribution, where we denote $A$ for either $A^0$ or $A^j$. For any $j \geq k' + C_2$, by Hölder's inequality

$$\begin{aligned} &\|\bar{P}_{k_0}\bar{Q}_{\geq j-5}\big(Q_jA_{k'}\partial\phi_k\big)\|_{\bar{X}_1^{-1/2}} \lesssim 2^{-\frac{1}{2}j}\|A_{k'}\|_{L^2L^\infty}\|\nabla\phi_k\|_{L^\infty L^2} \\ &\|\bar{P}_{k_0}\bar{Q}_{<j-5}\big(Q_jA_{k'}\bar{Q}_{\geq j-5}\partial\phi_k\big)\|_{L^1L^2} \lesssim \|A_{k'}\|_{L^2L^\infty}\|\bar{Q}_{\geq j-5}\nabla\phi_k\|_{L^2_{t,x}} \end{aligned} \tag{6.1.33}$$

Using (6.1.17), (6.1.9), (6.1.7), (6.1.8) and summing over $j \geq k' + C_2$, it follows that $\bar{P}_{k_0}\big(Q_{\geq k'+C_2}A_{k'}\partial\phi_k\big)$ is acceptable except

$$T = \sum_{j\geq k'+C_2} \bar{P}_{k_0}\bar{Q}_{<j-5}\big(Q_jA_{k'}\bar{Q}_{<j-5}\partial\phi_k\big)$$

By applying Lemma 5.5.2 (here we choose $C_2 > \frac{1}{2}C_0$) we see that the summand vanishes unless $j = k_{\max} + O(1)$. Then, by Lemma 5.6.1 we have

$$\|T\|_{L^1L^2} \lesssim \sum_{j=k+O(1)} \|Q_j A_{k'}\|_{L^2_{t,x}} \big( \sum_{\mathcal{C}_{k'}} \|P_{\mathcal{C}_{k'}} \nabla \phi_k\|^2_{L^2L^\infty} \big)^{\frac{1}{2}}$$

which is acceptable by (6.1.15), (6.1.8), (6.1.10). The terms

$$\bar{P}_{k_0}\bar{Q}_{\geq k'+C_2}\big(Q_{<k'+C_2}A_{k'}\partial\phi_k\big), \qquad \bar{P}_{k_0}\bar{Q}_{<k'+C_2}\big(Q_{<k'+C_2}A_{k'}\bar{Q}_{\geq k'+C_2}\partial\phi_k\big)$$

are treated in the same way as (6.1.33). We omit the details.

We continue with the low modulation terms. Since we are subtracting $\mathcal{H}^*$ we consider

$$I = \sum_{j<k'+C_2} \bar{P}_{k_0}\bar{Q}_j(Q_{<j}A^0_{k'} \cdot \partial_t \bar{Q}_{<j}\phi_k)$$

$$J = \sum_{j<k'+C_2} \bar{P}_{k_0}\bar{Q}_{\leq j}(Q_{\leq j}A^0_{k'} \cdot \partial_t \bar{Q}_j\phi_k)$$

and prove

$$\|I\|_{\bar{X}_1^{-1/2}} + \|J\|_{L^1L^2} \lesssim \|P_{k'}A_0\|_{Y^\sigma}\|\phi_k\|_{\bar{S}^1_k}.$$

by using (5.6.3) with (6.1.7) and (6.1.16) for $I$; we use (5.6.4) with (6.1.8) and (6.1.16) for $J$.

It remains to show that for $I_0, I_2$ from (6.1.28) and (6.1.30) (with summation over $j < k' + C_2$) we have

$$\|I_0\|_{\bar{X}_1^{-1/2}} + \|I_2\|_{L^1L^2} \lesssim \|A_{k'}\|_{S^\sigma_{k'}}\|\phi_k\|_{\bar{S}^1_k}.$$

These follow from (5.6.10) with (6.1.13), (6.1.7) and from (5.6.11) with (6.1.13), (6.1.8), respectively.

## Proof of (1.8.6).

This estimate follows from the next bound, for $k' < k - 5$

$$\|\mathcal{H}^*_{k'}\big(A^\alpha\partial_\alpha\phi_k\big)\|_{L^1L^2} \lesssim \|A_{k'}\|_{Z^\sigma_{k'}\times Z^\sigma_{ell,k'}}\|\phi_k\|_{\bar{S}^1_k}. \tag{6.1.34}$$

To prove (6.1.34), let $\ell = \frac{1}{2}(j-k')_- \geq -k - C$ and separate $A_0\partial_t\phi$ from $A^j\partial_j\phi$. We use (1.7.6) and denote by $\mathcal{N}(A,\phi)$ one of $A^0\partial_t\phi$ or $\mathcal{N}_{lm}(\Delta^{-1}\nabla^l A^m, \phi)$. We expand

$$\mathcal{H}^*_{k'}\mathcal{N}\big(A,\phi_k\big) = \sum_{j<k'+C_2^*}\sum_{\omega_1,\omega_2} \bar{Q}_{<j}\mathcal{N}\big(P^{\omega_1}_\ell Q_j A_{k'}, P^{\omega_2}_\ell \bar{Q}_{<j}\phi_k\big)$$

Splitting $\bar{Q}_{<j} = \bar{Q}^+_{<j} + \bar{Q}^-_{<j}$, $Q_j = Q^+_j + Q^-_j$, and applying Lemma 5.5.2 we see that the summand vanishes unless $|\angle(\omega_1, \pm\omega_2)| \lesssim 2^\ell$.

For $\mathcal{N} = \mathcal{N}_{lm}(\Delta^{-1}\nabla^l A^m, \phi)$ and $s_1, s \in \{+,-\}$, by Corollary 5.4.1 we have

$$\|\bar{Q}_{<j}\mathcal{N}\big(P_\ell^{\omega_1} Q_j^{s_1} A_{k'}, P_\ell^{\omega_2} \bar{Q}_{<j}^s \phi_k\big)\|_{L^1L^2} \lesssim 2^\ell \|P_\ell^{\omega_1} Q_j^{s_1} A_{k'}\|_{L^1L^\infty} \|P_\ell^{\omega_2} \bar{Q}_{<j}^s \nabla\phi_k\|_{L^\infty L^2}$$

For $\mathcal{N} = A^0 \partial_t \phi$ we have the same inequality but without the $2^\ell$ factor. This is compensated by the fact that the $Z_{ell}^\sigma$ norm is larger. Indeed, we have

$$Z_{ell}^\sigma = \Box^{\frac{1}{2}} \Delta^{-\frac{1}{2}} Z^\sigma \quad \text{and} \quad 2^\ell \|P_\ell^{\omega_1} Q_j^{s_1} A_{k'}^0\|_{L^1L^\infty} \simeq \|\Box^{\frac{1}{2}} \Delta^{-\frac{1}{2}} P_\ell^{\omega_1} Q_j^{s_1} A_{k'}^0\|_{L^1L^\infty}.$$

Note that for fixed $\omega_1$ there are only (uniformly) bounded number of $\omega_2$ such that the product is non-vanishing. Therefore, by Cauchy-Schwarz,

$$\|\mathcal{H}_{k'}^*\big(A^\alpha \partial_\alpha \phi_k\big)\|_{L^1L^2} \lesssim \sum_{\ell \leq 0} 2^{\frac{1}{2}\ell} \|A_{k'}\|_{Z_{k'}^\sigma \times Z_{ell,k'}^\sigma} \big(\sup_{\pm} \sum_{\omega_2} \|P_\ell^{\omega_2} \bar{Q}_{<j}^{\pm} \nabla\phi_k\|_{L^\infty L^2}^2\big)^{\frac{1}{2}}$$

which implies (6.1.34).

## Proof of (1.8.2), (1.8.7) and the $L^2H^{\sigma-\frac{3}{2}}$ part of (1.8.8)

One proceeds by dyadic decompositions. The $L_{t,x}^2$-type estimates follow easily by Hölder's inequality $L^\infty L^2 \times L^2 L^\infty \to L_{t,x}^2$ in the low-high/high-low cases and by Lemma 5.6.1 (eq. (5.6.1)) in the high-high to low case. One uses the norms (6.1.7), (6.1.14), (6.1.9), (6.1.17), (6.1.10), (6.1.13), (6.1.16).

The $L^\infty L^2$ estimate follows by Hölder's ( $L^\infty L^\infty \times L^\infty L^2 \to L^\infty L^2$ or $L^\infty L^2 \times L^\infty L^2 \to L^\infty L^1$) and Bernstein's inequalities ( $P_k L_x^2 \to 2^{\frac{d}{2}k} L_x^\infty$ or $P_k L_x^1 \to 2^{\frac{d}{2}k} L_x^2$), depending on which frequency (input or output) is the lowest.

## Proof of (1.8.8) for $\bar{N}$.

Suppose $k, k_2 \geq 0$, $k_1 \in \mathbb{Z}$. Let $r_0$ be the endpoint Strichartz exponent (i.e. $\frac{d-1}{r_0} = \sigma - \frac{1}{2}$). By Hölder's inequality and using Bernstein's inequality for the lowest frequency (input or output) we obtain

$$\|\bar{P}_k\big(P_{k_1} A \cdot \phi_{k_2}\big)\|_{L^1H^{\sigma-1}} \lesssim 2^{\frac{d}{r_0}(k - \max k_i)} 2^{-\frac{1}{r_0}|k_1 - k_2|} \|P_{k_1} A\|_{L^2\dot{H}^{\sigma-\frac{1}{2}}} \|\phi_{k_2}\|_{L^2W^{r_0,\rho}} \tag{6.1.35}$$

With $A = \partial_t A_0$ and $\|\phi_{k_2}\|_{L^2W^{r_0,\rho}} \lesssim \|\phi_{k_2}\|_{\bar{S}_{k_2}^\sigma}$, upon summation we obtain (1.8.8).

## Proof of (1.8.9) and (1.8.3)

We first prove the $L^1L^2$ part. For (1.8.9) we consider $k, k_2 \geq 0$, $k_1, k_3, k_4 \in \mathbb{Z}$. We apply (6.1.35) with $A = A_\alpha^1 A_\alpha^2$ together with

$$\|P_{k_1}(P_{k_3} A_\alpha^1 P_{k_4} A_\alpha^2)\|_{L^2\dot{H}^{\sigma-\frac{1}{2}}} \lesssim 2^{\frac{1}{2}(k_{\min} - k_{\max})} \|A_{\max\{k_3,k_4\}}\|_{L^\infty \dot{H}^\sigma} \|A_{\min\{k_3,k_4\}}\|_{L^2\dot{W}^{\infty,-\frac{1}{2}}}.$$

By summing we obtain (1.8.9). The same argument is used for $L^1L^2$ of (1.8.3).

To prove the $L^2\dot{H}^{\sigma-\frac{3}{2}}$ and $L^2H^{\sigma-\frac{3}{2}}$ estimates we write

$$\|P_k\big(P_{k_1}(fg)P_{k_2}h\big)\|_{L^2\dot{H}^{\sigma-\frac{3}{2}}} \lesssim 2^{\frac{1}{2}(k-\max k_i)}2^{-\frac{1}{2}|k_1-k_2|}\|P_{k_1}(fg)\|_{L^\infty\dot{H}^{\sigma-1}}\|P_{k_2}h\|_{L^2W^{r_0,\rho}}$$

and use $L^\infty\dot{H}^\sigma \times L^\infty\dot{H}^\sigma \to L^\infty\dot{H}^{\sigma-1}$ by Hölder and Sobolev embedding.

The $\ell^1L^\infty\dot{H}^{\sigma-2}$ part of (1.8.3) is similarly a consequence of Hölder and Bernstein inequalities.

## **Proof of** (6.1.5)

Recall that $\mathcal{H}$ subtracts terms only for high-high interactions. For $k' \leq k_2 - C_2 - 10$ we claim

$$\|(P_{k'} - \mathcal{H}_{k'})\mathbf{A}(\phi^1_{k_1},\phi^2_{k_2})\|_{Z^\sigma_{k'}\times Z^\sigma_{ell,k'}} \lesssim 2^{\frac{1}{2}(k'-k_2)}\|\phi^1_{k_1}\|_{\bar{S}^\sigma_{k_1}}\|\phi^2_{k_2}\|_{\bar{S}^\sigma_{k_2}}. \tag{6.1.36}$$

while the low-high interactions: for $k' \geq k_2 - C_2 - 10$

$$\|P_{k'}\mathbf{A}_x(\phi^1_{k_1},\phi^2_{k_2})\|_{Z^\sigma_{k'}} \lesssim 2^{-\frac{1}{2}|k_1-k_2|}\|\phi^1_{k_1}\|_{\bar{S}^\sigma_{k_1}}\|\phi^2_{k_2}\|_{\bar{S}^\sigma_{k_2}}. \tag{6.1.37}$$

Clearly, (6.1.36) and (6.1.37) imply (6.1.5). First we recall that

$$(\Box\mathbf{A}_x,\Delta\mathbf{A}_0)(\phi^1,\phi^2) = -\Im(\mathcal{P}_x(\phi^1\nabla_x\bar{\phi}^2),\phi^1\partial_t\bar{\phi}^2)$$

and the embedding from (2.1.16)

$$\big(\Box^{-1}\times\Delta^{-1}\big)P_{k'} : L^1L^2 \times L^1L^2 \to 2^{(\sigma-1)k'}Z^\sigma_{k'}\times Z^\sigma_{ell,k'}$$

**S**tep 1. Proof of (6.1.36). The terms

$$P_{k'}\mathbf{A}(\bar{Q}_{\geq k'+C}\phi^1_{k_1},\phi^2_{k_2}), \qquad P_{k'}\mathbf{A}(\bar{Q}_{\leq k'+C}\phi^1_{k_1},\bar{Q}_{\geq k'+C}\phi^2_{k_2})$$

are estimated using (6.1.19), (6.1.20) and (2.1.16). For $\mathbf{A}_0$ we note that (6.1.19), (6.1.20) still hold with $\mathcal{P}_j$ replaced by $\mathcal{L}$ [1] and $\nabla_x$ replaced by $\partial_t$. Recall that the $Z$ norms restrict modulation to $Q_{\leq k'+C}$. Thus it remains to consider

$$(P_{k'}Q_{\leq k'+C} - \mathcal{H}_{k'})\mathbf{A}(\bar{Q}_{\leq k'+C}\phi^1_{k_1},\bar{Q}_{\leq k'+C}\phi^2_{k_2})$$

For $\mathbf{A}_x$, using (1.7.5), we need to treat $\Box^{-1}I_1$, $\Box^{-1}I_2$ as defined in (6.1.22)-(6.1.24) (the $\Box^{-1}I_0$ term is subtracted by $\mathcal{H}_{k'}$). These are estimated using (6.1.25) and (2.1.16).

We turn to $\mathbf{A}_0$. By switching the roles of $\phi^1,\phi^2$ if needed, it remains to consider

$$J_j = P_{k'}Q_{\leq j}\mathbf{A}_0(\bar{Q}_j\phi^1_{k_1},\bar{Q}_{\leq j}\phi^2_{k_2}), \qquad j \leq k'+C.$$

[1] $\mathcal{L}$ denotes any translation invariant bilinear form with bounded mass kernel.

Using (2.1.17) we obtain

$$\begin{aligned}\|J_j\|_{Z^\sigma_{ell,k'}} &\lesssim \sum_{\pm; j'\leq j} 2^{\frac{1}{2}(j'-k')} \|P_{k'} Q^{\pm}_{j'} (\bar{Q}_j \phi^1_{k_1} \cdot \partial_t \bar{Q}_{\leq j} \phi^2_{k_2})\|_{L^1 \dot{H}^{\sigma-1}} \\ &\lesssim 2^{\frac{1}{2}(j-k')} 2^{\frac{1}{2}(k'-k_2)} \|\phi^1_{k_1}\|_{\bar{S}^\sigma_{k_1}} \|\phi^2_{k_2}\|_{\bar{S}^\sigma_{k_2}}\end{aligned}$$

For the last inequality we have used Prop. 5.6.2 together with (6.1.8) and (6.1.10). Summing in $j \leq k' + C$ completes the proof of (6.1.36).

**S**tep 2. Proof of (6.1.37). Due to skew-adjointness (see Remark 1.7.2), we may assume that $k_2 = k_{\min} + O(1)$. The terms

$$P_{k'} \mathbf{A}_x (\bar{Q}_{\geq k_2 - c} \phi^1_{k_1}, \phi^2_{k_2}), \qquad P_{k'} \mathbf{A}_x (\bar{Q}_{\prec k_2} \phi^1_{k_1}, \bar{Q}_{\geq k_2 - c} \phi^2_{k_2})$$

are estimated using (6.1.19), (6.1.20) and (2.1.16).

Note that the $Z$ norm restricts modulations to $Q_{\leq k'+C}$. Thus it remains to consider

$$P_{k'} Q_j \mathbf{A}_x (\bar{Q}_{\prec k_2} \phi^1_{k_1}, \bar{Q}_{\prec k_2} \phi^2_{k_2}) \tag{6.1.38}$$

for $j \leq k' + C$. When $j \geq k_2 + C$, by Lemma 5.5.2 the term vanishes unless $j = k' + O(1)$. In this case

$$\begin{aligned}&\|P_{k'} Q_j \mathbf{A}_x (\bar{Q}_{\prec k_2} \phi^1_{k_1}, \bar{Q}_{\prec k_2} \phi^2_{k_2})\|_{Z^\sigma_{k'}} \lesssim 2^{-2k'} \|\bar{Q}_{\prec k_2} \phi^1_{k_1} \nabla_x \bar{Q}_{\prec k_2} \phi^2_{k_2}\|_{L^1 L^\infty} \\ &\lesssim 2^{k_2 - 2k'} \|\bar{Q}_{\leq k_1} \phi^1_{k_1}\|_{L^2 L^\infty} \|\bar{Q}_{\prec k_2} \phi^2_{k_2}\|_{L^2 L^\infty} + 2^{k_2} \|\bar{Q}_{[k_2 - c, k_1]} \phi^1_{k_1}\|_{L^2 H^{\sigma-1}} \|\bar{Q}_{\prec k_2} \phi^2_{k_2}\|_{L^2 L^\infty}\end{aligned}$$

which is estimated using (6.1.9) and (6.1.8).

It remains to consider (6.1.38) for $j < k_2 + C$. Using (1.7.5) we decompose into sums of $\Box^{-1} I_i$, $(i = 0, 2)$ as defined in (6.1.22)-(6.1.24) (for $k_2 - C < j < k_2 + C$ with $\bar{Q}$ indices slightly adjusted). Then $\Box^{-1} I_1$ and $\Box^{-1} I_2$ are estimated using (6.1.25) and (2.1.16).

Now we consider $\Box^{-1} I_0$. Define $\ell := \frac{1}{2}(j - k_2)_- \geq \ell' := \frac{1}{2}(j - k')_-$ and for $s = \pm$ we decompose

$$P_{k'} Q^s_j \mathcal{N}_{lm} (\bar{Q}_{<j} \phi^1_{k_1}, \bar{Q}_{<j} \phi^2_{k_2}) = \sum_{s_2, \omega_i} P^{\omega_0}_{\ell'} P_{k'} Q^s_j \mathcal{N}_{lm} (P^{\omega_1}_{\ell'} \bar{Q}^s_{<j} \phi^1_{k_1}, P^{\omega_2}_{\ell} \bar{Q}^{s_2}_{<j} \phi^2_{k_2})$$

By Lemma 5.5.2, the summand on the RHS vanishes unless

$$\begin{aligned}|\angle(\omega_0, \omega_1)| \ &\lesssim 2^{\ell} 2^{k_2 - k'} + 2^{\ell'} \lesssim 2^{\ell'} \\ |\angle(s\omega_0, s_2 \omega_2)| &\lesssim 2^{\ell} + \max(2^{\ell'}, 2^{\ell}) \lesssim 2^{\ell}.\end{aligned}$$

Note that $P^{\omega_0}_{\ell'} P_{k'} Q^s_j$ and $2^{2\ell' + 2k'} \Box^{-1} P^{\omega_0}_{\ell'} P_{k'} Q^s_j$ are disposable. Corollary 5.4.1 implies

$$\|\mathcal{N}_{lm} (P^{\omega_1}_{\ell'} \bar{Q}^s_{<j} \phi^1_{k_1}, P^{\omega_2}_{\ell} \bar{Q}^{s_2}_{<j} \phi^2_{k_2})\|_{L^1 L^\infty} \lesssim 2^{\ell} \|P^{\omega_1}_{\ell'} \bar{Q}^s_{<j} \nabla \phi^1_{k_1}\|_{L^2 L^\infty} \|P^{\omega_2}_{\ell} \bar{Q}^{s_2}_{<j} \nabla \phi^2_{k_2}\|_{L^2 L^\infty} \tag{6.1.39}$$

For a fixed $\omega_0$ [resp. $\omega_1$], there are only (uniformly) bounded number of $\omega_1, \omega_2$ [resp. $\omega_0, \omega_2$] such that the summand is nonzero. Summing first in $\omega_2$ (finitely many terms), then the (essentially diagonal) summation in $\omega_0, \omega_1$, we obtain

$$\big(\sum_{\omega_0} \mathrm{LHS}(6.1.39)^2\big)^{\frac{1}{2}} \lesssim 2^{\ell}\big(\sum_{\omega_1} \|P_{\ell'}^{\omega_1} \bar{Q}^s_{<j} \nabla \phi^1_{k_1}\|^2_{L^2 L^\infty}\big)^{\frac{1}{2}} \sup_{\omega_2} \|P_{\ell}^{\omega_2} \bar{Q}^{s_2}_{<j} \nabla \phi^2_{k_2}\|_{L^2 L^\infty}$$

Keeping track of derivatives and dyadic factors, recalling the definition of $Z_{k'}$ and using (6.1.10) for $\phi^1, \phi^2$, we obtain

$$\|\Box^{-1} I_0\|_{Z^\sigma_{k'}} \lesssim \sum_{j<k_2} 2^{\frac{1}{4}(j-k_2)} 2^{k_2-k'} \|\phi^1_{k_1}\|_{\bar{S}^\sigma_{k_1}} \|\phi^2_{k_2}\|_{\bar{S}^\sigma_{k_2}}$$

This completes the proof of (6.1.37).

### Proof of (6.1.6)

The low-high part of the estimate for $\mathbf{A}_x(\phi^1, \phi^2)$ follows from (6.1.37). For the high-high parts of both $\mathbf{A}_x(\phi^1, \phi^2)$ and $\mathbf{A}_0(\phi^1, \phi^2)$ we fix the frequency and use (2.1.16), Hölder $L^2 L^4 \times L^2 L^4 \to L^1 L^2$ together with $L^2 L^4$ Strichartz inequalities. We gain the factor $2^{\frac{d-4}{2}(k_{\min}-k_{\max})}$ which suffices to do the summation in the present case $d \geq 5$.

## 6.2 Trilinear estimates for MKG

This section is devoted to the the proof of Proposition 1.8.3.

### Proof of Proposition 1.8.3

Our goal is to prove

$$\|\pi[\mathbf{A}(\phi^1, \phi^2)]\phi\|_{\bar{N}^{\sigma-1}} \lesssim \|\phi^1\|_{\bar{S}^\sigma} \|\phi^2\|_{\bar{S}^\sigma} \|\phi\|_{\bar{S}^\sigma}$$

First we note that (recalling definition (6.1.1)) (6.1.2) together with (6.1.3) implies

$$\|\pi[0, \mathbf{A}_0^{LH}(\phi^1, \phi^2)]\phi\|_{\bar{N}^{\sigma-1}} \lesssim \|\phi^1\|_{\bar{S}^\sigma} \|\phi^2\|_{\bar{S}^\sigma} \|\phi\|_{\bar{S}^\sigma}$$

Secondly, (6.1.4) and (1.8.5) imply

$$\|(I - \mathcal{H}^*)\pi[(\mathbf{A}_x, \mathbf{A}_0^{HH})(\phi^1, \phi^2)]\phi\|_{\bar{N}^{\sigma-1}} \lesssim \|\phi^1\|_{\bar{S}^\sigma} \|\phi^2\|_{\bar{S}^\sigma} \|\phi\|_{\bar{S}^\sigma}$$

For $d \geq 5$, (1.8.6) and (6.1.6) imply

$$\|\mathcal{H}^*\pi[(\mathbf{A}_x, \mathbf{A}_0^{HH})(\phi^1, \phi^2)]\phi\|_{\bar{N}^{\sigma-1}} \lesssim \|\phi^1\|_{\bar{S}^\sigma} \|\phi^2\|_{\bar{S}^\sigma} \|\phi\|_{\bar{S}^\sigma}$$

which concludes the proof in the case $d \geq 5$. In the remaining of this section we assume $d = 4$, $\sigma = 1$.

Next we use (1.8.6) together with (6.1.5) and obtain

$$\|\mathcal{H}^* \pi[(I - \mathcal{H})(\mathbf{A}_x, \mathbf{A}_0^{HH})(\phi^1, \phi^2)]\phi\|_{\bar{N}} \lesssim \|\phi^1\|_{\bar{S}^1} \|\phi^2\|_{\bar{S}^1} \|\phi\|_{\bar{S}^1}$$

Since $\mathcal{H}\mathbf{A}_0 = \mathcal{H}\mathbf{A}_0^{HH}$ it remains to consider $\mathcal{H}^* \pi[\mathcal{H}\mathbf{A}]\phi$ which we write using (1.7.8), (1.7.9) as

$$\mathcal{H}^* \pi[\mathcal{H}\mathbf{A}(\phi^1, \phi^2)]\phi = \mathcal{Q}^1 + \mathcal{Q}^2 + \mathcal{Q}^3$$

where

$$\begin{aligned}
\mathcal{Q}^1 :=& \mathcal{H}^*(\Box^{-1}\mathcal{H}\mathfrak{I}(\phi^1 \partial_\alpha \bar{\phi}^2) \cdot \partial^\alpha \phi), \\
\mathcal{Q}^2 :=& -\mathcal{H}^*(\mathcal{H}\Delta^{-1}\Box^{-1}\partial_t \partial_\alpha \mathfrak{I}(\phi^1 \partial_\alpha \bar{\phi}^2) \cdot \partial_t \phi), \\
\mathcal{Q}^3 :=& -\mathcal{H}^*(\mathcal{H}\Delta^{-1}\Box^{-1}\partial_\alpha \partial^i \mathfrak{I}(\phi^1 \partial_i \bar{\phi}^2) \cdot \partial^\alpha \phi).
\end{aligned}$$

and it remains to prove

$$\|\mathcal{Q}^i(\phi^1, \phi^2, \phi)\|_{\bar{N}} \lesssim \|\phi^1\|_{\bar{S}^1} \|\phi^2\|_{\bar{S}^1} \|\phi\|_{\bar{S}^1}, \qquad i = 1, 3;\ (d = 4). \tag{6.2.1}$$

## Proof of (6.2.1) for $\mathcal{Q}^1$

Fix $k, k_1, k_2 \geq 0$ and let $k_{\min} = \min(k, k_1, k_2) \geq 0$. The estimate follows from

$$\| \sum_{k' < k_{\min} - C} \sum_{j < k' + C} \mathcal{Q}^1_{j,k'}(\phi^1_{k_1}, \phi^2_{k_2}, \phi_k)\|_{N_k} \lesssim \|\phi^1_{k_1}\|_{\bar{S}^1_{k_1}} \|\phi^2_{k_2}\|_{\bar{S}^1_{k_2}} \|\phi_k\|_{\bar{S}^1_k} \tag{6.2.2}$$

by summing in $k_1 = k_2 + O(1)$, where

$$\mathcal{Q}^1_{j,k'}(\phi^1_{k_1}, \phi^2_{k_2}, \phi_k) = \bar{Q}_{<j}[P_{k'} Q_j \Box^{-1}(\bar{Q}_{<j}\phi^1_{k_1} \partial_\alpha \bar{Q}_{<j}\phi^2_{k_2}) \cdot \partial^\alpha \bar{Q}_{<j}\phi_k].$$

Define $l \in [-k_{\min}, C]$ by $j = k' + 2l$ which implies $\angle(\phi_k, P_{k'}A), \angle(\phi^2_{k_2}, P_{k'}A) \lesssim 2^l$. When $k_{\min} = 0$ we may set $l = 0$ and similarly for $l_0$ below.

In proving (6.2.2), we make the normalization

$$\|\phi^1_{k_1}\|_{\bar{S}^1_{k_1}} = 1, \qquad \|\phi^2_{k_2}\|_{\bar{S}^1_{k_2}} = 1, \qquad \|\phi_k\|_{\bar{S}^1_k} = 1. \tag{6.2.3}$$

Since we have a null form between $\phi^2$ and $\phi$ we use a bilinear partition of unity based on their angular separation:

$$\mathcal{Q}^1_{j,k'} = \sum_{l_0 + C < l' < l} \sum_{\substack{\omega_1, \omega_2 \in \Gamma(l') \\ \angle(\omega_1, \omega_2) \simeq 2^{l'}}} \mathcal{Q}^1_{j,k'}(\phi^1_{k_1}, P^{\omega_2}_{l'} \phi^2_{k_2}, P^{\omega_1}_{l'} \phi_k) + \sum_{\substack{\omega_1, \omega_2 \in \Gamma(l_0) \\ \angle(\omega_1, \omega_2) \lesssim 2^{l_0}}} \mathcal{Q}^1_{j,k'}(\phi^1_{k_1}, P^{\omega_2}_{l_0} \phi^2_{k_2}, P^{\omega_1}_{l_0} \phi_k)$$

where $l_0 := \max(-k_{\min}, l + k' - k_{\min}, \frac{1}{2}(j - k_{\min}))$ and the angle $\angle(\omega_1, \omega_2)$ is taken $\mod \pi$. Notice that the sums in $\omega_1, \omega_2$ are essentially diagonal. In each summand, we may insert $P_l^{[\omega_1]}$ in front of $P_{k'} Q_j \Box^{-1}$, where $P_l^{[\omega_1]}$ is uniquely (up to $O(1)$) defined by $\omega_1$ (or $\omega_2$).

For the first sum, for $k_{\min} > 0$, for any $l' \in [l_0 + C, l]$ we will prove

$$\sum_{\omega_1, \omega_2} \| \mathcal{Q}^1_{j,k'}(\phi^1_{k_1}, P^{\omega_2}_{l'} \phi^2_{k_2}, P^{\omega_1}_{l'} \phi_k) \|_{L^1 L^2} \lesssim 2^{\frac{1}{4}(l'+l)} 2^{\frac{1}{2}(k'-k_2)} \tag{6.2.4}$$

by employing the null-frame estimate in Corollary 5.2.2, which takes advantage of the angular separation. Summing in $l', j, k'$ we obtain part of (6.2.2).

At small angles however, one does not control the null-frame norms for Klein-Gordon waves and the null-form gives only a limited gain. We consider two cases.

For $j \geq -k_{\min}$ we sum the following in $j, k'$

$$\sum_{\omega_1, \omega_2} \| \mathcal{Q}^1_{j,k'}(\phi^1_{k_1}, P^{\omega_2}_{l_0} \phi^2_{k_2}, P^{\omega_1}_{l_0} \phi_k) \|_{L^1 L^2} \lesssim 2^l 2^{k' - k_{\min}} \tag{6.2.5}$$

When $j \leq -k_{\min}$ (thus $k' \leq -k_{\min} - 2l$ and $l_0 = -k_{\min}$) the operator $P_{k'} Q_j \Box^{-1}$ becomes more singular and we encounter a logarithmic divergence if we try to sum $k', j$ outside the norm in (6.2.2). We proceed as follows. We write

$$\mathcal{Q}^1_{j,k'}(\phi^1_{k_1}, P^{\omega_2}_{l_0} \phi^2_{k_2}, P^{\omega_1}_{l_0} \phi_k) = \bar{Q}_{<j} [P_l^{[\omega_1]} P_{k'} Q_j \Box^{-1} (\bar{Q}_{<j} \phi^1_{k_1} \partial_\alpha \bar{Q}_{<j} P^{\omega_2}_{l_0} \phi^2_{k_2}) \cdot \partial^\alpha \bar{Q}_{<j} P^{\omega_1}_{l_0} \phi_k]$$

We define

$$\tilde{\mathcal{Q}}^{\omega_i}_{j,k'} = P_l^{[\omega_1]} P_{k'} Q_j \Box^{-1} (\phi^1_{k_1} \partial_\alpha \bar{Q}_{<k_2 + 2l_0} P^{\omega_2}_{l_0} \phi^2_{k_2}) \cdot \partial^\alpha \bar{Q}_{<k+2l_0} P^{\omega_1}_{l_0} \phi_k$$

and we shall prove, using the embeddings in Prop. 2.2.2, that for any $l \in [-k_{\min}, C]$

$$\sum_{\omega_1, \omega_2} \| \sum_{k' \leq -k_{\min} - 2l} \tilde{\mathcal{Q}}^{\omega_i}_{k'+2l, k'} \|_{L^1 L^2} \lesssim 2^{-\frac{1}{2}(l + k_{\min})}, \tag{6.2.6}$$

which sums up (in $l$) towards the rest of (6.2.2) except for the remainders

$$\tilde{\mathcal{Q}}^{\omega_i}_{j,k'} - \mathcal{Q}^1_{j,k'}(\phi^1_{k_1}, P^{\omega_2}_{l_0} \phi^2_{k_2}, P^{\omega_1}_{l_0} \phi_k) = \mathcal{R}^{1,\omega_i}_{j,k'} + \mathcal{R}^{2,\omega_i}_{j,k'} + \mathcal{R}^{3,\omega_i}_{j,k'} + \mathcal{R}^{4,\omega_i}_{j,k'}$$

for which we have

$$\sum_{\omega_1, \omega_2} \| \mathcal{R}^{i,\omega}_{j,k'} \|_{N_k} \lesssim 2^{\frac{l}{2}} 2^{\frac{1}{2}(k'-k_2)}, \qquad i = 1, 4 \tag{6.2.7}$$

where

$$\begin{aligned}
\mathcal{R}^{1,\omega_i}_{j,k'} &:= \bar{Q}_{>j} [P_l^{[\omega_1]} P_{k'} Q_j \Box^{-1} (\bar{Q}_{<j} \phi^1_{k_1} \partial_\alpha \bar{Q}_{<j} P^{\omega_2}_{l_0} \phi^2_{k_2}) \cdot \partial^\alpha \bar{Q}_{<j} P^{\omega_1}_{l_0} \phi_k], \\
\mathcal{R}^{2,\omega_i}_{j,k'} &:= P_l^{[\omega_1]} P_{k'} Q_j \Box^{-1} (\bar{Q}_{<j} \phi^1_{k_1} \partial_\alpha \bar{Q}_{<j} P^{\omega_2}_{l_0} \phi^2_{k_2}) \cdot \partial^\alpha \bar{Q}_{[j, k-2k_{\min}]} P^{\omega_1}_{l_0} \phi_k, \\
\mathcal{R}^{3,\omega_i}_{j,k'} &:= P_l^{[\omega_1]} P_{k'} Q_j \Box^{-1} (\bar{Q}_{>j} \phi^1_{k_1} \partial_\alpha \bar{Q}_{<j} P^{\omega_2}_{l_0} \phi^2_{k_2}) \cdot \partial^\alpha \bar{Q}_{<k-2k_{\min}} P^{\omega_1}_{l_0} \phi_k, \\
\mathcal{R}^{4,\omega_i}_{j,k'} &:= P_l^{[\omega_1]} P_{k'} Q_j \Box^{-1} (\phi^1_{k_1} \partial_\alpha \bar{Q}_{[j, k_2 - 2k_{\min}]} P^{\omega_2}_{l_0} \phi^2_{k_2}) \cdot \partial^\alpha \bar{Q}_{<k-2k_{\min}} P^{\omega_1}_{l_0} \phi_k.
\end{aligned}$$

Summing in $j, k'$ we obtain the rest of (6.2.2).

## Proof of (6.2.4) and (6.2.5)

We are in the case $k_1 = k_2 + O(1)$, $k = \tilde{k} + O(1)$, $k' + C < k_{\min} = \min(k, k_1, k_2) > 0$, $j = k' + 2l$. We prove [2]

$$|\langle P_l^{[\omega_1]} P_{k'} Q_j \Box^{-1}(\bar{Q}_{<j}\phi^1_{k_1}\partial_\alpha \bar{Q}_{<j} P^{\omega_2}_{l'/l_0}\phi^2_{k_2}), \tilde{P}_{k'}\tilde{Q}_j(\partial^\alpha \bar{Q}_{<j} P^{\omega_1}_{l'/l_0}\phi_k \cdot \bar{Q}_{<j}\psi_{\tilde{k}}\rangle| \lesssim M_{\omega_1,\omega_2}\|\psi_{\tilde{k}}\|_{L^\infty L^2}$$

where $M_{\omega_1,\omega_2}$ will be defined below.

The two products above are summed over diametrically opposed boxes $\pm\mathcal{C}$ [resp. $\pm\mathcal{C}'$ ] of size $\simeq 2^{k'} \times (2^{k'+l})^3$ included in the angular caps $\mathcal{C}^{\omega_2}_{l'/l_0}$ [resp. $\mathcal{C}^{\omega_1}_{l'/l_0}$] where $P^{\omega_2}_{l'/l_0}$ [resp. $P^{\omega_1}_{l'/l_0}$] are supported (Lemma 5.5.2).

Note that $2^{j+k'} P_l^{[\omega_1]} P_{k'} Q_j \Box^{-1}$ acts by convolution with an integrable kernel. By a simple argument based on translation-invariance we may dispose of this operator (after first making the the $\mathcal{C}, \mathcal{C}'$ summation).

**S**tep 1: Proof of (6.2.4)

In this case the null form gains $2^{2l'}$. It suffices to show, having normalized (6.2.3)

$$2^{-j-k'}|\langle(\bar{Q}_{<j}\phi^1_{k_1}\partial_\alpha \bar{Q}^{\pm\pm'}_{<j} P^{\omega_2}_{l'}\phi^2_{k_2}), (\partial^\alpha \bar{Q}^{\pm}_{<j} P^{\omega_1}_{l'}\phi_k \cdot \bar{Q}_{<j}\psi_{\tilde{k}}\rangle| \lesssim M_{\omega_1,\omega_2}\|\psi_{\tilde{k}}\|_{L^\infty L^2} \tag{6.2.8}$$

$$\sum_{\omega_1,\omega_2} M_{\omega_1,\omega_2} \lesssim 2^{\frac{1}{4}(l'+l)} 2^{\frac{1}{2}(k'-k_2)} \tag{6.2.9}$$

where $\angle(\omega_1, \pm'\omega_2) \simeq 2^{l'}$. We write $2^{j+k'}\mathrm{LHS}(6.2.8) \lesssim$

$$\int \sum_{\substack{\mathcal{C}\subset\mathcal{C}^{\omega_2}_{l'} \\ \mathcal{C}'\subset\mathcal{C}^{\omega_1}_{l'}}} \|P_{-\mathcal{C}}\bar{Q}_{<j}\phi^1_{k_1}\|_{L^\infty_x}\|\partial_\alpha P_{\mathcal{C}}\bar{Q}^{\pm\pm'}_{<j}\phi^2_{k_2}\partial^\alpha P_{\mathcal{C}'}\bar{Q}^{\pm}_{<j}\phi_k\|_{L^2_x}\|P_{-\mathcal{C}'}\bar{Q}_{<j}\psi_{\tilde{k}}\|_{L^2_x}(t)\,\mathrm{d}t \lesssim$$

$$\int(\sum_{\mathcal{C}\subset\mathcal{C}^{\omega_2}_{l'}}\|P_{-\mathcal{C}}\bar{Q}_{<j}\phi^1_{k_1}\|^2_{L^\infty_x})^{\frac{1}{2}}(\sum_{\substack{\mathcal{C}\subset\mathcal{C}^{\omega_2}_{l'} \\ \mathcal{C}'\subset\mathcal{C}^{\omega_1}_{l'}}}\|\partial_\alpha P_{\mathcal{C}}\bar{Q}^{\pm\pm'}_{<j}\phi^2_{k_2}\partial^\alpha P_{\mathcal{C}'}\bar{Q}^{\pm}_{<j}\phi_k\|^2_{L^2_x})^{\frac{1}{2}}\|\bar{Q}_{<j}\psi_{\tilde{k}}\|_{L^2_x}(t)\,\mathrm{d}t$$

$$\lesssim (\sum_{\mathcal{C}\subset\mathcal{C}^{\omega_2}_{l'}}\|P_{-\mathcal{C}}\bar{Q}_{<j}\phi^1_{k_1}\|^2_{L^2L^\infty})^{\frac{1}{2}}\ \mathcal{I}_{\omega_1,\omega_2}(l')\|\bar{Q}_{<j}\ \psi_{\tilde{k}}\|_{L^\infty L^2}$$

where, using Corollary 5.2.2,

$$\mathcal{I}_{\omega_1,\omega_2}(l')^2 := \sum_{\mathcal{C}\subset\mathcal{C}^{\omega_2}_{l'}}\sum_{\mathcal{C}'\subset\mathcal{C}^{\omega_1}_{l'}}\|\partial_\alpha P_{\mathcal{C}}\bar{Q}^{\pm\pm'}_{<j}\phi^2_{k_2}\cdot\partial^\alpha P_{\mathcal{C}'}\bar{Q}^{\pm}_{<j}\phi_k\|^2_{L^2_{t,x}}$$
$$\lesssim 2^{l'}(\sum_{\mathcal{C}\subset\mathcal{C}^{\omega_2}_{l'}}\|P_{\mathcal{C}}\bar{Q}^{\pm\pm'}_{<j}\nabla_{t,x}\phi^2_{k_2}\|^2_{PW^{\pm\pm'}_{\mathcal{C}}})(\sum_{\mathcal{C}'\subset\mathcal{C}^{\omega_1}_{l'}}\|P_{\mathcal{C}'}\bar{Q}^{\pm}_{<j}\nabla_{t,x}\phi_k\|^2_{NE^{\pm}_{\mathcal{C}'}}).$$

[2]Notice that this case does not occur when $k_{\min} = 0$.

Thus in (6.2.8) we may take

$$M_{\omega_1,\omega_2} = 2^{-j-k'}(\sum_{\mathcal{C}\subset\mathcal{C}^{\omega_2}_{l'}} \|P_{-\mathcal{C}}\bar{Q}_{<j}\phi^1_{k_1}\|^2_{L^2L^\infty})^{\frac{1}{2}}\ \mathcal{I}_{\omega_1,\omega_2}(l')$$

and by Summing in $\omega_2$ (the $\omega_1$ sum is redundant) using C-S we have

$$\sum_{\omega_1,\omega_2} M_{\omega_1,\omega_2} \lesssim 2^{-2l-2k'}\cdot 2^{l'}\cdot 2^{\frac{1}{2}l}2^{k'}2^{-\frac{1}{2}k_1}\cdot 2^{\frac{3}{2}(k'+l)}$$

which implies (6.2.9).

**S**tep 2: Proof of (6.2.5) Here $j\geq -k_{\min}$. In this case the null form gains $2^{j-k_{\min}}$ and

$$2^{l_0} = \max(2^{-k_{\min}}, 2^l 2^{k'-k_{\min}}, 2^{\frac{1}{2}(j-k_{\min})}) \leq 2^l$$

By Prop. 5.2.1 and Remark 5.2.3 it suffices to prove, under (6.2.3)

$$2^{j-k_{\min}}|\langle P^{[\omega_1]}_l P_{k'}Q_j\Box^{-1}(\bar{Q}_{<j}\phi^1_{k_1}\nabla_{t,x}\bar{Q}_{<j}P^{\omega_2}_{l_0}\phi^2_{k_2}), \tilde{P}_{k'}\tilde{Q}_j(\nabla_{t,x}\bar{Q}_{<j}P^{\omega_1}_{l_0}\phi_k\cdot\bar{Q}_{<j}\psi_{\tilde{k}}\rangle| \lesssim M_{\omega_1,\omega_2}\|\psi_{\tilde{k}}\|_{L^\infty L^2}, \qquad \sum_{\omega_1,\omega_2} M_{\omega_1,\omega_2}\lesssim 2^l 2^{k'-k_{\min}}. \quad (6.2.10)$$

We have

$$\text{LHS (6.2.10)} \lesssim 2^{-k_{\min}-k'}2^{k_2}\sum_{\mathcal{C}\subset\mathcal{C}^{\omega_2}_{l_0}}\|P_{\mathcal{C}}\bar{Q}_{<j}\phi^1_{k_1}\|_{L^2L^\infty}\|P_{-\mathcal{C}}\bar{Q}_{<j}\phi^2_{k_2}\|_{L^2L^\infty} \times\sup_t\sum_{\mathcal{C}'\subset\mathcal{C}^{\omega_1}_{l_0}}\|P_{\mathcal{C}'}\bar{Q}_{<j}\nabla\phi_k(t)\|_{L^2_x}\|P_{-\mathcal{C}'}\bar{Q}_{<j}\psi_{\tilde{k}}(t)\|_{L^2_x}\lesssim M_{\omega_1,\omega_2}\|\psi_{\tilde{k}}\|_{L^\infty L^2}$$

where for each $t$ we have used Cauchy-Schwarz and orthogonality, where

$$M_{\omega_1,\omega_2} = 2^{-k_{\min}-k'}2^{k_2}(\sum_{\mathcal{C}\subset\mathcal{C}^{\omega_2}_{l_0}}\|P_{\mathcal{C}}\bar{Q}_{<j}\phi^1_{k_1}\|^2_{L^2L^\infty})^{\frac{1}{2}}(\sum_{\mathcal{C}\subset\mathcal{C}^{\omega_2}_{l_0}}\|P_{-\mathcal{C}}\bar{Q}_{<j}\phi^2_{k_2}\|^2_{L^2L^\infty})^{\frac{1}{2}}\|\nabla\phi_k\|_{L^\infty L^2}$$

Summing in $\omega_2$ (the $\omega_1$ sum is redundant) using C-S and (6.1.10), (6.1.7) we get (6.2.10).

## **Proof of** (6.2.6)

Recall that $l\in[-k_{\min}, C]$, $k_{\min}=\min(k,k_1,k_2)\geq 0$ and $k_1 = k_2 + O(1)$ are fixed. We are in the case $k'+2l = j\leq -k_{\min}$, thus $l_0 = -k_{\min}$, i.e. $\angle(\omega_1,\omega_2)\lesssim 2^{-k_{\min}}$.

By Prop. 5.2.1 and Remark 5.2.3 the null form gains $2^{-2k_{\min}}$. We can apply that proposition because $\bar{Q}_{<k_i-2k_{\min}} = Q_{<k_i-2k_{\min}+C}\bar{Q}_{<k_i-2k_{\min}}$.

We will apply Prop. 2.2.2. For $M=-k_{\min}-2l$, we write

$$\sum_{k'\leq M}\tilde{\mathcal{Q}}^{\omega_i}_{k'+2l,k'} = \sum_{\pm} T^{[\omega_1]}_l(\phi^1_{k_1}\partial_\alpha\bar{Q}_{<k_2+2l_0}P^{\omega_2}_{l_0}\phi^2_{k_2})\cdot\partial^\alpha\bar{Q}_{<k+2l_0}P^{\omega_1}_{l_0}\phi_k \quad (6.2.11)$$

We consider two cases.

**C**ase 1: $k_{\min} = k_2 + O(1)$. The null form gains $2^{-2k_2}$. We have

$$\|(6.2.11)\|_{L^1L^2} \lesssim 2^{-2k_2} \|T_l^{[\omega_1]}(\phi^1_{k_1} \bar{Q}_{<k_2+2l_0} P^{\omega_2}_{l_0} \nabla \phi^2_{k_2})\|_{L^1L^\infty} \|P^{\omega_1}_{l_0} \bar{Q}_{<k+2l_0} \nabla \phi_k\|_{L^\infty L^2}.$$

Using (2.2.5) and (2.2.2) we have

$$\|T_l^{[\omega_1]}(\phi^1_{k_1} \bar{Q}_{<k_2+2l_0} P^{\omega_2}_{l_0} \nabla \phi^2_{k_2})\|_{L^1L^\infty} \lesssim 2^{-\frac{1}{2}l} 2^{k_2} \|\phi^1_{k_1}\|_{L^2L^{4,2}} \|P^{\omega_2}_{l_0} \bar{Q}_{<k_2+2l_0} \phi^2_{k_2}\|_{L^2L^{4,2}}$$

Summing (diagonally) in $\omega_1, \omega_2$ we obtain (6.2.6) since $\|\phi^1_{k_1}\|_{L^2L^{4,2}} \lesssim 2^{-\frac{1}{4}k_1} \|\phi^1_{k_1}\|_{\bar{S}^1_{k_1}}$,

$$\big(\sum_\omega \|P^\omega_{l_0} \bar{Q}_{<k_2+2l_0} \phi^2_{k_2}\|^2_{L^2L^{4,2}}\big)^{\frac{1}{2}} \lesssim 2^{-\frac{1}{4}k_2} \|\phi^2_{k_2}\|_{\bar{S}^1_{k_2}}$$

$$\big(\sum_\omega \|P^\omega_{l_0} \bar{Q}_{<k+2l_0} \nabla_{t,x} \phi_k\|_{L^\infty L^2}\big)^{\frac{1}{2}} \lesssim \|\phi_k\|_{\bar{S}^1_k}. \tag{6.2.12}$$

**C**ase 2: $k_{\min} = k$. Now the null form gains $2^{-2k}$, so we can put $\phi_k$ in $L^2L^4$.

$$\|(6.2.11)\|_{L^1L^2} \lesssim 2^{-2k} \|T_l^{[\omega_1]}(\phi^1_{k_1} \bar{Q}_{<k_2+2l_0} P^{\omega_2}_{l_0} \nabla \phi^2_{k_2})\|_{L^2L^4} \|P^{\omega_1}_{l_0} \bar{Q}_{<k+2l_0} \nabla \phi_k\|_{L^2L^4}.$$

Using (2.2.4) and Hölder's inequality we have

$$\|T_l^{[\omega_1]}(\phi^1_{k_1} \bar{Q}_{<k_2+2l_0} P^{\omega_2}_{l_0} \nabla \phi^2_{k_2})\|_{L^2L^4} \lesssim 2^{-\frac{1}{2}l} 2^{k_2} \|\phi^1_{k_1}\|_{L^4L^{\frac{8}{3}}} \|P^{\omega_2}_{l_0} \bar{Q}_{<k_2+2l_0} \phi^2_{k_2}\|_{L^4L^{\frac{8}{3}}}$$

Summing (diagonally) in $\omega_1, \omega_2$ we obtain (6.2.6) since $\|\phi^1_{k_1}\|_{L^4L^{\frac{8}{3}}} \lesssim 2^{-\frac{5}{8}k_1} \|\phi^1_{k_1}\|_{\bar{S}^1_{k_1}}$,

$$\big(\sum_\omega \|P^\omega_{l_0} \bar{Q}_{<k_2+2l_0} \phi^2_{k_2}\|^2_{L^4L^{\frac{8}{3}}}\big)^{\frac{1}{2}} \lesssim 2^{-\frac{5}{8}k_2} \|\phi^2_{k_2}\|_{\bar{S}^1_{k_2}}$$

$$\big(\sum_\omega \|P^\omega_{l_0} \bar{Q}_{<k+2l_0} \nabla_{t,x} \phi_k\|_{L^2L^4}\big)^{\frac{1}{2}} \lesssim 2^{\frac{3}{4}k} \|\phi_k\|_{\bar{S}^1_k}.$$

## Proof of (6.2.7)

By Prop. 5.2.1 and Remark 5.2.3 the null form gains $2^{-2k_{\min}}$.

**S**tep 1: $\mathcal{R}^1$ and $\mathcal{R}^2$. Denoting

$$h^{\omega_i} = P_l^{[\omega_1]} P_{k'} Q_j \Box^{-1} (\bar{Q}_{<j} \phi^1_{k_1} \partial_\alpha \bar{Q}_{<j} P^{\omega_2}_{l_0} \phi^2_{k_2}),$$

we estimate using Bernstein and Prop. 5.6.2

$$\|h^{\omega_i}\|_{L^2L^\infty} \lesssim 2^{2k'+\frac{3}{2}l} \|h^{\omega_i}\|_{L^2_{t,x}} \lesssim 2^{-j-k'} 2^{2k'+\frac{3}{2}l} \big(\sum_{\mathcal{C}} \|P_{\mathcal{C}} \bar{Q}_{<j} \phi^1_{k_1}\|^2_{L^2L^\infty}\big)^{\frac{1}{2}} \|P^{\omega_2}_{l_0} \bar{Q}_{<j} \nabla \phi^2_{k_2}\|_{L^\infty L^2}$$

where $\mathcal{C} = C_{k'}(l)$. Using the $\bar{X}_1^{-\frac{1}{2}}$ norm, we have

$$\|\mathcal{R}^{1,\omega_i}_{j,k'}\|_{N_k} \lesssim 2^{-\frac{j}{2}} 2^{-2k_{\min}} \|h^{\omega_i}\|_{L^2L^\infty} \|P^{\omega_1}_{l_0} \bar{Q}_{<j} \nabla\phi_k\|_{L^\infty L^2}$$

$$\|\mathcal{R}^{2,\omega_i}_{j,k'}\|_{L^1L^2} \lesssim 2^{-2k_{\min}} \|h^{\omega_i}\|_{L^2L^\infty} \|P^{\omega_1}_{l_0} \bar{Q}_{[j,k-2k_{\min}]} \nabla\phi_k\|_{L^2_{t,x}}$$

Summing in $\omega_1, \omega_2$, we obtain (6.2.7) for $\mathcal{R}^1$, $\mathcal{R}^2$ by using (6.1.10) for $\phi^1$ and (6.2.12) for $\phi^2$ (first introducing $\bar{Q}_{<k_2+2l_0}$ and discarding $\bar{Q}_{<j}$), and (6.2.12), respectively (6.1.8) for $\phi$. We also use $2^{-k_{\min}} \lesssim 2^l$.

**S**tep 2: $\mathcal{R}^3$ and $\mathcal{R}^4$. We denote

$$\begin{aligned} h^{\omega_{1,2}}_3 &= P^{[\omega_1]}_l P_{k'} Q_j \Box^{-1} (\bar{Q}_{>j} \phi^1_{k_1} \partial_\alpha \bar{Q}_{<j} P^{\omega_2}_{l_0} \phi^2_{k_2}), \\ h^{\omega_{1,2}}_4 &= P^{[\omega_1]}_l P_{k'} Q_j \Box^{-1} (\phi^1_{k_1} \partial_\alpha \bar{Q}_{[j,k_2-2k_{\min}]} P^{\omega_2}_{l_0} \phi^2_{k_2}). \end{aligned}$$

For $i = 3, 4$ we have

$$\|\mathcal{R}^{i,\omega}_{j,k'}\|_{L^1L^2} \lesssim 2^{-2k_{\min}} \|h^{\omega_{1,2}}_i\|_{L^1L^\infty} \|P^{\omega_1}_{l_0} \bar{Q}_{<k-2k_{\min}} \nabla\phi_k\|_{L^\infty L^2}$$

We estimate using Prop. 5.6.2

$$\|h^{\omega_{1,2}}_3\|_{L^1L^\infty} \lesssim 2^{2k'+\frac{3}{2}l} \|h^{\omega_{1,2}}_3\|_{L^1L^2} \lesssim 2^{-j-k'} 2^{2k'+\frac{3}{2}l} \|\bar{Q}_{>j}\phi^1_{k_1}\|_{L^2_{t,x}} \Big(\sum_{\mathcal{C}} \|P_{\mathcal{C}} \bar{Q}_{<j} \nabla\phi^2_{k_2}\|^2_{L^2L^\infty}\Big)^{\frac{1}{2}}$$

where $\mathcal{C} = C_{k'}(0)$. Reversing the roles of $\phi^1, \phi^2$, $\|h^{\omega_{1,2}}_4\|_{L^1L^\infty}$ is also estimated.

Summing in $\omega_1, \omega_2$, we obtain (6.2.7) for $\mathcal{R}^3$, $\mathcal{R}^4$ by using (6.2.12) for $\phi$ and (6.1.8) (6.1.10), for $\phi^1, \phi^2$. We also use $2^{-k_{\min}} \lesssim 2^l$.

## Proof of (6.2.1) for $\mathcal{Q}^2$

Estimating in $L^1L^2$ we use Hölder's inequality with $\|\partial_t\phi_k\|_{L^\infty L^2} \lesssim \|\phi_k\|_{\bar{S}^1_k}$ and

$$\|\Delta^{-1}\Box^{-1}\partial_t Q_j P_{k'} \partial_\alpha (\bar{Q}_{<j}\phi^1_{k_1} \cdot \partial^\alpha \bar{Q}_{<j}\phi^2_{k_2})\|_{L^1L^\infty} \lesssim 2^l 2^{\frac{1}{2}(k'-k_1)} \|\phi^1_{k_1}\|_{\bar{S}^1_{k_1}} \|\phi^2_{k_2}\|_{\bar{S}^1_{k_2}} \tag{6.2.13}$$

The $\mathcal{Q}^2$ part of (6.2.1) follows by summing this in $k', j$, where

$$k_1 = k_2 + O(1), \quad k' + C < k_1, k, \quad j < k' + C, \quad l := \frac{1}{2}(j-k')_- \geq -k_1, k \tag{6.2.14}$$

To prove (6.2.13), first note that the product is summed over diametrically opposed boxes $\mathcal{C}_1, \mathcal{C}_2$ of size $\simeq 2^{k'} \times (2^{k'+l})^3$ (Lemma 5.5.2). Each term in the sum forces a localization $P^\omega_l$ in front of $Q_j P_{k'}$ and note that $2^{j+k'} P^\omega_l Q_j P_{k'} \Box^{-1}$ is disposable.

Now recall for (5.1.1) the decomposition (5.1.2)-(5.1.6). By Prop. 5.2.4, and the fact that here $\angle(\mathcal{C}_1, -\mathcal{C}_2) \lesssim 2^{l+k'-k_1}$ we have

$$\|\Delta^{-1}\Box^{-1}\partial_t Q_j P_{k'} \mathcal{N}_0(\bar{Q}_{<j}\phi^1_{k_1}, \bar{Q}_{<j}\phi^2_{k_2})\|_{L^1L^\infty} \lesssim 2^{-j-2k'} \times (2^{2l+2k'}) \times$$

$$\times \big( \sum_{\mathcal{C}_1} \|P_{\mathcal{C}_1} \bar{Q}_{<j} \phi^1_{k_1}\|^2_{L^2 L^\infty} \big)^{\frac{1}{2}} \big( \sum_{\mathcal{C}_2} \|P_{\mathcal{C}_2} \bar{Q}_{<j} \phi^2_{k_2}\|^2_{L^2 L^\infty} \big)^{\frac{1}{2}}$$

The same holds true for $\mathcal{M}_0$ , since now, by Prop. 5.1.2 we gain $2^{2k'-2k_1} \lesssim 2^{2k'+2l}$. Using (6.1.10) we obtain (6.2.13) for $\mathcal{N}_0, \mathcal{M}_0$. We turn to $\mathcal{R}_0^{\pm}$ and write

$$2^{k_2} \|\Delta^{-1} \Box^{-1} \partial_t Q_j P_{k'} \big( (\partial_t \mp i \langle D \rangle) \bar{Q}^{\pm}_{<j} \phi^1_{k_1} \cdot \bar{Q}^{\mp}_{<j} \phi^2_{k_2} \big) \|_{L^1 L^\infty} \lesssim 2^{k_2} \cdot 2^{-j-2k'} \times$$
$$\times \big( \sum_{\mathcal{C}_1} \|P_{\mathcal{C}_1} (\partial_t \mp i \langle D \rangle) \bar{Q}^{\pm}_{<j} \phi^1_{k_1}\|^2_{L^2 L^\infty} \big)^{\frac{1}{2}} \big( \sum_{\mathcal{C}_2} \|P_{\mathcal{C}_2} \bar{Q}^{\mp}_{<j} \phi^2_{k_2}\|^2_{L^2 L^\infty} \big)^{\frac{1}{2}}$$

Then we use (6.1.12), (6.1.10) to obtain $\mathcal{R}_0^{\pm}$-part of (6.2.13). The other parts of $\mathcal{R}_0^{\pm}$ follow by reversing the roles of $\phi^1, \phi^2$.

## Proof of (6.2.1) for $\mathcal{Q}^3$

Let $k_1, k_2, k, k', j, l$ as in (6.2.14) and $\tilde{k} = k + O(1)$, $k_{\min} := \min(k_1, k_2, k)$, $k' = k'' + O(1) < k_{\min} - C$, $j = j' + O(1) < k' + C$. We prove

$$\begin{aligned} &|\langle \frac{\partial Q_{j'} P_{k'}}{\Delta \Box} (\bar{Q}_{<j'} \phi^1_{k_1} \cdot \partial \bar{Q}_{<j'} \phi^2_{k_2}), Q_j P_{k''} \partial_\alpha (\partial^\alpha \bar{Q}_{<j} \phi_k \cdot \bar{Q}_{<j} \psi_{\tilde{k}}) \rangle| \lesssim \\ &\qquad \lesssim 2^{\frac{1}{2} l} 2^{\frac{1}{2}(k' - k_{\min})} \|\phi^1_{k_1}\|_{\bar{S}^1_{k_1}} \|\phi^2_{k_2}\|_{\bar{S}^1_{k_2}} \|\phi_k\|_{\bar{S}^1_k} \|\psi_{\tilde{k}}\|_{N^*_{\tilde{k}}} \end{aligned} \tag{6.2.15}$$

which, by duality, implies (6.2.1) for $\mathcal{Q}^3$. Like for $\mathcal{Q}^2$, we sum over diametrically opposed boxes $\pm \mathcal{C}$ of size $\simeq 2^{k'} \times (2^{k'+l})^3$ and introduce $P^\omega_l$ to bound $\Box^{-1}$.

First, using Prop. 5.6.2 and (6.1.7), (6.1.10), we estimate

$$\|\Delta^{-1} \Box^{-1} \partial Q_{j'} P_{k'} (\bar{Q}_{<j'} \phi^1_{k_1} \cdot \partial \bar{Q}_{<j'} \phi^2_{k_2}) \|_{L^2_{t,x}} \lesssim 2^{-j-2k'} (2^{\frac{1}{2} l} 2^{k'} 2^{-\frac{1}{2} k_1}) \|\phi^1_{k_1}\|_{\bar{S}^1_{k_1}} \|\phi^2_{k_2}\|_{\bar{S}^1_{k_2}}$$

For the second product, we recall the decomposition (5.1.1)-(5.1.6). By Prop. 5.2.4 and orthogonality, using the fact that $\angle(\phi, \psi) \lesssim 2^{l+k'-k}$, we have

$$\|Q_j P_{k''} \mathcal{N}_0 (\bar{Q}_{<j} \phi_k, \bar{Q}_{<j} \psi_{\tilde{k}}) \|_{L^2_{t,x}} \lesssim 2^{2l+2k'} \big( \sum_{\mathcal{C}} \|P_{\mathcal{C}} \bar{Q}_{<j} \phi_k\|^2_{L^2 L^\infty} \big)^{\frac{1}{2}} \|\psi_{\tilde{k}}\|_{L^\infty L^2}$$

The same holds true for $\mathcal{M}_0$ , since now, by Prop. 5.1.2 we gain $2^{2k'-2k} \lesssim 2^{2k'+2l}$.

For $\mathcal{R}_0^{\pm}$ we prove

$$\|Q_j P_{k''} \big( (\partial_t \mp i \langle D \rangle) \bar{Q}^{\pm}_{<j} \phi_k \cdot \bar{Q}^{\mp}_{<j} \psi_{\tilde{k}} \big) \|_{L^2_{t,x}} \lesssim \big( \sum_{\mathcal{C}} \|P_{\mathcal{C}} (\partial_t \mp i \langle D \rangle) \bar{Q}^{\pm}_{<j} \phi_k\|^2_{L^2 L^\infty} \big)^{\frac{1}{2}} \|\psi_{\tilde{k}}\|_{L^\infty L^2},$$

$$2^k \|Q_j P_{k''} (\bar{Q}^{\pm}_{<j} \phi_k \cdot (\partial_t \pm i \langle D \rangle) \bar{Q}^{\mp}_{<j} \psi_{\tilde{k}}) \|_{L^2_{t,x}} \lesssim 2^k \|\bar{Q}^{\pm}_{<j} \phi_k\|_{L^\infty L^2} \times$$
$$\times \big( \sum_{\mathcal{C}} \|P_{\mathcal{C}} (\partial_t \pm i \langle D \rangle) \bar{Q}^{\mp}_{<j} \psi_{\tilde{k}}\|^2_{L^2 L^\infty} \big)^{\frac{1}{2}} \lesssim 2^k 2^{2k' + \frac{3}{2} l} 2^{\frac{1}{2} j} \|\phi_k\|_{L^\infty L^2} \|\psi_{\tilde{k}}\|_{\bar{X}^{\frac{1}{2}}_\infty}.$$

where we have used Bernstein's inequality and orthogonality.

Putting all of the above together, using (6.1.10), (6.1.12) and (6.1.7), we obtain (6.2.15).

## 6.3 Bilinear estimates for MD

Here we prove Propositions 1.8.6–1.8.7 concerning bilinear estimates.

Unless otherwise stated, we restrict to the case $d = 4$; the general case of $d \geq 4$ is discussed in Remark 6.3.8 below.

### Preliminaries: Conventions and frequency envelope bounds

Henceforth, we use the shorthand $A$ to denote any $A_j$ ($j = 1, \dots, 4$). Unless otherwise stated, we normalize the frequency envelope norms of the inputs as follows:

$$\|B\|_{Y^1_a} = \|A\|_{S^1_a} = \|\psi\|_{(\tilde{S}^{1/2}_s)_b} = \|\varphi\|_{(\tilde{S}^{1/2}_{s'})_c} = 1. \tag{6.3.1}$$

Having control of the $S^1$ and $S^{1/2}_{\pm}$ norms through the frequency envelopes $a, b$ results in the following estimates, which we will use repeatedly in the proofs of the bilinear and trilinear estimates[3]:

$$\|A_k\|_{L^\infty L^2} \lesssim 2^{-k} a_k, \qquad \|\psi_k\|_{L^\infty L^2} \lesssim 2^{-\frac{1}{2}k} b_k, \tag{6.3.2}$$

$$\|Q_j A_k\|_{L^2 L^2} \lesssim 2^{-\frac{1}{2}\max\{j,k\}} 2^{-\frac{1}{2}j} 2^{-\frac{1}{2}k} a_k, \quad \|Q^s_j \psi_k\|_{L^2 L^2} \lesssim 2^{-\frac{1}{2}\max\{j,k\}} 2^{-\frac{1}{2}j} b_k. \tag{6.3.3}$$

For $k' < k$, we have

$$\begin{aligned} \Big( \sum_{\mathcal{C}_{k'}(0)} \|P_{\mathcal{C}_{k'}(0)} A_k\|^2_{L^2 L^\infty} \Big)^{1/2} &\lesssim 2^{k'} 2^{-\frac{1}{2}k} a_k, \\ \Big( \sum_{\mathcal{C}_{k'}(0)} \|P_{\mathcal{C}_{k'}(0)} \psi_k\|^2_{L^2 L^\infty} \Big)^{1/2} &\lesssim 2^{k'} b_k. \end{aligned} \tag{6.3.4}$$

For $k'$ such that $k' \leq k$ and $j \leq k' + C$, define $\ell = \frac{1}{2}(j - k')_-$. Then we have

$$\begin{aligned} \Big( \sum_{\mathcal{C}_{k'}(\ell)} \|P_{\mathcal{C}_{k'}(\ell)} Q_{<j} A_k\|^2_{L^2 L^\infty} \Big)^{1/2} &\lesssim 2^{k'} 2^{\frac{1}{2}\ell} 2^{-\frac{1}{2}k} a_k, \\ \Big( \sum_{\mathcal{C}_{k'}(\ell)} \|P_{\mathcal{C}_{k'}(\ell)} Q^s_{<j} \psi_k\|^2_{L^2 L^\infty} \Big)^{1/2} &\lesssim 2^{k'} 2^{\frac{1}{2}\ell} b_k. \end{aligned} \tag{6.3.5}$$

These bounds follow immediately from the definition of the norms $S^1_a$ and $S^{1/2}_b$.

The $\tilde{Z}^{1/2}_s$ component leads to the bound

$$\|Q^s_j \psi_k\|_{L^1 L^\infty} \lesssim 2^{\frac{1}{2}k} 2^{5(k-j)_+} b_k. \tag{6.3.6}$$

[3] Of course, the same estimates as $\psi$ hold for $\varphi$ with $(s, b_k)$ replaced by $(s', c_k)$.

Indeed, by (1.5.1) we have

$$\begin{aligned}\|Q_j^s\psi_k\|_{L^1L^\infty} \lesssim& 2^{-j}2^{4(k-j)_+}\|(i\partial_t+s|D|)\psi_k\|_{L^1L^\infty}\\ \lesssim& 2^{\frac{1}{2}k}2^{k-j}2^{4(k-j)_+}\|\psi_k\|_{\tilde{Z}_s^{1/2}},\end{aligned}$$

from which (6.3.6) follows.

Finally, the normalization $\|B\|_{Y_a^1}=1$ implies

$$\|B_k\|_{L^2L^2}\lesssim 2^{-\frac{3}{2}k}a_k, \qquad \|Q_jB_k\|_{L^2L^2}\lesssim 2^{-\max\{j,k\}}2^{-\frac{1}{2}k}a_k. \tag{6.3.7}$$

## Proof of Proposition 1.8.6

Here we prove (1.8.12)–(1.8.14).

### Step 0: Reduction to dyadic estimates

Under the normalization (6.3.1), we claim:

$$2^{-\frac{1}{2}k_0}\|P_{k_0}\mathcal{L}(\psi_{k_1},\varphi_{k_2})\|_{L^2L^2} \lesssim 2^{\frac{1}{2}(k_{\max}-k_{\min})}b_{k_1}c_{k_2}, \tag{6.3.8}$$

$$\|P_{k_0}\mathcal{N}^*(\psi_{k_1},\varphi_{k_2})\|_N \lesssim 2^{\delta_0(k_{\max}-k_{\min})}b_{k_1}c_{k_2}, \tag{6.3.9}$$

$$\|P_{k_0}\mathcal{N}_{ss'}(\psi_{k_1},\varphi_{k_2})\|_N \lesssim 2^{\delta_0(k_{\max}-k_{\min})}b_{k_1}c_{k_2}. \tag{6.3.10}$$

Proposition 1.8.6 follows from the above dyadic estimates. We begin with the proof of (1.8.12). Observe that $\mathcal{M}^E(\tilde{P}_{k_1}\cdot,\tilde{P}_{k_2}\cdot)=\mathcal{L}$ and $P_{k_0}\partial_t\mathcal{M}^E(\tilde{P}_{k_1}\cdot,\tilde{P}_{k_2}\cdot)=|D|P_{k_0}\mathcal{L}$. Therefore, (6.3.8) implies

$$\|P_{k_0}\mathcal{M}^E(\psi_{k_1},\varphi_{k_2})\|_{L^2\dot{H}^{-1/2}}+\|P_{k_0}\partial_t\mathcal{M}^E(\psi_{k_1},\varphi_{k_2})\|_{L^2\dot{H}^{-3/2}}\lesssim 2^{\frac{1}{2}(k_{\max}-k_{\min})}b_{k_1}c_{k_2}.$$

The LHS is non-vanishing only if $|k_{\max}-k_{\mathrm{med}}|\leq 5$ (Littlewood-Paley trichotomy). We now divide into cases $k_{\min}=k_0$, $k_1$ and $k_2$, which roughly correspond to (high-high), (low-high) and (high-low), respectively. In each case, summing up in $k_1,k_2$ using the exponential gain $2^{\frac{1}{2}(k_{\min}-k_{\max})}$ and the slow variance of $b,c$, we arrive at

$$\|P_{k_0}\mathcal{M}^E(\psi,\varphi)\|_{L^2\dot{H}^{-1/2}}+\|P_{k_0}\partial_t\mathcal{M}^E(\psi,\varphi)\|_{L^2\dot{H}^{-3/2}}\lesssim b_{k_0}c_{k_0},$$

which is precisely the desired estimate (1.8.12) under the normalization (6.3.1).

The proof of (1.8.13) and (1.8.14) proceeds similarly. By Proposition 5.4.4, we have $\mathcal{M}_x^R=\mathcal{N}^*$ and $\mathcal{M}_{x,s'}^S(\Pi_s\cdot,\cdot)=\mathcal{N}_{ss'}(\cdot,\cdot)$. Therefore, (6.3.9) and (6.3.10) imply

$$\|P_{k_0}\mathcal{M}^R(\psi_{k_1},\varphi_{k_2})\|_N+\|P_{k_0}\mathcal{M}_{s'}^S(\Pi_s\psi_{k_1},\varphi_{k_2})\|_N\lesssim 2^{\delta_0(k_{\min}-k_{\max})}b_{k_1}c_{k_2}.$$

On the other hand, application of (6.3.8) shows that

$$\|P_{k_0}\mathcal{M}^R(\psi_{k_1},\varphi_{k_2})\|_{L^2\dot{H}^{-1/2}}+\|P_{k_0}\mathcal{M}_{s'}^S(\Pi_s\psi_{k_1},\varphi_{k_2})\|_{L^2\dot{H}^{-1/2}}\lesssim 2^{\frac{1}{2}(k_{\min}-k_{\max})}b_{k_1}c_{k_2}.$$

Proceeding as before using Littlewood-Paley trichotomy, the exponential gain in $k_{\min}-k_{\max}$ and the slow variance of $b,c$, the desired estimates (1.8.13) and (1.8.14) follow.

The rest of this subsection is devoted to establishing (6.3.8)–(6.3.10).

**Step 1: Proof of** (6.3.8)

Without loss of generality, assume that $k_2 \le k_1$. Then (6.3.8) follows from application of (5.6.1) in Lemma 5.6.1 and the frequency envelope bounds (6.3.2) and (6.3.4).

**Step 2: Proof of** (6.3.9)

We first treat the high modulation contribution.

**Lemma 6.3.1.** *Assume the normalization* (6.3.1)*. For any* $k_0, k_1, k_2, j \in \mathbb{Z}$*, we have*

$$\begin{aligned} 2^{-\frac{1}{2}j}\|P_{k_0}Q_j\mathcal{L}(\psi_{k_1},\varphi_{k_2})\|_{L^2L^2} &\lesssim 2^{\frac{1}{2}(k_{\min}-j)}2^{\frac{1}{2}(k_{\min}-k_{\max})}b_{k_1}c_{k_2},\\ \|P_{k_0}\mathcal{L}(Q_j^{s_1}\psi_{k_1},\varphi_{k_2})\|_{L^1L^2} &\lesssim 2^{\frac{1}{2}(k_{\min}-j)}2^{\frac{1}{2}(k_{\min}-k_{\max})}b_{k_1}c_{k_2},\\ \|P_{k_0}\mathcal{L}(\psi_{k_1},Q_j^{s_2}\varphi_{k_2})\|_{L^1L^2} &\lesssim 2^{\frac{1}{2}(k_{\min}-j)}2^{\frac{1}{2}(k_{\min}-k_{\max})}b_{k_1}c_{k_2}. \end{aligned}$$

*Proof.* The lemma is a corollary of Lemma 5.6.1. Indeed, the first estimate follows from (5.6.1) and the frequency envelope bounds (6.3.2) and (6.3.4). Similarly, the second estimate follows from (5.6.2) and the frequency envelope bounds (6.3.3) and (6.3.4). The final estimate follows from the second one by symmetry. □

By Lemma 6.3.1 and (1.5.2), it follows that

$$\begin{aligned} \|P_{k_0}Q_{\ge k_{\min}-10}\mathcal{N}^*(\psi_{k_1},\varphi_{k_2})\|_{X_1^{0,-1/2}} &\lesssim 2^{\frac{1}{2}(k_{\min}-k_{\max})}b_{k_1}c_{k_2},\\ \|P_{k_0}Q_{<k_{\min}-10}\mathcal{N}^*(Q^s_{\ge k_{\min}-10}\psi_{k_1},\varphi_{k_2})\|_{L^1L^2} &\lesssim 2^{\frac{1}{2}(k_{\min}-k_{\max})}b_{k_1}c_{k_2},\\ \|P_{k_0}Q_{<k_{\min}-10}\mathcal{N}^*(Q^s_{<k_{\min}-10}\psi_{k_1},Q^{s'}_{\ge k_{\min}-10}\varphi_{k_2})\|_{L^1L^2} &\lesssim 2^{\frac{1}{2}(k_{\min}-k_{\max})}b_{k_1}c_{k_2}, \end{aligned}$$

which are all acceptable. Using the identity $P_{k_0}Q_{<k_{\min}-10} = \sum_{s_0} P_{k_0}Q^{s_0}_{<k_{\min}-10}$, the remainder can be written as

$$\sum_{s_0} P_{k_0}Q^{s_0}_{<k_{\min}-10}\mathcal{N}^*(Q^s_{<k_{\min}-10}\psi_{k_1},Q^{s'}_{<k_{\min}-10}\varphi_{k_2})$$

Summing according to the highest modulation, we decompose the remainder into $I_0 + I_1 + I_2$, where

$$I_0 = \sum_{s_0}\sum_{j<k_{\min}-10} P_{k_0}Q_j^{s_0}\mathcal{N}^*(Q^s_{<j}\psi_{k_1},Q^{s'}_{<j}\varphi_{k_2}), \tag{6.3.11}$$

$$I_1 = \sum_{s_0}\sum_{j<k_{\min}-10} P_{k_0}Q_{\le j}^{s_0}\mathcal{N}^*(Q^s_{j}\psi_{k_1},Q^{s'}_{<j}\varphi_{k_2}), \tag{6.3.12}$$

$$I_2 = \sum_{s_0}\sum_{j<k_{\min}-10} P_{k_0}Q_{\le j}^{s_0}\mathcal{N}^*(Q^s_{\le j}\psi_{k_1},Q^{s'}_{j}\varphi_{k_2}). \tag{6.3.13}$$

These sums can be estimated using Proposition 5.6.4. We split into three cases according to Littlewood-Paley trichotomy:

**S**tep 2.1: (high-high) interaction, $k_0 = k_{\min}$. Let $\ell = \frac{1}{2}(j - k_{\min})$. By Proposition 5.6.4, we have

$$\|I_0\|_{X_1^{0,-1/2}} \lesssim \sum_{j<k_0-10} 2^{-\frac{1}{2}j} 2^{\ell} \|\psi_{k_1}\|_{L^\infty L^2} \Big( \sum_{\mathcal{C}_{k_0}(\ell)} \|P_{\mathcal{C}_{k_0}(\ell)} Q^{s'}_{<j} \varphi_{k_2}\|^2_{L^2 L^\infty} \Big)^{1/2},$$

$$\|I_1\|_{L^1 L^2} \lesssim \sum_{j<k_0-10} 2^{\ell} \|Q^s_j \psi_{k_1}\|_{L^2 L^2} \Big( \sum_{\mathcal{C}_{k_0}(\ell)} \|P_{\mathcal{C}_{k_0}(\ell)} Q^{s'}_{<j} \varphi_{k_2}\|^2_{L^2 L^\infty} \Big)^{1/2},$$

$$\|I_2\|_{L^1 L^2} \lesssim \sum_{j<k_0-10} 2^{\ell} \Big( \sum_{\mathcal{C}_{k_0}(\ell)} \|P_{\mathcal{C}_{k_0}(\ell)} Q^s_{\leq j} \psi_{k_1}\|^2_{L^2 L^\infty} \Big)^{1/2} \|Q^{s'}_j \varphi_{k_2}\|_{L^2 L^2}.$$

Then by the frequency envelope bounds (6.3.2), (6.3.3) and (6.3.5), we obtain

$$\|I_0\|_{X_1^{0,-1/2}} + \|I_1\|_{L^1 L^2} + \|I_2\|_{L^1 L^2} \lesssim \sum_{j<k_0-10} 2^{\frac{1}{4}\ell} 2^{\frac{1}{2}(k_0-k_1)} b_{k_1} c_{k_2},$$

which is bounded by $2^{\frac{1}{2}(k_0-k_1)} b_{k_1} c_{k_2}$ and thus acceptable.

**S**tep 2.2: (high-low) interaction, $k_2 = k_{\min}$. As before, let $\ell = \frac{1}{2}(j - k_{\min})$. By Proposition 5.6.4, we have

$$\|I_0\|_{X_1^{0,-1/2}} \lesssim \sum_{j<k_2-10} 2^{\frac{1}{2}j} 2^{\ell} \|\psi_{k_1}\|_{L^\infty L^2} \Big( \sum_{\mathcal{C}_{k_2}(\ell)} \|P_{\mathcal{C}_{k_2}(\ell)} Q^{s'}_{<j} \varphi_{k_2}\|^2_{L^2 L^\infty} \Big)^{1/2},$$

$$\|I_1\|_{L^1 L^2} \lesssim \sum_{j<k_2-10} 2^{\ell} \|Q^s_j \psi_{k_1}\|_{L^2 L^2} \Big( \sum_{\mathcal{C}_{k_2}(\ell)} \|P_{\mathcal{C}_{k_2}(\ell)} Q^{s'}_{<j} \varphi_{k_2}\|^2_{L^2 L^\infty} \Big)^{1/2},$$

which are both bounded by $\lesssim 2^{\frac{1}{2}(k_2-k_1)} b_{k_1} c_{k_2}$ by the frequency envelope bounds (6.3.2), (6.3.3) and (6.3.5). However, a naive application of the same strategy to $I_2$ only yields

$$\|I_2\|_N \lesssim \sum_{j<k_2-10} 2^{\ell} \Big( \sum_{\mathcal{C}_{k_2}(\ell)} \|P_{\mathcal{C}_{k_2}(\ell)} Q^s_{\leq j} \psi_{k_1}\|^2_{L^2 L^\infty} \Big)^{1/2} \|Q_j \varphi_{k_2}\|_{L^2 L^2} \lesssim b_{k_1} c_{k_2}.$$

which lacks the necessary exponential gain in $k_1 - k_2$.

Here the idea is to use the $\tilde{Z}^{1/2}_{s'}$ bound (6.3.6). We introduce a small number $\delta_1 > 0$ to be determined later. We split the $j$-summation in $I_2$ to $I_2' = \sum_{j<k_2-10+\delta_1(k_2-k_1)}(\cdots)$ and $I_2'' = \sum_{j\in[k_2-10+\delta_1(k_2-k_1),k_2-10)}(\cdots)$. For the first sum $I_2'$, we use Proposition 5.6.3, (6.3.3) and (6.3.5) as before to estimate

$$\begin{aligned}\|I_2'\|_{L^1 L^2} &\lesssim \sum_{j<k_2-10+\delta_1(k_2-k_1)} 2^{\ell} \Big( \sum_{\mathcal{C}_{k_2}(\ell)} \|P_{\mathcal{C}_{k_2}(\ell)} Q^s_{\leq j} \psi_{k_1}\|^2_{L^2 L^\infty} \Big)^{1/2} \|Q^{s'}_j \varphi_{k_2}\|_{L^2 L^2} \\ &\lesssim \sum_{j<k_2-10+\delta_1(k_2-k_1)} 2^{\frac{1}{2}\ell} b_{k_1} c_{k_2} \lesssim 2^{\frac{\delta_1}{4}(k_2-k_1)} b_{k_1} c_{k_2},\end{aligned}$$

For the second sum $I_2''$, we use (1.5.2), Hölder's inequality and the frequency envelope bounds (6.3.2) and (6.3.6) to bound

$$\begin{aligned}\|I_2''\|_{L^1L^2} &\lesssim \sum_{j\in[k_2-10+\delta_1(k_2-k_1),k_2-10)} \|\psi_{k_1}\|_{L^\infty L^2}\|Q_j^{s'}\varphi_{k_2}\|_{L^1L^\infty}\\ &\lesssim \sum_{j\in[k_2-10+\delta_1(k_2-k_1),k_2-10)} 2^{-\frac{1}{2}k_1}2^{\frac{1}{2}k_2}2^{5(k_2-j)}b_{k_1}c_{k_2} \lesssim 2^{(\frac{1}{2}-5\delta_1)(k_2-k_1)}b_{k_1}c_{k_2}.\end{aligned}$$

In conclusion, we have

$$\|I_2\|_{L^1L^2} \lesssim 2^{\min\{\frac{\delta_1}{4},\frac{1}{2}-5\delta_1\}(k_2-k_1)}b_{k_1}c_{k_2},$$

which is acceptable once we choose $0 < \delta_1 < \frac{1}{10}$.

**S**tep 2.3: (low-high) interaction, $k_1 = k_{\min}$. This case is strictly easier than Step 2.2, thanks to the additional gain $2^{k_{\min}-\min\{k_0,k_2\}} \simeq 2^{k_1-k_2}$ in Proposition 5.6.4; in particular, the use of the $\tilde{Z}_s^{1/2}$ bound (6.3.6) is not necessary. We omit the details.

#### Step 3: Proof of (6.3.10)

We proceed similarly to Step 2, replacing the null form $\mathcal{N}^*$ by $\mathcal{N}_{ss'}$ and thus Proposition 5.6.4 by Proposition 5.6.3. The proof applies verbatim until reduction to the low modulation case (i.e., before Steps 2.1–2.3). A minor difference now is that the factor $2^{k_{\min}-\min\{k_1,k_2\}}$ does *not*[4] gain $2^{k_1-k_2}$ in the (low-high) interaction case (i.e., analogue of Step 2.3); however, the same proof as in the (high-low) case applies (Step 2.2).

## Proof of Proposition 1.8.7, part I: $N_{\pm}^{1/2}$-bounds for $\widetilde{\mathcal{N}}$

In this subsection, we prove (1.8.15)–(1.8.17) concerning the remainders $\widetilde{\mathcal{N}}^E$, $\widetilde{\mathcal{N}}^R$ and $\widetilde{\mathcal{N}}_s^S$.

#### Step 0: Reduction to dyadic estimates

Recall that $\mathcal{N}^E(\tilde{P}_{k_1}\cdot,\tilde{P}_{k_2}\cdot) = \mathcal{L}$, $\mathcal{N}^R = \mathcal{N}$ and $\Pi_{s'}\mathcal{N}_s^S = \mathcal{N}_{ss'}^*$, which vanish when applied to inputs $A_{k_1},\psi_{k_2}$ unless (say) $k_1 \geq k_2-20$. The condition $k_1 \geq k_2-20$ effectively eliminates the (low-high) interaction (i.e., $k_{\min} = k_1$). More precisely, if $k_1 = k_{\min}$ and $k_1 \geq k_2-20$, then all three frequencies must be comparable (i.e., $|k_{\max}-k_{\min}| \leq C$) thanks to the Littlewood-Paley trichotomy $|k_{\max}-k_{\rm med}| \leq 5$.

Under the normalization (6.3.1) and the condition $k_1 \geq k_2-20$, we claim:

$$\|P_{k_0}\mathcal{L}(B_{k_1},\psi_{k_2})\|_{N_{s'}^{1/2}} \lesssim 2^{\frac{1}{2}(k_{\min}-k_{\max})}a_{k_1}b_{k_2}, \tag{6.3.14}$$

$$\|P_{k_0}\mathcal{N}(A_{k_1},\psi_{k_2})\|_{N_{s'}^{1/2}} \lesssim 2^{\delta_0(k_{\min}-k_{\max})}a_{k_1}b_{k_2}, \tag{6.3.15}$$

$$\|P_{k_0}\mathcal{N}_{ss'}^*(A_{k_1},\psi_{k_2})\|_{N_{s'}^{1/2}} \lesssim 2^{\delta_0(k_{\min}-k_{\max})}a_{k_1}b_{k_2}. \tag{6.3.16}$$

[4]Now this factor gains another $2^{k_0-k_1}$ in the (high-high) interaction case (i.e., analogue of Step 2.1), which was already fine.

From these estimates, (1.8.15)–(1.8.17) follow as in the proof of (1.8.12)–(1.8.14) from the dyadic bounds (6.3.8)–(6.3.10) in Section 6.3; we omit the details.

**Step 1: Proof of** (6.3.14)

By (5.6.2) in Lemma 5.6.1 and the frequency envelope bounds (6.3.7) and (6.3.4), we have

$$\|P_{k_0}\mathcal{L}(B_{k_1}, \psi_{k_2})\|_{L^1 \dot{H}^{1/2}} \lesssim 2^{k_{\min}} 2^{\frac{1}{2}k_0} 2^{-\frac{3}{2}k_1} a_{k_1} b_{k_2},$$

which implies (6.3.14) under the condition $k_1 \geq k_2 - 20$.

**Step 2: Proof of** (6.3.15)

As before, we begin with the high modulation contribution.

**Lemma 6.3.2.** *Assume the normalization* (6.3.1). *For any* $k_0, k_1, k_2, j \in \mathbb{Z}$ *such that* $k_1 \geq k_2 - 20$*, we have*

$$\begin{aligned}
\|P_{k_0} Q^{s'}_j \mathcal{L}(A_{k_1}, \psi_{k_2})\|_{N^{1/2}_{s'}} \lesssim& 2^{\frac{1}{2}(k_{\min}-j)} 2^{\frac{1}{2}(k_{\min}-k_{\max})} a_{k_1} b_{k_2}, \\
\|P_{k_0} \mathcal{L}(Q_j A_{k_1}, \psi_{k_2})\|_{N^{1/2}_{s'}} \lesssim& 2^{\frac{1}{2}(k_{\min}-j)} 2^{\frac{1}{2}(k_{\min}-k_{\max})} a_{k_1} b_{k_2}, \\
\|P_{k_0} \mathcal{L}(A_{k_1}, Q^s_j \psi_{k_2})\|_{N^{1/2}_{s'}} \lesssim& 2^{\frac{1}{2}(k_{\min}-j)} 2^{\frac{1}{2}(k_{\min}-k_{\max})} a_{k_1} b_{k_2}.
\end{aligned}$$

*Proof.* Like Lemma 6.3.1, this lemma is a corollary of Lemma 5.6.1. Applying (5.6.1) with $(f, g) = (A_{k_1}, \psi_{k_2})$ and the frequency envelope bounds (6.3.2) and (6.3.4), we have

$$2^{\frac{1}{2}k_0} 2^{-\frac{1}{2}j} \|P_{k_0} Q^{s'}_j \mathcal{L}(A_{k_1}, \psi_{k_2})\|_{L^2 L^2} \lesssim 2^{k_{\min}} 2^{\frac{1}{2}k_0} 2^{-k_1} 2^{-\frac{1}{2}j} a_{k_1} b_{k_2},$$

which proves the first estimate under the condition $k_1 \geq k_2 - 20$. On the other hand, applying (5.6.1) in two different ways, then using the frequency envelope bounds (6.3.3) and (6.3.4), we have

$$\begin{aligned}
2^{\frac{1}{2}k_0} \|P_{k_0} \mathcal{L}(Q_j A_{k_1}, \psi_{k_2})\|_{L^1 L^2} \lesssim& 2^{k_{\min}} 2^{\frac{1}{2}k_0} 2^{-k_1} 2^{-\frac{1}{2}j} a_{k_1} b_{k_2}, \\
2^{\frac{1}{2}k_0} \|P_{k_0} \mathcal{L}(A_{k_1}, Q^s_j \psi_{k_2})\|_{L^1 L^2} \lesssim& 2^{k_{\min}} 2^{\frac{1}{2}k_0} 2^{-\frac{1}{2}k_1} 2^{-\frac{1}{2}k_2} 2^{-\frac{1}{2}j} a_{k_1} b_{k_2},
\end{aligned}$$

which imply the other two estimates under the condition $k_1 \geq k_2 - 20$. □

Proceeding as in Step 2 of Section 6.3, where we use Lemma 6.3.2 instead of Lemma 6.3.1, the proof of (6.3.15) is reduced to handling the contribution of

$$\sum_{s_1} P_{k_0} Q^{s'}_{<k_{\min}-10} \mathcal{N}(Q^{s_1}_{<k_{\min}-10} A_{k_1}, Q^s_{<k_{\min}-10} \psi_{k_2}) = I_0 + I_1 + I_2,$$

where

$$I_0 = \sum_{s_1} \sum_{j<k_{\min}-10} P_{k_0} Q_j^{s'} \mathcal{N}(Q_{\leq j}^{s_1} A_{k_1}, Q_{\leq j}^{s} \varphi_{k_2}), \tag{6.3.17}$$

$$I_1 = \sum_{s_1} \sum_{j<k_{\min}-10} P_{k_0} Q_{<j}^{s'} \mathcal{N}(Q_j^{s_1} A_{k_1}, Q_{<j}^{s} \varphi_{k_2}), \tag{6.3.18}$$

$$I_2 = \sum_{s_1} \sum_{j<k_{\min}-10} P_{k_0} Q_{<j}^{s'} \mathcal{N}(Q_{\leq j}^{s_1} A_{k_1}, Q_j^{s} \varphi_{k_2}). \tag{6.3.19}$$

We now split into two (slightly overlapping) cases, which roughly correspond to (high-high) and (high-low) interaction:

**S**tep 2.1: (high-high) interaction, $k_0 = k_{\min} + O(1)$. Let $\ell = \frac{1}{2}(j - k_{\min})$. Using Proposition 5.6.3 *neglecting* the gain $2^{k_{\min} - \min\{k_1, k_2\}} \simeq 2^{k_{\min} - k_{\max}}$, we have

$$\|I_0\|_{X_{s',1}^{1/2,-1/2}} \lesssim \sum_{j<k_{\min}-10} 2^{-\frac{1}{2}j} 2^{\frac{1}{2}k_0} 2^{\ell} \|A_{k_1}\|_{L^\infty L^2} \Big( \sum_{\mathcal{C}_{k_{\min}}(\ell)} \|P_{\mathcal{C}_{k_{\min}}(\ell)} Q_{<j}^{s} \psi_{k_2}\|_{L^2 L^\infty}^2 \Big)^{1/2},$$

$$\|I_1\|_{L^1 \dot{H}^{1/2}} \lesssim \sum_{s_1} \sum_{j<k_{\min}-10} 2^{\frac{1}{2}k_0} 2^{\ell} \|Q_j^{s_1} A_{k_1}\|_{L^2 L^2} \Big( \sum_{\mathcal{C}_{k_{\min}}(\ell)} \|P_{\mathcal{C}_{k_{\min}}(\ell)} Q_{<j}^{s} \psi_{k_2}\|_{L^2 L^\infty}^2 \Big)^{1/2},$$

$$\|I_2\|_{L^1 \dot{H}^{1/2}} \lesssim \sum_{s_1} \sum_{j<k_{\min}-10} 2^{\frac{1}{2}k_0} 2^{\ell} \Big( \sum_{\mathcal{C}_{k_{\min}}(\ell)} \|P_{\mathcal{C}_{k_{\min}}(\ell)} Q_{\leq j}^{s_1} A_{k_1}\|_{L^2 L^\infty}^2 \Big)^{1/2} \|Q_j^{s} \psi_{k_2}\|_{L^2 L^2}.$$

Then by the frequency envelope bounds (6.3.2), (6.3.3) and (6.3.5), we obtain

$$\|I_0\|_{N_{s'}^{1/2}} + \|I_1\|_{N_{s'}^{1/2}} + \|I_2\|_{N_{s'}^{1/2}} \lesssim \sum_{j<k_{\min}-10} 2^{\frac{1}{4}\ell} 2^{k_{\min}-k_{\max}} a_{k_1} b_{k_2} \lesssim 2^{k_{\min}-k_{\max}} a_{k_1} b_{k_2},$$

which is acceptable.

**S**tep 2.2: (high-low) interaction, $k_2 = k_{\min}$. As in Step 2.2 of Section 6.3, we need to use the $\tilde{Z}_s^{1/2}$ bound (6.3.6) in addition to Proposition 5.6.3. As before, let $\ell = \frac{1}{2}(j - k_{\min})$ and $\delta_1 \in (0, 1/10)$ be the small constant in Step 2.2 of Section 6.3. By Proposition 5.6.3 we have

$$\|I_0\|_{X_1^{1/2,-1/2}} \lesssim \sum_{j<k_2-10} 2^{\frac{1}{2}j} 2^{\frac{1}{2}k_0} 2^{\ell} \|A_{k_1}\|_{L^\infty L^2} \Big( \sum_{\mathcal{C}_{k_2}(\ell)} \|P_{\mathcal{C}_{k_2}(\ell)} Q_{<j}^{s} \psi_{k_2}\|_{L^2 L^\infty}^2 \Big)^{1/2},$$

$$\|I_1\|_{L^1 L^2} \lesssim \sum_{s_1} \sum_{j<k_2-10} 2^{\frac{1}{2}k_0} 2^{\ell} \|Q_j^{s_1} A_{k_1}\|_{L^2 L^2} \Big( \sum_{\mathcal{C}_{k_2}(\ell)} \|P_{\mathcal{C}_{k_2}(\ell)} Q_{<j}^{s} \psi_{k_2}\|_{L^2 L^\infty}^2 \Big)^{1/2},$$

which are bounded by $\lesssim 2^{\frac{1}{2}(k_2-k_1)} a_{k_1} b_{k_2}$ thanks to the frequency envelope bounds (6.3.2), (6.3.3) and (6.3.5). For $I_2$, we split the $j$-summation and write $I_2 = I_2' + I_2''$, where

$I_2' = \sum_{j<k_2-10+\delta_1(k_2-k_1)}(\cdots)$ and $I_2'' = \sum_{j\in[k_2-10+\delta_1(k_2-k_1),k_2-10)}(\cdots)$. For $I_2'$, we use Proposition 5.6.3 to obtain

$$\|I_2'\|_{L^1\dot{H}^{1/2}} \lesssim \sum_{s_1} \sum_{j<k_2-10+\delta_1(k_2-k_1)} 2^{\ell} 2^{\frac{1}{2}k_0} \Big( \sum_{\mathcal{C}_{k_2}(\ell)} \|P_{\mathcal{C}_{k_2}(\ell)} Q^{s_1}_{<j} A_{k_1}\|^2_{L^2L^\infty} \Big)^{1/2} \|Q^s_j \psi_{k_2}\|_{L^2L^2},$$

which, in turn, can be bounded by $\lesssim 2^{\frac{\delta_1}{4}(k_2-k_1)} a_{k_1} b_{k_2}$ using (6.3.3) and (6.3.5). For $I_2''$, we use (1.5.2), Hölder's inequality, (6.3.2) and (6.3.6) to bound

$$\begin{aligned}\|I_2''\|_{L^1\dot{H}^{1/2}} &\lesssim \sum_{j\in[k_2-10+\delta_1(k_2-k_1),k_2-10)} 2^{\frac{1}{2}k_0} \|A_{k_1}\|_{L^\infty L^2} \|Q^s_j \psi_{k_2}\|_{L^1L^\infty} \\ &\lesssim \sum_{j\in[k_2-10+\delta_1(k_2-k_1),k_2-10)} 2^{\frac{1}{2}k_0} 2^{-k_1} 2^{\frac{1}{2}k_2} 2^{5(k_2-j)} a_{k_1} b_{k_2},\end{aligned}$$

which is bounded by $2^{(\frac{1}{2}-5\delta_1)(k_2-k_1)} a_{k_1} b_{k_2}$ and thus acceptable (since $\delta_1 < 1/10$).

#### Step 3: Proof of (6.3.16)

The argument in Step 2 applies exactly, with $\mathcal{N}$ and Proposition 5.6.3 replaced by $\mathcal{N}^*_{ss'}$ and Proposition 5.6.4, respectively; note that this is possible since we have *not* used the extra gain $2^{k_{\min}-\min\{k_1,k_2\}}$ from Proposition 5.6.3 in Step 2.1 above. We omit the details.

*Remark* 6.3.3. In the course of Step 2, we have proved the bound

$$\|P_{k_0}\mathcal{N}(A_{k_1},\psi_{k_2})\|_{N^{1/2}_{s'}} \lesssim \|A_{k_1}\|_{S^1} \|\psi_{k_2}\|_{S^{1/2}_s} \tag{6.3.20}$$

when $k_0 = k_{\min} + O(1)$ and $k_1 > k_2 - 20$. In fact, the number 20 does not play any role, and the same bound holds (with an adjusted constant) when all three $k_0, k_1 k_2$ are within an $O(1)$-interval of each other.

## Proof of Proposition 1.8.7, part II: $N^{1/2}_{\pm}$-bounds for $\pi[A]$

Here we prove (1.8.18), (1.8.19) and (1.8.20) concerning the paradifferential terms $\pi^E[A_0]$, $\pi^R[A_x]$ and $\pi^S_s[A_x]$.

#### Step 0: Reduction to dyadic estimates

As before, note that $\mathcal{N}^E(\tilde{P}_{k_1}\cdot, \tilde{P}_{k_2}\cdot) = \mathcal{L}$, $\mathcal{N}^R = \mathcal{N}$ and $\Pi_{s'}\mathcal{N}^S_s = \mathcal{N}^*_{ss'}$, and $\pi^E[A_0], \pi^R[A_x], \Pi_{s'}\pi^S_s[A_x]$ vanish when applied to $A_{k_1}, \psi_{k_2}$ unless (say) $k_1 < k_2 - 5$. By Littlewood-Paley trichotomy ($|k_{\max} - k_{\text{med}}| \leq 5$), we only need to consider the (low-high) interaction, i.e., $k_{\min} = k_1$ and $k_0 = k_2 + O(1)$.

Under the normalization (6.3.1) and the condition $k_1 < k_2 - 5$, we claim:

$$\|P_{k_0}\mathcal{L}(B_{k_1}, \psi_{k_2})\|_{N^{1/2}_{-s}} \lesssim 2^{\frac{1}{4}(k_{\min}-k_{\max})} a_{k_1} b_{k_2}, \tag{6.3.21}$$

$$\|P_{k_0}\mathcal{L}(A_{k_1}, \psi_{k_2})\|_{N^{1/2}_{-s}} \lesssim 2^{\frac{1}{4}(k_{\min}-k_{\max})} a_{k_1} b_{k_2}, \tag{6.3.22}$$

$$\|P_{k_0}\mathcal{N}^*_+(A_{k_1}, \psi_{k_2})\|_{N^{1/2}_{s}} \lesssim 2^{\frac{1}{4}(k_{\min}-k_{\max})} a_{k_1} b_{k_2}. \tag{6.3.23}$$

We remind the reader that $\psi$ is assumed to be normalized in $(S^{1/2}_s)_b$; hence (6.3.21) and (6.3.22) concern the case when the output is estimated in the opposite-signed $N^{1/2}_{-s}$ space, whereas (6.3.23) is the same sign case.

The estimates (1.8.18) and (1.8.19) follow from (6.3.21) and (6.3.22), respectively, whereas (1.8.20) may be proved by combining (6.3.22) (opposite sign case) and (6.3.23) (same sign case). As the proof is similar to Step 0 of Section 6.3, we omit the details.

### Step 1: Case of opposite waves

Here we prove (6.3.21) and (6.3.22). Henceforth we write $f$ for either $B$ or $A$. We begin with the case when the output or $\psi$ has high modulation.

**Lemma 6.3.4.** *Assume the normalization* (6.3.1)*. For any $k_0, k_1, k_2, j \in \mathbb{Z}$ such that $k_1 < k_2 - 5$, we have*

$$2^{\frac{1}{2}k_0} 2^{-\frac{1}{2}j} \|P_{k_0} Q^{s'}_j \mathcal{L}(f_{k_1}, \psi_{k_2})\|_{L^2L^2} \lesssim 2^{\frac{1}{2}(k_1-j)} 2^{-\frac{1}{2}k_1} \|f_{k_1}\|_{L^2L^\infty} b_{k_2},$$
$$2^{\frac{1}{2}k_0} \|P_{k_0} \mathcal{L}(f_{k_1}, Q^s_j \psi_{k_2})\|_{L^1L^2} \lesssim 2^{\frac{1}{2}(k_1-j)} 2^{-\frac{1}{2}k_1} \|f_{k_1}\|_{L^2L^\infty} b_{k_2}.$$

*Proof.* The first estimate follows from the Hölder inequality $L^2L^\infty \times L^\infty L^2 \to L^2L^2$ and the frequency envelope bound (6.3.2). Similarly, the second estimate follows from the Hölder inequality $L^2L^\infty \times L^2L^2 \to L^1L^2$ and the frequency envelope bound (6.3.3). □

By the frequency envelope bounds (6.3.4) and (6.3.7), note that $f = B$ and $A$ yield the common bound

$$\|B_{k_1}\|_{L^2L^\infty} \lesssim 2^{2k_1} \|B_{k_1}\|_{L^2L^2} \lesssim 2^{\frac{1}{2}k_1} a_{k_1}, \qquad \|A_{k_1}\|_{L^2L^\infty} \lesssim 2^{\frac{1}{2}k_1} a_{k_1}. \tag{6.3.24}$$

Since $k_1 < k_2 - 5$, we have $k_{\min} = k_1$ and $k_0, k_2 = k_{\max} + O(1)$. Then from Lemma 6.3.4 and (1.5.2), it follows that

$$\|P_{k_0} Q^{-s}_{\geq k_0 + \frac{1}{2}(k_1-k_0) - C'_1} \mathcal{L}(f_{k_1}, \psi_{k_2})\|_{N^{1/2}_{-s}} \lesssim_{C'_1} 2^{\frac{1}{4}(k_{\min}-k_{\max})} a_{k_1} b_{k_2},$$
$$\|P_{k_0} Q^{-s}_{< k_0 + \frac{1}{2}(k_1-k_0) - C'_1} \mathcal{L}(f_{k_1}, Q^s_{\geq k_2 + \frac{1}{2}(k_1-k_2) - C'_1} \psi_{k_2})\|_{N^{1/2}_{-s}} \lesssim_{C'_1} 2^{\frac{1}{4}(k_{\min}-k_{\max})} a_{k_1} b_{k_2},$$

which are acceptable for any $C'_1 \geq 0$. It remains to treat the contribution of

$$I = P_{k_0} Q^{-s}_{< \frac{1}{2}(k_0+k_1) - C'_1} \mathcal{L}(f_{k_1}, Q^s_{< \frac{1}{2}(k_1+k_2) - C'_1} \psi_{k_2})$$

for some $C'_1 \geq 0$ to be determined. We now use the 'geometry of the cone' to force modulation localization of $f$.

**Lemma 6.3.5.** *Let $k_0, k_1, k_2, j_0, j_1, j_2 \in \mathbb{Z}$ be such that $|k_0 - k_2| \leq 5$ and $k_1 \leq \min\{k_0, k_2\} - 5$. Assume furthermore that $j_0 \leq k_0 - C_1'$ and $j_2 \leq k_2 - C_1'$ for a sufficiently large $C_1' > 0$. For any sign $s \in \{+, -\}$, the expression*

$$P_{k_0} Q_{j_0}^{-s} \mathcal{L}(P_{k_1} Q_{j_1} f, P_{k_2} Q_{j_2}^{s} g)$$

*vanishes unless $j_1 = k_{\max} + O(1)$.*

*Proof.* By duality, it suffices to consider the expression

$$\iint P_{k_0} Q_{j_0}^{+} h \mathcal{L}(P_{k_1} Q_{j_1} f, P_{k_2} Q_{j_2}^{+} g) \,\mathrm{d}t\mathrm{d}x.$$

We proceed as the proof of Lemma 5.5.4. If the expression does not vanish, there exists $\Xi^i$ ($i = 0, 1, 2$) such that $\sum_i \Xi^i = 0$ and $\Xi^i \in \{|\xi| \simeq 2^{k_i},\ |\tau - s_i|\xi|| \simeq 2^{j_i}\}$, where $s_0 = s_2 = s$ and $s_1$ is the sign of $\tau$. Consider the quantity $H = s_0|\xi^0| + s_1|\xi^1| + s_2|\xi^2|$. Subtracting $\sum_i \tau^i = 0$ and using the hypothesis on $k_0, k_2, j_0, j_2$, we have

$$|H| \lesssim 2^{j_1} + 2^{k_{\max} - C_1'}.$$

On the other hand, since $s_0 = s_2 = s$ and $k_1 \leq \min\{k_0, k_2\} - 5$, we have

$$|H| = |s|\xi^0| + s_1|\xi^1| + s|\xi^2|| \simeq 2^{k_{\max}}.$$

Taking $C_1'$ sufficiently large, it follows that $j_{\max} \geq k_{\max} - C$ for some constant $C$ independent of $C_1'$. Taking $C_1'$ even larger so that $j_1 \geq \max\{j_0, j_2\} + 5$, we have $|H| \simeq 2^{j_1}$ and the claim follows. ∎

Choosing $C_1' \geq 0$ to be sufficiently large, Lemma 6.3.5 is applicable to $I$. Hence

$$I = \sum_{j = k_{\max} + O(1)} P_{k_0} Q^{-s}_{< \frac{1}{2}(k_0 + k_1) - C_1'} \mathcal{L}(Q_j f_{k_1}, Q^{s}_{< \frac{1}{2}(k_1 + k_2) - C_1'} \psi_{k_2}),$$

By (1.5.2), (5.6.2) and the frequency envelope bound (6.3.4), we may estimate

$$\begin{aligned} 2^{\frac{1}{2}k_0} \|I\|_{L^1 L^2} &\lesssim \sum_{j = k_{\max} + O(1)} 2^{\frac{1}{2}k_0} \|Q_j f_{k_1}\|_{L^2 L^2} \Big( \sum_{\mathcal{C}_{k_{\min}}(0)} \|P_{\mathcal{C}_{k_{\min}}(0)} \psi_{k_2}\|^2_{L^2 L^\infty} \Big)^{1/2} \\ &\lesssim \sum_{j = k_{\max} + O(1)} 2^{k_{\min}} 2^{\frac{1}{2}k_0} \|Q_j f_{k_1}\|_{L^2 L^2} b_{k_2}. \end{aligned} \tag{6.3.25}$$

By the frequency envelope bounds (6.3.3) and (6.3.7), we have the following common bound for $f = B$ or $A$ when $j > k_1$:

$$\|Q_j B_{k_1}\|_{L^2 L^2} \lesssim 2^{-j} 2^{-\frac{1}{2}k_1} a_{k_1}, \quad \|Q_j A_{k_1}\|_{L^2 L^2} \lesssim 2^{-j} 2^{-\frac{1}{2}k_1} a_{k_1}. \tag{6.3.26}$$

Therefore,

$$2^{\frac{1}{2}k_0} \|I\|_{L^1 L^2} \lesssim 2^{\frac{1}{2}(k_{\min} - k_{\max})} a_{k_1} b_{k_2},$$

which completes the proof of (6.3.21) and (6.3.22).

**Step 2: Proof of** (6.3.23)

This is one of the key estimates showing that spinorial nonlinearities have better structure than the Riesz-transform parts. The idea is that the null form $\mathcal{N}_+^*$ gains an extra factor $2^{k_{\min}-k_{\max}}$ in the low-high case.

We begin with the high modulation bounds:

**Lemma 6.3.6.** *For any $k_0, k_1, k_2 \in \mathbb{Z}$ such that $|k_{\max} - k_{\text{med}}| \leq 5$ and $k_1 < k_2 - 5$, we have*

$$\|P_{k_0}\mathcal{N}_+^*(f_{k_1}, g_{k_2})\|_{L^2L^2} \lesssim 2^{k_1-k_2}\|f_{k_1}\|_{L^2L^\infty}\|g_{k_2}\|_{L^\infty L^2}, \tag{6.3.27}$$

$$\|P_{k_0}\mathcal{N}_+^*(f_{k_1}, g_{k_2})\|_{L^1L^2} \lesssim 2^{k_1-k_2}\|f_{k_1}\|_{L^2L^\infty}\|g_{k_2}\|_{L^2L^2}, \tag{6.3.28}$$

$$\|P_{k_0}\mathcal{N}_+^*(f_{k_1}, g_{k_2})\|_{L^1L^2} \lesssim 2^{k_1-k_2}\|f_{k_1}\|_{L^2L^2}\Big(\sum_{\mathcal{C}_{k_1}(0)}\|P_{\mathcal{C}_{k_1}(0)}g_{k_2}\|_{L^2L^\infty}^2\Big)^{1/2}. \tag{6.3.29}$$

*Proof.* The idea is to proceed as in the proof of Lemma 5.6.1 (where $\mathcal{L}$ is replaced by $\mathcal{N}_+^*$) with the following modification, to use:

$$|I_{\mathcal{C}^0,\mathcal{C}^1,\mathcal{C}^2}(t)| \lesssim \theta\|P_{\mathcal{C}^0}h_{k_0}(t)\|_{L^{q_0}}\|P_{\mathcal{C}^1}f_{k_1}(t)\|_{L^{q_1}}\|P_{\mathcal{C}^2}g_{k_2}(t)\|_{L^{q_2}}, \tag{6.3.30}$$

where $\theta = \max\{|\angle(\mathcal{C}^0, -\mathcal{C}^2)|, 2^{k_1-k_0}, 2^{k_1-k_2}\}$. This bound follows from Proposition 5.3.1; note that $2^{k_1-k_i}$ is the angular dimension of $\mathcal{C}^i$ for $i = 0, 2$. By Statement (2) of Lemma 5.5.1 and the hypothesis on $k_0, k_1, k_2$, it follows that $\theta \simeq 2^{k_1-k_2}$. Then proceeding as in the proof of Lemma 5.6.1, we directly obtain (6.3.29). The other two estimates (6.3.27) and (6.3.28) also follow from the same proof by switching the roles of $f, g$ and using the obvious bound

$$\Big(\sum_{\mathcal{C}_{k_1}(0)}\|P_{\mathcal{C}_{k_1}(0)}f_{k_1}\|_{L^2L^\infty}^2\Big)^{1/2} \simeq \|f_{k_1}\|_{L^2L^\infty}.$$

□

By Lemma 6.3.6 and the frequency envelop bounds (6.3.2), (6.3.3) and (6.3.4), we have

$$\|P_{k_0}Q_j^s\mathcal{N}_+^*(A_{k_1}, \psi_{k_2})\|_{N_s^{1/2}} \lesssim 2^{\frac{1}{2}(k_1-j)}2^{k_1-k_2}a_{k_1}b_{k_2},$$
$$\|P_{k_0}\mathcal{N}_+^*(A_{k_1}, Q_j^s\psi_{k_2})\|_{N_s^{1/2}} \lesssim 2^{\frac{1}{2}(k_1-j)}2^{k_1-k_2}a_{k_1}b_{k_2},$$
$$\|P_{k_0}\mathcal{N}_+^*(Q_jA_{k_1}, \psi_{k_2})\|_{N_s^{1/2}} \lesssim 2^{\frac{1}{2}(k_1-j)}2^{\frac{1}{2}(k_1-k_2)}a_{k_1}b_{k_2}.$$

Thanks to the exponential gain in $k_2 - k_1$ (as well as $j - k_1$), we may proceed as before (cf. Step 1 of Section 6.3 or 6.3) to reduce the proof of (6.3.23) to estimating the contribution of

$$I = \sum_{s_1} P_{k_0}Q_{<k_1-10}^s\mathcal{N}_+^*(Q_{<k_1-10}^{s_1}A_{k_1}, Q_{<k_1-10}^s\psi_{k_2}).$$

The norm $\|I\|_{N_s^{1/2}}$ can be bounded by the sum $\sum_{s_1}\sum_{j<k_1-10}$ of the terms

$$\|P_{k_0}Q_j^s\mathcal{N}_+^*(Q_{<j}^{s_1}A_{k_1}, Q_{<j}^s\psi_{k_2})\|_{N_s^{1/2}} \lesssim 2^{\frac{1}{4}(j-k_1)}2^{\frac{1}{2}(k_1-k_2)}a_{k_1}b_{k_2},$$
$$\|P_{k_0}Q_{\leq j}^s\mathcal{N}_+^*(Q_j^{s_1}A_{k_1}, Q_{<j}^s\psi_{k_2})\|_{N_s^{1/2}} \lesssim 2^{\frac{1}{4}(j-k_1)}2^{\frac{1}{2}(k_1-k_2)}a_{k_1}b_{k_2},$$
$$\|P_{k_0}Q_{\leq j}^s\mathcal{N}_+^*(Q_{\leq j}^{s_1}A_{k_1}, Q_j^s\psi_{k_2})\|_{N_s^{1/2}} \lesssim 2^{\frac{1}{4}(j-k_1)}2^{k_1-k_2}a_{k_1}b_{k_2},$$

where we used Proposition 5.6.4 and the frequency envelope bounds (6.3.2), (6.3.3) and (6.3.5) to derive the estimates. Observe the crucial exponential gain in $k_2 - k_1$, which arises from the factor $2^{k_{\min}-\min\{k_0,k_2\}}$ in Proposition 5.6.4. Summing up in $s_1 \in \{+,-\}$ and $j < k_1 - 10$, we obtain

$$\|I\|_{N_s^{1/2}} \lesssim 2^{\frac{1}{2}(k_1-k_2)} a_{k_1} b_{k_2},$$

which completes the proof of (6.3.23).

*Remark* 6.3.7. Repeating Step 2 with $\mathcal{N}^*$ replaced by $\mathcal{N}$ (hence Proposition 5.6.4 is replaced by Proposition 5.6.3), Lemma 1.8.10 can be proved. The key differences are the lack of the extra factor $2^{k_{\min}-k_{\max}}$ in Proposition 5.6.3, and that $Q_j A^{free} = 0$ for any $j \in \mathbb{Z}$. We omit the details.

## Proof of Proposition 1.8.7, part III: Completion of proof

We finish the proof of Proposition 1.8.7 by establishing the bounds (1.8.21)–(1.8.26). Here we do not need to utilize the null structure. Moreover, instead of the normalizing the $(\tilde{S}_s^{1/2})_b$ norm as in (6.3.1), we normalize the slightly weaker $(S_s^{1/2})_b$ norm, i.e., we assume

$$\|B\|_{Y_a^1} = \|A\|_{S_a^1} = \|\psi\|_{(S_s^{1/2})_b} = \|\varphi\|_{(S_{s'}^{1/2})_c} = 1.$$

Note that the bounds (6.3.2)–(6.3.5) and (6.3.7) still hold.

### Step 0: Reduction to dyadic estimates

Let $f$ denote either $B$ or $A$. Under the normalization (6.3.1), it clearly suffices to prove the following dyadic bounds:

$$\|P_{k_0}\mathcal{L}(f_{k_1}, \psi_{k_2})\|_{L^2L^2} \lesssim 2^{\frac{1}{2}(k_{\min}-k_{\max})} a_{k_1} b_{k_2}, \tag{6.3.31}$$

$$2^{-\frac{3}{2}k_0}\|P_{k_0}\mathcal{L}(f_{k_1}, \psi_{k_2})\|_{L^1L^\infty} \lesssim 2^{\frac{1}{2}(k_{\min}-k_{\max})} a_{k_1} b_{k_2}. \tag{6.3.32}$$

### Step 1: Proof of (6.3.31)

We first use Lemma 5.6.1 and (6.3.2) to estimate

$$\|P_{k_0}\mathcal{L}(f_{k_1}, \psi_{k_2})\|_{L^2L^2} \lesssim 2^{-\frac{1}{2}k_2}\Big(\sum_{\mathcal{C}_{k_{\min}}(0)} \|P_{\mathcal{C}_{k_{\min}}(0)} f_{k_1}\|_{L^2L^\infty}^2\Big)^{1/2} b_{k_2}.$$

By Bernstein's inequality, (6.3.4) and (6.3.7), we have

$$\begin{aligned}\Big(\sum_{\mathcal{C}_{k_{\min}}(0)} \|P_{\mathcal{C}_{k_{\min}}(0)} B_{k_1}\|_{L^2L^\infty}^2\Big)^{1/2} &\lesssim 2^{2k_{\min}} 2^{-\frac{3}{2}k_1} a_{k_1},\\ \Big(\sum_{\mathcal{C}_{k_{\min}}(0)} \|P_{\mathcal{C}_{k_{\min}}(0)} A_{k_1}\|_{L^2L^\infty}^2\Big)^{1/2} &\lesssim 2^{k_{\min}} 2^{-\frac{1}{2}k_1} a_{k_1}.\end{aligned} \tag{6.3.33}$$

In each case, it can be checked (using Littlewood-Paley trichotomy and dividing into cases $k_{\min} = k_0, k_1, k_2$) that (6.3.31) holds.

**Step 2: Proof of** (6.3.32)

We split into three cases.

**S**tep 2.1: (high-high) interaction, $k_0 = k_{\min}$. Here the factor $2^{-\frac{3}{2}k_0}$ on the LHS is detrimental, and we need to perform an orthogonality argument using Lemma 5.5.1. We claim that

$$\|P_{k_0}\mathcal{L}(f_{k_1}, g_{k_2})\|_{L^1L^\infty} \lesssim \Big(\sum_{\mathcal{C}_{k_0}(0)} \|P_{\mathcal{C}_{k_0}(0)} f_{k_1}\|_{L^2L^\infty}\Big)^{1/2} \Big(\sum_{\mathcal{C}_{k_0}(0)} \|P_{\mathcal{C}_{k_0}(0)} g_{k_2}\|_{L^2L^\infty}\Big)^{1/2} \tag{6.3.34}$$

Once (6.3.34) is proved, (6.3.32) would follow from (6.3.4) and (6.3.33).

To prove the claim, we follow the proof of Lemma 5.6.1. Let $\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2$, $I(t)$ and $I_{\mathcal{C}^0,\mathcal{C}^1,\mathcal{C}^2}(t)$ be as in the proof of Lemma 5.6.1, with $g$ replaced by $\psi$. Since there are only finitely many boxes $\mathcal{C}^0 = \mathcal{C}_{k_0}(0)$ in $\{|\xi| \simeq 2^{k_0}\}$, we have

$$|I(t)| \lesssim \|h_{k_0}(t)\|_{L^1} \Big(\sum_{\mathcal{C}^1} \|P_{\mathcal{C}^1} f_{k_1}(t)\|_{L^\infty}^2\Big)^{1/2} \Big(\sum_{\mathcal{C}^2} \|P_{\mathcal{C}^2} \psi_{k_2}(t)\|_{L^\infty}^2\Big)^{1/2}.$$

Then integrating and applying Hölder in $t$ appropriately, the desired claim (6.3.34) follows by duality.

**S**teps 2.2 & 2.3: (low-high) or (high-low) interaction, $k_1 = k_{\min}$ or $k_2 = k_{\min}$. These cases are easier thanks to the factor $2^{-\frac{3}{2}k_0}$ on the LHS, as $k_0 = k_{\max} + O(1)$ by Littlewood-Paley trichotomy. Indeed, by Hölder's inequality and the frequency envelope bounds (6.3.4) and (6.3.24) we have

$$2^{-\frac{3}{2}k_0}\|P_{k_0}\mathcal{L}(f_{k_1}, \psi_{k_2})\|_{L^1L^\infty} \lesssim 2^{-\frac{3}{2}k_0}\|f_{k_1}\|_{L^2L^\infty}\|\psi_{k_2}\|_{L^2L^\infty} \lesssim 2^{-\frac{3}{2}k_{\max}} 2^{\frac{1}{2}k_1} 2^{k_2} a_{k_1} b_{k_2},$$

which is acceptable.

*Remark* 6.3.8. In a general dimension $d \geq 4$, essentially every proof in this section is valid with substitutions as in Remark 1.8.5. The constant $\delta_0 > 0$ would change, since (6.3.6) must be replaced by

$$\|Q_j^s \psi_k\|_{L^1L^\infty} \lesssim 2^{\frac{5-d}{2}k} 2^{(d+1)(k-j)_+} \|\psi_k\|_{\tilde{Z}_{s,k}^{\frac{d-3}{2}}}.$$

## 6.4 Trilinear estimates for MD

In this section, we establish Proposition 1.8.8. We will first decompose the nonlinearity further and treat the part for which the bilinear null structure suffices. We will then be left with a part of the trilinear form

$$-\Delta^{-1}\langle \Pi_{s_1}\varphi^1, \mathcal{R}_0\Pi_{s_2}\varphi^2\rangle \mathcal{R}_0\psi + \Box^{-1}\mathcal{P}_i\langle \Pi_{s_1}\varphi^1, \mathcal{R}_x\Pi_{s_2}\varphi^2\rangle \mathcal{R}^i\psi$$

with certain restriction on the modulation and frequencies of the inputs and the output; for the precise expression, see (6.4.35). This nonlinearity exhibits a similar multilinear null structure as (1.7.8), (1.7.9) in the case of MKG. We thus complete the proof of Proposition 1.8.8 by reducing the present case to the multilinear null form estimate in [31].

As before, we restrict to the case $d = 4$ for most part of this section. The argument is simpler in the higher dimensional case $d \geq 5$; see Remark 6.4.3 below.

## Preliminaries: Conventions and definitions

Fix signs $s_1, s_2, s \in \{+, -\}$ and let $a, \tilde{a}, b, c, d$ be admissible frequency envelopes. In this section, we normalize the frequency envelope norms of the inputs as follows:

$$\begin{aligned} &\|A\|_{S^1_a} = \|A\|_{Z^1_{\tilde{a}}} = \|B\|_{Y^1_a} = \|B\|_{(Z^1_{ell})_{\tilde{a}}} = 1, \\ &\|\psi\|_{(\tilde{S}^{1/2}_s)_b} = \|\varphi^1\|_{(\tilde{S}^{1/2}_{s_1})_c} = \|\varphi^2\|_{(\tilde{S}^{1/2}_{s_2})_d} = 1. \end{aligned} \tag{6.4.1}$$

From (6.4.1), it follows that $A, B, \psi$ obey the frequency envelope bounds (6.3.2)–(6.3.7). Note that also $\psi$ obeys the bound

$$\sup_{\ell \leq 0} \Big( \sum_{\omega} \|P^{\omega}_{\ell} Q_{<k+2\ell} \psi\|^2_{L^\infty L^2} \Big)^{1/2} \lesssim 2^{-\frac{1}{2}k} b_k. \tag{6.4.2}$$

Moreover, $\varphi^1, \varphi^2$ obey the same estimates with $(s, b_k)$ replaced by $(s_1, c_k)$ and $(s_2, d_k)$, respectively. The normalizations $\|A\|_{Z^1_{\tilde{a}}} = 1$ and $\|B\|_{(Z^1_{ell})_{\tilde{a}}} = 1$ imply

$$\sup_{j<k+C} \Big( \sum_{\omega} \|P^{\omega}_{\ell} Q_j A_k\|^2_{L^1 L^\infty} \Big)^{\frac{1}{2}} \leq 2^{-\frac{1}{4}(j-k)} \tilde{a}_k, \tag{6.4.3}$$

$$\sup_{j<k+C} \Big( \sum_{\omega} \|P^{\omega}_{\ell} Q_j B_k\|^2_{L^1 L^\infty} \Big)^{\frac{1}{2}} \leq 2^{\frac{1}{4}(j-k)} \tilde{a}_k. \tag{6.4.4}$$

To identify the part that we cannot handle with only bilinear estimates, we borrow some definitions from [31]. Given $k \in \mathbb{Z}$ and a translation-invariant bilinear operator $L$, define

$$\mathcal{H}_k L(f, g) = \sum_{j<k+C_2} P_k Q_j L(Q_{<j} f, Q_{<j} g), \tag{6.4.5}$$

$$\mathcal{H}^*_k L(f, g) = \sum_{j<k+C^*_2} Q_{<j} L(P_k Q_j f, Q_{<j} g). \tag{6.4.6}$$

Here $C_2, C^*_2 > 0$ are universal constants such that

$$\frac{1}{2} C_0 < C^*_2 < C_1 < C_2 < C_0, \tag{6.4.7}$$

where $C_0$ is the constant in Lemma 5.5.4 and $C_1$ is the constant in the definitions (2.1.14)–(2.1.15) of $Z^r_k$ and $Z^r_{ell,k}$.

Given signs $s_1, s_2, s \in \{+,-\}$, we also define

$$\mathcal{H}_{s_1,s_2} L(f,g) = \sum_{k_0,k_1,k_2 \colon k_0 < k_2 - C_2 - 10} \mathcal{H}_{k_0} L(T_{s_1} f_{k_1}, P_{k_2} T_{s_2} g),$$
$$\mathcal{H}^*_{s',s} L(f,g) = \sum_{k_0,k_1,k_2 \colon k_1 < k_2 - C_2^* - 10} P_{k_0} T_{s'} \mathcal{H}^*_{k_1} L(f, T_s g_{k_2}).$$

## Further decomposition of $\mathbf{A}_x$ and $\pi^R$

Consider the trilinear operator

$$\mathcal{T}^R_{s_1,s_2,s}(\varphi^1,\varphi^2,\psi) = s s_2 \mathcal{H}^*_{s,s}\big(\mathcal{H}_{s_1,s_2} \Box^{-1} \mathcal{P}_i \langle \Pi_{s_1}\varphi^1, \mathcal{R}_x \Pi_{s_2}\varphi^2\rangle \mathcal{R}^i \psi\big), \tag{6.4.8}$$

where $\Box^{-1}$ denotes the Fourier multiplier[5] with symbol $(\tau^2 - |\xi|^2)^{-1}$. Our goal is to show that all of $\pi^R[\mathbf{A}(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)]\psi$ except $\mathcal{T}^R_{s_1,s_2,s}$ can be handled by applying bilinear estimates in tandem. We use the auxiliary $Z^1$ norm as an intermediary.

More precisely, under the normalization (6.4.1) and $f$ as in (1.8.27), we claim that

$$\| -s\pi^R[\mathbf{A}(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)]\psi - \mathcal{T}^R_{s_1,s_2,s}(\varphi^1,\varphi^2,\psi)\|_{(N^{1/2}_s)_f} \lesssim 1. \tag{6.4.9}$$

### Step 0: Reduction to bilinear estimates

Let $a,b,c,d$ be admissible frequency envelopes. Define $e_k = (\sum_{k'<k} a_{k'}) b_k$ and $\tilde{e}_k = (\sum_{k'<k} \tilde{a}_{k'}) b_k$. We claim that

$$\|(\mathrm{I}_{4\times 4} - \mathcal{H}^*_{s,s})\pi^R[A]\psi\|_{(N^{1/2}_s)_e} \lesssim \|A\|_{S^1_a} \|\psi\|_{(\tilde{S}^{1/2}_s)_b}, \tag{6.4.10}$$
$$\|\mathcal{H}^*_{s,s}\pi^R[A]\psi\|_{(N^{1/2}_s)_{\tilde{e}}} \lesssim \|A\|_{Z^1_{\tilde{a}}} \|\psi\|_{(\tilde{S}^{1/2}_s)_b}, \tag{6.4.11}$$
$$\|(I - \mathcal{H}_{s_1,s_2})\mathbf{A}^R(\varphi^1,\varphi^2)\|_{Z^1_{cd}} \lesssim \|\varphi^1\|_{(\tilde{S}^{1/2}_{s_1})_c} \|\varphi^2\|_{(\tilde{S}^{1/2}_{s_2})_d}, \tag{6.4.12}$$
$$\|\mathbf{A}^S_{s_2}(\Pi_{s_1}\varphi^1,\varphi^2)\|_{Z^1_{cd}} \lesssim \|\varphi^1\|_{(\tilde{S}^{1/2}_{s_1})_c} \|\varphi^2\|_{(\tilde{S}^{1/2}_{s_2})_d}. \tag{6.4.13}$$

Assuming these estimates, we first conclude the proof of (6.4.9). Assume the normalization (6.4.1). Note that $P_k \Pi_s$ is disposable for any $k \in \mathbb{Z}$ and $s \in \{+,-\}$. Hence, from the bilinear estimates (1.8.13)–(1.8.14) and (6.4.12)–(6.4.13), we obtain

$$\|\mathbf{A}^R(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)\|_{S^1_{cd}} + \|(1 - \mathcal{H}_{s_1,s_2})\mathbf{A}^R(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)\|_{Z^1_{cd}} \lesssim 1,$$
$$\|\mathbf{A}^S_{s_2}(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)\|_{(S^1 \cap Z^1)_{cd}} \lesssim 1.$$

Applying (6.4.10) and (6.4.11) with $a = \tilde{a} = cd$, $e = \tilde{e} = (\sum_{k'<k} c_{k'} d_{k'}) b_k$ and

$$A = \mathbf{A}(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2) = -s_2 \mathbf{A}^R(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2) + \mathbf{A}^S_{s_2}(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2),$$

[5] In general, this 'multiplier' is problematic near $\{\tau^2 - |\xi|^2 = 0\}$; however, thanks to the modulation projection $Q_j$ in the definition of $\mathcal{H}_{s_1,s_2}$, the expression $\mathcal{H}_{s_1,s_2}\Box^{-1}$ is well-defined and coincides with $\mathcal{H}_{s_1,s_2} K$, where $Kf$ denotes the solution $\phi$ to $\Box\phi = f$ with $\phi[0] = 0$.

we arrive at

$$\|s\pi^R[\mathbf{A}(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)]\psi - s\mathcal{H}^*_{s,s}\pi^R[\mathcal{H}_{s_1,s_2}(-s_2\mathbf{A}^R)(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)]\psi\|_{(N_s^{1/2})_e} \lesssim 1.$$

Recalling the definitions of $\mathbf{A}^R$ and $\pi^R$, observe that

$$s_2 s\mathcal{H}^*_{s,s}\pi^R[\mathcal{H}_{s_1,s_2}\mathbf{A}^R(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)]\psi = \mathcal{T}^R_{s_1,s_2,s}(\varphi^1, \varphi^2, \psi).$$

Moreover, by Cauchy-Schwarz, the frequency envelope $e$ is dominated by $f$ as in (1.8.27). The desired estimate (6.4.9) follows.

**Step 1: Proof of** (6.4.10)

Under the normalization (6.4.1) and the condition $k_1 < k_2 - C_2^* - 5$, we claim that:

$$\|P_{k_0}\mathcal{N}(A_{k_1}, \psi_{k_2}) - P_{k_0}T_s\mathcal{H}^*_{k_1}\mathcal{N}(A, T_s\psi_{k_2})\|_{N_s^{1/2}} \lesssim a_{k_1}b_{k_2}. \tag{6.4.14}$$

Since $\pi^R[A]\psi = \sum_k \mathcal{N}(P_{<k-10}A, \psi_k)$ by Proposition 5.4.4, (6.4.10) clearly follows from summing up (6.4.14) for $k_1 < k_2 - C_2^* - 10$ and (6.3.20) in Remark 6.3.3 for $k_1 \in [k_2 - C_2^* - 10, k_2 - 10)$.

The proof of (6.4.14) is identical to the proof of (1.8.5) and is omitted.

**Step 2: Proof of** (6.4.11)

Assuming (6.4.1) and $k_1 < k_2 - C_2^* - 5$, we claim:

$$\|P_{k_0}T_s\mathcal{H}^*_{k_1}\mathcal{N}(A, T_s\psi_{k_2})\|_{L^1\dot{H}^{1/2}} \lesssim \tilde{a}_{k_1}b_{k_2}. \tag{6.4.15}$$

As before, (6.4.11) clearly follows from (6.4.15).

The proof of (6.4.15) is the same as the proof of (1.8.6) and is omitted.

**Step 3: Proof of** (6.4.12)

For $k_0 \geq k_2 - C_2 - 20$, we claim that

$$\|P_{k_0}\mathcal{N}^*(\varphi^1_{k_1}, \varphi^2_{k_2})\|_{\Box Z^1} \lesssim 2^{\delta_0(k_{\max}-k_{\min})}c_{k_1}d_{k_2}. \tag{6.4.16}$$

Moreover, for $k_0 < k_2 - C_2 - 5$, we claim that

$$\|P_{k_0}\mathcal{N}^*(\varphi^1_{k_1}, \varphi^2_{k_2}) - \mathcal{H}_{k_0}\mathcal{N}^*(T_{s_1}\varphi^1_{k_1}, T_{s_2}\varphi^2_{k_2})\|_{\Box Z^1} \lesssim 2^{\delta_0(k_0-k_1)}c_{k_1}d_{k_2}. \tag{6.4.17}$$

Since $\Box\mathbf{A}^R = \mathcal{M}^R = \mathcal{N}^*$ by Proposition 5.4.4, (6.4.12) clearly follows from (6.4.16) and (6.4.17).

The proofs of (6.4.16) and (6.4.17) is very similar to the proof of (6.1.5) and is omitted.

**Step 4: Proof of** (6.4.13)

In this case, recall that $\Box \mathbf{A}^S_{s_2}(\Pi_{s_1}\cdot,\cdot) = \mathcal{M}^S_{s_2}(\Pi_{s_1}\cdot,\cdot) = \mathcal{N}_{s_1 s_2}(\cdot,\cdot)$ by Proposition 5.4.4. Repeating the argument in Step 3, the following analogues of (6.4.16) and (6.4.17) can be proved: For $k_0 \geq k_2 - C_2 - 20$, we have

$$\|P_{k_0}\mathcal{N}_{s_1 s_2}(\varphi^1_{k_1},\varphi^2_{k_2})\|_{\Box Z^1} \lesssim 2^{\delta_0(k_{\min}-k_{\max})} c_{k_1} d_{k_2}, \tag{6.4.18}$$

and for $k_0 < k_2 - C_2 - 5$, we have

$$\|P_{k_0}\mathcal{N}_{s_1 s_2}(\varphi^1_{k_1},\varphi^2_{k_2}) - \mathcal{H}_{k_0}\mathcal{N}_{s_1 s_2}(T_{s_1}\varphi^1_{k_1}, T_{s_2}\varphi^2_{k_2})\|_{\Box Z^1} \lesssim 2^{\delta_0(k_0-k_2)} c_{k_1} d_{k_2}. \tag{6.4.19}$$

We omit the straightforward details.

Under the condition $k_0 < k_2 - C_2 - 5$, we claim furthermore that

$$\|\mathcal{H}_{k_0}\mathcal{N}_{s_1 s_2}(\varphi^1_{k_1},\varphi^2_{k_2})\|_{\Box Z^1} \lesssim 2^{k_0-k_2} c_{k_1} d_{k_2}. \tag{6.4.20}$$

Clearly, (6.4.13) would follow from (6.4.18)–(6.4.20).

To prove (6.4.20), we need to estimate

$$I = P_{k_0} Q_j \mathcal{N}_{s_1 s_2}(Q^{s_1}_{<j}\varphi^1_{k_1}, Q^{s_2}_{<j}\varphi^2_{k_2})$$

in $\Box Z^1$. We proceed similarly to the proof of Proposition 5.6.3 and perform an orthogonality argument using Lemma 5.5.5.

Let $j \leq k_0 + C_2$ and $\ell = \frac{1}{2}(j-k_0)_-$. For $i = 0,1,2$, let $\mathcal{C}^i$ be a rectangular box of the form $\mathcal{C}_{k_0}(\ell)$. We split

$$I = \sum_{\mathcal{C}^0,\mathcal{C}^1,\mathcal{C}^2} P_{k_0} P_{-\mathcal{C}^0} Q_j \mathcal{N}_{s_1 s_2}(P_{\mathcal{C}^1} Q^{s_1}_{<j}\varphi^1_{k_1}, P_{\mathcal{C}^2} Q^{s_2}_{<j}\varphi^2_{k_2}).$$

Splitting $Q_j = Q^+_j T_+ + Q^-_j T_-$ and applying Lemma 5.5.5, we see that the summand on the RHS vanishes unless (5.5.7) is satisfied for $s_0 = +$ or $-$. In particular, by disposability of $P_k P_{\mathcal{C}_{k_0}(\ell)} Q_j = P_k P^\omega_\ell Q_j$ and Proposition 5.3.1, it follows that

$$\begin{aligned}\|P_{k_0} P_{-\mathcal{C}^0} Q_j \mathcal{N}_{s_1 s_2}(P_{\mathcal{C}^1} Q^{s_1}_{<j}\varphi^1_{k_1}, P_{\mathcal{C}^2} Q^{s_2}_{<j}\varphi^2_{k_2})\|_{L^1 L^\infty} \\ \lesssim 2^\ell 2^{k_0-k_2}\|P_{\mathcal{C}^1} Q^{s_1}_{<j}\varphi^1\|_{L^2 L^\infty}\|P_{\mathcal{C}^2} Q^{s_2}_{<j}\varphi^2\|_{L^2 L^\infty}.\end{aligned} \tag{6.4.21}$$

Moreover, by Lemma 5.5.5, note that for a fixed $\mathcal{C}^1$ [resp. $\mathcal{C}^2$], there are only (uniformly) bounded number of $\mathcal{C}^0,\mathcal{C}^2$ [resp. $\mathcal{C}^0,\mathcal{C}^1$] such that (5.5.7) is satisfied with $s_0 = +$ or $-$. Summing up first in $\mathcal{C}_0$ (for which there are only finitely many terms) and then applying Lemma 1.5.5 to the summation in $\mathcal{C}^1,\mathcal{C}^2$ (which is essentially diagonal), we obtain

$$\begin{aligned}&\sum_{\mathcal{C}^0}\|P_{k_0} P_{-\mathcal{C}^0} Q_j \mathcal{N}_{s_1 s_2}(Q^{s_1}_{<j}\varphi^1_{k_1}, Q^{s_2}_{<j}\varphi^2_{k_2})\|_{L^1 L^\infty} \\ &\lesssim 2^\ell 2^{k_0-k_2}\Big(\sum_{\mathcal{C}^1}\|P_{\mathcal{C}^1} Q^{s_1}_{<j}\varphi^1_{k_1}\|^2_{L^2 L^\infty}\Big)^{1/2}\Big(\sum_{\mathcal{C}^2}\|P_{\mathcal{C}^2} Q^{s_2}_{<j}\varphi^2_{k_2}\|^2_{L^2 L^\infty}\Big)^{1/2}.\end{aligned}$$

Recall the convention $P_k P_{\mathcal{C}_k(\ell)} = P_k P^{\omega}_{\ell}$. By (2.2.20) and (6.3.5), we have

$$\|P_{k_0} Q_j \mathcal{N}_{s_1 s_2}(P_{\mathcal{C}^1} Q^{s_1}_{<j} \varphi^1_{k_1}, P_{\mathcal{C}^2} Q^{s_2}_{<j} \varphi^2_{k_2})\|_{\Box Z^1} \lesssim 2^{\frac{1}{4}(j-k_0)} 2^{k_0-k_2} c_{k_1} d_{k_2}.$$

Summing up in $j < k_0 + C_2$, (6.4.20) follows.

## Further decomposition of $\mathbf{A}_0$ and $\pi^E$

We now deal with the term involving $A_0 = \mathbf{A}_0(\varphi^1, \varphi^2)$ in $\pi^E[A_0]\psi$. Consider the trilinear operator

$$\mathcal{T}^E_{s_1,s_2,s}(\varphi^1, \varphi^2, \psi) = s_2 s \mathcal{H}^* \big( \mathcal{H}_{s_1,s_2} \Delta^{-1} \langle \Pi_{s_1} \varphi^1, \mathcal{R}_0 \Pi_{s_2} \varphi^2 \rangle \mathcal{R}^0 \psi \big).$$

We will show that all of $\pi^E[\mathbf{A}_0(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)]\psi$ except $\mathcal{T}^E_{s_1,s_2,s}$ can be handled by bilinear estimates. The $Z^1_{ell}$ norm will be used as an intermediary.

Under the normalization (6.4.1), we claim that

$$\|\pi^E[\mathbf{A}_0(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)]\psi - \mathcal{T}^E_{s_1,s_2,s}(\varphi^1, \varphi^2, \psi)\|_{(N^{1/2}_s)_e} \lesssim 1. \tag{6.4.22}$$

### Step 0: Reduction to bilinear estimates

Let $a, b, c, d$ be admissible frequency envelopes. Define $e_k = (\sum_{k'<k} a_{k'}) c_k$ and $\tilde{e}_k = (\sum_{k'<k} \tilde{a}_{k'}) c_k$. We claim that

$$\|(\mathrm{I}_{4\times 4} - \mathcal{H}^*_{s,s}) \pi^E[A_0]\psi\|_{(N^{1/2}_s)_e} \lesssim \|A_0\|_{Y^1_a} \|\psi\|_{(\tilde{S}^{1/2}_s)_c}, \tag{6.4.23}$$

$$\|\mathcal{H}^*_{s,s} \pi^E[A_0](\mathrm{I}_{4\times 4} + s\mathcal{R}^0)\psi\|_{(N^{1/2}_s)_e} \lesssim \|A_0\|_{Y^1_a} \|\psi\|_{(\tilde{S}^{1/2}_s)_c}, \tag{6.4.24}$$

$$\|\mathcal{H}^*_{s,s} \pi^E[A_0] \mathcal{R}^0 \psi\|_{(L^1 \dot{H}^{1/2})_{\tilde{e}}} \lesssim \|A_0\|_{(Z^1_{ell})_{\tilde{a}}} \|\psi\|_{(\tilde{S}^{1/2}_s)_c}, \tag{6.4.25}$$

$$\|(1 - \mathcal{H}_{s_1,s_2}) \mathbf{A}_0(\varphi^1, \varphi^2)\|_{(Z^1_{ell})_{cd}} \lesssim \|\varphi^1\|_{(\tilde{S}^{1/2}_{s_1})_b} \|\varphi^2\|_{(\tilde{S}^{1/2}_{s_2})_c}, \tag{6.4.26}$$

$$\|\mathcal{H}_{s_1,s_2}(\mathbf{A}_0 + s_2 \mathbf{A}^R_0)(\varphi^1, \varphi^2)\|_{(Z^1_{ell})_{cd}} \lesssim \|\varphi^1\|_{(\tilde{S}^{1/2}_{s_1})_b} \|\varphi^2\|_{(\tilde{S}^{1/2}_{s_2})_c}. \tag{6.4.27}$$

where

$$\mathbf{A}^R_0(\varphi^1, \varphi^2) := \Delta^{-1} \langle \varphi^1, \mathcal{R}_0 \varphi^2 \rangle = -\Delta^{-1} \langle \varphi^1, \mathcal{R}^0 \varphi^2 \rangle.$$

Assuming these estimates, we now prove (6.4.22). Assume the normalization (6.4.1). By (1.8.12), (6.4.26) and (6.4.27), as well as disposability of $P_k \Pi_s$, we have

$$\|\mathbf{A}_0(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)\|_{Y^1_{cd}} + \|(\mathbf{A}_0 - \mathcal{H}_{s_1,s_2}(-s_2 \mathbf{A}^R_0))(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)\|_{(Z^1_{ell})_{cd}} \lesssim 1.$$

Applying (6.4.23)–(6.4.25) with $a = \tilde{a} = cd$, $e = \tilde{e} = (\sum_{k'<k} c_{k'} d_{k'}) b_k$ and $A_0 = \mathbf{A}_0(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)$, we obtain

$$\|\pi^E[\mathbf{A}_0(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)]\psi + s\mathcal{H}^*_{s,s} \pi^E[\mathcal{H}_{s_1,s_2}(-s_2 \mathbf{A}^R_0)(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)] \mathcal{R}^0 \psi\|_{(N^{1/2}_s)_e} \lesssim 1.$$

By definition, observe that

$$s_2 s \mathcal{H}^*_{s,s} \pi^E[\mathcal{H}_{s_1,s_2} \mathbf{A}^R_0(\Pi_{s_1}\varphi^1, \Pi_{s_2}\varphi^2)] \mathcal{R}^0 \psi = \mathcal{T}^E_{s_1,s_2,s}(\varphi^1, \varphi^2, \psi).$$

As before, the frequency envelope $e$ is dominated by $f$ as in (1.8.27) by Cauchy-Schwarz; this completes the proof of (6.4.22).

**Step 1: Proof of** (6.4.23)

Assuming (6.4.1) and $k_1 < k_2 - C_2^* - 5$, it suffices to have:

$$\|P_{k_0}\mathcal{L}(B_{k_1}, \psi_{k_2}) - P_{k_0}T_s\mathcal{H}^*_{k_1}\mathcal{L}(B, T_s\psi_{k_2})\|_{N_s^{1/2}} \lesssim a_{k_1}b_{k_2}. \tag{6.4.28}$$

For this, we refer to the proof of (1.8.5).

**Step 2: Proof of** (6.4.24)

Assuming (6.4.1) and $k_1 < k_2 - C_2^* - 5$, we claim:

$$\|P_{k_0}T_s\mathcal{H}^*_{k_1}\mathcal{L}(B, T_s(\mathrm{I}_{4\times 4} + s\mathcal{R}^0)\psi_{k_2}))\|_{N_s^{1/2}} \lesssim 2^{k_1-k_2}a_{k_1}b_{k_2}. \tag{6.4.29}$$

Note that (6.4.29) is more than enough to prove (6.4.24) (i.e., the gain $2^{k_1-k_2}$ is unnecessary).

Fix $j < k_1 + C_2^*$ and introduce the shorthand $\tilde{\psi} = (\mathrm{I}_{4\times 4} + s\mathcal{R}^0)\psi$. By (1.5.2), Hölder's inequality $L^2L^\infty \times L^2L^2 \to L^1L^2$, Bernstein's inequality and (6.3.7), we have

$$\|P_{k_0}Q^s_{<j}\mathcal{L}(Q_jB_{k_1}, Q^s_{<j}\tilde{\psi}_{k_2}))\|_{L^1\dot{H}^{1/2}} \lesssim 2^{\frac{1}{2}k_0+\frac{1}{2}k_1}a_{k_1}\|Q^s_{<j}\tilde{\psi}_{k_2}\|_{L^2L^2}$$

By (1.7.13) and (6.3.3), we have

$$\begin{aligned}\|Q_{<j}\tilde{\psi}_{k_2}\|_{L^2L^2} =& \|Q_{<j}\frac{i\partial_t + s|D|}{|D|}\psi_{k_2}\|_{L^2L^2}\\ &\lesssim \sum_{j'<j} 2^{j'-k_2}\|Q_{j'}\psi_{k_2}\|_{L^2L^2} \lesssim 2^{\frac{1}{2}j}2^{-\frac{3}{2}k_2}b_{k_2}.\end{aligned} \tag{6.4.30}$$

It follows that

$$\|P_{k_0}Q^s_{<j}\mathcal{L}(Q_jB_{k_1}, Q^s_{<j}(\mathrm{I}_{4\times 4} + s\mathcal{R}^0)\psi_{k_2}))\|_{L^1\dot{H}^{1/2}} \lesssim 2^{\frac{1}{2}(j-k_1)}2^{k_1-k_2}a_{k_1}b_{k_2}.$$

Summing up in $j < k_1 + C_2^*$, we obtain (6.4.29) as desired.

**Step 3: Proof of** (6.4.25)

Assuming (6.4.1) and $k_1 < k_2 - C_2^* - 5$, it suffices to have:

$$\|P_{k_0}T_s\mathcal{H}^*_{k_1}\mathcal{L}(B, T_s\psi_{k_2})\|_{L^1\dot{H}^{1/2}} \lesssim \tilde{a}_{k_1}b_{k_2}. \tag{6.4.31}$$

See the proof of (1.8.6).

**Step 4: Proof of** (6.4.26)

Under the normalization (6.4.1), it suffices to prove the following dyadic bounds: For $k_0 \geq k_2 - C_2 - 20$, we claim that

$$\|P_{k_0}\mathcal{N}^*(\varphi^1_{k_1}, \varphi^2_{k_2})\|_{\Delta Z^1_{ell}} \lesssim 2^{\delta_0(k_0-k_1)}c_{k_1}d_{k_2}, \tag{6.4.32}$$

and for $k_0 < k_2 - 5$, we claim that

$$\|P_{k_0}\mathcal{N}^*(\varphi^1_{k_1}, \varphi^2_{k_2}) - \mathcal{H}_{k_0}\mathcal{N}^*(T_{s_1}\varphi^1_{k_1}, T_{s_2}\varphi^2_{k_2})\|_{\Delta Z^1_{ell}} \lesssim 2^{\delta_0(k_0-k_1)}c_{k_1}d_{k_2}. \tag{6.4.33}$$

We refer to the proof of (6.1.5).

**Step 5: Proof of** (6.4.27)

Assuming (6.4.1) and $k_0 < k_2 - C_2 - 5$, it suffices to prove

$$\|\mathcal{H}_{k_0}\mathcal{L}(T_{s_1}\varphi^1_{k_1}, T_{s_2}(\mathrm{I}_{4\times 4} + s_2\mathcal{R}^0)\varphi^2_{k_2}\|_{\Delta Z^1_{ell}} \lesssim 2^{\frac{3}{2}(k_0-k_2)} c_{k_1} d_{k_2}. \tag{6.4.34}$$

In order to ensure that the projections $Q_{<j}$ in $\mathcal{H}_{k_0}$ are disposable, we perform an orthogonality argument as before. Fix $j < k_0 + C_2$ and introduce the shorthands $\ell = \frac{1}{2}(j - k_0)_-$ and $\tilde{\varphi} := (\mathrm{I}_{4\times 4} + s_2\mathcal{R}^0)\varphi^2$. For $i = 0, 1, 2$, let $\mathcal{C}^i$ be a rectangular box of the form $\mathcal{C}_{k_0}(\ell)$. We expand

$$P_{k_0} Q_j \mathcal{L}(Q^{s_1}_{<j}\varphi^1_{k_1}, Q^{s_2}_{<j}\tilde{\varphi}_{k_2}) = \sum_{\mathcal{C}^0,\mathcal{C}^1,\mathcal{C}^2} P_{k_0} P_{-\mathcal{C}^0} Q_j \mathcal{L}(P_{\mathcal{C}^1} Q^{s_1}_{<j}\varphi^1_{k_1}, P_{\mathcal{C}^2} Q^{s_2}_{<j}\tilde{\varphi}_{k_2}).$$

By (1.5.9), we have

$$\|P_{k_0} P_{-\mathcal{C}^0} Q_j \mathcal{L}(P_{\mathcal{C}^1} Q^{s_1}_{<j}\varphi^1_{k_1}, P_{\mathcal{C}^2} Q^{s_2}_{<j}\tilde{\varphi}_{k_2})\|_{L^1L^2} \lesssim \|P_{\mathcal{C}^1} Q^{s_1}_{<j}\varphi^1_{k_1}\|_{L^2L^\infty} \|Q^{s_2}_{<j}\tilde{\varphi}\|_{L^2L^2}$$

Moreover, splitting $Q_j = Q^+_j T_+ + Q^-_j T_-$ and applying Lemma 5.5.5, we see that the LHS vanishes unless (5.5.7) holds with $s_0 = +$ or $-$. Thus for a fixed $\mathcal{C}^1$ [resp. $\mathcal{C}^2$], there are only (uniformly) bounded number of $\mathcal{C}^0, \mathcal{C}^2$ [resp. $\mathcal{C}^0, \mathcal{C}^1$] such that LHS does not vanish. Summing up first in $\mathcal{C}_0$ and then applying Lemma 1.5.5 to the (essentially diagonal) summation in $\mathcal{C}^1, \mathcal{C}^2$, we obtain

$$\|P_{k_0} Q_j \mathcal{L}(Q^{s_1}_{<j}\varphi^1_{k_1}, Q^{s_2}_{<j}\tilde{\varphi}_{k_2})\|_{L^1L^2} \lesssim \Big(\sum_{\mathcal{C}^1} \|P_{\mathcal{C}^1} Q^{s_1}_{<j}\varphi^1_{k_1}\|^2_{L^2L^\infty}\Big)^{1/2} \|Q^{s_2}_{<j}\tilde{\varphi}_{k_2}\|_{L^2L^2}$$

By (2.2.24), (6.3.5) and (6.4.30), we have

$$\|P_{k_0} Q_j \mathcal{L}(Q^{s_1}_{<j}\varphi^1_{k_1}, Q^{s_2}_{<j}\tilde{\varphi}_{k_2})\|_{\Delta Z^1_{ell}} \lesssim 2^{\frac{5}{2}(j-k_0)} 2^{\frac{3}{2}(k_0-k_2)} c_{k_1} d_{k_2}$$

Summing up in $j < k_0 + C_2$, the desired estimate (6.4.34) follows.

## Genuinely multilinear null form estimate

To complete the proof of Proposition 1.8.8, it remains to estimate

$$\begin{aligned}
\mathcal{T}_{s_1,s_2,s}(\varphi^1,\varphi^2,\psi) =& \mathcal{T}^E_{s_1,s_2,s}(\varphi^1,\varphi^2,\psi) + \mathcal{T}^R_{s_1,s_2,s}(\varphi^1,\varphi^2,\psi) \\
=& s_2 s\Big( - \mathcal{H}^*_{s,s}\big(\mathcal{H}_{s_1,s_2}\Delta^{-1}\langle \Pi_{s_1}\varphi^1, \mathcal{R}_0\Pi_{s_2}\varphi^2\rangle \mathcal{R}_0\psi\big) \\
&+ \mathcal{H}^*_{s,s}\big(\mathcal{H}_{s_1,s_2}\Box^{-1}\mathcal{P}_i\langle \Pi_{s_1}\varphi^1, \mathcal{R}_x\Pi_{s_2}\varphi^2\rangle \mathcal{R}^i\psi\big)\Big).
\end{aligned} \tag{6.4.35}$$

This part has a multilinear null structure akin to (1.7.8), (1.7.9) for MKG. In fact, thanks to the way we have set things up, it is possible to directly borrow the relevant estimates in

[31]. We introduce the trilinear operator

$$\begin{aligned}\mathcal{T}^{MKG}_{k,k'}(f^1_{k_1}, f^2_{k_2}, f^3_{k_3}) = &- \mathcal{H}^*_k\Big(\mathcal{H}_{k'}\Delta^{-1}\mathcal{L}(f^1_{k_1}, \partial_t f^2_{k_2})\partial_t f^3_{k_3}\Big)\\ &+ \mathcal{H}^*_k\Big(\mathcal{H}_{k'}\Box^{-1}\mathcal{P}_i\mathcal{L}(f^1_{k_1}, \partial_x f^2_{k_2})\partial^i f^3_{k_3}\Big),\end{aligned}$$

where $\mathcal{L}$ on both lines represent a single bilinear operator. Note that $\mathcal{T}^{MKG}_{k,k'}$ vanishes unless $|k-k'|<3$. Moreover, in [31, Eq. (136), (137) and (138); Appendix], the following estimate was proved [6], which is the massless analogue of (6.2.1) :

**Proposition 6.4.1.** *For $k < \min\{k_0, k_1, k_2, k_3\} - C$ and $|k'-k| < 3$, we have*

$$\|P_{k_0}\mathcal{T}^{MKG}_{k,k'}(f^1, f^2, f^3)\|_{N_{k_0}} \lesssim 2^{\delta_0(k-k_1)} 2^{\frac{1}{2}k_0} 2^{k_1} 2^{k_2} 2^{k_3} \prod_{i=1}^{3} \|f^i_{k_i}\|_{\tilde{S}_{k_i}}. \tag{6.4.36}$$

*Remark* 6.4.2. The proof of (6.4.36) exploits the trilinear null structure (1.7.8), (1.7.9) originally uncovered in [34], which is sometimes referred to as the secondary null structure of Maxwell–Klein–Gordon.

Plugging in

$$f^1_{k_1} = \Pi_{s_1} Q_{<k_1-3} T_{s_1}\varphi^1_{k_1}, \quad f^2_{k_2} = \frac{1}{i|D|}\Pi_{s_1} Q_{<k_2-3} T_{s_2}\varphi^2_{k_2}, \quad f^3_{k_3} = \frac{1}{i|D|} Q_{<k_3-3} T_s \psi_{k_3},$$

observe that

$$P_{k_0}\mathcal{T}_{s_1,s_2,s}(\varphi^1_{k_1}, \varphi^2_{k_2}, \psi_{k_3}) = -s_2 s \sum_{\substack{k<k_3-C_2^*-10\\ k'<k_2-C_2-10}} P_{k_0}\mathcal{T}^{MKG}_{k,k'}(f^1_{k_1}, f^2_{k_2}, f^3_{k_3}).$$

By Proposition 6.4.1 and the facts that $k_1 = k_2 + O(1)$, $k_3 = k_0 + O(1)$, we have

$$\|P_{k_0}\mathcal{T}_{s_1,s_2,s}(\varphi^1_{k_1}, \varphi^2_{k_2}, \psi_{k_3})\|_{N_s^{1/2}} \lesssim 2^{\delta_0(\min\{k_1,k_3\}-k_1)} c_{k_1} d_{k_2} b_{k_3}. \tag{6.4.37}$$

Keeping $k_0$ fixed and summing up in $k_1, k_2, k_3$, we obtain

$$\|P_{k_0}\mathcal{T}_{s_1,s_2,s}(\varphi^1, \varphi^2, \psi)\|_{N_s^{1/2}} \lesssim (\sum_{k'<k_0} c^2_{k'})^{1/2} (\sum_{k'<k_0} d^2_{k'})^{1/2} b_{k_0}$$

which completes the proof.

[6] We remark that in [31], this estimate is stated with the exponential factor $2^{\delta(k-k_{\min})}$ instead of $2^{\delta(k-k_1)}$. A closer inspection of the proofs of [31, Eq. (136), (137) and (138)], however, reveals that (6.4.36) holds.

*Remark* 6.4.3. In the higher dimensional case $d \geq 5$, all proofs in this section are valid with the substitutions as in Remark 6.3.8, as well as $Z^1 \to Z^{\frac{d-2}{2}}$ and $Z^1_{ell} \to Z_{ell}^{\frac{d-2}{2}}$. Moreover, the multilinear null form estimate in Proposition 6.4.1 is unnecessary. We claim that the following additional estimates hold:

$$\|\mathcal{H}_{s_1,s_2}\mathbf{A}^R_x(\varphi^1,\varphi^2)\|_{(Z^{\frac{d-2}{2}})_{bc}} \lesssim \|\varphi^1\|_{(\tilde{S}_{s_1}^{\frac{d-3}{2}})_b}\|\varphi^2\|_{(\tilde{S}_{s_2}^{\frac{d-3}{2}})_c}, \tag{6.4.38}$$

$$\|\mathcal{H}_{s_1,s_2}\mathbf{A}_0(\varphi^1,\varphi^2)\|_{(Z_{ell}^{\frac{d-2}{2}})_{bc}} \lesssim \|\varphi^1\|_{(\tilde{S}_{s_1}^{\frac{d-3}{2}})_b}\|\varphi^2\|_{(\tilde{S}_{s_2}^{\frac{d-3}{2}})_c}, \tag{6.4.39}$$

where the space $\tilde{S}_s^{\frac{d-3}{2}}$ does *not* involve the null frame spaces $PW^{\mp}_{\omega}(l)$ and $NE$; Combined with (the higher dimensional analogues of) (6.4.11) and (6.4.25), we obtain an analogue of Proposition 6.4.1 without relying on the null structure of $\mathcal{T}_{s_1,s_2,s}$ discussed in Remark 6.4.2.

One can prove (6.4.38) and (6.4.39) by following the argument of (6.4.20). Alternatively, one could use $L^2L^4$ Strichartz estimates like in the proof of (6.1.6).

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