A theorem on the rank of a product of matrices*

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Abstract

This paper develops a theorem that facilitates computing the degrees of freedom of an asymptotic χ^2 goodness-of-fit test for moment restrictions under rank deficiency of key matrices involved in the definition of the test.

Introduction

It frequently happens that for testing a null hypothesis of restrictions on the parameters of a model an asymptotic χ^2 test based on the "Wald method" (Moore, 1977) is used. The test involves an asymptotically normally distributed statistic whose variance matrix is of the form $C'VC$, where V is the variance matrix of a vector of sample moments and C is a matrix associated with the restrictions being tested. The rank of the above matrix product gives the degrees of freedom of the test. When V is nonsingular and C is of full column rank, the computation of this rank is very simple. However, in the case of a possibly singular matrix V, and possibly non-full column rank of C, deriving the rank is not trivial. An example of this rank deficiency arises in Satorra and Neudecker (2003), where an asymptotic χ^2 goodness-of-fit test statistic for moment structure models is derived in the setting of possibly singular variance matrix of the vector of sample moments. The purpose of the paper is to present a theorem that facilitates the computation of the rank of the above product of matrices in the general setting of rank deficiency. This is accomplished in the next section. Throughout, we use $r(.)$ and $\mathcal{M}(.)$ to denote respectively rank and column space of the corresponding matrix.

The fundamental rank theorem

This section contains the fundamental theorem of the paper. For proving the theorem we require the following lemma.

LEMMA 1: *Consider the matrices $H\,(n,m)$ and $X\,(m,m)$, where H is of full column rank. Then*

$$r(HXH') = r(X). \tag{1}$$

PROOF: The result follows from the (in)equality chain

$$r(X) \geq r(HXH') \geq r(H'HXH'H) = r(X),$$

We have used the following well-known properties:

1. $r(H'H) = r(H)$ and $r(HT) \leq \min\left[r(H), r(T)\right]$ for arbitrary matrices H and T. See, e.g., Magnus & Neudecker (1999, Ch. 1.7).

2. $r(UBU') = r(B)$ when U is nonsingular.

■

We move on to the theorem.

THEOREM 1: *Consider the matrices $A\;(p,q)$ and $C\;(p,m)$ satisfying*

$$r(A) + r(C) = p \tag{2}$$

and

$$C'A = 0; \tag{3}$$

let V *(p,p) be a positive semidefinite matrix. Then*

$$r\,(C'VC) = r(V) - r(A) \tag{4}$$

iff

$$\mathcal{M}(A) \subset \mathcal{M}(V), \tag{5}$$

In its present form the theorem is a generalized version of a result posed in problem form by Satorra and Neudecker (2003). A solution of that problem was subsequently provided by Puntanen, Styan, and Werner (2003). In the present version the matrix C is less specific.

PROOF: Equation (3) has as general solution

$$C' = QM$$

with symmetric idempotent $M = I_p - AA^+$ and arbitrary $(m,p)\,Q$. See, e.g., Magnus & Neudecker (1999, p. 38, Exerc. 4). As usual superscript + denotes the Moore-Penrose generalized inverse. It is clear that $r(M) = \operatorname{tr} M = p - r(A) = h$, say. Clearly, $MA = 0$.

Let us write $M = LL'$ with $(p,h)\,L$ and $L'L = I_h$. It follows that $C'VC = QMVMQ' = (QL)L'VL(L'Q')$ with $(m,h)\,QL$. As given, $r(A)+r(C) = p$, hence $r(C) = p-r(A) = h$. From the (in)equality chain

$$h = r(C) = r(C') = r(QM) = r(QLL') \leq r(QL) \leq r(L) = h$$

we get

$$r(QL) = h \tag{6}$$

Hence QL is of full column rank. We used the (in)equality $r(FG) \leq \min\,[r(F), r(G)]$ and the property that the rank of a matrix cannot exceed its lowest dimension. Using the lemma we conclude that

$$r(C'VC) = r(L'VL). \tag{7}$$

Let now $r(C'VC) = r(V) - r(A)$. This can be rephrased on the strength of (7) as

$$r(L'VL) = r(V) - r(A). \tag{8}$$

We shall then show that $\mathcal{M}(A) \subset \mathcal{M}(V)$.

SUBPROOF: (1) $MVM = LL'VLL'$, hence $r(MVM) = r(LL'VLL') = r(L'VL)$ as L has full column rank. We can now rephrase (8) in its turn as

$$r(MVM) + r(A) = r(V) \tag{9}$$

Clearly

$$r(MV^{\frac{1}{2}}, A) = r[(MV^{\frac{1}{2}}, A)'(MV^{\frac{1}{2}}, A)]$$

$$= r\left[\begin{pmatrix} V^{\frac{1}{2}}M \\ A' \end{pmatrix}\begin{pmatrix} MV^{\frac{1}{2}}, A \end{pmatrix}\right] = r\left(\begin{bmatrix} V^{\frac{1}{2}}MV^{\frac{1}{2}} & V^{\frac{1}{2}}MA \\ A'MV^{\frac{1}{2}} & A'A \end{bmatrix}\right)$$

$$= r\left(\begin{bmatrix} V^{\frac{1}{2}}MV^{\frac{1}{2}} & 0 \\ 0 & A'A \end{bmatrix}\right) = r(V^{\frac{1}{2}}M^2V^{\frac{1}{2}}) + r(A'A)$$

$$= r(MVM) + r(A).$$

We used the idempotence of M and the definitional equality $MA = 0$.

So we have shown that

$$r(MV^{\frac{1}{2}}, A) = r(V). \tag{10}$$

(2) Consider then an arbitrary vector $x = V^{\frac{1}{2}}a$. Hence $Mx = MV^{\frac{1}{2}}a$. This is a consistent equation with general solution

$$x = M^+MV^{\frac{1}{2}}a + (I_p - M^+M)b$$

(b arbitrary)

$$= MV^{\frac{1}{2}}a + (I_p - M)b$$

$$= MV^{\frac{1}{2}}a + AA^+b$$

$$= (MV^{\frac{1}{2}}, A)\begin{pmatrix} a \\ A^+b \end{pmatrix},$$

see, e.g., Magnus & Neudecker (1999, p.37, Theorem 12). Hence $x \in \mathcal{M}(MV^{\frac{1}{2}}, A)$. By consequence

$$\mathcal{M}(V^{\frac{1}{2}}) = \mathcal{M}(MV^{\frac{1}{2}}, A)$$

and

$$\mathcal{M}(A) \subset \mathcal{M}(V^{\frac{1}{2}}) = \mathcal{M}(V). \tag{11}$$

So we have established (5), viz.

$$\mathcal{M}(A) \subset \mathcal{M}(V).$$

Finally, we shall start from (5) and prove (4). Consider the partitioned $(p, p+q)$ matrix $(MV^{\frac{1}{2}}, A)$. Write then $A = V^{\frac{1}{2}}P$ for expressing (5). So

$$MV^{\frac{1}{2}} = (I_p - AA^+)V^{\frac{1}{2}} = V^{\frac{1}{2}} - V^{\frac{1}{2}}PA^+V^{\frac{1}{2}} = V^{\frac{1}{2}}(I_p - PA^+V^{\frac{1}{2}})$$

and

$$(MV^{\frac{1}{2}}, A)\begin{pmatrix} I \\ A^+V^{\frac{1}{2}} \end{pmatrix} = MV^{\frac{1}{2}} + AA^+V^{\frac{1}{2}} = (M + AA^+)V^{\frac{1}{2}} = V^{\frac{1}{2}}.$$

(We used $M = I_p - AA^+$.) From this follows

$$r(V) = r(V^{\frac{1}{2}}) \leq r(MV^{\frac{1}{2}}, A) \tag{12}$$

Further

$$(MV^{\frac{1}{2}}, A) = (V^{\frac{1}{2}} - V^{\frac{1}{2}}PA^{+}V^{\frac{1}{2}}, V^{\frac{1}{2}}P) = V^{\frac{1}{2}}(I_p - PA^{+}V^{\frac{1}{2}}, P).$$

Hence

$$r(MV^{\frac{1}{2}}, A) \leq r(V^{\frac{1}{2}}) = r(V). \tag{13}$$

So by (12) and (13) we find

$$r(MV^{\frac{1}{2}}, A) = r(V). \tag{14}$$

As shown earlier,

$$r(MV^{\frac{1}{2}}, A) = r(MVM) + r(A).$$

So,

$$r(MV^{\frac{1}{2}}, A) = r(L'VL) + r(A) = r(C'VC) + r(A),$$

and

$$r(C'VC) = r(V) - r(A).$$

Hence (4) has now been proved. ■

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