Title

Topics in Financial Math (Uncertain Volatility, Ross Recovery and Mean Field Games on Random Graph)

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Publication Date

2018

University of California
Santa Barbara

Topics in Financial Math (Uncertain Volatility, Ross Recovery and Mean Field Games on Random Graph)

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Statistics and Applied Probability

by

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June 2018

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All the glory dedicated to the mighty God.

Acknowledgements

I am a very lucky and blessed person. One of the biggest blessings is having Distinguished Professor Jean-Pierre Fouque as my Ph.D. adviser. Prof. Fouque is very famous and successful as being the Editor-in-Chief of SIAM Journal on Financial Mathematics, the president of Bachelier Finance Society, and others. However, he never perform as many big people as being arrogant, instead he has the best temper and always super nice to me. Can you believe a Ph.D. adviser never mad at students and never lose temper? That's my adviser.

When the time I was still a teenager at Shandong University in China, from a Mathematics History course, I learned in old times great French Mathematicians are mainly born and raised up in noble families, and they do Mathematics just for fun. The elegant and dedicated to details style that the way great French Mathematicians do research, can be exactly seen on my adviser. I asked him many naive questions in research, journal peer reviewing and career development, and he has always been patiently answered all my questions in all these years Ph.D. study. Doing research is very hard and you just fail many times until you succeed. Every time I felt very frustrated or I was even not satisfied with my work myself, my adviser encouraged me and never blamed me. Even sometimes I did not think I deserve such a good adviser, but he just never left me for a second. In fact, my adviser was one of the main reasons that I chose the career path as a scholar and keep doing research. I am not confident that I can be such a perfect Ph.D. adviser in all aspects to my future students, but I will learn from my adviser and will always remind myself on that.

I really want to thank our department of Statistics and Applied Probability. Every faculty in our department is excel at both research and teaching. Prof. Tomoyuki Ichiba is a precious asset in our department; he is very knowledgeable and truly responsible on graduate student affairs that he has been in charge of these year; he directly helped me on my research of uncertain volatility models with stochastic bounds and very luckily I learned local time in Stochastic Calculus from him, one of the leading experts on that. Thanks a lot to Tomoyuki's Teaching Assistant assignments, I was able to teach Time Series for many quarters and half of students in our department was in my sections, and amazingly I was able to work with and have a long time super good relationship with Prof. Raya Feldman on Time Series teaching. In fact, I really like her since the exact first quarter at UCSB by taking her course of PSTAT 213A (Introduction to Probability Theory and Stochastic Processes). At that time, I just came back from industry and the only one A+ in the whole class greatly encouraged me. She has been more like a friend to me and kindly asked me to lunches at many Actuarial Sciences events. I also met and had a great time chatting with her daughter, who is the same as me being USC Alumni, and currently a Ph.D. candidate in Engineering at Caltech, a super excellent girl.

Prof. Mike Ludkovski is kind and generous, and I had a great time working as his Teaching Assistant on Financial Math. Mike graduated from Princeton under the supervision of Prof. Carmona, and they both clearly carry the merit of caring about others' feeling. Mike is an expert in Simulation and Algorithm Design; in my transition period from purely theoretical research to both theoretical and practical, he always answered my elementary level questions in a very gentle way. Our department hired Prof. Nils Detering from 2016 and I received A+ on his random graph Ph.D. level course in Spring 2017, before my research of Mean Field Games on Random Graph started. There maybe other good candidates for the tenure track positions in our department, but the decision on Nils is absolutely correct, for the reason that he is a great person to work with and always has smiles on his face and super nice to everyone, exactly in the spirit of our department.

All the Statistics I can do is learned from Distinguished Professor S. Rao Jammalamadaka. After more than three years full time working in Data Science and Programming in industry, Dr. J as the Admission Committee Chair offered me the opportunity. I took his PSTAT 207 ABC (Mathematical Statistics) series and passed the Ph.D. qualification exam at the first try. Because of my excellent grades in Statistics, I had the chance to start a Probability and Bayesian Statistical Inference project "Reconstruction Problem" together with my long time best friend Wenjian Liu. Later, based on my good reputation in Time Series in my department, I was recommended to and then successfully developed the Multivariate Bayesian Structural Time Series (MBSTS) model together with my another long time best friend Jinwen Qiu, and fully tested this Machine Learning model on Finance.

My coauthor Peter Carr is someone I really need to thank to. Peter is absolutely a very good friend, but actually he is very famous and currently work as the Chair of the department of Finance and Risk Engineering at NYU Tandon School of Engineering, after he headed various top quant groups in the financial industry for the last twenty years. He taught me a lot of Finance, especially what is really happening in the Wall Street. He met me whenever I was in New York and have been meeting with our other Ross Recovery team members every week. I have always been truly impressed by how knowledgeable he is in Finance, Economic, Financial Math, Financial Engineer, and even Physics (probably he learned from his smart and pretty daughter majoring Physics).

In these years, I received a lot of help and encouragements from many colleagues in Probability and Financial Math, which I really appreciated. I am also really grateful for the research financial support from the following sources, which include but not limited to: NSF grant DMS-1409434 (PI: Jean-Pierre Fouque), Graduate Fellowship from PSTAT department as Research Travel Grant, Travel Support from annual conferences (Northeast Probability Seminar, Frontier Probability Days, Seminar on Stochastic Processes, Finger Lakes Probability Seminar, Western Conference on Mathematical Finance, Eastern Conference on Mathematical Finance). All research talk opportunities in seminars and conferences are truly appreciated.

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  with P. Carr et al., preprint, 2018.
- *Large Degree Asymptotics and the Reconstruction Threshold of the Asymmetric Binary Channels*
  with W. Liu, submitted, 2018.
- *Evolution of Regional Innovation with Spatial Knowledge Spillover: Convergence or Divergence?*
  with J. Qiu and W. Liu, submitted, 2018.
- *Multivariate Bayesian Structural Time Series Model*
  with R. Jammalamadaka and J. Qiu, submitted, 2018.
- *Uncertain Volatility Models with Stochastic Bounds*
  with J. P. Fouque, submitted, 2017.
- *The Tightness of the Kesten-Stigum Reconstruction Bound of Symmetric Model with Multiple Mutations*
  with R. Jammalamadaka, W. Liu, *Journal of Statistical Physics*, 2018.

Abstract

Topics in Financial Math (Uncertain Volatility, Ross Recovery and Mean Field Games on Random Graph)

by

Ning Ning

In this thesis, we discuss three new topics in Financial Mathematics using partial differential equation (PDE): uncertain volatility with stochastic bounds, Ross recovery with multivariate driving states and mean field games under the Erdös Rényi random graph, in three chapters respectively.

In Chapter 1, we study a class of uncertain volatility models with stochastic bounds, over which volatility stays between two bounds, but instead of using two deterministic bounds, the uncertain volatility fluctuates between two stochastic bounds generated by its inherent stochastic volatility process. We then apply a regular perturbation analysis upon the worst-case scenario price, and derive the first order approximation in the regime of slowly varying stochastic bounds. The original problem which involves solving a fully nonlinear PDE in dimension two for the worst-case scenario price, is reduced to solving a nonlinear PDE in dimension one and a linear PDE with source, which gives a tremendous computational advantage.

In Chapter 2, we address the problem of recovering the real world probability distribution from observed option prices by avoiding the intensively debated transition independence, through placing structure on the dynamics of the numeraire portfolio in a preference-free manner. We firstly utilize the Itô and Feynman–Kac theorem to derive a a uniformly elliptic operator, whose inverse is a compact linear operator, based on boundary conditions, and then apply the Krein-Rutman theorem to guarantee the uniqueness of

the positive eigenfunction, which happens to generate the physical transition probability.

In Chapter 3, we analyze a model of inter-bank lending and borrowing, by means of mean field games on the Erdös Rényi random graph. An open-loop Nash equilibrium is obtained using a system of fully coupled forward backward stochastic differential equations (FBSDEs), whose unique solution leads to the master equation. We explore the approximation to the finite player game equilibrium through a decoupled system of diffusion equations generated by the master equation under frozen graph, and through a weakly interacting particle system on random graph generated by the master equation under random graph, respectively.

Contents

Chapter 1

Topic in Uncertain Volatility

In this chapter, we study a class of uncertain volatility models with stochastic bounds. Like in the regular uncertain volatility models, we know only that volatility stays between two bounds, but instead of using two deterministic bounds, the uncertain volatility fluctuates between two stochastic bounds generated by its inherent stochastic volatility process. This brings better accuracy and is consistent with the observed volatility path such as for the VIX as a proxy for instance. We apply a regular perturbation analysis upon the worst-case scenario price, and derive the first order approximation in the regime of slowly varying stochastic bounds. The original problem which involves solving a fully nonlinear PDE in dimension two for the worst-case scenario price, is reduced to solving a nonlinear PDE in dimension one and a linear PDE with source, which gives a tremendous computational advantage. Numerical experiments show that this approximation procedure performs very well, even in the regime of moderately slow varying stochastic bounds. This chapter is based on the paper [1].

1.1 Overview of Uncertain Volatility Models with Deterministic Bounds

In the standard Black–Scholes model of option pricing ([2]), volatility is assumed to be known and constant over time. Since then, it has been widely recognized and well-documented that this assumption is not realistic. Extensions of the Black–Scholes model to model ambiguity have been proposed, such as the stochastic volatility approach ([3], [4]), the jump diffusion model ([5], [6]), and the uncertain volatility model ([7], [8]). Among these extensions, the uncertain volatility model has received intensive attention in Mathematical Finance for risk management purpose.

In the uncertain volatility models (UVMs), volatility is not known precisely and is assumed to lie between constant upper and lower bounds $\underline{\sigma}$ and $\overline{\sigma}$. These bounds could be inferred from extreme values of the implied volatilities of the liquid options, or from high-low peaks in historical stock- or option-implied volatilities. Under the risk-neutral measure, the price process of the risky asset satisfies the following stochastic differential equation (SDE):

$$dX_t = rX_t dt + \alpha_t X_t dW_t, \tag{1.1}$$

where r is the constant risk-free rate, (W_t) is a Brownian motion and the volatility process (α_t) belongs to a family $\mathcal{A}$ of progressively measurable and $[\underline{\sigma}, \overline{\sigma}]$-valued processes.

When pricing a European derivative written on the risky asset with maturity T and nonnegative payoff $h(X_T)$, the seller of the contract is interested in the worst-case scenario. By assuming the worst case, sellers are guaranteed coverage against adverse market behavior, if the realized volatility belongs to the candidate set. [9] showed that the seller of the derivative can superreplicate it with initial wealth $\operatorname{ess\,sup}_{\alpha \in \mathcal{A}} \mathbb{E}_t[h(X_T)]$, whatever the true volatility process is. The worst-case scenario price at time $t < T$ is given by

$$P(t, X_t) := \exp(-r(T-t))\, \mathrm{ess\,sup}_{\alpha\in\mathcal{A}}\, \mathbb{E}_t[h(X_T)], \tag{1.2}$$

where $\mathbb{E}_t[\cdot]$ is the conditional expectation given $\mathcal{F}_t$ with respect to the risk neutral measure.

Following the arguments in stochastic control theory, $P(t,x)$ is the viscosity solution to the following Hamilton-Jacobi-Bellman (HJB) equation, which is the generalized Black–Scholes–Barenblatt (BSB) nonlinear equation in Financial Mathematics,

$$\begin{aligned} \partial_t P + r(x\partial_x P - P) + \sup_{\alpha\in[\underline{\sigma},\overline{\sigma}]}\left[\frac{1}{2}x^2\alpha^2\partial^2_{xx}P\right] &= 0, \\ P(T,x) &= h(x). \end{aligned} \tag{1.3}$$

It is well known that the worst-case scenario price is equal to its Black–Scholes price with constant volatility $\overline{\sigma}$ (resp. $\underline{\sigma}$) for convex (resp. concave) payoff function (see [10] for instance).

For general terminal payoff functions, an asymptotic analysis of the worst-case scenario option prices as the volatility interval degenerates to a single point is derived in [11]. That is, in a small volatility interval $[\sigma, \sigma+\epsilon]$, the worst case scenario price $P^\epsilon(t, X_t)$ solves the following Black-Scholes-Barenblatt equation:

$$\begin{aligned} \partial_t P^\epsilon + r(x\partial_x P^\epsilon - P^\epsilon) + \sup_{\alpha\in[\sigma,\sigma+\epsilon]}\{\frac{1}{2}\alpha^2x^2\partial^2_{xx}P^\epsilon\} &= 0, \\ P^\epsilon(T) &= h. \end{aligned}$$

Fouque and Ren showed that, in [11], assume that the payoff function $h \in C^4_p(\mathbb{R}+)$, h is Lipschitz, and its derivatives up to order 4 have polynomial growth, and the second derivative of h has a finite number of zero points, then pointwise,

$$\lim_{\epsilon \downarrow 0} \frac{P^{\epsilon} - (P_0 + \epsilon P_1)}{\epsilon} = 0,$$

where P_0 is the solution of the following Black-Scholes equation:

$$\begin{aligned} \partial_t P_0 + r(x\partial_x P_0 - P_0) + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 P_0 = 0, \\ P_0(T) = h. \end{aligned}$$

P_1 is the solution of the following equation:

$$\begin{aligned} \partial_t P_1 + r(x\partial_x P_1 - P_1) + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 P_1 + \sup_{g \in [0,1]} g\sigma x^2 \partial_{xx}^2 P_0 = 0, \\ P_1(T) = 0. \end{aligned}$$

1.2 Uncertain Volatility Models with Stochastic Bounds

Looking at the VIX over years, which is a popular measure of the implied volatility of SP500 index options, you will see that for longer time-horizons, it is no longer consistent with observed volatility to assume that the bounds are constant. Therefore, instead of modeling α_t fluctuating between two deterministic bounds i.e. $\underline{\sigma} \leq \alpha_t \leq \overline{\sigma}$, it is reasonable to consider the case that the uncertain volatility moves between two stochastic bounds i.e. $\underline{\sigma}_t \leq \alpha_t \leq \overline{\sigma}_t$. [12] introduced the notion of random G-expectation, which successfully extended the G-expectation (see [13]) by allowing the range of the volatility uncertainty to be stochastic. Later [14] established the duality formula for the super-replication price, in a setting of volatility uncertainty including random G-expectation. [15] consolidated the foundation of this new area, by providing a general construction of time-consistent sublinear expectations on the space of continuous paths, which yields the existence of the conditional G-expectation of a Borel-measurable random variable and an optional sampling theorem. [16] further provided the PDE characterization of the superreplication price in a jump diffusion setting, in which the link between the worst-case scenario price under stochastic bounds and its associated BSB equation is established for the first time.

In this chapter, we study a class of models where the bounds are stochastic and slowly moving. The "center" of the bounds follows a stochastic process $F(Z_t)$, where F is a positive increasing and differentiable function on the domain of a regular diffusion of the form

$$dZ_t = \delta\mu(Z_t)dt + \sqrt{\delta}\beta(Z_t)dW_t^Z. \tag{1.4}$$

Here, W^Z is a Brownian motion possibly correlated to the Brownian motion W driving the stock price, with $d\langle W, W^Z\rangle_t = \rho dt$ for $|\rho| \leq 1$. The parameter $\delta > 0$ represents the reciprocal of the time-scale of the process Z and will be small in the asymptotics that we consider in the chapter. The volatility bound itself is given by

$$\underline{\sigma}_t := d\,F(Z_t) \leq \alpha_t \leq \overline{\sigma}_t := u\,F(Z_t), \quad \text{for} \quad 0 \leq t \leq T, \tag{1.5}$$

with u and d two constants such that $0 < d < 1 < u$. In the following, we will use the popular CIR process for Z, that is $\mu(z) = \kappa(\theta - z)$ and $\beta(z) = \sqrt{z}$ in (1.4), under the Feller condition $2\kappa\theta \geq 1$ to ensure that Z_t stays positive:

$$dZ_t = \delta\kappa(\theta - Z_t)dt + \sqrt{\delta}\sqrt{Z_t}dW_t^Z, \quad Z_0 = z > 0. \tag{1.6}$$

Our asymptotic analysis will reveal that, to the order $\sqrt{\delta}$, only the vol-vol value $\beta(z)$, the volatility level $F(z)$ and its slope $F'(z)$ are involved, but not the drift function μ. In the spirit of the Heston model we will use $F(z) = \sqrt{z}$ on $(0, \infty)$, and we will also give the corresponding formulas for our approximation in terms of a general function F. We denote $\alpha_t := q_t\sqrt{Z_t}$ so that the uncertainty in the volatility can be absorbed in the uncertain adapted slope as follows

$$d \leq q_t \leq u, \quad \text{for} \quad 0 \leq t \leq T.$$

One realization of the bounds is shown in Figure 1.1 with $\delta = .05$.

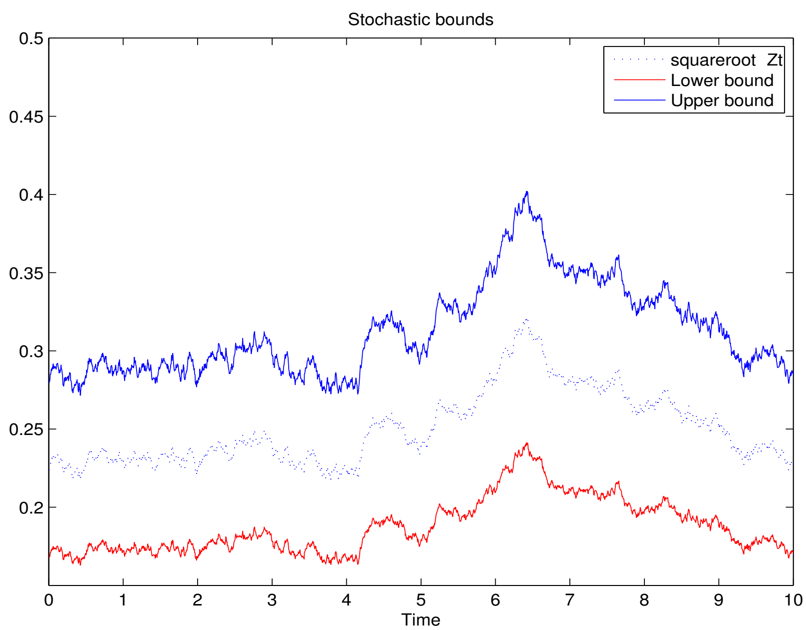

Figure 1.1: Simulated stochastic bounds $[0.75\sqrt{Z_t}, 1.25\sqrt{Z_t}]$ where Z_t is the (slow) mean-reverting CIR process (1.6).

In order to study the asymptotic behavior, we emphasize the importance of δ and reparameterize the SDE of the risky asset price process as

$$dX_t^\delta = rX_t^\delta dt + q_t\sqrt{Z_t}X_t^\delta dW_t. \tag{1.7}$$

When $\delta = 0$, note that the CIR process Z_t is frozen at z, and then the risky asset price process follows the dynamic

$$dX_t^0 = rX_t^0 dt + q_t\sqrt{z}X_t^0 dW_t, \tag{1.8}$$

both X_t^δ and X_t^0 starting at the same point x.

We denote the smallest riskless selling price (worst-case scenario) at time $t < T$ as

$$P^\delta(t,x,z) := \exp(-r(T-t))\,\mathrm{ess\,sup}_{q_\cdot\in[d,u]}\,\mathbb{E}_{(t,x,z)}[h(X_T^\delta)], \tag{1.9}$$

where $\mathbb{E}_{(t,x,z)}[\cdot]$ is the conditional expectation given $\mathcal{F}_t$ with $X_t^\delta = x$ and $Z_t = z$. When $\delta = 0$, we represent the smallest riskless selling price as

$$P_0(t,x,z) = \exp(-r(T-t))\,\mathrm{ess\,sup}_{q_\cdot\in[d,u]}\,\mathbb{E}_{(t,x,z)}[h(X_T^0)], \tag{1.10}$$

where the subscripts in $\mathbb{E}_{(t,x,z)}[\cdot]$ also means that $X_t^0 = x$ and $Z_t = z$ given the same filtration $\mathcal{F}_t$. Notice that $P_0(t,X_t,z)$ corresponds to $P(t,X_t)$ in (1.2) with constant volatility bounds given by $d\sqrt{z}$ and $u\sqrt{z}$.

Before displaying our result, it is worth mentioning some related new literatures. The result of [17] can be used to derive a robust superhedging duality and the existence of an optimal superhedging strategy for general contingent claims. [18] studied a robust portfolio optimization problem under model uncertainty for an investor with logarithmic or power utility, where uncertainty is specified by a set of possible Lévy triplets. [19] analyzed the formation of derivative prices in equilibrium between risk neutral agents with heterogeneous beliefs, in the spirit of uncertain volatility with stochastic bounds.

1.3 Asymptotic Analysis by Perturbation Method

In this section, we first prove the Lipschitz continuity of the worst-case scenario price P^δ with respect to the parameter δ. Then, we derive the main BSB equation that the worst-case scenario price should follow and further identify the first order approximation when δ is small enough. We reduce the original problem of solving the fully nonlinear PDE (1.11) in dimension two to solving the nonlinear PDE (1.13) in dimension one and a linear PDE (1.18) with source. The accuracy of this approximation is given in Theorem 4, the main theorem of this chapter.

1.3.1 Convergence of P^δ

It is established in Appendix A.1 that X_t^δ and Z_t have finite moments for δ sufficiently small, which leads to the following result:

Proposition 1. *Let X^δ satisfies the SDE* (1.7) *and X^0 satisfies the SDE* (1.8)*, then, uniformly in* (*q.*)*,*

$$\mathbb{E}_{(t,x,z)}(X_T^\delta - X_T^0)^2 \leq C_0\delta$$

where C_0 is a positive constant independent of δ.

Proof. See Appendix A.2. □

In order to carry out our asymptotic analysis, we need to impose some regularity of the payoff function h. Note that our numerical example in Section 1.5, a "butterfly" profile, does not satisfy these assumptions but we mention there a possible regularization step.

Assumption 1. *We assume that the terminal function h is four times differentiable, with a bounded first derivative and polynomial growth of the fourth derivative:*

$$\begin{cases} |h'(x)| \le K_1, \\ |h^{(4)}(x)| \le K_4(1+|x|^l), \end{cases}$$

for constants K_1 and K_4, and an integer l.

Remark 1. *The polynomial growth condition on $h^{(4)}$ implies polynomial growth of h'' and h''', and the bounded first derivative assumption implies that h is Lipschitz.*

Remark 2. *Note that for convex or concave payoff functions, such as for vanilla European Calls or Puts, if $h(\cdot)$ is convex (resp. concave), for the reason that supremum and expectation preserves convexity (resp. concavity), one can see that the worst-case scenario price*

$$P^\delta(t,x,z) = \exp(-r(T-t))\,\mathrm{ess\,sup}_{q.\in[d,u]}\,\mathbb{E}_{(t,x,z)}[h(X_T)],$$

is convex (resp. concave) with $\partial^2_{xx}P^\delta > 0$ (resp. < 0), and thus $q^{*,\delta} = u$ (resp. $= d$). In these two cases, we are back to perturbations around Black–Scholes prices which have been treated in [20]. In this chapter, we work with general terminal payoff functions, neither convex nor concave, therefore the signs of the second derivatives of the option prices cannot be easily determined. In order to proceed we impose regularity conditions on the payoff functions (Assumption 1) as in [11].

Theorem 1. *Under Assumption 1, $P^\delta(\cdot,\cdot,\cdot)$, as a family of functions of (t,x,z) indexed by δ, converges to $P_0(\cdot,\cdot,\cdot)$ with rate $\sqrt{\delta}$, uniformly in $(t,x,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^+$.*

Proof. For P^δ given by (1.9) and P_0 given by (1.10), using the Lipschitz continuous of

$h(\cdot)$ and by the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|P^\delta - P_0| =& \exp(-r(T-t)) \left| \operatorname{ess\,sup}_{q.\in[d,u]} \mathbb{E}_{(t,x,z)}[h(X_T^\delta)] - \operatorname{ess\,sup}_{q.\in[d,u]} \mathbb{E}_{(t,x,z)}[h(X_T^0)] \right| \\
\leq& \exp(-r(T-t)) \left| \operatorname{ess\,sup}_{q.\in[d,u]} \mathbb{E}_{(t,x,z)}[h(X_T^\delta)] - \mathbb{E}_{(t,x,z)}[h(X_T^0)] \right| \\
\leq& \exp(-r(T-t)) \operatorname{ess\,sup}_{q.\in[d,u]} \left| \mathbb{E}_{(t,x,z)}[h(X_T^\delta) - h(X_T^0)] \right| \\
\leq& K_0 \exp(-r(T-t)) \operatorname{ess\,sup}_{q.\in[d,u]} \mathbb{E}_{(t,x,z)} \left| X_T^\delta - X_T^0 \right| \\
\leq& K_0 \exp(-r(T-t)) \operatorname{ess\,sup}_{q.\in[d,u]} \left[\mathbb{E}_{(t,x,z)} (X_T^\delta - X_T^0)^2 \right]^{1/2}.
\end{aligned}
$$

Therefore, by Proposition 1, we have

$$
|P^\delta - P_0| \leq C_1 \sqrt{\delta}
$$

where C_1 is a positive constant independent of δ, as desired. □

1.3.2 Pricing Nonlinear PDEs

We now derive P_0 and P_1, the leading order term and the first correction for the approximation of the worst-case scenario price P^δ, which is the solution to the HJB equation associated to the corresponding control problem given by the generalized BSB nonlinear equation:

$$
\begin{aligned}
\partial_t P^\delta + r(x\partial_x P^\delta - P^\delta) + \sup_{q\in[d,u]} \{ \frac{1}{2} q^2 z x^2 \partial_{xx}^2 P^\delta + \sqrt{\delta}(q\rho z x \partial_{xz}^2 P^\delta) \} \\
+ \delta(\frac{1}{2} z \partial_{zz}^2 P^\delta + \kappa(\theta - z)\partial_z P^\delta) = 0,
\end{aligned} \tag{1.11}
$$

with terminal condition $P^\delta(T,x,z) = h(x)$. For simplicity and without loss of generality, $r = 0$ is assumed for the rest of chapter.

In this section, we use the regular perturbation approach to formally expand the value function $P^\delta(t,x,z)$ as follows:

$$P^\delta = P_0 + \sqrt{\delta} P_1 + \delta P_2 + \cdots . \tag{1.12}$$

Inserting this expansion into the main BSB equation (1.11), by Theorem 1, the leading order term P_0 is the solution to

$$\begin{aligned} \partial_t P_0 + \sup_{q\in[d,u]} \{\frac{1}{2} q^2 z x^2 \partial^2_{xx} P_0\} = 0, \\ P_0(T,x,z) = h(x). \end{aligned} \tag{1.13}$$

In this case, z is just a positive parameter, and we have existence and uniqueness of a smooth solution to (1.13) (we refer to [10]). Note that in the general model given by (1.4) and (1.5), the equation for P_0 would be

$$\partial_t P_0 + \sup_{q\in[d,u]} \{\frac{1}{2} q^2 \sigma^2 x^2 \partial^2_{xx} P_0\} = 0, \quad \sigma := F(z).$$

1.3.3 Convergence of $\partial^2_{xx} P^\delta$

In what follows, we will assume regularity of the solution P^δ of the nonlinear PDE (1.11).

Assumption 2. *Throughout the chapter, we make the following assumptions on P^δ:*
(i) $P^\delta(\cdot,\cdot,\cdot)$ belongs to $C_p^{1,2,2}$ (p for polynomial growth), for δ fixed.
(ii) $\partial_x P^\delta(\cdot,\cdot,\cdot)$ and $\partial^2_{xx} P^\delta(\cdot,\cdot,\cdot)$ are uniformly bounded in δ.

Remark 3. *In the present chapter, we are concerned with a practical approximation of the superreplication problem viewed as a perturbation around the classical case of fixed volatility bounds. Our starting point is a superreplication price given as the classical solution of a nonlinear PDE. Regarding the link between the worst-case scenario option price* *with its associated BSB equation, as well as regularity conditions and uniform boundedness of derivatives, we refer to [14] and Lemma 3.2 in [19] in a different context.*

Then, under Assumption 2, we have the following Proposition:

Proposition 2. *Under Assumptions 1 and 2, the family* $\partial_{xx}^2 P^\delta(\cdot,\cdot,\cdot)$ *of functions of* (t,x,z) *indexed by* δ, *converges to* $\partial_{xx}^2 P_0(\cdot,\cdot,\cdot)$ *as* δ *tends to* 0 *with rate* $\sqrt{\delta}$, *uniformly on compact sets in* (x,z) *and* $t \in [0,T]$.

Proof. Under Assumptions 1 and 2, and by Theorem 1, the Proposition can be obtained by following the arguments in Theorem 5.2.5 of [21]. □

Denote the zero sets of $\partial_{xx}^2 P_0$ as

$$S_{t,z}^0 := \{x = x(t,z) \in \mathbb{R}^+ | \partial_{xx}^2 P_0(t,x,z) = 0\}.$$

Define the set where $\partial_{xx}^2 P^\delta$ and $\partial_{xx}^2 P_0$ take different signs as

$$\begin{aligned} A_{t,z}^\delta :=& \{x = x(t,z) | \partial_{xx}^2 P^\delta(t,x,z) > 0, \partial_{xx}^2 P_0(t,x,z) < 0\} \\ & \cup \{x = x(t,z) | \partial_{xx}^2 P^\delta(t,x,z) < 0, \partial_{xx}^2 P_0(t,x,z) > 0\}. \end{aligned} \tag{1.14}$$

Assumption 3. *We make the following assumptions:*

(i) There is a finite number of zero points of $\partial_{xx}^2 P_0(t,x,z)$, *for any* $t \in [0,T]$ *and* $z > 0$, *that is,* $S_{t,z}^0 = \{x_1 < x_2 < \cdots < x_{m(t,z)}\}$, *where we assume that the number* $m(t,z)$ *is uniformly bounded in* $t \le T$ *and* $z \in \mathbb{R}$.

(ii) There exists a constant C *such that the set* $A_{t,z}^\delta$ *defined in* (1.14) *is included in* $\cup_{i=1}^{m(t,z)} I_i^\delta$, *where*

$$I_i^\delta := [x_i - C\sqrt{\delta}, x_i + C\sqrt{\delta}], \quad \textit{for } x_i \in S_{t,z}^0 \textit{ and } 1 \le i \le m(t,z).$$

Furthermore, we assume that for every $M > 0$ there exists $B > 0$ such that $|x_i| \leq B$ for any $x_i \in S^0_{t,z}$, $1 \leq i \leq m(t,z)$, $z \leq M$.

Remark 4. *Here we explain the rationale for Assumption 3 (ii).*

Suppose P_0 has a third derivative with respect to x, which does not vanish on the set $S^0_{t,z}$. By Proposition 2, $\partial^2_{xx}P^\delta$ converges to $\partial^2_{xx}P_0$ with rate $\sqrt{\delta}$, therefore we conclude that there exists a constant C such that on the set $(\cup_{i=1}^{m(t,z)} I_i^\delta)^c$, $\partial^2_{xx}P^\delta(t,x,z)$ and $\partial^2_{xx}P_0(t,x,z)$ have the same sign, and Assumption 3 (ii) would follow. This is illustrated in Figure 1.5.3 by an example with two zero points for $\partial^2_{xx}P_0(t,x,z)$.

Otherwise, I_i^δ would have a larger radius of order $\mathcal{O}(\delta^\alpha)$ for $\alpha \in (0, \frac{1}{2})$, and then the accuracy in the main Theorem 4 would be $\mathcal{O}(\delta^{\alpha+1/2})$, but in any case of order $o(\sqrt{\delta})$.

In the sequel, in order to simplify the expressions, we denote

$$P_0 := P_0(t,x,z) \quad \text{and} \quad P^\delta := P^\delta(t,x,z),$$

and similar notations apply to the corresponding derivatives.

1.3.4 Optimizers

The optimal control in the nonlinear PDE (1.13) for P_0, denoted as

$$q^{*,0}(t,x,z) := \arg\max_{q\in[d,u]}\{\frac{1}{2}q^2 z x^2 \partial^2_{xx}P_0\},$$

is given by

$$q^{*,0}(t,x,z) = \begin{cases} u, \partial^2_{xx}P_0 \geq 0 \\ d, \partial^2_{xx}P_0 < 0 \end{cases}. \tag{1.15}$$

The optimizer to the main BSB equation (1.11) is given in the following lemma:

Lemma 1. *Under Assumption 3, for δ sufficiently small and for $x \notin S^0_{t,z}$, the optimal control in the nonlinear PDE* (1.11) *for P^δ, denoted as*

$$q^{*,\delta}(t,x,z) := \arg\max_{q\in[d,u]}\{\frac{1}{2}q^2zx^2\partial^2_{xx}P^\delta + \sqrt{\delta}(q\rho zx\partial^2_{xz}P^\delta)\},$$

is given by

$$q^{*,\delta}(t,x,z) = \begin{cases} u, \partial^2_{xx}P^\delta \geq 0 \\ d, \partial^2_{xx}P^\delta < 0 \end{cases}. \tag{1.16}$$

Proof. To find the optimizer $q^{*,\delta}$ to

$$\sup_{q\in[d,u]}\{\frac{1}{2}q^2zx^2\partial^2_{xx}P^\delta + \sqrt{\delta}(q\rho zx\partial^2_{xz}P^\delta)\},$$

we firstly relax the restriction $q \in [d,u]$ to $q \in \mathbb{R}$.

Denote

$$f(q) := \frac{1}{2}q^2zx^2\partial^2_{xx}P^\delta + \sqrt{\delta}(q\rho zx\partial^2_{xz}P^\delta).$$

By the result of Proposition 2 that $\partial^2_{xx}P^\delta$ uniformly converge to $\partial^2_{xx}P_0$ as δ goes to 0, for $x \notin S^0_{t,z}$, the optimizer of $f(q)$ is given by

$$\hat{q}^{*,\delta} = -\frac{\rho\sqrt{\delta}\partial^2_{xz}P^\delta}{x\partial^2_{xx}P^\delta}.$$

Since X_t and Z_t are strictly positive, the sign of the coefficient of q^2 in $f(q)$ is determined by the sign of $\partial^2_{xx}P^\delta$. We have the following cases represented in Figure 1.2, from which we can see that for δ sufficiently small such that $|\hat{q}^{*,\delta}| \leq d$, the optimizer is given by

$$q^{*,\delta} = u\mathbb{1}_{\{\partial^2_{xx}P^\delta\geq 0\}} + d\mathbb{1}_{\{\partial^2_{xx}P^\delta<0\}}.$$

□

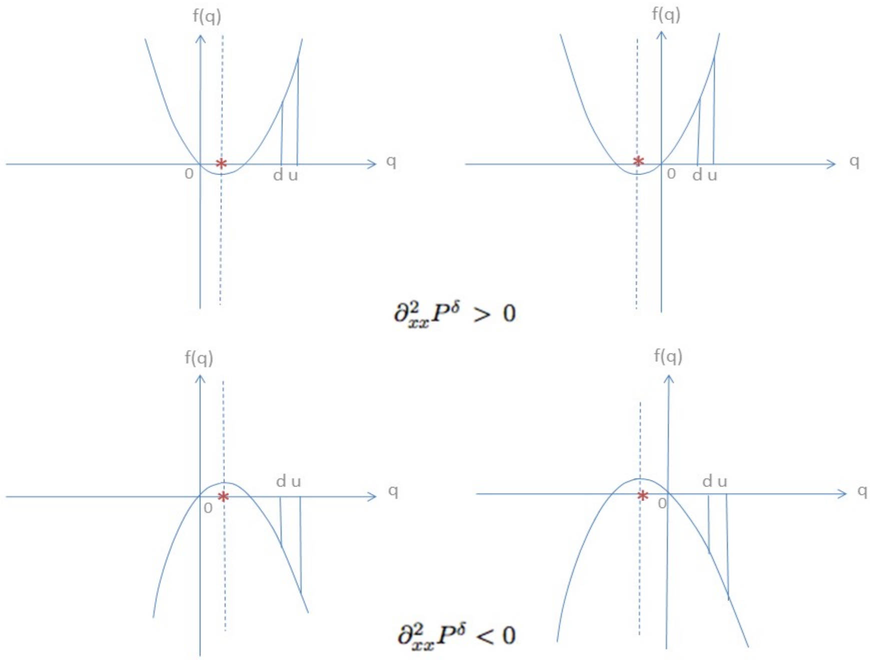

Figure 1.2: Illustration of the derivation of $q^{*,\delta}$: if $\partial^2_{xx}P^\delta > 0$, whether $\hat{q}^{*,\delta}$ is positive or negative, with the requirement $q \in [d, u]$, $q^{*,\delta} = u$; otherwise $q^{*,\delta} = d$.

Plugging the optimizer $q^{*,\delta}$ given by Lemma 1, the BSB equation (1.11) can be rewritten as

$$\partial_t P^\delta + \frac{1}{2}(q^{*,\delta})^2 z x^2 \partial^2_{xx} P^\delta + \sqrt{\delta}(q^{*,\delta}\rho z x \partial^2_{xz} P^\delta) + \delta(\frac{1}{2} z \partial^2_{zz} P^\delta + \kappa(\theta - z)\partial_z P^\delta) = 0, \quad (1.17)$$

with terminal condition $P^\delta(T, x, z) = h(x)$.

1.3.5 Heuristic Expansion

We insert the expansion (1.12) into the main BSB equation (1.17) and collect terms in successive powers of $\sqrt{\delta}$. Under Assumption 3 that $q^{*,\delta} \to q^{*,0}$ as $\delta \to 0$, without loss of accuracy, the first order correction term P_1 is chosen as the solution to the linear equation:

$$\begin{aligned} \partial_t P_1 + \frac{1}{2}(q^{*,0})^2 z x^2 \partial^2_{xx} P_1 + q^{*,0} \rho z x \partial^2_{xz} P_0 = 0, \\ P_1(T,x,z) = 0, \end{aligned} \tag{1.18}$$

where $q^{*,0}$ is given by (1.15).

Since (1.18) is linear, the existence and uniqueness result of a smooth solution P_1 can be achieved by firstly change the variable $x \to \ln x$, and then use the classical result of [22] for the parabolic equation (1.18) with diffusion coefficient bounded below by $d^2 z > 0$.

Note that in the general model given by (1.4) and (1.5), using the chain rule, the equation for P_1 would be

$$\partial_t P_1 + \frac{1}{2}(q^{*,0})^2 \sigma^2 x^2 \partial^2_{xx} P_1 + q^{*,0} \rho \sigma \sigma' \beta x \partial^2_{x\sigma} P_0 = 0, \quad \sigma = F(z), \sigma' := F'(z), \beta := \beta(z).$$

We shall show in the following that under additional regularity conditions imposed on the derivatives of P_0 and P_1, the approximation error $|P^\delta - P_0 - \sqrt{\delta} P_1|$ is of order $\mathcal{O}(\delta)$.

Assumption 4. *The following derivatives of P_0 and P_1 are of polynomial growth:*

$$\begin{cases} |\partial^2_{xx} P_0(t,x,z)| \le a_{20}(1 + x^{b_{20}} + z^{c_{20}}) \\ |\partial^2_{xz} P_0(t,x,z)| \le a_{11}(1 + x^{b_{11}} + z^{c_{11}}) \\ |\partial_z P_0(t,x,z)| \le a_{01}(1 + x^{b_{01}} + z^{c_{01}}) \\ |\partial^2_{xx} P_1(t,x,z)| \le \bar{a}_{20}(1 + x^{\bar{b}_{20}} + z^{\bar{c}_{20}}) \\ |\partial_z P_1(t,x,z)| \le \bar{a}_{01}(1 + x^{\bar{b}_{01}} + z^{\bar{c}_{01}}) \\ |\partial^2_{zz} P_1(t,x,z)| \le \bar{a}_{02}(1 + x^{\bar{b}_{02}} + z^{\bar{c}_{02}}) \end{cases} \tag{1.19}$$

where $a_i, b_i, c_i, \bar{a}_i, \bar{b}_i, \bar{c}_i$ are positive integers for $i \in (20, 11, 01, 02)$.

Remark 5. *As explained at the beginning of Section 1.3.3, regularity of P_0 and subsequently of P_1 given by (1.18), can be obtained from the assumed regularity of the payoff h (Assumption 1). The proof being outside the scope of this chapter, we list these properties as assumptions and we introduce the notation for the constants needed later.*

1.3.6 Formal Expansion

Define the following operator

$$\begin{aligned} \mathcal{L}^{\delta}(q) :&= \partial_t + \frac{1}{2}q^2 z x^2 \partial^2_{xx} + \sqrt{\delta} q \rho z x \partial^2_{xz} + \delta(\frac{1}{2} z \partial^2_{zz} + \kappa(\theta - z)\partial_z) \\ &= \mathcal{L}_0(q) + \sqrt{\delta}\mathcal{L}_1(q) + \delta \mathcal{L}_2, \end{aligned} \tag{1.20}$$

where $\mathcal{L}_0(q)$ contains the time derivative and is the Black–Scholes operator $\mathcal{L}_{BS}(q\sqrt{z})$, $\mathcal{L}_1(q)$ contains the mixed derivative due to the covariation between X and Z, and $\delta\mathcal{L}_2$ is the infinitesimal generator of the process Z, also denoted by $\delta\mathcal{L}_{CIR}$.

The main equation (1.17) can be rewritten as

$$\begin{aligned} \mathcal{L}^{\delta}(q^{*,\delta})P^{\delta} &= 0, \\ P^{\delta}(t,x,z) &= h(x). \end{aligned} \tag{1.21}$$

Equation (1.13) becomes

$$\begin{aligned} \mathcal{L}_0(q^{*,0})P_0 &= 0, \\ P_0(T,x,z) &= h(x). \end{aligned} \tag{1.22}$$

Equation (1.18) becomes

$$\begin{aligned}\mathcal{L}_0(q^{*,0})P_1 + \mathcal{L}_1(q^{*,0})P_0 &= 0,\\ P_1(T,x,z) &= 0.\end{aligned} \tag{1.23}$$

Applying the operator $\mathcal{L}^\delta(q^{*,\delta})$ to the error term, it follows that

$$\begin{aligned}
\mathcal{L}^\delta(q^{*,\delta})E^\delta =& \mathcal{L}^\delta(q^{*,\delta})(P^\delta - P_0 - \sqrt{\delta}P_1)\\
=& \underbrace{\mathcal{L}^\delta(q^{*,\delta})P^\delta}_{=0,\text{ eq. (1.21).}} - \mathcal{L}^\delta(q^{*,\delta})(P_0 + \sqrt{\delta}P_1)\\
=& -\left(\mathcal{L}_0(q^{*,\delta}) + \sqrt{\delta}\mathcal{L}_1(q^{*,\delta}) + \delta\mathcal{L}_{CIR}\right)(P_0 + \sqrt{\delta}P_1)\\
=& -\mathcal{L}_0(q^{*,\delta})P_0 - \sqrt{\delta}\left[\mathcal{L}_1(q^{*,\delta})P_0 + \mathcal{L}_0(q^{*,\delta})P_1\right] - \delta\left[\mathcal{L}_1(q^{*,\delta})P_1 + \mathcal{L}_{CIR}P_0\right]\\
& - \delta^{\frac{3}{2}}\left[\mathcal{L}_{CIR}P_1\right]\\
=& -\underbrace{\mathcal{L}_0(q^{*,0})P_0}_{=0,\text{ eq. (1.22).}} - (\mathcal{L}_0(q^{*,\delta}) - \mathcal{L}_0(q^{*,0}))P_0 - \sqrt{\delta}\Bigg[\underbrace{\mathcal{L}_1(q^{*,0})P_0 + \mathcal{L}_0(q^{*,0})P_1}_{=0,\text{ eq. (1.23).}}\\
& + (\mathcal{L}_1(q^{*,\delta}) - \mathcal{L}_1(q^{*,0}))P_0 + (\mathcal{L}_0(q^{*,\delta}) - \mathcal{L}_0(q^{*,0}))P_1\Bigg]\\
& - \delta\left[\mathcal{L}_1(q^{*,\delta})P_1 + \mathcal{L}_{CIR}P_0\right] - \delta^{\frac{3}{2}}(\mathcal{L}_{CIR}P_1)\\
=& -(\mathcal{L}_0(q^{*,\delta}) - \mathcal{L}_0(q^{*,0}))P_0 - \sqrt{\delta}\Bigg[(\mathcal{L}_1(q^{*,\delta}) - \mathcal{L}_1(q^{*,0}))P_0\\
& + (\mathcal{L}_0(q^{*,\delta}) - \mathcal{L}_0(q^{*,0}))P_1\Bigg] - \delta\left[\mathcal{L}_1(q^{*,\delta})P_1 + \mathcal{L}_{CIR}P_0\right] - \delta^{\frac{3}{2}}(\mathcal{L}_{CIR}P_1)\\
=& -\frac{1}{2}[(q^{*,\delta})^2 - (q^{*,0})^2]zx^2\partial^2_{xx}P_0\\
& - \sqrt{\delta}\left[\rho(q^{*,\delta} - q^{*,0})zx\partial^2_{xz}P_0 + \frac{1}{2}\left((q^{*,\delta})^2 - (q^{*,0})^2\right)zx^2\partial^2_{xx}P_1\right]\\
& - \delta\left[\rho(q^{*,\delta})zx\partial^2_{xz}P_1 + \frac{1}{2}z\partial^2_{zz}P_0 + \kappa(\theta - z)\partial_z P_0\right]\\
& - \delta^{\frac{3}{2}}\left[\frac{1}{2}z\partial^2_{zz}P_1 + \kappa(\theta - z)\partial_z P_1\right],
\end{aligned}$$

where $q^{*,0}$ and $q^{*,\delta}$ are given in (1.15) and (1.16) respectively.

The terminal condition of E^δ is given by

$$E^\delta(T,x,z) = P^\delta(T,x,z) - P_0(T,x,z) - \sqrt{\delta}P_1(T,x,z) = 0.$$

1.4 Accuracy of the Approximation

1.4.1 Feynman–Kac representation of the error term

For δ sufficiently small, the optimal choice $q^{*,\delta}$ to the main BSB equation (1.11) is given explicitly in Lemma 1. Correspondingly, the asset price in the worst-case scenario is a stochastic process which satisfies the SDE (1.1) with $(q_t) = (q^{*,\delta})$ and $r = 0$, i.e.,

$$dX_t^{*,\delta} = q^{*,\delta}\sqrt{Z_t}X_t^{*,\delta}dW_t. \tag{1.24}$$

Given the existence and uniqueness result of $X_t^{*,\delta}$ proved in Appendix A.3, we have the following probabilistic representation of $E^\delta(t,x,z)$ by Feynman–Kac formula:

$$E^\delta(t,x,z) = I_0 + \delta^{\frac{1}{2}}I_1 + \delta I_2 + \delta^{\frac{3}{2}}I_3,$$

where

$$
\begin{aligned}
I_0 &:= \mathbb{E}_{(t,x,z)}\left[\int_t^T \frac{1}{2}\left((q^{*,\delta})^2-(q^{*,0})^2\right) Z_s (X_s^{*,\delta})^2 \partial_{xx}^2 P_0(s,X_s^{*,\delta},Z_s)ds\right],\\
I_1 &:= \mathbb{E}_{(t,x,z)}\left[\int_t^T \left((q^{*,\delta}-q^{*,0})\rho Z_s X_s^{*,\delta}\partial_{xz}^2 P_0(s,X_s^{*,\delta},Z_s)\right.\right.\\
&\qquad \left.\left.+\frac{1}{2}\left((q^{*,\delta})^2-(q^{*,0})^2\right) Z_s (X_s^{*,\delta})^2 \partial_{xx}^2 P_1(s,X_s^{*,\delta},Z_s)\right)ds\right],\\
I_2 &:= \mathbb{E}_{(t,x,z)}\left[\int_t^T \left(q^{*,\delta}\rho Z_s X_s^{*,\delta}\partial_{xz}^2 P_1(s,X_s^{*,\delta},Z_s)+\frac{1}{2}Z_s\partial_{zz}^2 P_0(s,X_s^{*,\delta},Z_s)\right.\right.\\
&\qquad \left.\left.+\kappa(\theta-Z_s)\partial_z P_0(s,X_s^{*,\delta},Z_s)\right)ds\right],\\
I_3 &:= \mathbb{E}_{(t,x,z)}\left[\int_t^T \left(\frac{1}{2}Z_s\partial_{zz}^2 P_1(s,X_s^{*,\delta},Z_s)+\kappa(\theta-Z_s)\partial_z P_1(s,X_s^{*,\delta},Z_s)\right)ds\right].
\end{aligned}
$$

Note that for $q^{*,0}$ given in (1.15) and $q^{*,\delta}$ given in (1.16) , we have

$$
q^{*,\delta}-q^{*,0}=(u-d)(\mathbb{1}_{\{\partial_{xx}^2 P^\delta\geq 0\}}-\mathbb{1}_{\{\partial_{xx}^2 P_0\geq 0\}}), \tag{1.25}
$$

and

$$
(q^{*,\delta})^2-(q^{*,0})^2=(u^2-d^2)(\mathbb{1}_{\{\partial_{xx}^2 P^\delta\geq 0\}}-\mathbb{1}_{\{\partial_{xx}^2 P_0\geq 0\}}). \tag{1.26}
$$

Also note that $\{q^{*,\delta}\neq q^{*,0}\}=A_{t,z}^\delta$ defined in (1.14).

In order to show that E^δ is of order $\mathcal{O}(\delta)$, it suffices to show that I_0 is of order $\mathcal{O}(\delta)$, I_1 is of order $\mathcal{O}(\sqrt{\delta})$, and I_2 and I_3 are uniformly bounded in δ. Clearly, I_0 is the main term that directly determines the order of the error term E^δ.

1.4.2 Control of the term I_0

In this section, we are going to handle the dependence in δ of the process $X^{*,\delta}$ by a time-change argument.

Theorem 2. *Under Assumptions 1, 2 and 3, there exists a positive constant M_0, such that*

$$|I_0| \leq M_0\delta$$

where M_0 may depend on (t,x,z) but not on δ. That is, I_0 is of order $\mathcal{O}(\delta)$.

Proof. Since $0 < d \leq q^{*,\delta}, q^{*,0} \leq u$, we have

$$\begin{aligned}
I_0 =& \mathbb{E}_{(t,x,z)}\left[\int_t^T \frac{1}{2}\left((q^{*,\delta})^2 - (q^{*,0})^2\right) Z_s (X_s^{*,\delta})^2 \partial_{xx}^2 P_0(s, X_s^{*,\delta}, Z_s) ds\right] \\
\leq& \frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A^\delta_{s,Z_s}\}}(q^{*,\delta})^2 Z_s (X_s^{*,\delta})^2 |\partial_{xx}^2 P_0(s, X_s^{*,\delta}, Z_s)| ds\right] \\
=& \frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A^\delta_{s,Z_s}\}}(q^{*,\delta})^2 Z_s (X_s^{*,\delta})^2 |\partial_{xx}^2 P_0(s, X_s^{*,\delta}, Z_s)| \mathbb{1}_{\{\sup_{t\leq s'\leq T} Z_{s'}\leq M\}} ds\right] \\
&+ \frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A^\delta_{s,Z_s}\}}(q^{*,\delta})^2 Z_s (X_s^{*,\delta})^2 |\partial_{xx}^2 P_0(s, X_s^{*,\delta}, Z_s)| \mathbb{1}_{\{\sup_{t\leq s'\leq T} Z_{s'}> M\}} ds\right] \\
:=& \mathcal{M}_1 + \mathcal{M}_2.
\end{aligned}$$

In the following, we are going to show that both terms $\mathcal{M}_1$ and $\mathcal{M}_2$ are of order $\mathcal{O}(\delta)$.

Step 1. Control of term $\mathcal{M}_1$

Recall that, under Assumption 3, the set $A^\delta_{t,z}$ defined in (1.14) is included in $\cup_{i=1}^{m(t,z)} I_i^\delta$, which is included in a compact set for $z \leq M$. From Proposition 2, there exists a constant C_0 such that

$$|\partial_{xx}^2 P_0(s, X_s^{*,\delta}, Z_s)| \leq C_0\sqrt{\delta}, \text{ for } t \leq s \leq T,\ X_s^{*,\delta} \in A^\delta_{s,Z_s} \text{ and } \sup_{t\leq s\leq T} Z_s \leq M,$$

which yields

$$\begin{aligned}\mathcal{M}_1 \leq & \frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A_{s,Z_s}^\delta\}}(q^{*,\delta})^2 Z_s (X_s^{*,\delta})^2 C_0\sqrt{\delta}\mathbb{1}_{\{\sup_{t\leq s'\leq T} Z_{s'}\leq M\}}ds\right] \\ \leq & \frac{u^2}{2d^2}C_0\sqrt{\delta}\,\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A_{s,Z_s}^\delta\}}(q^{*,\delta})^2 Z_s (X_s^{*,\delta})^2\mathbb{1}_{\{\sup_{t\leq s'\leq T} Z_{s'}\leq M\}}ds\right].\end{aligned} \tag{1.27}$$

In order to show that $\mathcal{M}_1$ is of order $\mathcal{O}(\delta)$, it suffices to show that there exists a constant C_1 such that

$$\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{\sup_{t\leq s'\leq T} Z_{s'}\leq M\}}\mathbb{1}_{\{X_s^{*,\delta}\in A_{s,Z_s}^\delta\}}\sigma^2(X_s^{*,\delta})ds\right]\leq C_1\sqrt{\delta}, \tag{1.28}$$

where $\sigma(X_s^{*,\delta}) := q^{*,\delta}\sqrt{Z_s}X_s^{*,\delta}$ and $dX_s^{*,\delta} = \sigma(X_s^{*,\delta})dW_s$ by (1.24). Define the stopping time

$$\tau(v) := \inf\{s > t; \langle X^{*,\delta}\rangle_s > v\},$$

where $\langle X^{*,\delta}\rangle_s = \int_t^s \sigma^2(X_u^{*,\delta})du$. Then according to Theorem 4.6 (time-change for martingales) in [23], we know that $B_v := X_{\tau(v)}^{*,\delta}$ is a standard one-dimensional Brownian motion. In particular, the filtration $\mathcal{F}_v^B := \mathcal{F}_{\tau(v)}$ satisfies the usual condition and we have $\mathbb{Q}$-a.s. $X_t^{*,\delta} = B_{\langle X^{*,\delta}\rangle_t}$.

From the definition of $\tau(v)$ given above, we have

$$\int_t^{\tau(v)} \sigma^2(X_s^{*,\delta})ds = v,$$

which tells us that the inverse function of $\tau(\cdot)$ is

$$\tau^{-1}(T) = \int_t^T \sigma^2(X_s^{*,\delta})ds. \tag{1.29}$$

Next use the substitution $s = \tau(v)$ and for any $i \in [1, m(v,z)]$, we have

$$
\begin{aligned}
&\int_t^T \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}} \mathbb{1}_{\{|X_s^{*,\delta}-x_i|<C\sqrt{\delta}\}} \sigma^2(X_s^{*,\delta})ds \\
=&\int_0^{\tau^{-1}(T)} \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}} \mathbb{1}_{\{|X_{\tau(v)}^{*,\delta}-x_i|<C\sqrt{\delta}\}} \sigma^2(X_{\tau(v)}^{*,\delta}) \frac{1}{\sigma^2(X_{\tau(v)}^{*,\delta})} dv \\
=&\int_0^{\tau^{-1}(T)} \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}} \mathbb{1}_{\{|X_{\tau(v)}^{*,\delta}-x_i|<C\sqrt{\delta}\}} dv \\
=&\int_0^{\tau^{-1}(T)} \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}} \mathbb{1}_{\{|B_v+x-x_i|<C\sqrt{\delta}\}} dv.
\end{aligned}
\tag{1.30}
$$

Note that on the set $\{|B_v + x - x_i| < C\sqrt{\delta}\} \cap \{\sup_{t\le s'\le T} Z_{s'} \le M\}$, we have $(X_s^{*,\delta})^2 \le (x_i + C\sqrt{\delta})^2 \le D$, where D is a positive constant, and then by (1.29) we have

$$
\tau^{-1}(T) = \int_t^T (q^{*,\delta}\sqrt{Z_s}X_s^{*,\delta})^2 ds \le Du^2TM. \tag{1.31}
$$

From (1.30) and (1.31), we have

$$
\begin{aligned}
&\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}} \mathbb{1}_{\{|X_s^{*,\delta}-x_i|<C\sqrt{\delta}\}} \sigma^2(X_s^{*,\delta})ds\right] \\
\le&\int_0^{Du^2TM} \mathbb{Q}^B\{|B_v + x - x_i| < C\sqrt{\delta}\}dv \\
\le&\int_0^{Du^2TM} \frac{2C\sqrt{\delta}}{\sqrt{2\pi v}}dv \\
\le&\sqrt{\delta}\left(\frac{4C}{\sqrt{2\pi}}\sqrt{Du^2TM}\right).
\end{aligned}
$$

By finite union over the x_i's we deduce (1.28) and $\mathcal{M}_1 = \mathcal{O}(\sqrt{\delta})$ follows.

Step 2. Control of term $\mathcal{M}_2$

By the polynomial growth condition imposed in Assumption 4, one has

$$\begin{aligned}\mathcal{M}_2 =& \frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A^\delta_{s,Z_s}\}}(q^{*,\delta})^2 Z_s (X_s^{*,\delta})^2 |\partial^2_{xx}P_0(s,X_s^{*,\delta},Z_s)|\mathbb{1}_{\{\sup_{t\leq s'\leq T} Z_{s'}>M\}}ds\right]\\ \leq& \frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A^\delta_{s,Z_s}\}}(q^{*,\delta})^2 Z_s (X_s^{*,\delta})^2 |a_{20}(1+(X_s^{*,\delta})^{b_{20}}+Z_s^{c_{20}})|\mathbb{1}_{\{\sup_{t\leq s'\leq T} Z_{s'}>M\}}ds\right].\end{aligned} \tag{1.32}$$

In order to show $\mathcal{M}_2 = \mathcal{O}(\delta)$, it suffices to show that, for any power $m, n \in \mathbb{N}$,

$$\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{\sup_{t\leq s'\leq T} Z_{s'}>M\}}(X_s^{*,\delta})^m Z_s^n ds\right] = \mathcal{O}(\delta). \tag{1.33}$$

By Cauchy-Schwarz inequality and the result established in Appendix A.1 that X_t^δ and Z_t have finite moments for δ sufficiently small, we know that

$$\begin{aligned}&\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{\sup_{t\leq s'\leq T} Z_{s'}>M\}}(X_s^{*,\delta})^m Z_s^n ds\right]\\ \leq& \int_t^T \mathbb{E}^{1/2}_{(t,x,z)}\left((X_s^{*,\delta})^{2m} Z_s^{2n}\right)\cdot \mathbb{Q}^{1/2}\left(\sup_{t\leq s'\leq T} Z_{s'} > M\right) ds\\ \leq& C\mathbb{Q}^{1/2}\left(\sup_{t\leq s'\leq T} Z_{s'} > M\right),\end{aligned} \tag{1.34}$$

where C may depend on (t, x, z) and (m, n) but not on δ and we allow C to vary from line to line in the sequel. Integrating the SDE of the process Z over $[t, s]$ for $s \in [t, T]$, yields

$$Z_s = z + \int_t^s \delta\kappa(\theta - Z_v)dv + \Gamma_s,$$

with $\Gamma_s = \int_t^s \sqrt{\delta}\sqrt{Z_v}dW_v^Z$. Since $Z_t \geq 0$ and $0 \leq \delta \leq 1$, we have

$$\sup_{t\leq s\leq T} Z_s \leq (z + \kappa\theta T) + \sup_{t\leq s\leq T} \Gamma_s, \tag{1.35}$$

and then let $M = z + \kappa\theta T + 1$, we have

$$\mathbb{1}_{\{\sup_{t\leq s\leq T} Z_s > M\}} \leq \mathbb{1}_{\{\sup_{t\leq s\leq T} \Gamma_s > 1\}}. \tag{1.36}$$

Therefore, from (1.35) and (1.36), by Chebyshev inequality, we obtain

$$\mathbb{Q}^{1/2}\left(\sup_{t\leq s\leq T} Z_s > M\right) \leq \mathbb{Q}^{1/2}\left(\sup_{t\leq s\leq T} \Gamma_s > 1\right) \leq \mathbb{E}^{1/2}\left(\sup_{t\leq s\leq T} \Gamma_s^4\right). \tag{1.37}$$

Note that Γ_s is a martingale and then Γ_s^4 is a nonnegative submartingale, thus by Doob's maximal inequality ([23], page 14) and the result that the process Z has finite moments uniformly in δ, we have

$$\begin{aligned}
\mathbb{E}^{1/2}\left(\sup_{t\leq s\leq T} \Gamma_s^4\right) \leq & C\mathbb{E}^{1/2}\left(\Gamma_T^4\right) \\
= & C\delta\mathbb{E}^{1/2}\left(\int_t^T \sqrt{Z_v} dW_v^Z\right)^4 \\
\leq & C\delta\left(6T\mathbb{E}\int_t^T {Z_v}^2 dv\right)^{1/2} \\
\leq & C\delta,
\end{aligned} \tag{1.38}$$

where the second inequality established by the Martingale Moment Inequalities ([23], page 163).

Now, we have (1.33) as desired, which completes the proof. □

1.4.3 Control of the term I_1

Theorem 3. *Under Assumptions 1, 2, 3 and 4, there exists a constant M_1, such that*

$$|I_1| \leq M_1\sqrt{\delta}$$

where M_1 may depend on (t,x,z) but not on δ. That is, I_1 is of order $\mathcal{O}(\sqrt{\delta})$.

Proof. Under Assumption 4 and $0 < d \leq q^{*,\delta}, q^{*,0} \leq u$, we have

$$\begin{aligned}
|I_1| = &\Bigg|\mathbb{E}_{(t,x,z)}\Bigg[\int_t^T \Bigg((q^{*,\delta} - q^{*,0})\rho Z_s X_s^{*,\delta}\partial_{xz}^2 P_0(s, X_s^{*,\delta}, Z_s) \\
&+ \frac{1}{2}\left((q^{*,\delta})^2 - (q^{*,0})^2\right) Z_s (X_s^{*,\delta})^2 \partial_{xx}^2 P_1(s, X_s^{*,\delta}, Z_s)\Bigg) ds\Bigg]\Bigg| \\
\leq &\, \mathbb{E}_{(t,x,z)}\Bigg[\int_t^T \Bigg(|q^{*,\delta} - q^{*,0}| Z_s X_s^{*,\delta} |\partial_{xz}^2 P_0(s, X_s^{*,\delta}, Z_s)| \\
&+ \frac{1}{2}|(q^{*,\delta})^2 - (q^{*,0})^2| Z_s (X_s^{*,\delta})^2 |\partial_{xx}^2 P_1(s, X_s^{*,\delta}, Z_s)|\Bigg) ds\Bigg] \\
\leq &\, \frac{u}{d^2}\mathbb{E}_{(t,x,z)}\Bigg[\int_t^T \mathbb{1}_{\{X_s^{*,\delta} \in A_{s,Z_s}^\delta\}} (q^{*,\delta})^2 Z_s X_s^{*,\delta} a_{11}(1 + (X_s^{*,\delta})^{b_{11}} + Z_s^{c_{11}}) ds\Bigg] \\
&+ \frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\Bigg[\int_t^T \mathbb{1}_{\{X_s^{*,\delta} \in A_{s,Z_s}^\delta\}} (q^{*,\delta})^2 Z_s (X_s^{*,\delta})^2 \bar{a}_{20}(1 + (X_s^{*,\delta})^{\bar{b}_{20}} + Z_s^{\bar{c}_{20}}) ds\Bigg].
\end{aligned}$$

Using the same techniques in proving Theorem 2, the result that $X_s^{*,\delta}$ and Z_s have finite moments for δ sufficiently small, and $X_s^{*,\delta} \leq C(X_s^{*,\delta})^2$ on $\{X_s^{*,\delta} \in A_{s,Z_s}^\delta\}$, we can deduce that I_1 is of order $\mathcal{O}(\sqrt{\delta})$. □

1.4.4 Approximation Accuracy Result

Theorem 4. *Under Assumptions 1, 3 and 4, the residual function $E^\delta(t,x,z)$ defined by*

$$E^\delta(t,x,z) := P^\delta(t,x,z) - P_0(t,x,z) - \sqrt{\delta}P_1(t,x,z) \tag{1.39}$$

is of order $\mathcal{O}(\delta)$. In other words, $\forall (t,x,z) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}^+$, there exists a positive constant C, such that $|E^\delta(t,x,z)| \leq C\delta$, where C may depend on (t,x,z) but not on δ.

Proof. With the result of theorem 2 that I_0 is of order $\mathcal{O}(\delta)$, the result of theorem 3 that

I_1 is of order $\mathcal{O}(\sqrt{\delta})$, and the result that I_2 and I_3 are uniformly bounded in δ where derivation of these bounds are given in the appendix A.4, we can see that

$$E^{\delta}(t,x,z) = I_0 + \delta^{\frac{1}{2}} I_1 + \delta I_2 + \delta^{\frac{3}{2}} I_3,$$

is of order $\mathcal{O}(\delta)$, which completes the proof.. □

1.5 Numerical Illustration

In this section, we use the nontrivial example in [11], and consider a symmetric European butterfly spread with the payoff function

$$h(x) = (x-90)^+ - 2(x-100)^+ + (x-110)^+. \tag{1.40}$$

Although this payoff function does not satisfy the conditions imposed in this chapter, we could consider a regularization of it, that is to introduce a small parameter for the regularization and then remove this small parameter asymptotically without changing the accuracy estimate. This can be achieved by considering $P_0(T-\epsilon, x)$ as the regularized payoff (see [24] for details on this regularization procedure in the context of the Black–Scholes equation).

The original problem is to solve the fully nonlinear PDE (1.11) in dimension two for the worst-case scenario price, which is not analytically solvable in practice. In the following, we use the Crank–Nicolson version of the weighted finite difference method in [25], which corresponds to the case of solving P_0 in one dimension. To extend the original algorithm to our two dimensional case, we apply discretization grids on time and two state variables. Denote $u^n_{i,j} := P_0(t_n, x_i, z_j)$, $v^n_{i,j} := P_1(t_n, x_i, z_j)$ and $w^n_{i,j} := P^{\delta}(t_n, x_i, z_j)$,

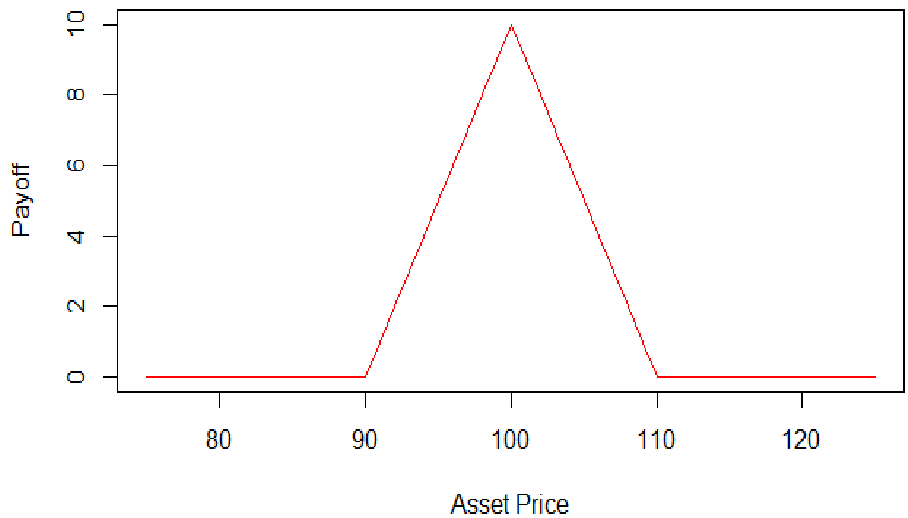

Figure 1.3: The payoff function of a symmetric European butterfly spread.

where $n = 0, 1, \cdots, N$ stands for the index of time, $i = 0, 1, \cdots, I$ stands for the index of the asset price process, and $j = 0, 1, \cdots, J$ stands for the index of the volatility process. In the following, we build a uniform grid of size 100×100 and use 20 time steps.

We use the classical discrete approximations to the continuous derivatives:

$$\partial_x(w^n_{i,j}) = \frac{w^n_{i+1,j} - w^n_{i-1,j}}{2\Delta x} \qquad \partial^2_{zz}(w^n_{i,j}) = \frac{w^n_{i,j+1} + w^n_{i,j-1} - 2w^n_{i,j}}{\Delta z^2}$$

$$\partial^2_{xx}(w^n_{i,j}) = \frac{w^n_{i+1,j} + w^n_{i-1,j} - 2w^n_{i,j}}{\Delta x^2} \qquad \partial_z(w^n_{i,j}) = \frac{w^n_{i,j+1} - w^n_{i,j-1}}{2\Delta z}$$

$$\partial^2_{xz}(w^n_{i,j}) = \frac{w^n_{i+1,j+1} + w^n_{i-1,j-1} - w^n_{i-1,j+1} - w^n_{i+1,j-1}}{4\Delta x \Delta z} \qquad \partial_t(w^n_{i,j}) = \frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t}$$

To simplify our algorithms and facilitate the implementation by matrix operations, we denote the following operators without any parameters:

$$L_{xx} = zx^2\partial^2_{xx} \qquad L_{zz} = z\partial^2_{zz} \qquad L_{xz} = xz\partial^2_{xz}$$

$$L_x = x\partial_x \qquad L_{z1} = \partial_z \qquad L_{z2} = z\partial_z$$

1.5.1 Simulation of P_0 and P_1

Note that in the PDE (1.18) for P_1 , $q^{*,0}$ must be solved in the PDE (1.13) for P_0 . Therefore, we solve P_0 and P_1 together in each 100×100 space grids and iteratively back to the starting time.

Set $u^N_{i,j} = h(x_I)$ and $v^N_{i,j} = 0$.

Solve $u^n_{i,j}$ (predictor)

$$\frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} + [\frac{1}{2}(q^{n+1}_{i,j})^2 L_{xx}](\frac{u^{n+1}_{i,j} + u^n_{i,j}}{2}) = 0$$

with

$$q^{n+1}_{i,j} = u\mathbb{1}_{\{u^2 L_{xx}(u^{n+1}_{i,j}) \geq d^2 L_{xx}(u^{n+1}_{i,j})\}} + d\mathbb{1}_{\{u^2 L_{xx}(u^{n+1}_{i,j}) < d^2 L_{xx}(u^{n+1}_{i,j})\}}$$

Solve $u^n_{i,j}$ (corrector)

$$\frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} + [\frac{1}{2}(q^n_{i,j})^2 L_{xx}](\frac{u^{n+1}_{i,j} + u^n_{i,j}}{2}) = 0$$

with

$$q_{i,j}^n = u\mathbb{1}_{\{u^2 L_{xx}(\frac{u_{i,j}^{n+1}+u_{i,j}^n}{2}) \geq d^2 L_{xx}(\frac{u_{i,j}^{n+1}+u_{i,j}^n}{2})\}} + d\mathbb{1}_{\{u^2 L_{xx}(\frac{u_{i,j}^{n+1}+u_{i,j}^n}{2}) < d^2 L_{xx}(\frac{u_{i,j}^{n+1}+u_{i,j}^n}{2})\}}$$

4: Solve $v_{i,j}^n$

$$\frac{v_{i,j}^{n+1} - v_{i,j}^n}{\Delta t} + \frac{1}{2}(q_{i,j}^n)^2 L_{xx}(\frac{v_{i,j}^{n+1} + v_{i,j}^n}{2}) + \rho(q_{i,j}^n) L_{xz}(\frac{u_{i,j}^{n+1} + u_{i,j}^n}{2}) = 0$$

Throughout all the experiments, we set $X_0 = 100$, $Z_0 = 0.04$, $T = 0.25$, $r = 0$, $d = 0.75$, and $u = 1.25$. Therefore, the two deterministic bounds for P_0 are given by $\underline{\sigma} = d\sqrt{Z_0} = 0.15$ and $\overline{\sigma} = u\sqrt{Z_0} = 0.25$, which are standard Uncertain Volatility model bounds setup. From Figure 1.5.1, we can see that P_0 is above the Black–Scholes prices with constant volatility 0.15 and 0.25 all the time, which corresponds to the fact that we need extra cash to superreplicate the option when facing the model ambiguity. As expected, the Black–Scholes prices with constant volatility 0.25 (resp. 0.15) is a good approximation when P_0 is convex (resp. concave).

1.5.2 Simulation of P_δ

Considering the main BSB equation given by (1.11), if we relax the restriction $q \in [d, u]$ to $q \in \mathbb{R}$, the optimizer of

$$f(q) := \frac{1}{2}q^2 z x^2 \partial_{xx}^2 P^\delta + q\rho z x\sqrt{\delta}\partial_{xz}^2 P^\delta$$

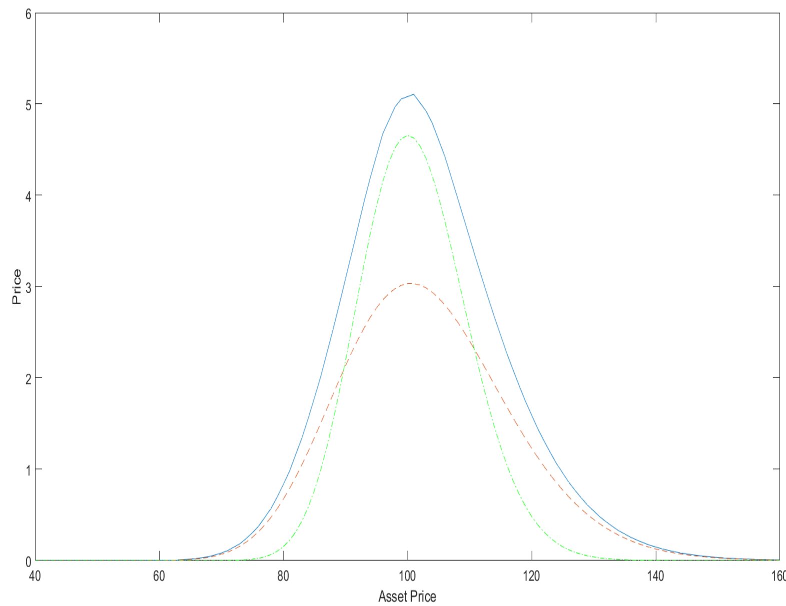

Figure 1.4: The blue curve represents the usual uncertain volatility model price P_0 with two deterministic bounds 0.15 and 0.25, the red curve marked with "- -" represents the Black–Scholes prices with $\sigma = 0.25$, the green curve marked with "-." represents the Black–Scholes prices with $\sigma = 0.15$.

is given by $\hat{q}^{*,\delta} = -\frac{\rho\sqrt{\delta}\partial_{xz}^2 P^\delta}{x\partial_{xx}^2 P^\delta}$, and the maximum value of $f(q)$ is given by $f(\hat{q}^{*,\delta}) = -\frac{\rho^2\delta z(\partial_{xz}^2 P^\delta)^2}{2\partial_{xx}^2 P^\delta}$. Therefore,

$$\sup_{q\in[d,u]} f(q) = f(u) \vee f(d) \vee f(\hat{q}^{*,\delta}).$$

To simplify the algorithm, we denote

$$L_A = \frac{1}{2}u^2 L_{xx} + u\rho\sqrt{\delta}L_{xz}, \qquad L_B = \frac{1}{2}d^2 L_{xx} + d\rho\sqrt{\delta}L_{xz}, \qquad L_C = -\frac{\rho^2\delta(L_{xz})^2}{2L_{xx}}.$$

1: Set $w_{i,j}^N = h(x_I)$.

2: Predictor:

$$\frac{w_{i,j}^{n+1} - w_{i,j}^n}{\Delta t} + [\frac{1}{2}(q_{i,j}^{n+1})^2 L_{xx} + (q_{i,j}^{n+1})\rho\sqrt{\delta}L_{xz} + \delta(\frac{1}{2}L_{zz} + \kappa\theta L_{z1} - \kappa L_{z2})](\frac{w_{i,j}^{n+1} + w_{i,j}^n}{2}) = 0$$

with

$$\begin{aligned} q_{i,j}^{n+1} =& u\mathbb{1}_{\{L_A(w_{i,j}^{n+1})=\max\{L_A,L_B,L_C\}(w_{i,j}^{n+1})\}} + d\mathbb{1}_{\{L_B(w_{i,j}^{n+1})=\max\{L_A,L_B,L_C\}(w_{i,j}^{n+1})\}} \\ &- \frac{\rho\sqrt{\delta}L_{xz}}{L_{xx}}(w_{i,j}^{n+1})\mathbb{1}_{\{L_C(w_{i,j}^{n+1})=\max\{L_A,L_B,L_C\}(w_{i,j}^{n+1})\}} \end{aligned}$$

3: Corrector:

$$\frac{w_{i,j}^{n+1} - w_{i,j}^n}{\Delta t} + [\frac{1}{2}(q_{i,j}^n)^2 L_{xx} + (q_{i,j}^n)\rho\sqrt{\delta}L_{xz} + \delta(\frac{1}{2}L_{zz} + \kappa\theta L_{z1} - \kappa L_{z2})](\frac{w_{i,j}^{n+1} + w_{i,j}^n}{2}) = 0$$

with

$$\begin{aligned}q_{i,j}^{n} =&u\mathbb{1}_{\{L_A(\frac{w_{i,j}^{n+1}+w_{i,j}^{n}}{2})=\max\{L_A,L_B,L_C\}(\frac{w_{i,j}^{n+1}+w_{i,j}^{n}}{2})\}}+d\mathbb{1}_{\{L_B(\frac{w_{i,j}^{n+1}+w_{i,j}^{n}}{2})=\max\{L_A,L_B,L_C\}(\frac{w_{i,j}^{n+1}+w_{i,j}^{n}}{2})\}}\\&-\frac{\rho\sqrt{\delta}L_{xz}}{L_{xx}}(\frac{w_{i,j}^{n+1}+w_{i,j}^{n}}{2})\mathbb{1}_{\{L_C(\frac{w_{i,j}^{n+1}+w_{i,j}^{n}}{2})=\max\{L_A,L_B,L_C\}(\frac{w_{i,j}^{n+1}+w_{i,j}^{n}}{2})\}}\end{aligned}$$

We set $\kappa = 15$ and $\theta = 0.04$, which satisfies the Feller condition required in this chapter.

1.5.3 Error analysis

To visualize the approximation as δ vanishes, we plot P^δ, P_0 and $P_0 + \sqrt{\delta}P_1$ with ten equally spaced values of δ from 0.05 to 0, and consider a typical case of correlation $\rho = -0.9$ (see [26]). In Figure 1.5.3, we see that the first order prices capture the main feature of the worst-case scenario prices for different values of δ. As can be seen, for δ very small, the approximation performs very well and it worth noting that, even for δ not very small such as 0.1, it still performs well.

To investigate the convergence of the error of our approximation as δ decrease, we compute the error of the approximation for each δ as following

$$\text{error}(\delta) = \sup_{x,z} |P^\delta(0, x, z) - P_0(0, x, z) - \sqrt{\delta}P_1(0, x, z)|.$$

As shown in Figure 1.5.3, the error decreases linearly as δ decreases (at least for δ small enough), as predicted by our Main Theorem 4.

Remark 6. *In Remark 4, for the case that P_0 has a third derivative with respect to x, which does not vanish on the set $S^0_{t,z}$, we have Assumption 3 (ii) as a direct result. In Figure 1.5.3, we can see that the slopes at the zero points of $\partial^2_{xx}P^\delta$ and $\partial^2_{xx}P_0$ are not* 0*, hence for this symmetric butterfly spread, Assumption 3 (ii) is satisfied.*

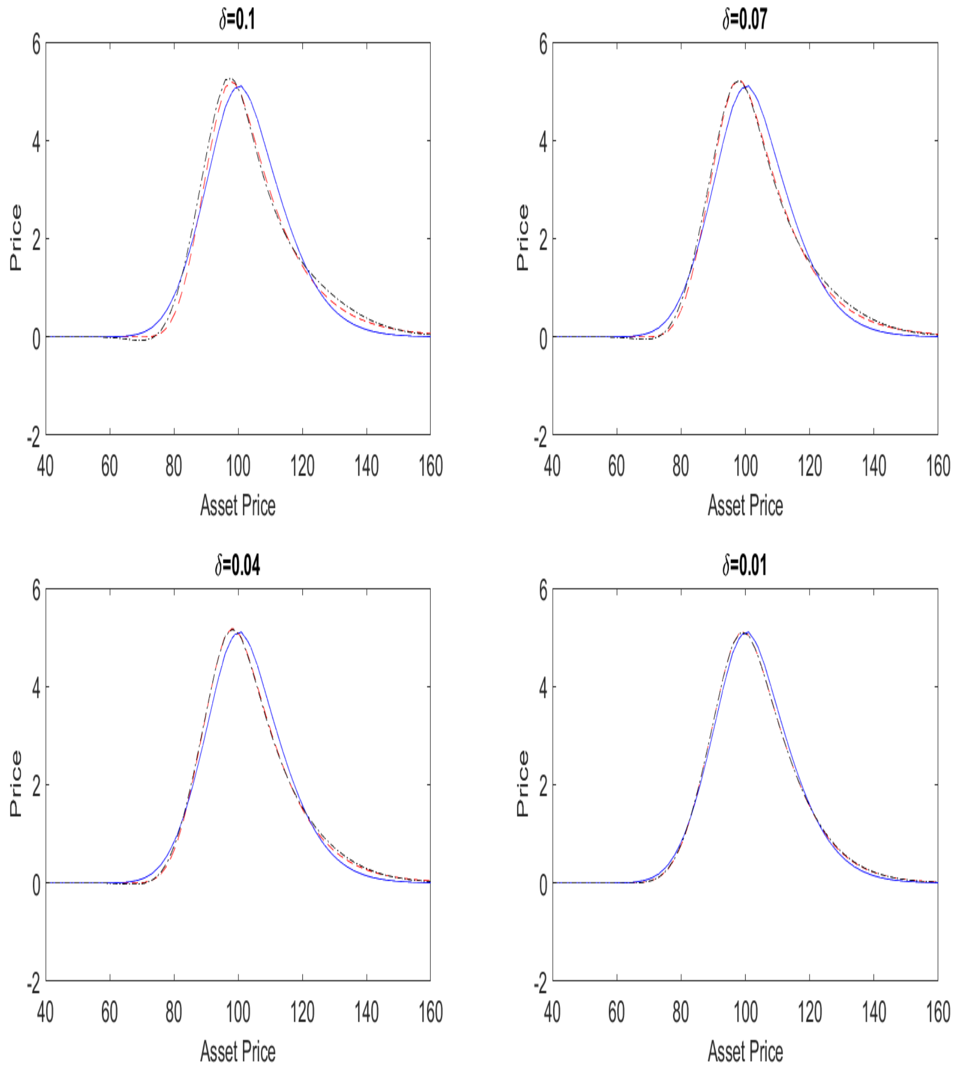

Figure 1.5: The red curve marked with "- -" represents the worst-case scenario prices P^{δ}; the blue curve represents the leading term P_0; the black curve marked with "-." represents the approximation $P_0 + \sqrt{\delta} P_1$.

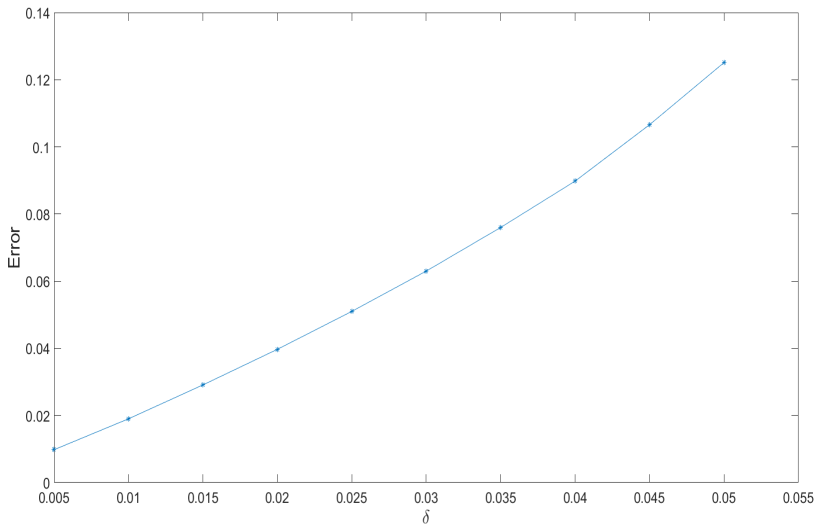

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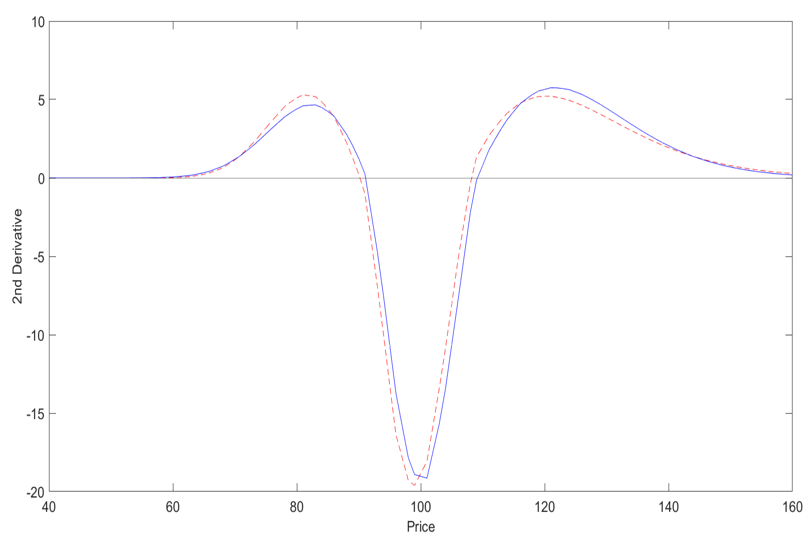
Figure 1.6: Error for different values of δ



1.6 Concluding Remarks

In this chapter, we discussed the uncertain volatility models with stochastic bounds driven by a CIR process. Our method is not limited to the CIR process and can be used with any other positive stochastic processes such as positive functions of an OU process. We further studied the asymptotic behavior of the worst-case scenario option prices in the regime of slowly varying stochastic bounds. This study not only helps understanding that uncertain volatility models with stochastic bounds are more flexible than uncertain volatility models with constant bounds for option pricing and risk management, but also provides an approximation procedure for worst-case scenario option prices when the bounds are slowly varying. From the numerical results, we see that the approximation procedure works really well even when the payoff function does not satisfy the requirements enforced in this chapter, and even when δ is not so small such as $\delta = 0.1$.

Note that as risk evaluation in a financial institution requires more accuracy and efficiency nowadays, our approximation procedure highly improves the estimation and still maintains the same efficiency level as the regular uncertain volatility models. Moreover, the worst-case scenario price P^δ (1.11) has to be recomputed for any change in its parameters κ, θ and δ. However, the PDEs (1.13) and (1.18) for P_0 and P_1 are independent of these parameters, so the approximation requires only to compute P_0 and P_1 once for all values of κ, θ and δ.

Chapter 2

Topic in Ross Recovery

The real-world probability distribution is of wide interest in many aspects, including policy decisions making, market surveillance of central banks, risk management, and portfolio optimization, and then is highly valuable to investors, policy makers and society in general. It is widely acknowledged that the risk-neutral transition probabilities can be determined from option prices. Also, in financial economics, it is widely agreed that the risk-neutral pricing distribution blends the real-world distribution and the pricing kernel, which conveys risk preferences. In this chapter, we are going to show that the real world transition probabilities of a multidimensional Markovian and bounded diffusion can be recovered by its risk-neutral transition probabilities, by placing the structure on the dynamics of the numeraire portfolio. In a direct application to the European call option pricing on private equity under multiple economy uncertainties, although the theoretical price of the numeraire portfolio is unique up to positive scaling, the associated theoretical option price is unique without scaling. This chapter is based on the paper [27].

2.1 Overview of Ross Recovery

Decoding the dynamics of the physical density and the pricing kernel using historical option or equity market data, has been extensively conducted, see [28, 29, 30, 31, 32, 33, 34] for references. Recently, [35] contributed a revolutionary breakthrough against the conventional wisdom, by showing that enforcing a restriction on preferences and applying the Perron-Frobenius theorem, option prices forecast not only the average return, but also the entire distribution. Precisely, conditioning on the time homogeneity of the risk-neutral process of a Markovian state variable X which determines aggregate consumption, one can uniquely determine a positive matrix whose elements are Arrow–Debreu security prices, from the option prices on X. Then by placing sufficient structure on preferences, i.e. existence of a representative agent when utilities are state independent and additively separable, the real-world transition probabilities of X can be uniquely determined.

Let us firstly summarize the recovery theory in the continuous setting exactly as in [35], and still keep in mind that the setup is a discrete-time model with a finite number of states: under the budget constraint

$$c(\theta_i) + \int c(\theta)p(\theta_i, \theta)d\theta = w,$$

the agent seeks

$$\max_{c(\theta_i), c(\theta)_{\theta\in\Omega}} \left\{ U(c(\theta_i)) + \delta \int U(c(\theta))f(\theta_i, \theta)d\theta \right\},$$

where θ_i denotes the current state, θ is the state of nature in the next period, $c(\theta)$ is the consumption as a function of the state and $U(c(\theta))$ is the utility of this consumption. According to [35], the first order condition for the optimum allows one to interpret the kernel $\phi(\cdot, \cdot)$, which is the agent's marginal rate of substitution as a function of aggregate consumption (see [36]), as

$$\phi(\theta_i, \theta_j) = \frac{p(\theta_i, \theta_j)}{f(\theta_i, \theta_j)} = \frac{\delta U'(c(\theta_j))}{U'(c(\theta_i))}. \tag{2.1}$$

Assume the kernel to be transition independent (see, Definition 1 in [35]), namely, a function of the ending state and depends on the beginning state only through dividing to normalize it, as (2.1). The Perron-Frobenius theorem implies that there exists exactly one positive eigenvector, which is unique up to positive scaling, and its corresponding principal eigenvalue is positive. Therefore, by setting the representative agent's discount factor δ equal to this principal root, and the vector of $U'(c(\theta))$ equal to any positive multiple of the principal eigenvector, the real world transition probabilities $\phi(\cdot, \cdot)$ can be uniquely determined.

However, this transition independence assumption has been intensively debated. [37] pointed out the relation between Ross' recovery result and the pricing kernel factorization in [38], which used Perron–Frobenius Theory to identify a probability measure that reflects the long-term implications for risk pricing. [37] showed that the pricing kernel can be decomposed into a transition independent component which absorbs long-term risk prices, and a martingale component which must be constant to recover the real-world distribution. The resulting misspecification of Ross' recovery as theoretically proved in [37], is confirmed in [39], which showed that transition independence and degeneracy of the martingale component are implausible assumptions in the bond market, and further verified in [40], whose empirical results undermine the implications of the recovery theorem. Another drawback is that, in the representative agent framework, under transition independence, the asset considered must be able to serve as a proxy for the wealth of a representative agent, which rules out many assets, for example futures as assets in zero net supply.

To tackle this potential misspecified recovery, several generalizations have been proposed. [41] firstly proposed the idea to price future payoffs condtioned on the current time instead of enforcing the independent dynamical probability transition, in a continuous setting. [42] tackled the recovery problem in a discrete setting, similarly by starting directly with the state prices for all future times given only the current state (see their Figure 1 in Appendix D for a clear understanding), which is followed with successful empirical tests. [43] proposed to incorporate recursive preferences of Epstein-Zin type, which do not necessarily produce transition independent pricing kernels.

Next, we follow [41] in recovering the real-world transition probabilities in a preference-free manner, by placing structure on the dynamics of the numeraire portfolio rather than enforcing restrictions on the form of the pricing kernel, where the numeraire portfolio is a self-financing portfolio with positive price (see [44] for reference). In a multivariate setting and under arbitrage free markets assumption, Long showed that the numeraire portfolio always exists, and deflating each asset's price by the value of the numeraire portfolio yields a martingale under the real-world probability measure. Therefore, the more concrete and economically grounded notion of the numeraire portfolio could be used to take the role of the rather abstract probabilistic notion of an equivalent martingale measure. To be more specific, the risk neutral probability measure $\mathbb{Q}$ equivalent to the real world probability measure $\mathbb{P}$ can be generated by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = M_T,$$

where M is a positive $\mathbb{P}$ local martingale whose existence is implied by the first fundamental theorem of asset pricing. Equivalently, the real world probability density function can be obtained via:

$$d\mathbb{P}|_{\mathcal{F}_T} = \frac{1}{M_T}\, d\mathbb{Q}|_{\mathcal{F}_T}\,. \tag{2.2}$$

Given the the risk neutral probability measure $\mathbb{Q}$, [41] achieved $d\mathbb{P}|_{\mathcal{F}_T}$ by restricting the form and dynamics of the numeraire portfolio through

$$\frac{1}{M_T} = e^{-\int_0^T r_t dt} L_T,$$

where r_t is the interest rate process.

As pioneers in Recovery theory, [41] followed [35] in assuming that there is a single Markov process X driving all asset prices, and showed the uniqueness of the positive eigenfunction by virtue of the Sturm–Liouille theory. Since one usually assumes that multiple Markovian processes drive some curve or surface, therefore it is meaningful to explore the multiple drivers case. However the S-L theory only applicable in the case of a single variable, not in the multidimensional case. In this chapter, with the assistance of a refined spectrum analysis of the elliptic operator, we extend the single driver recovery problem to the general $n \in \mathbb{N}$ driving state variables case. To be specific, our purpose is to show that with the multivariate driver diffusion process, it is still possible to uniquely determine, the volatility vector process of Long's numeraire portfolio and then the real world probability measure.

Before displaying our result, it is worth mentioning some excellent related literatures. The unbounded domain extension as suggested in [41] was firstly attempted by [45] given transition independence, which explored the recovery problem in a representative agent economy where the state evolves in continuous time according to a time homogeneous diffusion process on an unbounded domain. Also under the debatable transition independence, [46] generalized the recovery theorem to unbounded continuous state spaces using Perron-Frobenius operator theory, with the help of the Jentzsch's theorem of integral operators, and they showed recovery misspecification at the end. [47] extended the Recovery Theorem from discrete time, finite state irreducible Markov chains to recurrent Borel right processes. [48] later dropped the Markovian structure in the continuous state space. [49] discussed several conditions under which the recovery of the objective measure from the risk-neutral measure is possible in a continuous-time model. A partial list of wonderful empirical studies applying Ross' recovery result and its generalizations includes, [50, 51, 52, 53, 54].

2.2 The Basic Framework

2.2.1 Assumptions of the model

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the probability measure $\mathbb{P}$ unknown ex ante. We make the following assumptions, whose one dimensional counterparts are justified in [41]:

Assumption 5. *The market is free of arbitrage and complete, with $d \in \mathbb{N}$ random sources. There exists an empirically observable time-homogeneous and bounded multivariate diffusion $X = (X_1, X_2, \ldots, X_n)^*$, where the notation $*$ denotes the transpose of the matrix. The process X evolves according to the following $\mathbb{Q}$-dynamic*

$$dX(t) = \mu(X(t))dt + A(X(t))dW(t), \tag{2.3}$$

where $W = (W_1, \ldots, W_d)^*$ is a standard d-dimensional $\mathbb{Q}$-Wiener process. The drift vector $\mu(\cdot) = (\mu_1(\cdot), \cdots, \mu_n(\cdot))^*$ and the $n \times d$-dimensional diffusion matrix

$$A(\cdot) = \begin{pmatrix} a_{11}(\cdot) & a_{12}(\cdot) & \cdots & a_{1d}(\cdot) \\ a_{21}(\cdot) & a_{22}(\cdot) & \cdots & a_{2d}(\cdot) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(\cdot) & a_{n2}(\cdot) & \cdots & a_{nd}(\cdot) \end{pmatrix}$$

are both known ex ante. The support region of X denoted as $\Omega \subset \mathbb{R}^n$ is bounded and $\partial\Omega$ is $C^{2,\alpha}$ for some $\alpha \in (0,1]$.

Following [41], the multi–dimensional process X is referred as the drivers rather than the state variables, for the reason that the state of entire economy is not required to be determined by X.

With d sources of randomness, by means of the Meta Theorem ([55]), we can generically specify the price processes of d different "benchmark" claims, conditioning on the market being arbitrage free and complete. The price processes of all other claims will then be uniquely determined by the prices of the benchmarks. Therefore, we assume that there are d risky assets and their prices depend on the drivers and time t.

Assumption 6. *There exists a money market account (MMA) with balance*

$$S_{0t} = \exp\left\{ \int_0^t r_s ds \right\},$$

with the interest rate process $r_t = r(X_t) \in \mathbb{R}^+$ known ex ante; there exists d different risky securities whose spot prices $S_{it} = S_i(t, X_t)$ evolve as continuous real-valued semi-martingales. For each risky asset, its initial spot price is observed, there is no dividends or holding costs for simplicity, and the local martingale part is non-trivial.

Under the complete market assumption, the risk–neutral probability measure $\mathbb{Q}$ is unique, and each r-discounted security price $e^{-\int_0^t r_s ds} S_{it}$ evolves as a $\mathbb{Q}$-martingale, i.e.

$$\mathbf{E}^{\mathbb{Q}} \left\{ \frac{S_{iT}}{S_{0T}} \,\middle|\, \mathcal{F}_t \right\} = \frac{S_{it}}{S_{0t}}, \quad t \in [0,T], \; i = 0, 1, \ldots, d. \tag{2.4}$$

Assumption 7. *Under $\mathbb{Q}$, the price process of the numeraire portfolio $L_t = L(t, X(t))$*

satisfies the following stochastic differential equation

$$\frac{dL_t}{L_t} = r(X(t))dt + \sigma(X(t))dW(t). \tag{2.5}$$

Note that the above diffusion assumption is a traditional setting of option pricing. The goal of this analysis is mainly to find $\sigma(\cdot)$ by knowledge of the risk–neutral dynamics of X, and then uniquely determine the $\mathbb{P}$-dynamics of X and the real–world probability measure $\mathbb{P}$ itself.

2.2.2 Preliminary Analysis

Let us follow [41] to impose structure on the price process of Long's numeraire portfolio:

$$L_t = L(t, X(t)) = \frac{S_{0t}}{M_t}, \tag{2.6}$$

where M is the positive martingale used to create the martingale measure $\mathbb{Q}$ in (2.2). We know that $L(t,x)$ solves the following linear parabolic partial differential equation:

$$\begin{cases} \mathcal{G}L(t,x) + \frac{\partial L}{\partial t} = r(x)L(t,x) \\ \\ L(t,x) \in C^1([0,T]) \times (C^2(\Omega) \cap C(\overline{\Omega})) \\ \\ L(t,x) > 0 \text{ on } [0,T] \times \Omega \end{cases} \tag{2.7}$$

where $\mathcal{G}$ is the infinitesimal generator given by

$$\mathcal{G}L(t,x) = \sum_{i=1}^{n} \mu_i(x) \frac{\partial L}{\partial x_i}(t,x) + \frac{1}{2} \sum_{i,j=1}^{n} C_{ij}(x) \frac{\partial^2 L}{\partial x_i \partial x_j}(t,x),$$

with

$$C(x) := (C_{ij}(x))_{n\times n} = A(x)A^*(x).$$

Let us firstly apply the separation of variables and write

$$L(t,x) = u(x)p(t). \tag{2.8}$$

Note that $L(t,x)$ is always positive, therefore we are able to suppose $u(x), p(t) \in \mathbb{R}$ without loss of generality. Hence, we have

$$p'(t)u(x) + p(t)\sum_{i=1}^{n} \mu_i(x)\frac{\partial u}{\partial x_i}(x) + p(t)\frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = r(x)u(x)p(t).$$

Dividing by $u(x)p(t)$ on both sides implies:

$$\frac{1}{u}\left\{\frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\mu_i(x)\frac{\partial u}{\partial x_i} - r(x)u(x)\right\} = -\frac{p'(t)}{p(t)}$$

The two sides can only be equal if they are constants, say $-\lambda \in \mathbb{R}$. Then the original PDE becomes two separate ones:

$$\frac{p'(t)}{p(t)} = \lambda \tag{2.9}$$

and

$$\frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\mu_i(x)\frac{\partial u}{\partial x_i} - r(x)u(x) = -\lambda u(x)$$

Without loss of generality, set $p(0) = 1$ and then (2.9) has a unique solution

$$p(t) = e^{\lambda t}.$$

Then the parabolic equation (2.7) becomes a second-order elliptic partial differential equation with the general boundary condition as following:

$$\begin{cases} \mathcal{L}(u) = \lambda u, \quad \lambda \in \mathbb{R} \\ \\ u \in C^2(\Omega) \cap C(\overline{\Omega}) \\ \\ u > 0 \text{ in } \Omega \quad \text{and} \quad \mathcal{B}u \equiv 0 \text{ on } \partial\Omega \end{cases} \tag{2.10}$$

where $\mathcal{B}$ is a Dirichlet, Neumann or Robin boundary operator, and

$$\mathcal{L}(u) = -\frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \mu_i(x)\frac{\partial u}{\partial x_i} + r(x)u(x) \tag{2.11}$$

is a uniformly elliptic operator defined on Ω, since $C = AA^*$ is a positive-definite matrix. In the following context, we attempt to establish the uniqueness of the solution to (2.7) by exploring the spectrum theory of the elliptic operator (2.11), and then carve out a way to recover of the real world probability measure.

2.3 Spectrum Analysis of the Elliptic Operator

Note that $r(x) > 0$ in Ω. Therefore, the theory of the boundedness of inverse ([56], Theorem 6.14, 6.31) implies that the inverse operator of $\mathcal{L}$ does exist on $C^\alpha(\overline{\Omega})$ based on boundary conditions, say, $\mathcal{L}^{-1} : C^\alpha(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega})$. For any $f \in C^\alpha(\overline{\Omega})$, then $\mathcal{L}^{-1}(f) \in C^{2,\alpha}(\Omega)$, $\mathcal{B}\mathcal{L}^{-1}(f) = 0$ on $\partial\Omega$, and

$$\|\mathcal{L}^{-1}(f)\|_{2,\alpha} \leq C\|f\|_\alpha \leq C_1\|f\|_{1,\alpha}$$

where $C_1 > 0$ is independent of f. It follows that $\mathcal{L}^{-1}$ is a compact linear operator. Define a *cone* K consisting all nonnegative functions of $C^{1,\alpha}(\overline{\Omega})$. It follows from the weak maximum principle ([56], Theorem 3.1) that $\mathcal{L}^{-1}(K) \subset K$.

Before displaying Theorem 6, we need a preliminary lemma which will be used intensively in the current framework.

Lemma 2. *Assume that $u \in C^{2,\alpha}(\overline{\Omega})$ satisfies $\mathcal{L}u \geq 0$.*

1. *If it satisfies Neumann or Robin boundary condition, $\mathcal{B}u|_{\partial\Omega} = 0$, where $\mathcal{B}u = \gamma(x)u + D_\nu u = 0$ with $\gamma(x) \geq 0, \gamma(x) \in C^{1,\alpha}(\partial\Omega)$, then $u > 0$ on $\overline{\Omega}$ unless $u \equiv 0$.*

2. *If it has Dirichlet boundary condition, $u = 0$ on $\overline{\Omega}$, then $u > 0$ in Ω. Furthermore, for any $v \in C^2(\overline{\Omega})$ with $v|_{\partial\Omega} = 0$, there exists an $\epsilon > 0$ such that $w \geq \epsilon v$. If u is not identically 0, then $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, where ν is the exterior unit normal of $\partial\Omega$.*

Proof. See Appendix. □

For any nonzero $f \in C^{1,\alpha}(\overline{\Omega})$, there exists a small constant $r > 0$ such that $w = u + r\frac{f}{\|f\|_{1,\alpha}} > 0$ on $\partial\Omega$, that is, $w \in K$. It follows $f = (\|f\|_{1,\alpha}/r)(w - u) \in K - K$. Hence $K - K = C^{1,\alpha}(\overline{\Omega})$, i.e. K is a *total cone* (actually it is reproducing). Then the Krein-Rutman theorem [57] yields

Theorem 5. *Let X be a Banach space, and let $K \subset X$ be a convex cone such that $K - K$ is dense in X. Let $T : X \to X$ be a non-zero compact operator which is positive, meaning that $T(K) \subset K$, and assume that its spectral radius $\varrho(T)$ is strictly positive. Then $\varrho(T)$ is an eigenvalue of T with positive eigenvector, meaning that there exists $u \in K \setminus \{0\}$ such that $T(u) = \varrho(T)u$.*

The following spectrum result of the elliptic operator is the footstone of our multi-dimensional case, playing the similarly crucial role as Perron–Frobenius theorem in [35] and Sturm–Liouville theorem in [41], much more complicated though.

Theorem 6. *There is a unique positive eigenfunction $\phi(x) > 0$ in Ω, up to positive scaling, of the operator $\mathcal{L}$ with the boundary condition $\mathcal{B}\phi|_{\partial\Omega} = 0$. The eigenvalue ρ corresponding to $\phi(x)$ is positive and simple. Furthermore, for any other eigenvalues of $\mathcal{L}$, say, $\lambda \neq \rho$, it must satisfy*

$$\Re(\lambda) > \rho.$$

Proof. We will display our proof in following 4 steps:

(a) Choose $v \in K, v > 0$ on $\overline{\Omega}$ and any $w \in C^{1,\alpha}(\overline{\Omega})$. The preceding discussion gives $\mathcal{L}^{-1}v > 0$ on $\overline{\Omega}$. It is easy to see that there exists $\varepsilon > 0$, independent of w, and $\eta > 0$ such that $w \leq \|w\|v/\varepsilon$ and $\eta\mathcal{L}^{-1}v \geq v$ on $\overline{\Omega}$. Therefore, for any integer $n \geq 1$,

$$v \leq \eta\mathcal{L}^{-1}v \leq \cdots \leq \left(\eta\mathcal{L}^{-1}\right)^n v \leq \varepsilon^{-1}\left\|\left(\eta\mathcal{L}^{-1}\right)^n v\right\| v \leq \varepsilon^{-1}\left\|\left(\eta\mathcal{L}^{-1}\right)^n\right\| \|v\| v$$

It implies $\left\|\left(\eta\mathcal{L}^{-1}\right)^n\right\| \geq \varepsilon/\|v\|$, and hence Gelfand's Formula yields the spectral radius of $\eta\mathcal{L}^{-1}$, $\varrho\left(\eta\mathcal{L}^{-1}\right) = \lim_{n\to\infty}\left\|\left(\eta\mathcal{L}^{-1}\right)^n\right\|^{1/n} > 0$. Therefore Theorem 5 asserts that $\varrho\left(\eta\mathcal{L}^{-1}\right)$ is an eigenvalue of $\eta\mathcal{L}^{-1}$, and there exists its positive eigenvector $\phi \in K$, that is

$$\eta\mathcal{L}^{-1}\phi = \varrho\left(\eta\mathcal{L}^{-1}\right)\phi \Longleftrightarrow \mathcal{L}\phi = \rho\phi$$

where $\rho = \eta/\varrho\left(\eta\mathcal{L}^{-1}\right) > 0$.

(b) Next show that ρ is simple. Set $\mu = 1/\rho$, and part (a) claims $\mathcal{L}^{-1}\phi = \mu\phi$. Suppose $\mathcal{L}^{-1}w = \mu w$ with $w \neq 0$. If w is complex, we could discuss its real part $\Re w$ and imaginary part $\Im w$ respectively, and hence suppose w is real. Replacing w by $-w$ if needs be, suppose also $w > 0$ somewhere in Ω.

Then $\mathcal{L}^{-1}(\phi - sw) = \mu(\phi - sw)$ for all $s > 0$. Denote by

$$\mathcal{S} = \{\eta > 0 \mid \phi - \eta w \geq 0\}.$$

and hence, $\mathcal{S}$ is nonempty and bounded. Next let $\xi = \sup \mathcal{S}$ and we claim that $\phi - \xi w \equiv 0$. Otherwise, we can conclude from the fact $\phi - \xi w \geq 0$ and Lemma 5 that there exists $\epsilon > 0$ such that

$$\mathcal{L}^{-1}(\phi - \xi w) \geq \epsilon w \tag{2.12}$$

On the other side

$$\mathcal{L}^{-1}(\phi - \xi w) = \mu\phi - \xi\mu w = \mu(\phi - \xi w). \tag{2.13}$$

Combining (2.20) and (2.21) yields

$$\phi - \left(\xi + \frac{\epsilon}{\mu}\right) w \geq 0,$$

contradicting $\xi = \sup \mathcal{S}$. This proves $w \in \text{span}\{\phi\}$.

Secondly let $(\mu - \mathcal{L}^{-1})^2 w = 0$. The preceding discussion yields $(\mu - \mathcal{L}^{-1}) w = c\phi$ for some constant $c \in \mathbb{R}$. We want to show $c = 0$. Apply reductio ad absurdum and assume $c > 0$ (otherwise change w to $-w$). Then for all $s > 0$,

$$\mathcal{L}^{-1}(\phi + sw) = \mu(\phi + sw) - sc\phi < \mu(\phi + sw) \quad \text{on } \overline{\Omega} \tag{2.14}$$

Note that for sufficiently small $s \geq 0$, $\phi + sw > 0$ on $\overline{\Omega}$, so we could deduce that it is true for all $s \geq 0$. Otherwise assume $\zeta = \sup\{s > 0 \mid \phi + sw > 0 \text{ on } \overline{\Omega}\} < \infty$, that is, $\phi + \zeta w$ must be nonnegative and attain zero somewhere on $\overline{\Omega}$. Hence, (2.14) gives $\phi + \zeta w > 1/\mu\mathcal{L}^{-1}(\phi + \zeta w) \geq 0$ on $\overline{\Omega}$, a contradiction. This implies $w \geq 0$, and thus $w = 1/\mu\,(\mathcal{L}^{-1}w + c\phi) > 0$. Next repeat the same trick above by setting $\kappa = \sup\{s > 0 \mid w - s\phi \geq 0\}$, and the previous discussion implies

$$0 < \kappa < \infty; \quad w - \kappa\phi \text{ achieves zero somewhere on } \overline{\Omega}. \tag{2.15}$$

But then

$$\mu w - c\phi - \kappa\mu\phi = \mathcal{L}^{-1}(w - \kappa\phi) \geq 0$$

that is, $w - \kappa\phi \geq c\phi/\mu > 0$, contadicting (2.15). Therefore, $c = 0$ and hence $w \in \operatorname{span}\{\phi\}$. In sum, μ is a simple eigenvalue, so is ρ.

(c) In this context we will show that $\Re(\lambda) > \rho$ for any other eigenvalue $\lambda \neq \rho$. Suppose that the corresponding eigenfunction to λ is u, i.e. $\mathcal{L}(u) = \lambda u$. Next we will discuss boundary condition respectively. If $\mathcal{B}$ is Neumann or Robin, then set $v = u/\phi$. Thus

$$\lambda v = \frac{1}{\lambda}\mathcal{L}(v\phi) = \mathcal{L}v - rv - \frac{1}{\phi}\sum_{i,j=1}^{n} C_{ij}\frac{\partial\phi}{\partial x_j}\frac{\partial v}{\partial x_i} + \frac{v}{\phi}\mathcal{L}\phi \tag{2.16}$$

Define a new operator

$$\mathcal{K} = \mathcal{L} - r - \frac{1}{\phi}\sum_{i,j=1}^{n} C_{ij}\frac{\partial\phi}{\partial x_j}\frac{\partial}{\partial x_i} = -\frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{n}\left(\mu_i + \frac{1}{\phi}\sum_{j=1}^{n} C_{ij}\frac{\partial\phi}{\partial x_j}\right)\frac{\partial}{\partial x_i}$$

Then (2.16) becomes

$$\mathcal{K}v + (\rho - \lambda)v = 0, \quad \mathcal{K}\bar{v} + (\rho - \bar{\lambda})\bar{v} = 0 \tag{2.17}$$

Moreover, note that $(C_{ij})_{n\times n}$ is positive definite, and hence

$$\sum_{i,j=1}^{n} C_{ij}\frac{\partial v}{\partial x_i}\frac{\partial \bar{v}}{\partial x_j} = \sum_{i,j=1}^{n} C_{ij}\left(\Re\left(\frac{\partial v}{\partial x_i}\right)\Re\left(\frac{\partial v}{\partial x_j}\right) + \Im\left(\frac{\partial v}{\partial x_i}\right)\Im\left(\frac{\partial v}{\partial x_j}\right)\right) \geq 0.$$

Consequenly, this implies

$$\mathcal{K}\left(|v|^2\right) = \bar{v}\mathcal{K}v + v\mathcal{K}\bar{v} - \sum_{i,j=1}^{n} C_{ij}\frac{\partial v}{\partial x_i}\frac{\partial \bar{v}}{\partial x_j} \leq \bar{v}\mathcal{K}v + v\mathcal{K}\bar{v} = 2(\Re(\lambda) - \rho)|v|^2$$

Since both ϕ and u satisfy Neumann or Robin boundary condition,

$$D_\nu \phi + \gamma \phi = 0, \quad D_\nu u + \gamma u = 0 \quad \text{on } \partial\Omega$$

it leads to

$$D_\nu v = \frac{1}{\phi^2}\left(\phi D_\nu u - u D_\nu \phi\right) = 0$$

and hence $D_\nu |v|^2 = 0$. Next assume $\Re(\lambda) \leq \rho$, then

$$\mathcal{K}\left(|v|^2\right) \leq 0, \quad D_\nu |v|^2 = 0$$

Thus the strong maximum principle and Hopf boundary lemma imply that $|v|^2$ is constant. Thus $v = ce^{if(x)}$ where c is a real constant and $f(x)$ is a real function on $\overline{\Omega}$, and then (2.17) implies

$$\frac{1}{2}\sum_{i,j=1}^{n} C_{ij} f_{x_i} f_{x_j} + (\rho - \Re(\lambda)) = 0$$

By virtue of the assumption $\Re(\lambda) \leq \rho$ and the positive definiteness of $C(x) = (C_{ij}(x))_{n\times n}$, it follows $\rho = \Re(\lambda)$ and $f_{x_i} = 0$ for all i, that is, f is a constant. As a consequence, $u \in \operatorname{span}\phi$, and hence $\lambda = \rho$, a contradiction. Therefore we must have $\Re(\lambda) > \rho$.

To prove the Dirichlet case, apply the same trick in Lemma 5 to choose local coordinates $\{x_1, x_2, \ldots, x_n\}$ on a sufficiently small open set U such that $U \cap \partial\Omega = U \cap \{x_n = 0\}$ and $U \cap \Omega = \{x_n > 0\}$. Then Lemma 5 admits $\left.\frac{\partial \phi}{\partial x_n}\right|_{\partial\Omega} < 0$. Thus the Malgrange preparation theorem ([58]) indicates that $\phi = f \cdot x_n$ and $u = g \cdot x_n$ hold locally, where $f, g \neq 0$ on $U \cap \partial\Omega$. Therefore,

$$v = \frac{g \cdot x_n}{f \cdot x_n} = \frac{g}{f}$$

is well defined on $U \cap \overline{\Omega}$. Despite of the singularity of $\frac{1}{\phi}\frac{\partial \phi}{\partial x_i}$ on $\partial\Omega$, the regularity of $|v|^2$

in (2.17) implies that

$$\frac{1}{\phi}\sum_{i,j=1}^{n} C_{ij}\frac{\partial \phi}{\partial x_i}\frac{\partial |v|^2}{\partial x_j} < \infty. \tag{2.18}$$

Therefore a new operator $\mathcal{K}$ is well defined on $\overline{\Omega}$ and also an elliptic operator. Apply the same trick as reductio ad absurdum and assume $\Re(\lambda) \leq \rho$, which leads to $\mathcal{K}(|v|^2) \leq 0$. Then by the strong maximum principle, we know that $|v|^2$ can only attain its maximum at some point $x_0 \in \partial\Omega$, and thus Hopf's lemma ([59], Theorem 2.5) asserts that

$$\frac{\partial |v|^2}{\partial N} > 0 \quad \text{at} \quad x_0, \tag{2.19}$$

where $\frac{\partial |v|^2}{\partial N} = \sum_{i,j=1}^{n} C_{ij}\frac{\partial |v|^2}{\partial x_i}\nu_j$, and $\nu_j = \langle \nu, \frac{\partial}{\partial x_j}\rangle$ with ν being the exterior unit normal of $\partial\Omega$. On the other side, we can conclude from (2.18) and the fact of $\nu^*\nu = 1$ that

$$\frac{1}{\phi}\frac{\partial \phi}{\partial \nu}\frac{\partial |v|^2}{\partial N} = \frac{1}{\phi}\sum_{i,j=1}^{n} C_{ij}\frac{\partial \phi}{\partial x_i}\frac{\partial |v|^2}{\partial x_j} < \infty.$$

Considering $\mathcal{L}\phi = \rho\phi \geq 0$ on $\overline{\Omega}$ and $\phi = 0$ on the boundary, Lemma 5 asserts $\frac{\partial \phi}{\partial \nu}\big|_{\partial\Omega} < 0$, and hence $\frac{\partial |v|^2}{\partial N}\Big|_{\partial\Omega} = 0$, a contradiction to (2.19). This means that $\Re(\lambda) > \rho$.

(d) Last show the uniqueness of the positive eigenfunction. Assume there is a positive eigenfunction $u > 0$ in Ω and $\mathcal{B}u|_{\partial\Omega} = 0$ such that $\mathcal{L}^{-1}(u) = \mu u$. Part (b) asserts $0 < \mu \leq 1/\rho$. Define

$$\mathcal{T} = \{\eta > 0 \mid u - \eta\phi \geq 0\}.$$

We can see that $\mathcal{T}$ is nonempty and bounded. Next let $\chi = \sup \mathcal{T}$ and we claim that $u - \chi\phi \equiv 0$. Otherwise, we can conclude from the fact $u - \chi\phi \geq 0$ and Lemma 5 that there exists $\epsilon > 0$ such that

$$\mathcal{L}^{-1}(u - \chi\phi) \geq \epsilon\phi \tag{2.20}$$

On the other side

$$\mathcal{L}^{-1}(u-\chi\phi)=\mu u-\frac{\chi}{\rho}\phi\leq\frac{1}{\rho}(u-\chi\phi). \tag{2.21}$$

Combining (2.20) and (2.21) yields

$$u\geq(\chi+\epsilon\rho)\,\phi,$$

contradicting $\chi=\sup\mathcal{T}$. Therefore the positive eigenfunction is unique up to a positive scalar constant.

□

Note that, although $L(t,x)=u(x)p(t)$ can be easily established in one-dimensional case (see equation (40) in Section Analysis in [41]), it is not easy when it comes to multidimension, which can be seen from equation (2.26) following. But with the help of Theorem 6, we can still validate that and further uniquely determine the solution to (2.7) by obtaining ρ and $\phi(x)$ in Theorem 6.

Corollary 1. *The value function of the numeraire portfolio, i.e. the solution to* (2.7)*, is in the form*

$$L(t,x)=ce^{\rho t}\phi(x),\quad \textit{for } x\in\overline{\Omega},\ t\in[0,T], \tag{2.22}$$

with c a positive scalar.

Proof. Apply reductio ad absurdum. By means of the separation of variables, the general solution to (2.7) can be written in the form

$$L(t,x)=ce^{\rho t}\phi(x)+\sum_{k=1}^{\infty}e^{\lambda_k t}u_k(x) \tag{2.23}$$

where ρ,λ_k are eigenvalues, c is a positive constant and $\phi(x),u_k(x)$ are corresponding eigenfunctions.

Firstly assume that $L(t, x)$ is in the form other than (2.22), i.e. there exists at least one eigenvalue different to ρ, and hence the eigenfunction associated to it is not straightly positive by Theorem 6.

Considering $L(t, x) > 0$ in $[0, T] \times \Omega$ and eigenfunctions $u_k(x)$ for $k \geq 1$ are linearly independent, we know that all eigenvalues involved in (2.23) must be real. Without loss of generality, we can assume that

$$0 < \rho < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots .$$

Note that $L(t, x) > 0$ in $[0, T] \times \Omega$, thus there exists a positive integer $m \in \mathbb{Z}^+$, such that

$$L_m(t, x) = ce^{\rho t}\phi(x) + \sum_{k=1}^{m} e^{\lambda_k t} u_k(x)$$

is also a solution to (2.7). Next according to Theorem 6, $u_m(x)$ is supposed to switch signs inside Ω. Therefore, suppose that $u_m(x) < 0$ in some region $\mathcal{D} \subset \Omega$, by choosing some point $x_0 \in \mathcal{D}$ and taking t large enough, we can achieve that

$$L_m(t, x_0) = e^{\lambda_m t} \left\{ ce^{(\rho - \lambda_m)t}\phi(x_0) + \sum_{k=1}^{m-1} e^{(\lambda_k - \lambda_m)t} u_k(x_0) + u_m(x_0) \right\} < 0,$$

for the reason that $\rho - \lambda_m < 0$ and $\lambda_k - \lambda_m < 0$ for any $1 \leq k < m$. However, this is a contradiction to the assumption that $L(t, x) > 0$ in $[0, T] \times \Omega$. As a result, the unique solution to (2.7) can only be expressed in the form of (2.22). $\square$

2.4 Recovery and Application

2.4.1 Recovery

Now, let us back to the real probability measure $\mathbb{P}$. By (2.4), we know that for any fixed time $T > 0$, it follows that

$$\mathbf{E}^{\mathbb{P}}\left(\frac{M_T}{M_t}\frac{S_{iT}}{S_{0T}}\,\Big|\,\mathcal{F}_t\right) = \frac{S_{it}}{S_{0t}}, \quad t \in [0, T],\ i = 0, 1, \ldots, d,$$

which implies that

$$\mathbf{E}^{\mathbb{P}}\left(\frac{S_{iT}}{L_T}\,\Big|\,\mathcal{F}_t\right) = \frac{S_{it}}{L_t}. \tag{2.24}$$

In other words, (2.24) asserts that the real-world probability measure $\mathbb{P}$ becomes the martingale measure, if Long's portfolio is taken as numeraire. Specially (2.24) includes

$$\mathbf{E}^{\mathbb{P}}\left\{\frac{S_{0T}}{L_T}\,\Big|\,\mathcal{F}_t\right\} = \frac{S_{0t}}{L_t},$$

and then the martingale condition implies that

$$\frac{d(S_{0t}/L_t)}{S_{0t}/L_t} = -\sigma(X(t))dB(t),$$

where $B(t)$ is a standard Brownian motion vector under $\mathbb{P}$ and $\sigma(X(t))$ is the lognormal volatility vector of L_t.

Noting (2.6) and the fact that M is a martingale, it follows from the Itô's formula that

$$\frac{d(L_t/S_{0t})}{L_t/S_{0t}} = \sigma(X(t))\sigma^*(X(t))dt + \sigma(X(t))dB(t)$$

and the dynamics of L_t under $\mathbb{P}$ is

$$\frac{dL_t}{L_t} = \Big(r(X(t)) + \sigma(X(t))\sigma^*(X(t))\Big)dt + \sigma(X(t))dB(t). \tag{2.25}$$

With the explicit representation of the numeraire portfolio together with (2.5), we are able to determine the conditional volatility vector function, that is

$$\sigma(x) = \sum_{i=1}^{n} \frac{\partial \log L}{\partial x_i} a_i = \sum_{i=1}^{n} \frac{\partial \log \phi}{\partial x_i} a_i, \tag{2.26}$$

where the row vector a_i is the i-th row of the matrix A, i.e. $a_i = (a_{i1}, \dots, a_{id})$. Furthermore, (2.25) and the preceding result assert that the risk premium of the numeraire portfolio is

$$\sigma(x)\sigma^*(x) = \left(\sum_{i=1}^{n} \frac{\partial \log \phi}{\partial x_i} a_i\right)\left(\sum_{i=1}^{n} \frac{\partial \log \phi}{\partial x_i} a_i\right)^*$$

and the market price of risk vector process, as well as, the Girsanov kernel is uniquely determined.

At last, let us turn to determine the real world transition density of X. The change of numeraire theorem ([60]) asserts that the Radon–Nikodym derivative is:

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{S_{00}}{S_{0T}}\frac{L_T}{L_0} = e^{-\int_0^T r(X(t))dt}\frac{L(T, X(T))}{L(0, X(0))} = e^{-\int_0^T r(X(t))dt}\frac{\phi(X(T))}{\phi(X(0))}e^{\rho T}.$$

Therefore, the real world density function is given by

$$d\mathbb{P} = e^{-\int_0^T r(X(t))dt} e^{\rho T}\frac{\phi(X(T))}{\phi(X(0))} d\mathbb{Q}.$$

2.4.2 Application to Option Pricing on Private Equity

In finance, private equity is the type of equity that consists of equity securities and debt in operating companies, which are not publicly traded on a stock exchange, hence there is no market price associated with it. Since there is no time series of the underlying stock prices, it is impossible to calculate a historical volatility. Even worse, there is no cross section of related derivatives prices, and then one cannot calculate an implied volatility neither.

Considering the missing underlying stock price being the present value of its future payouts, one can still project future payouts and obtain the stock price by the discounted cash flow (DCF) technique. [61] explored option pricing on private equity in an unbounded domain by placing structure on the dynamics of the numeraire portfolio rather than on the preferences of the representative agent, using the similar techniques as [62]. They showed that the volatility of the private equity can be uniquely determined by the specification of a risk–neutral diffusion process for dividend yields, using the solution of a Sturm Liouville problem. In this section, we explore the case that there are n underlying uncertainties in the economy, with one of these uncertainties can be treated as the dividend yield.

We consider the problem of valuing an European call option written on private equity, with payoff

$$C_T = \left(\frac{S_T}{S_0} - K\right)^+,$$

where S_t denotes the unknown spot price of one share of private equity at time $t \in [0, T]$, K is defined as the strike ratio and T is the maturity date. Suppose the interest rate r is constant and there is a money market account as the numeraire whose balance is e^{rt}

at time t. In this case, the arbitrage-free option price is given by

$$C_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left(\frac{S_T}{S_0} - K\right)^+. \tag{2.27}$$

We assume there are n uncertainties $X(t) := (X_i(t))_{i=1,\dots,n}$ in the economy, evolving according to

$$dX(t) = \mu(X(t))dt + A(X(t))dW(t), \tag{2.28}$$

where $W = (W_1, \dots, W_d)^*$ is a standard d-dimensional $\mathbb{Q}$-Wiener process. The drift vector $\mu(\cdot) = (\mu_1(\cdot), \dots, \mu_n(\cdot))^*$ and the $n \times d$-dimensional diffusion matrix $A(\cdot)$, as well as the initial condition $X(0)$ are all known ex ante. The price of the private equity solely depends on the uncertainties $(X_i)_{i=1,\dots,n}$ and the time t, i.e. $S_t = S(t, X_1(t), \cdots, X_n(t))$. Suppose S evolves as the following under the risk–neutral measure $\mathbb{Q}$:

$$\frac{dS_t}{S_t} = r(X(t))dt + \sigma(X(t))dW(t), \tag{2.29}$$

where the function $r(\cdot)$ is assumed to be known ex ante, and the lognormal volatility function $\sigma(\cdot)$ is not.

Theorem 6 and Corollary 1 imply that $S(t, x)$ can be represented as

$$S(t, x) = e^{\rho t}\phi(x),$$

with $\phi(x)$ the unique positive eigenfunction up to positive scaling and ρ the corresponding positive eigenvalue. Therefore, the call option on \$1 of the notional can be priced as

$$\begin{aligned} C_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left(\frac{S(T, X(T))}{S(0, X(0))} - K\right)^+ \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left(\frac{\phi(X(T))}{\phi(X(0))}e^{\rho t} - K\right)^+. \end{aligned} \tag{2.30}$$

Note that, the unknown scale factor dropped out of the ratio $\frac{\phi(X(T))}{\phi(X(0))}$, hence the call option value can be uniquely determined.

Chapter 3

Topic in Mean Field Games on Random Graph

The Mean Field Games theory fall in the category of large population stochastic control, which analyzes the asymptotic equilibrium among a large population of controlled players with mean field interaction and subject to minimization constraints, while distinguish itself by tackling the Nash equilibrium which describe consensus among the players that each of them make the best decision by taking into account the current states of others. In this chapter, we introduce a random graph based inter-bank borrowing and lending model of Mean Field Games type, analyze the forward backward stochastic differential equation (FBSDE) of the McKean-Vlasov type at Mean Field Equilibrium, derive the corresponding master equation, and then address the convergence problem by means of a weakly interacting particle system on random graph generated by the master equation. This chapter is based on the paper [63].

3.1 Overview of Mean Field Games

The theory of Mean Field Games (MFGs) was introduced and further developed in the seminal work independently by [64, 65, 66] and by [67, 68]. The term "mean field" was borrowed from statistical physics, in the sense that individual players have statistically similar behaviors, and their mutual interactions are through average quantities in such a way that each player has a very small impact on the outcome. This subject is widely recognized as an important methodology to analyze large systems such as financial markets, crowd dynamics, social networks, etc. Two approaches are used in the formulation and analysis of Mean Field Games. One is based on the solution of a fully-coupled forward-backward system of nonlinear partial differential equations(PDEs) which include a forward Fokker-Planck equation describing the dynamics of the population and a backward Hamilton-Jacobi-Bellman equation describing the optimization constraints. The other relies on the solution of a forward-backward stochastic differential system of equations of McKean-Vlasov type, see [69, 70, 71]. Mean field game system has been widely studied and new applications of Mean Field Games include but not limited to, major and minor players [72], optimal investment under relative performance criteria [73], robust mean field games [74], rare Nash equilibrium and the price of anarchy in large static games [75], and etc.

An amazing new tool in Mean Field Games credit to the development of the so-called "master equation". Lions introduced this infinite dimensional nonlinear Partial Differential Equation in his lectures at Collège de France, whose characteristic trajectories interpret the flow of measures solving the forward Fokker-Planck equation, and whose solution contains all the necessary information to entirely describe the equilibrium of the game. [76] discussed the well-posedness of the master equation, and established the convergence of the value functions and empirical measure of the finite players to the value function and the distribution of the state of the MFG by the means of the master equation. The solution of the master equation as a function with time, state, and measure as arguments, approximates the value function of an arbitrary player from the n-player game at a given time when one takes the arguments as the players' state and the empirical distribution including other players. [77] derived the master equation for Mean Field Games and the control of McKean-Vlasov SDEs, and discussed the similarities and differences between the corresponding two sets of results.

Our model is based on the Carmona-Fouque-Sun model proposed in [78] which models inter-bank borrowing and lending, where the evolution of the log-monetary reserves of N banks is described by a system of diffusion processes coupled through their drifts in such a way that stability of the system depends on the rate of inter-bank borrowing and lending. That model incorporates a game feature where each bank controls its rate of borrowing/lending to a central bank, which acts as a clearing house, adding liquidity to the system without affecting its systemic risk. The optimization reflects the desire of each bank to borrow from the central bank when its monetary reserve falls below a critical level or lend if it rises above this critical level which is chosen as the average monetary reserve. The difference is that, we model that bank i minimizes its finite horizon objective function, taking into account a quadratic cost for lending or borrowing and a linear incentive to borrow if the reserve is low or lend if the reserve is high, relative to the average capitalization of bank i's neighborhood modeled through the Erdös Rényi random graph.

An open-loop mean field equilibrium of Nash type is obtained using a system of fully coupled forward backward stochastic differential equation (FBSDE), whose unique solution leads to the master equation. Using master equation to approximate the equilibrium states of the finite player game is a very new direction in Mean Field Games, especially with random graph involved in. A pioneer and beautiful work in this field credit to

[79], which studies the quenched convergence of the equilibrium towards the solution of a Mean Field Game when the graph connection between players is of the Erdös Rényi type, by the means of the strong solution of the master equation. In contrast to [79], our running cost function and terminal cost function are not bounded, and it cannot be split into two terms: one term with control solely and the other with empirical distribution of the population only, in the sense that there is a cross term with control and empirical distribution of the population. Also without the specific form enforced ($c_f = c_g = 1$ in [79]), our McKean Vlasov FBSDE in Mean Field Equilibrium is impossible to be transferred to a deterministic system, also it is impossible to show the convergence of the Z_t^{ij} process and the corresponding one generated using the master equation by means of taking differential as in [79]. We finished this "impossible" mission by firstly writing the FBSDEs of the finite player games in equilibrium to its associated quasilinear parabolic system of PDEs, and then by using the uniqueness and wellposedness of its solution, we give the precise bound of the convergence of the state processes of the finite player game to a decoupled system of diffusion equations generated by the master equation. We achieved the same result as [79] under frozen graph, and we also achieved a functional central limit theorem of a coupled diffusion processes system with unbounded drift functions generated by the master equation with random graph, which is the same result as [80] under boundedness imposed on the drift function and positive self loop.

A very new and beautiful paper [81] establishes a functional central limit theorem that characterizes the limiting fluctuations around the law of large numbers limit, whose proof of convergence relies on the master equation for the value function of the MFG. In this paper, our model involves the Erdös Rényi random graph, with this mixed discrete and continuous probability nature in the probability space setup, and then the difficulties in all the analyses are significantly increased. The open loop control the same as in [79] is used, instead of closed loop in [81], and the master equation corresponding to the first derivative of the value function with respect to the states then. Furthermore, we built up the equivalence between the the FBSDEs of the finite player games in equilibrium and its associated quasilinear parabolic system of PDEs, and analyzed the convergence by means of the uniqueness of its solutions to another quasilinear parabolic system of PDEs generated by the master equation; to the authors' best knowledge, this technique is the first time used in Mean Field Games with open loop controls. At last, [81] used the basic Carmona-Fouque-Sun model without graph as an example to show that the strong boundedness conditions imposed, can be relaxed with the help of the explicit solution of the Carmona-Fouque-Sun model; However, in this paper, when random graph is involved in, there is no explicit solution for the finite player game in equilibrium.

As the first few papers in this new direction, there are multiple extensions worth exploring. For simplicity, we model the state process of each player X_t^i with independent standard Brownian motions W_t^i, namely *individual noises*. In fact, the state process X_t^i induced by an infinite sequence of Brownian motions $(\widetilde{W^i})_{i\geq 1}$ can be defined as $\widetilde{W_t^i} = \rho W_t^0 + \sqrt{1-\rho^2} W_t^i$, for $0 \leq t \leq T$, with $|\rho| \leq 1$ and W_t^0 the same for all the equations, namely the *common noise*. Further reading on common noise, we refer to [76], [82], [83], especially [81] and [84]. In [84], the master equation is used to construct an associated McKean-Vlasov interacting n-particle system that is exponentially close to the Nash equilibrium dynamics of the n-player game for large n, and then a weak large deviation principle for McKean-Vlasov systems is established in the presence of common noise and a full large deviation principle is established in the absence of common noise. At the end of this Chapter, we provide the connection between the CLT and LDP analysis in this setting, which obviously requires more delicate analysis and better concentration estimates, and we leave the extension to both the common noise and LDP for future research.

3.2 Mean Field Games on Random Graph

In this section, we propose a random graph based model of inter-bank borrowing and lending, where the evolutions of the log-monetary reserves of N banks are described by a system of diffusion processes coupled through their drift terms. We model through the Erdös Rényi [85] $G(N,p)$ model and we consider the nontrivial case that $p \in (0,1)$. For the trivial cases that $p \in \{0,1\}$, see Section 2 of [78] for reference and simulation result.

Each edge is included in the graph with probability p independent of every other edge. We define the relation as edge g_{ij} between the vertex bank i and the vertex bank j, on the probability space $(\Omega_g, \mathcal{F}_g, \mathbf{P}_g)$. We consider the undirected graph, that is $g_{ij} = g_{ji}$; we assume $g_{ii} = 0$ for all i, since self-links or loops do not have real meaning in this framework. The real-valued matrix $g := (g_{ij})_{N\times N}$ is often referred as the adjacency matrix, as it lists which nodes (banks) are adjacent to one another. In this section, we only consider the standard case $g_{ij} \in \{0,1\}$, and leave the case in which the entries of g take on more than two values and can track the intensity level of relationships to future research.

Denote the degree N_i of a node i as the number of links that involves node i, which is the cardinality of its neighborhood:

$$N_i := \#\{j : g_{ij} = 1\} = \sum_{j=1}^{N} g_{ij}. \tag{3.1}$$

The distribution of the degree of any particular vertex is binomial:

$$\mathbf{P}_g(N_i = k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}. \tag{3.2}$$

The log-monetary reserves of N banks lending to and borrowing from each other are represented through the diffusion processes $(X_t^i)_{0\le t\le T}$ for $i = 1, \cdots, N$. We model X^i

on another stochastic basis $(\Omega_x, \mathcal{F}_x, \mathbf{P}_x)$, where the corresponding filtration supports an infinite sequence of independent standard Brownian motions $(W^i)_{i\geq 1}$ corresponding to individual noises. With each index $i \in 1, \cdots, N$, we associate a particle (player), whose dynamic satisfies the following form:

$$dX_t^i = \alpha_t^i dt + \sigma dW_t^i, \tag{3.3}$$

where the diffusion coefficient is assumed as constant and identical, and the control process $(\alpha_t^i)_{0\leq t\leq T}$ is assumed to be progressively-measurable with respect to the filtration generated by all the noises and satisfy the square-integrability condition

$$\mathbf{E}_x \int_0^T (\alpha_t^i)^2 dt < \infty.$$

The system starts at time $t = 0$ from i.i.d. $\mathcal{F}_0$ measurable and square-integrable random variables $X_0^i = \xi_0^i$ independent of the Brownian motions, such that $\mathbf{E}_x(\xi_0^i) = 0$. Note that all the X_i for $i \in \{1, 2, \cdots, N\}$ are statistically identical, in which each player's influence on the whole system vanishes as the number of players grows unboundedly.

Denote by $\mathcal{F} = \mathcal{F}_x \otimes \mathcal{F}_g$, the sigma algebra on the Cartesian product $\Omega = \Omega_x \otimes \Omega_g$, which is called the tensor product σ-algebra product space. Therefore, the Hahn-Kolmogorov theorem can guarantee the existence and uniqueness of the product measure

$$\mathbf{P} = \mathbf{P}_g \times \mathbf{P}_x,$$

such that for all $A_1 \in \mathcal{F}_g$ and $A_2 \in \mathcal{F}_x$,

$$\mathbf{P}(A_1 \times A_2) = \mathbf{P}_g(A_1) \cdot \mathbf{P}_x(A_2).$$

For any random variable $G \in \mathcal{F}_g$, $X \in \mathcal{F}_x$, induce its extension on $\mathcal{F}$, say $\bar{G} \in \mathcal{F}$ and $\bar{X} \in \mathcal{F}$, by

$$\bar{G}(\omega_g, \omega_x) = G(\omega_g), \quad \bar{X}(\omega_g, \omega_x) = X(\omega_x).$$

Under this definition, for any Borel set B, we have

$$\{\bar{G} \in B\} = \{G \in B\} \times \Omega_x, \quad \{\bar{X} \in B\} = \Omega_g \times \{X \in B\}.$$

Therefore,

$$\mathbf{P}\{\bar{G} \in B\} = \mathbf{P}_g\{G \in B\}, \quad \mathbf{P}\{\bar{X} \in B\} = \mathbf{P}_x\{X \in B\}.$$

Then for any measurable function f, one has

$$\mathbf{E}f(\bar{G}) = \int f(\bar{G}) d\mathbf{P} = \int f(G) d\mathbf{P}_g = \mathbf{E}_g f(G),$$

$$\mathbf{E}f(\bar{X}) = \int f(\bar{X}) d\mathbf{P} = \int f(X) d\mathbf{P}_x = \mathbf{E}_x f(X).$$

In sum, on this filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\})$, the edge random variables g_{ij} and random variable for the initial value of the state processes X_0^i are given, as well as an infinite collection of standard Brownian motions W^i, such that $\{W^i, X_0^i, g_{ij}\}$ are mutually independent.

Note that on this Erdös Rényi graph framework, for each player i, other players are indistinguishable. In this manner, we model bank i controls its rate of lending and borrowing at time t by choosing the control α_t^i in order to minimize the following cost function in quadratic form:

$$J^i(\alpha^1, \cdots, \alpha^N) = \mathbf{E}\Big[\int_0^T f_i(\mathbf{X}_t, \alpha^i) dt + g_i(\mathbf{X}_T)\Big] \tag{3.4}$$

with running cost function f_i defined as

$$f_i(\mathbf{X}_t, \alpha^i) = \frac{1}{2}(\alpha_t^i)^2 - q\alpha_t^i(\bar{X}_t^i - X_t^i) + \frac{\epsilon}{2}(\bar{X}_t^i - X_t^i)^2,$$

terminal cost function g_i defined as

$$g_i(\mathbf{X}_t) = \frac{c}{2}(\bar{X}_T^i - X_T^i)^2,$$

and interaction modeled through realization of the random graph

$$\bar{X}_t^i = \frac{1}{N_i}\sum_{j=1}^{N} g_{ij}X_t^j = \int_{\mathbb{R}} x d\bar{\mu}_t^{N,i},$$

where $\bar{\mu}_t^{N,i}$ is the empirical distributions of the particles connected to i, in the Wasserstein space of probability measures on $\mathbb{R}$ with a finite second-order moment, say $\mathcal{P}_2(\mathbb{R})$:

$$\bar{\mu}_t^{N,i} = \frac{1}{N_i}\sum_{j=1}^{N} g_{ij}\delta_{X_t^j}.$$

Whenever $N_i = 0$, we let $\bar{\mu}_t^{N,i}$ be the null measure.

Here, the effect of the parameter $q > 0$ is to control the incentive to borrowing or lending: the bank i will want to borrow ($\alpha_t^i > 0$) if X_t^i is smaller than the empirical mean ($\bar{X}_t$) and lend ($\alpha_t^i < 0$) if X_t^i is larger than $\bar{X}_t$, with q large meaning low fees. The effect of the parameter $\epsilon > 0$ with the quadratic term $(\bar{X}_t^i - X_t^i)^2$ and $c > 0$ in the terminal cost are both to penalize the departure from the average. The condition $q^2 \leq \epsilon$ is imposed, such that $f_i(x, \alpha^i)$ is convex in (x, α^i).

3.3 Mean Field Equilibrium and the Master Equation

In this section, we firstly introduce the concept of Mean Field Equilibrium and derive its associated forward backward stochastic differential equation (FBSDE) of the McKean-Vlasov type. The construction of the decoupling field is based on the existence and uniqueness of the solution to this FBSDE of McKean-Vlasov type. By means of a chain rule for functions defined on the space $\mathcal{P}_2(\mathbb{R})$, we derive the master equation. At the end, we show that the explicit form of the decoupling field is the unique classical solution of the master equation.

3.3.1 Mean Field Equilibrium and its associated FBSDE

We introduce the notion of optimality by the concept of Nash equilibrium.

Definition 1 (Definition 5.2 in [86]). *A set of admissible strategies*

$$\alpha^* = (\alpha^{*1}, \cdots, \alpha^{*N}) \in \mathbb{A}$$

is said to be a Nash equilibrium for the game if for any $i \in \{1, \cdots, N\}$ *and for any* $\alpha^i \in \mathbb{A}^i$,

$$J^i(\alpha^*) \leq J^i(\alpha^{*-i}, \beta^i),$$

where (α^{*-i}, β^i) *is the collective set of strategies such that just player* i *switches from action* α^{*i} *to* β^i *while others stay the same.*

The optimal strategy and the existence and uniqueness of the equilibrium, strongly depend upon the information available to the players, and the way that they are able to react. Open loop Nash equilibrium (OLNE), deterministic Nash equilibrium (DNE), closed loop Nash equilibrium (CLNE), closed loop Nash equilibrium in feedback form (CLFFNE) are the most popular notions of admissibility for strategy profiles, with definitions given in [86] (see, Definition 5.3 - 5.6).

In the sequel, we focus on the open loop Nash equilibrium, where the controls are of the form $\alpha_t^i = \phi^i(t, X_0, W_{[0,t]})$, where ϕ^i is a deterministic function and $W_{[0,t]}$ is the paths of Brownian motions between time 0 and t. In contrast to the closed loop Nash equilibrium (CLNE), where the controls are of the form $\alpha_t^i = \phi^i(t, X_{[0,t]})$, that is at each time all past plays are common knowledges, we note that in the open loop model, players cannot observe the actions of their opponents. Note that although the Mean Field Equilibrium (MFE), which will be defined next, is the same no matter open loop strategies or closed loop strategies, its associated Master equation is different in these two cases.

Definition 2. *A deterministic measure flow $(\mu_t)_{t\in[0,T]} \in C([0,T];\mathcal{P}_2(\mathbb{R}))$ is a Mean Field Equilibrium, if there exists an optimal strategy α^* such that*

$$J((\alpha_t^*)_{0\le t\le T}) = \inf_{(\alpha_t)_{0\le t\le T}} J((\alpha_t)_{0\le t\le T}),$$

where $J((\alpha_t)_{0\le t\le T})$ is the cost function with the flow of marginal laws of the optimal process

$$J((\alpha_t)_{0\le t\le T}) = \boldsymbol{E}\Big[\int_0^T \Big(\frac{1}{2}(\alpha_t)^2 - q\alpha_t(\int_{\mathbb{R}} x d\mu_t - X_t) + \frac{\epsilon}{2}(\int_{\mathbb{R}} x d\mu_t - X_t)^2\Big)dt + \frac{c}{2}(\int_{\mathbb{R}} x d\mu_T - X_T)^2\Big],$$

X_t solves the SDE

$$dX_t = \alpha_t dt + \sigma dW_t.$$

and the marginal law of the optimal process which evolves according to

$$dX_t = \alpha_t^* dt + \sigma dW_t$$

is exactly $(\mu_t)_{t\in[0,T]}$ *itself.*

The associated reduced Hamiltonian (when the control α only appear in the drift term of the state process), is given by

$$H(X_t, Y_t, \alpha_t) = \alpha_t Y_t + \frac{1}{2}(\alpha_t)^2 - q\alpha_t(\int_{\mathbb{R}} x d\mu_t - X_t) + \frac{\epsilon}{2}(\int_{\mathbb{R}} x d\mu_t - X_t)^2.$$

By the necessary condition of the Pontryagin stochastic maximum principle, the optimal choice can be obtained by minimizing H over α_t:

$$\alpha_t^* = -Y_t + q(\mathbf{E}[X_t] - X_t). \tag{3.5}$$

By the probabilistic approach in [69], solutions of the Mean Field Game may be characterized through the forward backward stochastic differential equation (FBSDE) of the McKean-Vlasov type. Plugging in the optimal choice α^*, the dynamic of the state of the system is given by

$$\begin{aligned} dX_t &= \Big[-Y_t + q(\mathbf{E}[X_t] - X_t)\Big]dt + \sigma dW_t, \\ dY_t &= \Big[qY_t - (\epsilon - q^2)(X_t - \mathbf{E}[X_t])\Big]dt + Z_t dW_t, \\ X_0 &= \xi_0, \quad Y_T = c(X_T - \mathbf{E}[X_T]), \end{aligned} \tag{3.6}$$

with derivation details given in Appendix C.1. One explicit solution of the above FBSDE (3.6) with common noise can be found in [78]. In this research, the corresponding explicit solution is given by

$$Y_t = \eta_t(X_t - \mathbf{E}[X_t]) \quad \text{and} \quad Z_t = \sigma\eta_t,$$

where η_t is given by

$$\eta_t = \frac{-(\epsilon - q^2)(e^{(\delta^+ - \delta^-)(T-t)} - 1) - c(\delta^+ e^{(\delta^+ - \delta^-)(T-t)} - \delta^-)}{(\delta^- e^{(\delta^+ - \delta^-)(T-t)} - \delta^+) - c(e^{(\delta^+ - \delta^-)(T-t)} - 1)}, \tag{3.7}$$

with

$$\delta^\pm = -q \pm \sqrt{\epsilon}.$$

The readers just interested in this simpler case can find the detailed derivation in the Appendix C.2.

Plugging in the explicit solution of Y_t to the dynamics of X, we have

$$dX_t = \Big[- \eta_t(X_t - \mathbf{E}[X_t]) + q(\mathbf{E}[X_t] - X_t) \Big] dt + \sigma dW_t. \tag{3.8}$$

Noting that $\mathbf{E}[X_t] = 0$, the above dynamic can be rewritten as

$$dX_t = -(\eta_t + q)X_t dt + \sigma dW_t. \tag{3.9}$$

The explicit solution of this time-dependent Ornstein–Uhlenbeck process is given by

$$X_t = \phi_0(t)\left\{ \int_0^t \frac{\sigma}{\phi_0(s)} dW_s \right\}, \tag{3.10}$$

and the law of X_t denoted as μ_t is $N(0, \mathrm{Var}(X_t))$, where

$$\mathrm{Var}(X_t) = \phi_0^2(t)\left\{ \int_0^t \frac{\sigma^2}{\phi_0^2(s)} ds \right\} \tag{3.11}$$

with

$$\phi_0(t) = \exp\left(\int_0^t (\eta_s + q)\, ds\right).$$

3.3.2 Derivation of the master equation

With the Lipschitz continuity of coefficients, by the classical results of [70] and [87], the desired wellposedness of the solution follows.

Proposition 3. *There exists a unique solution $(X_s, Y_s, Z_s)_{s\in[0,T]}$ to the system* (3.6)*, such that*

$$\sup_{s\in[0,T]} |X_s|^2, \quad \sup_{s\in[0,T]} |Y_s|^2, \quad \int_0^T |Z_s|^2 ds$$

are integrable.

With the help of the above Proposition, the decoupling field U can be defined, following the classical probabilistic approach used in [88]. The decoupling field of the forward-backward system (3.6) can be expressed as the function $U : [0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$, such that

$$U(t, X_t, \mu) = Y_t,$$

where μ is the law of X_t.

One of the amazing features of the master equation which contains all the necessary information of the FBSDE in Mean Field Equilibrium in both forward and backward directions, is that, it involves derivatives with respect to measure. In the following, we firstly introduce the definitions of these derivatives and give brief explanations.

Definition 3 ([76])**.** *1. We say that $V : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is C^1 if there exists a continuous mapping $\frac{\delta V}{\delta \mu} : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$, such that, for any μ, $\mu' \in \mathcal{P}_2(\mathbb{R})$,*

$$\lim_{s\to 0^+} \frac{V((1-s)\mu + s\mu') - V(\mu)}{s} = \int_{\mathbb{R}} \frac{\delta V}{\delta \mu}(\mu, v) d(\mu' - \mu)(v).$$

2. *If $\frac{\delta V}{\delta \mu}$ is of class C^1 with respective to the second variable, the intrinsic derivative $D_\mu V : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is defined by*

$$D_\mu V(\mu, v) := D_v \frac{\delta V}{\delta \mu}(\mu, v)$$

3. *If, for a fixed $v \in \mathbb{R}$, the map $\mu \to \frac{\delta V}{\delta \mu}(\mu, v)$ is C^1, we denote $\frac{\delta^2 V}{\delta \mu^2}$ as its derivative and say that V is C^2. If $\frac{\delta^2 V}{\delta \mu^2} = \frac{\delta^2 V}{\delta \mu^2}(\mu, v, v')$ is C^2 in the variables v and v', then we set*

$$D^2_\mu V(\mu, v, v') := D^2_{v,v'} \frac{\delta^2 V}{\delta \mu^2} V(\mu, v, v').$$

As [79], in this literature, we use $\partial_\mu V(v, \mu)(v)$ to denote the first order Wasserstein derivative with respective to measure $D_\mu V(\mu, v)$, and use $\partial^2_\mu V(t, v, \mu)(v, v')$ to denote the second order Wasserstein derivative with respective to measure $D^2_\mu V(\mu, v, v')$. Note that v' accounts for the second order derivative in the direction v'. Differentiating in v when μ is fixed in $\partial_\mu V(v, \mu)(v)$ gives the cross derivative $\partial_v \partial_\mu V(v, \mu)(v)$.

In the following, we are going to show that these Wasserstein derivatives in our model have the very desired properties.

Proposition 4. *The function $U(t, v, \mu) : [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ has bounded and continuous first and second order derivatives with respect to v. The function U is differentiable with respect to the measure argument μ and the function*

$$[0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \ni (t, v, \mu) \to \partial_\mu U(t, v, \mu)(v)$$

is bounded and continuous, which is further differentiable with respect to μ and v separately, and the functions

$$[0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \ni (t, v, \mu) \to \partial_v \partial_\mu U(t, v, \mu)(v),$$

$$[0,T]\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R})\times\mathbb{R}\ni(t,v,\mu,v')\to\partial_\mu^2U(t,v,\mu)(v,v'),$$

are bounded and continuous as well.

Proof. Recalling that $Y_t=\eta_t(X_t-\int_{\mathbb{R}}vd\mu(v))=U(t,X_t,\mu)$, we have

$$\partial_xU(t,x,\mu)=\eta_t,\quad \partial_x^2U(t,x,\mu)=0,$$

and we have

$$U(t,v,(1-s)\mu+s\mu')-U(t,v,\mu)=-s\eta_t\int_{\mathbb{R}}vd(\mu'(v)-\mu(v)),$$

$$\frac{\delta U}{\delta\mu}(t,v,\mu)=-\eta_tv$$

and

$$\partial_\mu U(t,v,\mu)(v)=-\eta_t,$$

where η_t is given in (3.7) and is bounded and continuous.

Next, fixing v and differentiating further with respect to μ gives,

$$\partial_v\partial_\mu U(t,v,\mu)(v)=0.$$

fixing μ and differentiating further with respect to v in the direction v' yields,

$$\partial_\mu^2U(t,v,\mu)(v,v')=0.$$

□

The following lemma is one result taken from [88]:

Lemma 3. *Let $(X_t)_{t\in[0,T]}$ be a real-valued Itô process evolving according to*

$$dX_t = \alpha_t dt + \sigma dW_t,$$

where α_t is a real-valued adapted process satisfying

$$\boldsymbol{E}\int_0^T \alpha_t^2 dt < \infty.$$

Let $U : [0,T]\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be twice differentiable with respect to the state variable and the measure, and the derivatives be bounded and continuous, then one has

$$\begin{aligned}
&d(U(t,X_t,\mathcal{L}(X_t))) \\
=&\Big(\partial_t U(t,X_t,\mathcal{L}(X_t)) + \partial_x U(t,X_t,\mathcal{L}(X_t))\alpha_t + \frac{\sigma^2}{2}\partial_x^2 U(t,X_t,\mathcal{L}(X_t)) \\
&+ \boldsymbol{E}\left(\partial_\mu U(t,x,\mathcal{L}(X_t))(X_t)\alpha_t\right)_{x=X_t} + \frac{\sigma^2}{2}\boldsymbol{E}\left(\partial_v\partial_\mu U(t,x,\mathcal{L}(X_t))(X_t)\right)_{x=X_t}\Big)dt \\
&+ \partial_x U(t,X_t,\mathcal{L}(X_t))\sigma dW_t,
\end{aligned} \tag{3.12}$$

where $\mathcal{L}(X_t)$ is the law of X_t.

Proposition 5. *The decoupling field function $U : [0,T]\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ satisfies the PDE*

$$\begin{aligned}
&\partial_t U(t,x,\mu) + \left[-U(t,x,\mu) + q\left(\int_{\mathbb{R}} v d\mu(v) - x\right)\right]\partial_x U(t,x,\mu) + \frac{\sigma^2}{2}\partial_x^2 U(t,x,\mu) \\
&+ \int_{\mathbb{R}}\left[-U(t,v,\mu) + q\left(\int_{\mathbb{R}} v d\mu(v) - v\right)\right]\partial_\mu U(t,x,\mu)(v)d\mu(v) \\
&+ \frac{\sigma^2}{2}\int_{\mathbb{R}}\partial_v\partial_\mu U(t,x,\mu)(v)d\mu(v) - qU(t,x,\mu) + (\epsilon - q^2)\left(x - \int_{\mathbb{R}} v d\mu(v)\right) = 0,
\end{aligned} \tag{3.13}$$

with $U(T,x,\mu) = c\left(x - \int_{\mathbb{R}} v d\mu(v)\right)$ as terminal condition.

Remark 7. *Equation* (3.13) *is the so-called master equation of the system* (3.6).

Proof. By Lemma 3, Proposition 4 and the forward SDE of X_t in the FBSDE (3.6)

$$dX_t = \left[-Y_t + q\left(\int_{\mathbb{R}} v d\mu(v) - x\right)\right] dt + \sigma dW_t,$$

the master equation can be derived in the form of (3.13), where the term

$$-qU(t,x,\mu) + (\epsilon - q^2)(x - \int_{\mathbb{R}} v d\mu(v))$$

is achieved by plugging in the optimal value of the control

$$\alpha^* = -Y_t + q(\int_{\mathbb{R}} v d\mu(v) - X_t)$$

to the corresponding running cost term, i.e. $q\alpha^* + \epsilon(x - \int_{\mathbb{R}} v d\mu(v))$. □

By the results of Proposition 4 and considering the constant volatility term, a direct application of Theorem 2.8 in [88] gives the following Proposition.

Proposition 6. *The function* $U(t,x,\mu) = \eta_t(x - \int_{\mathbb{R}} v d\mu(v))$*, is the unique classical solution to the master equation* (3.13)*.*

Remark 8. *Let us take a look at the master equation (32) in [77], which is achieved by passing to infinity of the explicit solution of the finite player game. With* $a = 0$ *and* $\rho = 0$ *corresponding to the case in this chapter, their master equation is given by*

$$\begin{aligned}&\partial_t V(t,x,m) + q(m-x)\partial_x V(t,x,m) + \frac{1}{2}(\epsilon - q^2)(m-x)^2 \\ &- \frac{1}{2}[\partial_x V(t,x,m)]^2 + \frac{\sigma^2}{2}\partial_x^2 V(t,x,m) = 0,\end{aligned} \tag{3.14}$$

where m *is the mean and the function* V *is the value function. Now, let us take derivative* *of V with respect to x and denote $\widetilde{V}(t,x,\mu)=\partial_x V(t,x,m)$, we have*

$$\begin{aligned}&\partial_t\widetilde{V}(t,x,\mu)+q(m-x)\partial_x\widetilde{V}(t,x,\mu)-q\widetilde{V}(t,x,\mu)+(\epsilon-q^2)(x-m)\\&-\widetilde{V}(t,x,\mu)\partial_x\widetilde{V}(t,x,\mu)+\frac{\sigma^2}{2}\partial_x^2\widetilde{V}(t,x,\mu)=0,\end{aligned}\tag{3.15}$$

One can find that the function $U(t,x,\mu)=\eta_t(x-\int_{\mathbb{R}} v d\mu(v))$ in this chapter, is a classical solution to equation (3.15).

3.4 Analysis of the Finite Player Game in Equilibrium

3.4.1 FBSDE in the finite player game and the corresponding quasilinear parabolic system

Now, let us come back to the finite player game. Recall that $(\Omega,\mathcal{F},\mathbf{P},\{\mathcal{F}_t\})$ is a stochastic basis, where the edge random variables g_{ij} and random variable for the initial value of the state processes X_0^i are given, as well as an infinite collection of standard Brownian motions W^i, such that $\{W^i, X_0^i, g_{ij}\}$ are mutually independent. The log-monetary reserves of N banks lending to and borrowing from each other are represented through the diffusion processes X_t^i for $i=1,\cdots,N$. The system starts at time $t=0$ from i.i.d. random variables $X_0^i=\xi_0^i$, such that $\mathbf{E}(\xi_0^i)=0$ and $\mathbf{E}(\xi_0^i)^2<\infty$.

The reduced Hamiltonian for bank i is given by

$$\begin{aligned}&H^i(x^1,\cdots,x^N,y^{i,1},\cdots,y^{i,N},\alpha^1,\cdots,\alpha^N)\\=&\sum_{k=1}^N\alpha^k y^{i,k}+\frac{1}{2}(\alpha^i)^2-q\alpha^i(\bar{x}^i-x^i)+\frac{\epsilon}{2}(\bar{x}^i-x^i)^2.\end{aligned}$$

The necessary condition of the game version of the Pontryagin principle suggests that one minimizes H^i with respect to α^i to achieve the optimal control

$$\alpha^{*,i} = -y^{i,i} + q(\bar{x}^i - x^i).$$

The adjoint processes $(Y_t^{i,j})_{i,j=1,\cdots,N}$ and $(Z_t^{i,j,k})_{i,j,k=1,\cdots,N}$ are defined as the solutions of the following backward stochastic differential equations (BSDEs):

$$dY_t^{i,j} = -\partial_{x^j} H^i(X_t^1, \cdots, X_t^N, Y_t^{i,1}, \cdots, Y_t^{i,N}, \alpha^1, \cdots, \alpha^N)\ dt + \sum_{k=1}^N Z_t^{i,j,k} dW_t^k.$$

Any equilibrium taken over open loop strategies must satisfy the following fully coupled system of FBSDEs:

$$\begin{aligned} dX_t^i &= \left[-Y_t^{i,i} + q\left(\int_{\mathbb{R}} x d\bar{\mu}_t^{N,i}(x) - X_t^i\right)\right] dt + \sigma dW_t^i, \\ dY_t^{i,i} &= \left[qY_t^{i,i} - (\epsilon - q^2)\left(X_t^i - \int_{\mathbb{R}} x d\bar{\mu}_t^{N,i}(x)\right)\right] dt + \sum_{j=1}^N Z_t^{i,i,j} dW_t^j, \\ X_0^i &= \xi_0^i, \quad Y_T^{i,i} = c\left(X_T^i - \int_{\mathbb{R}} x d\bar{\mu}_T^{N,i}(x)\right), \end{aligned} \tag{3.16}$$

with the empirical distributions

$$\bar{\mu}_t^{N,i} = \frac{1}{N_i} \sum_{j=1}^N g_{ij} \delta_{X_t^j}.$$

With a slight abuse of notation, for simplicity of the analysis following, we denote $Y^i :=$

Y^{ii} and $Z_t^{i,j} := Z_t^{i,i,j}$, and study the following system of FBSDEs:

$$
\begin{aligned}
dX_t^i &= \left[-Y_t^i + q\left(\frac{1}{N_i}\sum_{j=1}^N g_{ij}X_t^j - X_t^i\right)\right]dt + \sigma dW_t^i,\\
dY_t^i &= \left[qY_t^i - (\epsilon - q^2)\left(X_t^i - \frac{1}{N_i}\sum_{j=1}^N g_{ij}X_t^j\right)\right]dt + \sum_{j=1}^N Z_t^{i,j}dW_t^j, \qquad (3.17)\\
X_0^i &= \xi_0^i, \quad Y_T^i = c\left(X_T^i - \frac{1}{N_i}\sum_{j=1}^N g_{ij}X_T^j\right).
\end{aligned}
$$

3.4.2 The wellposedness of the system of FBSDEs (3.17)

The system of FBSDEs (3.17) can be written in matrix form,

$$
\begin{aligned}
d\overrightarrow{X_t} &= \left[A\overrightarrow{X_t} + B\overrightarrow{Y_t}\right]dt + \sigma d\overrightarrow{W_t},\\
d\overrightarrow{Y_t} &= \left[\hat{A}\overrightarrow{X_t} + \hat{B}\overrightarrow{Y_t}\right]dt + Z_t d\overrightarrow{W_t}, \qquad (3.18)\\
X_0^N &= x, \quad \overrightarrow{Y_T} = G\overrightarrow{X_T},
\end{aligned}
$$

where $\overrightarrow{X_t} := (X_t^1, \cdots, X_t^N)^T$, $\overrightarrow{Y_t} := (Y_t^1, \cdots, Y_t^N)^T$ and $\overrightarrow{W_t} := (W_t^1, \cdots, W_t^N)^T$ are all valued in $\mathbb{R}^N$, $Z_t := (Z_t^{i,j})$ is valued in $\mathbb{R}^{N\times N}$, and the coefficient matrices are given by $A = qM$, $B = (-1)I_N$, $\hat{A} = (\epsilon - q^2)M$, $\hat{B} = qI_N$, $G = (-c)M$ with

$$
M = \begin{pmatrix} -1 & \frac{1}{N_1}g_{12} & \cdots & \frac{1}{N_1}g_{1N} \\ \frac{1}{N_2}g_{21} & -1 & \cdots & \frac{1}{N_2}g_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N_N}g_{N1} & \frac{1}{N_N}g_{N2} & \cdots & -1 \end{pmatrix}_{N\times N}, \quad I_N = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}_{N\times N}. \qquad (3.19)
$$

With the linearity of coefficients of this fully coupled FBSDE, by the Theorem 2.1 in [70], we have the following existence result of a finite solution, which is crucial in the convergence analysis later.

Proposition 7. *the FBSDE system* (3.18) *has a solution* $(\overrightarrow{X_t}, \overrightarrow{Y_t}, Z_t)$ *such that*

$$\boldsymbol{E}(\sup_{0\le t\le T} |\overrightarrow{X_t}|^2) < \infty, \quad \boldsymbol{E}(\sup_{0\le t\le T} |\overrightarrow{Y_t}|^2) < \infty, \quad \boldsymbol{E}\int_0^T |Z_t|^2 dt < \infty.$$

With the linearity in the drift terms and the constant volatility term of X_t^i and Y_t^i processes, the uniqueness of solution is a direct result of [87].

Proposition 8. *(i) the FBSDE system* (3.17) *admits a unique solution* $(X_t^i, Y_t^i, Z_t^{i,j})_{i,j=1,\cdots,N}$.

(ii) The function $v^{N,i}(t,\boldsymbol{x})$ *for* $(t,\boldsymbol{x}) \in [0,T]\times\mathbb{R}^N$ *and* $i = 1,2,\cdots,N$, *defined through* $v^{N,i}(t,\boldsymbol{X}_t) := Y_t^i$ *with* $\boldsymbol{X}_t = (X_t^1,\cdots,X_t^N)$, *is continuous and is a viscosity solution of the quasilinear parabolic systems of PDEs*

$$\begin{aligned}
&\partial_t v^{N,i}(t,\boldsymbol{x}) + \sum_{j=1}^N \left(-v^{N,j}(t,\boldsymbol{x}) + q\left(\frac{1}{N_j}\sum_{k=1}^N g_{jk}x^k - x^j\right)\right)\partial_{x^j} v^{N,i}(t,\boldsymbol{x}) \\
&+ \frac{\sigma^2}{2}\sum_{j=1}^N \partial_{x^j}^2 v^{N,i}(t,\boldsymbol{x}) - qv^{N,i}(t,\boldsymbol{x}) + (\epsilon - q^2)\left(x^i - \frac{1}{N_i}\sum_{j=1}^N g_{ij}x^j\right) = 0, \qquad (3.20)\\
&v^{N,i}(T,\boldsymbol{x}) = c\left(x^i - \frac{1}{N_i}\sum_{j=1}^N g_{ij}x^j\right).
\end{aligned}$$

3.5 Convergence Results under Frozen Graph

For $(t,x) \in [0,T]\times\mathbb{R}^N$, and $i = 1,2,\cdots,N$, we set

$$\bar{u}^i(t,\mathbf{X}_t) = U(t, X_t^i, \bar{\mu}_t^N), \quad \bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}.$$

3.5.1 Closeness of $\bar{u}^i(t,\mathbf{X}_t)$ to $v^{N,i}(t,\mathbf{X}_t)$

Before we prove the convergence of $\bar{u}^i(t,\mathbf{X}_t)$ to $v^{N,i}(t,\mathbf{X}_t)$, let us first give the PDEs $\bar{u}^i(t,\mathbf{X}_t)$ satisfies.

Proposition 9. *One has for any $i \in 1, \cdots, N$,*

$$\begin{aligned}&\partial_t \bar{u}^i(t,\boldsymbol{x}) + \sum_{j=1}^N \Bigg[-\bar{u}^j(t,\boldsymbol{x}) + q\left(\frac{1}{N}\sum_{k=1}^N x^k - x^j\right)\Bigg]\partial_{x^j}\bar{u}^i(t,\boldsymbol{x}) \\ &+ \frac{\sigma^2}{2}\sum_{j=1}^N \partial^2_{x^j}\bar{u}^i(t,\boldsymbol{x}) - q\bar{u}^i(t,\boldsymbol{x}) + (\epsilon - q^2)\left(x^i - \frac{1}{N}\sum_{k=1}^N x^k\right) = 0.\end{aligned} \tag{3.21}$$

Proof. Recall that the master equation is given by

$$\begin{aligned}&\partial_t U(t,x,\mu) + \Bigg[-U(t,x,\mu) + q\left(\int_{\mathbb{R}} v d\mu(v) - x\right)\Bigg]\partial_x U(t,x,\mu) + \frac{\sigma^2}{2}\partial_x^2 U(t,x,\mu) \\ &+ \int_{\mathbb{R}} \Bigg[-U(t,v,\mu) + q\left(\int_{\mathbb{R}} v d\mu(v) - v\right)\Bigg]\partial_\mu U(t,x,\mu)(v) d\mu(v) \\ &+ \frac{\sigma^2}{2}\int_{\mathbb{R}} \partial_v\partial_\mu U(t,x,\mu)(v)d\mu(v) - qU(t,x,\mu) + (\epsilon - q^2)\left(x - \int_{\mathbb{R}} v d\mu(v)\right) = 0,\end{aligned} \tag{3.22}$$

therefore, one has at a point $(t, x^i, \bar{\mu}^N_t)$:

$$\begin{aligned}&\partial_t \bar{u}^i(t,\mathbf{x}) + \Bigg[-\bar{u}^i(t,\mathbf{x}) + q\left(\int_{\mathbb{R}} v d\bar{\mu}^N_t(v) - x^i\right)\Bigg]\partial_x U(t,x^i,\bar{\mu}^N_t) + \frac{\sigma^2}{2}\partial_x^2 U(t,x^i,\bar{\mu}^N_t) \\ &+ \int_{\mathbb{R}} \Bigg[-U(t,v,\bar{\mu}^N_t) + q\left(\int_{\mathbb{R}} v d\bar{\mu}^N_t(v) - v\right)\Bigg]\partial_\mu U(t,x^i,\bar{\mu}^N_t)(v) d\bar{\mu}^N_t(v) \\ &+ \frac{\sigma^2}{2}\int_{\mathbb{R}} \partial_v\partial_\mu U(t,x^i,\bar{\mu}^N_t)(v)d\bar{\mu}^N_t(v) - q\bar{u}^i(t,\mathbf{x}) + (\epsilon - q^2)\left(x^i - \int_{\mathbb{R}} v d\bar{\mu}^N_t(v)\right) = 0.\end{aligned} \tag{3.23}$$

With the explicit form

$$U(t,x^i,\bar{\mu}^N_t) = \eta_t(x^i - \frac{1}{N}\sum_{j=1}^N x^j),$$

we have

$$
\begin{aligned}
\partial_{x^i}\bar{u}^i(t,x^1,\cdots,x^N) =& \partial_x U(t,x^i,\bar{\mu}_t^N) + \frac{1}{N}\partial_\mu U(t,x^i,\bar{\mu}_t^N)(x^i),\\
\partial_{x^j}\bar{u}^i(t,x^1,\cdots,x^N) =& \frac{1}{N}\partial_\mu U(t,x^i,\bar{\mu}_t^N)(x^j), \quad j\neq i,\\
\partial^2_{x^i}\bar{u}^i(t,x^1,\cdots,x^N) =& \frac{1}{N}\partial_v\partial_\mu U(t,x^i,\bar{\mu}_t^N)(x^i),\\
\partial^2_{x^j}\bar{u}^i(t,x^1,\cdots,x^N) =& \frac{1}{N}\partial_v\partial_\mu U(t,x^i,\bar{\mu}_t^N)(x^j), \quad j\neq i,
\end{aligned}
$$

and $\partial_x^2 U(t,x^i,\bar{\mu}_t^N)=0$. Plugging in equation, we have

$$
\begin{aligned}
&\partial_t\bar{u}^i(t,\mathbf{x}) + \left[-\bar{u}^i(t,\mathbf{x}) + q\left(\frac{1}{N}\sum_{k=1}^N x^k - x^i\right)\right]\left[\partial_{x^i}\bar{u}^i(t,\mathbf{x}) - \cancel{\frac{1}{N}\partial_\mu U(t,x^i,\bar{\mu}_t^N)}(x^i)\right]\\
&+\sum_{j=1,j\neq i}^N\left[-\bar{u}^j(t,\mathbf{x}) + q\left(\frac{1}{N}\sum_{k=1}^N x^k - x^j\right)\right]\partial_{x^j}\bar{u}^i(t,\mathbf{x})\\
&+\cancel{\left[-U(t,x^i,\bar{\mu}_t^N) + q\left(\frac{1}{N}\sum_{k=1}^N x^k - x^i\right)\right]\partial_\mu U(t,x^i,\bar{\mu}_t^N)(x^i)}\\
&+\frac{\sigma^2}{2}\sum_{j=1}^N\partial^2_{x^j}\bar{u}^i(t,\mathbf{x}) - q\bar{u}^i(t,\mathbf{x}) + (\epsilon-q^2)\left(x^i - \frac{1}{N}\sum_{k=1}^N x^k\right) = 0.
\end{aligned}
\tag{3.24}
$$

Therefore,

$$
\begin{aligned}
&\partial_t\bar{u}^i(t,\mathbf{x}) + \sum_{j=1}^N\left[-\bar{u}^j(t,\mathbf{x}) + q\left(\frac{1}{N}\sum_{k=1}^N x^k - x^j\right)\right]\partial_{x^j}\bar{u}^i(t,\mathbf{x})\\
&+\frac{\sigma^2}{2}\sum_{j=1}^N\partial^2_{x^j}\bar{u}^i(t,\mathbf{x}) - q\bar{u}^i(t,\mathbf{x}) + (\epsilon-q^2)\left(x^i - \frac{1}{N}\sum_{k=1}^N x^k\right) = 0.
\end{aligned}
\tag{3.25}
$$

□

By Itô's formula and using the parabolic system equation (3.20), we have

$$
\begin{aligned}
&dv^{N,i}(t,\mathbf{X}_t)\\
=&\Bigg\{\partial_t v^{N,i}(t,\mathbf{X}_t)+\sum_{j=1}^N\left[-v^{N,j}(t,\mathbf{X}_t)+q\left(\frac{1}{N_j}\sum_{k=1}^N g_{jk}X_t^k-X_t^j\right)\right]\partial_{x^j}v^{N,i}(t,\mathbf{X}_t)\\
&+\frac{\sigma^2}{2}\sum_{j=1}^N\partial_{x^j}^2 v^{N,i}(t,\mathbf{X}_t)\Bigg\}dt+\sigma\sum_{j=1}^N\partial_{x^j}v^{N,i}(t,\mathbf{X}_t)dW_t^j\\
=&\left\{qv^{N,i}(t,\mathbf{X}_t)-(\epsilon-q^2)\left(X_t^i-\frac{1}{N_i}\sum_{j=1}^N g_{ij}X_t^j\right)\right\}dt+\sigma\sum_{j=1}^N\partial_{x^j}v^{N,i}(t,\mathbf{X}_t)dW_t^j.
\end{aligned}
\tag{3.26}
$$

By Itô's formula and using the result of Proposition 9, we have

$$
\begin{aligned}
&d\bar{u}^i(t,\mathbf{X}_t)\\
=&\Bigg\{\partial_t\bar{u}^i(t,\mathbf{X}_t)+\sum_{j=1}^N\left[-v^{N,j}(t,\mathbf{X}_t)+q\left(\frac{1}{N_j}\sum_{k=1}^N g_{jk}X_t^k-X_t^j\right)\right]\partial_{x^j}\bar{u}^i(t,\mathbf{X}_t)\\
&+\frac{\sigma^2}{2}\sum_{j=1}^N\partial_{x^j}^2\bar{u}^i(t,\mathbf{X}_t)\Bigg\}dt+\sigma\sum_{j=1}^N\partial_{x^j}\bar{u}^i(t,\mathbf{X}_t)dW_t^j\\
=&\bar{r}^i(t,\mathbf{X}_t)dt+\left\{q\bar{u}^i(t,\mathbf{X}_t)-(\epsilon-q^2)\left(X_t^i-\frac{1}{N}\sum_{j=1}^N X_t^j\right)\right\}dt+\sigma\sum_{j=1}^N\partial_{x^j}\bar{u}^i(t,\mathbf{X}_t)dW_t^j
\end{aligned}
\tag{3.27}
$$

with

$$\begin{aligned}
\bar{r}^i(t,\mathbf{X}_t) =& \sum_{j=1}^{N}\left[-v^{N,j}(t,\mathbf{X}_t) + q\left(\frac{1}{N_j}\sum_{k=1}^{N} g_{jk}X_t^k - X_t^j\right)\right]\partial_{x^j}\bar{u}^i(t,\mathbf{X}_t) \\
&- \sum_{j=1}^{N}\left[-\bar{u}^j(t,\mathbf{X}_t) + q\left(\frac{1}{N}\sum_{k=1}^{N} X_t^k - X_t^j\right)\right]\partial_{x^j}\bar{u}^i(t,\mathbf{X}_t) \\
=&\left[(\bar{u}^i(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)) + q\left(\frac{1}{N_i}\sum_{k=1}^{N} g_{ik}X_t^k - \frac{1}{N}\sum_{k=1}^{N} X_t^k\right)\right]\eta_t(1-\frac{1}{N}) \\
&- \sum_{j\neq i}\left[(\bar{u}^j(t,\mathbf{X}_t) - v^{N,j}(t,\mathbf{X}_t)) + q\left(\frac{1}{N_j}\sum_{k=1}^{N} g_{jk}X_t^k - \frac{1}{N}\sum_{k=1}^{N} X_t^k\right)\right]\frac{\eta_t}{N}.
\end{aligned} \tag{3.28}$$

Taking difference of the two equations above, we have

$$\begin{aligned}
& d\left(\bar{u}^i(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right) \\
=& q\left(\bar{u}^i(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right)dt + \bar{r}^i(t,\mathbf{X}_t)dt - (\epsilon - q^2)\left(\frac{1}{N_i}\sum_{j=1}^{N} g_{ij}X_t^j - \frac{1}{N}\sum_{j=1}^{N} X_t^j\right)dt \\
&+ \sigma\sum_{j=1}^{N}\left(\partial_{x^j}\bar{u}^i(t,\mathbf{X}_t) - \partial_{x^j}v^{N,i}(t,\mathbf{X}_t)\right)dW_t^j.
\end{aligned} \tag{3.29}$$

Taking the square and applying Itô's formula, we have

$$\begin{aligned}
& d\left[\bar{u}^i(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right]^2 \\
=& 2q\left(\bar{u}^i(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right)^2 dt \\
& + 2\left(\bar{u}^i(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right)\bar{r}^i(t,\mathbf{X}_t)dt \\
& - 2(\epsilon - q^2)\left(\bar{u}^i(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right)\left(\frac{1}{N_i}\sum_{j=1}^N g_{ij}X_t^j - \frac{1}{N}\sum_{j=1}^N X_t^j\right)dt \\
& + \sigma^2\sum_{j=1}^N(\partial_{x^j}u^{N,i}(t,X_t) - \partial_{x^j}v^{N,i}(t,\mathbf{X}_t))^2 dt \\
& + 2\sigma\sum_{j=1}^N\left(\bar{u}^i(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right)(\partial_{x^j}\bar{u}^i(t,\mathbf{X}_t) - \partial_{x^j}v^{N,i}(t,\mathbf{X}_t))dW_t^j.
\end{aligned} \tag{3.30}$$

The terminal conditions are given by

$$\bar{u}^i(T,\mathbf{x}) = c(x^i - \frac{1}{N}\sum_{j=1}^N x^j), \quad v^{N,i}(T,\mathbf{x}) = c\left(x^i - \frac{1}{N_i}\sum_{j=1}^N g_{ij}x^j\right).$$

3.5.2 Preliminary Convergence Analysis

Now let us see the convergence of $\bar{u}^i(t,\mathbf{X}_t)$ to $v^{N,i}(t,\mathbf{X}_t)$, when the realization of the graph is frozen.

Proposition 10. *There exists a constant C, independent of N and of the realization of the Erdös Rényi graph, such that*

$$\begin{aligned}
& \boldsymbol{E}_x\left[\bar{u}^i(t,\boldsymbol{X}_t) - v^{N,i}(t,\boldsymbol{X}_t)\right]^2 \\
\leq & C\boldsymbol{E}_x\left(\frac{1}{N_i}\sum_{j=1}^N g_{ij}X_T^j - \frac{1}{N}\sum_{j=1}^N X_T^j\right)^2 + C\int_t^T \boldsymbol{E}_x\left(\frac{1}{N_i}\sum_{k=1}^N g_{ik}X_s^k - \frac{1}{N}\sum_{k=1}^N X_s^k\right)^2 ds
\end{aligned}$$

Proof. Integrating from t to T and taking expectation, we have

$$
\begin{aligned}
&\mathbf{E}_x \left[\bar{u}^i(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right]^2 \\
=& c^2 \mathbf{E}_x \left(\frac{1}{N_i}\sum_{j=1}^N g_{ij} X_T^j - \frac{1}{N}\sum_{j=1}^N X_T^j \right)^2 - 2q \int_t^T \mathbf{E}_x \left(\bar{u}^i(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right)^2 ds \\
&- 2\int_t^T \mathbf{E}_x \left[\left(\bar{u}^i(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right) \bar{r}^i(s,\mathbf{X}_s)\right] ds \\
&+ 2(\epsilon - q^2) \int_t^T \mathbf{E}_x \left[\left(\bar{u}^i(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right) \left(\frac{1}{N_i}\sum_{j=1}^N g_{ij} X_s^j - \frac{1}{N}\sum_{j=1}^N X_s^j \right) \right] ds \\
&- \sigma^2 \sum_{j=1}^N \int_t^T \mathbf{E}_x (\partial_{x^j}\bar{u}^i(s,\mathbf{X}_s) - \partial_{x^j} v^{N,i}(s,\mathbf{X}_s))^2 ds.
\end{aligned} \tag{3.31}
$$

Notice that

$$
\sigma^2 \sum_{j=1}^N \int_t^T \mathbf{E}_x \left(\partial_{x^j}\bar{u}^i(s,\mathbf{X}_s) - \partial_{x^j} v^{N,i}(s,\mathbf{X}_s)\right)^2 ds \geq 0,
$$

we have

$$
\begin{aligned}
&\mathbf{E}_x \left[\bar{u}^i(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right]^2 \\
\leq & C \mathbf{E}_x \left(\frac{1}{N_i}\sum_{j=1}^N g_{ij} X_T^j - \frac{1}{N}\sum_{j=1}^N X_T^j \right)^2 + C \int_t^T \mathbf{E}_x \left(\bar{u}^i(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right)^2 ds + \mathcal{M} + \mathcal{N}.
\end{aligned} \tag{3.32}
$$

For term $\mathcal{M}$, by convexity argument and by symmetry, we have

$$\begin{aligned}
\mathcal{M} :=& 2\int_t^T \mathbf{E}_x\left[\left|\bar{u}^i(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right| \cdot |\bar{r}^i(s,\mathbf{X}_s)|\right] ds \\
\le& C\int_t^T \mathbf{E}_x\left(\bar{u}^i(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right)^2 ds \\
&+ C\int_t^T \mathbf{E}_x\left[\left|\bar{u}^i(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right| \cdot \left|\frac{1}{N_i}\sum_{k=1}^N g_{ik}X_s^k - \frac{1}{N}\sum_{k=1}^N X_s^k\right|\right] ds \\
&+ \frac{C}{N}\sum_{j\neq i}\int_t^T \mathbf{E}_x\left[\left|\bar{u}^i(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right| \cdot |\bar{u}^j(t,\mathbf{X}_s) - v^{N,j}(s,\mathbf{X}_s)|\right] ds \\
&+ \frac{C}{N}\sum_{j\neq i}\int_t^T \mathbf{E}_x\left[\left|\bar{u}^i(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right| \cdot \left|\frac{1}{N_j}\sum_{k=1}^N g_{jk}X_s^k - \frac{1}{N}\sum_{k=1}^N X_s^k\right|\right] ds \\
\le& C\int_t^T \mathbf{E}_x\left(\bar{u}^i(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right)^2 ds + C\int_t^T \mathbf{E}_x\left(\frac{1}{N_i}\sum_{k=1}^N g_{ik}X_s^k - \frac{1}{N}\sum_{k=1}^N X_s^k\right)^2 ds
\end{aligned} \tag{3.33}$$

For term $\mathcal{N}$, by convexity argument and by symmetry, we have

$$\begin{aligned}
\mathcal{N} :=& C\int_t^T \mathbf{E}_x\left[\left(\bar{u}^i(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right)\left(\frac{1}{N_i}\sum_{j=1}^N g_{ij}X_s^j - \frac{1}{N}\sum_{j=1}^N X_s^j\right)\right] ds \\
\le& C\int_t^T \mathbf{E}_x\left(\bar{u}^i(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right)^2 ds + C\int_t^T \mathbf{E}_x\left(\frac{1}{N_i}\sum_{k=1}^N g_{ik}X_s^k - \frac{1}{N}\sum_{k=1}^N X_s^k\right)^2 ds
\end{aligned} \tag{3.34}$$

In sum, by Gronwall's lemma, we have

$$\begin{aligned}
&\mathbf{E}_x\left[\bar{u}^i(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right]^2 \\
\le& C\mathbf{E}_x\left(\frac{1}{N_i}\sum_{j=1}^N g_{ij}X_T^j - \frac{1}{N}\sum_{j=1}^N X_T^j\right)^2 + C\int_t^T \mathbf{E}_x\left(\frac{1}{N_i}\sum_{k=1}^N g_{ik}X_s^k - \frac{1}{N}\sum_{k=1}^N X_s^k\right)^2 ds
\end{aligned}$$

□

Proposition 11. *There exists a sequence of random variables* $(\delta_N)_{N\geq 1}$ *constructed on the probability space* $(\Omega_g, \mathcal{F}_g, \boldsymbol{P}_g)$*, such that*

$$\sup_{0\leq t\leq T} \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{E}_x \left[\bar{u}^i(t,\boldsymbol{X}_t) - v^{N,i}(t,\boldsymbol{X}_t)\right]^2 \leq \delta_N$$

and

$$\mathbf{P}_g[\lim_{N\to\infty} \delta_N = 0] = 1.$$

Proof. Applying the result of Section 4.2 on Delarue [79], and the unique bounded second moment of $(X^i_s)_{s\in[0,T],i=1,\cdots,N}$ given in Proposition 7, we know that

$$\frac{1}{N}\sum_{i=1}^{N} \mathbf{E}_x \left[\bar{u}^i(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right]^2 \leq \delta_N,$$

where

$$\delta_N = \frac{C}{N^2}\sum_{j,l=1}^{N}\left|1 - \frac{1}{N}\sum_{i=1}^{N}\frac{N^2}{N_i^2}\mathbb{1}_{N_i\geq 1} g_{ij}g_{il}\right| + \frac{C}{N}\sum_{j=1}^{N}\left|1-\frac{1}{N}\sum_{i=1}^{N}\frac{N}{N_i}\mathbb{1}_{N_i\geq 1}g_{ij}\right|.$$

and $\lim_{N\to\infty}\delta_N = 0$, $\mathbf{P}_g$ almost surely. □

3.5.3 Law of Large Numbers

We denote by $\mathcal{P}_p(\mathbb{R})$ the subspace of $\mathcal{P}(\mathbb{R})$ of the probability measures of order p, namely those elements of $\mathcal{P}(\mathbb{R})$ which integrate the p-th power of the distance to a fixed point. For each $p \geq 1$, if μ and μ' are probability measures of order p, $W_p(\mu,\mu')$ denotes the p-Wasserstein's distance defined as

$$W_p(\mu,\mu') = \inf\left\{\left[\int |x-y|^p_{\mathbb{R}}\pi(dx,dy)\right]^{1/p}; \pi\in\mathcal{P}_p(\mathbb{R}\times\mathbb{R}) \text{ with marginals } \mu \text{ and } \mu'\right\}.$$

Notice that if X and X' are random variables of order 2 taking values in $\mathbb{R}$, and with law μ and μ' respectively, then we have

$$W_2(\mu, \mu') \leq \left[\mathbf{E}_x|X - X'|_{\mathbb{R}}^2\right]^{1/2}.$$

Proposition 12. $\mathbf{P}_g$ *almost surely,*

$$\lim_{N\to\infty} \sup_{0\leq t\leq T} \boldsymbol{E}_x[W_2(\bar{\mu}_t^N, \mu_t)^2] = 0,$$

where W_2 is the 2-Wasserstein distance, $\bar{\mu}_t^N$ is the empirical distribution of state process of the finite player game i.e. $\bar{\mu}_t^N = \frac{1}{N}\sum_{j=1}^N X_t^j$, and μ_t is the law of the state process at Mean Field Equilibrium.

Proof. Inspired by Delarue [79], we create copies of $(X_t)_{0\leq t\leq T}$, which are driven by the $(X_0^i, (W_t^i)_{0\leq t\leq T})$ as $(X_t^i)_{0\leq t\leq T}$, instead of driving by the $(X_0, (W_t)_{0\leq t\leq T})$, namely

$$d\hat{X}_t^i = \left[-U(t, \hat{X}_t^i, \mathcal{L}(\hat{X}_t^i)) + q\left(\mathbf{E}_x[\hat{X}_t^i] - \hat{X}_t^i\right)\right] dt + \sigma dW_t^i.$$

That is, $(\hat{X}_t^i)_{i\geq 1}$ are i.i.d with the same dynamics as (X_t), and the law of $\hat{X}_t^i$ is μ_t. The dynamics of the difference of the processes X_t^i and $\hat{X}_t^i$ are given by

$$\begin{aligned} &d(X_t^i - \hat{X}_t^i) \\ &= \left[-Y_t^i + U(t, \hat{X}_t^i, \mathcal{L}(\hat{X}_t^i)) + q\left(\frac{1}{N_i}\sum_{j=1}^N g_{ij}X_t^j - \mathbf{E}_x[\hat{X}_t^i]\right) - q(X_t^i - \hat{X}_t^i)\right] dt \end{aligned} \tag{3.35}$$

Note that, by the explicit form of the U function, we have

$$\begin{aligned}&\left(-Y_s^i+U(s,\hat{X}_s^i,\mathcal{L}(\hat{X}_s^i))\right)^2\\ \leq&2\left(Y_s^i-U(t,X_s^i,\bar{\mu}_s^N)\right)^2+2\left(U(s,X_s^i,\bar{\mu}_s^N)-U(s,\hat{X}_s^i,\mathcal{L}(\hat{X}_s^i))\right)^2\\ \leq&C\left(Y_s^i-U(s,X_s^i,\bar{\mu}_s^N)\right)^2+C(X_s^i-\hat{X}_s^i)^2+C\left(\frac{1}{N}\sum_{j=1}^N X_s^j-\mathbf{E}_x[\hat{X}_s^i]\right)^2.\end{aligned}\tag{3.36}$$

Therefore, considering $X_0^i=\hat{X}_0^i$, integrating from 0 to t, and taking square and then expectation, we have

$$\begin{aligned}&\mathbf{E}_x(X_t^i-\hat{X}_t^i)^2\\ =&2\int_0^t\mathbf{E}_x\left[(X_s^i-\hat{X}_s^i)\left(-Y_s^i+U(s,\hat{X}_s^i,\mathcal{L}(\hat{X}_s^i))\right)\right]ds\\ &+2q\int_0^t\mathbf{E}_x\left[(X_s^i-\hat{X}_s^i)\left(\frac{1}{N_i}\sum_{j=1}^N g_{ij}X_s^j-\mathbf{E}_x[\hat{X}_s^i]\right)\right]ds-2q\int_0^t\mathbf{E}_x(X_s^i-\hat{X}_s^i)^2ds\\ \leq&C\int_0^t\mathbf{E}_x\left(-Y_s^i+U(s,X_s^i,\bar{\mu}_s^N)\right)^2ds\\ &+C\int_0^t\mathbf{E}_x\left(\frac{1}{N}\sum_{j=1}^N X_s^j-\mathbf{E}_x[\hat{X}_s^i]\right)^2ds+C\int_0^t\mathbf{E}_x\left(\frac{1}{N_i}\sum_{j=1}^N g_{ij}X_s^j-\mathbf{E}_x[\hat{X}_s^i]\right)^2ds,\end{aligned}$$

where in the last two inequalities we applied convexity argument and Gronwall's Lemma. Recalling that the 2-Wasserstein distance W_2 is given by

$$W_2(\mu,\mu')=\inf_\gamma\left\{\int_{\mathbb{R}\times\mathbb{R}}|u-u'|^2\gamma(du,du');\quad\gamma(\cdot\times\mathbb{R})=\mu,\quad\gamma(\mathbb{R}\times\cdot)=\mu'\right\}^{1/2},$$

we know there exists a constant C independent of N and t such that

$$\mathbf{E}_x[\hat{X}_t^i-X_t^i]^2\leq C\int_0^t\mathbf{E}_x[Y_s^i-\bar{Y}_s^i]^2ds+C\int_0^t\mathbf{E}_x[W_2(\bar{\mu}_s^N,\mu_s)^2]ds+C\int_0^t\mathbf{E}_x[W_2(\bar{\mu}_s^{N,i},\mu_s)^2]ds$$

Taking the mean over i yields,

$$\mathbf{E}_x[W_2(\hat{\mu}_t^N, \bar{\mu}_t^N)^2] \leq \delta_N + C\int_0^t \mathbf{E}_x[W_2(\bar{\mu}_s^N, \mu_s)^2]ds + C\int_0^t \mathbf{E}_x[W_2(\bar{\mu}_s^{N,i}, \mu_s)^2]ds, \quad (3.37)$$

where $\hat{\mu}_t^N$ is defined as the empirical distribution of an independent and identically distributed sample of law, i.e.

$$\hat{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\hat{X}_t^i}.$$

For this reason, apparently,

$$\lim_{N\to\infty} \mathbf{E}_x[W_2(\hat{\mu}_t^N, \mu_t)^2] = 0,$$

and

$$\sup_{0\leq t\leq T} \mathbf{E}_x[W_2(\hat{\mu}_t^N, \mu_t)^2] \leq 2\sup_{0\leq t\leq T} \mathbf{E}_x[X_t^2] < \infty.$$

By Lebesgue's dominated convergence theorem and equicontinuity argument as in Delarue [79], one can show that

$$\lim_{N\to\infty} \sup_{0\leq t\leq T} \mathbf{E}_x[W_2(\hat{\mu}_t^N, \mu_t)^2] = 0. \quad (3.38)$$

The triangle inequality implies

$$\mathbf{E}_x[W_2(\bar{\mu}_s^{N,i}, \mu_s)^2] \leq \mathbf{E}_x[W_2(\bar{\mu}_s^{N,i}, \bar{\mu}_s^N)^2] + \mathbf{E}_x[W_2(\bar{\mu}_s^N, \mu_s)^2] \quad (3.39)$$

and

$$\mathbf{E}_x[W_2(\bar{\mu}_s^N, \mu_s)^2] \leq \mathbf{E}_x[W_2(\hat{\mu}_s^N, \bar{\mu}_s^N)^2] + \mathbf{E}_x[W_2(\hat{\mu}_s^N, \mu_s)^2]. \quad (3.40)$$

Therefore, by Gronwall's lemma, inequality (3.37) gives

$$\mathbf{E}_x[W_2(\hat{\mu}_t^N, \bar{\mu}_t^N)^2] \leq \delta_N + C\int_0^t \mathbf{E}_x[W_2(\hat{\mu}_s^N, \mu_s)^2]ds + C\int_0^t \mathbf{E}_x[W_2(\bar{\mu}_s^{N,i}, \bar{\mu}_s^N)^2]ds. \quad (3.41)$$

Recalling that

$$\bar{\mu}_t^{N,i} = \frac{1}{N_i}\sum_{j=1}^N g_{i,j}\delta_{X_t^j}, \quad \bar{\mu}_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}.$$

Note that

$$\sup_{0\leq t\leq T} \mathbf{E}_x[W_2(\bar{\mu}_t^{N,i}, \bar{\mu}_t^N)^2] \leq \sup_{0\leq t\leq T} \mathbf{E}_x[\frac{1}{N_i}\sum_{j=1}^N g_{i,j}X_t^j - \frac{1}{N}\sum_{j=1}^N X_t^j]^2 < \infty.$$

Therefore, we have

$$\lim_{N\to\infty}\sup_{0\leq t\leq T} \mathbf{E}_x[W_2(\bar{\mu}_t^{N,i}, \bar{\mu}_t^N)^2] = 0. \quad (3.42)$$

By equation (3.41), we know that

$$\lim_{N\to\infty}\sup_{0\leq t\leq T} \mathbf{E}_x[W_2(\hat{\mu}_t^N, \bar{\mu}_t^N)^2] = 0. \quad (3.43)$$

By equation (3.40), we obtain

$$\lim_{N\to\infty}\sup_{0\leq t\leq T} \mathbf{E}_x[W_2(\bar{\mu}_t^N, \mu_t)^2] = 0. \quad (3.44)$$

as desired. □

3.6 Weakly Interacting Particle System on Random Graph

Recall that on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\})$, the edge random variables g_{ij} and random variable for the initial value of the state processes ξ^i are given, as well as an infinite collection of standard Brownian motions W^i, such that $\{W^i, \xi^i, g_{ij}\}$ are mutually independent, over where the state processes of finite player games are given by

$$dX_t^i = \left[-Y_t^i + q\left(\frac{1}{N_i}\sum_{j=1}^N g_{ij}X_t^j - X_t^i\right)\right]dt + \sigma dW_t^i.$$

Now, in order to achieve a higher level analysis (from the law of large numbers) in terms of the central limit theorem, it is necessary to incorporate graph in the master equation. Specifically, on $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\})$, we define a new system of SDEs, starting at the same point as X_t^i i.e. $X_0^i = \widetilde{X}_0^i = \xi^i$, with $\mathbf{E}\xi^i = 0$ for simplicity,

$$d\widetilde{X}_t^{i,N} = \Big[- U(t, \widetilde{X}_t^{i,N}, \widetilde{\mu}_t^{N,i}) + q(\frac{1}{N_i}\sum_{j=1}^N g_{ij}\widetilde{X}_t^{j,N} - \widetilde{X}_t^{i,N})\Big]dt + \sigma dW_t^i, \tag{3.45}$$

where

$$\widetilde{\mu}_t^{N,i} = \frac{1}{N_i}\sum_{j=1}^N g_{ij}\delta_{\widetilde{X}_t^{j,N}}.$$

That is,

$$d\widetilde{X}_t^{i,N} = (\eta_t + q)(\frac{1}{N_i}\sum_{j=1}^N g_{ij}\widetilde{X}_t^{j,N} - \widetilde{X}_t^{i,N})dt + \sigma dW_t^i,$$

in matrix notation simply

$$d\widetilde{X}_t^N = -G \cdot \widetilde{X}_t^N dt + \sigma dW_t, \quad i = 1, \cdots, N \tag{3.46}$$

where

$$\widetilde{X}_t = \begin{pmatrix} \widetilde{X}_t^{1,N} \\ \vdots \\ \widetilde{X}_t^{N,N} \end{pmatrix}, \quad G = a \begin{pmatrix} 1 & -\frac{g_{12}}{N_1} & \cdots & -\frac{g_{1N}}{N_1} \\ -\frac{g_{21}}{N_2} & 1 & \cdots & -\frac{g_{2N}}{N_2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{g_{N1}}{N_N} & -\frac{g_{N2}}{N_N} & \cdots & 1 \end{pmatrix}, \quad dW_t = \begin{pmatrix} dW_t^1 \\ \vdots \\ dW_t^N \end{pmatrix}.$$

For simplicity, we firstly consider the case $(\eta_t + q)$ equals to a constant a and focus on the following weakly interacting particle system on random graph, which consists of a large number of nodes in which the state of each node is governed by a diffusion process that is influenced by the neighboring nodes:

$$\widetilde{X}_t^{i,N} = \xi^i + a\int_0^t \left(\sum_{j=1}^N \frac{g_{ij}}{N_i} \mathbb{1}_{\{N_i>0\}} \widetilde{X}_s^{j,N} - \widetilde{X}_s^{i,N} \right) ds + \int_0^t \sigma dW_s^i, \quad i = 1, \cdots, N. \quad (3.47)$$

Note that, here the "strength" of the interaction between a node and its neighbor is inversely proportional to the total number of neighbors of that node, which gives the so-called "weakly interacting". We also note that M or $M_.$ denotes a constant independent of N whose definition may change from one proof to another, in the sequel.

3.6.1 Law of Large Numbers

We create an infinite particle system, which are driven by the $(\xi^i, (W_t^i)_{0\le t\le T})$ as $(\widetilde{X}_t^{i,N})_{0\le t\le T}$, and evolves according to:

$$d\widetilde{X}_t^i = a(\int_{\mathbb{R}} v d\mu_t(v) - \widetilde{X}_t^i)dt + \sigma dW_t^i,$$

where $(\widetilde{X}_t^i)_{i\geq 1}$ are i.i.d and the law of $\widetilde{X}_t^i$ is μ_t. That is, $(\widetilde{X}_t^i)_{i\geq 1}$ have the same dynamics in the McKean-Vlasov type, simply

$$\widetilde{X}_t^i = \xi^i - a\int_0^t \widetilde{X}_s^i ds + \int_0^t \sigma dW_s^i, \quad i = 1, \cdots, N. \tag{3.48}$$

The following theorem will show that $\mathbf{E}\left[\max_{0\leq u\leq t} |\widetilde{X}_u^{i,N} - \widetilde{X}_u^i|\right]$ is of order at most $N^{-1/2}$, and the law of large numbers and propagation of chaos result holds following a standard argument [89].

Theorem 7. *One has*

$$\sup_{N\geq 1} \sqrt{N}\boldsymbol{E}\left[\max_{0\leq u\leq t} |\widetilde{X}_u^{i,N} - \widetilde{X}_u^i|\right] < \infty. \tag{3.49}$$

Proof. We have

$$\begin{aligned}
&\mathbf{E}\left[\max_{0\leq u\leq t} |\widetilde{X}_u^{i,N} - \widetilde{X}_u^i|\right] \\
\leq& a\int_0^t \mathbf{E}\left|\sum_{j=1}^N \frac{g_{ij}}{N_i}\mathbb{1}_{\{N_i>0\}}\widetilde{X}_s^{j,N} - \widetilde{X}_s^{i,N} + \widetilde{X}_s^i\right| ds \\
\leq& a\int_0^t \mathbf{E}\left|\sum_{j=1}^N \frac{g_{ij}}{N_i}\mathbb{1}_{\{N_i>0\}}(\widetilde{X}_s^{j,N} - \widetilde{X}_s^j)\right| + \mathbf{E}\left|\sum_{j=1}^N \frac{g_{ij}}{N_i}\mathbb{1}_{\{N_i>0\}}\widetilde{X}_s^j\right| + \mathbf{E}|\widetilde{X}_s^{i,N} - \widetilde{X}_s^i| ds \\
=& a\int_0^t ① + ② + ③ ds.
\end{aligned} \tag{3.50}$$

Let us use the fact that

$$\mathcal{L}(g_{ij}, N_i, \widetilde{X}_s^{i,N}, \widetilde{X}_s^i) = \mathcal{L}(g_{ji}, N_j, \widetilde{X}_s^{j,N}, \widetilde{X}_s^j)$$

to firstly tackle ①:

$$\begin{aligned}
① =&\mathbf{E}\left|\sum_{j=1}^{N}\frac{g_{ij}}{N_i}\mathbb{1}_{\{N_i>0\}}(\widetilde{X}_s^{j,N}-\widetilde{X}_s^{j})\right| \\
\leq&\mathbf{E}\left[\sum_{j=1}^{N}\frac{g_{ji}}{N_j}\mathbb{1}_{\{N_j>0\}}\left|\widetilde{X}_s^{i,N}-\widetilde{X}_s^{i}\right|\right] \\
=&\mathbf{E}\left[\left(\sum_{j=1}^{N}\frac{g_{ji}}{N_j}\mathbb{1}_{\{N_j>0\}}-1\right)\left|\widetilde{X}_s^{i,N}-\widetilde{X}_s^{i}\right|\right]+\mathbf{E}\left|\widetilde{X}_s^{i,N}-\widetilde{X}_s^{i}\right| \\
\leq&\left[\mathbf{E}\left(\sum_{j=1}^{N}\frac{g_{ji}}{N_j}\mathbb{1}_{\{N_j>0\}}-1\right)^2\mathbf{E}\left|\widetilde{X}_s^{i,N}-\widetilde{X}_s^{i}\right|^2\right]^{1/2}+\mathbf{E}\left|\widetilde{X}_s^{i,N}-\widetilde{X}_s^{i}\right| \\
\leq&(M_1④)^{1/2}+\mathbf{E}\left|\widetilde{X}_s^{i,N}-\widetilde{X}_s^{i}\right|,
\end{aligned} \tag{3.51}$$

where M_1 a constant independent of N thanks to the finite second moment of $\widetilde{X}_s^{i,N}$ and $\widetilde{X}_s^{i}$, and

$$④ = \mathbf{E}\left(\sum_{j=1}^{N}\frac{g_{ji}}{N_j}\mathbb{1}_{\{N_j>0\}}-1\right)^2.$$

Next, we are going to use the independence of (g_{ji}, N_j) and (g_{ki}, N_k) for $j \neq k$ to tackle ④.

$$
\begin{aligned}
④ =&\mathbf{E}\left(\sum_{j=1}^{N}\frac{g_{ji}}{N_j}\mathbb{1}_{\{N_j>0\}}-1\right)^2\\
=&\mathbf{E}\left[\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{g_{ji}}{N_j}\frac{g_{ki}}{N_k}\mathbb{1}_{\{N_j>0\}}\mathbb{1}_{\{N_k>0\}}\right]-2\mathbf{E}\left[\sum_{j=1}^{N}\mathbb{1}_{\{N_j>0\}}\frac{g_{ji}}{N_j}\right]+1\\
=&\sum_{j=1}^{N}\mathbf{E}\left[\frac{g_{ji}}{N_j^2}\mathbb{1}_{\{N_j>0\}}\right]+\mathbf{E}\left[\sum_{j=1}^{N}\sum_{k=1,k\neq j}^{N}\frac{g_{ji}}{N_j}\frac{g_{ki}}{N_k}\mathbb{1}_{\{N_j>0\}}\mathbb{1}_{\{N_k>0\}}\right]\\
&-2\sum_{j=1}^{N}\mathbf{E}\left[\mathbb{1}_{\{N_j>0\}}\frac{g_{ji}}{N_j}\right]+1\\
=&\sum_{j=1}^{N}\mathbf{E}\left[\frac{g_{ji}}{N_j^2}\mathbb{1}_{\{N_j>0\}}\right]+\left[\sum_{j=1}^{N}\sum_{k=1,k\neq j}^{N}\mathbf{E}\left[\frac{g_{ji}}{N_j}\mathbb{1}_{\{N_j>0\}}\right]\mathbf{E}\left[\frac{g_{ki}}{N_k}\mathbb{1}_{\{N_k>0\}}\right]\right]\\
&-2\mathbf{P}(N_i>0)+1\\
=&\mathbf{E}\left[\frac{\mathbb{1}_{\{N_i>0\}}}{N_i}\right]+⑤-2\mathbf{P}(N_i>0)+1.
\end{aligned}
\tag{3.52}
$$

Let us firstly take care of the first term in (3.52).

$$
\begin{aligned}
\mathbf{E}\left[\frac{\mathbb{1}_{\{N_i>0\}}}{N_i}\right]\leq&\mathbf{E}\left[\frac{2}{1+N_i}\right]\\
=&\sum_{k=0}^{N-1}\frac{2}{1+k}\begin{pmatrix}N-1\\k\end{pmatrix}p^k(1-p)^{N-1-k}\\
=&2\sum_{k=0}^{N-1}\frac{(N-1)!}{(k+1)!(N-1-k)!}p^k(1-p)^{N-1-k}\\
=&\frac{2}{p}\sum_{m=1}^{N}\frac{1}{N}\begin{pmatrix}N\\m\end{pmatrix}p^m(1-p)^{N-m}\\
\leq&\frac{2}{pN}.
\end{aligned}
\tag{3.53}
$$

Then, for ⑤ in (3.52), we have

$$
\begin{aligned}
⑤ =& \sum_{j=1}^{N} \sum_{k=1, k\neq j}^{N} \mathbf{E}\left[\frac{g_{ji}}{N_j}\mathbb{1}_{\{N_j>0\}}\right]\mathbf{E}\left[\frac{g_{ki}}{N_k}\mathbb{1}_{\{N_k>0\}}\right] \\
\leq& \sum_{j=1}^{N} \sum_{k=1}^{N} \mathbf{E}\left[\frac{g_{ji}}{N_j}\mathbb{1}_{\{N_j>0\}}\right]\mathbf{E}\left[\frac{g_{ki}}{N_k}\mathbb{1}_{\{N_k>0\}}\right] \\
=& \left(\sum_{j=1}^{N} \mathbf{E}\left[\frac{g_{ji}}{N_j}\mathbb{1}_{\{N_j>0\}}\right]\right)^2 \\
=& \left(\sum_{j=1}^{N} \mathbf{E}\left[\frac{g_{ij}}{N_i}\mathbb{1}_{\{N_i>0\}}\right]\right)^2 \\
=& \mathbf{P}^2(N_i>0).
\end{aligned}
\tag{3.54}
$$

Plug in the above two results to in (3.52), we have

$$
\begin{aligned}
④ \leq& \frac{2}{pN} + (\mathbf{P}(N_i>0)-1)^2 \\
=& \frac{2}{pN} + \mathbf{P}^2(N_i=0) \\
=& \frac{2}{pN} + (1-p)^{2(N-1)}
\end{aligned}
\tag{3.55}
$$

So far, we have finished the analysis of ① in (3.50).

In the following, we analyze ② in (3.50), using the fact that the graph is independent of $\widetilde{X}_s^i$ and $\mathbf{E}\widetilde{X}_s^i = 0$.

$$
\begin{aligned}
②^2 \leq & \mathbf{E}\left(\sum_{j=1}^{N} \frac{g_{ij}}{N_i} \mathbb{1}_{\{N_i>0\}} \widetilde{X}_s^j\right)^2 \\
= & \mathbf{E}\left(\sum_{j=1}^{N} \frac{g_{ji}}{N_i} \mathbb{1}_{\{N_i>0\}} \widetilde{X}_s^j\right)^2 \\
= & \mathbf{E}\sum_{j=1}^{N}\sum_{k=1}^{N} \frac{g_{ji}}{N_i}\frac{g_{ki}}{N_i} \mathbb{1}_{\{N_i>0\}} \widetilde{X}_s^j \widetilde{X}_s^k \\
= & \sum_{j=1}^{N}\sum_{k=1}^{N} \mathbf{E}\left(\frac{g_{ji}}{N_i}\frac{g_{ki}}{N_i} \mathbb{1}_{\{N_i>0\}}\right) \mathbf{E}\left(\widetilde{X}_s^j \widetilde{X}_s^k\right) \\
= & \sum_{j=1}^{N} \mathbf{E}\left(\frac{g_{ji}^2}{N_i^2} \mathbb{1}_{\{N_i>0\}}\right) \mathbf{E}\left(\widetilde{X}_s^j\right)^2 \\
& + \sum_{j=1}^{N}\sum_{k=1,k\neq j}^{N} \mathbf{E}\left(\frac{g_{ji}}{N_i}\frac{g_{ki}}{N_i} \mathbb{1}_{\{N_i>0\}}\right) \mathbf{E}\left(\widetilde{X}_s^j \widetilde{X}_s^k\right) \\
= & \sum_{j=1}^{N} \mathbf{E}\left(\frac{g_{ji}}{N_i^2} \mathbb{1}_{\{N_i>0\}}\right) \mathbf{E}(\widetilde{X}_s^j)^2 \\
\leq & M_2 \mathbf{E}\left(\frac{\mathbb{1}_{\{N_i>0\}}}{N_i}\right) \\
\leq & \frac{2M_2}{pN},
\end{aligned}
\tag{3.56}
$$

where the last two inequalities are achieved by the fact that $\widetilde{X}_s^i$ has finite second moment and the last inequality in equation (3.53).

Plug the above results into equation (3.50), we have

$$\begin{aligned}
&\mathbf{E} \max_{0\le u\le t} |\widetilde{X}_u^{i,N} - \widetilde{X}_u^{i}| \\
\le& 2a \int_0^t \mathbf{E} \max_{0\le u\le t} \left| \widetilde{X}_u^{j,N} - \widetilde{X}_u^{j} \right| ds + \left(M_1 \left[\frac{2}{pN} + (1-p)^{2(N-1)} \right] \right)^{1/2} Ta + \sqrt{\frac{2M_2}{pN}} Ta \\
\le& 2a \int_0^t \mathbf{E} \max_{0\le u\le t} \left| \widetilde{X}_u^{j,N} - \widetilde{X}_u^{j} \right| ds + \frac{M}{\sqrt{pN}},
\end{aligned} \tag{3.57}$$

where M depends on a, p and T.

By Gronwall's lemma, we have

$$\mathbf{E}\left[\max_{0\le u\le t} |\widetilde{X}_u^{i,N} - \widetilde{X}_u^{i}| \right] \le \frac{M}{\sqrt{pN}} e^{2at}. \tag{3.58}$$

Therefore,

$$\sup_{N\ge 1} \sqrt{N} \mathbf{E}\left[\max_{0\le u\le t} |\widetilde{X}_u^{i,N} - \widetilde{X}_u^{i}| \right] < \infty. \tag{3.59}$$

□

Note that, here the result naturally generate to $\max_{i\in\mathbb{N}}$, since for all i,

$$\mathbf{E}\left[\max_{0\le u\le t} |\widetilde{X}_u^{i,N} - \widetilde{X}_u^{i}| \right]$$

are the same by symmetry.

3.6.2 Fluctuation and Central Limit Theorem

In this section, we establish a functional central limit theorem that characterizes the limiting fluctuations of $\widetilde{X}^{i,N}$ around its law of large numbers limit. Specifically, we show that, the family $\{\mathcal{U}^N(\phi)\}$ converges weakly to a mean 0 Gaussian field $\{\mathcal{U}(\phi)\}$, as N goes to infinity, in the sense of convergence of finite dimensional distributions, where

$$\mathcal{U}^N(\phi) = \frac{\phi(\widetilde{X}_T^{1,N}) + \phi(\widetilde{X}_T^{2,N}) + \cdots + \phi(\widetilde{X}_T^{N,N})}{\sqrt{N}},$$

for $\phi \in L_c^2(\mathscr{C}, \mu)$, a family of functions on the path space that are suitably centered and have appropriate integrability properties, which will be defined precisely in the proof of this functional central limit theorem (Theorem 8).

The proof relies on a change of measure technique using Girsanov's theorem, which goes back to the classical works of [90] and [91]. Let us firstly recall that on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\})$, the edge random variables g_{ij} and random variable for the initial value of the state processes $\widetilde{X}_0^{i,N}$ are given, as well as an infinite collection of standard Brownian motions W^i, such that $\{W^i, \widetilde{X}_0^{i,N}, g_{ij}\}$ are mutually independent. Let $V^i = (W^i, \widetilde{X}^i)$,

$$\widetilde{\mathcal{F}}_t^N = \sigma\{V^1(s), V^2(s), \cdots, V^N(s), \{g_{ij}, 1 \le i, j \le N\}, 0 \le s \le t\}$$

and

$$\mathbf{P}^N = \mathcal{L}(V^1, V^2, \cdots, V^N, \{g_{ij}, 1 \le i, j \le N\}).$$

Define a new probability measure $\mathbf{Q}^N$ by

$$\frac{d\mathbf{Q}^N}{d\mathbf{P}^N} = \exp(J^N(T)),$$

where $\exp(J^N(T))$ is an $\widetilde{\mathcal{F}}_t^N$-martingale under $\mathbf{P}^N$ given by

$$J^N(t) = J^{N,1}(t) - \frac{1}{2}J^{N,2}(t), \quad t \in [0,T],$$

with

$$J^{N,1}(t) = \frac{a}{\sigma}\sum_{i=1}^N \int_0^t \left(\frac{1}{N_i}\sum_{j=1}^N g_{ij}\mathbb{1}_{\{N_i>0\}}\widetilde{X}_s^j\right) dW_s^i,$$

$$J^{N,2}(t) = \frac{a^2}{\sigma^2}\sum_{i=1}^N \int_0^t \left(\frac{1}{N_i}\sum_{j=1}^N g_{ij}\mathbb{1}_{\{N_i>0\}}\widetilde{X}_s^j\right)^2 ds.$$

By Girsanov Theorem, $(\widetilde{X}^1, \cdots, \widetilde{X}^N, g_{ij})$ has the same distribution under $\mathbf{Q}^N$ as $(\widetilde{X}^{1,N}, \cdots, \widetilde{X}^{N,N}, g_{ij})$ under $\mathbf{P}$. Therefore, define

$$U^N(\phi) = \frac{\phi(\widetilde{X}_T^1) + \phi(\widetilde{X}_T^2) + \cdots + \phi(\widetilde{X}_T^N)}{\sqrt{N}}$$

for $\phi \in L_c^2(\mathscr{C}, \mu)$, and we have

$$\begin{aligned}\mathbf{E}\exp(i\mathcal{U}^N(\phi)) =& \mathbf{E}^{\mathbf{Q}^N}\exp\left[iU^N(\phi)\right]\\ =& \mathbf{E}^{\mathbf{P}^N}\exp\left[iU^N(\phi) + J^{N,1}(T) - \frac{1}{2}J^{N,2}(T)\right]\end{aligned}$$

where i is the imaginary number.

Now, we can see that the original CLT term $\mathcal{U}^N(\phi)$ with graph involved in is replaced with the corresponding term $U^N(\phi)$ generated by the i.i.d. system without graph, however the tradeoff is the additional martingale term $J^{N,1}(T)$ and quadratic variation term $J^{N,2}(T)$, which both contain graph. In the following, we firstly analyze their asymptotic behaviors as N goes to infinity.

Preliminary Analysis of $J^{N,1}(t)$:

In the following proposition, we give the preliminary result regarding the term $J^{N,1}(t)$.

Proposition 13. *For*

$$J^{N,1}(t) = \frac{a}{\sigma}\sum_{i=1}^{N}\int_0^t \left(\frac{1}{N_i}\sum_{j=1}^{N} g_{ij}\mathbb{1}_{\{N_i>0\}}\widetilde{X}_s^j\right) dW_s^i,$$

we have

$$\begin{aligned}J^{N,1}(T) =& \frac{a}{\sigma}\frac{1}{N}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\int_0^T \widetilde{X}_s^j dW_s^i \\ &+ \frac{a}{\sigma}\frac{1}{pN}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}(g_{ij}-p)\widetilde{X}_s^j dW_s^i + R_1\end{aligned} \tag{3.60}$$

with

$$\boldsymbol{E}^{\boldsymbol{P}^N} R_1^2 \leq \frac{M\log N}{N}.$$

Proof.

$$\begin{aligned}J^{N,1}(t) =& \frac{a}{\sigma}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^t \left(\frac{1}{N_i}g_{ij}\mathbb{1}_{\{N_i>0\}}\widetilde{X}_s^j\right) dW_s^i \\ =& \frac{a}{\sigma}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\int_0^t \left(\frac{1}{pN}\right) g_{ij}\widetilde{X}_s^j dW_s^i \\ &+ \underbrace{\frac{a}{\sigma}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\int_0^t \left(\frac{\mathbb{1}_{\{N_i>0\}}}{N_i} - \frac{1}{pN}\right) g_{ij}\widetilde{X}_s^j dW_s^i}_{①}\end{aligned} \tag{3.61}$$

Because $\left(\frac{\mathbb{1}_{\{N_i>0\}}}{N_i}-\frac{1}{pN}\right)g_{ij}$ and $\widetilde{X}^j_s$ are independent, we have

$$\begin{aligned}
\mathbf{E}^{\mathbf{P}^N}①^2 =& \frac{a^2}{\sigma^2}\sum_{i=1}^N\sum_{\substack{j=1\\j\neq i}}^N\int_0^T \mathbf{E}^{\mathbf{P}^N}\left[\left(\frac{\mathbb{1}_{\{N_i>0\}}}{N_i}-\frac{1}{pN}\right)g_{ij}\widetilde{X}^j_s\right]^2 ds \\
=& \frac{a^2}{\sigma^2}\sum_{i=1}^N\sum_{\substack{j=1\\j\neq i}}^N\int_0^T \mathbf{E}^{\mathbf{P}^N}\left[\left(\frac{\mathbb{1}_{\{N_i>0\}}}{N_i}-\frac{1}{pN}\right)^2 g_{ij}^2\right]\mathbf{E}^{\mathbf{P}^N}\left[\widetilde{X}^j_s\right]^2 ds \\
\leq& M_1\sum_{i=1}^N\sum_{\substack{j=1\\j\neq i}}^N \mathbf{E}^{\mathbf{P}^N}\left(\frac{\mathbb{1}_{\{N_i>0\}}}{N_i}-\frac{1}{pN}\right)^2 \qquad (3.62)\\
\leq& M_1\mathbf{E}^{\mathbf{P}^N}\frac{(pN\mathbb{1}_{\{N_i>0\}}-N_i)^2}{p^2N_i^2} \\
=& M_1\left[\frac{1}{p^2}\mathbf{P}^N(N_i=0)+\underbrace{\mathbf{E}^{\mathbf{P}^N}\frac{(pN-N_i)^2}{p^2N_i^2}\mathbb{1}_{\{N_i>0\}}}_{②}\right],
\end{aligned}$$

where

$$\begin{aligned}
② =& \mathbf{E}^{\mathbf{P}^N}\frac{(pN-N_i)^2}{p^2N_i^2}\mathbb{1}_{\{N_i>0\}}\mathbb{1}_{\{|N_i-pN|>\sqrt{2(N-1)\log N}\}} \\
&+\mathbf{E}^{\mathbf{P}^N}\frac{(pN-N_i)^2}{p^2N_i^2}\mathbb{1}_{\{N_i>0\}}\mathbb{1}_{\{|N_i-pN|\leq\sqrt{2(N-1)\log N}\}} \\
\leq& \frac{N^2}{p^2}\mathbf{P}^N(|N_i-pN|>\sqrt{2(N-1)\log N}) \\
&+\frac{2(N-1)\log N}{p^2(pN-\sqrt{2(N-1)\log N})^2} \qquad (3.63)\\
\leq& \underbrace{\frac{N^2}{p^2}\frac{2}{N^4}}_{\text{by Hoeffding's inequality}}+\frac{2(N-1)\log N}{p^2\frac{p^2}{4}N^2} \\
\leq& M_2\frac{\log N}{N}.
\end{aligned}$$

Note that, in the first inequality above,

$$\frac{(pN - N_i)^2}{p^2 N_i^2} \mathbb{1}_{\{N_i > 0\}} \leq \frac{N^2}{p^2}$$

achieved by the fact that $pN \leq N$ and $1 \leq N_i \leq N$. Therefore,

$$\begin{aligned} \mathbf{E}^{\mathbf{P}^N} ①^2 &\leq M_1 \left[\frac{1}{p^2}(1-p)^{N-1} + \frac{M_2 \log N}{N} \right] \\ &\leq \frac{M_3 \log N}{N}. \end{aligned} \tag{3.64}$$

In sum, we have

$$\begin{aligned} J^{N,1}(T) =& \frac{a}{\sigma} \frac{1}{N} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_0^T \widetilde{X}_s^j dW_s^i \\ &+ \frac{a}{\sigma} \frac{1}{pN} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} (g_{ij} - p) \widetilde{X}_s^j dW_s^i + R_1 \end{aligned}$$

with

$$\mathbf{E}^{\mathbf{P}^N} R_1^2 \leq \frac{M \log N}{N}.$$

□

Next, define

$$U_1 = \frac{a}{\sigma} \frac{1}{pN} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} (g_{ij} - p) \widetilde{X}_s^j dW_s^i.$$

Note that, here $\mathbf{E}^{\mathbf{P}^N}(U_1 | \mathbf{W}, \mathbf{X}) = 0$.

Proposition 14. *One has*

$$\boldsymbol{E}_{\boldsymbol{W},\boldsymbol{X}}^{\boldsymbol{P}^N} \left[e^{itU_1} \right] \to e^{-\frac{1}{2}t^2 \sigma_1^2} \to e^{-\frac{1}{2}t^2 \tilde{\sigma}^2},$$

where

$$\sigma_1^2 := \boldsymbol{E}_{\mathbf{W},\boldsymbol{X}}^{\boldsymbol{P}^N}\left[U_1^2\right]$$

and

$$\widetilde{\sigma}^2 := \frac{1-p}{p}\frac{a^2}{\sigma^2}\int_0^T \sigma_s^2 ds, \quad \sigma_s^2 = \boldsymbol{E}^{\boldsymbol{P}^N}(\widetilde{X}_s^1)^2.$$

Proof. Firstly, rewrite U_1 as

$$U_1 = \frac{a}{\sigma}\frac{1}{pN}\sum\sum_{i<j}\left[\int_0^T (g_{ij}-p)\widetilde{X}_s^j dW_s^i + \int_0^T (g_{ji}-p)\widetilde{X}_s^i dW_s^j\right].$$

Note that, for different pairs (i,j) with $i<j$, (g_{ij}) are independent. Then

$$\begin{aligned}
\sigma_1^2 =& \mathbf{E}_{\mathbf{W},\widetilde{\mathbf{X}}}^{\mathbf{P}^N}\left[U_1^2\right] \\
=& \frac{a^2}{\sigma^2}\frac{1}{p^2N^2}\mathbf{E}_{\mathbf{W},\mathbf{X}}^{\mathbf{P}^N}\sum\sum_{i<j}\left[\int_0^T (g_{ij}-p)\widetilde{X}_s^j dW_s^i + \int_0^T (g_{ji}-p)\widetilde{X}_s^i dW_s^j\right]^2.
\end{aligned}$$

Hence, by Itô Isometry, we have

$$\begin{aligned}
\mathbf{E}^{\mathbf{P}^N}\left[\sigma_1^2\right] =& \frac{a^2}{\sigma^2}\frac{1}{p^2N^2}\sum\sum_{i<j}\left[\mathbf{E}^{\mathbf{P}^N}\left(\int_0^T (g_{ij}-p)\widetilde{X}_s^j dW_s^i\right)^2 + \mathbf{E}^{\mathbf{P}^N}\left(\int_0^T (g_{ji}-p)\widetilde{X}_s^i dW_s^j\right)^2\right]^2 \\
=& \frac{a^2}{\sigma^2}\frac{2}{p^2N^2}\sum\sum_{i<j}\int_0^T \mathbf{E}^{\mathbf{P}^N}\left[(g_{12}-p)\widetilde{X}_s^1\right]^2 ds \\
=& \frac{a^2}{\sigma^2}\frac{2}{p^2N^2}\binom{N}{2}p(1-p)\int_0^T \mathbf{E}^{\mathbf{P}^N}(\widetilde{X}_s^1)^2 ds \\
=& \frac{1-p}{p}\frac{a^2}{\sigma^2}\frac{N-1}{N}\int_0^T \sigma_s^2 ds \\
=& \frac{N-1}{N}\widetilde{\sigma}^2,
\end{aligned}$$

where

$$\widetilde{\sigma}^2 = \frac{1-p}{p}\frac{a^2}{\sigma^2}\int_0^T \sigma_s^2 ds.$$

Next, let us analyze the variance

$$\begin{aligned}
\mathbf{Var}^{\mathbf{P}^N}\left[\sigma_1^2\right] =& \mathbf{E}^{\mathbf{P}^N}\left[\sigma_1^2 - \mathbf{E}^{\mathbf{P}^N}\left(\sigma_1^2\right)\right]^2 \\
=&\frac{a^4}{\sigma^4}\frac{1}{p^4N^4}\sum\sum_{i<j}\mathbf{E}^{\mathbf{P}^N}\left\{\mathbf{E}^{\mathbf{P}^N}_{\mathbf{W},\mathbf{X}}\left[\int_0^T(g_{ij}-p)\widetilde{X}_s^j dW_s^i + \int_0^T(g_{ji}-p)\widetilde{X}_s^i dW_s^j\right]^2\right. \\
&\left. - \mathbf{E}^{\mathbf{P}^N}\left[\int_0^T(g_{ij}-p)\widetilde{X}_s^j dW_s^i + \int_0^T(g_{ji}-p)\widetilde{X}_s^i dW_s^j\right]^2\right\} \\
\leq& M\frac{a^4}{\sigma^4}\frac{1}{p^4N^4}\binom{N}{2}.
\end{aligned}$$

That is, $\mathbf{Var}^{\mathbf{P}^N}\left[\sigma_1^2\right] \to 0$ as $N \to \infty$, which implies $\sigma_1^2 \to \widetilde{\sigma}^2$ as $N \to \infty$ in L^2.

Therefore, by Lyapunov CLT, we have the result as desired. □

Define

$$U_2 = \frac{a}{\sigma}\frac{1}{N}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\int_0^T \widetilde{X}_s^j dW_s^i.$$

By Theorem 1 in [92], we have

$$U_2 \Longrightarrow \frac{a}{\sigma}I_2(h_2),$$

where

$$h_2(\omega,\omega') = \frac{1}{2}(h(\omega,\omega') + h((\omega',\omega))),$$

$$h(\omega,\omega') = \int_0^T \widetilde{X}_s(\omega)dW_s(\omega')$$

and $I_2(h_2)$ is h_2's multiple wiener integral. Here, the symbol " $\Longrightarrow$ " is used to denote convergence in distribution.

Preliminary Analysis of $J^{N,2}(t)$:

Before we proceed with the analysis regarding the term $J^{N,2}(t)$, let us firstly prove the following Lemma, which will be used later.

Lemma 4. *One has*

$$\boldsymbol{E}^{\boldsymbol{P}^N}\left(\frac{N^2\mathbb{1}_{\{N_1>0\}}}{N_1^2}-\frac{1}{p^2}\right)^2 \leq M\frac{\log N}{N}. \tag{3.65}$$

Proof. Conditional on the value of N_1, one has

$$\begin{aligned}
&\mathbf{E}^{\mathbf{P}^N}\left(\frac{N^2\mathbb{1}_{\{N_1>0\}}}{N_1^2}-\frac{1}{p^2}\right)^2\\
=&\mathbf{E}^{\mathbf{P}^N}\left(\left(\frac{N^2\mathbb{1}_{\{N_1>0\}}}{N_1^2}-\frac{1}{p^2}\right)^2\middle|\{N_1>0\}\right)\mathbf{P}^N(N_1>0)\\
&+\mathbf{E}^{\mathbf{P}^N}\left(\left(\frac{N^2\mathbb{1}_{\{N_1>0\}}}{N_1^2}-\frac{1}{p^2}\right)^2\middle|\{N_1=0\}\right)\mathbf{P}^N(N_1=0)\\
=&\underbrace{\mathbf{E}^{\mathbf{P}^N}\left(\left(\frac{N^2}{N_1^2}-\frac{1}{p^2}\right)^2\middle|\{N_1>0\}\right)}_{\mathcal{A}}\mathbf{P}^N(N_1>0)+\frac{1}{p^4}\mathbf{P}^N(N_1=0).
\end{aligned} \tag{3.66}$$

In the following, we assume $N_1>0$ and work on term $\mathcal{A}$.

$$\begin{aligned}
\mathcal{A}=&\mathbf{E}^{\mathbf{P}^N}\left(\frac{(p^2N^2-N_i^2)^2}{p^4N_i^4}\right)\mathbb{1}_{\{|N_i-pN|>\sqrt{3(N-1)\log N}\}}\\
&+\mathbf{E}^{\mathbf{P}^N}\left(\frac{(p^2N^2-N_i^2)^2}{p^4N_i^4}\right)\mathbb{1}_{\{|N_i-pN|\leq\sqrt{3(N-1)\log N}\}}\\
\leq&\frac{N^4}{p^4}\mathbf{P}^N\left(|N_i-pN|>\sqrt{3(N-1)\log N}\right)\\
&+\frac{3(N-1)\log N\cdot((p+1)N)^2}{p^4\cdot\left(pN-\sqrt{3(N-1)\log N}\right)^4}\\
\leq&\frac{2N^4}{p^4N^6}+\frac{(3\times 4)N^3\log N}{p^4\cdot\left(\frac{pN}{2}\right)^4}\\
\leq&M\frac{\log N}{N},
\end{aligned} \tag{3.67}$$

where we used Hoeffding's inequality and the quantity that

$$\begin{aligned}(p^2N^2 - N_i^2)^2 =&(pN - N_i)^2 \cdot (pN + N_i)^2 \\ \leq& 3(N-1)\log N \cdot ((1+p)N)^2.\end{aligned} \tag{3.68}$$

□

Now we are ready to give the analysis result regarding the term $J^{N,2}(t)$ in the following proposition.

Proposition 15. *For*

$$J^{N,2}(t) = \frac{a^2}{\sigma^2}\sum_{i=1}^{N}\int_0^t \left(\frac{1}{N_i}\sum_{j=1}^{N} g_{ij}\mathbb{1}_{\{N_i>0\}}\widetilde{X}_s^j\right)^2 ds,$$

we have

$$J^{N,2}(T) = \frac{a^2}{p\sigma^2}\int_0^T (\sigma_s)^2 ds + \frac{a^2}{\sigma^2}\cdot\frac{N-2}{N^2}\sum\sum_{i\neq j}\int_0^T \widetilde{X}_s^i\widetilde{X}_s^j ds + R_2 \tag{3.69}$$

with

$$\boldsymbol{E}^{\boldsymbol{P}^N} R_2^2 \leq \frac{M\log N}{N},$$

where

$$\sigma_s^2 = \boldsymbol{E}^{\boldsymbol{P}^N}(\widetilde{X}_s^i)^2, \quad i = 1, \cdots, N.$$

Proof.

$$
\begin{aligned}
J^{N,2}(T) =& \frac{a^2}{\sigma^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\int_0^T\left(\frac{\mathbb{1}_{\{N_i>0\}}}{N_i^2}g_{ij}g_{ik}\widetilde{X}_s^j\widetilde{X}_s^k\right)ds \\
=& \underbrace{\frac{a^2}{\sigma^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^T\left(\frac{\mathbb{1}_{\{N_i>0\}}}{N_i^2}\right)g_{ij}(\widetilde{X}_s^j)^2 ds}_{③} \\
&+ \underbrace{\frac{a^2}{\sigma^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}\int_0^T\frac{\mathbb{1}_{\{N_i>0\}}}{N_i^2}g_{ij}g_{ik}\widetilde{X}_s^j\widetilde{X}_s^k ds}_{④}
\end{aligned}
\tag{3.70}
$$

Step 1. Let us firstly deal with term ③

$$
\begin{aligned}
③ =& \sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^T\left(\frac{\mathbb{1}_{\{N_i>0\}}}{N_i^2}\right)g_{ij}\mathbf{E}^{\mathbf{P}^N}(\widetilde{X}_s^j)^2 ds \\
&+ \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^T\left(\frac{\mathbb{1}_{\{N_i>0\}}}{N_i^2}\right)g_{ij}\left[(\widetilde{X}_s^j)^2-\mathbf{E}^{\mathbf{P}^N}(\widetilde{X}_s^j)^2\right]ds}_{⑤}
\end{aligned}
\tag{3.71}
$$

Now let us work on the major term of ③

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbb{1}_{\{N_i>0\}}}{N_i^2}g_{ij}\int_0^T\mathbf{E}^{\mathbf{P}^N}(\widetilde{X}_s^j)^2 ds \\
=&\int_0^T\sigma_s^2 ds\cdot\sum_{i=1}^{N}\frac{\mathbb{1}_{\{N_i>0\}}}{N_i} \\
=&\frac{1}{p}\int_0^T\sigma_s^2 ds + \underbrace{\frac{a^2}{\sigma^2}\int_0^T\sigma_s^2 ds\sum_{i=1}^{N}\left(\frac{\mathbb{1}_{\{N_i>0\}}}{N_i}-\frac{1}{Np}\right)}_{\mathcal{B}},
\end{aligned}
\tag{3.72}
$$

where

$$
\sigma_s^2 = \mathbf{E}^{\mathbf{P}^N}(\widetilde{X}_s^i)^2, \quad i=1,\cdots,N.
$$

By previous results in (3.62) and (3.63), we have

$$\mathbf{E}^{\mathbf{P}^N}\mathcal{B}^2 \le \frac{M\log N}{N}. \tag{3.73}$$

Next, we work on the term ⑤.

$$\begin{aligned}\mathbf{E}^{\mathbf{P}^N}⑤^2 =&\mathbf{E}^{\mathbf{P}^N}\sum_{i_1=1}^N\sum_{j_1=1}^N\sum_{i_2=1}^N\sum_{j_2=1}^N\int_0^T\left(\frac{\mathbb{1}_{\{N_{i_1}>0\}}}{N_{i_1}^2}\right)g_{i_1j_1}\left[(\widetilde{X}_s^{j_1})^2-\mathbf{E}^{\mathbf{P}^N}(\widetilde{X}_s^{j_1})^2\right]ds\\ &\times\int_0^T\left(\frac{\mathbb{1}_{\{N_{i_2}>0\}}}{N_{i_2}^2}\right)g_{i_2j_2}\left[(\widetilde{X}_t^{j_2})^2-\mathbf{E}^{\mathbf{P}^N}(\widetilde{X}_t^{j_2})^2\right]dt.\end{aligned} \tag{3.74}$$

Note that, the $j_1 \neq j_2$ terms will disappear, for the reason that $\widetilde{X}_s$ and g_{ij} are independent and

$$\mathbf{E}^{\mathbf{P}^N}\left[(\widetilde{X}_t^{j_2})^2-\mathbf{E}^{\mathbf{P}^N}(\widetilde{X}_t^{j_2})^2\right]=0.$$

Denote

$$\lambda=\mathbf{E}^{\mathbf{P}^N}\left\{\int_0^T\left[(\widetilde{X}_t^1)^2-\mathbf{E}^{\mathbf{P}^N}(\widetilde{X}_t^1)^2\right]ds\right\}^2$$

$$
\begin{aligned}
\mathbf{E}^{\mathbf{P}^N} ⑤^2 =&\lambda \sum_{i_1=1}^N \sum_{i_2=1}^N \sum_{j=1}^N \mathbf{E}^{\mathbf{P}^N} \left[\frac{\mathbb{1}_{\{N_{i_1}>0\}}}{N_{i_1}^2} \frac{\mathbb{1}_{\{N_{i_2}>0\}}}{N_{i_2}^2} g_{i_1 j} g_{i_2 j} \right] \\
=&\lambda \sum_{i=1}^N \sum_{j=1}^N \mathbf{E}^{\mathbf{P}^N} \left[\frac{\mathbb{1}_{\{N_i>0\}}}{N_i^4} g_{ij} \right] \\
&+ \lambda \sum_{i_1=1}^N \sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^N \sum_{j=1}^N \mathbf{E}^{\mathbf{P}^N} \left[\frac{\mathbb{1}_{\{N_{i_1}>0\}}}{N_{i_1}^2} \frac{\mathbb{1}_{\{N_{i_2}>0\}}}{N_{i_2}^2} g_{i_1 j} g_{i_2 j} \right] \\
\leq&\lambda \sum_{i=1}^N \mathbf{E}^{\mathbf{P}^N} \left[\frac{\mathbb{1}_{\{N_i>0\}}}{N_i^3} \right] + \lambda \sum_{i_1=1}^N \sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^N \sum_{j=1}^N \mathbf{E}^{\mathbf{P}^N} \left[\frac{\mathbb{1}_{\{N_{i_1}>0\}}}{N_{i_1}^2} \frac{\mathbb{1}_{\{N_{i_2}>0\}}}{N_{i_2}^2} \right] \\
\leq&N\lambda \mathbf{E}^{\mathbf{P}^N} \left[\frac{2^3}{(N_1+1)^3} \right] + N^3 \lambda \mathbf{E}^{\mathbf{P}^N} \left[\frac{\mathbb{1}_{\{N_1>0\}}}{N_1^2} \frac{\mathbb{1}_{\{N_2>0\}}}{N_2^2} \right] \\
\leq& \underbrace{8N\lambda \frac{3^3}{N^3 p^3}}_{\text{Lemma 5.1 in [80]}} + N^3 \lambda \mathbf{E}^{\mathbf{P}^N} \left[\frac{4}{(\widetilde{N}_1+1)^2} \frac{4}{(\widetilde{N}_2+1)^2} \right] \\
=&\frac{8 \cdot 3^3 \lambda}{N^2 p^3} + \underbrace{16 N^3 \lambda \left(\frac{2^2}{(N-1)^2 p^2} \right)^2}_{\text{Lemma 5.1 in [80]}} \\
\leq&\frac{M_4}{N},
\end{aligned}
\tag{3.75}
$$

where $\widetilde{N}_1 := N_1 - g_{12}$ and $\widetilde{N}_2 := N_2 - g_{21}$, and thus $\widetilde{N}_1$ and $\widetilde{N}_2$ are independent, in addition $\widetilde{N}_1, \widetilde{N}_2 \sim \text{Binomial}(N-2, p)$. Therefore,

$$
③ = \frac{1}{p} \int_0^T \sigma_s^2 ds + ⑤ + \mathcal{B}, \tag{3.76}
$$

where

$$
\mathbf{E}^{\mathbf{P}^N} [⑤^2 + \mathcal{B}^2] \leq \frac{M \log N}{N}.
$$

Step 2. Now, let us handle term ④

$$
\begin{aligned}
④ =& \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}\frac{1}{p^2N^2}g_{ij}g_{ik}\int_0^T \widetilde{X}_s^j\widetilde{X}_s^k ds \\
&+ \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}\left(\frac{\mathbb{1}_{\{N_i>0\}}}{N_i^2}-\frac{1}{p^2N^2}\right)g_{ij}g_{ik}\int_0^T \widetilde{X}_s^j\widetilde{X}_s^k ds}_{⑥}.
\end{aligned}
\tag{3.77}
$$

Next, let us firstly take care of term ⑥ in term ④:

$$
\begin{aligned}
\mathbf{E}^{\mathbf{P}^N}⑥^2 \leq& \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}\mathbf{E}^{\mathbf{P}^N}\left[\left(\frac{\mathbb{1}_{\{N_{i_1}>0\}}}{N_{i_1}^2}-\frac{1}{p^2N^2}\right)\left(\frac{\mathbb{1}_{\{N_{i_2}>0\}}}{N_{i_2}^2}-\frac{1}{p^2N^2}\right)g_{i_1j}g_{i_1k}g_{i_2j}g_{i_2k}\right] \\
&\times \underbrace{\mathbf{E}^{\mathbf{P}^N}\left(\int_0^T \widetilde{X}_s^j\widetilde{X}_s^k ds\right)^2}_{\mu:=} \\
\leq&\frac{\mu}{N^2}\mathbf{E}^{\mathbf{P}^N}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\left(\frac{N^2\mathbb{1}_{\{N_{i_1}>0\}}}{N_{i_1}^2}-\frac{1}{p^2}\right)\left(\frac{N^2\mathbb{1}_{\{N_{i_2}>0\}}}{N_{i_2}^2}-\frac{1}{p^2}\right) \\
\leq&\frac{\mu}{2N^2}\mathbf{E}^{\mathbf{P}^N}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\left[\left(\frac{N^2\mathbb{1}_{\{N_{i_1}>0\}}}{N_{i_1}^2}-\frac{1}{p^2}\right)^2+\left(\frac{N^2\mathbb{1}_{\{N_{i_2}>0\}}}{N_{i_2}^2}-\frac{1}{p^2}\right)^2\right] \\
=&\frac{\mu}{N^2}N^2\mathbf{E}^{\mathbf{P}^N}\left(\frac{N^2\mathbb{1}_{\{N_1>0\}}}{N_1^2}-\frac{1}{p^2}\right)^2 \\
\leq&\frac{M\log N}{N},
\end{aligned}
\tag{3.78}
$$

where the last inequality follows by Lemma 4.

Next, we go back to the major term of ④

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}\frac{1}{p^2N^2}g_{ij}g_{ik}\int_0^T \widetilde{X}_s^j\widetilde{X}_s^k ds\\
=&\sum_{j\neq i}\sum_{k\neq i,j}\frac{1}{N^2}\int_0^T \widetilde{X}_s^j\widetilde{X}_s^k ds+\underbrace{\sum_{j\neq i}\sum_{k\neq i,j}\frac{g_{ij}g_{ik}-p^2}{p^2N^2}\int_0^T \widetilde{X}_s^j\widetilde{X}_s^k ds}_{⑦},
\end{aligned}
\tag{3.79}
$$

where ⑦ can be bounded as the following

$$
\begin{aligned}
\mathbf{E}^{\mathbf{P}^N}⑦^2=&\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{\substack{j=1\\j\neq i_1,i_2}}^{N}\sum_{\substack{k=1\\k\neq i_1,i_2,j}}^{N}\mathbf{E}^{\mathbf{P}^N}\left[\frac{g_{i_1j}g_{i_1k}-p^2}{p^2N^2}\cdot\frac{g_{i_2j}g_{i_2k}-p^2}{p^2N^2}\right]\\
&\qquad\times\mathbf{E}^{\mathbf{P}^N}\left(\int_0^T \widetilde{X}_s^j\widetilde{X}_s^k ds\right)^2\\
=&\mu\sum_{j\neq i}\sum_{k\neq i,j}\mathbf{E}^{\mathbf{P}^N}\left(\frac{g_{ij}g_{ik}-p^2}{p^2N^2}\right)^2\\
\leq&\frac{M}{N}.
\end{aligned}
\tag{3.80}
$$

Therefore,

$$
④=\sum_{j\neq i}\sum_{k\neq i,j}\frac{1}{N^2}\int_0^T \widetilde{X}_s^j\widetilde{X}_s^k ds+⑥+⑦, \tag{3.81}
$$

where

$$
\mathbf{E}^{\mathbf{P}^N}[⑥^2+⑦^2]\leq\frac{M\log N}{N}.
$$

In sum of (3.76) and (3.81), we have

$$
J^{N,2}(T)=\frac{a^2}{p\sigma^2}\int_0^T(\sigma_s)^2ds+\frac{a^2}{\sigma^2}\cdot\frac{N-2}{N^2}\sum_{j\neq k}\int_0^T \widetilde{X}_s^j\widetilde{X}_s^k ds+R_2
$$

with

$$\mathbf{E}^{\mathbf{P}^N} R_2^2 \le \frac{M \log N}{N}.$$

□

Next, define

$$U_3 = \frac{a^2}{\sigma^2} \cdot \frac{N-2}{N^2} \sum \sum_{j \neq i} \int_0^T \widetilde{X}_s^i \widetilde{X}_s^j ds.$$

By Theorem 1 in [92], we have

$$U_3 \Longrightarrow \frac{a^2}{\sigma^2} I_2(\widetilde{h}_2),$$

where

$$\widetilde{h}_2(\omega, \omega') = \int_0^T \widetilde{X}_s(\omega) \widetilde{X}_s(\omega') ds$$

and $I_2(\widetilde{h}_2)$ is $\widetilde{h}_2$'s multiple wiener integral.

Derivation of Central Limit Theorem:

Denote $\mathscr{C} := \mathbb{C}([0,T] : \mathbb{R})$, the space of continuous functions from $[0,T]$ to $\mathbb{R}$, endowed with the uniform topology; denote $\mathscr{C}_2 := \mathscr{C} \times \mathscr{C}$; denote $\nu \in \mathcal{P}(\mathscr{C}_2)$, the common law of $(W^i, \widetilde{X}^i)$ for $i = 1, \cdots, N$, where the dynamics of $\widetilde{X}^i$ are given by (3.48); recall that μ is the law of $\widetilde{X}^i$ and let $L^2(\mathscr{C}, \mu)$ be the space of measurable functions ϕ such that $\int_{\mathscr{C}} \phi^2(x)\mu(dx) < \infty$; denote $L_c^2(\mathscr{C}, \mu)$ as the space of all functions ϕ such that $\int_{\mathscr{C}} \phi(x)\mu(dx) = 0$; define the canonical processes $V_* := (W_*(\omega), X_*(\omega)) := (\omega_1, \omega_2)$ for $\omega = (\omega_1, \omega_2) \in \mathscr{C}_2$.

Recall that

$$U^N(\phi) = \frac{\phi(\widetilde{X}_T^1) + \phi(\widetilde{X}_T^2) + \cdots + \phi(\widetilde{X}_T^N)}{\sqrt{N}},$$

where $\phi \in L_c^2(\mathscr{C}, \mu)$. Denote

$$\varphi(t, s_1, s_2) = \exp\left(-\frac{1}{2}t^2\widetilde{\sigma}^2\right)\psi(s_1, s_2),$$

where $\psi(s_1, s_2)$ is the characteristic function of $\left(Z(\phi), \frac{a}{\sigma}I_2(h_2)) - \frac{a^2}{2\sigma^2}I_2(\widetilde{h}_2)\right)$, and

$$Z(\phi) \sim N\left(0, \mathbf{E}^{\mathbf{P}^N}\left(\phi(\widetilde{X}_T^1)\right)^2\right).$$

Denote $\varphi_U(t, s_1, s_2)$ as the characteristic function of $\left(U_1, U^N(\phi), U_2 - \frac{1}{2}U_3\right)$. It follows from Theorem 1 in [92] that

$$\mathbf{E}^{\mathbf{P}^N}\exp\left(i\left[[s_1U^N(\phi) + s_2\left(U_2 - \frac{1}{2}U_3\right)\right]\right) \to \psi(s_1, s_2),$$

as $N \to \infty$. Therefore,

$$\begin{aligned}
&\varphi_U(t, s_1, s_2) - \varphi(t, s_1, s_2)\\
=&\mathbf{E}^{\mathbf{P}^N}\left\{\exp\left(itU_1 + i\left[s_1U^N(\phi) + s_2\left(U_2 - \frac{1}{2}U_3\right)\right]\right) - \exp\left(-\frac{1}{2}t^2\widetilde{\sigma}^2\right)\psi(s_1, s_2)\right\}\\
=&\mathbf{E}^{\mathbf{P}^N}\left\{\exp\left(i\left[s_1U^N(\phi) + s_2\left(U_2 - \frac{1}{2}U_3\right)\right]\right)\mathbf{E}^{\mathbf{P}^N}_{\mathbf{W},\widetilde{\mathbf{X}}}\exp(itU_1) - \exp\left(-\frac{1}{2}t^2\widetilde{\sigma}^2\right)\psi(s_1, s_2)\right\}\\
=&\mathbf{E}^{\mathbf{P}^N}\left\{\exp\left(i\left[s_1U^N(\phi) + s_2\left(U_2 - \frac{1}{2}U_3\right)\right]\right)\left(\mathbf{E}^{\mathbf{P}^N}_{\mathbf{W},\widetilde{\mathbf{X}}}\exp(itU_1) - \exp\left(-\frac{1}{2}t^2\widetilde{\sigma}^2\right)\right)\right\}\\
&+\left\{\mathbf{E}^{\mathbf{P}^N}\exp\left(i\left[s_1U^N(\phi) + s_2\left(U_2 - \frac{1}{2}U_3\right)\right]\right) - \psi(s_1, s_2)\right\}\exp\left(-\frac{1}{2}t^2\widetilde{\sigma}^2\right),
\end{aligned}$$

which converges to 0 as N goes to infinity.

Therefore,

$$(U_1, U^N(\phi), U_2 - \frac{1}{2}U_3) \Longrightarrow (Z, Z(\phi), \frac{a}{\sigma}I_2(h_2)) - \frac{a^2}{2\sigma^2}I_2(\widetilde{h}_2)),$$

with $Z \sim N(0, \widetilde{\sigma}^2)$. Hence,

$$\begin{aligned}(U^N(\phi), J^{N,1}(T) - \frac{1}{2}J^{N,2}(T)) \Longrightarrow &(Z(\phi), Z + \frac{a}{\sigma}I_2(h_2)) - \frac{a^2}{2\sigma^2}I_2(\widetilde{h}_2)) + \frac{a^2}{2p\sigma^2}\int_0^T \sigma_s^2 ds) \\ =&(Z(\phi), Z + \frac{1}{2}I_2(\eta) + \frac{a^2}{2p\sigma^2}\int_0^T \sigma_s^2 ds)\end{aligned}$$

where

$$\eta(\omega,\omega') = \frac{a}{\sigma}\left(h(\omega,\omega') + h(\omega',\omega)\right) - \frac{a^2}{2\sigma^2}\psi_2(\omega,\omega').$$

Define integral operator A on $\mathcal{L}^2(\mathscr{C}_2, \nu)$ as

$$Af(\omega) = \frac{a}{\sigma}\int_{\mathscr{C}_2}\left(\int_0^T X(\omega)dW(\omega')\right) f(\omega')v(d\omega'),$$

for $f \in \mathcal{L}^2(\mathscr{C}_2, \nu)$ and $\omega \in \mathscr{C}_2$. Then

$$\begin{aligned}\text{Trace}(AA^*) =&\frac{a^2}{\sigma^2}\int_{\mathscr{C}_2\times\mathscr{C}_2}\left(\int_0^T X(\omega)dW(\omega')\right)^2 v(d\omega)v(d\omega') \\ =&\frac{a^2}{\sigma^2}\int_0^T \mathbf{E}^{\mathbf{P}^N}\widetilde{X}_s^2 ds \\ =&\frac{a^2}{\sigma^2}\int_0^T \sigma_s^2 ds.\end{aligned}$$

Note that

$$\widetilde{\sigma}^2 = \frac{a^2}{2p\sigma^2}\int_0^T \sigma_s^2 ds - \text{Trace}(AA^*).$$

Next, recall that

$$\mathcal{U}^N(\phi) = \frac{\phi(\widetilde{X}_T^{1,N}) + \phi(\widetilde{X}_T^{2,N}) + \cdots + \phi(\widetilde{X}_T^{N,N})}{\sqrt{N}}.$$

Also recall that, by Girsanov Theorem, $(\widetilde{X}^1, \cdots, \widetilde{X}^N, g_{ij})$ has the same distribution under

$\mathbf{Q}^N$ as $(\widetilde{X}^{1,N}, \cdots, \widetilde{X}^{N,N}, g_{ij})$ under $\mathbf{P}$, we have

$$\begin{aligned}
\mathbf{E}\exp(i\mathcal{U}^N(\phi)) =& \mathbf{E}^{\mathbf{P}^N}\exp\left[iU^N(\phi) + J^{N,1}(T) - \frac{1}{2}J^{N,2}(T)\right] \\
\to& \mathbf{E}^{\mathbf{P}^N}\exp\left[iZ(\phi) + \frac{1}{2}I_2(\eta) - \frac{1}{2}\text{Trace}(AA^*) + Z - \frac{1}{2}\widetilde{\sigma}^2\right] \\
=& \mathbf{E}^{\mathbf{P}^N}\exp\left[iZ(\phi) + \frac{1}{2}I_2(\eta) - \frac{1}{2}\text{Trace}(AA^*)\right] \cdot \mathbf{E}^{\mathbf{P}^N}\exp\left[Z - \frac{1}{2}\widetilde{\sigma}^2\right] \\
=& \mathbf{E}^{\mathbf{P}^N}\exp\left[iZ(\phi) + \frac{1}{2}I_2(\eta) - \frac{1}{2}\text{Trace}(AA^*)\right] \\
=& \exp\left[-\frac{1}{2}\left\|(I-A)^{-1}\boldsymbol{\phi}\right\|_{L^2(\mathscr{C}_2,\nu)}\right],
\end{aligned}$$

for $\boldsymbol{\phi} = \phi(X_*) \in L^2(\mathscr{C}_2, \nu)$, where we have used the property that Z and $(Z(\phi), I_2(\eta))$ are independent,

$$\mathbf{E}^{\mathbf{P}^N}\exp[Z] = \exp\left[\frac{1}{2}\widetilde{\sigma}^2\right]$$

and the last equality follows by the classical result in Shiga–Tanaka [90].

Theorem 8 (Central Limit Theorem). *$\mathcal{U}^N(\phi)$ converges to a mean 0 Gaussian field $\{\mathcal{U}(\phi) : \phi \in L^2_c(\mathscr{C}, \mu)\}$ in the sense of finite dimensional distribution such that,*

$$\boldsymbol{E}(\mathcal{U}(\phi)\mathcal{U}(\psi)) = \langle (I-A)^{-1}\boldsymbol{\phi}, (I-A)^{-1}\boldsymbol{\psi}\rangle_{L^2(\mathscr{C}_2,\nu)},$$

for $\phi, \psi \in L^2_c(\mathscr{C}, \mu)$ and $\boldsymbol{\phi}, \boldsymbol{\psi} \in L^2(\mathscr{C}_2, \nu)$, as N goes to infinity.

3.7 Convergence Results under Random Graph

3.7.1 Preliminary result for convergence

For $(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^N$ and $i = 1, 2, \cdots, N$, we set

$$u^{N,i}(t, \mathbf{x}) := U(t, x^i, \bar{\mu}_x^{N,i}), \quad \bar{\mu}_x^{N,i} = \frac{1}{N_i} \sum_{j=1}^{N} g_{ij} \delta_{x^j}.$$

In the following, we show that $u^{N,i}$ is almost a solution to the quasilinear parabolic system (3.20):

Proposition 16. *One has for any $i \in 1, \cdots, N$,*

$$\begin{aligned}
&\partial_t u^{N,i}(t, \boldsymbol{x}) + \sum_{j=1}^{N} \left(-u^{N,j}(t, \boldsymbol{x}) + q \left(\frac{1}{N_j} \sum_{k=1}^{N} g_{jk} x^k - x^j \right) \right) \partial_{x^j} u^{N,i}(t, \boldsymbol{x}) \\
&+ \frac{\sigma^2}{2} \sum_{j=1}^{N} \partial_{x^j}^2 u^{N,i}(t, \boldsymbol{x}) - q u^{N,i}(t, \boldsymbol{x}) + (\epsilon - q^2) \left(x^i - \frac{1}{N_i} \sum_{j=1}^{N} g_{ij} x^j \right) + r^{N,i}(t, \boldsymbol{x}) = 0, \\
&u^{N,i}(T, \boldsymbol{x}) = c \left(x^i - \frac{1}{N_i} \sum_{j=1}^{N} g_{ij} x^j \right),
\end{aligned} \tag{3.82}$$

with

$$r^{N,i}(t, \boldsymbol{x}) = \eta_t(\eta_t + q) \left(-\frac{1}{N_i} \sum_{k=1}^{N} g_{ik} x^k + \frac{1}{N_i} \sum_{j=1}^{N} g_{ij} \frac{1}{N_j} \sum_{l=1}^{N} g_{jl} x^l \right).$$

Remark 9. *Note that even under complete graph,*

$$
\begin{aligned}
r^{N,i}(t,\boldsymbol{x}) =& \eta_t(\eta_t+q)\left(-\frac{1}{N-1}\sum_{\substack{k=1\\k\neq i}}^{N}x^k+\frac{1}{(N-1)^2}\sum_{\substack{j=1\\j\neq i}}^{N}\sum_{\substack{l=1\\l\neq j}}^{N}x^l\right)\\
\neq& 0.
\end{aligned}
$$

Proof. Recall that the master equation is given by

$$
\begin{aligned}
&\partial_t U(t,x,\mu)+\left[-U(t,x,\mu)+q\left(\int_{\mathbb{R}} v d\mu(v)-x\right)\right]\partial_x U(t,x,\mu)+\frac{\sigma^2}{2}\partial_x^2 U(t,x,\mu)\\
&+\int_{\mathbb{R}}\left[-U(t,v,\mu)+q\left(\int_{\mathbb{R}} v d\mu(v)-v\right)\right]\partial_\mu U(t,x,\mu)(v)d\mu(v)\\
&+\frac{\sigma^2}{2}\int_{\mathbb{R}}\partial_v\partial_\mu U(t,x,\mu)(v)d\mu(v)-qU(t,x,\mu)+(\epsilon-q^2)\left(x-\int_{\mathbb{R}} v d\mu(v)\right)=0,
\end{aligned}
$$

therefore, one has at a point $(t,x^i,\bar{\mu}_x^{N,i})$:

$$
\begin{aligned}
&\partial_t u^{N,i}+\left[-u^{N,i}+q\left(\int_{\mathbb{R}} v d\bar{\mu}_x^{N,i}(v)-x^i\right)\right]\partial_x U(t,x^i,\bar{\mu}_x^{N,i})+\frac{\sigma^2}{2}\partial_x^2 U(t,x^i,\bar{\mu}_x^{N,i})\\
&+\int_{\mathbb{R}}\left[-U(t,v,\bar{\mu}_x^{N,i})+q\left(\int_{\mathbb{R}} v d\bar{\mu}_x^{N,i}(v)-v\right)\right]\partial_\mu U(t,x^i,\bar{\mu}_x^{N,i})(v)d\bar{\mu}_x^{N,i}(v)\\
&+\frac{\sigma^2}{2}\int_{\mathbb{R}}\partial_v\partial_\mu U(t,x^i,\bar{\mu}_x^{N,i})(v)d\bar{\mu}_x^{N,i}(v)-qu^{N,i}+(\epsilon-q^2)\left(x^i-\int_{\mathbb{R}} v d\bar{\mu}_x^{N,i}(v)\right)=0.
\end{aligned}
$$

With the explicit form

$$
U(t,x^i,\bar{\mu}_x^{N,i})=\eta_t(x^i-\frac{1}{N_i}\sum_{j=1}^{N}g_{ij}x^j),
$$

we have

$$\begin{aligned}
\partial_{x^i} u^{N,i}(t,\mathbf{x}) =& \partial_x U(t,x^i,\bar{\mu}_x^{N,i}),\\
\partial_{x^j} u^{N,i}(t,\mathbf{x}) =& \frac{g_{ij}}{N_i}\partial_\mu U(t,x^i,\bar{\mu}_x^{N,i})(x^j), \quad j\neq i,\\
\partial^2_{x^i} u^{N,i}(t,\mathbf{x}) =& \partial^2_x U(t,x^i,\bar{\mu}_x^{N,i}),\\
\partial^2_{x^j} u^{N,i}(t,\mathbf{x}) =& \frac{g_{ij}}{N_i}\partial_v\partial_\mu U(t,x^i,\bar{\mu}_x^{N,i})(x^j), \quad j\neq i.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}}\Big[-U(t,v,\bar{\mu}_x^{N,i})+q\left(\int_{\mathbb{R}} v d\bar{\mu}_x^{N,i}(v)-v\right)\Big]\partial_\mu U(t,x^i,\bar{\mu}_x^{N,i})(v)d\bar{\mu}_x^{N,i}(v)\\
=&\frac{1}{N_i}\sum_{j\neq i} g_{ij}\Big[-U(t,x^j,\bar{\mu}_x^{N,i})+q\left(\int_{\mathbb{R}} v d\bar{\mu}_x^{N,i}(v)-x^j\right)\Big]\partial_\mu U(t,x^i,\bar{\mu}_x^{N,i})(x^j)\\
=&\sum_{j\neq i}\Big[-U(t,x^j,\bar{\mu}_x^{N,i})+q\left(\int_{\mathbb{R}} v d\bar{\mu}_x^{N,i}(v)-x^j\right)\Big]\partial_{x^j} u^{N,i}(t,\mathbf{x})\\
=&\sum_{j\neq i}\Big[-U(t,x^j,\bar{\mu}_x^{N,j})+q\left(\int_{\mathbb{R}} v d\bar{\mu}_x^{N,j}(v)-x^j\right)\Big]\partial_{x^j} u^{N,i}(t,\mathbf{x})+r^{N,i}(t,\mathbf{x}),
\end{aligned} \tag{3.83}$$

where

$$\begin{aligned}
r^{N,i}(t,\mathbf{x}) :=& \sum_{j\neq i}\Big[-U(t,x^j,\bar{\mu}_x^{N,i})+q\left(\int_{\mathbb{R}} v d\bar{\mu}_x^{N,i}(v)-x^j\right)\\
&+U(t,x^j,\bar{\mu}_x^{N,j})-q\left(\int_{\mathbb{R}} v d\bar{\mu}_x^{N,j}(v)-x^j\right)\Big]\partial_{x^j} u^{N,i}(t,\mathbf{x}).
\end{aligned} \tag{3.84}$$

By the explicit form of $u^{N,i}(t,\mathbf{x})$, we know that, for $j\neq i$,

$$\partial_{x^j} u^{N,i}(t,\mathbf{x}) = -\frac{\eta_t}{N_i} g_{ij},$$

and

$$
\begin{aligned}
& - U(t, x^j, \bar{\mu}_x^{N,i}) + q\left(\int_{\mathbb{R}} v d\bar{\mu}_x^{N,i}(v) - x^j\right) + U(t, x^j, \bar{\mu}_x^{N,j}) - q\left(\int_{\mathbb{R}} v d\bar{\mu}_x^{N,j}(v) - x^j\right) \\
=& (\eta_t + q)\left(\frac{1}{N_i}\sum_{k=1}^N g_{ik}x^k - \frac{1}{N_j}\sum_{l=1}^N g_{jl}x^l\right).
\end{aligned}
\tag{3.85}
$$

Therefore, we know that

$$
\begin{aligned}
r^{N,i}(t,\mathbf{x}) =& -\frac{\eta_t(\eta_t+q)}{N_i}\sum_{j=1}^N g_{ij}\left(\frac{1}{N_i}\sum_{k=1}^N g_{ik}x^k - \frac{1}{N_j}\sum_{l=1}^N g_{jl}x^l\right) \\
=& \eta_t(\eta_t+q)\left(-\frac{1}{N_i}\sum_{k=1}^N g_{ik}x^k + \frac{1}{N_i}\sum_{j=1}^N g_{ij}\frac{1}{N_j}\sum_{l=1}^N g_{jl}x^l\right).
\end{aligned}
\tag{3.86}
$$

Next, note that

$$
\begin{aligned}
& \frac{\sigma^2}{2}\int_{\mathbb{R}} \partial_v\partial_\mu U(t, x^i, \bar{\mu}_x^{N,i})(v) d\bar{\mu}_x^{N,i}(v) \\
=& \frac{\sigma^2}{2N_i}\sum_{j\neq i} g_{ij}\partial_v\partial_\mu U(t, x^i, \bar{\mu}_x^{N,i})(x^j) = \sum_{j\neq i}\frac{\sigma^2}{2}\partial_{x^j}^2 u^{N,i}(t,\mathbf{x}).
\end{aligned}
\tag{3.87}
$$

Therefore,

$$
\begin{aligned}
& \partial_t u^{N,i}(t,\mathbf{x}) + \sum_{j=1}^N\left(-u^{N,j}(t,\mathbf{x}) + q\left(\frac{1}{N_j}\sum_{k=1}^N g_{jk}x^k - x^j\right)\right)\partial_{x^j}u^{N,i}(t,\mathbf{x}) \\
& + \frac{\sigma^2}{2}\sum_{j=1}^N \partial_{x^j}^2 u^{N,i}(t,\mathbf{x}) - qu^{N,i}(t,\mathbf{x}) + (\epsilon - q^2)\left(x^i - \frac{1}{N_i}\sum_{j=1}^N g_{ij}x^j\right) + r^{N,i}(t,\mathbf{x}) = 0,
\end{aligned}
\tag{3.88}
$$

with

$$r^{N,i}(t,\mathbf{x}) = \eta_t(\eta_t + q)\left(-\frac{1}{N_i}\sum_{k=1}^{N} g_{ik}x^k + \frac{1}{N_i}\sum_{j=1}^{N} g_{ij}\frac{1}{N_j}\sum_{l=1}^{N} g_{jl}x^l\right).$$

□

Let us firstly suppose the rate of convergence of term $r^{N,i}(t,\mathbf{x})$ is $\mathcal{C}(N)$, a function $\mathcal{C}(\cdot)$ depending on N. The importance of the bound of this term will be revealed in the following analysis. As an appetizer, we firstly give a preliminary analysis of bound $\mathcal{O}(N^{-1/2})$ in the following Remark.

Remark 10. *One has*

$$\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{E}\left(-\frac{1}{N_i}\sum_{k=1}^{N} g_{ik}X_s^k + \frac{1}{N_i}\sum_{j=1}^{N} g_{ij}\frac{1}{N_j}\sum_{l=1}^{N} g_{jl}X_s^l\right)^2 \le \frac{C}{\sqrt{N}}$$

and then

$$\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{E}\left(r^{N,i}(t,\boldsymbol{x})\right)^2 \le \frac{C}{\sqrt{N}}.$$

This can be seen from below:

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{E}\left(-\frac{1}{N_i}\sum_{k=1}^{N} g_{ik}X_s^k + \frac{1}{N_i}\sum_{j=1}^{N} g_{ij}\frac{1}{N_j}\sum_{l=1}^{N} g_{jl}X_s^l\right)^2\\
=&\boldsymbol{E}\frac{1}{N}\sum_{i=1}^{N}\left\{\frac{\mathbb{1}_{\{N_i\ge 1\}}}{N_i^2}\sum_{j=1}^{N}\sum_{l=1}^{N} g_{ij}g_{il}X_s^jX_s^l - \frac{2\mathbb{1}_{\{N_i\ge 1\}}}{N_i^2}\sum_{k=1}^{N} g_{ik}X_s^k\sum_{j=1}^{N} g_{ij}\frac{1}{N_j}\sum_{l=1}^{N} g_{il}X_s^l\right.\\
&\left.+\frac{\mathbb{1}_{\{N_i\ge 1\}}}{N_i^2}\left(\sum_{j=1}^{N} g_{ij}\frac{1}{N_j}\sum_{l=1}^{N} g_{jl}X_s^l\right)^2\right\}\\
=&\boldsymbol{E}\left(① - 2\times ② + ③\right),
\end{aligned} \tag{3.89}$$

with

$$
① = \frac{1}{N^2}\sum_{j,l=1}^{N}\left[\frac{1}{N}\sum_{i=1}^{N}\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}g_{ij}g_{il}-1\right]X_s^jX_s^l, \tag{3.90}
$$

$$
② = \frac{1}{N^2}\sum_{k,l=1}^{N}\left[\frac{1}{N}\sum_{i=1}^{N}\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}g_{ik}g_{il}\left(\frac{1}{N}\sum_{j=1}^{N}\frac{N}{N_j}\mathbb{1}_{\{N_j\geq 1\}}g_{ij}\right)-1\right]X_s^kX_s^l, \tag{3.91}
$$

Firstly, let's work on term ①. By Cauchy–Schwartz inequality, we have

$$
\begin{aligned}
\boldsymbol{E}(①) \leq & \frac{1}{N^2}\left\{\sum_{j,l=1}^{N}\boldsymbol{E}\left[\frac{1}{N}\sum_{i=1}^{N}\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}g_{ij}g_{il}-1\right]^2\right\}^{1/2}\cdot\left[\sum_{j,l=1}^{N}\boldsymbol{E}(X_s^jX_s^l)^2\right]^{1/2} \\
= & \left\{\frac{1}{N^2}\sum_{j,l=1}^{N}\boldsymbol{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}-\frac{1}{p^2}\right)g_{ij}g_{il}+\frac{1}{p^2}\left(\frac{1}{N}\sum_{i=1}^{N}g_{ij}g_{il}-p^2\right)\right]^2\right\}^{1/2} \\
& \cdot\frac{1}{N}\left[\sum_{j=1}^{N}\boldsymbol{E}(X_s^j)^2\right]^{1/2} \\
\leq & C\left\{\frac{1}{N^2}\sum_{j,l=1}^{N}\boldsymbol{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}-\frac{1}{p^2}\right)\right]+\frac{1}{p^4}\boldsymbol{E}\left(\frac{1}{N}\sum_{i=1}^{N}g_{ij}g_{il}-p^2\right)^2\right\}^{1/2} \\
\leq & C\left[\boldsymbol{E}\frac{1}{N}\sum_{i=1}^{N}\left(\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}-\frac{1}{p^2}\right)^2+\frac{C}{N}\right]^{1/2} \\
\leq & \frac{C}{\sqrt{N}}.
\end{aligned} \tag{3.92}
$$

Term ② can be tackled in the same way, using Cauchy–Schwartz inequality.

$$\begin{aligned}
\boldsymbol{E}(②) \leq & C\left\{\frac{1}{N^2}\sum_{k,l=1}^{N}\boldsymbol{E}\left[\frac{1}{N}\sum_{i=1}^{N}\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}g_{ik}g_{il}\left(\frac{1}{N}\sum_{j=1}^{N}\frac{N}{N_j}\mathbb{1}_{\{N_j\geq 1\}}g_{ij}\right)-1\right]^2\right\}^{1/2} \\
\leq & C\Bigg\{\frac{1}{N^2}\sum_{k,l=1}^{N}\boldsymbol{E}\Bigg[\frac{1}{N}\sum_{i=1}^{N}\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}g_{ik}g_{il}\left(\frac{1}{N}\sum_{j=1}^{N}\frac{N}{N_j}\mathbb{1}_{\{N_j\geq 1\}}g_{ij}-1\right) \\
& +\left(\frac{1}{N}\sum_{i=1}^{N}\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}g_{ik}g_{il}-1\right)\Bigg]^2\Bigg\}^{1/2} \\
\leq & C\Bigg\{\frac{1}{N^2}\sum_{k,l=1}^{N}\boldsymbol{E}\Bigg[\frac{1}{N}\sum_{i=1}^{N}\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}g_{ik}g_{il}\left(\frac{1}{N}\sum_{j=1}^{N}\frac{N}{N_j}\mathbb{1}_{\{N_j\geq 1\}}g_{ij}-1\right)+\frac{C}{\sqrt{N}}\Bigg]^2\Bigg\}^{1/2} \\
\leq & C\Bigg\{\frac{1}{N^2}\sum_{k,l=1}^{N}\boldsymbol{E}\Bigg[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}-\frac{1}{p^2}\right)\left(\frac{1}{N}\sum_{j=1}^{N}\frac{N}{N_j}\mathbb{1}_{\{N_j\geq 1\}}g_{ij}-1\right) \\
& +\frac{1}{p^2}\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N}\frac{N}{N_j}\mathbb{1}_{\{N_j\geq 1\}}g_{ij}-1\right)\Bigg]^2\Bigg\}^{1/2}+\frac{C}{\sqrt{N}} \\
= & \boldsymbol{E}\Bigg[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}-\frac{1}{p^2}\right)\left(\frac{1}{N}\sum_{j=1}^{N}\frac{N}{N_j}\mathbb{1}_{\{N_j\geq 1\}}g_{ij}-1\right)\Bigg]^2 \\
\leq & \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{E}\left(\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}-\frac{1}{p^2}\right)^2\left(\frac{1}{N}\sum_{j=1}^{N}\frac{N}{N_j}\mathbb{1}_{\{N_j\geq 1\}}g_{ij}-1\right)^2 \\
\leq & \left[\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{E}\left(\frac{N^2}{N_i^2}\mathbb{1}_{\{N_i\geq 1\}}-\frac{1}{p^2}\right)^4\right]^{1/2}\cdot\left[\left(\frac{1}{N}\sum_{j=1}^{N}\frac{N}{N_j}\mathbb{1}_{\{N_j\geq 1\}}g_{ij}-1\right)^4\right]^{1/2} \\
= & \text{Ⓘ}\cdot\text{Ⓘⓘ},
\end{aligned} \tag{3.93}$$

where

$$\text{Ⓘ} \leq \left(\frac{C}{N^2}\right)^{1/2} = \frac{C}{N}$$

and

$$
\begin{aligned}
① =&\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{E}\left[\frac{1}{N}\sum_{j=1}^{N}\left(\frac{N}{N_j}\mathbb{1}_{\{N_j\geq 1\}}-\frac{1}{p}\right)g_{ij}+\frac{1}{pN}\sum_{j=1}^{N}(g_{ij}-p)\right]^4\\
\leq&\frac{C}{N}\sum_{i=1}^{N}\boldsymbol{E}\left[\frac{1}{N}\sum_{j=1}^{N}\left(\frac{N}{N_j}\mathbb{1}_{\{N_j\geq 1\}}-\frac{1}{p}\right)\right]^4+\frac{C}{N}\sum_{i=1}^{N}\boldsymbol{E}\left[\frac{1}{pN}\sum_{j=1}^{N}(g_{ij}-p)\right]^4\\
\leq&\frac{C}{N}\sum_{i=1}^{N}\frac{1}{N}\sum_{j=1}^{N}\boldsymbol{E}\left(\frac{N}{N_j}\mathbb{1}_{\{N_j\geq 1\}}-\frac{1}{p}\right)^4+\frac{C}{Np^4}\sum_{i=1}^{N}\boldsymbol{E}\left[\frac{N_i}{N}-p\right]^4\\
\leq&\frac{C}{N^2}.
\end{aligned}
$$

Now, we know that

$$② \leq \frac{C}{\sqrt{N}}.$$

Similarly, we can prove that

$$③ \leq \frac{C}{\sqrt{N}}.$$

Proposition 17. *One has,*

$$\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{E}\left[u^{N,i}(t,\boldsymbol{X}_t)-v^{N,i}(t,\boldsymbol{X}_t)\right]^2\leq \mathcal{C}(N),$$

where $\mathcal{C}(N)$ is a function depending on N , the same as the bound of $r^{N,i}(t,\boldsymbol{x})$.

Proof. We prove this Proposition by firstly analyze under the realization of the random graph, that is we use the notation $\mathbb{E}$ to denote conditional expectation under graph, and then we take expectation $\mathbf{E}$ with respect to probability measure $\mathbf{P}$.

Recall that the system of FBSDEs of finite player games (3.17) is given by

$$
\begin{aligned}
dX_t^i &= \left[-Y_t^i + q\left(\frac{1}{N_i}\sum_{j=1}^N g_{ij}X_t^j - X_t^i\right)\right]dt + \sigma dW_t^i,\\
dY_t^i &= \left[qY_t^i - (\epsilon - q^2)\left(X_t^i - \frac{1}{N_i}\sum_{j=1}^N g_{ij}X_t^j\right)\right]dt + \sum_{j=1}^N Z_t^{i,j}dW_t^j,\\
X_0^i &= \xi_0^i, \quad Y_T^i = c\left(X_T^i - \frac{1}{N_i}\sum_{j=1}^N g_{ij}X_T^j\right).
\end{aligned}
$$

By Itô's formula and using the parabolic system equation (3.20), we have

$$
\begin{aligned}
&dv^{N,i}(t,\mathbf{X}_t)\\
=&\Bigg\{\partial_t v^{N,i}(t,\mathbf{X}_t) + \sum_{j=1}^N\left[-v^{N,j}(t,\mathbf{X}_t) + q\left(\frac{1}{N_j}\sum_{k=1}^N g_{jk}X_t^k - X_t^j\right)\right]\partial_{x^j}v^{N,i}(t,\mathbf{X}_t)\\
&+\frac{\sigma^2}{2}\sum_{j=1}^N \partial_{x^j}^2 v^{N,i}(t,\mathbf{X}_t)\Bigg\}dt + \sigma\sum_{j=1}^N \partial_{x^j}v^{N,i}(t,\mathbf{X}_t)dW_t^j\\
=&\left\{qv^{N,i}(t,\mathbf{X}_t) - (\epsilon - q^2)\left(X_t^i - \frac{1}{N_i}\sum_{j=1}^N g_{ij}X_t^j\right)\right\}dt + \sigma\sum_{j=1}^N \partial_{x^j}v^{N,i}(t,\mathbf{X}_t)dW_t^j.
\end{aligned}
\tag{3.94}
$$

By Itô's formula and using the result of Proposition 16, we have

$$
\begin{aligned}
&du^{N,i}(t,\mathbf{X}_t)\\
=&\Bigg\{\partial_t u^{N,i}(t,\mathbf{X}_t)+\sum_{j=1}^{N}\left[-v^{N,j}(t,\mathbf{X}_t)+q\left(\frac{1}{N_j}\sum_{k=1}^{N}g_{jk}X_t^k-X_t^j\right)\right]\partial_{x^j}u^{N,i}(t,\mathbf{X}_t)\\
&+\frac{\sigma^2}{2}\sum_{j=1}^{N}\partial_{x^j}^2u^{N,i}(t,\mathbf{X}_t)\Bigg\}dt+\sigma\sum_{j=1}^{N}\partial_{x^j}u^{N,i}(t,\mathbf{X}_t)dW_t^j\\
=&\Bigg\{qu^{N,i}(t,\mathbf{X}_t)-(\epsilon-q^2)\left(X_t^i-\frac{1}{N_i}\sum_{j=1}^{N}g_{ij}X_t^j\right)-r^{N,i}(t,\mathbf{X}_t)\Bigg\}dt\\
&+\sum_{j=1}^{N}\left[-v^{N,j}(t,\mathbf{X}_t)+u^{N,j}(t,\mathbf{X}_t)\right]\partial_{x^j}u^{N,i}(t,\mathbf{X}_t)dt\\
&+\sigma\sum_{j=1}^{N}\partial_{x^j}u^{N,i}(t,\mathbf{X}_t)dW_t^j.
\end{aligned}
\tag{3.95}
$$

Taking difference of the two equations above, we have

$$
\begin{aligned}
&du^{N,i}(t,\mathbf{X}_t)-v^{N,i}(t,\mathbf{X}_t)\\
=&q\left(u^{N,i}(t,\mathbf{X}_t)-v^{N,i}(t,\mathbf{X}_t)\right)dt-r^{N,i}(t,\mathbf{X}_t)dt\\
&+\sum_{j=1}^{N}\left[-v^{N,j}(t,\mathbf{X}_t)+u^{N,j}(t,\mathbf{X}_t)\right]\partial_{x^j}u^{N,i}(t,\mathbf{X}_t)dt\\
&+\sigma\sum_{j=1}^{N}\left(\partial_{x^j}u^{N,i}(t,\mathbf{X}_t)-\partial_{x^j}v^{N,i}(t,\mathbf{X}_t)\right)dW_t^j.
\end{aligned}
\tag{3.96}
$$

Taking the square and applying Itô's formula, we have

$$\begin{aligned}
& d\left[u^{N,i}(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right]^2 \\
=& 2q\left(u^{N,i}(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right)^2 dt \\
& - 2\left(u^{N,i}(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right) r^{N,i}(t,\mathbf{X}_t)dt \\
& + 2\left(u^{N,i}(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right)\sum_{j=1}^{N}\left[-v^{N,j}(t,\mathbf{X}_t) + u^{N,j}(t,\mathbf{X}_t)\right]\partial_{x^j}u^{N,i}(t,\mathbf{X}_t)dt \\
& + \sigma^2\sum_{j=1}^{N}(\partial_{x^j}u^{N,i}(t,\mathbf{X}_t) - \partial_{x^j}v^{N,i}(t,\mathbf{X}_t))^2 dt \\
& + 2\sigma\left(u^{N,i}(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right)\sum_{j=1}^{N}(\partial_{x^j}u^{N,i}(t,\mathbf{X}_t) - \partial_{x^j}v^{N,i}(t,\mathbf{X}_t))dW_t^j.
\end{aligned} \tag{3.97}$$

Integrating from t to T, taking expectation and considering the terminal condition

$$u^{N,i}(T,x) = v^{N,i}(T,x),$$

we have

$$\begin{aligned}
& \mathbb{E}\left[u^{N,i}(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right]^2 \\
=& -2q\int_t^T \mathbb{E}\left(u^{N,i}(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right)^2 ds \\
& + \underbrace{2\int_t^T \mathbb{E}\left[\left(u^{N,i}(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right) r^{N,i}(s,\mathbf{X}_s)\right] ds}_{\mathcal{M}} \\
& \underbrace{-2\int_t^T \mathbb{E}\left[\left(u^{N,i}(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right)\sum_{j=1}^{N}\left[-v^{N,j}(s,\mathbf{X}_s) + u^{N,j}(s,\mathbf{X}_s)\right]\partial_{x^j}u^{N,i}(s,\mathbf{X}_s)\right] ds}_{\mathcal{N}} \\
& - \sigma^2\sum_{j=1}^{N}\int_t^T \mathbb{E}(\partial_{x^j}u^{N,i}(s,\mathbf{X}_s) - \partial_{x^j}v^{N,i}(s,\mathbf{X}_s))^2 ds.
\end{aligned} \tag{3.98}$$

Recalling that, for $s \in [t, T]$,

$$u^{N,i}(s, \mathbf{X}_s) = \eta_s(X_s^i - \frac{1}{N_i} \sum_{j=1}^N g_{ij} X_s^j),$$

we have

$$\partial_{x^i} u^{N,i}(s, x) = \eta_s, \quad \partial_{x^j} u^{N,i}(s, x) = -\frac{\eta_s}{N_i} g_{ij}, \quad j \neq i.$$

Recalling that

$$r^{N,i}(t, x) = \eta_t(\eta_t + q) \left(-\frac{1}{N_i} \sum_{k=1}^N g_{ik} x^k + \frac{1}{N_i} \sum_{j=1}^N g_{ij} \frac{1}{N_j} \sum_{l=1}^N g_{jl} x^l \right),$$

and notice that

$$\sigma^2 \sum_{j=1}^N \int_t^T \mathbb{E} |\partial_{x^j} u^{N,i}(s, \mathbf{X}_s) - \partial_{x^j} v^{N,i}(s, \mathbf{X}_s)|^2 ds \geq 0$$

and

$$2q \int_t^T \mathbb{E} \left(u^{N,i}(s, \mathbf{X}_s) - v^{N,i}(s, \mathbf{X}_s) \right)^2 ds \geq 0,$$

we have

$$\mathbb{E} \left[u^{N,i}(t, \mathbf{X}_t) - v^{N,i}(t, \mathbf{X}_t) \right]^2 \leq \mathcal{M} + \mathcal{N}. \tag{3.99}$$

For term $\mathcal{M}$, by convexity argument and by symmetry, we have

$$
\begin{aligned}
\mathcal{M} \leq & C \int_t^T \mathbb{E}\left[\left|u^{N,i}(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right| \cdot \left| -\frac{1}{N_i}\sum_{k=1}^N g_{ik} X_s^k + \frac{1}{N_i}\sum_{j=1}^N g_{ij}\frac{1}{N_j}\sum_{l=1}^N g_{jl} X_s^l \right|\right] ds \\
\leq & C \int_t^T \mathbb{E}\left(u^{N,i}(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right)^2 ds \\
& + C \int_t^T \mathbb{E}\left(-\frac{1}{N_i}\sum_{k=1}^N g_{ik} X_s^k + \frac{1}{N_i}\sum_{j=1}^N g_{ij}\frac{1}{N_j}\sum_{l=1}^N g_{jl} X_s^l \right)^2 ds.
\end{aligned}
$$

For term $\mathcal{N}$, by convexity argument and by symmetry, and by the fact that $N_i < N$ and $g_{ij} \leq 1$, we have

$$
\mathcal{N} \leq \frac{C}{N_i} g_{ij} \int_t^T \mathbb{E}\left[\sum_{j=1}^N \left|u^{N,i}(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right| \cdot \left|u^{N,j}(s,\mathbf{X}_s) - v^{N,j}(s,\mathbf{X}_s)\right|\right] ds
$$

$$
\begin{aligned}
\leq & \frac{C}{N_i} g_{ij} \sum_{j=1}^N \int_t^T \mathbb{E}\left[\frac{1}{2}\left|u^{N,i}(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right|^2 + \frac{1}{2}\left|u^{N,j}(s,\mathbf{X}_s) - v^{N,j}(s,\mathbf{X}_s)\right|^2\right] ds \\
\leq & C \int_t^T \mathbb{E}\left|u^{N,i}(s,\mathbf{X}_s) - v^{N,i}(s,\mathbf{X}_s)\right|^2 ds.
\end{aligned}
$$

By Gronwall's lemma, we have

$$
\mathbb{E}\left[u^{N,i}(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right]^2 \leq C \int_t^T \mathbb{E}\left(-\frac{1}{N_i}\sum_{k=1}^N g_{ik} X_s^k + \frac{1}{N_i}\sum_{j=1}^N g_{ij}\frac{1}{N_j}\sum_{l=1}^N g_{jl} X_s^l \right)^2 ds.
$$

Taking expectation under $\mathbf{P}$, we have

$$
\frac{1}{N}\sum_{i=1}^N \mathbf{E}\left[u^{N,i}(t,\mathbf{X}_t) - v^{N,i}(t,\mathbf{X}_t)\right]^2 \leq \mathcal{C}(N),
$$

where $\mathcal{C}(N)$ is a function depending on N , the same as the bound of $r^{N,i}(t,\mathbf{x})$. □

Proposition 18. *One has,*

$$\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{E}(X_t^i - \widetilde{X}_t^{i,N})^2 \leq \mathcal{C}(N),$$

where $\mathcal{C}(N)$ *is a function depending on* N *, the same as the bound of* $r^{N,i}(t,\boldsymbol{x})$.

Proof. Recall that the state processes of finite player games are given by

$$dX_t^i = \left[-Y_t^i + q\left(\frac{1}{N_i}\sum_{j=1}^{N} g_{ij}X_t^j - X_t^i\right)\right]dt + \sigma dW_t^i.$$

Recall that the weakly interacting particle system on random graph $\widetilde{X}_t^{i,N}$, starting at the same point as X_t^i, i.e. $X_0^i = \widetilde{X}_0^{i,N} = \xi_0^i$, evolves according to

$$d\widetilde{X}_t^{i,N} = \left[-U(t,\widetilde{X}_t^{i,N},\widetilde{\mu}_t^{N,i}) + q(\frac{1}{N_i}\sum_{j=1}^{N} g_{ij}\widetilde{X}_t^{j,N} - \widetilde{X}_t^{i,N})\right]dt + \sigma dW_t^i, \tag{3.100}$$

where

$$\widetilde{\mu}_t^{N,i} = \frac{1}{N_i}\sum_{j=1}^{N} g_{ij}\delta_{\widetilde{X}_t^{j,N}}.$$

The dynamics of the difference of the processes X_t^i and $\widetilde{X}_t^{i,N}$ are given by

$$
\begin{aligned}
& d(X_t^i - \widetilde{X}_t^{i,N}) \\
&= \left[-Y_t^i + U(t, \widetilde{X}_t^{i,N}, \widetilde{\mu}_t^{N,i}) + \frac{q}{N_i} \sum_{j=1}^N g_{ij}(X_t^j - \widetilde{X}_t^{j,N}) - q(X_t^i - \widetilde{X}_t^{i,N}) \right] dt \\
&= \left[\Big(U(t, \widetilde{X}_t^{i,N}, \widetilde{\mu}_t^{N,i}) - U(t, X_t^i, \bar{\mu}_t^{N,i}) \Big) + \Big(U(t, X_t^i, \bar{\mu}_t^{N,i}) - Y_t^i \Big) \right] dt \\
&\quad + \left[\frac{q}{N_i} \sum_{j=1}^N g_{ij}(X_t^j - \widetilde{X}_t^{j,N}) - q(X_t^i - \widetilde{X}_t^{i,N}) \right] dt \\
&= \left[\frac{(\eta_t + q)}{N_i} \sum_{j=1}^N g_{ij}(X_t^j - \widetilde{X}_t^{j,N}) - (\eta_t + q)(X_t^i - \widetilde{X}_t^{i,N}) \right] dt \\
&\quad + \Big(u^{N,i}(t, \mathbf{X}_t) - v^{N,i}(t, \mathbf{X}_t) \Big) \, dt
\end{aligned}
\tag{3.101}
$$

Therefore, considering $X_0^i = \widetilde{X}_0^{i,N}$, a standard estimate using Gronwall's lemma yields

$$
\frac{1}{N} \sum_{i=1}^N \mathbf{E}(X_t^i - \widetilde{X}_t^{i,N})^2 \leq \frac{C}{N} \sum_{i=1}^N \mathbf{E} \left(u^{N,i}(t, \mathbf{X}_t) - v^{N,i}(t, \mathbf{X}_t) \right)^2,
$$

and then

$$
\frac{1}{N} \sum_{i=1}^N \mathbf{E}(X_t^i - \widetilde{X}_t^{i,N})^2 \leq \mathcal{C}(N)
$$

follows. □

Proposition 19. *When*

$$
\frac{1}{N} \sum_{i=1}^N \boldsymbol{E}(X_t^i - \widetilde{X}_t^{i,N})^2 \leq C N^{-1-\epsilon},
$$

for $\epsilon > 0$, X_t^i *and* $\widetilde{X}_t^{i,N}$ *have the same central limit theorem result.*

Proof. Let us focus on the difference $\mathcal{R}$ between the characteristic functions of $\frac{1}{\sqrt{N}} \sum_{i=1}^N X_t^i$

and $\frac{1}{\sqrt{N}}\sum_{i=1}^N \widetilde{X}_t^{i,N}$. That is

$$
\begin{aligned}
|\mathcal{R}| =& \left|\mathbf{E}\exp\left(ia\frac{\sum_{i=1}^N X_t^i}{\sqrt{N}}\right) - \mathbf{E}\exp\left(ia\frac{\sum_{i=1}^N \widetilde{X}_t^{i,N}}{\sqrt{N}}\right)\right| \\
=& \left|\mathbf{E}\left(1-\exp\left(ia\frac{\sum_{i=1}^N\left(\widetilde{X}_t^{i,N}-X_t^i\right)}{\sqrt{N}}\right)\right)\exp\left(ia\frac{\sum_{i=1}^N X_t^i}{\sqrt{N}}\right)\right| \\
\leq& \mathbf{E}\left|\exp\left(ia\frac{\sum_{i=1}^N\left(\widetilde{X}_t^{i,N}-X_t^i\right)}{\sqrt{N}}\right)-1\right| \\
\leq& \mathbf{E}\left\{\left|ia\frac{\sum_{i=1}^N\left(\widetilde{X}_t^{i,N}-X_t^i\right)}{\sqrt{N}}\right| + \left|\frac{\left(ia\frac{\sum_{i=1}^N\left(\widetilde{X}_t^{i,N}-X_t^i\right)}{\sqrt{N}}\right)^2}{2}\right| + \left(a\frac{\sum_{i=1}^N\left(\widetilde{X}_t^{i,N}-X_t^i\right)}{\sqrt{N}}\right)^2\right\} \\
\leq& a\sqrt{\frac{\mathbf{E}\left(\sum_{i=1}^N\left(\widetilde{X}_t^{i,N}-X_t^i\right)\right)^2}{N}} + \frac{5}{2}a^2\frac{\mathbf{E}\left(\sum_{i=1}^N\left(\widetilde{X}_t^{i,N}-X_t^i\right)\right)^2}{N} \\
\leq& a\sqrt{\frac{N\sum_{i=1}^N\mathbf{E}\left(\widetilde{X}_t^{i,N}-X_t^i\right)^2}{N}} + \frac{5}{2}a^2\frac{N\sum_{i=1}^N\mathbf{E}\left(\widetilde{X}_t^{i,N}-X_t^i\right)^2}{N}
\end{aligned}
$$

where the second inequality is by Lemma 3.3.7 in [93]. Now we see that when

$$
\frac{1}{N}\sum_{i=1}^N \mathbf{E}(X_t^i - \widetilde{X}_t^{i,N})^2 \leq CN^{-1-\epsilon},
$$

for $\epsilon > 0$, $|\mathcal{R}|$ goes to 0 as N goes to infinity, and hence X_t^i and $\widetilde{X}_t^{i,N}$ have the same central limit theorem result. □

Therefore, we see that to transfer the CLT result of $\widetilde{X}_T^{i,N}$ to X_T^i, it suffices to show that the bound of $r^{N,i}(t,\mathbf{x})$ is $\mathcal{O}(N^{-1-\epsilon})$, which is covered in the paper [63].

3.8 Open Problems and Future Research

Systemic risk is the risk of collapse of an entire system or market, which refers to the risks imposed by interlinkages and interdependencies among different parties in the system, and related to areas such as Statistics, Finance, Mathematical Finance, Behavioral Finance, Networks, Counterparty Risk, and etc (see [94]). The rich network of interconnections among firms is one of the most pervasive aspects of the contemporary financial environment. Linkages between firms can be cyclical in the sense that a default by firm i on its obligations to firm j through their linkage modeled through g_{ij}, may lead firm j to default on its obligations to firm k through their linkage modeled through g_{jk}, and eventually a default by firm k may have a feedback effect on firm i. Thus, financial system architectures may exhibit cyclical dependence in interfirm obligations. For system risk analyzed in the case without control, we refer to [95], [96], [97] and [98]. There are limited literatures which tackled system risk by means of mean field games, see [78, 99].

Mathematically formulate this problem is that, we want to analysis the asymptotic behavior of $\frac{1}{N}\log\mathbf{P}(\frac{1}{N}\sum_{i=1}^N \widetilde{X}_T^{i,N} \le D)$, as N goes to infinity. There is a close connection to the analysis in the proof of the functional central limit theorem we have done, which can be seen through using the Chebyshev's exponential inequality as the following:

$$
\begin{aligned}
&\frac{1}{N}\log\mathbf{P}(\frac{1}{N}\sum_{i=1}^N \widetilde{X}_T^{i,N} \le D)\\
=&\frac{1}{N}\log\mathbf{P}(-\frac{1}{N}\sum_{i=1}^N \widetilde{X}_T^{i,N} \ge -D)\\
\le&\frac{1}{N}\log\frac{\mathbf{E}^{\mathbf{P}}[\exp(-\frac{1}{N}\sum_{i=1}^N \widetilde{X}_T^{i,N})]}{\exp(-D)}\\
=&\frac{1}{N}\log\frac{\mathbf{E}^{\mathbf{Q}^N}[\exp(-\frac{1}{N}\sum_{i=1}^N \widetilde{X}_T^{i})]}{\exp(-D)}\\
=&\frac{1}{N}\log\frac{\mathbf{E}^{\mathbf{P}^N}[\exp(-\frac{1}{N}\sum_{i=1}^N \widetilde{X}_T^{i}+J^{N,1}(T)-\frac{1}{2}J^{N,2}(T))]}{\exp(-D)}.
\end{aligned}
$$

Two steps are necessary to finish the system risk analysis: one is to finish the LDP analysis regarding the weakly interacting particle system $\widetilde{X}_T^{i,N}$, the other is to establish the so-called "Exponential Equivalence" By Theorem 4.2.13 in [100], we know that as far as the LDP is concerned, exponentially equivalent measures are indistinguishable. That is, if an LDP with a good rate function $I(\cdot)$ holds for the probability measures $\{\overline{\widetilde{\mu}}_t^N\}$, which are exponentially equivalent to $\{\bar{\mu}_t^N\}$, then the same LDP holds for $\{\bar{\mu}_t^N\}$. By Definition 4.2.10 in [100], the exponential equivalence is equivalent to

$$\lim_{n\to\infty} \frac{1}{N} \log \mathbf{P}(\sup_{t\in[0,T]} W_2(\bar{\mu}_t^N, \overline{\widetilde{\mu}}_t^N) > \epsilon) = -\infty, \quad \forall \epsilon > 0, \tag{3.102}$$

with

$$\bar{\mu}_t^N = \sum_{i=1}^N X_T^i, \quad \overline{\widetilde{\mu}}_t^N = \sum_{i=1}^N \widetilde{X}_T^{i,N}$$

in our case, which is to be established in order to transfer the LDP result of $\widetilde{X}_T^{i,N}$ to X_T^i.

Appendix A

Appendix to Chapter 1

A.1 Moments of Z_t and X_t

A.1.1 Moments of Z_t

Proposition 20. *The process Z_t given by* (1.6) *has finite moments of any order uniformly in $0 \le \delta \le 1$ for $t \le T$.*

The proof is given by Lemma 4.9 in [20]. Thus, for $k \in \mathbb{Z}$,

$$\mathbf{E}_{(0,z)}\left[\int_0^T |Z_s|^k ds\right] \le C_k(T,z), \quad Z_0 = z, \tag{A.1}$$

where $C_k(T,z)$ may depend on (k,T,z) but not on δ.

A.1.2 Moments of X_t

In this subsection, we consider the process X_t evolving according to the SDE,

$$dX_t = rX_t dt + q_t\sqrt{Z_t}X_t dW_t, \quad X_0 = x, \tag{A.2}$$

where $q_t \in [d, u]$ and Z_t is the CIR process given by (1.6). In order to show that X_t has finite moments of any order, we will use a change of measure which will give rise to the following CIR process

$$d\widetilde{Z}_t = \left(\delta\kappa\theta - (\delta\kappa - nq_t\sqrt{\delta})\widetilde{Z}_t\right)dt + \sqrt{\delta}\sqrt{\widetilde{Z}_t}d\widetilde{W}_t^Z, \tag{A.3}$$

where the parameters κ, θ and δ are the same as the ones in the CIR process given by (1.6) and $n \in \mathbb{N}$.

Denote the moment generating function of the integrated $\widetilde{Z}_t$ process given $\widetilde{Z}_s|_{s=0} = z$ by

$$\widetilde{M}_z^\delta(\eta) := \mathbf{E}_{(0,z)}\left[\exp(\eta \int_0^t \widetilde{Z}_s ds)\right], \quad \text{for } \eta \in \mathbb{R}.$$

Then, we have the following preliminary result:

Proposition 21. *For $\eta \in \mathbb{R}$, $\widetilde{M}_z^\delta(\eta)$ is bounded uniformly, for δ sufficiently small and for all $t \in [0, T]$, that is, there exists $\epsilon = \epsilon(n, u, d, \kappa, T, \eta) > 0$ and $\widetilde{N}(\kappa, \theta, T, z, \eta) < \infty$ such that $|\widetilde{M}_z^\delta(\eta)| \leq \widetilde{N}(\kappa, \theta, T, z, \eta) < \infty$, for all $\delta \leq \epsilon$.*

Proof. Note that under the Feller condition $2\kappa\theta \geq 1$, the $\widetilde{Z}_t$ process is strictly positive as it is the original CIR process given by (1.6) in the case $n = 0$, and for $n \geq 1$ the drift is positive for δ small enough. Therefore, $\widetilde{M}_z^\delta(\eta) \leq 1$ for $\eta \leq 0$, and we only need to focus on $\eta > 0$ in the following. Also, since $t = 0$ is a trivial case, we concentrate on $t \in (0, T]$ in the proof. By Corollary 3 of [101], we know that

$$\widetilde{M}_z^\delta(\eta) = \Psi(\eta, t)e^{-z\Xi(\eta,t)},$$

where

$$\Psi(\eta,t) = \left(\frac{\bar{b} e^{bt/2}}{\bar{b}\frac{e^{\bar{b}t/2}+e^{-\bar{b}t/2}}{2} + b\frac{e^{\bar{b}t/2}-e^{-\bar{b}t/2}}{2}} \right)^{2\kappa\theta},$$

$$\Xi(\eta,t) = -2\eta \left(\frac{\frac{e^{\bar{b}t/2}-e^{-\bar{b}t/2}}{2}}{\bar{b}\frac{e^{\bar{b}t/2}+e^{-\bar{b}t/2}}{2} + b\frac{e^{\bar{b}t/2}-e^{-\bar{b}t/2}}{2}} \right),$$

and

$$\bar{b} = \sqrt{b^2 - 2\eta\delta}, \quad b = \delta\kappa - nq_t\sqrt{\delta}.$$

Note that the sign of b depends on the value of $n \in \mathbb{N}$. That is, when $n \geq 1$, b is negative for δ sufficiently small, while when $n = 0$, b is always positive. We also need to discuss the sign of the term $b^2 - 2\eta\delta$, which determines whether $\bar{b}$ is a real number or a complex number.

Case $n \geq 1$ ($b < 0$).

- If $b^2 - 2\eta\delta \geq 0$, then $\bar{b} \geq 0$. Note that when $\delta < (nd/\kappa)^2$, we have

$$\bar{b}t = t\sqrt{(nq_t\sqrt{\delta} - \delta\kappa)^2 - 2\eta\delta} \leq |b|t \leq nq_t\sqrt{\delta}t \leq nu\sqrt{\delta}T,$$

  and there exists $\epsilon_1 = \epsilon_1(n,u,d,\kappa,T)$ such that when $\delta < \epsilon_1$, we have $\bar{b}t \leq 1$ and $|bt + \mathcal{O}\left[(bt)^2\right]| < \frac{1}{2}$. Therefore, by the fact that $e^{bt/2} \leq 1$ and $\frac{e^{\bar{b}t/2}+e^{-\bar{b}t/2}}{2} \geq 1$, we have

$$\Psi(\eta,t) = \left(\frac{e^{bt/2}}{\frac{e^{\bar{b}t/2}+e^{-\bar{b}t/2}}{2} + b\frac{\bar{b}t+\mathcal{O}\left[(\bar{b}t)^2\right]}{\bar{b}}}\right)^{2\kappa\theta} = \left(\frac{e^{bt/2}}{\frac{e^{\bar{b}t/2}+e^{-\bar{b}t/2}}{2} + bt + \mathcal{O}\left[(bt)^2\right]}\right)^{2\kappa\theta}$$
$$\leq \left(\frac{1}{1-\frac{1}{2}}\right)^{2\kappa\theta} = 2^{2\kappa\theta}$$

and

$$|\Xi(\eta,t)| = \left|-2\eta\left(\frac{\bar{b}t+\mathcal{O}\left[(\bar{b}t)^2\right]}{\bar{b}\frac{e^{\bar{b}t/2}+e^{-\bar{b}t/2}}{2} + b\{\bar{b}t+\mathcal{O}\left[(\bar{b}t)^2\right]\}}\right)\right| = \left|-2\eta\left(\frac{t+\mathcal{O}\left[\bar{b}t^2\right]}{\frac{e^{\bar{b}t/2}+e^{-\bar{b}t/2}}{2} + bt + \mathcal{O}\left[(bt)^2\right]}\right)\right|$$
$$\leq 2\eta\frac{t+t}{1-\frac{1}{2}} = 8\eta t.$$

Therefore, for $\delta < \epsilon_1$, we have

$$\widetilde{M}_z^\delta(\eta) \leq 2^{2\kappa\theta}e^{8\eta T z}.$$

- If $b^2 - 2\eta\delta < 0$, then $\bar{b} = iv$, where $v = \sqrt{2\eta\delta - b^2}$. Note that $0 < vt \leq \sqrt{2\eta\delta}T$ and $\left|\frac{\sin(vt/2)}{vt/2}\right| \leq 1$. There exists $\epsilon_2 = \epsilon_2(n,u,T,\eta)$ such that when $\delta < \epsilon_2$, we have $\cos(vt/2) \geq \frac{3}{4}$ and $|bt| \leq 1$. Therefore,

$$\Psi(\eta,t) = \left(\frac{ive^{bt/2}}{iv\cos(vt/2) + ib\sin(vt/2)}\right)^{2\kappa\theta} = \left(\frac{e^{bt/2}}{\cos(vt/2) + \frac{bt}{2}\left(\frac{\sin(vt/2)}{vt/2}\right)}\right)^{2\kappa\theta},$$

$$\Xi(\eta,t) = -2\eta\left(\frac{\sin(vt/2)}{v\cos(vt/2) + b\sin(vt/2)}\right) = -2\eta\left(\frac{t}{2\cos(vt/2)\frac{vt/2}{\sin(vt/2)} + bt}\right)$$

and

$$\widetilde{M}_z^\delta(\eta) = \Psi(\eta,t)e^{-z\Xi(\eta,t)} \leq \left(\frac{1}{\frac{3}{4}-\frac{1}{2}}\right)^{2\kappa\theta}\exp\left(\frac{2\eta T z}{2\times\frac{3}{4}-1}\right) = 4^{2\kappa\theta}e^{4\eta T z}.$$

Case $n = 0$ ($b > 0$).

- If $b^2 - 2\eta\delta \geq 0$, then $\bar{b} \geq 0$. We have $\bar{b}t = t\sqrt{\delta^2\kappa^2 - 2\eta\delta} \leq \delta\kappa T$, and there exists $\epsilon_3 = \epsilon_3(\kappa, T)$, such that when $\delta < \epsilon_3$, we have $\bar{b}t \leq 1$,

$$\Psi(\eta, t) \leq \left(\frac{\bar{b}e^{\delta\kappa t/2}}{\bar{b}\frac{e^{\bar{b}t/2} + e^{-\bar{b}t/2}}{2}} \right)^{2\kappa\theta} \leq \left(e^{\delta\kappa t/2}\right)^{2\kappa\theta} \leq e^{\kappa^2\theta T},$$

$$|\Xi(\eta, t)| \leq 2\eta \left(\frac{e^{\bar{b}t/2} - e^{-\bar{b}t/2}}{2\bar{b}} \right) \leq \eta(1 + \mathcal{O}(\bar{b}t))T \leq 2\eta T$$

and

$$\widetilde{M}_z^\delta(\eta) \leq e^{\kappa^2\theta T} e^{2\eta T z}.$$

- If $b^2 - 2\eta\delta < 0$, then $\bar{b} = iv$, where $v = \sqrt{2\eta\delta - \delta^2\kappa^2}$. Note that $0 < vt \leq \sqrt{2\eta\delta}T$ and there exists $\epsilon_4 = \epsilon_4(n, u, T, \eta)$ such that when $\delta < \epsilon_4$, we have $\cos(vt/2) \geq \frac{3}{4}$ and $\sin(vt/2) \geq 0$. Therefore,

$$\Psi(\eta, t) = \left(\frac{ive^{\delta\kappa t/2}}{iv\cos(vt/2) + i\delta\kappa\sin(vt/2)} \right)^{2\kappa\theta} = \left(\frac{e^{\delta\kappa t/2}}{\cos(vt/2) + \frac{\delta\kappa t}{2}\left(\frac{\sin(vt/2)}{vt/2}\right)} \right)^{2\kappa\theta},$$

$$\Xi(\eta, t) = -2\eta \left(\frac{\sin(vt/2)}{v\cos(vt/2) + \delta\kappa\sin(vt/2)} \right) = -2\eta \left(\frac{t}{2\cos(vt/2)\frac{vt/2}{\sin(vt/2)} + \delta\kappa t} \right)$$

and

$$\widetilde{M_z^\delta}(\eta) = \Psi(\eta,t)e^{-z\Xi(\eta,t)} \leq \left(\frac{4e^{\kappa T/2}}{3}\right)^{2\kappa\theta} \exp(\frac{2\eta T z}{2\times\frac{3}{4}}) = \left(\frac{4e^{\kappa T/2}}{3}\right)^{2\kappa\theta} \exp(\frac{4\eta T z}{3}).$$

In sum, there exists $\epsilon = \epsilon(n,u,d,\kappa,T,\eta)$ and $\widetilde{N} = \widetilde{N}(\kappa,\theta,T,z,\eta)$ which is independent of δ and t, such that when $\delta < \epsilon$, we have $\widetilde{M_z^\delta}(\eta) \leq \widetilde{N}$ as desired. □

Proposition 22. *The process X_t given by* (A.2)*, has finite moments of any order, for $t \leq T$ and $\delta < \epsilon(n,u,d,\kappa,T,\eta)$ given in Proposition 21, where n is a positive integer.*

Proof. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} X_t^n =& x^n \exp\left(nrt - \frac{n}{2}\int_0^t (q_s\sqrt{Z_s})^2 ds + n\int_0^t q_s\sqrt{Z_s}dW_s\right) \\ =& x^k \exp\left(nrt + \frac{n^2-n}{2}\int_0^t (q_s\sqrt{Z_s})^2 ds\right) \times \Lambda_t, \end{aligned}$$

where

$$\Lambda_t = \exp\left(-\frac{n^2}{2}\int_0^t (q_s\sqrt{Z_s})^2 ds + n\int_0^t q_s\sqrt{Z_s}dW_s\right)$$

is a martingale, whose Novikov condition is satisfied thanks to Proposition 21, i.e.

$$\mathbf{E}_{(0,x,z)}\left[\exp\left(\frac{1}{2}\int_0^t (nq\sqrt{Z_s})^2 ds\right)\right] \leq \mathbf{E}_{(0,x,z)}\left[\exp\left(\frac{n^2u^2}{2}\int_0^t Z_s ds\right)\right] < \infty.$$

By the corresponding change of measure and the inequality $q_s \leq u$, we get

$$\mathbf{E}_{(0,x,z)}\left[X_t^n\right] \leq x^n \exp(nrt)\widetilde{\mathbf{E}}_{(0,x,z)}\left[\exp\left(\frac{(n^2-n)u^2}{2}\int_0^t \widetilde{Z}_s ds\right)\right] \tag{A.4}$$

where, under the new measure $\widetilde{\mathbf{Q}}$, the process $\widetilde{Z}_t$ driven by a $\widetilde{\mathbf{Q}}$-Brownian motion $\widetilde{W}_t^Z$

evolves according to (A.3). Hence, by Proposition 21, we have

$$\mathbf{E}_{(0,x,z)}\left[X_t^n\right] \le x^n \exp(nrT)\widetilde{M}_z^\delta\left(\frac{(n^2-n)u^2}{2}\right) \le x^n \exp(nrT)\widetilde{N},$$

where the upper bound $x^n \exp(nrT)\widetilde{N}$ is independent of δ and t. □

Therefore, for δ sufficiently small,

$$\mathbf{E}_{(0,x,z)}\left[\int_0^T |X_s|^n ds\right] \le N_n, \tag{A.5}$$

where N_n does not on δ and $t \in [0,T]$.

A.2 Proof of Proposition 1

Integrating over $[t,T]$ the SDE (1.7) and the SDE (1.8), we have

$$X_T^\delta = x + \int_t^T rX_s^\delta ds + \int_t^T q_s\sqrt{Z_s}X_s^\delta dW_s \tag{A.6}$$

and

$$X_T^0 = x + \int_t^T rX_s^0 ds + \int_t^T q_s\sqrt{z}X_s^0 dW_s. \tag{A.7}$$

The difference of (A.6) and (A.7) is given by

$$\begin{aligned} X_T^\delta - X_T^0 &= \int_t^T r(X_s^\delta - X_s^0)ds + \int_t^T q_s(\sqrt{Z_s}X_s^\delta - \sqrt{z}X_s^0)dW_s \\ &= \int_t^T r(X_s^\delta - X_s^0)ds + \int_t^T q_s\sqrt{z}(X_s^\delta - X_s^0)dW_s + \int_t^T q_s(\sqrt{Z_s} - \sqrt{z})X_s^\delta dW_s. \end{aligned} \tag{A.8}$$

Let $Y_s = X_s^\delta - X_s^0$, then $Y_t = 0$ and

$$Y_T = \int_t^T rY_s ds + \int_t^T q_s\sqrt{z}Y_s dW_s + \int_t^T q_s(\sqrt{Z_s} - \sqrt{z})X_s^\delta dW_s. \tag{A.9}$$

Therefore,

$$\begin{aligned}
&\mathbf{E}_{(t,x,z)}\left[Y_T^2\right] \\
\leq &3\mathbf{E}_{(t,x,z)}\left[\left(\int_t^T rY_s ds\right)^2 + \left(\int_t^T q_s\sqrt{z}Y_s dW_s\right)^2 + \left(\int_t^T q_s(\sqrt{Z_s} - \sqrt{z})X_s^\delta dW_s\right)^2\right] \\
\leq &\int_t^T \left(3Tr^2 + 3u^2 z\right)\mathbf{E}_{(t,x,z)}\left[Y_s^2\right] ds + \underbrace{3u^2\int_t^T \mathbf{E}_{(t,x,z)}\left[(\sqrt{Z_s} - \sqrt{z})^2(X_s^\delta)^2\right] ds}_{R(\delta)}.
\end{aligned} \tag{A.10}$$

Notice that only the upper bound of q is used, which gives the uniform convergence in q. Also note that using the result that X_t and Z_t have finite moments for δ sufficiently small, we can show that $|R(\delta)| \leq C\delta$, where $C = C(T, \theta, u, d, z)$ is independent of δ.

Denote $f(T) = \mathbf{E}_{(t,x,z)}(Y_T^2)$ and $\lambda = 3Tr^2 + 3u^2 z > 0$, and by Gronwall inequality, equation (A.10) can be written as

$$f(T) \leq \int_t^T \lambda f(s)ds + C\delta \leq \delta\int_t^T C\lambda e^{\lambda(T-s)}ds + C\delta$$

Therefore,

$$\mathbf{E}_{(t,x,z)}(X_T^\delta - X_T^0)^2 = \mathbf{E}_{(t,x,z)}Y_T^2 = f(T) \leq C'\delta,$$

and the Proposition follows.

A.3 Existence and uniqueness of $(X_t^{*,\delta})$

For the existence and uniqueness of $X_t^{*,\delta}$, we consider the transformation $Y_t^{*,\delta} := \log X_t^{*,\delta}$, which is well defined for any $t < \tau^\epsilon$, where for any $\epsilon > 0$,

$$\begin{aligned}\tau^\epsilon :&= \inf\{t > 0 | X_t^{*,\delta} = \epsilon \quad \text{or} \quad X_t^{*,\delta} = 1/\epsilon\} \\ &= \inf\{t > 0 | Y_t^{*,\delta} = \log \epsilon \quad \text{or} \quad Y_t^{*,\delta} = -\log \epsilon\}.\end{aligned}$$

By Itô's formula, the process $Y_t^{*,\delta}$ satisfies the following SDE:

$$dY_t^{*,\delta} = -\frac{1}{2}(q^{*,\delta})^2 Z_t dt + q^{*,\delta}\sqrt{Z_t}dW_t. \tag{A.11}$$

Note that the diffusion coefficient satisfies $q^{*,\delta}\sqrt{Z_t} \geq d\sqrt{Z_t} > 0$, and is bounded away from 0, hence by Theorem 1 in section 2.6 of [102] and the result 7.3.3 of [103], the SDE (A.11) has a unique weak solution. Consequently, we have a unique solution to the SDE (1.24) until τ^ϵ for any $\epsilon > 0$. In order to show (1.24) has a unique solution, it suffices to prove that, for any $T > 0$,

$$\lim_{\epsilon \downarrow 0} \mathbb{Q}(\tau^\epsilon < T) = 0.$$

Note that the contribution of $Y_0^{*,\delta}(= \log x)$ is trivial on the term $\lim_{\epsilon\downarrow 0} \log(\frac{\epsilon}{x})$, for simplicity, we consider $Y_0^{*,\delta} = 0$ in the following. For any $t \in [0, T]$, one has

$$Y_t^{*,\delta} = \int_0^t -\frac{1}{2}(q^{*,\delta})^2 Z_s ds + \int_0^t q^{*,\delta}\sqrt{Z_s}dW_s.$$

Then

$$
\begin{aligned}
&\mathbb{Q}(\sup_{t\in[0,T]}|Y_t^{*,\delta}|>|\log\epsilon|)\\
\leq&\mathbb{Q}\left(\sup_{t\in[0,T]}\left[\int_0^t\frac{1}{2}u^2Z_sds+\left|\int_0^t q^{*,\delta}\sqrt{Z_s}dW_s\right|\right]>|\log\epsilon|\right)\\
\leq&\mathbb{Q}\left(\frac{1}{2}u^2\int_0^T Z_sds+\sup_{t\in[0,T]}\left|\int_0^t q^{*,\delta}\sqrt{Z_s}dW_s\right|>|\log\epsilon|\right)\\
\leq&\mathbb{Q}\left(\frac{1}{2}u^2\int_0^T Z_sds>\frac{|\log\epsilon|}{2}\right)+\mathbb{Q}\left(\sup_{t\in[0,T]}\left|\int_0^t q^{*,\delta}\sqrt{Z_s}dW_s\right|>\frac{|\log\epsilon|}{2}\right)\\
:=&\mathcal{A}+\mathcal{B}.
\end{aligned}
$$

By Markov inequality and by (A.1), we have

$$
\mathcal{A}\leq\frac{u^2\mathbf{E}\int_0^T Z_sds}{|\log\epsilon|}\leq\frac{u^2TC_1(T,z)}{|\log\epsilon|}.
$$

By Doob's martingale inequality and by (A.1), we have

$$
\mathcal{B}\leq\frac{\mathbf{E}(\int_0^t q^{*,\delta}\sqrt{Z_s}dW_s)^2}{\left(\frac{\log\epsilon}{2}\right)^2}\leq\frac{\int_0^t\mathbf{E}\{(q^{*,\delta})^2Z_s\}ds}{\left(\frac{\log\epsilon}{2}\right)^2}\leq\frac{u^2\int_0^T\mathbf{E}Z_sds}{\left(\frac{\log\epsilon}{2}\right)^2}\leq\frac{u^2TC_1(T,z)}{\left(\frac{\log\epsilon}{2}\right)^2}.
$$

Therefore,

$$
\lim_{\epsilon\downarrow 0}\mathcal{A}=\lim_{\epsilon\downarrow 0}\mathcal{B}=0.
$$

Finally, we can conclude that

$$
\lim_{\epsilon\downarrow 0}\mathbb{Q}(\tau^\epsilon<T)=\lim_{\epsilon\downarrow 0}\mathbb{Q}(\sup_{t\in[0,T]}|Y_t^{*,\delta}|>|\log\epsilon|)=0,
$$

for any $T>0$, as desired.

A.4 Proof of Uniform Boundedness of I_2 and I_3 on δ

With the help of Assumption 4, Cauchy–Schwarz inequality and the uniformly bounded moments of Z_t and X_t processes given in (A.1) and (A.5) respectively, we are going to prove that I_2 and I_3 are uniformly bounded in δ.

First recall that

$$
\begin{aligned}
I_2 =& \mathbf{E}_{(t,x,z)}\Bigg[\int_t^T \Bigg(\rho(q^{*,\delta})Z_s X_s^{*,\delta}\partial^2_{xz}P_1(s,X_s^{*,\delta},Z_s) \\
&+\frac{1}{2}Z_s\partial^2_{zz}P_0(s,X_s^{*,\delta},Z_s)+\kappa(\theta-Z_s)\partial_z P_0(s,X_s^{*,\delta},Z_s)\Bigg)ds\Bigg] \\
\doteq& I_2^{(1)}+I_2^{(2)}+I_2^{(3)}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
I_2^{(1)} \leq& \mathbf{E}_{(t,x,z)}\left[\int_t^T \rho u Z_s X_s^{*,\delta}|\partial^2_{xz}P_1(s,X_s^{*,\delta},Z_s)|ds\right] \\
\leq& \rho u \mathbf{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(Z_s X_s^{*,\delta}\right)^2 ds\right]\cdot \mathbf{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(\partial^2_{xz}P_1(s,X_s^{*,\delta},Z_s)\right)^2 ds\right] \\
\leq& \rho u \mathbf{E}^{1/4}_{(t,x,z)}\left[\int_t^T (Z_s)^4 ds\right]\cdot \mathbf{E}^{1/4}_{(t,x,z)}\left[\int_t^T (X_s^{*,\delta})^4 ds\right] \\
&\cdot \bar{a}_{11}^2\mathbf{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(1+|X_s^{*,\delta}|^{\bar{b}_{11}}+|Z_s|^{\bar{c}_{11}}\right)^2 ds\right] \\
\leq& \rho u\left(C_4\right)^{1/4}\cdot\left(N_4\right)^{1/4}\cdot \bar{A}_{11}\left[C_{2\bar{b}_{11}}+N_{2\bar{c}_{11}}\right]^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
I^{(3)} \leq& \frac{1}{2}\mathbf{E}^{1/2}_{(t,x,z)}\left[\int_t^T (Z_s)^2 ds\right]\cdot \mathbf{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(\partial^2_{zz}P_0(s,X_s^{*,\delta},Z_s)\right)^2 ds\right] \\
\leq& \frac{1}{2}\left(C_2\right)^{1/2}\cdot A_{02}\left[C_{2b_{02}}+N_{2c_{02}}\right]^{1/2}
\end{aligned}
$$

and

$$\begin{aligned}
I^{(4)} &\leq \kappa \mathbf{E}^{1/2}_{(t,x,z)}\left[\int_t^T (\theta - Z_s)^2 ds\right] \cdot \mathbf{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(\partial_z P_0(s, X_s^{*,\delta}, Z_s)\right)^2 ds\right] \\
&\leq \kappa \mathbf{E}^{1/2}_{(t,x,z)}\left[\int_t^T \theta^2 + Z_s^2 ds\right] \cdot \mathbf{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(\partial_z P_0(s, X_s^{*,\delta}, Z_s)\right)^2 ds\right] \\
&\leq \frac{1}{2}\left(C_2 + \theta^2 T\right)^{1/2} \cdot A_{01}\left[C_{2b_{01}} + N_{2c_{01}}\right]^{1/2},
\end{aligned}$$

where $\bar{A}_{01}$, $\bar{A}_{11}$ and A_{02} are positive constants.

Next recall that

$$I_3 = \mathbf{E}_{(t,x,z)}\left[\int_t^T \frac{1}{2} Z_s \partial^2_{zz} P_1(s, X_s^{*,\delta}, Z_s) + \kappa(\theta - Z_s)\partial_z P_1(s, X_s^{*,\delta}, Z_s) ds\right] \doteq I_3^{(1)} + I_3^{(2)}.$$

Then we have

$$\begin{aligned}
I_3^{(1)} \leq& \frac{1}{2}\mathbf{E}^{1/2}_{(t,x,z)}\left[\int_t^T (Z_s)^2 ds\right] \cdot \mathbf{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(\partial^2_{zz} P_1(s, X_s^{*,\delta}, Z_s)\right)^2 ds\right] \\
\leq& (C_2)^{1/2} \cdot \bar{A}_{02}\left[C_{2\bar{b}_{02}} + N_{2\bar{c}_{02}}\right]^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
I_3^{(2)} &\leq 2\kappa \mathbf{E}^{1/2}_{(t,x,z)}\left[\int_t^T \theta^2 + Z_s^2 ds\right] \cdot \mathbf{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(\partial_z P_1(s, X_s^{*,\delta}, Z_s)\right)^2 ds\right] \\
&\leq 2\kappa\left[\theta^2 T + C_2\right]^{1/2} \cdot \bar{A}_{01}\left[C_{2\bar{b}_{01}} + N_{2\bar{c}_{01}}\right]^{1/2},
\end{aligned}$$

where $\bar{A}_{01}$, $\bar{A}_{02}$ are positive constants.

Appendix B

Appendix to Chapter 2

Lemma 5. *Assume that* $u \in C^{2,\alpha}(\overline{\Omega})$ *satisfies* $\mathcal{L}u \geq 0$.

1. *If it satisfies Neumann or Robin boundary condition,* $\mathcal{B}u|_{\partial\Omega} = 0$*, where* $\mathcal{B}u = \gamma(x)u + D_\nu u = 0$ *with* $\gamma(x) \geq 0, \gamma(x) \in C^{1,\alpha}(\partial\Omega)$*, then* $u > 0$ *on* $\overline{\Omega}$ *unless* $u \equiv 0$.

2. *If it has Dirichlet boundary condition,* $u = 0$ *on* $\overline{\Omega}$*, then* $u > 0$ *in* Ω*. Furthermore, for any* $v \in C^2(\overline{\Omega})$ *with* $v|_{\partial\Omega} = 0$*, there exists an* $\epsilon > 0$ *such that* $w \geq \epsilon v$*. If* u *is not identically 0, then* $\frac{\partial u}{\partial \nu} < 0$ *on* $\partial\Omega$*, where* ν *is the exterior unit normal of* $\partial\Omega$.

Proof. (1) Suppose now $v \in K \setminus \{0\}$ and denote by $u = \mathcal{L}^{-1}v$, then $v = \mathcal{L}u \geq 0$ in Ω. Therefore the strong maximum principle ([104], Theorem 4, Section 6.4.2, [56], Theorem 3.5) implies that $u > 0$ in Ω, and thus, $u \geq 0$ on $\overline{\Omega}$. Furthermore, assume $u(x_0) = 0$ for some $x_0 \in \partial\Omega$. Then the Hopf boundary lemma ([56], Lemma 3.4) asserts that $D_\nu u(x_0) < 0$ and hence $\mathcal{B}u(x_0) < 0$, contradicting Robin boundary condition. Therefore $u > 0$ on $\overline{\Omega}$.

(2) It follows from the maximum principle that $w > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$. Next for any $x \in \partial\Omega$, after a local change coordinates, we may assume that there is a neighborhood U of x on which is defined a coordinate system $x = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1})$, such

that $U \cap \partial\Omega$ is defined by $x_n = 0$ and $U \cap \Omega$ is expressed by $x_n > 0$. Then the maximum principle implies that $\frac{\partial w}{\partial x_n} > 0$. On the other side, expanding the Taylor series of w and v yields

$$w(x', x_n) = w(x', 0) + \frac{\partial w}{\partial x_n}(x', 0)x_n + o(x_n)$$

and

$$v(x', x_n) = v(x', 0) + \frac{\partial v}{\partial x_n}(x', 0)x_n + o(x_n)$$

The boundary conditions indicates that

$$w(x', 0) = v(x', 0) = 0 \quad \text{and} \quad \frac{\partial w}{\partial x_n}(x', 0) > 0.$$

Therefore, the preceding equations imply that in a small neighborhood of $\partial\Omega$, say, $\mathcal{A}$, there exist a small $\epsilon_1 > 0$ such that $w > \epsilon_1 v$, and the selection of $\mathcal{A}$ can be chosen to guarantee that $\mathcal{D} = \Omega \cap \mathcal{A}^c$ is compact. Therefore considering the continuity of w and v, as well as, the positivity of w in Ω, there exists $\epsilon_2 > 0$ such that

$$\min_{x \in \mathcal{D}} w(x) > \epsilon_2 \max_{x \in \mathcal{D}} v(x).$$

Taking $\epsilon = \min\{\epsilon_1, \epsilon_2\} > 0$ completes the proof of the lemma. $\square$

Appendix C

Appendix to Chapter 3

C.1 Derivation of FBSDE

$$
\begin{aligned}
dX_t &= \alpha_t^* dt + \sigma dW_t = \left[-Y_t + q(\mathbf{E}\left[X_t\right] - X_t)\right] dt + \sigma dW_t \\
dY_t &= -\partial_x H(t, X_t, Y_t, Z_t, \alpha_t^*) dt + Z_t dW_t \\
&= \left[-q\alpha_t^* - \epsilon(X_t - \mathbf{E}\left[X_t\right])\right] dt + Z_t dW_t \\
&= \left[-q\left[-Y_t + q(\mathbf{E}\left[X_t\right] - X_t)\right] - \epsilon(X_t - \mathbf{E}\left[X_t\right])\right] dt + Z_t dW_t \\
&= \left[qY_t - (\epsilon - q^2)(X_t - \mathbf{E}\left[X_t\right])\right] dt + Z_t dW_t \\
Y_T &= \partial_x \left[\frac{c}{2}(\int_{\mathbb{R}} x d\mu_t - X_T)^2\right] = c(X_T - \mathbf{E}\left[X_T\right])
\end{aligned}
\tag{C.1}
$$

C.2 Explicit Solution of FBSDE (3.6)

We make the following ansatz

$$Y_t = \eta_t(X_t - m_t) + \mu_t,$$

where m_t is the expectation of X_t, η_t and μ_t are deterministic functions satisfying $\eta_T = c$ and $\mu_T = 0$. By

$$dX_t = [-Y_t + q(\mathbf{E}[X_t] - X_t)]\,dt + \sigma dW_t,$$

we have

$$\begin{aligned} X_t - X_0 &= \int_0^t [-Y_s + q(\mathbf{E}[X_s] - X_s)]\,ds + \sigma W_t \\ d\mathbf{E}[X_t] &= -\mathbf{E}[Y_t]\,dt \end{aligned} \tag{C.2}$$

Similarly, we have

$$d\mathbf{E}[Y_t] = q\mathbf{E}[Y_t]\,dt \tag{C.3}$$

Considering $\mathbf{E}[Y_T] = 0$, we know that $\dot{m}_t = 0$, for all $t \in [0, T]$. Therefore,

$$dY_t = \dot{\eta}_t(X_t - m_t)dt + \eta_t \left\{ [-Y_t + q(m_t - X_t)]\,dt + \sigma dW_t \right\} + \dot{\mu}_t dt \tag{C.4}$$

Compare to the SDE of Y_t in (3.6), we know that

$$Z_t = \sigma\eta_t$$

$$qY_t - (\epsilon - q^2)(X_t - m_t) = \dot{\eta}_t(X_t - m_t) + \eta_t\{[-Y_t + q(m_t - X_t)]\} + \dot{\mu}_t \tag{C.5}$$

Given

$$Y_t = \eta_t(X_t - m_t) + \mu_t,$$

we have

$$\left[\dot{\eta}_t - \eta_t^2 - 2q\eta_t + (\epsilon - q^2)\right](X_t - m_t) + [\dot{\mu}_t - (\eta_t + q)\mu_t] = 0. \tag{C.6}$$

Therefore,

$$\begin{aligned} \dot{\eta}_t - \eta_t^2 - 2q\eta_t + (\epsilon - q^2) = 0 \\ \eta_T = c. \end{aligned} \tag{C.7}$$

The solution of this scalar Riccati equation is given by (3.7). The ODE for μ_t is given by

$$\begin{aligned} \dot{\mu}_t - (q + \eta_t)\mu_t = 0 \\ \mu_T = 0, \end{aligned} \tag{C.8}$$

which admits the solution $\mu_t = 0$, for all $t \in [0, T]$. Therefore,

$$Y_t = \eta_t(X_t - m_t) \quad \text{and} \quad Z_t = \sigma\eta_t,$$

is one solution of the FBSDE, with η_t given in (3.7).

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