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Two Studies of Topological Quantum Field Theory in Two Dimensions

by

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Abstract

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Take powers of the determinant line bundles on the relative moduli spaces (or stacks) of principal G-bundles over relative curves $C \to B$, and then push them down to the base space B — the resulting sheaves over B, which are in fact vector bundles, are known as the *Verlinde bundles*. They satisfy certain "gluing" properties [55] and yield a structure known as (K-theoretic) cohomological field theory, which is a type of families 2-dimensional topological quantum field theory [62]. In the first part of this thesis, we carry out an investigation of the *higher twisted Verlinde bundles*, as defined by Teleman–Woodward in [63], for the case of the abelian group $G = \mathbb{C}^*$. In particular, we show that their Chern characters can be written in terms of tautological classes. This generalizes the known fact that the ordinary Verlinde bundles have tautological Chern characters; indeed we also include in our study an explicit computation of the Chern characters of the $\mathbb{C}^*$-Verlinde bundles. From this computation we are able to explicitly demonstrate the gluing properties in action.

By results of Costello [13] and of Konstevich and collaborators [32, 36], cohomological field theories can also arise from categories that are homologically smooth, proper, and Calabi–Yau. The state space of such a theory is given by the Hochschild (co)homology of the category. In the second part of this thesis, then, we study categories of matrix factorizations for Landau–Ginzburg models (X, W), where X is a variety over $\mathbb{C}$. We show that when X is smooth and Calabi–Yau as a variety and W has a proper critical locus, the corresponding category of matrix factorizations is homologically smooth, proper, and Calabi–Yau as a category. Furthermore, we compute the Hochschild cohomology of these matrix factorization categories, and we get the result predicted by Kontsevich. These results, which are joint with Daniel Pomerleano [37], generalize the results of Dyckerhoff [15] for the case when X is affine local and W has an isolated singularity.

We conclude with some brief remarks on forthcoming work, also joint with Daniel Pomerleano, in which we propose mirror partners to certain Fano non-toric 3-folds which can be degenerated to nodal toric varieties. Using our results on matrix factorization categories, we are able to prove some homological mirror symmetry results for these proposed mirror pairs.

Dedicated to my mother and the memory of my father

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I would like thank my advisor Constantin Teleman for taking me on as a student and for introducing me to the rich and fascinating worlds of quantum field theory and mirror symmetry. I have gained so much from his perspective, which is as awesomely deep as it is awesomely broad. I am very grateful to have had the opportunity to glean some of his insight and vision during my time in grad school.

The work in the second part of this thesis is based on joint work with Daniel Pomerleano. I cannot understate my gratitude to Dan. I am deeply thankful for his encouragement and for our countless mathematical discussions over the years. He has been a huge inspiration and a great friend who has lifted me up during many of my struggles.

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Contents

Chapter 1

Introduction

Since at least the 1980s, quantum field theory and string theory have had a tremendous influence on mathematics, leading to many new discoveries and vistas. Insights and ideas from physics have given mathematicians new lens with which to view classical objects, as well as many new notions not previously studied or considered by mathematicians. An important such notion is that of a topological quantum field theory, as well as the various extensions and generalizations thereof. It turns out that these structures, perhaps surprisingly, arise in many different areas of mathematics. In this thesis, we will study two interesting examples: higher twists of abelian gauge theory in families and matrix factorizations for non-affine Landau–Ginzburg models. In this first chapter we introduce some of the background concepts and motivation, and present the statements of our main results.

1.1 Topological quantum field theories

We start with most basic underlying theme: topological quantum field theories. We will give a mathematical definition and refrain from any discussion of the physics underpinnings. For the physics background the reader should refer to [5] and the references therein.

Definition 1.1.1. A *symmetric monoidal category* is a category C equipped with a *tensor product*, that is, a functor $\otimes : C \times C \to C$, and a distinguished *unit object* $\mathbf{1}$, together with natural isomorphisms $M \otimes \mathbf{1} \cong M$, $M \otimes N \cong N \otimes M$, and $(M \otimes N) \otimes L \cong M \otimes (N \otimes L)$.

Definition 1.1.2. The category Bord_d is the category whose objects are (possibly disconnected) closed oriented $(d-1)$-manifolds. A morphism between two objects M, N is a *bordism*, that is, a (possibly disconnected) compact oriented d-manifold B together with an orientation-preserving diffeomorphism $\partial B \cong \overline{M} \coprod N$, where $\overline{M}$ denotes the manifold M with the opposite orientation. We take bordisms up to boundary-preserving diffeomorphism. Composition of morphisms is given by gluing of bordisms. For any object M, the identity morphism $M \to M$ is given by the cylinder $M \times I$.

Moreover, the category Bord_d can be given the structure of a symmetric monoidal category; the tensor product is the disjoint union of manifolds, and the unit object is the empty manifold.

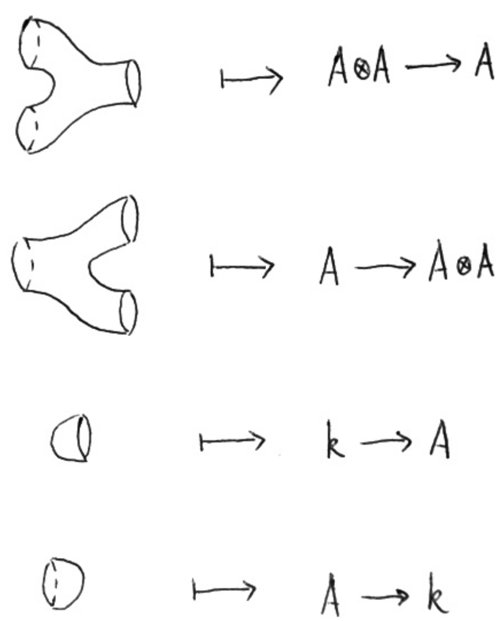
Definition 1.1.3 (Atiyah–Segal [5, 56])**.** Fix a field k. A *d-dimensional topological quantum field theory*, or *TQFT*, is a functor of symmetric monoidal categories $Z : \mathrm{Bord}_d \to \mathrm{Vect}(k)$, where $\mathrm{Vect}(k)$ is the category of vector spaces over the field k with symmetric monoidal structure given by the tensor product of vector spaces.

In this thesis, we will focus on the case $d = 2$, and so from now on all TQFTs will be assumed to be 2-dimensional ones. Then we have the following well-known folk theorem.

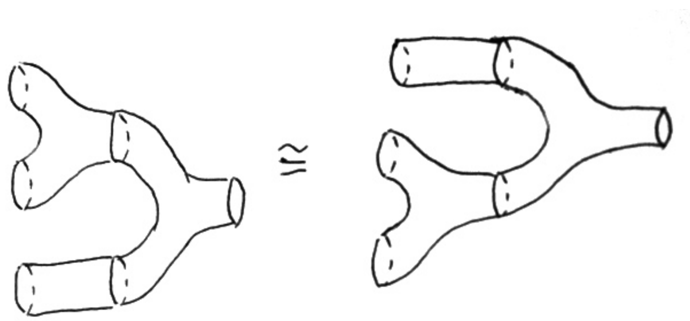
Theorem 1.1.4. *A TQFT Z determines and is uniquely determined by a commutative Frobenius algebra A over the field k.*

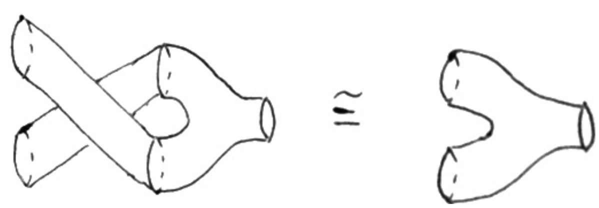
Proof. We will just give a sketch of the proof here. A full proof can be found in [2].

If Z is a TQFT, then underlying vector space of the Frobenius algebra is $A = Z(S^1)$. Its product is given by applying Z to the "pants", its coproduct is given by applying Z to the "copants", its unit is given by applying Z to the "cup", and its counit is given by applying Z to the "cap":



One can check that A with these operations is indeed a commutative Frobenius algebra. For example, the multiplication is associative and commutative by the following pictures:





The other properties can be proven by similar pictures.

Conversely, suppose A is a commutative Frobenius algebra. Then it determines a TQFT Z whose values on the circle S^1, the pants, the copants, the cup, and the cap are, respectively, the underlying vector space of A, the product, the coproduct, the unit, and the counit. The value of Z on gluings of pants, copants, cups, and caps is then determined. By Morse theory, any surface can be obtained by such a gluing, so Z is determined on all surfaces. □

As we will see, not only do TQFTs lead to interesting mathematical structures, but they can arise in many interesting and seemingly disparate mathematical contexts.

1.2 Cohomological field theories

Beyond 2-dimensional topological quantum field theories, there is also the related notion of *cohomological field theory*, or *CohFT*, which is more situated in the realm of algebraic geometry. Let us spell this out precisely, as it will be very important to us. We work over the ground field $\mathbb{C}$. Let A be the *state space* of the CohFT — a vector space with a metric g. Suppose $X \to B$ is a relative stable curve[1] over a space B with n ordered marked points, that is, sections $p_i : B \to X$. In other words, $X \to B$ is a *family* of stable curves — the fibers are stable curves over $\mathbb{C}$. Then a CohFT, by definition, assigns to the family $X \to B$ a map of vector spaces

$$Z(X \to B) : A^{\otimes n} \to \mathrm{H}^*(B)$$

[1] A stable curve is a possibly nodal curve with at most finitely many automorphisms [41, 24]. In this thesis, "curve" will always mean complex curve, whereas "surface" will always mean real surface, so that the underlying real space of a curve is a surface.

where H^* is a cohomology theory, e.g. singular cohomology[2].

We require that this assignment Z be natural in the sense that if X' is the base change of $X \to B$ along a map $f : B' \to B$, then

$$Z(X' \to B') = f^* \circ Z(X \to B).$$

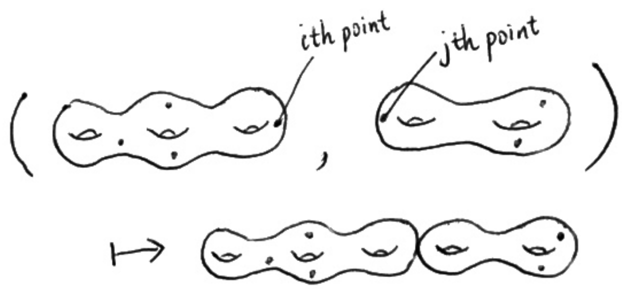
Given this naturality requirement, we see that Z determines and is determined by the single map

$$Z(\overline{C}_{g,n} \to \overline{M}_{g,n}) : A^{\otimes n} \to \mathrm{H}^*(\overline{M}_{g,n})$$

where $\overline{C}_{g,n}$ is the universal stable curve of genus g with n ordered marked points over the moduli $\overline{M}_{g,n}$ of stable curves of genus g with n ordered marked points.[3] For brevity we write $Z_{g,n}$ instead of $Z(\overline{C}_{g,n} \to \overline{M}_{g,n})$.

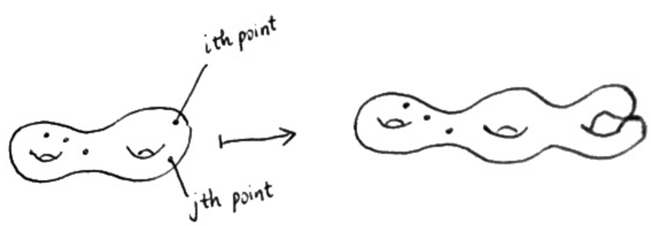
Now, given two stable curves of genus g_1 and g_2 respectively and with n_1 and n_2 ordered marked points respectively, we can glue the ith marked point of the first curve to the jth marked point of the second curve to obtain a new stable curve of genus $g_1 + g_2$ and with $n_1 + n_2 - 2$ ordered marked points. This can be done naturally in families and so we have maps

$$G_{i,j} : \overline{M}_{g_1,n_1} \times \overline{M}_{g_2,n_2} \to \overline{M}_{g_1+g_2,n_1+n_2-2}.$$



Similarly, we can take a stable curve of genus g with $n \geq 2$ marked points, and glue the ith point to the jth point to obtain a new stable curve of genus $g + 1$ and $n - 2$ marked points. This can be done naturally in families as well, and we have maps

$$H_{i,j} : \overline{M}_{g,n} \to \overline{M}_{g+1,n-2}.$$



[2]One could also use Chow rings [23].

[3]See [24] for background on moduli spaces and moduli stacks.

The $G_{i,j}$ and the $H_{i,j}$ are algebraic analogues of the gluing of bordisms described in the previous section.

Definition 1.2.1 (Kontsevich–Manin [35])**.** A *cohomological field theory*, or *CohFT*, is an assignment Z as above which is compatible with the maps $G_{i,j}$ and $H_{i,j}$. More precisely, the diagrams

$$\begin{array}{ccc} A^{\otimes(n_1+n_2-2)} & \xrightarrow{\Delta_{i,j}} & A^{\otimes(n_1+n_2)} \\ \downarrow{\scriptstyle Z_{g_1+g_2,n_1+n_2-2}} & & \downarrow{\scriptstyle Z_{g_1,n_1}\otimes Z_{g_2,n_2}} \\ \mathrm{H}^*(\overline{M}_{g_1+g_2,n_1+n_2-2}) & \xrightarrow{G^*_{i,j}} & \mathrm{H}^*(\overline{M}_{g_1,n_1}) \otimes \mathrm{H}^*(\overline{M}_{g_2,n_2}) \end{array}$$

and

$$\begin{array}{ccc} A^{\otimes(n-2)} & \xrightarrow{\Delta_{i,j}} & A^{\otimes n} \\ \downarrow{\scriptstyle Z_{g+1,n-2}} & & \downarrow{\scriptstyle Z_{g,n}} \\ \mathrm{H}^*(\overline{M}_{g+1,n-2}) & \xrightarrow{H^*_{i,j}} & \mathrm{H}^*(\overline{M}_{g,n}) \end{array}$$

must commute. The map $\Delta_{i,j}$ is given as follows. Let Δ_a be a basis of A and let $g_{ab} = g(\Delta_a, \Delta_b)$. Then we define

$$\Delta_{i,j}(a_1 \otimes \cdots \otimes a_k) = \sum_{a,b} g_{ab} \cdot a_1 \otimes \cdots \otimes \Delta_a \otimes \cdots \otimes \Delta_b \otimes \cdots \otimes a_n$$

where Δ_a and Δ_b are respectively always in the ith and jth tensor positions.

If we take some n_1 of the n points to be incoming points and the remaining $n_2 = n - n_1$ points to be outgoing points, and if we use the metric g to identify A with its dual, then we can think of the map

$$Z_{g,n} : A^{\otimes n} \to \mathrm{H}^*(\overline{M}_{g,n})$$

as being a map

$$A^{\otimes n_1} \to \mathrm{H}^*(\overline{M}_{g,n}) \otimes A^{\otimes n_2}.$$

Via Poincaré duality, this can also be thought of as a map

$$\mathrm{H}_*(\overline{M}_{g,n}) \otimes A^{\otimes n_1} \to \otimes A^{\otimes n_2},$$

which can then be interpreted from an *operadic* point of view as an action of the homology of the moduli spaces $\overline{M}_{g,n}$ on A [41].

Specializing to the case where X is a single curve rather than a family of curves, i.e. the case where B is a point, we see that we have assigned to a single curve a single map

$$A^{\otimes n_1} \to A^{\otimes n_2}.$$

Such a map is exactly what a TQFT assigns to (the underlying real surface of) the curve if we think of the marked points as boundary circles, and the commutativity of the diagrams in Definition 1.2.1 correspond to the TQFT functoriality property. So a CohFT can be thought of as a kind of *families TQFT*. One can also consider other kinds of families TQFTs, for different types of families of surfaces[4] — we will see some examples in the next section.

Note that since $\overline{M}_{0,3}$ is a single point, we have

$$Z_{0,3} : A^{\otimes 3} \to \mathrm{H}^*(\overline{M}_{0,3}) \cong k,$$

which, via the metric g, can be viewed as a map

$$m : A^{\otimes 2} \to A.$$

It is not hard to prove that:

Proposition 1.2.2. *The map m together with the metric g gives A the structure of a commutative Frobenius algebra.*

There are many interesting examples of CohFTs.

Example 1.2.3 (Gromov–Witten theory)**.** Let (X, ω) be a compact symplectic manifold, and suppose J is a compatible almost complex structure. Gromov–Witten theory concerns (J-)holomorphic maps of complex curves C into X. Let $\beta \in \mathrm{H}_2(X)$ and let $\overline{M}_{g,n}(X, \beta)$ denote the moduli space of stable maps $f : C \to X$ of genus g curves with n ordered marked points into X such that $f_*([C]) = \beta$. Then for each point p_i we have the evaluation map

$$e_{i,\beta} : \overline{M}_{g,n}(X, \beta) \to X$$

which sends a map $f : C \to X$ to $f(p_i) \in X$. We also have the forgetful map

$$\phi_\beta : \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n}.$$

Letting $A = \mathrm{H}^*(X)$ with Poincaré duality metric g, we thus can define

$$Z_{g,n,\beta} : A^{\otimes n} \to \mathrm{H}^*(\overline{M}_{g,n})$$

as

$$Z_{g,n,\beta}(a_1 \otimes \cdots \otimes a_n) = \phi_{\beta,*}(e^*_{1,\beta}(a_1) \wedge \cdots \wedge e^*_{n,\beta}(a_n))$$

[4]The best reference for families TQFTs is [62] but see also [56, 13].

where the pushforward is the virtual pushforward. Finally, put

$$Z_{g,n} = \sum_{\beta \in \mathrm{H}_2(X)} \exp(\textstyle\int_\beta \omega) \cdot Z_{g,n,\beta}.$$

Then it is a theorem that this defines a CohFT [35], at least if we assume convergence of the series. Furthermore, the induced Frobenius algebra structure on $A = \mathrm{H}^*(X)$ coincides with the Frobenius algebra structure given by ordinary cup product multiplication.

Example 1.2.4 (Verlinde bundles)**.** Instead of considering moduli of maps of curves into a space X, we can consider moduli of principal G-bundles on curves. These two situations could be considered as special cases of the same general construction — note that a principal G-bundle on a curve C is the same as a map from C to the classifying stack $\mathrm{B}G$.

For concreteness, a smooth curve C with n ordered marked points, and put $G = \mathrm{SL}(2)$. Let $M_G(C)$ be the moduli stack of principal G-bundles on C with universal bundle $U \to C \times M_G(C)$. We fix a *level* $k \in \mathbb{N}$. Then the state space of our theory will be the vector space A generated by the irreducible representations of $G = \mathrm{SL}(2)$ of dimension $\leq k+1$, considered as a subspace of the representation ring of $\mathrm{SL}(2)$; note that such representations can be considered as vector bundles over the classifying stack $\mathrm{BSL}(2)$. We have the important *inverse determinant line bundle*

$$\mathcal{L} := \det{}^{-1} \mathbb{R}\pi_{2,*}(U)$$

on $M_G(C)$, where the pushforward is the derived pushforward or the K-theoretic pushforward.

As in the previous Gromov–Witten theory example, there are evaluation maps

$$E_i : M_G(C) \to \mathrm{B}G$$

corresponding to the marked points. Hence given $V \in A$, we can consider V as a vector bundle over $\mathrm{B}G$ and thus we can take the pullback $E_i^*(V)$ to get a vector bundle over $M_G(C)$.

Additionally, all of this can be done in families, so if $C \to B$ is a relative curve of genus g and with n ordered marked points, then we have the relative moduli stack $M_G(C/B) \to B$. In particular we have the forgetful map

$$\phi : M_G(C_{g,n}/M_{g,n}) \to M_{g,n}.$$

So given $V_i \in A$, we define $Z^{(k)}_{g,n}(V_1 \otimes \cdots \otimes V_n)$ as the pushforward

$$\phi_*(\mathcal{L}^k \otimes e_1^*(V_1) \otimes \cdots \otimes e_n^*(V_n)). \tag{1.1}$$

This is a vector bundle over $M_{g,n}$. We remark also that the higher direct images are zero. Note that this construction does not give a vector bundle over $\overline{M}_{g,n}$. The reason is that moduli stacks of principal G-bundles over nodal curves are not complete. In order to extend this construction over $\overline{M}_{g,n}$, one needs to take a completion of those moduli stacks; however, there is no canonical choice of completion. See for example [18] and [60] for more references and discussion on such questions.

Because of the $\mathcal{L}^k$ term, this construction is different from the traditional construction of Gromov–Witten theory. We will be working in K-theory, but we remark that an analogue of this modified construction in the context of the Gromov–Witten *cohomology* theory of ordinary spaces was studied by Coates and Givental and is known as *twisted* Gromov–Witten theory [12].

In any case, we will refer to the bundles $Z^{(k)}_{g,n}(V_1 \otimes \cdots \otimes V_n)$ as *level k Verlinde bundles.* It is a theorem that these Verlinde bundles can be naturally extended to $\overline{M}_{g,n}$ in such a way that these extensions satisfy the CohFT gluing properties and thus give a K-*theoretic* CohFT. In this context, these gluing properties are also known as *the fusion rules* [55]. The ranks of these bundles is of mathematical interest and their CohFT properties allow us to derive a formula for these ranks. This formula is known as the Verlinde formula [8].

The Verlinde bundles are nontrivial, and so it is of interest to study not only their ranks but also their Chern classes and Chern characters. It is known that the Verlinde bundles carry projectively flat connections [29], and from this it follows that their Chern characters can be written in terms of the tautological classes in $\mathrm{H}^*(M_{g,n})$.

We will present some computations which identify explicitly the Chern characters of the Verlinde bundles for the case of $G = \mathbb{C}^*$. Furthermore, one can also consider *higher twisted* Verlinde bundles, which are defined by incorporating certain additional tensor factor terms in the expression (1.1). One of the main results of this thesis will be the following:

Theorem 1.2.5. *For $G = \mathbb{C}^*$, the Chern characters of the higher twisted Verlinde bundles can be expressed in terms of tautological classes on $M_{g,n}$.*

For now we just mention that this theorem will be proven by exploiting the fact that a sufficiently large symmetric power $C^{(N)}$ of a curve C is, via the Abel–Jacobi map, a projective space bundle over the Jacobian of the curve [3, 49]. Although we focus on the simplest case $G = \mathbb{C}^*$, our hope is that we will eventually be able to use the results from this case to do computations for other groups, such as $G = \mathrm{SL}(2)$, via the Teleman–Woodward localization theorem [63].

1.3 Topological conformal field theories

In the first section, we considered fixed surfaces with parameterized boundary circles. In the second section, we considered stable curves in families. Now we consider fixed, or families of, surfaces with parameterized boundary circles, some or all of which are divided into parameterized intervals that are either designated as *open* boundaries or as *free* boundaries. Boundary circles that are not so divided are *closed* boundaries. Moreover, each of the free boundaries is decorated with an element of a set of labels Λ, known as *D-branes*.

An *open conformal field theory*, as defined by Costello in [13], assigns a vector space $\mathrm{Hom}(\lambda, \lambda')$ to each ordered pair $(\lambda, \lambda') \in \Lambda \times \Lambda$ of D-branes[5]; and to a (Riemann) surface with incoming and outgoing open boundaries (and no closed boundaries) it assigns a morphism from the tensor product of the Hom spaces corresponding to the incoming open boundaries to that of the outgoing open boundaries.

These assignments should all be compatible with respect to gluing of surfaces along open boundaries with matching D-branes. Finally, we have the notion of *open topological conformal field theory*, or *open TCFT*, in which the Hom spaces are dg vector spaces (i.e. chain complexes) instead of ordinary vector spaces, and where the singular chains on the moduli spaces of surfaces with open boundaries act on the Hom spaces in a way that is compatible with the gluing maps (the analogues of the maps $G_{i,j}$ and $H_{i,j}$ from before). This action is in the appropriate operadic sense. Similarly we have *closed TCFTs* and *open-closed TCFTs*, which involve respectively surfaces with only closed boundaries and surfaces with both kinds of boundaries.

Costello [13] then proves that open TCFTs correspond to Calabi–Yau A_∞-categories — more precisely, that the category of open TCFTs is homotopy equivalent to the category of Calabi–Yau A_∞-categories. The Calabi–Yau condition on a category should thought of as a categorical analogue of the Frobenius condition on an algebra; and Costello's theorem can be thought of as a categorical analogue of Theorem 1.1.4. Given an open TCFT, the objects of the corresponding A_∞-category correspond to the D-branes, and the morphism spaces correspond to the dg vector spaces $\mathrm{Hom}(\lambda, \lambda')$ (which thus also explains the notation for those vector spaces).

Additionally, Costello proves that given a Calabi–Yau A_∞-category $\mathcal{C}$, there is an action of the chains $\mathrm{C}_*(\widetilde{M})$ on moduli spaces of surfaces with closed boundaries on the Hochschild chain complex of $\mathcal{C}$, and thus an action of the homology $\mathrm{H}_*(\widetilde{M})$ of those moduli spaces on the Hochschild homology $\mathrm{HH}_*(\mathcal{C})$.[6] Accordingly, this space is called the *closed state space* of the theory.

Remark 1.3.1. TCFTs are related to the notion of *extended 2-dimensional TQFTs*. The results of Costello mentioned above are discussed in relation to the Baez–Dolan cobordism hypothesis in §4.2 of Lurie's paper on extended TQFTs [39].

It is natural to ask if this action of $\mathrm{H}_*(\widetilde{M})$ can be extended to give a CohFT — that is to say, whether the action can be extended to an action of the homology $\mathrm{H}_*(\overline{M})$ of the Deligne–Mumford moduli spaces of stable curves. Kontsevich and collaborators have claimed the following:

Theorem 1.3.2 (Kontsevich–Soibelman [36], Katzarkov–Kontsevich–Pantev [32])**.** *Suppose $\mathcal{C}$ is a Calabi–Yau A_∞-category satisfying the Hodge-to-de Rham degeneration. Then a choice*

[5] The D stands for Dirichlet [30].

[6] An alternate account of these results can be found in [36].

of Calabi–Yau structure on $\mathcal{C}$ together with a choice of splitting of the Hodge filtration determines a CohFT with state space $A = \mathrm{HH}_*(\mathcal{C})$ and metric g induced by the Calabi–Yau structure. This CohFT extends the action of the homology $\mathrm{H}_*(\widetilde{M})$ of moduli spaces of surfaces with closed boundaries on A. Different choices of splitting of the Hodge filtration determine different extensions.

Conjecture 1.3.3 ("Degeneration conjecture" [36]). *If $\mathcal{C}$ is an A_∞-category that is homologically smooth and proper, then it satisfies the Hodge-to-de Rham degeneration.*

Example 1.3.4 (Homological mirror symmetry for Calabi–Yau manifolds). *Mirror symmetry*[7] was the term coined for an observation, first made by physicists, that every smooth compact Calabi–Yau manifold has a smooth compact Calabi–Yau manifold partner such that the Hodge diamond of the first is a reflection of that of the second. Such manifolds were called *mirror manifolds*. It was then later discovered and proved that mirror manifolds are related in much deeper ways. For instance, the genus 0 Gromov–Witten theory of a Calabi–Yau manifold is related to the variation of Hodge structure of the mirror Calabi–Yau manifold [14]. This is the "mirror conjecture" first proved by Givental and Lian–Liu–Yau.

At the 1994 ICM, Kontsevich proposed the *homological* mirror symmetry conjecture [34], which posited further relations between mirror manifolds, namely that the Fukaya category[8] [21, 22], respectively the derived category, of a Calabi–Yau is equivalent to the derived category, respectively the Fukaya category, of its mirror Calabi–Yau.

The derived category of a smooth compact Calabi–Yau manifold is known to be homologically smooth, proper, and Calabi–Yau as a category. The same should be true for the Fukaya category of such a manifold. Thus such categories yield CohFTs, by the results of Costello and Kontsevich et al. above. It is believed that the CohFTs arising in this fashion from Fukaya categories should be related to the CohFTs coming from Gromov–Witten theory.

Mirror symmetry and homological mirror symmetry have since been extended beyond Calabi–Yau manifolds. Let Y be, for example, a Fano variety. Then its mirror should be not a manifold or a variety, but a *Landau–Ginzburg model* [6]. A Landau–Ginzburg model is a pair (X, W), where X is a Kähler manifold or variety, and W is a holomorphic function $X \to \mathbb{C}$ called the *superpotential*. Associated to a Landau–Ginzburg model one can define a category $\mathrm{MF}(X, W)$ of *matrix factorizations*, which is a Landau–Ginzburg model analogue of the derived category of coherent sheaves on an ordinary variety. Then homological mirror symmetry in this new context states that $\mathrm{MF}(X, W)$ is equivalent to the Fukaya category of Y. It also states that the Fukaya–Seidel category [57] of (X, W) is equivalent to the derived category of Y.

[7]For a comprehensive treatment of mirror symmetry and homological mirror symmetry, see [30]. We give only a brief outline here.

[8]Strictly speaking Fukaya categories are not categories but A_∞-categories.

If $X = \operatorname{Spec} R$ for a commutative finite type $\mathbb{C}$-algebra R and if $W \in R$ has a single critical value, which we assume without loss of generality to be $0 \in \mathbb{C}$, then the differential $\mathbb{Z}/2\mathbb{Z}$-graded category of matrix factorizations $\mathrm{MF}(X, W)$ is defined as follows.

Definition 1.3.5. The category $\mathrm{MF}(X, W)$ of *matrix factorizations* has as objects

$$P = (P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0)$$

where the P_i are finitely generated projective R-modules, and the p_i are R-module morphisms satisfying $p_{i+1} \circ p_i = W \cdot \mathrm{id}_{P_i}$.[9] For the morphisms between two matrix factorizations P and P', one takes the $\mathbb{Z}/2\mathbb{Z}$-graded complex of all R-module morphisms

$$\mathrm{Hom}(P, P') = \bigoplus_{i,j} \mathrm{Hom}_R(P_i, P'_j)$$

with grading given by $i + j$ (modulo 2), and with the differential

$$\partial : f \longmapsto p' \circ f - (-1)^{|f|} f \circ p$$

for homogeneous f. For more details, see [48, 47].

Matrix factorizations are also known as *curved $\mathbb{Z}/2\mathbb{Z}$-graded complexes of finitely generated projective R-modules with curvature W* [52]. We may truncate this in various ways: *curved complexes of projective modules*, *curved projective modules*, etc. More generally, we will consider *curved modules* of various kinds, not necessarily finitely generated or projective.

When R is regular and local, and if W has an isolated singularity at the unique closed point of $\operatorname{Spec} R$, Dyckerhoff [15] has proved that the category of matrix factorizations of $(\operatorname{Spec} R, W)$ is homologically smooth, proper, Calabi–Yau, and dg affine. We generalize these results as follows:

Theorem 1.3.6 (Lin–Pomerleano [37])**.** *If X is any smooth Calabi–Yau variety — not necessarily affine — and $W : X \to \mathbb{C}$ has a proper singular locus — not necessarily isolated — then one can associate to the Landau–Ginzburg model (X, W) an appropriate category of matrix factorizations that is dg affine, homologically smooth, proper, and Calabi–Yau.*

We are also able to identify compact generators for our matrix factorization categories. Let us denote these categories as $\mathrm{MF}(X, W)$ for now; we will use different notation later. Then, assuming the degeneration conjecture, with a choice of Calabi–Yau structure and splitting of the Hodge filtration, such a Landau–Ginzburg model should yield cohomological field theories with state space $\mathrm{HH}_*(\mathrm{MF}(Y, W))$. We will compute this state space:

[9] The nomenclature "matrix factorization" is due to Eisenbud [16] and comes from the fact that when the P_i are free modules, the d_i can be thought of as matrices with entries in A that factorize the scalar matrices $W \cdot \mathrm{id}_{P_i}$.

Theorem 1.3.7 (Lin–Pomerleano [37])**.** *Under the same hypotheses of the previous theorem, we have*

$$\mathrm{HH}^*(\mathrm{MF}(X, W)) \cong \mathrm{H}^*(\Lambda^\bullet T_X, [W, -])$$

where $\Lambda^\bullet T_X$ *is the sheaf of polyvector fields on* X *and* $[-,-]$ *is the Schouten–Nijenhuis bracket. The Hochschild homology and Hochschild cohomology agree up to a shift of grading by the dimension of* X*.*

More precise statements of these theorems will be given in the last chapter of this thesis. In the final section, we will also briefly discuss some applications of these results to new examples of homological mirror symmetry.

Chapter 2

Higher twists of abelian gauge theory in families

We begin our study of higher twists of abelian gauge theory by first recalling some basic requisite notions and definitions. We will always work over the base field $\mathbb{C}$, and we work with the multiplicative group of complex numbers $G = \mathbb{C}^*$. In §2.2 and §2.3 we will compute the Chern characters of all of the Verlinde bundles without higher twists over the moduli stack of smooth curves $M_{g,n}$ [24]. Then in §2.4 we will show that they can be naturally extended to the Deligne–Mumford moduli stack $\overline{M}_{g,n}$ of stable curves in such a way that they yield a CohFT.

The remainder of the chapter is dedicated to higher twists, as defined by Teleman–Woodward [63], and showing that the invariants in that case are still tautological. This is done by employing the classical fact that a sufficiently large symmetric power $C^{(N)}$ of a curve is a projective bundle over the Jacobian. Let us briefly recall the classical argument for this fact [3]. The Abel–Jacobi map u sends an effective divisor $x_1 + \cdots + x_N$, i.e. a point of the symmetric power $C^{(N)}$, to the point of the Jacobian Jac^N corresponding to the line bundle $\mathcal{O}(x_1 + \cdots + x_N)$. Given a degree N line bundle $L \in \mathrm{Jac}^N$, the fiber $u^{-1}(L)$ consists of those degree N effective divisors D such that $\mathcal{O}(D) \cong L$. The set of such effective divisors is naturally identified with the projectivization $\mathbb{P}\mathrm{H}^0(L)$ of the space of global sections of L [26]. It is in this way, roughly speaking, that we can view $u : C^{(N)} \to \mathrm{Jac}^N$ as a bundle of projective spaces.

In any case, the index of a vector bundle over Jac agrees with the index of the vector bundle after pulling back to $C^{(N)}$. Then by pulling back again to the Cartesian power C^N, the Grothendieck–Riemann–Roch formula allows us to express the Chern character of the higher twisted Verlinde bundles in terms of tautological classes.

2.1 Relative Jacobians

We start by reviewing Jacobians of smooth curves, i.e. moduli spaces of line bundles, i.e. moduli spaces of principal $\mathbb{C}^*$-bundles. This section is largely based on [33].

Definition 2.1.1. Let $f : X \to S$ be a flat and projective morphism with integral geometric fibers. Given any S-scheme $T \to S$, we let X_T denote the base change $X \times_S T$, and let f_T denote the corresponding projection $X_T \to T$.

Definition 2.1.2. The (*relative*) *Picard functor* Pic_f is the functor from the category of S-schemes to the category of sets given on objects by

$$T \mapsto \operatorname{Pic}(X_T)/\operatorname{Pic}(T).$$

The Picard group $\operatorname{Pic}(X_T)$ of isomorphism classes of line bundles is naturally a module over $\operatorname{Pic}(T)$ via the pullback f_T^*.

Definition 2.1.3. The (*relative*) *Picard stack* $\underline{\operatorname{Pic}}_f$ is the functor which takes an S-scheme T and sends it to the groupoid of line bundles on X_T, or equivalently, the groupoid of principal $\mathbb{C}^*$-bundles on X_T.

It is a theorem of Grothendieck that Pic_f is represented by a scheme which is separated and locally of finite type over S; hence we will use the same notation Pic_f to refer to this representing object, which we will call the (*relative*) *Picard scheme*.

In the rest of this paper, any family $f : X \to S$ will always be equipped with at least one section $s : S \to X$. Then we can define the *s-rigidified Picard functor* $\operatorname{Pic}_{f,s}$ to be the functor from the category of S-schemes to the category of sets given on objects by

$$T \mapsto \{(L, u) : L \in \operatorname{Pic}(X_T), u : s_T^* L \xrightarrow{\cong} \mathcal{O}_T\}/\sim .$$

The relation that we quotient out by is the obvious one: we say that $(L, u) \sim (L', u')$ if there is an isomorphism $\psi : L \to L'$ such that $u = u' \circ s_T^* \psi$.

The isomorphism u is called an *s-rigidification* of L.

Proposition 2.1.4. *There is an isomorphism* $\operatorname{Pic}_{f,s} \xrightarrow{\cong} \operatorname{Pic}_f$.

Proof. The map sends the equivalence class of a pair (L, u) consisting of a line bundle L on X_T and an s-rigidification $u : s_T^* L \xrightarrow{\cong} \mathcal{O}_T$ to the equivalence class of L in $\operatorname{Pic}(X_T)/\operatorname{Pic}(T)$. Given any $[N] \in \operatorname{Pic}(X_T)/\operatorname{Pic}(T)$, the line bundle $N' = N \otimes f_T^* s_T^* N^{-1}$ is in the same equivalence class as N and is trivial when pulled back to T via the section s_T, so the map is surjective. It is clearly injective. □

Proposition 2.1.5. *There is an isomorphism*

$$\operatorname{Pic}_{f,s} \times \mathrm{B}\mathbb{C}^* \xrightarrow{\cong} \underline{\operatorname{Pic}}_f,$$

where $\mathrm{B}\mathbb{C}^*$ *denotes the classifying stack of line bundles or equivalently the classifying stack of principal* $\mathbb{C}^*$*-bundles.*

Proof. The map assigns a pair (L, u) together with a line bundle E on T to the line bundle $L \otimes f_T^* E$ on X_T. The inverse map sends a line bundle N on X_T to the pair $(N \otimes f_T^* s_T^* N^{-1}, u)$, where u is the canonical isomorphism $s_T^*(N \otimes f_T^* s_T^* N^{-1}) \cong \mathcal{O}_T$, together with the line bundle $s_T^* N$ on T. □

Let P denote a universal (i.e. Poincaré) line bundle over $X \times_S \mathrm{Pic}_f \cong X \times_S \mathrm{Pic}_{f,s}$, and let $\mathcal{O}(1)$ be the line bundle over $\mathrm{B}\mathbb{C}^*$ corresponding to the standard 1-dimensional irreducible representation of $\mathbb{C}^*$. It is not hard to see that $P \boxtimes \mathcal{O}(1)$ gives a universal line bundle over

$$X \times_S \underline{\mathrm{Pic}}_f \cong X \times_S \mathrm{Pic}_{f,s} \times \mathrm{B}\mathbb{C}^*.$$

Definition 2.1.6. In the case that the fibers of $f : X \to S$ are smooth curves, which is the situation of interest in this paper, the Picard scheme Pic_f (resp. $\mathrm{Pic}_{f,s}$, resp. $\underline{\mathrm{Pic}}_f$) breaks into countably infinitely many components indexed by the integers. The d-th component corresponds to line bundles of (relative) degree d. It is the *degree d (relative) Jacobian scheme* Jac_f^d (resp. *s-rigidified Jacobian scheme* $\mathrm{Jac}_{f,s}^d$, resp. *Jacobian stack* $\underline{\mathrm{Jac}}_f^d$).

The degree 0 Jacobian scheme is an abelian scheme and each degree d Jacobian is a torsor over the degree 0 Jacobian and, at least in the presence of a section of f, the different degree Jacobians are non-canonically isomorphic.

Now let $M_{g,n}$ denote the moduli stack of genus g curves with $n > 0$ marked points, and let

$$C_{g,n} \to M_{g,n}$$

be the universal curve, with sections (i.e. marked points) denoted by $p_1, \ldots, p_n$. Then we have the corresponding universal degree d Jacobian scheme (resp. stack) $\mathrm{Jac}_{g,n}^d$ (resp. $\underline{\mathrm{Jac}}_{g,n}^d$) over $M_{g,n}$, and an isomorphism $\underline{\mathrm{Jac}}_{g,n}^d \cong \mathrm{Jac}_{g,n}^d \times \mathrm{B}\mathbb{C}^*$. Letting P_d be a Poincaré bundle over

$$C_{g,n} \times_{M_{g,n}} \mathrm{Jac}_{g,n}^d,$$

we see that $P_d \boxtimes \mathcal{O}(1)$ gives a Poincaré bundle over

$$C_{g,n} \times_{M_{g,n}} \underline{\mathrm{Jac}}_{g,n}^d \cong C_{g,n} \times_{M_{g,n}} \mathrm{Jac}_{g,n}^d \times \mathrm{B}\mathbb{C}^*.$$

In order to make the Poincaré bundle unique (up to canonical isomorphism), we require that P_d is trivial along the first section p_1; more precisely we require P_d to be trivial when we pull it back along the map

$$p_1 \times \mathrm{id} : \mathrm{Jac}_{g,n}^d \to C_{g,n} \times_{M_{g,n}} \mathrm{Jac}_{g,n}^d.$$

If E is a sheaf on $C_{g,n} \times_{M_{g,n}} \mathrm{Jac}_{g,n}^d$, then for convenience we will write

$$E|_{p_i} := (p_i \times \mathrm{id})^* E. \tag{2.1}$$

Concretely, we can set $\mathrm{Jac}_{g,n}^d = \mathrm{Jac}_{g,n}^0$ and we can set

$$P_d = P_0 \otimes \pi_1^* \mathcal{O}_C(d \cdot p_1) \otimes \mathcal{L}_i^{\otimes d}, \tag{2.2}$$

where p_1 here denotes by abuse of notation the image of the first section $p_1 : M_{g,n} \to C_{g,n}$ considered as a relative effective divisor; letting ω denote the relative canonical bundle of $C_{g,n} \to M_{g,n}$, the *i-th cotangent line* Ψ_i refers to the line bundle $p_i^*\omega$ over $M_{g,n}$. Throughout this paper we will also abusively write Ψ_i to refer to the lift of Ψ_i along any map to the base $M_{g,n}$. We do the same for the *Hodge bundle* $\mathbb{H}_g := \Pi_*\omega$; Π will always denote projection to the base stack $M_{g,n}$. Note that thusly defined, P_d satisfies $P_d|_{p_1} \cong \mathcal{O}$ as required.

The moduli stack $M_{g,n}(\mathrm{B}\mathbb{C}^*)$ of maps of curves to $\mathrm{B}\mathbb{C}^*$ and the union $\underline{\mathrm{Jac}}^\bullet_{g,n} := \coprod_d \underline{\mathrm{Jac}}^d_{g,n}$ of universal Jacobian stacks are isomorphic as stacks over $M_{g,n}$. We will freely switch between these two whenever convenient. The former notation allows us to think in terms of Gromov–Witten theory; the latter notation allows us to think in terms of gauge theory.

2.2 The inverse determinant line

Let V be a representation of $\mathbb{C}^*$. Let U be the universal principal $\mathbb{C}^*$-bundle over

$$C_{g,n} \times_{M_{g,n}} M_{g,n}(\mathrm{B}\mathbb{C}^*)$$

associated to the universal principal $\mathbb{C}^*$-bundle. Let E^*V denote the vector bundle $U \times_{\mathbb{C}^*} V$.

Definition 2.2.1. We define the (*Dolbeault*) *index bundle*

$$E_C^*V := \mathbb{R}\pi_{2,*}E^*V, \tag{2.3}$$

the *inverse determinant line*

$$D_C(V) := \det{}^{-1} E_C^*V, \tag{2.4}$$

and the *evaluation bundles*

$$E_i^*V, \tag{2.5}$$

where E_i is the map $M_{g,n}(\mathrm{B}\mathbb{C}^*) \to \mathrm{B}\mathbb{C}^*$ corresponding to evaluating at the i-th marked point and V is considered as a vector bundle over $\mathrm{B}\mathbb{C}^*$. Finally, let V_1 be the standard representation of $\mathbb{C}^*$. Then we define

$$\mathcal{L} := D_C(V_1) \tag{2.6}$$

Remark 2.2.2. Note that the bundle E^*V_1 is just the Poincaré bundle over $C_{g,n} \times_{M_{g,n}} \underline{\mathrm{Jac}}^\bullet_{g,n}$.

Theorem 2.2.3. *We have* $\mathbb{R}\Pi_*\mathcal{L} = \mathcal{O}_{M_{g,n}}$.

Proof. Restricting to the d-th connected component

$$C_{g,n} \times_{M_{g,n}} \mathrm{Jac}^d_{g,n} \times \mathrm{B}\mathbb{C}^*,$$

the bundle E^*V_1 is $P_d \boxtimes \mathcal{O}(1)$, and so we have

$$\mathcal{L} = (\det{}^{-1}\mathbb{R}\pi_{2,*}P_d) \boxtimes \mathcal{O}(-d + g - 1)$$

by the Riemann–Roch theorem.

Since the index along $\underline{\mathrm{Jac}}^\bullet = \mathrm{Jac}^\bullet \times \mathrm{B}\mathbb{C}^* \to M_{g,n}$ is the $\mathbb{C}^*$-invariant part of the index along $\mathrm{Jac}^\bullet \to M_{g,n}$, and since

$$-d + g - 1 = 0 \Rightarrow d = g - 1,$$

we see that only the $(g-1)$-st Jacobian gives a nonzero contribution to $\mathbb{R}\Pi_*\mathcal{L}$. So working over the degree $(g-1)$ Jacobian, consider the short exact sequence

$$0 \to P_{g-1} \xrightarrow{i} P_{g-1} \otimes \pi_1^*\mathcal{O}_C(N \cdot p_1) \xrightarrow{\alpha} Q \to 0,$$

where N is large and Q is $\mathrm{coker}(i)$. Then $\mathbb{R}\pi_{2,*}P_{g-1}$ is represented by the complex of vector bundles

$$\pi_{2,*}(P_{g-1} \otimes \pi_1^*\mathcal{O}_C(N \cdot p_1)) \xrightarrow{\pi_{2,*}(\alpha)} \pi_{2,*}Q.$$

The two vector bundles are of the same rank, and therefore we can take the determinant of the morphism $\pi_{2,*}(\alpha)$, which then gives us a nonvanishing section of

$$\mathcal{L} = \det{}^{-1}\mathbb{R}\pi_{2,*}P_{g-1} = \det \pi_{2,*}Q \otimes \det{}^{-1}\pi_{2,*}(P_{g-1} \otimes \pi_1^*\mathcal{O}_C(N \cdot p_1)).$$

This section corresponds to the classical *theta function* [3].

If we work over a fixed curve rather than a family of curves, then $\mathcal{L}$ gives a principal polarization of the Jacobian. Thus we see that $R^i\Pi_*\mathcal{L}$ is zero for each $i > 0$ and is of rank 1 for $i = 0$. But we have shown that $\Pi_*\mathcal{L}$ has a nonvanishing section, hence it is the trivial line bundle. □

Definition 2.2.4. For each postive integer k, we let ρ^k denote the multiplicative K-theory operation which is by the splitting principle defined by the property that

$$\rho^k(L) = 1 + L + L^2 + \cdots + L^{k-1}$$

for line bundles L. Similarly, ψ^k denotes the k-th Adams operation on K-theory which is by the splitting principle defined by the property $\psi^k(L) = L^k$ for line bundles L.

Theorem 2.2.5. *For each positive integer k, we have $\mathbb{R}\Pi_*(\mathcal{L}^k) = \rho^k(\mathbb{H}_g)$ in $\mathrm{K}(M_{g,n})[1/k]$.*

Proof. This follows by the Adams–Riemann–Roch theorem, which states that, after inverting k, we have in K-theory

$$\mathbb{R}\Pi_*(\psi^k(\mathcal{L}) \otimes \rho^k(\Omega_\Pi)^{-1}) = \psi^k(\mathbb{R}\Pi_*\mathcal{L})$$

where Ω_Π is the relative cotangent bundle of Π. But recall that Ω_Π is just the Hodge bundle $\mathbb{H}_g$ and hence by the projection formula we have

$$\mathbb{R}\Pi_*(\mathcal{L}^k) = \mathbb{R}\Pi_*(\psi^k\mathcal{L}) = \rho^k(\mathbb{H}_g) \otimes \psi^k(\mathcal{O}_{M_{g,n}}) = \rho^k(\mathbb{H}_g) \otimes \mathcal{O}_{M_{g,n}} = \rho^k(\mathbb{H}_g)$$

as claimed. □

Definition 2.2.6. Recall that the Chern character $\mathrm{ch}(E)$ of a vector bundle E is given as follows. Let $a_1, \ldots, a_n$ be the Chern roots of E. Then

$$\mathrm{ch}(E) = \sum_{i=1}^{n} \exp(a_i).$$

The bundle $\mathbb{R}\Pi_* \mathcal{L}^k$ is equipped with a natural projectively flat connection [29]. As a consequence, its Chern character satisfies the relation

$$\mathrm{ch} = \mathrm{rank} \cdot \exp(\mathrm{c}_1 / \mathrm{rank}).$$

We will refer to the ratio $\mathrm{c}_1/\mathrm{rank}$ as the *Chern slope*. Let $\lambda_i := \mathrm{c}_i(\mathbb{H}_g)$.

Theorem 2.2.7. *The Chern slope of* $\mathbb{R}\Pi_* \mathcal{L}^k$ *is* $\frac{k-1}{2}\lambda_1$.

Proof. By the splitting principle, let us write $\mathbb{H}_g = L_1 + \cdots + L_g$ and $a_i = c_1(L_i)$, i.e. the a_i are the Chern roots. Then we have

$$\rho^k(\mathbb{H}_g) = \prod_{i=1}^{g} (1 + L_i + \cdots + L_i^{k-1}).$$

Then an inductive argument on g shows that

$$\mathrm{c}_1(\rho^k(\mathbb{H}_g)) = \frac{k^g(k-1)}{2}(a_1 + \cdots + a_g),$$

and we have $\lambda_1 = a_1 + \cdots + a_g$. We remark that $\frac{k^g(k-1)}{2}$ is integral for all k. It is easy to see that $\mathrm{rank}\, \rho^k(\mathbb{H}_g) = k^g$. □

2.3 Evaluation bundles

In this section we will extend the calculation of the previous section to indexes of products of determinant line bundles tensored with products of evaluation bundles.

Theorem 2.3.1. *Let* $A \to S$ *be an abelian scheme. Let* P *be the functor sending an* S*-scheme* T *to the group of isomorphism classes of rigidified line bundles on* A_T*. Let* L *be a line bundle on* A*. Then there is a map*

$$\phi_L : A \to P, a \mapsto t_a^* L \otimes L^{-1} \otimes L|_e^{-1} \otimes L|_a^{-1}$$

and it is a homomorphism.

Proof. It is a standard consequence of the theorem of the cube [43]. □

Corollary 2.3.2. *We have*

$$t_{x+y}^*\det{}^{-1}\mathbb{R}\pi_{2,*}E \otimes \det{}^{-1}\mathbb{R}\pi_{2,*}E \cong t_x^*\det{}^{-1}\mathbb{R}\pi_{2,*}E \otimes t_y^*\det{}^{-1}\mathbb{R}\pi_{2,*}E$$

for any bundle E on $C \times_M \mathrm{Jac}$ and any sections $x, y : M \to \mathrm{Jac}$.

For any positive integer k and any line bundle K on the universal curve $C_{g,n} \to M_{g,n}$, Jarvis [31] constructs an algebraic stack $\mathfrak{S}_{g,n}^{1/k}(K)$ of triples (X, L, b) consisting of a smooth curve X, a line bundle L on X, and an isomorphism $b : L^{\otimes r} \to K$. In other words it is the moduli stack of curves together with a k-th root of K. When K is the relative canonical bundle $\omega_{C_{g,n}/M_{g,n}}$ then we get the moduli stack of k-spin curves.

Theorem 2.3.3 ([31])**.** *$\mathfrak{S}_{g,n}^{1/k}(K)$ is a smooth Deligne–Mumford stack, and the natural forgetful morphism*

$$F : \mathfrak{S}_{g,n}^{1/k}(K) \to M_{g,n}$$

is finite.

Theorem 2.3.4. *Fix a positive integer k. Let V_i be the i-th tensor power of the standard representation V_1 of $\mathbb{C}^*$. Let*

$$\mathcal{E} = \mathcal{L}^k \otimes E_1^* V_{w_1} \otimes \cdots \otimes E_n^* V_{w_n}$$

where $0 \le w_i \le k-1$. If $\sum_i w_i$ is a multiple of k, then $\mathbb{R}\Pi_*\mathcal{E} = \rho^k(\mathbb{H}_g)$ in rational K-theory. Otherwise, $\mathbb{R}\Pi_*\mathcal{E} = 0$.

Proof. Restricting to the degree d Jacobian, we have

$$\mathcal{L}^k = L^k \boxtimes \mathcal{O}((-d+g-1)k),$$

where

$$L = \det{}^{-1}\mathbb{R}\pi_{2,*}P_d.$$

For any i, j, we have

$$E_j^* V_i = P_d^{\otimes i}|_{p_j} \boxtimes \mathcal{O}(i)$$

and therefore it is clear that $\mathbb{R}\Pi_*\mathcal{E} = 0$ unless $\sum_i w_i$ is a multiple of k.

So suppose that $\sum_i w_i = N \cdot k$ for some integer N. Then $\mathbb{R}\Pi_*\mathcal{E}$ gets a nonzero contribution only from the degree

$$D := g - 1 - N$$

Jacobian. We have already covered the case $N = 0$; so assume $N > 0$. Then, working over the degree D Jacobian, we can write

$$\mathcal{E} = \mathcal{L}^k \otimes P_D^{\otimes w_1}|_{p_1} \otimes \cdots \otimes P_D^{\otimes w_n}|_{p_n} = (\det{}^{-1}\mathbb{R}\pi_{2,*}P_D)^k \otimes P_D^{\otimes w_1}|_{p_1} \otimes \cdots \otimes P_D^{\otimes w_n}|_{p_n}.$$

For any point p_i, consider the short exact sequence

$$0 \to \mathcal{O}_C(-p_i) \to \mathcal{O} \to \mathcal{O}_{p_i} \to 0.$$

Pulling back and then tensoring by P_D, we get

$$0 \to P_D(-p_i) \to P_D \to P_D|_{p_i} \to 0.$$

Pushing down and then taking determinants, we see that

$$\det{}^{-1}\mathbb{R}\pi_{2,*}P_D(-p_i) \cong \det{}^{-1}\mathbb{R}\pi_{2,*}P_D \otimes P_D|_{p_i} = \mathcal{L} \otimes P_D|_{p_i}.$$

We can inductively apply the above argument to conclude that

$$\mathcal{L} \otimes P_D|_{p_{i_1}} \otimes \cdots \otimes P_D|_{p_{i_N}} \cong \det{}^{-1}\mathbb{R}\pi_{2,*}P_D(-p_{i_1} - p_{i_2} - \cdots - p_{i_N}),$$

for pairwise distinct indices $i_1, \ldots, i_N$.

Recalling our definition (2.2) of the Poincaré bundle P_D over the Jacobian $\mathrm{Jac}^D_{g,n} = \mathrm{Jac}^0_{g,n}$, a straightforward argument shows that

$$\det{}^{-1}\mathbb{R}\pi_{2,*}P_D(-p_{i_1} - \cdots - p_{i_N}) \cong t_x^*\det{}^{-1}\mathbb{R}\pi_{2,*}P_{g-1},$$

a translate of the principal polarization, for some section $x : M_{g,n} \to \mathrm{Jac}^0_{g,n}$. For example, if $1 \notin \{i_1, \ldots, i_N\}$, then the section x is the one corresponding to the line bundle

$$\mathcal{O}_C(Np_1 - p_{i_1} - p_{i_2} - \cdots - p_{i_N}) \otimes \Psi_1^{\otimes N}$$

on the universal curve $C_{g,n}$. On the other hand if, say, $i_1 = 1$, then the section x corresponds to the line bundle

$$\mathcal{O}_C((N-1)p_1 - p_{i_2} - \cdots - p_{i_N}) \otimes \Psi_1^{\otimes N-1}$$

on $C_{g,n}$. In this case we have

$$\begin{aligned} t_x^*\det{}^{-1}\mathbb{R}\pi_{2,*}P_{g-1} &\cong \det{}^{-1}\mathbb{R}\pi_{2,*}P_{D-1}(-p_{i_2} - \cdots - p_{i_N}) \\ &\cong \det{}^{-1}\mathbb{R}\pi_{2,*}P_D(-p_{i_1} - \cdots - p_{i_N}) \otimes \Psi_1^{-1} \\ &\cong \det{}^{-1}\mathbb{R}\pi_{2,*}P_D(-p_{i_1} - \cdots - p_{i_N}) \end{aligned}$$

where the last equality is by base change and the fact that $\mathbb{R}\pi_{2,*}P_D(-p_{i_1} - \cdots - p_{i_N})$ is a virtual bundle of rank 0 by Riemann–Roch.

Working inductively, we can conclude that over the degree D Jacobian, $\mathcal{E}$ is isomorphic to a product of translates of the principal polarization:

$$\mathcal{E} \cong t_{x_1}^*\det{}^{-1}\mathbb{R}\pi_{2,*}P_{g-1} \otimes \cdots \otimes t_{x_k}^*\det{}^{-1}\mathbb{R}\pi_{2,*}P_{g-1}$$

for some sections x_i. By the theorem of the cube, we have

$$\mathcal{E} \cong t_x^*\det{}^{-1}\mathbb{R}\pi_{2,*}P_{g-1} \otimes (\det{}^{-1}\mathbb{R}\pi_{2,*}P_{g-1})^{\otimes k-1}$$

where $x = \sum_i x_i$. Let L_x be the line bundle on $C_{g,n}$ which corresponds to the section x. Now consider the Cartesian square

$$\begin{array}{ccc} \mathrm{Jac}' & \longrightarrow & \mathrm{Jac} \\ \big\downarrow{\scriptstyle \Pi'} & & \big\downarrow{\scriptstyle \Pi} \\ \mathfrak{S}^{1/k}_{g,n}(L_x) & \xrightarrow{F} & M_{g,n}. \end{array}$$

Working over $\mathfrak{S}^{1/k}_{g,n}(L_x)$ we have a kth root of L_x so it makes sense to speak of a section "x/k" of Jac$'$. Then, decomposing x as k copies of x/k, we have, again by the theorem of the cube,

$$\mathcal{E} \cong t_x^* \mathrm{det}^{-1} \mathbb{R}\pi_{2,*} P_{g-1} \otimes (\mathrm{det}^{-1} \mathbb{R}\pi_{2,*} P_{g-1})^{\otimes k-1} \cong (t_{x/k}^* \mathrm{det}^{-1} \mathbb{R}\pi_{2,*} P_{g-1})^{\otimes k}.$$

Therefore we have $\mathbb{R}\Pi'_* \mathcal{E} = \Pi'_* \mathcal{E} = \rho^k(\mathbb{H}_g)$ by repeating the exact same argument as in the proof of Theorem 2.2.5. Since F is a finite cover, it follows that

$$F^* : \mathrm{H}^*(M_{g,n}; \mathbb{Q}) \to \mathrm{H}^*(\mathfrak{S}^{1/k}_{g,n}(L_x); \mathbb{Q})$$

is injective, and thus F^* is injective on rational K-theory as well, via the Chern character. Thus we have $\Pi_* \mathcal{E} = \rho^k(\mathbb{H}_g)$ in rational K-theory, since $F^* \mathbb{H}_g = \mathbb{H}_g$. $\square$

Definition 2.3.5. We will think of the classes calculated by the above theorem as the *level k Gromov–Witten $\mathrm{K}_{\mathbb{Q}}$-classes of $\mathrm{B}\mathbb{C}^*$ over the locus $M_{g,n}$ of smooth curves* (or the *level k Verlinde bundles for gauge group $G = \mathbb{C}^*$*). Let us thus introduce the suggestive notation

$$\begin{aligned} Z^{(k)}_{g,n}(V_{w_1}, \ldots, V_{w_n}) &= \mathbb{R}\Pi_*(\mathcal{L}^k \otimes E_1^* V_{w_1} \otimes \cdots \otimes E_n^* V_{w_n}) \\ &= \Pi_*(\mathcal{L}^k \otimes E_1^* V_{w_1} \otimes \cdots \otimes E_n^* V_{w_n}) \end{aligned}$$

for $0 \leq w_i \leq k-1$.

2.4 Extension and factorization

The Hodge bundle $\mathbb{H}_g$ on $M_{g,n}$ has a natural extension to the Deligne–Mumford compactification $\overline{M}_{g,n}$, the moduli stack of *stable curves*: curves of genus g with n marked points and with at worst nodal singularities and finite automorphism groups. Namely, consider the universal curve $\overline{C}_{g,n}$ over $\overline{M}_{g,n}$, and let $\overline{\omega}$ be its relative dualizing sheaf, i.e. its relative Serre duality sheaf [26]. Then $\Pi_* \overline{\omega}$ is an extension of $\mathbb{H}_g$ from $M_{g,n}$ to $\overline{M}_{g,n}$, where Π as before denotes the map to the base, which is now $\overline{M}_{g,n}$. By abuse of notation and terminology, we also will refer to $\Pi_* \overline{\omega}$ as the *Hodge bundle* and also denote it by $\mathbb{H}_g$.

The bundles resulting from our previous calculations over the open moduli space $M_{g,n}$ thus can be extended to the compactification $\overline{M}_{g,n}$.

Definition 2.4.1. Let $0 \leq w_i \leq k-1$. Then the *extended level k Gromov–Witten $\mathrm{K}_{\mathbb{Q}}$-classes of* $\mathrm{B}\mathbb{C}^*$ are given by

$$\overline{Z}^{(k)}_{g,n}(V_{w_1}, \dots, V_{w_n}) := 0 \in \mathrm{K}_{\mathbb{Q}}(\overline{M}_{g,n})$$

if $Z^{(k)}_{g,n}(V_{w_1}, \dots, V_{w_n}) = 0$, and

$$\overline{Z}^{(k)}_{g,n}(V_{w_1}, \dots, V_{w_n}) := \rho^k(\mathbb{H}_g) \in \mathrm{K}_{\mathbb{Q}}(\overline{M}_{g,n})$$

if $Z^{(k)}_{g,n}(V_{w_1}, \dots, V_{w_n}) = \rho^k(\mathbb{H}_g)$. As we have shown, these are the only two possible cases.

Recall that we have the "gluing" maps

$$G_{i,j} : \overline{M}_{g,n} \times \overline{M}_{g',n'} \to \overline{M}_{g+g',n+n'-2}$$

and

$$H_{i,j} : \overline{M}_{g,n} \to \overline{M}_{g+1,n-2}.$$

Lemma 2.4.2. *We have*

$$G^*_{i,j}(\mathbb{H}_{g+g'}) \cong \mathbb{H}_g \boxplus \mathbb{H}_{g'} \tag{2.7}$$

and

$$H^*_{i,j}(\mathbb{H}_{g+1}) \cong \mathbb{H}_g + 1. \tag{2.8}$$

Proof. Let C be a nodal curve. Then its dualizing sheaf $\overline{\omega}$ has a concrete description [41]. For simplicity assume that C has a single node at p. Let $f : \widetilde{C} \to C$ be the normalization of C, let p', p'' denote the two preimages of p, and let D be the divisor $p' + p''$. Then we can define $\overline{\omega}$ to be the sheaf on C which is given by

$$\Gamma(U, \overline{\omega}) = \{\nu \in \Gamma(f^{-1}(U), \Omega_{\widetilde{C}}(\log D)) : \mathrm{res}_{p'}\nu + \mathrm{res}_{p''}\nu = 0\}.$$

Here $\Omega_{\widetilde{C}}(\log D)$ denotes the sheaf of meromorphic 1-forms on $\widetilde{C}$ with at worst logarithmic poles along the divisor D.

For $i = 1, 2$ let C_i be a curve with a marked (nonsingular) point p_i. Let C be the nodal curve gotten by C_1 to C_2 along their respective marked points, and let $p \in C$ denote the resulting node. Then the dualizing sheaves fit into the short exact sequence

$$0 \to \overline{\omega}_{C_1} \oplus \overline{\omega}_{C_2} \to \overline{\omega}_C \to \overline{\omega}_C|_p \to 0,$$

and so the corresponding long exact sequence in cohomology

$$0 \to \mathrm{H}^0(\overline{\omega}_{C_1}) \oplus \mathrm{H}^0(\overline{\omega}_{C_2}) \to \mathrm{H}^0(\overline{\omega}_C) \to \mathrm{H}^0(\overline{\omega_C}|_p) \to \mathrm{H}^1(\omega_{C_1}) \oplus \mathrm{H}^1(\omega_{C_2}) \to \mathrm{H}^1(\omega_C) \to 0.$$

By Serre duality, it follows that $\mathrm{H}^1(\overline{\omega}_{C_1}) = \mathrm{H}^1(\overline{\omega}_{C_2}) = \mathrm{H}^1(\overline{\omega}_C)$ is canonically $\mathbb{C}$. It can be shown that $\overline{\omega}_C|p = \overline{\omega}_{C_i}(p_i)|_{p_i}$, hence $\mathrm{H}^0(\overline{\omega}_C|_p)$ is canonically $\mathbb{C}$. So in K-theory, we have

$$\mathrm{H}^0(\overline{\omega}_{C_1}) + \mathrm{H}^0(\overline{\omega}_{C_2}) = \mathrm{H}^0(\overline{\omega}_C).$$

This same argument for a family of curves gives us the first equation.

Now let C be a curve with two marked (nonsingular) points p_1, p_2, and let C' be the nodal curve gotten by gluing p_1 to p_2. Let $p \in C'$ be the resulting node. Then we have the short exact sequence

$$0 \to \overline{\omega}_C \to \overline{\omega}_{C'} \to \overline{\omega}_{C'}|_p \to 0,$$

and using a similar argument to the one above, we get the second equation. □

Theorem 2.4.3. *The extended Gromov–Witten* $\mathrm{K}_{\mathbb{Q}}$*-classes of* $\mathrm{B}\mathbb{C}^*$ *satisfy the TQFT factorization properties*

$$\sum_{m=0}^{k-1} \overline{Z}^{(k)}_{g,n}(V_{w_1}, \ldots, V_{w_{i-1}}, V_m, V_{w_{i+1}}, \ldots, V_{w_n}) \boxtimes \overline{Z}^{(k)}_{g',n'}(V_{u_1}, \ldots, V_{u_{j-1}}, V_{k-m \bmod k}, V_{u_{j+1}}, \ldots V_{u_{n'}}) = G^*_{i,j} \overline{Z}^{(k)}_{g+g',n+n'-2}(V_{w_1}, \ldots, \widehat{V}_{w_i}, \ldots, V_{w_n}, V_{u_1}, \ldots, \widehat{V}_{u_j}, \ldots, V_{u_{n'}})$$

and

$$\sum_{m=0}^{k-1} \overline{Z}^{(k)}_{g,n}(V_{w_1}, \ldots, V_{w_{i-1}}, V_m, V_{w_{i+1}}, \ldots, V_{w_{j-1}}, V_{k-m \bmod k}, V_{w_{j+1}}, \ldots, V_{w_{n-2}}) = H^*_{i,j} \overline{Z}^{(k)}_{g+1,n-2}(V_{w_1}, \ldots, \widehat{V}_{w_i}, \ldots, \widehat{V}_{w_j}, \ldots, V_{w_{n-2}})$$

where $0 \leq w_\ell \leq k-1$ *and* $0 \leq u_\ell \leq k-1$ *for all* ℓ*.*

Proof. By (2.7) we have

$$G^*_{i,j}(\rho^k(\mathbb{H}_{g+g'})) = \rho^k(\mathbb{H}_g \boxplus \mathbb{H}_{g'}) = \rho^k(\mathbb{H}_g) \boxtimes \rho^k(\mathbb{H}_{g'}),$$

which implies the first equation.

By (2.8) we have

$$H^*_{i,j}(\rho^k(\mathbb{H}_{g+1})) = \rho^k(\mathbb{H}_g + 1) = k\rho^k(\mathbb{H}_g),$$

which implies the second equation. □

2.5 Projective bundles and K-theory

In this section, we prove some basic results about projective bundles and K-theory. Atiyah's book [4] and Hatcher's book [28] are also good references for this material.

Definition 2.5.1. As usual, $\Lambda^i E$ denotes the i-th exterior power of a vector bundle E. Let λ^i be the operation $\mathrm{K}(X) \to \mathrm{K}(X)$ which is defined by $\lambda^i E = \Lambda^i E$ if E is a vector bundle. Furthermore we have $\lambda_t : \mathrm{K}(X) \to 1 + \mathrm{K}(X)[\![t]\!]$ which is defined by the formula $\lambda_t E = \sum_i t^i \lambda^i E$ if E is a vector bundle.

Similarly, we have the symmetric power operation $S^i : \mathrm{K}(X) \to \mathrm{K}(X)$ and $S_t : \mathrm{K}(X) \to 1 + \mathrm{K}(X)[\![t]\!]$ defined by $S_t E = \sum_i t^i S^i E$ if E is a vector bundle.

Proposition 2.5.2. *If E and F are vector bundles, then we have*

$$\lambda^k(E \oplus F) = \sum_{i+j=k} \lambda^i E \otimes \lambda^j F$$

and

$$S^k(E \oplus F) = \sum_{i+j=k} S^i E \otimes S^j F.$$

Consequently, given a short exact sequence $0 \to E' \to E \to E'' \to 0$ we have

$$\lambda_t E = \lambda_t E' \cdot \lambda_t E''$$

and

$$S_t E = S_t E' \cdot S_t E''$$

We note that if L is a line bundle, then $\lambda_{-t} L = 1 - tL$ and so

$$(\lambda_{-t} L)^{-1} = 1 + tL + t^2 L^2 + \cdots = S_t L.$$

Clearly, then, the same relation holds if L is a sum of line bundles. By the splitting principle, it follows that

$$(\lambda_{-t} E)^{-1} = S_t E$$

for any vector bundle E.

Proposition 2.5.3. *Let $\pi : E \to X$ be a rank n vector bundle and let $H = \mathcal{O}(-1)$ be the tautological line bundle over the projective space bundle $\mathbb{P}(E)$. Then we have*

$$\mathrm{K}(\mathbb{P}(E)) \cong \frac{\mathrm{K}(X)[H]}{(\sum_{i=0}^{n} (-1)^i H^i \cdot \lambda^{n-i} E = 0)}.$$

Equivalently,

$$\mathrm{K}(\mathbb{P}(E)) \cong \frac{\mathrm{K}(X)[H^{-1}]}{(\sum_i (-1)^i H^{-i} \cdot \lambda^i E = 0)}.$$

Proof. It follows by the Leray–Hirsch theorem that $\mathrm{K}(\mathbb{P}(E))$ is a free $\mathrm{K}(X)$-module with basis

$$1, H, H^2, \ldots, H^{n-1}.$$

Let $\alpha : \mathcal{O}(-1) \to \pi^* E$ be the tautological map and let $Q = \operatorname{coker}(\alpha)$. Then we have

$$\lambda_t \pi^* E = \lambda_t \mathcal{O}(-1) \lambda_t Q$$

and so

$$\lambda_t \pi^* E \cdot S_{-t} \mathcal{O}(-1) = \lambda_t Q.$$

Note that the coefficient of t^n on the right hand side is zero, since Q is of rank $n-1$. The coefficient of t^n on the left hand side is

$$\sum_i (-1)^i \mathcal{O}(-i) \cdot \lambda^{n-i} E,$$

and so the desired relation follows. □

We remark that the sheaf Q in the above proof is in fact nothing other than $T_\pi \otimes \mathcal{O}(-1)$ where T_π is the relative tangent bundle of π.

Proposition 2.5.4. *Let $E \to X$ be a vector bundle of rank n and let $\pi : \mathbb{P}(E) \to X$ be the associated projective bundle. Then for each $i \geq 0$ we have $\mathbb{R}\pi_* \mathcal{O}(i) = S^i E$.*

Proof. Again consider the short exact sequence

$$0 \to \mathcal{O}(-1) \to \pi^* E \to Q \to 0.$$

The dual short exact sequence is

$$0 \to Q^\vee \to \pi^* E^\vee \to \mathcal{O}(1) \to 0.$$

It follows that $\lambda_t Q^\vee = \lambda_t \pi^* E^\vee \cdot (\lambda_t \mathcal{O}(1))^{-1} = \lambda_t \pi^* E^\vee \cdot S_{-t}\mathcal{O}(1)$. By the projection formula, we have

$$\mathbb{R}\pi_* \lambda_t Q^\vee = \lambda_t E \cdot \mathbb{R}\pi_* S_{-t} \mathcal{O}(1).$$

We first claim that $\mathbb{R}\pi_* \lambda_t Q^\vee = 1$. As we have stated above, $Q \cong T_\pi \otimes \mathcal{O}(-1)$, and so $Q^\vee \cong \Omega_\pi \otimes \mathcal{O}(1)$. Thus, to prove this claim, we show that $h^j(\mathbb{P}^{n-1}; \Omega^k \otimes \mathcal{O}(k)) = 0$ for all j and all $k \geq 1$. So consider the Euler sequence

$$0 \to \Omega^1 \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O} \to 0.$$

Then we have the corresponding Koszul complex

$$0 \to \Omega^k \to \Lambda^k \mathcal{O}(-1)^{\oplus(n+1)} \to \Lambda^{k-1}\mathcal{O}(-1)^{\oplus(n+1)} \to \cdots \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O} \to 0,$$

which can be rewritten

$$0 \to \Omega^k \to \mathcal{O}(-k)^{\oplus\binom{n+1}{k}} \to \mathcal{O}(-(k-1))^{\oplus\binom{n+1}{k-1}} \to \cdots \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O} \to 0.$$

Tensoring by $\mathcal{O}(k)$, we get

$$0 \to \Omega^k \otimes \mathcal{O}(k) \to \mathcal{O}^{\oplus\binom{n+1}{k}} \to \mathcal{O}(1)^{\oplus\binom{n+1}{k-1}} \to \cdots \to \mathcal{O}(k-1)^{\oplus(n+1)} \to \mathcal{O}(k) \to 0.$$

This is an acyclic resolution of $\Omega^k \otimes \mathcal{O}(k)$. From here it follows easily that $h^j(\mathbb{P}^{n-1}; \Omega^k \otimes \mathcal{O}(k)) = 0$ and the claim is proven.

Thus we have

$$1 = \lambda_t E \cdot \mathbb{R}\pi_* S_{-t} \mathcal{O}(1)$$

and so

$$S_{-t} E = S_{-t} \mathcal{O}(1).$$

We are finished. □

2.6 Abel–Jacobi map and higher twistings

Suppose C is a smooth curve over the base field $\mathbb{C}$ with n marked points $p_1, \dots, p_n$. Again we will always assume that $n > 0$ and we will treat the first marked point p_1 as distinguished. Let $C^{(N)} = C^N/S_N$ denote the N-th symmetric power of C. For each N and each degree $d \in \mathbb{Z}$ we have the *Abel–Jacobi map* $u_{N,d} : C^{(N)} \to \mathrm{Jac}^d$ which sends the effective divisor $x_1 + \cdots + x_N$ to the degree d line bundle

$$\mathcal{O}(x_1 + \cdots + x_N + (d - N)p_1).$$

Remark 2.6.1. Given a Riemann surface C with a marked point p_1, we can exhibit the Jacobian variety as a complex torus

$$\mathrm{Jac}^0 \cong \frac{\mathrm{H}^0(C; \omega_C)^\vee}{\mathrm{H}_1(C; \mathbb{Z})},$$

and the Abel–Jacobi map can be described as the map that assigns to the effective divisor $x_1 + \cdots + x_n$ the map

$$\alpha \mapsto \sum_{i=1}^{n} \int_{p_1}^{x_i} \alpha$$

for $\alpha \in \mathrm{H}^0(C; \omega_C)$. In fact this was the classical formulation of the Abel–Jacobi map [3]. The integrals $\int_{p_1}^{x_i}$ depend on the choice of paths from p_1 to x_i, but this dependency disappears once we quotient out by $\mathrm{H}_1(C; \mathbb{Z})$.

The Abel–Jacobi map can also be defined in families. It suffices to just describe the universal case, as follows:

Definition 2.6.2. Let $C_{g,n} \to M_{g,n}$ be the universal curve. Then we write

$$C_{g,n}^N = \underbrace{C_{g,n} \times_{M_{g,n}} \cdots \times_{M_{g,n}} C_{g,n}}_{N \text{ times}},$$

and we write

$$C_{g,n}^{(N)} = C_{g,n}^N / S_N.$$

We will omit the g, n subscripts when it is convenient and will not lead to any confusion or ambiguity.

For $i, j \in \{1, 2, \dots, N+1\}$ with $i \neq j$, let $\Delta_{i,j} \subset C^N \times_M C$ be (the relative version of) the diagonal

$$\{(x_1, \dots, x_{i-1}, x, x_{i+1} \dots, x_{j-1}, x, x_{j+1}, \dots, x_{N+1})\}.$$

Then we have a divisor $D_N \subset C^{(N)} \times_M C$ coming from the descent of the S_N-invariant divisor

$$\sum_{i=1}^{N} \Delta_{i,n+1} \subset C^N \times_M C.$$

For each point $p_i : M \to C$, we have the divisor $R_i^N \subset C^{(N)}$. More precisely, R_i^N is the restriction of D_N to $C^{(N)}$ via the map $C^{(N)} \to C^{(N)} \times_M C$ induced by the section $p_i : M \to C$.

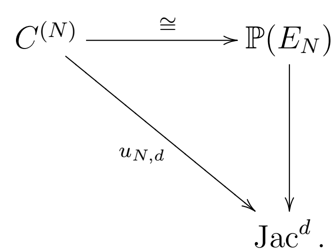
The Abel–Jacobi map $u_{N,d}$ is the map $C^{(N)} \to \mathrm{Jac}^d$ corresponding to the line bundle

$$\mathcal{O}_{C^{(N)} \times_M C}(D_N) \otimes \mathcal{O}_C((-N+d)p_1) \otimes \mathcal{O}_{C^{(N)}}(-R_1^N) \otimes \Psi_1^{-N+d} \tag{2.9}$$

over $C^{(N)} \times_M C$. In other words, the line bundle above is the pullback of the Poincaré line bundle P_d along the map $u_{N,d} \times \mathrm{id} : C^{(N)} \times_M C \to \mathrm{Jac}^d \times_M C$. The last two tensor factors in the expression above are there because Jac^d is defined to be the moduli of degree d line bundles that are *trivialized* along the point p_1.

Proposition 2.6.3. *We can interpret $C^{(N)}$ as the moduli of degree N effective divisors on C. Then D_N is the universal degree N effective divisor.*

Theorem 2.6.4. *For sufficiently large N, we have an isomorphism $C^{(N)} \cong \mathbb{P}(E_N)$ for some vector bundle $E_N \to \mathrm{Jac}^d$. This isomorphism is compatible with the Abel–Jacobi map in the sense that the diagram*



commutes. Under this isomorphism, the sheaf $\mathcal{O}(1)$ on $\mathbb{P}(E_N)$ corresponds to the sheaf $\mathcal{O}_{C^{(N)}}(R_1^N) \otimes \Psi^{N-d}$ on $C^{(N)}$.

Proof. We let $E_N = \pi_{1,*}(P_d \otimes \mathcal{O}_C((N-d)p_1)$. If N is large, there is no higher cohomology, and so the bundle E_N is indeed a vector bundle of rank $N - g + 1$ by Riemann–Roch.

By base change we have

$$u_{N,d}^* E_N = \pi_{1,*}(\mathcal{O}_{C^{(N)} \times_M C}(D_N)) \otimes \mathcal{O}_{C^{(N)}}(-R_1^N) \otimes \Psi_1^{-N+d}.$$

The embedding

$$\mathcal{O}_{C^{(N)}}(-R_1^N) \otimes \Psi_1^{-N+d} \to u_{N,d}^* E_N$$

corresponds to the embedding $\mathcal{O}(-1) \to u_{N,d}^* E_N$. □

Definition 2.6.5 (Teleman–Woodward [63]). Let k be a positive integer. Let $V_1, \ldots, V_\ell$ be representations of $\mathbb{C}^*$. Let E be the product of

$$\mathcal{L}^k \otimes \exp(t_1 E_C^* V_1) \otimes \cdots \otimes \exp(t_\ell E_C^* V_\ell)$$

together with any number of evaluation bundles. Here the t_i's are just formal parameters. Recall the notation from Definition 2.2.1. The derived pushforward

$$\mathbb{R}\Pi_* E \in \mathrm{K}(M_{g,n})[\![t_1, \ldots, t_\ell]\!]$$

is a *higher twisted level k Gromov–Witten* K-*class of* $\mathrm{B}\mathbb{C}^*$, and the components of its Chern character are the *higher twisted level k Gromov–Witten classes of* $\mathrm{B}\mathbb{C}^*$.

Remark 2.6.6. Although $\underline{\mathrm{Jac}}^\bullet$ has infinitely many components, the higher twisted Gromov–Witten classes are still well-defined, because each coefficient in the series has a nonzero contribution from only finitely many of the components.

Now let us recall the definition of the tautological classes of $M_{g,n}$.

Definition 2.6.7. Let $\Pi : C_{g,n} \to M_{g,n}$ be the universal curve with sections $p_i : M_{g,n} \to C_{g,n}$, with relative canonical bundle ω. Then define the *kappa classes*

$$\kappa_i := \Pi_*(\mathrm{c}_1(\omega)^{i+1}), \tag{2.10}$$

the *lambda classes*

$$\lambda_i := \mathrm{c}_i(\Pi_*\omega) = \mathrm{c}_i(\mathbb{H}_g), \tag{2.11}$$

and the *psi classes*

$$\psi_i := \mathrm{c}_1(\Psi_i) = \mathrm{c}_1(p_i^*\omega). \tag{2.12}$$

We can consider these classes as being elements of the rational cohomology ring $\mathrm{H}^*(M_{g,n})$. We let $R^*(M_{g,n}) \subset \mathrm{H}^*(M_{g,n})$ be the subrings generated by the tautological classes. We call these the *tautological rings.*

Remark 2.6.8. The kappa, lambda, and psi classes can also be defined for the Deligne–Mumford moduli $\overline{M}_{g,n}$ in an analogous way using the relative dualizing sheaf $\overline{\omega}$ for the universal family $\overline{C}_{g,n} \to \overline{M}_{g,n}$.

Remark 2.6.9. One can also consider the tautological classes as elements of the rational Chow ring [23]. All of the results that we discuss in this section, except for Theorem 2.6.12 and Theorem 2.6.13, hold in either case.

Proposition 2.6.10. *The lambda classes are polynomial in the kappa classes.*

Proof. By Grothendieck–Riemann–Roch, we have

$$\mathrm{ch}(\mathbb{H}_g) = \Pi_*(\mathrm{ch}(\omega) \cdot \mathrm{td}(\omega^\vee)).$$

The LHS is clearly a polynomial in the kappa classes. So we have

$$\mathrm{ch}_1(\mathbb{H}_g) = \lambda_1 = \text{polynomial in kappa classes.}$$

Similarly we have

$$\mathrm{ch}_2(\mathbb{H}_g) = \frac{1}{2}(\lambda_1^2 - 2\lambda_2) = \text{polynomial in kappa classes}$$

and so λ_2 is polynomial in kappa classes as well. Proceeding by induction it follows that each λ_i is polynomial in kappa classes. □

Remark 2.6.11. One can prove the following elegant identity:

$$\sum_{i=0}^{\infty} \lambda_i t^i = \exp\left(\sum_{i=1}^{\infty} \frac{B_{2i}\kappa_{2i-1}}{2i(2i-1)} t^{2i-1}\right).$$

Here B_{2i} denote the Bernoulli numbers. So in fact the lambda classes are polynomial in the *odd* kappa classes.

So the tautological rings can be alternatively defined as the rings generated by just the kappa classes and the psi classes. In a sense — in the "large genus limit" — these are the only classes that matter and moreover they have no nontrivial relations. To be precise:

Theorem 2.6.12 (Mumford, Madsen–Weiss [40])**.** *In the stable range of total degree $< g/2$ we have*

$$\mathrm{H}^*(M_g;\mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \dots].$$

Theorem 2.6.13 (Looijenga [38])**.** *In the stable range of total degree $< g/2$ we have*

$$\mathrm{H}^*(M_{g,n};\mathbb{Z}) \cong \mathrm{H}^*(M_g;\mathbb{Z})[\psi_1, \dots, \psi_n].$$

But for a fixed genus g, the tautological ring is not so simple, and there are many interesting relations between the tautological classes. For instance, it is a fact proved by Mumford [44] that the tautological ring $R^*(M_g)$ is generated by the $g-2$ classes $\kappa_1, \dots, \kappa_{g-2}$. See [17] for more on tautological relations and the rich and interesting structure of the tautological rings.

With the importance and interest of the tautological classes thus established, the rest of this section will be devoted to the following theorem:

Theorem 2.6.14. *The higher twisted Gromov–Witten classes of* $\mathrm{B}\mathbb{C}^*$ *are tautological.*

This will be proven as follows. It suffices to consider, one by one, each connected component $\mathrm{Jac}^d \times \mathrm{B}\mathbb{C}^*$. The bundle E is a sum of products of the form $F \boxtimes \mathcal{O}(m)$, where $m \in \mathbb{Z}$ and where F is a bundle over Jac^d. Thus it suffices to prove the statement for the indexes of such F's along $\mathrm{Jac}^d \to M$; the statement for E follows by looking at just those terms with $m = 0$, i.e. looking at the $\mathbb{C}^*$-invariant part.

We will pull the F's back to $C^{(N)}$, where N is a sufficiently large integer. More precisely, we need to have $N > d - g + 1$. Since $C^{(N)}$ is a projective bundle over Jac^d, the index along $C^{(N)} \to M$ agrees with the index along $\mathrm{Jac}^d \to M$, by Proposition 2.5.4. The Chern character of the result can be computed, via the Grothendieck–Riemann–Roch formula, by computing a cohomology pushforward along $C^{(N)} \to M$ after multiplying by a Todd class.

Definition 2.6.15. Recall that the Todd class $\mathrm{td}(E)$ of a vector bundle E is defined to be

$$\prod_{i=1}^{n} \frac{a_i}{1 - \exp(-a_i)}$$

where a_i are the Chern roots of E.

To be precise, we want to show that the Grothendieck–Riemann–Roch expression

$$\mathrm{ch}(\mathbb{R}\Pi_* F) = \Pi_*(\mathrm{ch}(u^* F) \cdot \mathrm{td}(T_{C^{(N)}/M}))$$

is tautological, where u is the Abel–Jacobi map and $T_{C^{(N)}/M}$ is the relative tangent bundle. Since the canonical map $\phi : C^N \to C^{(N)}$ is an $N!$-fold cover, the above expression is equivalent to

$$\frac{1}{N!}\Pi_*(\mathrm{ch}(\phi^* u^* F) \cdot \mathrm{td}(\phi^* T_{C^{(N)}/M})). \tag{2.13}$$

Therefore we just need to show that this expression is tautological. It follows by the remaining three results in this section.

Definition 2.6.16. Again let C denote the universal curve over $M_{g,n}$ and let $p_i : M_{g,n} \to C$ be the sections. Let m be a positive integer. Let $\pi_i : C^m \to C$ be the i-th projection map. As before, let $\Delta_{i,j}$ be the diagonals in C^m, now considered as a divisor or as a cohomology class. Let $K_i := \pi_i^*(\mathrm{c}_1(\omega))$ considered as a divisor or as a cohomology class. Finally let $p_i^j : C^{m-1} \to C^m$ be (the relative version of) the map

$$(x_1, \ldots, \widehat{x}_j, \ldots, x_m) \mapsto (x_1, \ldots, x_{j-1}, p_i, x_{j+1}, \ldots, x_m)$$

By abuse of notation, we also let p_i^j denote the image of this map considered as a divisor or as a cohomology class. We will refer classes that are polynomials in these classes and (pullbacks of) the tautological classes on $M_{g,n}$ as *tautological classes on* C^m.

Lemma 2.6.17. *Let α be a tautological class on C^m. Let $\rho^m : C^m \to C^{m-1}$ be (the relative version of) the map*

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_{m-1}).$$

Then $\rho^m_* \alpha$ is a tautological class on C^{m-1}. In particular, $\Pi_*(\alpha) = (\rho^1 \circ \rho^2 \circ \cdots \circ \rho^m)_*(\alpha)$ is a tautological class on $M_{g,n}$.

Proof. See page 55 of [25]. □

Proposition 2.6.18. *The term $\mathrm{ch}(\phi^* u^* F)$ in (2.13) is tautological.*

Proof. It suffices to check the cases where F is an index bundle, a determinant line, and an evaluation bundle. Using (2.9), we see that the pullback of the Poincaré bundle P_d along the map

$$(u \circ \phi) \times \mathrm{id} : C^N \times_M C \to \mathrm{Jac}^d \times_M C$$

is

$$\mathcal{O}_{C^N \times_M C}(\textstyle\sum_{i=1}^N \Delta_{i,N+1}) \otimes \mathcal{O}_C((-N+d)p_1) \otimes \mathcal{O}_{C^N}(-\sum_{i=1}^N p_1^i) \otimes \Psi_1^{-N+d}.$$

Let us write P'_d for the above line bundle. Then the statement for F an index bundle or determinant line follows by base change and the previous lemma.

To prove the statement for F an evaluation bundle, it suffices to prove that (the Chern character of) $P'_d|_{p_j}$ is tautological for each j. Here we are using the same "restriction" notation as in equation (2.1). Concretely, we have $P'_d|_{p_1} = \mathcal{O}_{C^N}$ and for $j \neq 1$ we have

$$P'_d|_{p_j} = \mathcal{O}_{C^N}(\textstyle\sum_{i=1}^N p_j^i - p_1^i) \otimes \Psi_1^{-N+d},$$

and we are done. □

Proposition 2.6.19. *We have*

$$\phi^* T_{C^{(N)}/M} = \sum_{i=1}^N T_i(\textstyle\sum_{j=i+1}^N \Delta_{i,j})$$

in K*-theory and so the term* $\mathrm{td}(\phi^* T_{C^{(N)}/M})$ *in* (2.13) *is tautological. Here* T_i *denotes* $\pi_i^*(T_{C/M})$ *where* $\pi_i : C^N \to C$ *is the projection and* $T_{C/M}$ *is the relative anticanonical bundle (i.e. the relative tangent bundle).*

Proof. Let π_1 be the projection $C^{(N)} \times_M C \to C^{(N)}$. Proposition 19.2 of [49] states that we have a canonical isomorphism

$$T_{C^{(N)}/M} \cong \pi_{1,*}(\mathcal{O}_{D_N}(D_N)).$$

By base change and the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(D_N) \to \mathcal{O}_{D_N}(D_N) \to 0$$

we have

$$\phi^*(T_{C^{(N)}/M}) = \mathbb{R}\pi_{1,*}(-\mathcal{O} + \mathcal{O}(\textstyle\sum_{i=1}^N \Delta_{1,N+1})).$$

We have

$$0 \to \mathcal{O} \to \mathcal{O}(\Delta_{1,N+1}) \to \mathcal{O}_{\Delta_{1,N+1}}(\Delta_{1,N+1}) \to 0,$$

and so

$$0 \to \mathcal{O}(\textstyle\sum_{i=2}^N \Delta_{i,N+1}) \to \mathcal{O}(\sum_{i=1}^N \Delta_{i,N+1}) \to \mathcal{O}_{\Delta_{1,N+1}}(\sum_{i=1}^N \Delta_{i,N+1}) \to 0.$$

Since

$$\mathbb{R}\pi_{1,*}\mathcal{O}_{\Delta_{1,N+1}}(\textstyle\sum_{i=1}^N \Delta_{i,N+1}) \cong T_1(\sum_{j=2}^N \Delta_{1,j}),$$

it follows that

$$\phi^*(T_{C^{(N)}/M}) = \mathbb{R}\pi_{1,*}(-\mathcal{O} + \mathcal{O}(\textstyle\sum_{i=2}^N \Delta_{i,N+1})) + T_1(\sum_{j=2}^N \Delta_{1,j}).$$

The result follows by induction. □

This completes the proof of Theorem 2.6.14.

2.7 Further directions

Let us make a few comments on further directions of research. We have dealt with the simplest case, $G = \mathbb{C}^*$. We have shown that the Gromov–Witten cohomology classes in this case are tautological, and we have found elegant and explicit formulas for those classes when there are no higher twistings. However, we have not given explicit formulas for the classes in the presence of higher twistings. By unwinding the proofs of the results in the previous section, one should be able to derive such a formula, and it would be interesting to do so.

Additionally, one may be interested in other groups such as $G = \mathrm{SL}(2)$. One can similarly define and study the Gromov–Witten (K-)classes for $\mathrm{BSL}(2)$, with or without higher twists. We will omit their definition here, but we note that they are defined as indexes of certain *admissible* bundles E along $M_{g,n}(\mathrm{BSL}(2)) \to M_{g,n}$; see [63, 18] for a full treatment.

Note that the inclusion $\mathbb{C}^* \to \mathrm{SL}(2)$ induces a natural morphism

$$j : M_{g,n}(\mathrm{B}\mathbb{C}^*) \to M_{g,n}(\mathrm{BSL}(2))$$

which sends a principal $\mathbb{C}^*$-bundle F to the principal $\mathrm{SL}(2)$-bundle $F \times_{\mathbb{C}^*} \mathrm{SL}(2)$. Then the Teleman–Woodward localization theorem [63] states that the index of E along the map $M_{g,n}(\mathrm{BSL}(2)) \to M_{g,n}$ can be computed in terms of the indexes of $j^*E \otimes \lambda_{-1}(\nu^\vee)^{-1}$ along the maps $\mathrm{Jac}^d \to M_{g,n}$, where ν is the virtual normal bundle for the morphism j. It should follow from the results in the previous section that the latter indexes are tautological and so therefore the former index is as well. It would again be interesting to derive explicit formulas.

We also remark that, since our Gromov–Witten cohomology classes are tautological classes in $\mathrm{H}^*(M_{g,n})$, they can be clearly extended to $\mathrm{H}^*(\overline{M}_{g,n})$ (cf. for example Remark 2.6.8). Note, however, that the extensions are not unique. One could then study the different extensions and the corresponding CohFTs [62, 32].

Chapter 3

Global matrix factorizations

We now turn our attention to matrix factorizations for the case of Landau–Ginzburg models (X, W) where X is not necessarily affine. The results in this chapter are joint with Daniel Pomerleano, and many have already appeared in the preprint [37]. The word "global" in the title of this chapter is used to emphasize the fact that we will be working with general varieties and not just affine local ones.

Let X be a smooth variety over $\mathbb{C}$ and $W \in \Gamma(X)$ a regular function such that the corresponding morphism $X \to \mathbb{A}^1_{\mathbb{C}}$ is flat. Now replacing R-modules with $\mathcal{O}_X$-modules, Definition 1.3.5 still makes sense — to be precise, *matrix factorizations* are now defined to be curved complexes of *locally free sheaves of finite type*. However, as is briefly discussed in §3.2 of Katzarkov–Kontsevich–Pantev [32], the "correct" definition of the matrix factorization category in the non-affine situation should take into account the non-vanishing of higher sheaf cohomology. Roughly speaking, this means that we should replace the complex $\mathrm{Hom}(P, P')$ with some form of a "derived complex" $\mathbb{R}\mathrm{Hom}(P, P')$, for instance via a Čech or Dolbeault resolution of the sheaf complex $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(P, P')$. This is analogous to the relation between the derived categories and ordinary categories of sheaves.

To make the above precise, we consider in §3.1 the category $\mathrm{QCoh}(X, W)$ of curved complexes of *quasi-coherent* sheaves. We equip this category with a model category structure for which fibrant objects are curved complexes of injective sheaves. This gives rise to the dg category $\mathrm{Inj}(X, W)$, which is a dg enhancement of the absolute derived category $\mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X, W)$. Via fibrant replacement, that is, replacing curved quasi-coherent complexes with curved injective complexes, we define the derived complex $\mathbb{R}\mathrm{Hom}(P, P')$ of morphisms and thusly the "correct" matrix factorization dg category $\mathrm{MF}_{\mathrm{dg}}(X, W)$. This is completely analogous to the standard procedure of taking injective resolutions in ordinary homological algebra (i.e. the uncurved situation). Furthermore, we show that matrix factorizations are compact when considered as objects of $\mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X, W)$ and that the idempotent completion of the subcategory thereof recovers $\mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X, W)_{\mathrm{c}}$, the full subcategory of all compact objects. The results in this section are adaptations of results from Positselski's homological theory of curved modules [52]; this section forms the technical foundation of our study of curved sheaves and matrix factorizations.

In §3.2, we will compute the Hochshild cohomology of $\mathrm{Inj}(X, W)$ and hence that of $\mathrm{MF}_{\mathrm{dg}}(X, W)$, yielding a result which was anticipated in [32]. To this end, like Dyckerhoff [15] in the affine case, we identify a compact generator of the category — all of our subsequent results rely on this important observation. Dyckerhoff identifies generators for matrix factorization categories in the local situation, but we identify generators in the global situation for the corresponding *derived categories of singularities* [47] of the zero fibers using the work of Rouquier [54]. But the two categories are equivalent. Subsequently, the abstract identification of the category with the dg derived category of the endomorphism dg algebra of the generator is enough to apply the derived Morita theory of Toën [64]. To reach our Hochschild cohomology result, we take a detour into the work of Yekutieli [65] on the global Hochschild–Kostant–Rosenberg theorem, and we employ the calculations of Caldararu–Tu [10]. It also follows from results in this section that $\mathrm{MF}_{\mathrm{dg}}(X, W)$ is smooth, and that it is proper when the singular locus of W is proper.

In §3.3, using Grothendieck duality [27], we show that if the Landau–Ginzburg model (X, W) satisfies the condition that X is Calabi–Yau, then $\mathrm{MF}_{\mathrm{dg}}(X, W)$ is a Calabi–Yau category. We remark that our proof of the Calabi–Yau condition on the category mimics Dyckerhoff's proof in the affine case; however, we are able to identify explicitly how the Calabi–Yau condition on the space X comes into play. This is not immediately transparent in Dyckerhoff's proof, since in his local situation the Calabi–Yau condition on X is automatic.

For ease of notation and exposition, we always assume that our superpotentials W have a single critical value $0 \in \mathbb{C}$. If there are multiple critical values c_i, then all results discussed will still hold by considering the product $\prod_i \mathrm{MF}(X, W - c_i)$ instead of $\mathrm{MF}(X, W)$, etc. Furthermore, unless specified otherwise, when we say *dg category* we will always mean differential $\mathbb{Z}/2\mathbb{Z}$-graded category, that is, a category enriched over the category of $\mathbb{Z}/2\mathbb{Z}$-graded complexes of $\mathbb{C}$-vector spaces. More generally, all of our graded objects are $\mathbb{Z}/2\mathbb{Z}$-graded objects. We work over the field $\mathbb{C}$, since this is the situation of primary interest in applications, but we remark that all results still hold over any field of characteristic zero.

We note that Anatoly Preygel has independently proved results similar to ours, using an exciting new and different approach involving derived algebraic geometry [53].

3.1 Curved quasi-coherent sheaves and matrix factorizations

In this section we study categories of curved sheaves, derived categories of singularities, and the various important technical results and constructions that are needed to prove our main results of interest. We begin by adapting some standard homological theory of sheaves to the curved situation.

We consider the dg category of curved complexes of quasi-coherent sheaves $\mathrm{QCoh}(X, W)$, that is, the category with objects

$$E = (E_1 \underset{e_0}{\overset{e_1}{\rightleftarrows}} E_0)$$

where the E_i are quasi-coherent sheaves of $\mathcal{O}_X$-modules and the e_i are morphisms of $\mathcal{O}_X$-modules satisfying, as before, $e_{i+1} \circ e_i = W \cdot \mathrm{id}_{E_i}$. The morphism complexes are defined exactly as before except with $\mathrm{Hom}_{\mathcal{O}_X}$ rather than Hom_R. We will denote by $E[1]$ the curved complex

$$(E_0 \underset{-e_1}{\overset{-e_0}{\rightleftarrows}} E_1).$$

Furthermore, one can define the cone of a morphism and a class of exact triangles in $\mathrm{QCoh}(X, W)$ which together with the shift functor $E \mapsto E[1]$ makes the homotopy category $[\mathrm{QCoh}(X, W)] = H^0(\mathrm{QCoh}(X, W))$ a triangulated category. For more details on the triangulated category structure, refer to [48, 47].

More generally, given any dg category $\mathcal{C}$ of curved objects, we will let $[\mathcal{C}]$ denote the homotopy category of $\mathcal{C}$ with triangulated category structure defined in the same way. Also we will use the functor $E \mapsto E^{\#}$ sending a curved object to the underlying graded object gotten by forgetting the maps e_0 and e_1.

Definition 3.1.1. Denote by $\mathrm{Acycl}^{\mathrm{abs}}[\mathrm{QCoh}(X, W)] \subset [\mathrm{QCoh}(X, W)]$ the thick triangulated subcategory generated by[1] the total curved complexes of exact triples of curved quasi-coherent $\mathcal{O}_X$-modules. Objects of $\mathrm{Acycl}^{\mathrm{abs}}[\mathrm{QCoh}(X, W)]$ are called *acyclic*. The triangulated category $\mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X, W)$ is defined to be the quotient triangulated category

$$\frac{[\mathrm{QCoh}(X, W)]}{\mathrm{Acycl}^{\mathrm{abs}}[\mathrm{QCoh}(X, W)]}.$$

We call this category the *absolute derived category.* This definition is also used in [52, 47].

Remark 3.1.2. Note that in our curved situation, we are unable to define the derived category in the usual way by inverting quasi-isomorphisms. This is because the usual notion of quasi-isomorphism as a morphism which is an isomorphism after taking cohomology no longer makes sense — we cannot speak of cohomology of a *curved* complex, since we don't have "$d^2 = 0$". Similarly, the usual notion of acyclicity does not make sense. However, note that in the case of ordinary uncurved complexes of sheaves, the total complex of an exact sequence of complexes is acyclic. This motivates the definitions of acyclicity and absolute derived category.

Lemma 3.1.3. *Let H be a triangulated category and A, F be full triangulated subcategories. Then the natural functor $F/(A \cap F) \to H/A$ is an equivalence of triangulated categories if for any object $X \subset H$ there exists an object $Y \in F$ together with a morphism $X \to Y$ in H such that a cone of that morphism belongs to A.*

[1] This means that we take recursively all shifts, cones, and direct summands [54].

Proposition 3.1.4. *Denote by* $\operatorname{Inj}(X, W)$ *the full subcategory of* $\operatorname{QCoh}(X, W)$ *consisting of curved complexes of injective quasi-coherent sheaves. The natural functor*

$$[\operatorname{Inj}(X, W)] \to \mathbf{D}^{\mathrm{abs}}\operatorname{QCoh}(X, W)$$

is an equivalence of triangulated categories. Therefore, we see that the category $\operatorname{Inj}(X, W)$ *defines a dg enhancement of* $\mathbf{D}^{\mathrm{abs}}\operatorname{QCoh}(X, W)$.

Proof. This is a scheme theoretic version of Theorems 3.5 and 3.6 of [52] and is proved in exactly the same way. We give a sketch of the proof. We wish to apply the previous lemma and so we proceed in two steps.

The first step is very general — we claim that if $B \in \operatorname{Acycl}^{\mathrm{abs}}[\operatorname{QCoh}(X, W)]$ and I is a curved complex of injective sheaves, then $\operatorname{Hom}(B, I)$ is an acyclic complex. Indeed if B is the total curved module of an exact sequence of curved modules

$$0 \to L \to M \to N \to 0,$$

then $\operatorname{Hom}(B, I)$ is the total complex of the exact sequence of complexes

$$0 \to \operatorname{Hom}(N, I) \to \operatorname{Hom}(M, I) \to \operatorname{Hom}(L, I) \to 0,$$

so it is acyclic. Since $\operatorname{Acycl}^{\mathrm{abs}}[\operatorname{QCoh}(X, W)]$ is the thick triangulated subcategory generated by such B, the claim follows. We see immediately that

$$\operatorname{Acycl}^{\mathrm{abs}}[\operatorname{QCoh}(X, W)] \cap \operatorname{Inj}(X, W) = 0.$$

It remains to show that for each $B \in \operatorname{QCoh}(X, W)$, there is a morphism $r : B \to J$ such that $J \in \operatorname{Inj}(X, W)$ and $\operatorname{Cone}(r) \in \operatorname{Acycl}^{\mathrm{abs}}[\operatorname{QCoh}(X, W)]$. Indeed, there is an embedding of any curved complex of quasi-coherent sheaves B into a curved complex of injectives G_0. To see this, note that the underlying graded sheaf $B^{\#}$ embeds into an injective graded quasi-coherent sheaf I_0, as the category of quasi-coherent sheaves has enough injectives. One then takes the curved complex $G^{-}(I_0)$ of quasi-coherent sheaves cofreely cogenerated by I_0 (see the proof of Theorem 3.6 of [52]), and one checks that B embeds into $G_0 := G^{-}(I_0)$ and that $G^{-}(I_0)^{\#}$ is injective. Let H_0 be the cokernel G_0/B, and similarly we construct a curved complex $G_1 = G^{-}(I_1)$ of injectives into which H_0 embeds. Proceeding inductively, we obtain a resolution

$$0 \to B \to G_0 \to G_1 \to \cdots$$

of B by curved complexes of injectives. However, since X is smooth, the category $\operatorname{QCoh}(X)$ of quasi-coherent sheaves has finite homological dimension, and hence for some finite n we must have that the underlying graded sheaf $H_n^{\#}$ of the cokernel $H_n = G_n/G_{n-1}$ is injective. Let J be the total curved module of the exact complex of curved modules

$$G_0 \to G_1 \to \cdots \to G_n \to H_n.$$

We are finished. □

Remark 3.1.5. Dyckerhoff [15] considers a regular local k-algebra R with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = k$. He takes a superpotential $W \in R$ with isolated singularity at the closed point $\mathfrak{m}$, and he considers the category $\mathrm{MF}^\infty(R, W)$ consisting of curved complexes of *projective* R-modules of arbitrary rank. In [52], it is proved that $[\mathrm{MF}^\infty(R, W)]$ and $\mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(R, W)$ are equivalent as triangulated categories. In our case, we are forced to use curved complexes of *injective* modules because there are not enough projectives in the global situation.

Theorem 3.1.6. *There is a model category structure on* $\mathrm{QCoh}(X, W)$ *where a morphism is a weak equivalence if its cone is acyclic; a morphism is cofibrant if it is monic and it is fibrant if it is epic and its kernel is a curved complex of injective sheaves; fibrant objects are curved complexes of injective sheaves.*

In the language of Toën [64], we have $\mathrm{Inj}(X, W) = \mathrm{Int}(\mathrm{QCoh}(X, W))$, where $\mathrm{Int}(-)$ denotes the full subcategory consisting of objects which are both fibrant and cofibrant. The proof of the above theorem is again similar to Positselski's discussion in the affine case and we omit it since we will not need the full strength of the theorem.

Now, given two curved complexes of quasi-coherent sheaves F and F', we have an *uncurved* complex of sheaves $\mathcal{H}\mathrm{om}(F, F')$ which is defined by

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(F|_U, F'|_U).$$

Definition 3.1.7. We have a derived functor

$$\mathbb{R}\mathcal{H}\mathrm{om} : \mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X, W) \times \mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X, W) \to \mathbf{D}\mathrm{Mod}(\mathcal{O}_X).$$

It is defined by first doing at least one of the following:

1. replacing the second argument by a weakly equivalent curved complex of injectives
2. if possible, replacing the first argument by a weakly equivalent curved complex of locally free sheaves of finite rank

and then taking $\mathcal{H}\mathrm{om}$. We have another derived functor

$$\mathbb{R}\mathrm{Hom} : \mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X, W) \times \mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X, W) \to \mathbf{D}\mathrm{Mod}(\mathbb{C})$$

defined by first replacing the second argument by a weakly equivalent curved complex of injectives and then taking Hom.

With the basic homological theory out of the way, the rest of this section will be devoted to results about compact objects and relationships between categories of curved sheaves and derived categories of singularities. We now define two different categories of matrix factorizations (same objects, different morphisms):

Definition 3.1.8. Define $\mathrm{mf}(X,W)$, $\mathrm{Acycl}^{\mathrm{abs}}[\mathrm{mf}(X,W)]$, and $\mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X,W)$ in the same way as we defined the analogous respective QCoh entities above, except here the objects are curved complexes of locally free sheaves of finite type, i.e. curved vector bundles, i.e. matrix factorizations.

Definition 3.1.9. Denote by $\mathrm{MF}_{\mathrm{dg}}(X,W)$ the full dg subcategory of $\mathrm{Inj}(X,W)$ consisting of objects weakly equivalent to matrix factorizations.

Remark 3.1.10. What we call $\mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X,W)$ agrees with what Orlov calls $\mathrm{MF}_0(X,W)$ in [47]. Recall that we are assuming, without loss of generality, that W has only one singular value $0 \in \mathbb{C}$.

In some arguments it will be convenient to use a third construction — a Čech model of $\mathrm{MF}_{\mathrm{dg}}(X,W)$. Let $\mathfrak{U} = \{U_i = \mathrm{Spec}\, A_i\}$ be a finite covering of X by affine subsets. We follow the notation of §III.4 of [26], and we write $\mathcal{C}^\bullet(\mathfrak{U}, F)$ for the sheaf Čech complex of a sheaf F. We define the dg category $\mathrm{MF}_{\mathrm{Cech}}(X,W)$ as follows: The objects are matrix factorizations; the morphisms $\mathrm{Hom}_{\mathrm{MF}_{\mathrm{Cech}}}(P,P')$ are given by the global sections of the total complex of the double complex $\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{H}\mathrm{om}(P,P'))$ with the first differential being the Čech differential and the second differential induced by that of $\mathcal{H}\mathrm{om}(P,P')$. Although $\mathrm{MF}_{\mathrm{Cech}}(X,W)$ depends on the covering $\mathfrak{U}$, we suppress this from the notation because different coverings yield weakly equivalent dg categories[2]. It is a tedious but standard consideration to see the following:

Proposition 3.1.11. *We have a weak equivalence* $\mathrm{MF}_{\mathrm{Cech}}(X,W) \to \mathrm{MF}_{\mathrm{dg}}(X,W)$ *of dg categories.*

Definition 3.1.12 ([48])**.** For any variety Y over $\mathbb{C}$, we denote by $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(Y)$ the bounded derived category of coherent sheaves on Y, and we denote by $\mathfrak{Perf}(Y)$ the full triangulated subcategory of perfect complexes. We set

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(Y) = \frac{\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(Y)}{\mathfrak{Perf}(Y)}.$$

Proposition 3.1.13. *The categories* $[\mathrm{MF}_{\mathrm{dg}}(X,W)]$ *and* $\mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X_0)$ *are equivalent. Here* X_0 *denotes the fiber* $W^{-1}(0)$*.*

Remark 3.1.14. Note that the statement resembles the well-known result of Orlov [47, 48] that $\mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X,W) \cong \mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X_0)$; but note also that it is not exactly the same statement.

Proof. There is a natural triangulated functor $\mathrm{coker} : [\mathrm{mf}(X,W)] \to \mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X_0)$ given by

$$P = (\ P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0\ ) \longmapsto \mathrm{coker}(p_1) =: \mathrm{coker}(P).$$

[2]The category of dg categories has a model category structure for which weak equivalences are quasi-equivalences of dg categories.

Let $\{U_i\}$ be as above an affine open cover of X. Consider an object P of $\mathrm{mf}(X,W)$ whose image in the homotopy category lies in the subcategory $\mathrm{Acycl}^{\mathrm{abs}}[\mathrm{mf}(X,W)]$. Then $P|_{U_i}$ is in $\mathrm{Acycl}^{\mathrm{abs}}[\mathrm{mf}(U_i,W)]$. By an argument similar to the first part of the proof of 3.1.4 (except in this situation consider projectives instead of injectives), this subcategory is 0, which means that $P|_{U_i}$ is contractible and hence its cokernel is locally free [48]. Since this holds for each U_i we conclude that $\mathrm{coker}(P)$ is locally free and therefore vanishes in $\mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X_0)$. Thus the coker functor factors through $\mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X,W)$, and Orlov [47] proves that the induced functor $\mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X,W) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X_0)$ is an equivalence of triangulated categories.

To prove the proposition, it suffices to show that the natural functor $\mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X,W) \to \mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X,W)$ is fully faithful. For this purpose, it is useful to consider the categories $\mathrm{Coh}(X,W)$, $\mathrm{Acycl}^{\mathrm{abs}}[\mathrm{Coh}(X,W)]$, and $\mathbf{D}^{\mathrm{abs}}\mathrm{Coh}(X,W)$ defined in the same way as the respective mf and QCoh entities. By Exercise II.5.15 of [26], it follows that any morphism from a curved coherent sheaf $F \in \mathrm{Coh}(X,W)$ to an acyclic curved quasi-coherent sheaf $A \in \mathrm{Acycl}^{\mathrm{abs}}[\mathrm{QCoh}(X,W)]$ factors through an acyclic curved coherent sheaf $A' \in \mathrm{Acycl}^{\mathrm{abs}}[\mathrm{Coh}(X,W)]$. From this it follows that $\mathbf{D}^{\mathrm{abs}}\mathrm{Coh}(X,W) \to \mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X,W)$ is fully faithful.

It remains to show that $\mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X,W) \to \mathbf{D}^{\mathrm{abs}}\mathrm{Coh}(X,W)$ is fully faithful. To see this, note that since we are on a smooth scheme, any coherent sheaf has a finite resolution by vector bundles. Therefore, following an argument similar to that of Proposition 3.1.4, for any curved coherent sheaf C we can produce a triangle $F \to C \to A$ where F is a matrix factorization and A is acyclic. It follows from the dual version of Lemma 3.1.3 that

$$[\mathrm{mf}(X,W)]/([\mathrm{mf}(X,W)] \cap \mathrm{Acycl}^{\mathrm{abs}}[\mathrm{Coh}(X,W)]) \cong \mathbf{D}^{\mathrm{abs}}\mathrm{Coh}(X,W).$$

The thick subcategory $[\mathrm{mf}(X,W)] \cap \mathrm{Acycl}^{\mathrm{abs}}[\mathrm{Coh}(X,W)]$ can be identified with the thick subcategory $\mathrm{Acycl}^{\mathrm{abs}}[\mathrm{mf}(X,W)]$. We see this as follows. Taking $\mathrm{coker}(P)$ of objects P of the former category gives the zero object in $\mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X_0)$ by the same local argument explained at the beginning of this proof. By Orlov's result mentioned above, it follows that P must have been equivalent to an object in $\mathrm{Acycl}^{\mathrm{abs}}[\mathrm{mf}(X,W)]$ to begin with. □

Proposition 3.1.15. *Objects of* $[\mathrm{MF}_{\mathrm{dg}}(X,W)]$ *are compact as objects of* $\mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X,W)$.

Proof. Let P be a matrix factorization and Q an arbitrary curved quasi-coherent sheaf. It is a standard consideration to see that $\mathbb{R}\mathrm{Hom}(P,Q)$ can be computed using the complex $\Gamma\,\mathrm{Tot}\,\mathcal{C}^\bullet(\mathfrak{U},\mathcal{H}\mathrm{om}(P,Q))$.

Since the U_i (and their intersections U_{ij}, etc.) are affine, it follows that the restrictions $P|_{U_i}$ are compact in $\mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(U_i,W)$ by [52] (and analogously for the intersections U_{ij}, etc.). Let $Q = \bigoplus_i Q_i$ be a direct sum of curved quasi-coherent sheaves. We have that

$$\Gamma\,\mathrm{Tot}\,\mathcal{C}^\bullet(\mathfrak{U},\mathcal{H}\mathrm{om}(P,\bigoplus_i Q_i)) \cong \Gamma\,\mathrm{Tot}\,\mathcal{C}^\bullet(\mathfrak{U},\bigoplus_i \mathcal{H}\mathrm{om}(P,Q_i)),$$

because the restrictions $P|_{U_i}$, $P|_{U_{ij}}$, etc. are compact, and finally we have

$$\Gamma\,\mathrm{Tot}\,\mathcal{C}^\bullet(\mathfrak{U},\bigoplus_i \mathcal{H}\mathrm{om}(P,Q_i)) = \bigoplus_i \Gamma\,\mathrm{Tot}\,\mathcal{C}^\bullet(\mathfrak{U},\mathcal{H}\mathrm{om}(P,Q_i)),$$

which completes the proof. ☐

Lemma 3.1.16 ([9])**.** *Let $\mathcal{T}$ be a triangulated category with arbitrary direct sums and which is compactly generated by a set of objects C. Then the set of compact objects of $\mathcal{T}$ is C^{thk}, the thick closure of C.*

Proposition 3.1.17. $\overline{[\mathrm{MF}_{\mathrm{dg}}(X,W)]} \cong \mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X,W)_{\mathrm{c}}$*, where the notation $\overline{\mathcal{C}}$ means idempotent completion of a category $\mathcal{C}$, and the notation $\mathcal{C}_{\mathrm{c}}$ means the full subcategory of $\mathcal{C}$ consisting of objects whose image in the triangulated category $[\mathcal{C}]$ is compact.*

Proof. By the above lemma it suffices to prove that $[\mathrm{MF}_{\mathrm{dg}}(X,W)] \cong \mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X,W) \cong \mathbf{D}^{\mathrm{abs}}\mathrm{Coh}(X,W)$ generates $\mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X,W)$. What we want to prove is the global version of Theorem 2 on page 43 of [52] and the proof is very similar. Let J be an object of $\mathrm{Inj}(X,W)$. By the standard Bousfield localization argument, what we have to show is that if $\mathrm{Hom}(B,J)$ is acyclic for every coherent curved module B, then J is contractible, meaning that it is weakly equivalent to the zero object.

Consider the ordered set of pairs (C,h), where C is a curved quasi-coherent subsheaf of J and h is a contracting homotopy for the inclusion $C \hookrightarrow J$. Using Zorn's lemma, let (M,h) be a maximal such pair. We show that if $M \neq J$, then $M \hookrightarrow J$ factors through some $M' \hookrightarrow J$, and the contracting homotopy h extends to a contracting homotopy h' for $M' \hookrightarrow J$. From here the result follows.

So suppose $M \neq J$. Then again using Exercise II.5.15 of [26], we can find a curved quasi-coherent subsheaf M' of J such that M' strictly contains M and the quotient M'/M is coherent. Producing the contracting homotopy proceeds exactly as in [52]. ☐

We conclude the section with one last lemma, which will need useful later:

Lemma 3.1.18. *Let F be a coherent sheaf on $W^{-1}(0) = X_0$ considered as an object of* $\mathrm{Coh}(X,W)$*. Suppose P is a matrix factorization and $f : P \to F$ is a morphism of curved sheaves such that* $\mathrm{Cone}(f)$ *is acyclic. Then* $\mathrm{coker}(P) \cong F$ *in* $\mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X_0)$*. Moreover, such a P exists.*

Proof. We know that $P \cong F$ in $\mathbf{D}^{\mathrm{abs}}\mathrm{Coh}(X,W)$. First we check that the result holds if F, as a coherent sheaf, is *maximal Cohen–Macaulay*, which means that $\mathcal{E}\mathrm{xt}^i(F,\mathcal{O}_{X_0}) = 0$ for $i > 0$. To see this, note that there is a length two resolution of F by locally free sheaves on X (see the proof of Theorem 3.9 in [48])

$$0 \to Q_1 \to Q_0 \to F \to 0.$$

Let $G^+(Q_0)$ be the free curved module generated by Q_0 (see again Theorem 3.6 of [52]). We have a surjection of curved sheaves $G^+(Q_0) \to F$ whose kernel is isomorphic to $Q[1]$, where

$$Q = (\ Q_1 \underset{q_0}{\overset{q_1}{\rightleftarrows}} Q_0\ ),$$

with q_1 the inclusion map and q_0 the homotopy expressing the fact that W kills F.

We clearly have $\operatorname{coker}(Q) = \operatorname{coker}(F) = F$. Since $G^+(Q_0)$ is contractible, we have an isomorphism $Q \cong F$ in $\mathbf{D}^{\mathrm{abs}}\mathrm{Coh}(X, W)$. Hence we have $P \cong F \cong Q$ in $\mathbf{D}^{\mathrm{abs}}\mathrm{Coh}(X, W)$. Previously we checked that the functor $\mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X, W) \to \mathbf{D}^{\mathrm{abs}}\mathrm{Coh}(X, W)$ is fully faithful, and hence $P \cong Q$ in $\mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X, W)$. Thus $\operatorname{coker}(P) \cong \operatorname{coker}(Q) = F$ in $\mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X_0)$.

For the general case, for any coherent sheaf F, there is a resolution

$$0 \to F' \to F_r \to \cdots \to F_2 \to F_1 \to F \to 0$$

where the F_i are locally free and F' is maximal Cohen–Macaulay. We assume without loss of generality that r is even; if it is odd we simply consider an additional syzygy $F_{r+1} \to F_r$ and take F' to be its kernel. Then we conclude that $F \cong F'$ in $\mathbf{D}^{\mathrm{abs}}\mathrm{Coh}(X, W)$ as well as in $\mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X_0)$ and so the lemma is proven. □

Remark 3.1.19. After receiving an early version of our preprint [37], Positselski has expanded on our ideas to produce a new proof of Orlov's equivalence [51].

3.2 Compact generators and Hochschild (co)homology

Let X be as above a smooth variety over $\mathbb{C}$, and let W be an arbitrary superpotential. The purpose of this section is to prove the following theorem:

Theorem 3.2.1. *The Hochschild cohomology of the category* $\mathrm{MF}_{\mathrm{dg}}(X, W)$ *is*

$$\mathbb{R}\Gamma(\Lambda^\bullet T_X, [W, -]),$$

where $[-, -]$ *denotes the Schouten–Nijenhuis bracket.*

Recall that the Hochschild cohomology of a dg category can be defined as the derived endomorphisms of the identity functor of the category. We have the following two basic results of Toën [64] concerning the *uncurved* situation:

Theorem 3.2.2. *Let X and Y be quasi-compact and separated k-schemes, where k is the underlying field. Suppose one of them is flat over k. For any scheme Z, let $\mathrm{L}_{\mathrm{qcoh}}(Z)$ denote the dg category of (fibrant) quasi-coherent complexes. Then we have a natural quasi-equivalence of dg categories*

$$\mathrm{L}_{\mathrm{qcoh}}(X \times_k Y) \cong \mathbb{R}\mathrm{Hom}_{\mathrm{c}}(\mathrm{L}_{\mathrm{qcoh}}(X), \mathrm{L}_{\mathrm{qcoh}}(Y)).$$

Here $\mathbb{R}\mathrm{Hom}_{\mathrm{c}}$ denotes the full dg subcategory of the dg category of morphisms consisting of continuous functors, i.e. functors commuting with infinite direct sums. If $X = Y$, then the structure sheaf of the diagonal $\Delta \subset X \times_k X$ corresponds to the identity endofunctor $\mathrm{L}_{\mathrm{qcoh}}(X) \to \mathrm{L}_{\mathrm{qcoh}}(X)$.

Theorem 3.2.3. *Let A and B be dg algebras. Then we have a natural quasi-equivalence of dg categories*

$$\widehat{A^{\mathrm{op}} \otimes B} \cong \mathbb{R}\mathrm{Hom}_{\mathrm{c}}(\widehat{A}, \widehat{B})$$

For a dg category T, the notation $\widehat{T} = \mathrm{Int}(T^{\mathrm{op}}\text{-}\mathrm{Mod})$ denotes the full dg subcategory of $T^{\mathrm{op}}\text{-}\mathrm{Mod}$ consisting of those T^{op}-modules that are both fibrant and cofibrant. If $A = B$, then A considered as an $A \otimes A^{\mathrm{op}}$-module corresponds to the identity endofunctor $\widehat{A} \to \widehat{A}$.

Consequently, the Hochschild cohomology of $\mathrm{L}_{\mathrm{qcoh}}(X)$ is given by the derived endomorphisms of the structure sheaf of the diagonal $\Delta \subset X \times_k X$. The Hochschild–Kostant–Rosenberg theorem [65] states that this is given by the derived global sections $\mathbb{R}\Gamma(\Lambda^\bullet T_X)$ of the sheaf of polyvector fields.

Now, going back to matrix factorizations, the Hochschild cohomology of $\mathrm{MF}_{\mathrm{dg}}(X, W)$ is the same as that of $\mathrm{Inj}(X, W)$. Consider the category of endofunctors of $\mathrm{Inj}(X, W)$, and consider the full subcategory consisting of continuous functors. We will identify this full subcategory with the category $\mathrm{Inj}(X \times X, \widetilde{W})$, where

$$\widetilde{W} := \pi_1^*(W) - \pi_2^*(W).$$

Furthermore, we will identify the identity endofunctor with the diagonal curved complex $\Delta \in \mathrm{Inj}(X \times X, \widetilde{W})$, that is, the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $X \hookrightarrow X \times X$ considered as a curved complex. We then compute the Hochschild cohomology of $\mathrm{MF}_{\mathrm{dg}}(X, W)$ by computing the derived endomorphisms of Δ. In other words, we will prove and employ curved analogues of Toën's basic uncurved results cited above.

Lemma 3.2.4. $\mathbb{R}\mathrm{Hom}(\Delta, \Delta) \cong \mathbb{R}\Gamma(\Lambda^\bullet T_X, [W, -])$.

Proof. We have a functor [50]

$$\mathcal{E}\mathrm{xt}^{\mathrm{II}} : \mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X, W) \times \mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X, W) \to \mathbf{D}\mathrm{QCoh}(X).$$

which is defined as follows — first do at least one of the following two things:

1. replace the second argument with a complex $I^\bullet$ of curved complexes of injective sheaves,
2. if possible, replace the first argument with a complex $P^\bullet$ of curved complexes of locally free sheaves of finite rank

then take their $\mathcal{H}\mathrm{om}$, and then finally take the direct sum total complex $\mathrm{Tot}^\oplus$ of the resulting double or triple complex. Because X is smooth and so $\mathrm{QCoh}(X)$ has finite homological dimension, we can choose such resolutions to have finite length, and thus we have that $\mathcal{E}\mathrm{xt}^{\mathrm{II}}(\Delta, \Delta)$ and $\mathbb{R}\mathcal{H}\mathrm{om}^\bullet(\Delta, \Delta)$ agree.

Our proof is essentially a curved adaptation of the work of Yekutieli [65] on the global Hochschild–Kostant–Rosenberg theorem. Let $\mathfrak{X}^q$ be the formal completion [26] of

$$X^q = X \times \cdots \times X$$

along the diagonal $X \hookrightarrow X^q$. For a commutative algebra A, denote by $B_q(A)$ the qth term $A \otimes A^{\otimes q} \otimes A$ in the standard bar complex $B(A)$. Let $\widehat{\mathcal{B}}_q(A)$ be the I_q-adic completion of $B_q(A)$, where I_q is the kernel of the map $B_q(A) \to A$ defined by

$$a_0 \otimes \cdots \otimes a_{q+1} \mapsto a_0 \cdots a_{q+1}.$$

On $B(A)$ we have the usual bar complex differential ∂_B, and we also have the "curved" differential ∂_W which is defined by

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto \sum_{i=0}^{n-1} (-1)^{i+1} a_0 \otimes a_1 \otimes \cdots \otimes a_i \otimes W \otimes a_{i+1} \otimes \cdots \otimes a_n.$$

Check that $(\partial_B + \partial_W)^2 = (\widetilde{W} \cdot -)$. It is easy to check that ∂_B and ∂_W are continuous with respect to the I-adic topologies.

Define $\widehat{\mathcal{B}}_q(X) := \mathcal{O}_{\mathfrak{X}^{q+2}}$. On an open affine $U = \operatorname{Spec} A \subset X$, we have $\Gamma(U, \widehat{\mathcal{B}}_q(X)) = \widehat{\mathcal{B}}_q(A)$. The ∂_B and ∂_W sheafify to give maps $\partial_B : \widehat{\mathcal{B}}_q(X) \to \widehat{\mathcal{B}}_{q-1}(X)$ and $\partial_W : \widehat{\mathcal{B}}_q(X) \to \widehat{\mathcal{B}}_{q+1}(X)$. Now let M be a curved $\mathcal{O}_{X^2}$-module with curvature $\widetilde{W}$. We denote the *Hochschild cohomology complex of $\mathcal{O}_X$ with coefficients in M* by $\mathcal{H}\mathrm{och}^{\oplus}(\mathcal{O}_X, M)$, and we define it as follows. It is a $\mathbb{Z}/2\mathbb{Z}$-graded complex with ith component given by

$$\bigoplus_{p+q=i} \mathcal{H}\mathrm{om}^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}_q(X), M_p).$$

This complex has differential $\partial + \partial_B + \partial_W$, where ∂ is induced from M, and ∂_B and ∂_W are induced by the respective maps defined above (see page 24 of [50]). The superscript "cont" denotes continuous morphisms, where we have the adic topology on $\widehat{\mathcal{B}}(X)$ and the discrete topology on M.

The category $\mathrm{Mod}_{\mathrm{disc}}\, \mathcal{O}_{\mathfrak{X}^2}$ (see §2 of [65]) has enough injectives. In fact, it is straightforward to see that these injectives can be chosen to be quasi-coherent as sheaves on $\mathcal{O}_{X^2}$. Consider $\mathcal{O}_X$ as an object of this category. Using the construction of proposition 2.4, we can then construct a resolution $I^\bullet$ of $\mathcal{O}_X$ by curved injective quasi-coherent sheaves on $\mathcal{O}_{X^2}$.

Therefore, we have

$$\mathcal{E}\mathrm{xt}^{\mathrm{II}}_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X) = \operatorname{Tot}^{\oplus} \mathcal{H}\mathrm{om}_{\mathcal{O}_{X^2}}(\mathcal{O}_X, I^\bullet).$$

Since all of the sheaves involved are discrete, we have

$$\operatorname{Tot}^{\oplus} \mathcal{H}\mathrm{om}_{\mathcal{O}_{X^2}}(\mathcal{O}_X, I^\bullet) = \operatorname{Tot}^{\oplus} \mathcal{H}\mathrm{om}^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\mathcal{O}_X, I^\bullet).$$

Consider the bicomplex $\mathcal{H}\mathrm{och}^{\oplus}(\mathcal{O}_X, I^\bullet)$ and the total complex obtained by taking direct sums of the diagonals of this bicomplex. Then we have two maps to this total complex:

$$\operatorname{Tot}^{\oplus} \mathcal{H}\mathrm{om}^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\mathcal{O}_X, I^\bullet) \to \operatorname{Tot}^{\oplus} \mathcal{H}\mathrm{och}^{\oplus}(\mathcal{O}_X, I^\bullet) \leftarrow \left(\bigoplus_q \mathcal{H}\mathrm{om}^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}_q(X), \mathcal{O}_X), \partial_B + \partial_W \right).$$

The second map is induced by the morphism $\widehat{\mathcal{B}}(X) \to \mathcal{O}_X$ and is a quasi-isomorphism by a spectral sequence argument. The first map is induced by $\mathcal{O}_X \to I^\bullet$ and is a quasi-isomorphism by Lemma 2.7 of [65], which states that that when X is smooth over $\mathbb{C}$, the functor

$$\mathcal{H}\mathrm{om}^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}_q(X), -) : \mathrm{Mod}_{\mathrm{disc}}\, \mathcal{O}_{\mathfrak{X}^2} \to \mathrm{Mod}_{\mathrm{disc}}\, \mathcal{O}_{\mathfrak{X}^2}$$

is exact. Our argument here parallels the argument on page 25 of [50].

Define $\widehat{\mathcal{C}}(X) := \widehat{\mathcal{B}}(X) \otimes_{\mathcal{O}_{\mathfrak{X}^2}} \mathcal{O}_X$, with induced differential also denoted by $\partial_B + \partial_W$. Then we have an identification of complexes

$$\left(\bigoplus_q \mathcal{H}\mathrm{om}^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}_q(X), \mathcal{O}_X), \partial_B + \partial_W\right) \cong \left(\bigoplus_q \mathcal{H}\mathrm{om}^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}_q(X), \mathcal{O}_X), \partial_B + \partial_W\right).$$

We now note that there is a quasi-isomorphism

$$\pi : (\Lambda^\bullet T_X, 0) \to \left(\bigoplus_q \mathcal{H}\mathrm{om}^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}_q(X), \mathcal{O}_X), \partial_B\right).$$

On an affine subscheme $\operatorname{Spec} A$, each graded component of the right hand side can be identified with polydifferential operators, namely the subcomplex of $\mathrm{Hom}_k(A^{\otimes q}, A)$ consisting of maps that are differential operators in each factor, and the isomorphism has the form

$$\pi(v_i \wedge \cdots \wedge v_q)(a_1 \otimes \cdots \otimes a_q) = \frac{1}{q!} \sum_{\sigma \in S_q} \mathrm{sgn}(\sigma) v_{\sigma(1)}(a_1) \cdots v_{\sigma(q)}(a_q).$$

One computes explicity in these local coordinates that $\pi([W, -]) = \partial_W(\pi(-))$ and thus we get an induced map of complexes

$$\pi : (\Lambda^\bullet T_X, [W, -]) \to \left(\bigoplus_q \mathcal{H}\mathrm{om}^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}_q, \mathcal{O}_X), \partial_B + \partial_W\right).$$

We conclude using exactly the same spectral sequence argument as in [10] in the affine case that this is a quasi-isomorphism. □

To complete the proof of Theorem 3.2.1, we must prove:

Theorem 3.2.5. *We have*

$$\mathbb{R}\mathrm{Hom}_{\mathrm{c}}(\mathrm{Inj}(X_1, W_1), \mathrm{Inj}(X_2, W_2)) \cong \mathrm{Inj}(X_1 \times X_2, \pi_1^*(W_1) - \pi_2^*(W_2)).$$

When $X_1 = X_2$ and $W_1 = W_2$, then the induced equivalence of homotopy categories identifies the identity functor with the diagonal curved sheaf Δ as an object of $\mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X \times X, \widetilde{W})$, where

$$\widetilde{W} := \pi_1^*(W) - \pi_2^*(W).$$

We will need some auxiliary results. We need the following theorem, which follows by results in §7 of [54].

Theorem 3.2.6. *If Z is a generator for $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(\mathrm{Sing}(X))$ and Y is a generator of $\mathfrak{Perf}(X)$, then $i_*Z \oplus Y$ generates $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(X)$. Here $\mathrm{Sing}(X)$ denotes the singular locus of X, and i denotes the inclusion $\mathrm{Sing}(X) \hookrightarrow X$.*

It follows that generators of $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(\mathrm{Sing}(X))$ are also generators of $\mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X)$. We note that this gives a new proof of a result of Dyckerhoff:

Corollary 3.2.7 ([15])**.** *If W has exactly one isolated singularity, then the residue field $\mathbb{C}$ of the singularity is a generator of the category $\mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(W^{-1}(0)) \cong [\mathrm{MF}_{\mathrm{dg}}(X, W)]$.*

Proof. The structure sheaf is a generator of $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(\mathrm{Spec}\,\mathbb{C})$. □

The proof of the following theorem was outlined to us by Raphaël Rouquier.

Theorem 3.2.8. *If E is a generator of $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(X)$ and F is a generator of $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(Y)$ then $E \otimes F$ is a generator of $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(X \times Y)$).*

Proof. First we observe that if $X = S \cup T$ is the union of two closed subvarieties, and if A generates $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(S)$ and B generates $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(T)$, then $A \oplus B$ generates $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(X)$. We show that $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(S)$ and $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(T)$ together generate $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(X)$. Let I_S be the sheaf of ideals of S and let I_T be the sheaf of ideals of T. Let F be a coherent sheaf on X. Then we have the short exact sequence

$$0 \to I_S F \to F \to F/I_S F \to 0.$$

Since $I_S I_T = 0$, we see that $I_S F$ is a coherent sheaf on T and $F/I_S F$ is a coherent sheaf on S. The claim follows.

Now to prove the theorem, we proceed by induction on $\dim X + \dim Y$. Let E' be a generator of $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(\mathrm{Sing}(X))$ and F' a generator of $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(\mathrm{Sing}(Y))$. By induction, we have that $E' \otimes F$ generates $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(\mathrm{Sing}(X) \times Y)$ and $E \otimes F'$ generates $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(X \times \mathrm{Sing}(Y))$. Let

$$Z = (\mathrm{Sing}(X) \times Y) \cup (X \times \mathrm{Sing}(Y))$$

which, because we are working over $\mathbb{C}$, is the same as $\mathrm{Sing}(X \times Y)$. Then $(E' \otimes F) \oplus (E \otimes F')$ generates $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(Z)$. Let E'' be a generator of $\mathfrak{Perf}(X)$ and F'' a generator of $\mathfrak{Perf}(Y)$. Then, $E'' \otimes F''$ generates $\mathfrak{Perf}(X \times Y)$ by Lemma 3.4.1 and Theorem 2.1.2 of [9], and hence

$$(E'' \otimes F'') \oplus (E' \otimes F) \oplus (E \otimes F')$$

generates $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(X \times Y)$. The claim follows. □

Lemma 3.2.9. *We have a functor D which takes a matrix factorization P to the matrix factorization $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(P, \mathcal{O}_X)$ and which induces an equivalence between $[\mathrm{MF}_{\mathrm{dg}}(X, W)]$ and $[\mathrm{MF}_{\mathrm{dg}}(X, -W)^{\mathrm{op}}]$. The functor*

$$\mathbb{R}\mathcal{H}\mathrm{om}(-, \mathcal{O}_{X_0}[1]) : \mathbf{D}^{\mathrm{b}}\mathrm{Coh}(X_0) \to \mathbf{D}^{\mathrm{b}}\mathrm{Coh}(X_0)^{\mathrm{op}}$$

induces a functor

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X_0) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X_0)^{\mathrm{op}}$$

which we will also denote by $\mathbb{R}\mathcal{H}\mathrm{om}(-, \mathcal{O}_{X_0}[1])$. Then finally we have the following commutative diagram

$$\begin{array}{ccc}
\mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X, W) & \longrightarrow & \mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X_0) \\
\downarrow D & & \downarrow \mathbb{R}\mathcal{H}\mathrm{om}(-, \mathcal{O}_{X_0}[1]) \\
\mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X, -W)^{\mathrm{op}} & \longrightarrow & \mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(X_0)^{\mathrm{op}}.
\end{array}$$

Proof of Theorem 3.2.5. Let E be a generator of $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(\mathrm{Sing}(W_1^{-1}(0)))$ and let F be a generator of $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}(\mathrm{Sing}(W_2^{-1}(0)))$. Let P be a matrix factorization of (X_1, W_1) such that we have a triangle

$$P \to E \to C$$

with C acyclic, and similarly let Q be a matrix factorization of $(X_2, -W_2)$ such that we have a triangle

$$Q \to F \to C'$$

with C' acyclic — we can do this by Lemma 3.1.18. Let A and B^{op} denote $\mathbb{R}\mathrm{Hom}(P, P)$ and $\mathbb{R}\mathrm{Hom}(Q, Q)$ respectively. Following the same argument as the proof of Theorem 4.2 of [15], we know that $\mathrm{Inj}(X_1, W_1) \cong \widehat{A}$ and we know that $\mathrm{Inj}(X_2, -W_2) \cong \widehat{B^{\mathrm{op}}}$. We also have $\mathrm{Inj}(X_2, W_2) \cong \widehat{B}$.

The cone of $P \otimes Q \to E \otimes F$ is acyclic. By Theorem 3.2.8, $E \otimes F$ generates the category $\mathbf{D}^{\mathrm{b}}_{\mathrm{Sing}}(W^{-1}(0))$ where $W = \pi_1^*(W_1) - \pi_2^*(W_2)$, because we have

$$\mathrm{Sing}(W^{-1}(0)) = \mathrm{Sing}(W_1^{-1}(0)) \times \mathrm{Sing}(W_2^{-1}(0)).$$

Therefore by Lemma 3.1.18, it follows that $P \otimes Q$ generates the matrix factorization category $\mathrm{MF}_{\mathrm{dg}}(X_1 \times X_2, W)$.

Since P and Q are curved vector bundles, we have a canonical isomorphism

$$\mathcal{H}\mathrm{om}(P, P) \otimes \mathcal{H}\mathrm{om}(Q, Q) \cong \mathcal{H}\mathrm{om}(P \otimes Q, P \otimes Q).$$

Therefore, we then have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{MF}_{\mathrm{Cech}}}(P \otimes Q, P \otimes Q) &= \Gamma\,\mathrm{Tot}\,\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{H}\mathrm{om}(P \otimes Q, P \otimes Q)) \\
&\cong \Gamma\,\mathrm{Tot}\,\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{H}\mathrm{om}(P, P) \otimes \mathcal{H}\mathrm{om}(Q, Q)) \\
&\cong \mathrm{Hom}_{\mathrm{MF}_{\mathrm{Cech}}}(P, P) \otimes \mathrm{Hom}_{\mathrm{MF}_{\mathrm{Cech}}}(Q, Q).
\end{aligned}$$

We see that

$$\operatorname{Inj}(X_1 \times X_2, W) \cong \widehat{A \otimes B^{\mathrm{op}}}.$$

The following string of isomorphisms follows by one of the theorems of Toën cited at the beginning of this section:

$$\operatorname{Inj}(X_1 \times X_2, W) \cong \widehat{A \otimes B^{\mathrm{op}}} \cong \mathbb{R}\mathrm{Hom}_{\mathrm{c}}(\widehat{A}, \widehat{B}) \cong \mathbb{R}\mathrm{Hom}_{\mathrm{c}}(\operatorname{Inj}(X_1, W_1), \operatorname{Inj}(X_2, W_2)).$$

In the case of $X_1 = X_2$, the claimed identification of the identity functor with Δ comes from the fact that $\mathbb{R}\mathrm{Hom}(P \otimes D(P), \Delta) \cong \mathbb{R}\mathrm{Hom}(P, P) = A$. The proof of this is the same as the proof of Proposition 6.3 of [15]. □

3.3 Smoothness, properness, and the Calabi–Yau property

By the calculations from the previous section and Corollary 1.24 of [48], we have the following:

Proposition 3.3.1. *When the critical locus of W is proper, the category* $\operatorname{Inj}(X, W)$ *is dg affine, homologically smooth, and proper as a differential* $\mathbb{Z}/2\mathbb{Z}$*-graded category.*[3]

Proof. We just need to show homological smoothness. The diagonal curved sheaf is compact, and so the identity endofunctor is compact. Writing $\operatorname{Inj}(X, W) \cong \widehat{A}$, the identity endofunctor corresponds to A as an $A \otimes A^{\mathrm{op}}$-module. So A is compact as an $A \otimes A^{\mathrm{op}}$-module as well, and therefore $\operatorname{Inj}(X, W)$ is homologically smooth. □

Lemma 3.3.2. *Suppose X_1 and X_2 are smooth with superpotentials W_1 and W_2 respectively. Suppose that the critical loci of W_1 and W_2 are proper. Then we have*

$$\mathbb{R}\mathrm{Hom}(\operatorname{Inj}(X_1, W_1)_{\mathrm{c}}, \operatorname{Inj}(X_2, W_2)_{\mathrm{c}}) \cong \operatorname{Inj}(X_1 \times X_2, \pi_1^*(W_1) - \pi_2^*(W_2))_{\mathrm{c}}.$$

Proof. $\operatorname{Inj}(X_1, W_1)$ and $\operatorname{Inj}(X_2, W_2)$ are respectively equivalent to $\widehat{A}$ and $\widehat{B}$, where A and B are smooth and proper dg algebras. What we need to know is that if M is an $A \otimes B^{\mathrm{op}}$-module such that for any perfect A-module P, in particular A itself, $P \otimes M$ is perfect as a B-module, then M is perfect. This follows immediately from the following well-known lemma, see e.g. Proposition 3.4 of [58]. □

Lemma 3.3.3. *A module N over a smooth and proper dg algebra over k is perfect if and only if* $\dim_k H^\bullet(N)$ *is finite.*

The goal of the rest of this section is to prove the following theorem:

[3]For the definitions of dg affineness, homological smoothness, and properness of categories, see Example 2.21 and Definition 2.23 of [32].

Theorem 3.3.4. *Let (X, W) be as above and, in addition, suppose X is Calabi–Yau. Then the category $\mathrm{Inj}(X, W)_{\mathrm{c}}$ is a Calabi–Yau category of dimension n, where n is the dimension of X.*[4]

As above, let $\widetilde{W}$ be the function $\pi_1^*(W) - \pi_2^*(W)$ on $X \times X$. Denote $\widetilde{W}^{-1}(0)$ by S. In the previous section we have proved that $\mathrm{Inj}(X, W) \cong \widehat{A}$, where $A = \mathbb{R}\mathrm{Hom}(P, P)$ and P is a compact generator. Let $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$ and recall that the inverse Serre bimodule is defined as

$$A^! = \mathbb{R}\mathrm{Hom}_{(A^{\mathrm{e}})^{\mathrm{op}}}(A, A^{\mathrm{e}}).$$

Thus to prove the Calabi–Yau property it suffices to prove that $A^! \cong A[n]$ in $[\mathrm{Int}(A^{\mathrm{e}}\text{-}\mathrm{Mod})]$.

We need to recall some theory from [27]. First we recall that given a closed immersion $i : X \to Y$ there is a functor

$$i^\flat := \mathbb{R}\mathcal{H}\mathrm{om}_Y(i_*\mathcal{O}_X, -) : \mathbf{D}^{\mathrm{b}}\mathrm{Coh}(Y)) \to \mathbf{D}^{\mathrm{b}}\mathrm{Coh}(X)).$$

It is easy to check that this functor has the property that given two morphisms i and j, we have $(j \circ i)^\flat \cong i^\flat \circ j^\flat$. Now we can factor the diagonal morphism $\Delta : X \to X \times X$ as the composition of $i : X \to S$ and $j : S \to X \times X$, so by the Fundamental Lemma on page 179 of [27],

$$\Delta^\flat(\mathcal{O}_{X\times X}) = \mathbb{R}\mathcal{H}\mathrm{om}_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_{X\times X}) = (\mathcal{O}_\Delta) \otimes \omega_{X/\mathbb{C}}^\vee[n],$$

where $\omega_{X/\mathbb{C}}$ is the canonical sheaf. The right hand side is $\mathcal{O}_\Delta[n]$ when X is Calabi–Yau. A simple calculation shows that $j^\flat(\mathcal{O}_{X\times X}) = \mathcal{O}_S[1]$. Thus we conclude that

$$\mathbb{R}\mathrm{Hom}_S(\mathcal{O}_\Delta, \mathcal{O}_S[1]) = \mathcal{O}_\Delta[n].$$

From here this argument follows exactly the argument of Lemma 6.8 of [15]. We repeat it here to show how to adapt it to our situation. Consider $D(P) \otimes P$, which is a generator for the category $\mathrm{MF}_{\mathrm{dg}}(X \times X, \widetilde{W})$. For any Z, we have

$$\mathbb{R}\mathrm{Hom}(D(P) \otimes P, Z) \cong \mathbb{R}\mathrm{Hom}(D(Z), P \otimes D(P)).$$

Now we let Z be the diagonal shifted by (the parity of) the dimension of X. By the discussion above and Lemma 3.2.9, $D(Z)$ corresponds to the diagonal Δ. We conclude with the sequence of isomorphisms:

$$\begin{aligned} A[n] &\cong \mathbb{R}\mathrm{Hom}(D(P) \otimes P, Z) \qquad (\Delta \text{ corresponds to the identity}) \\ &\cong \mathbb{R}\mathrm{Hom}(D(Z), P \otimes D(P)) \qquad (D \text{ is a contravariant equivalence}) \\ &\cong \mathbb{R}\mathrm{Hom}_{(A^{\mathrm{e}})^{\mathrm{op}}}(\mathbb{R}\mathrm{Hom}(P \otimes D(P), D(Z)), A^{\mathrm{e}}) \qquad (P \otimes D(P) \text{ is a generator}) \\ &\cong \mathbb{R}\mathrm{Hom}_{(A^{\mathrm{e}})^{\mathrm{op}}}(A, A^{\mathrm{e}}) \qquad (\Delta \text{ corresponds to the identity}) \\ &= A^!. \end{aligned}$$

This completes the proof of the Calabi–Yau condition.

[4] For the definition of Calabi–Yau category, see Definition 4.28 of [32].

3.4 Further directions

In this section, we discuss forthcoming work, which is again joint with Daniel Pomerleano, in which we construct Landau–Ginzburg models which we claim are mirror to certain smooth Fano 3-folds which can be degenerated to toric varieties with *nodal singularities*, also known as *ordinary double point* or *conifold singularities* [42, 59]. Our construction is based on the Strominger–Yau–Zaslow point of view on mirror symmetry, which we rapidly outline below, following the surveys of Auroux [6, 7]. With the aid of our results on matrix factorizations, we are also able to prove at least one new example of (one direction of) homological mirror symmetry. Much of this work came about in discussions and correspondence with Denis Auroux, whom we very gratefully acknowledge. We were also inspired by the work of Nishinou–Nohara–Ueda [45, 46]. We omit all proofs in this section for the sake of brevity.

Let X be a smooth Kähler manifold of complex dimension n with Kähler form ω, and let $D \subset X$ be an anticanonical divisor. Suppose that the complement $X \setminus D$ carries a nonvanishing holomorphic form Ω. Furthermore suppose that $X \setminus D$ admits a fibration by special Lagrangian tori. Such a fibration is called an *SYZ fibration*, after Strominger–Yau–Zaslow [61].

Given such a fibration on X, the total space of the mirror Landau–Ginzburg model is given by the moduli space M of pairs (L, ∇) where $L \subset X \setminus D$ is a special Lagrangian torus fiber and ∇ is (a gauge equivalence class of) a flat unitary connection on the trivial complex line bundle over L.

Lemma 3.4.1. *Let $A \in \mathrm{H}_2(X, L; \mathbb{Z})$ be a relative homology class with boundary $\partial A \neq 0 \in H_1(L; \mathbb{Z})$. Then the function*

$$z_A = \exp(-\textstyle\int_A \omega)\mathrm{hol}_\nabla(\partial A) : M \to \mathbb{C}^*$$

is holomorphic. This function may only be locally defined.

Lemma 3.4.2. *Let $A_i \in \mathrm{H}_2(X, L; \mathbb{Z})$ be a collection of relative homology classes such that ∂A_i forms a basis of $\mathrm{H}_1(L; \mathbb{Z})$. Suppose that the map $\mathrm{H}_1(L) \to \mathrm{H}_1(X)$ induced by the inclusion $L \hookrightarrow X$ is injective. Then the A_i yield a set of local holomorphic coordinates z_{A_i} on M.*

Given a holomorphic disc $\beta : (D^2, \partial D^2) \to (X, L)$, we let $\mu(\beta)$ denote the Maslov index of β, and we let $M(L, \beta)$ denote the moduli space of holomorphic discs in M with boundary on L and having the same relative homotopy class in $\pi_2(X, L)$ as β. In many cases, Maslov index can be easily computed by the formula in the lemma below.

Lemma 3.4.3. *If $L \subset X \setminus D$ is special Lagrangian, then $\mu(\beta)$ is twice the algebraic intersection number $\beta \cdot [D]$.*

Ignoring possible convergence issues, the superpotential W is then defined as follows.

Definition 3.4.4. For each point $(L, \nabla) \in M$, we let

$$W(L, \nabla) = \sum_{\beta} n_\beta(L) \exp(-\textstyle\int_\beta \omega) \mathrm{hol}_\nabla(\partial\beta) : M \to \mathbb{C}$$

where the sum is over all $\beta \in \pi_2(X, L)$ such that $\mu(\beta) = 2$, and where $n_\beta(L)$ is the number of holomorphic discs in the class β whose boundary passes through a generic point $p \in L$.

Remark 3.4.5. The general theory of counting holomorphic discs is quite subtle and technical, and we will not discuss it in depth. The reader is referred to the monumental works of Fukaya–Oh–Ohta–Ono for the complete details [21, 22, 19, 20].

We will assume that

1. all holomorphic discs of Maslov index 2 in (X, L) are regular, and
2. there are no non-constant holomorphic spheres in X with $\mathrm{c}_1(T_X) \cdot [S^2] \leq 0$.

In the cases that we consider, these assumptions actually do hold.

Over the locus of (L, ∇)'s in M such that L has no non-constant holomorphic discs of Maslov index 0, the number $n_\beta(L)$ is locally constant, and therefore the superpotential W as given above is well-defined and holomorphic. However, this locus is not connected, and the different components are separated by *walls*. In order to get a well-defined superpotential W on all of M, we then need to "glue" the connected components together by appropriate changes of variables so that the superpotentials agree when moving across the walls. These changes of variables — the "quantum corrections" — are determined by studying how Maslov index 2 discs change as L passes through the walls separating the different components. More precisely, bubbling of Maslov index 0 discs may occur, and this must be taken into consideration in order to get a correct mirror Landau–Ginzburg model.

Example 3.4.6. The best-known example is the case where X is an n-dimensional toric variety and D is the divisor consisting of all degenerate toric orbits. We write T^n for the torus acting on X that gives it the structure of a toric variety. Let $\mu : X \to \mathbb{R}^n$ be the moment map of the toric variety X and let $\Delta \subset \mathbb{R}^n$ be the moment polytope, which is the image of the moment map. We have $X \setminus D \cong (\mathbb{C}^*)^n$ with coordinates $x_1, \ldots x_n$. Under these coordinates, we let $\Omega = d \log x_1 \wedge \cdots d \log x_n$ be the holomorphic volume form. We let M be the complexified moduli space of pairs (L, ∇), where L is a T^n-orbit in the open stratum $X \setminus D$ and ∇ is as before. Then L is a special Lagrangian submanifold, and it is a fact that it is actually equal to a product of circles $S^1(r_1) \times \cdots \times S^1(r_n)$. It is also a fiber of the moment map over the interior of the moment polytope; so we can think of the moment map as an SYZ fibration on $X \setminus D$ with base space $\Delta \setminus \partial\Delta$.

Theorem 3.4.7. *The complexified moduli space M is biholomorphic to an open subset of $(\mathbb{C}^*)^n$ with coordinates z_i corresponding to the ith generator the first homology of*

$$L = S^1(r_1) \times \cdots \times S^1(r_n).$$

When X is Fano, it is customary to *inflate* M to be all of $(\mathbb{C}^*)^n$. We have the following well-known formula for the superpotential W in the case that X is both toric and Fano. Furthermore, we note that in the toric situation, there will be no Maslov index 0 discs, and so we will not have any wall-crossing phenomena to take into consideration.

Theorem 3.4.8 (Hori–Vafa, Cho–Oh [11]). *When X is a smooth toric Fano n-fold, its mirror Landau–Ginzburg model is given by $M = (\mathbb{C}^*)^n$ with superpotential*

$$W = \sum_F e^{-2\pi\alpha(F)} z^{\nu(F)},$$

where the sum is taken over all facets F of the moment polytope Δ of F, $\nu(F)$ is the primitive integer normal vector to F pointing into the interior of Δ, and $\alpha(F)$ is the real constant such that the equation of F is $\langle \nu(F), \phi \rangle + \alpha(F) = 0$.

We now consider the variety V_ε, which is not toric but can be degenerated to a singular toric variety V_0.

Definition 3.4.9. For $\varepsilon \in \mathbb{C}$ a constant, we define

$$V_\varepsilon = \{xy - zw = \varepsilon\} \subset \mathbb{C}^4.$$

The variety V_0 is called the *conifold*, or *node*, and when ε is nonzero V_ε is the *smoothed conifold.*

Remark 3.4.10. The literature on conifolds is vast. We simply point the reader to [42, 59] and make no further attempt to survey the topic.

Besides the smoothing, one can take a small resolution of the conifold singularity to obtain a different desingularization. It has been claimed in [42, 59] that the smoothing of the conifold should be mirror to the small resolution of the conifold. We are able to make this precise as follows:

Claim 3.4.11. *The SYZ mirror of the smoothed conifold V_ε is the Landau–Ginzburg model (M, W) where*

$$M = \{(u, v, w_1, w_2, [x_1, x_2]) \in \mathbb{C}^4 \times \mathbb{P}^1 : ux_2 = (1 + w_2)x_1, (1 + w_1)x_2 = vx_1\}$$

is the small resolution of the conifold singularity and where

$$W = u.$$

Let $\delta \in \mathbb{C}$ be a constant. We justify the claim by exhibiting an explicit SYZ fibration on $V_\varepsilon \setminus D_\delta$, where $D_\delta = \{xy = \delta\} \subset V_\varepsilon$ is an anticanonical divisor, and by doing explicit counts of holomorphic discs. Our method is very similar to the method used in the $\mathbb{P}^2$ example studied by Auroux in [6, 7] — see also the recent preprint [1]. The basic idea is that, despite the fact that V_ε is not toric, it nevertheless admits the following T^2 action:

$$(x, y, z, w) \mapsto (e^{i(t+s)}x, e^{-is}y, e^{i(t+r)}z, e^{-ir}w).$$

We can make use of the moment map corresponding to this action in order to construct a non-toric SYZ fibration on $V_\varepsilon \setminus D_\delta$.

We are then able to prove one direction of homological mirror symmetry for this candidate mirror pair:

Theorem 3.4.12. *The smoothed conifold V_ε is symplectomorphic to the cotangent bundle T^*S^3 of the 3-sphere S^3, and the wrapped Fukaya category of T^*S^3 is equivalent to* $\mathrm{MF}_{\mathrm{dg}}(M, W)$*, where (M, W) is as above.*

According to general philosophy, taking a compactification $\overline{X}$ of a variety X should correspond to including additional terms in the mirror superpotential. From the SYZ perspective, these additional terms arise due to new holomorphic discs in $\overline{X}$ which do not exist in X. For instance, the smoothed conifold can be compactified to obtain a smooth quadric 3-fold in $\mathbb{P}^4$. Then we claim that the mirror Landau–Ginzburg model is (M, W) where M is as above and

$$W = u + \frac{Qv}{w_1 w_2},$$

where Q is a nonzero constant. In future work we plan on addressing homological mirror symmetry for this example and related examples of non-toric Fano 3-folds.

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