

UC Irvine

UC Irvine Previously Published Works

Title

Axisymmetric Euler- α Equations without Swirl: Existence, Uniqueness, and Radon Measure Valued Solutions

Permalink

<https://escholarship.org/uc/item/1cj1c07d>

Authors

Jiu, Quansen
Niu, Dongjuan
Titi, Edriss S
et al.

Publication Date

2009-07-14

Peer reviewed

**AXISYMMETRIC EULER- α EQUATIONS WITHOUT SWIRL:
EXISTENCE, UNIQUENESS, AND RADON MEASURE VALUED
SOLUTIONS**

QUANSEN JIU¹

School of Mathematical Sciences, Capital Normal University
Beijing 100048, P. R. China
Email: jiuqs@mail.cnu.edu.cn

DONGJUAN NIU²

School of Mathematical Sciences, Capital Normal University
Beijing 100048, P. R. China
and
Departamento de Matematica, IMECC-UNICAMP Caixa Postal 6065,
Campinas, SP 13081-970, Brasil
Email: niudongjuan@gmail.com

EDRISS S. TITI³

Department of Computer Science and Applied Mathematics
Weizmann Institute of Science, Rehovot 76100, Israel
and
Department of Mathematics of Mechanical and Aerospace Engineering
University of California, Irvine, California 92697, USA
Email: etiti@math.uci.edu

ZHOUPING XIN⁴

The Institute of Mathematical Sciences, The Chinese University of Hong Kong
Shatin, N.T., Hong Kong
Email: zpxin@ims.cuhk.edu.hk

Date: July 14, 2009.

¹The research is partially supported by National Natural Sciences Foundation of China (No. 10871133) and Project of Beijing Education Committee.

²The research is partly supported by FAPESP grant # 2008/09473-0.

³The research is supported in part by the NSF grant no. DMS-0708832 and the ISF grant no. 120/06.

⁴The research is partially supported by Zheng Ge Ru Funds, Hong Kong RGC Earmarked Research Grant CUHK4042/08P and CUHK4040/06P, RGC Central Allocation Grant CA 05/06. SC01, and a grant from Northwest University, Xian, China.

ABSTRACT. The global existence of weak solutions for the three-dimensional axisymmetric Euler- α (also known as Lagrangian-averaged Euler- α) equations, without swirl, is established, whenever the initial unfiltered velocity v_0 satisfies $\frac{\nabla \times v_0}{r}$ is a finite Radon measure with compact support. Furthermore, the global existence and uniqueness, is also established in this case provided $\frac{\nabla \times v_0}{r} \in L_c^p(\mathbb{R}^3)$ with $p > \frac{3}{2}$. It is worth mention that no such results are known to be available, so far, for the three-dimensional Euler equations of ideal incompressible flows.

KEY WORDS: Euler- α equations; Lagrangian-averaged Euler- α equations; Axisymmetric fluids; Existence and uniqueness; Axisymmetric vortex sheets.

2000 MATHEMATICS SUBJECT CLASSIFICATION. 76B47; 35Q30.

In honor of Professor John Gibbon on his 60th birthday

1. INTRODUCTION

The Euler- α (also known as the Lagrangian-averaged Euler- α) equations, introduced by Holm, Marsden and Ratiu [19], [20], are a regularization of the Euler equations that describe the motion of an ideal incompressible fluid. Moreover, these same equations also govern the motion of inviscid (non-viscous) three-dimensional second-grade incompressible non-Newtonian fluid (see, e.g., [15], [16]). The equations are given by the system:

$$\begin{cases} \partial_t v + u \cdot \nabla v + \sum_j v_j \nabla u_j + \nabla p = 0, \\ v = (1 - \alpha^2 \Delta)u, \\ \operatorname{div} u = 0 \end{cases} \quad (1.1)$$

in \mathbb{R}^3 , where u, p represent the velocity vector field and the pressure of the fluid, respectively. The length scale $\alpha > 0$ is the regularization parameter, and one recovers, formally, the Euler equations when $\alpha = 0$.

The initial data for (1.1) are imposed as

$$u(t = 0, x) = u_0(x). \quad (1.2)$$

Denoting by $q = \operatorname{curl} v$, the vorticity, it follows from (1.1) that q satisfies the system:

$$\begin{cases} \partial_t q + u \cdot \nabla q = q \cdot \nabla u, \\ u = K_\alpha * q, \\ q(t = 0, x) = q_0(x) \end{cases} \quad (1.3)$$

in \mathbb{R}^3 , where K_α is the integral kernel of the inverse of the operator $(1 - \alpha^2 \Delta) \operatorname{curl}$. Equation (1.3) resembles the equation of motion of the vorticity in the three-dimensional Euler equations. However, the vorticity stretching term, which is the main obstacle for proving the global regularity for the Euler equations, is replaced above by the milder term $q \cdot \nabla u$. Despite this mollification of the vorticity stretching term the question of global regularity for the three-dimensional Euler- α is still, as in the case of the three-dimensional Euler equations, a challenging open problem.

In the following, $\operatorname{curl} u_0$ and $\operatorname{curl} v_0$ will denote the vorticity of u_0 and of $v_0 = (1 - \alpha^2 \Delta) u_0$ respectively. $\operatorname{curl} v_0$ is also called the initial unfiltered vorticity, while $\operatorname{curl} u_0$ is called the filtered initial vorticity.

There has been a lot of mathematical progress made on the Euler- α equations recently. For the two-dimensional case, Kouranbaeva and Oliver [27] obtained global existence and uniqueness of (1.1)-(1.2) for the initial unfiltered vorticity of class L^2 . The artificial viscosity method is applied in [27]. Furthermore, Oliver and Shkoller [37] proved the global existence and uniqueness of weak solutions for initial unfiltered vorticity $\operatorname{curl} v_0$ in $M(\mathbb{R}^2)$, which is the space of finite Radon measures. For the three-dimensional axisymmetric case, Busuioc and Ratiu [7] proved the global existence and uniqueness of classical solutions for the three-dimensional axisymmetric Euler- α equations without swirl. In particular, in the case of the whole space \mathbb{R}^3 , the restrictions on the initial data in [7] are: $u_0 \in H^3(\mathbb{R}^3)$, $\operatorname{curl} v_0 / r \in L^2(\mathbb{R}^3)$ and $\operatorname{curl} v_0 \in L^p(\mathbb{R}^3)$, for some $p \in [1, 2]$. For the general three-dimensional case, a blow up criterion of the smooth solutions, in the spirit of the Beale-Kato-Majda [5], was presented in [21] (see also [38]). The local existence and a blowup criterion in Besov spaces for the 3D Lagrangian averaged Euler equations was given in [30].

Formally, and as we have already mentioned above, when $\alpha = 0$ in (1.1), the Euler- α equations become the classical Euler equations. A natural question is whether solutions of the Euler equations can be approximated properly by those of the corresponding Euler- α equations, especially for the vortex-sheets initial data. It is well-known that when the initial data are a vortex-sheets data, *i.e.*, the initial vorticity is a finite Radon measure and the initial velocity is locally square-integrable, the two-dimensional time-dependent Euler equations have global (in time) weak solutions when the initial vorticity $\operatorname{curl} u_0$ is of one-sign. This result was first proved by Delort [12] by regularizing the initial data to construct the approximate solutions (see also [32] for a slight generalization). Then the result of Delort was proved by different approaches (see, e.g., [18], [31], [33], [34], [39]). Specifically, the Navier-Stokes approximations were used in [33], [36], and the vortex-method approximations were applied in [31]. Very recently, the

Euler- α equations were proposed as an inviscid approximation to the three-dimensional Euler equations. Specifically, it was shown in [1] and [2] the global regularity of the vortex sheet problem of the 2D Euler- α equations for a wider class of vortex-sheet. Moreover, it was shown that these solutions converge to one of the Delort solutions of the vortex sheet for the 2D Euler equations, as a subsequence of the regularization parameter $\alpha_j \rightarrow 0$. However, the vortex-sheet problem for the three-dimensional Euler equations remain unsolved even for the case of one-signed measure. In a recent work of [4] it has been shown, by an example of 3D shear flow of Euler equations, the existence, and persistence for all time, of singular vortex sheet solutions. This is a fundamentally different behavior than in the 3D case (see, e.g., [8], [14], [28], [29], [43]). For some other work concerning the vortex sheet problem the reader is referred to [3], [6], [9], [10], [13], [23]-[25], and further studies will be reported in the forthcoming paper [22] by using the Euler- α approximations in the spirit of [1] and [2].

The purpose of this paper is to prove the global existence and uniqueness of weak solutions for the three-dimensional axisymmetric Euler- α equations without swirl. We will first obtain the global existence and uniqueness of the solutions when the initial unfiltered vorticity $\frac{\text{curl}v_0}{r}$ belongs to $L^p_c(\mathbb{R}^3)$ with $p > \frac{3}{2}$, which is the usual L^p Lebesgue space with compact support. Then we will prove the existence of global weak solutions when the initial unfiltered vorticity $\frac{\text{curl}v_0}{r}$ belongs to $M_c(\mathbb{R}^3)$, which is the space of finite Radon measures with compact support. The uniqueness of the solutions is still not clear in the case of weak Radon measures valued solutions. In our analysis, the velocity $u(t, x)$ will be recovered from the unfiltered vorticity $q = \text{curl } v$ by the expression $u = K_\alpha * q$, where K_α is the integral kernel of the inverse of $(1 - \alpha^2 \Delta)\text{curl}$ (see (3.3) for more details). As in [1], [2] and [37], properties of the kernel K_α near the origin and at the infinity would be essential to apply the singular integral estimates approach. More precisely, when $\frac{\text{curl}v_0}{r} \in L^p_c(\mathbb{R}^3)$ with $p > \frac{3}{2}$, we will be able to prove that the velocity $u(t, x)$ is Lipschitz continuous with respect to the space variables and uniformly continuous with respect to the time variable so that we can prove the global existence and uniqueness of the solution. In the case where the initial unfiltered vorticity $\frac{\text{curl}v_0}{r}$ belongs to $M_c(\mathbb{R}^3)$ we obtain the global existence of the weak solutions. This is done by establishing appropriate bounds for the gradient and the Hessian of the approximate solutions and using standard compactness arguments. It should be noted that in this case the Lipschitz continuity of $u(t, x)$ on the spatial variables and the uniform continuity with respect to the time variable are still open.

We will state our main results in Section 2 and the proofs of our main results will be given in Section 3.

2. MAIN RESULTS

In the cylindrical coordinates r, θ, z , the velocity u is written as $u = u_r e_r + u_\theta e_\theta + u_z e_z$, where $e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0)$, $e_\theta = (\frac{x_2}{r}, -\frac{x_1}{r}, 0)$, $e_z = (0, 0, 1)$, $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$. Axisymmetric flows without swirl are solutions for which the azimuthal (angular) component of the velocity field satisfies $u_\theta \equiv 0$, and u_r, u_z are independent of θ . In this case, we also have the unfiltered velocity $v = v_r e_r + v_z e_z$ in the cylindrical coordinate systems. It should be mentioned that the space of axisymmetric flows is invariant under the solution of three-dimensional Euler- α equations.

Let $q = \text{curl } v$ be the unfiltered vorticity. Then only the azimuthal (angular) component of q is non-zero in the cylindrical system, i.e.,

$$q = q^\theta(t, r, z)e_\theta, \quad (2.1)$$

where $q^\theta = \partial_z v_r - \partial_r v_z$. It follows that q solves equation (1.3) and direct calculations show that

$$\partial_t \left(\frac{q^\theta}{r} \right) + u \cdot \nabla \left(\frac{q^\theta}{r} \right) = 0. \quad (2.2)$$

That is, the scalar function q^θ/r satisfies the transport equation. This is a key and important property of the invariant family of axisymmetric flows without swirl (see also [13], [23], [24], [17], [42]). As observed above, for this invariant family of flows the vorticity stretching term, i.e. the right-hand side of equation (1.3), $q \cdot \nabla u \equiv 0$. Consequently, it follows, formally, from the transport equation (2.2) that

$$\left\| \frac{q^\theta}{r} \right\|_{L^r} = \left\| \frac{q_0^\theta}{r} \right\|_{L^r}, \quad (2.3)$$

for $r \in [1, \infty]$.

Before stating our main results, we introduce the following definition of weak solutions. Let \mathcal{G} be the group of all homeomorphisms ϕ of \mathbb{R}^3 which preserve the Lebesgue measure.

Definition 2.1 (The case of $L^p(\mathbb{R}^3)$ ($p \geq 1$)). Let $T > 0$, the vector field $u \in C([0, T]; Lip(\mathbb{R}^3))$ and $q = \text{curl } v \in L^p(\mathbb{R}^3)$ with $p \geq 1$ are said to be a weak solution of (1.3) if there exists a unique Lagrangian trajectory $y(t, x) \in C([0, T]; \mathcal{G})$ satisfying

$$y(t, x_0) = x_0 + \int_0^t u(s, y(s, x_0)) ds \quad x_0 \in \mathbb{R}^3, \quad (2.4)$$

$$u(t, x) = K_\alpha * q, \quad (2.5)$$

$$u(t=0, x) = u_0(x), \quad (2.6)$$

and the first equation of (1.3) is satisfied in the sense of distributions. In (2.5), K_α is, as before, the integral kernel representation of the inverse operator of $(1 - \alpha^2 \Delta) \text{curl}$ (see (3.3) for more details).

Remark 2.1 For the axisymmetric Euler- α equations without swirl, the first equation of (1.3) becomes (2.2) and in Definition 2.1 it should be satisfied in the following sense:

$$\int_{\mathbb{R} \times \mathbb{R}^3} (\partial_t \varphi + u \cdot \nabla \varphi) \frac{q^\theta}{r} dx dt = 0 \quad (2.7)$$

for any $\varphi \in C_0^\infty((0, T) \times \mathbb{R}^3)$.

Definition 2.2 (The case of $M(\mathbb{R}^3)$). Let $T > 0$, and let $\text{curl}v_0/r \in M_c(\mathbb{R}^3)$. The vector field $u \in L^\infty([0, T] \times \mathbb{R}^3)$ is said to be a weak solution of (1.1) if

- (1) $\nabla u \in L^2([0, T]; L_{loc}^2(\mathbb{R}^3)); D^2 u \in L^1([0, T]; L_{loc}^1(\mathbb{R}^3))$.
- (2) For every test function $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^3)$ with $\text{div} \varphi = 0$, equation (1.1) is satisfied in the following sense

$$\begin{aligned} & \int_{[0, T] \times \mathbb{R}^3} [u(t, x) \cdot (1 - \alpha^2 \Delta) \partial_t \varphi(t, x) + (u \cdot \nabla) \varphi \cdot (1 - \alpha^2 \Delta) u] dx dt \\ & + \alpha^2 \int_{[0, T] \times \mathbb{R}^3} (\nabla \varphi : D^2) u \cdot u dx dt = - \int_{\mathbb{R}^3} u_0 \cdot (1 - \alpha^2 \Delta) \varphi(0, x) dx, \end{aligned} \quad (2.8)$$

where $\nabla \varphi : D^2 = \sum_{i, k=1}^n \partial_i \varphi_k \partial_k \partial_i$.

Our main results are stated as:

Theorem 2.1. *Assume that the initial velocity is divergence free, axisymmetric without swirl and $\text{curl}v_0/r \in L_c^p$ with $p > \frac{3}{2}$. Then for any $T > 0$, there exists a unique solution of (1.3) in the sense of Definition 2.1, over the interval $[0, T]$.*

Theorem 2.2. *Assume that the initial velocity is divergence free, axisymmetric without swirl and $\text{curl}v_0/r \in M_c(\mathbb{R}^3)$. Then for any $T > 0$, there exists a global weak solution $u \in L^\infty([0, T] \times \mathbb{R}^3)$ of (1.1) in the sense of Definition 2.2. Moreover, we have that $\nabla u \in L^\infty((0, T); L^a + L^\infty)$ with $1 \leq a < 3$ and $D^2 u \in L^\infty((0, T); L^b + L^\infty)$ with $1 \leq b < \frac{3}{2}$.*

3. PROOF OF THE MAIN RESULTS

In this section, we will give the proofs of Theorems 2.1 and 2.2.

For a smooth solution to (1.3), one can define a particle trajectory $y(t, x)$ by

$$\partial_t y(t, x_0) = u(t, y(t, x_0)),$$

$$y(0, x_0) = x_0, \quad (3.1)$$

where $x_0 \in \mathbb{R}^3$. It is noted that the transformation $y(t, x_0)$ on \mathbb{R}^3 preserves the measure due to the divergence free condition of u (see [35]). Moreover, one can recover the velocity $u(t, x)$ from the unfiltered vorticity $q = \text{curl } v$ through a precise expression of the integral kernel K_α in (3.3) as follows. Due to the divergence free condition of u , there exists a potential vector Ψ such that $\text{div} \Psi = 0$ and $u = \nabla \times \Psi$. Then by applying the curl operator to the second equation of (1.1) yields that

$$q = -(1 - \alpha^2 \Delta) \Delta \Psi. \quad (3.2)$$

Direct calculations show that the Green function associated with the operator $(1 - \alpha^2 \Delta) \Delta$ is (see, e.g., [26]) $G_\alpha(|x - y|) = \frac{1 - e^{-\frac{|x-y|}{\alpha}}}{4\pi|x-y|}$. Thus, we deduce that

$$\begin{aligned} u(t, x) &= \nabla \times \int_{\mathbb{R}^3} G_\alpha(|x - y|) q(t, y) dy \\ &= \int_{\mathbb{R}^3} f_\alpha(|x - y|) \frac{x - y}{|x - y|} \times q(t, y) dy, \end{aligned} \quad (3.3)$$

where $f_\alpha(|x - y|) := \frac{1}{\alpha^2} f\left(\frac{|x-y|}{\alpha}\right)$, and $f(z) := \frac{(1+z)e^{-z} - 1}{4\pi z^2}$. Obviously, $f(z)$, $\nabla f(z)$, and $zf(z)$ are continuous and bounded functions for $z \in (0, +\infty)$. In addition, the kernel K_α in the second equation of (1.3) can be represented as $K_\alpha(x, y) = \nabla \times G_\alpha(|x - y|)$.

In view of (2.1), we obtain that for the axisymmetric Euler- α equations without swirl the velocity is recovered by

$$u(t, x) = \int_{\mathbb{R}^3} f_\alpha(|x - y|) \frac{x - y}{|x - y|} \times \frac{q^\theta(t, y)}{r} (y_2, -y_1, 0) dy. \quad (3.4)$$

Now we are ready to prove Theorem 2.1. Motivated by [35] and [37], we will utilize the particle trajectory (3.1) to construct the approximate solutions and prove the existence and uniqueness of solutions of the axisymmetric Euler- α equations without swirl. The key ingredient is to prove the Lipschitz continuity of the vector fields u with respect to the spatial variables.

Proof of Theorem 2.1. The sequences of the approximate solutions of (2.2) can be constructed as follows

$$\partial_t y^n(t, x_0) = u^n(t, y^n(t, x_0)), \quad (3.5)$$

$$y^n(0, x_0) = x_0, \quad (3.6)$$

$$y^0(t, x_0) = x_0, \quad (3.7)$$

$$u^n(t, x) = \int_{\mathbb{R}^3} K_\alpha(x, y) q^{n-1}(t, y) dy, \quad (3.8)$$

$$\frac{(q^n)^\theta}{r}(y^n(t, x_0), t) = \frac{q_0^\theta}{r}(x_0), \quad (3.9)$$

for $n \in \mathbb{N}$.

Thanks to (3.4), we have

$$\begin{aligned} |u^n(t, x)| &= \left| \int_{\mathbb{R}^3} f_\alpha(|x - y^{n-1}(t, z)|) \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} \right. \\ &\quad \times (-x_2 + y_2^{n-1}(t, z), x_1 - y_1^{n-1}(t, z), 0) \frac{q_0^\theta}{r}(z) dz \\ &\quad \left. + \int_{\mathbb{R}^3} f_\alpha(|x - y^{n-1}(t, z)|) \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} \times (x_2, -x_1, 0) \frac{q_0^\theta}{r}(z) dz \right| \\ &\leq \int_{\mathbb{R}^3} |x - y^{n-1}(t, z)| |f_\alpha(|x - y^{n-1}(t, z)|)| \left| \frac{q_0^\theta}{r}(z) \right| dz \\ &\quad + \int_{\mathbb{R}^3} |f_\alpha(|x - y^{n-1}(t, z)|)| \left| \frac{q_0^\theta}{r}(z) \right| dz |x| \\ &\leq \frac{C}{\alpha^2} \left\| \frac{q_0^\theta}{r} \right\|_{L^1} (1 + |x|). \end{aligned} \quad (3.10)$$

In the last inequality above, we have used the facts that $|x|f_\alpha(|x|)$ and $f_\alpha(|x|)$ in (3.3) are continuous and bounded functions in \mathbb{R}^3 . By the assumption that $\text{curl}v_0/r \in L_c^p \subset L^1$, for $p > 3/2$, and (3.10), one has

$$|u^n(t, x)| \leq \frac{C}{\alpha^2} (1 + |x|), \quad (3.11)$$

where C is a constant depending on the L^1 norm of $\frac{q_0^\theta}{r}$ and the bounds of $f_\alpha(x)$ and $xf_\alpha(x)$ in (3.3), for $x \in (0, +\infty)$.

Integrating (3.5) from 0 to t gives

$$y^n(t, x_0) = x_0 + \int_0^t u^n(s, y^n(s, x_0)) ds. \quad (3.12)$$

(3.11) and the Gronwall inequality imply that

$$|y^n(t, x_0)| \leq (Ct + |x_0|) e^{Ct} := L(t, |x_0|), \quad (3.13)$$

for any $t \in [0, T]$. Using (3.4) again yields

$$\begin{aligned} |u^n(t, x)| &= \left| \int_{\mathbb{R}^3} f_\alpha(|x - y^{n-1}(t, z)|) \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} \right. \\ &\quad \left. \times (y_2^{n-1}(t, z), -y_1^{n-1}(t, z), 0) \frac{q_0^\theta}{r}(z) dz \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{C}{\alpha^2} \int_{\mathbb{R}^3} L(t, |z|) \left| \frac{q_0^\theta}{r} \right|(z) dz \\
 &\leq \frac{C}{\alpha^2} \left\| \frac{q_0^\theta}{r} \right\|_{L^1}, \tag{3.14}
 \end{aligned}$$

where C is a constant depending on $\sup\{|z| : z \in \text{supp } \frac{q_0^\theta}{r}\}$.

Next we prove that u^n is Lipschitz continuous with respect to the spatial variables and is uniformly continuous in time. For any $x, x' \in \mathbb{R}^3$, it follows from (3.4)-(3.9) that

$$\begin{aligned}
 &|u^n(t, x) - u^n(t, x')| \\
 &= \left| \int_{\mathbb{R}^3} \left[f_\alpha(|x - y^{n-1}(t, z)|) \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} - f_\alpha(|x' - y^{n-1}(t, z)|) \right. \right. \\
 &\quad \left. \left. \cdot \frac{x' - y^{n-1}(t, z)}{|x' - y^{n-1}(t, z)|} \right] \times (y_2^{n-1}(t, z), -y_1^{n-1}(t, z), 0) \frac{q_0^\theta}{r}(z) dz \right| \\
 &= \left| \int_{\mathbb{R}^3} \left[(f_\alpha(|x - y^{n-1}(t, z)|) - f_\alpha(|x' - y^{n-1}(t, z)|)) \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} \right. \right. \\
 &\quad \left. \left. + f_\alpha(|x' - y^{n-1}(t, z)|) \left(\frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} - \frac{x' - y^{n-1}(t, z)}{|x' - y^{n-1}(t, z)|} \right) \right] \right. \\
 &\quad \left. \times (y_2^{n-1}(t, z), -y_1^{n-1}(t, z), 0) \frac{q_0^\theta}{r}(z) dz \right|. \tag{3.15}
 \end{aligned}$$

By the mean value theorem, there exists a point $x'' \in \mathbb{R}^3$ such that

$$\begin{aligned}
 &|u^n(t, x) - u^n(t, x')| \\
 &\leq |x - x'| \int_{\mathbb{R}^3} \left[|\nabla f_\alpha(|x'' - y^{n-1}(t, z)|)| + \frac{|f_\alpha(|x' - y^{n-1}(t, z)|)|}{|x' - y^{n-1}(t, z)|} \right] \\
 &\quad \cdot |y^{n-1}(t, z)| \left| \frac{q_0^\theta}{r}(z) \right| dz \\
 &\leq \frac{c}{\alpha^3} |x - x'| \int_{\{z|z \in \text{supp}\{\frac{q_0^\theta}{r}\}\}} \left(1 + \frac{1}{|x' - y^{n-1}(t, z)|} \right) L(t, |z|) \frac{q_0^\theta}{r}(z) |dz| \\
 &\leq \frac{c}{\alpha^3} |x - x'| \left(\left\| \frac{q_0^\theta}{r} \right\|_{L^1} + \left\| \frac{q_0^\theta}{r} \right\|_{L^p} \left\| \frac{1}{|x' - y^{n-1}(t, z)|} \right\|_{L^{\frac{p}{p-1}}} \right) \\
 &\leq \frac{c}{\alpha^3} |x - x'| \left(\left\| \frac{q_0^\theta}{r} \right\|_{L^1} + \left\| \frac{q_0^\theta}{r} \right\|_{L^p} \right), \tag{3.16}
 \end{aligned}$$

where $p > \frac{3}{2}$, and c depends on the bounds of $\nabla f(|x|)$ and $L(T, |z|)$ (see (3.13)) if $z \in \text{supp}\{\frac{q_0^\theta}{r}\}$ and is independent of α .

To show the uniform continuity on the time variable, we have, for any $t, t' \in [0, T]$,

$$\begin{aligned}
& |u^n(t, x) - u^n(t', x)| \\
&= \left| \int_{\mathbb{R}^3} [f_\alpha(|x - y^{n-1}(t, z)|) \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} \times (y_2^{n-1}(t, z), -y_1^{n-1}(t, z), 0) \right. \\
&\quad \left. - f_\alpha(|x - y^{n-1}(t', z)|) \frac{x - y^{n-1}(t', z)}{|x - y^{n-1}(t', z)|} \times (y_2^{n-1}(t', z), -y_1^{n-1}(t', z), 0)] \frac{q_0^\theta}{r}(z) dz \right| \\
&\leq \int_{\mathbb{R}^3} |f_\alpha(|x - y^{n-1}(t, z)|) - f_\alpha(|x - y^{n-1}(t', z)|)| \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} \|y^{n-1}(t, z)\| \frac{q_0^\theta}{r}(z) dz \\
&\quad + \int_{\mathbb{R}^3} |f_\alpha(|x - y^{n-1}(t', z)|)| \left| \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} - \frac{x - y^{n-1}(t', z)}{|x - y^{n-1}(t', z)|} \right| \|y^{n-1}(t, z)\| \frac{q_0^\theta}{r}(z) dz \\
&\quad + |f_\alpha(|x - y^{n-1}(t', z)|)| \left| \frac{x - y^{n-1}(t', z)}{|x - y^{n-1}(t', z)|} \|y^{n-1}(t, z) - y^{n-1}(t', z)\| \right| \frac{q_0^\theta}{r}(z) dz \\
&\leq \|\nabla f_\alpha\|_{L^\infty} \int_{\mathbb{R}^3} |y^{n-1}(t, z) - y^{n-1}(t', z)| \|y^{n-1}(t, z)\| \frac{q_0^\theta}{r}(z) dz \\
&\quad + \int_{\mathbb{R}^3} |f_\alpha(|x - y^{n-1}(t', z)|)| \left| \frac{y^{n-1}(t, z) - y^{n-1}(t', z)}{|x - y^{n-1}(t', z)|} \|y^{n-1}(t', z)\| \right| \frac{q_0^\theta}{r}(z) dz \\
&\quad + \int_{\mathbb{R}^3} |f_\alpha(|x - y^{n-1}(t', z)|)| \|y^{n-1}(t, z) - y^{n-1}(t', z)\| \frac{q_0^\theta}{r}(z) dz \\
&\leq \frac{C}{\alpha^3} \int_{\mathbb{R}^3} |y^{n-1}(t, z) - y^{n-1}(t', z)| \left[\frac{1 + |x - y^{n-1}(t', z)|}{|x - y^{n-1}(t', z)|} L(T, z) + 1 \right] \frac{q_0^\theta}{r}(z) dz \\
&\leq \frac{C}{\alpha^3} |t - t'| \int_{\mathbb{R}^3} \left| \sup_{s \in [0, T]} |u^{n-1}(s, z)| \left[\frac{1 + |x - y^{n-1}(t', z)|}{|x - y^{n-1}(t', z)|} + 1 \right] \right| \frac{q_0^\theta}{r}(z) dz \\
&\leq \frac{C}{\alpha^3} |t - t'| \left(\left\| \frac{q_0^\theta}{r} \right\|_{L^1} + \left\| \frac{q_0^\theta}{r} \right\|_{L^p} \right) \tag{3.17}
\end{aligned}$$

for any $p > \frac{3}{2}$, where constant C is independent of α and depends on the bounds of $u^{n-1}(t, z)$ and $y^{n-1}(t, z)$ if z lies in the support of $\frac{q_0^\theta}{r}$ in view of (3.13) and (3.14) respectively. Since u^n is Lipschitz continuous with respect to the spatial variables and is uniformly continuous with respect to time, and u^n is uniformly bounded by (3.14), then the map $y^n \in C^1([0, T]; C(\mathbb{R}^3))$, for every $T > 0$. Moreover, we have that $y^n(t, x) - x \in C^1([0, T]; C_B(\mathbb{R}^3))$ in which $C_B(\mathbb{R}^3)$ means the space of bounded and continuous functions.

Similar to the estimates in (3.16) and (3.17), one will prove that $\{y^n(t, x) - x\}$ is a Cauchy sequence in $C([0, T]; C_B(\mathbb{R}^3))$ and furthermore $\{y^n\}$ is a Cauchy sequence in $C([0, T]; \mathcal{G})$. For simplicity, the time dependence of sequences $\{u^n\}$ and $\{y^n\}$, $n \in \mathbb{N}$, is dropped in the following estimates.

Note that

$$\begin{aligned}
 & |y^n(t, x) - y^{n-1}(t, x)| \\
 & \leq \int_0^t |u^n(s, y^n(s, x)) - u^{n-1}(s, y^{n-1}(s, x))| ds \\
 & \leq \int_0^t \left| \int_{\mathbb{R}^3} f_\alpha(|y^n(x) - y^{n-1}(z)|) \frac{y^n(x) - y^{n-1}(z)}{|y^n(x) - y^{n-1}(z)|} \times (y_2^{n-1}, -y_1^{n-1}, 0) \frac{q_0^\theta}{r} \right. \\
 & \quad \left. - f_\alpha(|y^{n-1}(x) - y^{n-2}(z)|) \frac{y^{n-1}(x) - y^{n-2}(z)}{|y^{n-1}(x) - y^{n-2}(z)|} \times (y_2^{n-2}, -y_1^{n-2}, 0) \frac{q_0^\theta}{r} dz \right| ds \\
 & \leq \frac{C}{\alpha^3} \int_0^t \int_{\mathbb{R}^3} |y^n(x) - y^{n-1}(x)| |y^{n-1}(z)| \frac{q_0^\theta}{r} |dz ds \\
 & \quad + \frac{C}{\alpha^2} \int_0^t \int_{\mathbb{R}^3} \frac{|y^n(x) - y^{n-1}(x)|}{|y^{n-1}(x) - y^{n-1}(z)|} |y^{n-1}(z)| \frac{q_0^\theta}{r} |dz ds \\
 & \quad + \frac{C}{\alpha^2} \int_0^t \int_{\mathbb{R}^3} |y^{n-1}(x) - y^{n-2}(x)| \frac{q_0^\theta}{r} |dz ds \\
 & \leq \frac{C}{\alpha^3} (\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p}) \int_0^t (\sup_{x \in \mathbb{R}^3} |y^n - y^{n-1}| + \sup_{x \in \mathbb{R}^3} |y^{n-1} - y^{n-2}|) ds
 \end{aligned} \tag{3.18}$$

for $p > \frac{3}{2}$, where $C > 0$ is a constant depending on $\sup\{|z| : z \in \text{supp } \frac{q_0^\theta}{r}\}$ and is independent of α . Define $g^N(t) = \sup_{n \geq N} \sup_{x \in \mathbb{R}^3} |y^n(x, t) - y^{n-1}(x, t)|$.

Then (3.14) implies that $g^N(t) \leq \frac{C}{\alpha^3} \| \frac{q_0^\theta}{r} \|_{L^1} t < \infty$ for $t \in [0, T]$, where $C > 0$ is a constant depending on $\sup\{|z| : z \in \text{supp } \frac{q_0^\theta}{r}\}$. Choose $T_1 > 0$ small enough such that $\frac{C}{\alpha^3} (\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p}) T_1 \leq 1/2$. Then it follows from (3.18) that

$$g^N(t) \leq k \int_0^t g^{N-1}(s) ds, \quad t \in [0, T_1], \tag{3.19}$$

where $k := \frac{2C}{\alpha^3} (\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p})$, with $C > 0$ a constant depending on $\sup\{|z| : z \in \text{supp } \frac{q_0^\theta}{r}\}$. Similar as Lemma 3.2 of Chapter 2 in [35], we obtain that

$$\lim_{N \rightarrow \infty} g^N(t) \rightarrow 0 \tag{3.20}$$

uniformly on $[0, T_2]$, for some $T_2 \in (0, T_1]$ sufficient small. Actually, it can be proved that $g^N(t)$ can be bounded by the terms of a convergent geometrical series for $t \in [0, T_2]$. This implies that $\{y^n - x\}$ is a Cauchy sequence in $C([0, T_0]; C_B(\mathbb{R}^3))$, where $T_0 = \min\{T_1, T_2\}$ depends on $\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p}$

and $\sup\{|z| : z \in \text{supp } \frac{q_0^\theta}{r}\}$. To continue the procedure above to the interval $[T_0, 2T_0]$, we have

$$\begin{aligned} & |y^n(t, x) - y^{n-1}(t, x)| \\ & \leq \int_{T_0}^t |u^n(y^n) - u^{n-1}(y^{n-1})| ds + |y^n(T_0, x) - y^{n-1}(T_0, x)| \end{aligned} \quad (3.21)$$

for $t \geq T_0$. Due to (3.20), for any $\epsilon > 0$, there exists a $N_1 > 0$ such that $\sup_{x \in \mathbb{R}^3} |y^n(T_0, x) - y^{n-1}(T_0, x)| < \epsilon$, for $n \geq N_1$. Therefore, similar to (3.18), for all $n \geq N_1$, one has

$$\begin{aligned} & |y^n(t, x) - y^{n-1}(t, x)| \\ & \leq \int_{T_0}^t |u^n(y^n) - u^{n-1}(y^{n-1})| ds + \epsilon \\ & \leq \frac{C}{\alpha^3} (\|\frac{q_0^\theta}{r}\|_{L^1} + \|\frac{q_0^\theta}{r}\|_{L^p}) \int_{T_0}^t (\sup_{x \in \mathbb{R}^3} |y^n - y^{n-1}| + \sup_{x \in \mathbb{R}^3} |y^{n-1} - y^{n-2}|) ds + \epsilon, \end{aligned} \quad (3.22)$$

for $t \in [T_0, 2T_0]$. It follows that

$$\begin{aligned} & \int_{T_0}^t \sup_{x \in \mathbb{R}^3} |y^n(s, x) - y^{n-1}(s, x)| ds \\ & \leq \frac{C}{\alpha^3} (\|\frac{q_0^\theta}{r}\|_{L^1} + \|\frac{q_0^\theta}{r}\|_{L^p}) (t - T_0) \int_{T_0}^t (\sup_{x \in \mathbb{R}^3} |y^n - y^{n-1}| + \sup_{x \in \mathbb{R}^3} |y^{n-1} - y^{n-2}|) ds + \epsilon(t - T_0), \end{aligned}$$

for $t \in [T_0, 2T_0]$. This implies that

$$\begin{aligned} & \int_{T_0}^t \sup_{x \in \mathbb{R}^3} |y^n(s, x) - y^{n-1}(s, x)| ds \\ & \leq \int_{T_0}^t \sup_{x \in \mathbb{R}^3} |y^{n-1} - y^{n-2}| ds + 2\epsilon(t - T_0), \end{aligned} \quad (3.23)$$

Putting (3.23) into (3.22), we get

$$\begin{aligned} & \sup_{x \in \mathbb{R}^3} |y^n(t, x) - y^{n-1}(t, x)| \\ & \leq 2 \frac{C}{\alpha^3} (\|\frac{q_0^\theta}{r}\|_{L^1} + \|\frac{q_0^\theta}{r}\|_{L^p}) \int_{T_0}^t (\sup_{x \in \mathbb{R}^3} |y^n - y^{n-1}| + \sup_{x \in \mathbb{R}^3} |y^{n-1} - y^{n-2}|) ds \\ & + 2\epsilon(t - T_0) \frac{C}{\alpha^3} (\|\frac{q_0^\theta}{r}\|_{L^1} + \|\frac{q_0^\theta}{r}\|_{L^p}) + \epsilon. \end{aligned} \quad (3.24)$$

Since $\frac{C}{\alpha^3}(\|\frac{q_0^\theta}{r}\|_{L^1} + \|\frac{q_0^\theta}{r}\|_{L^p})(t - T_0) \leq 1/2$, for all $t \in [T_0, 2T_0]$, it follows that

$$g^N(t) \leq k \int_{T_0}^t g^{N-1}(s) ds + 2\epsilon, \quad t \in [T_0, 2T_0], \quad (3.25)$$

where k is same as in (3.19). By the arbitrariness of ϵ , similar to the previous procedure (see also Lemma 3.2 of Chapter 2 in [35]), we can prove that $\{y^n - x\}$ is a Cauchy sequence in $C([T_0, 2T_0]; C_B(\mathbb{R}^3))$, and furthermore in $C([0, T]; C_B(\mathbb{R}^3))$; and $\{y^n\}$ is a Cauchy sequence in $C([0, T]; \mathcal{G})$, for every $T > 0$. The limit of $\{y^n(t, x)\}$ in $C([0, T]; \mathcal{G})$ is denoted by $y(t, x)$.

Define $q(t, x) = \frac{q^\theta}{r}(x, t)(x_2, -x_1, 0)$ satisfying $\frac{q^\theta}{r}(y(t, x), t) = \frac{q_0^\theta}{r}(x, t)$. Then it is straightforward to prove that

$$q^n(t, x) = \frac{(q^n)^\theta}{r}(x, t)(x_2, -x_1, 0) \rightharpoonup q(t, x)$$

in the sense of weakly-* convergence in $L^\infty([0, T]; L^p(\mathbb{R}^3))$, with $p > 3/2$, and

$$u^n(t, x) \rightarrow u(t, x)$$

in $C_B([0, T] \times \mathbb{R}^3)$. Moreover, y, u, q solve (3.22), (2.5) and satisfy

$$\int_{\mathbb{R} \times \mathbb{R}^3} (\partial_t \varphi + u \cdot \nabla \varphi) \frac{q^\theta}{r} dx dt = 0 \quad (3.26)$$

for any $\varphi \in C_0^\infty((0, T) \times \mathbb{R}^3)$.

The uniqueness can be shown by the direct estimate on the difference of two flow maps. The estimates are similar to the previous ones and we omit the details here. The proof of the theorem is finished.

Proof of Theorem 2.2. Mollifying the initial potential vorticity, we can construct the approximate solutions of (1.3) by solving the following problem:

$$\begin{cases} \partial_t q^\epsilon + u^\epsilon \cdot \nabla q^\epsilon = q^\epsilon \cdot \nabla u^\epsilon, \\ u^\epsilon = K_\alpha * q^\epsilon, \\ q^\epsilon(t=0, x) = q_0^\epsilon, \end{cases} \quad (3.27)$$

where q_0^ϵ is a smooth vector with compact support which converges to q_0 in $M(\mathbb{R}^3)$ and $\|q_0^\epsilon\|_{L^1} \leq \|q_0\|_M$. Then there exists a unique smooth solution (u^ϵ, q^ϵ) to (3.27) (see [7] and references therein). Moreover, there is a smooth pressure p^ϵ such that u^ϵ satisfies

$$\begin{cases} \partial_t v^\epsilon + u^\epsilon \cdot \nabla v^\epsilon + \sum_j v_j^\epsilon \nabla u_j^\epsilon + \nabla p^\epsilon = 0, \\ v^\epsilon = (1 - \alpha^2 \Delta) u^\epsilon, \\ \operatorname{div} u^\epsilon = 0. \end{cases} \quad (3.28)$$

Therefore, for any test function $\varphi \in C_0^\infty([0, T], \mathbb{R}^3)$, satisfying $\operatorname{div} \varphi = 0$, integration by parts yields

$$\begin{aligned} & \int_{[0, T] \times \mathbb{R}^3} [u^\epsilon(t, x)(1 - \alpha^2 \Delta) \partial_t \varphi(t, x) + (u^\epsilon \cdot \nabla) \varphi \cdot (1 - \alpha^2 \Delta) u^\epsilon] dx dt \\ & + \alpha^2 \int_{[0, T] \times \mathbb{R}^3} (\nabla \varphi : D^2) u^\epsilon \cdot u^\epsilon dx dt = - \int_{\mathbb{R}^3} u_0^\epsilon (1 - \alpha^2 \Delta) \varphi(0, x) dx. \end{aligned} \quad (3.29)$$

Similar analysis as for (3.14) shows that

$$\|u^\epsilon\|_{L^\infty([0, T] \times \mathbb{R}^3)} \leq \frac{C}{\alpha^2} \left\| \frac{q_0^\theta}{r} \right\|_M, \quad (3.30)$$

where $C > 0$ is a constant depending on T and the Lebesgue measure of the support of $\frac{q_0^\theta}{r}$.

We now estimate ∇u^ϵ and $D^2 u^\epsilon$. Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a smooth functions satisfying

$$\chi(s) = \begin{cases} 1, & |s| < 1, \\ 0, & |s| > 2. \end{cases}$$

Then it follows from (3.4) that

$$\begin{aligned} & \partial_{x_i} u^\epsilon(t, x) \\ & = \int_{\mathbb{R}^3} \partial_{x_i} [f_\alpha(|x - y^\epsilon|) \frac{x - y^\epsilon}{|x - y^\epsilon|}] (1 - \chi(|x - y^\epsilon|)) \times \frac{q^\theta(t, y^\epsilon)}{r}(y_2^\epsilon, -y_1^\epsilon, 0) dy \\ & \quad + \int_{\mathbb{R}^3} \partial_{x_i} [f_\alpha(|x - y^\epsilon|) \frac{x - y^\epsilon}{|x - y^\epsilon|}] \chi(|x - y^\epsilon|) \times \frac{q^\theta(t, y^\epsilon)}{r}(y_2^\epsilon, -y_1^\epsilon, 0) dy \\ & \equiv G_1(t, x) + G_2(t, x), \end{aligned} \quad (3.31)$$

for $i = 1, 2, 3$. It is clear that

$$\begin{aligned} |G_1(t, x)| & \leq \frac{C}{\alpha^3} \int_{\mathbb{R}^3} \frac{1}{|x - y^\epsilon(t, z)|} (1 - \chi(|x - y^\epsilon(t, z)|)) \left| \frac{(q_0^\epsilon)^\theta}{r}(z) \right| |y^\epsilon(t, z)| dz \\ & \leq \frac{C}{\alpha^3} \left\| \frac{q_0^\theta}{r} \right\|_M, \end{aligned}$$

for $(t, x) \in [0, T] \times \mathbb{R}^3$. One can use Young's inequality for convolutions to obtain

$$\begin{aligned} & \|G_2(t, x)(t, x)\|_{L^\infty((0, T); L^a)} \\ & \leq \sup_{t \in [0, T]} \left\| \int_{\mathbb{R}^3} \frac{1}{|x - y^\epsilon(t, z)|} \chi(|x - y^\epsilon(t, z)|) \left| \frac{(q_0^\epsilon)^\theta}{r}(z) \right| |y^\epsilon(t, z)| dz \right\|_{L^a} \\ & \leq \frac{C}{\alpha^3} \left\| \frac{(q_0^\epsilon)^\theta}{r} \right\|_{L^1} \leq \frac{C}{\alpha^3} \left\| \frac{q_0^\theta}{r} \right\|_M \end{aligned} \quad (3.32)$$

for $1 \leq a < 3$. Thus $\nabla u^\epsilon(t, x)$ is bounded in $L^\infty((0, T); L^a + L^\infty)$ and

$$\|\nabla u^\epsilon(t, x)\|_{L^\infty((0, T); L^a + L^\infty)} \leq \frac{C}{\alpha^3} \left\| \frac{q_0^\theta}{r} \right\|_M \quad (3.33)$$

for $1 \leq a < 3$. Similarly, we can prove that $D^2 u^\epsilon(t, x)$ is bounded in $L^\infty((0, T); L^b + L^\infty)$ and that

$$\|D^2 u^\epsilon(t, x)\|_{L^\infty((0, T); L^b + L^\infty)} \leq \frac{C}{\alpha^4} \left\| \frac{q_0^\theta}{r} \right\|_M \quad (3.34)$$

for $1 \leq b < \frac{3}{2}$. In (3.33) and (3.34), the constant C is a positive constant depending on $\left\| \frac{q_0^\theta}{r} \right\|_M$ and the support of $\frac{q_0^\theta}{r}$. In view of (3.30), (3.33) and (3.34), it is easy to obtain that the terms $u^\epsilon \cdot \nabla v^\epsilon$, $\sum_j v_j^\epsilon \nabla u_j^\epsilon$ and ∇p^ϵ

are bounded in $L^\infty((0, T); W_{loc}^{-2,2}(\mathbb{R}^3))$ and hence it follows from (3.28) that $\partial_t u^\epsilon$ is bounded in $L^\infty((0, T); L_{loc}^2(\mathbb{R}^3))$. Note that u^ϵ is bounded in $L^\infty((0, T); W_{loc}^{1,a}(\mathbb{R}^3))$ for any $1 \leq a < 3$. By the Aubin-Lions Lemma (see, e.g., [11], [40]), we obtain that (up to a subsequence) that $u^\epsilon \rightarrow u$ strongly in $C([0, T]; L_{loc}^c(\mathbb{R}^3))$, with $1 \leq c < \frac{3a}{3-a}$, where $u \in L^\infty([0, T] \times \mathbb{R}^3)$ is a function satisfying $\nabla u \in L^\infty((0, T); L^a + L^\infty)$, with $1 \leq a < 3$, and $D^2 u \in L^\infty((0, T); L^b + L^\infty)$, with $1 \leq b < \frac{3}{2}$. Thanks to (3.30), we have that $u^\epsilon \rightarrow u$ strongly in $L^\sigma([0, T]; L_{loc}^\sigma(\mathbb{R}^3))$, for any $\sigma \in (1, \infty)$. Moreover, it is clear that $D^2 u^\epsilon \rightharpoonup D^2 u$ weakly- $*$ convergence in $L^\infty((0, T); L_{loc}^b)$, for any $1 \leq b < 3/2$. Thus, it follows from (3.29) that

$$\begin{aligned} & \int_{[0, T] \times \mathbb{R}^3} [u(t, x)(1 - \alpha^2 \Delta) \partial_t \varphi(t, x) + (u \cdot \nabla) \varphi \cdot (1 - \alpha^2 \Delta) u] dx dt \\ & + \alpha^2 \int_{[0, T] \times \mathbb{R}^3} (\nabla \varphi : D^2) u \cdot u dx dt = - \int_{\mathbb{R}^3} u_0 (1 - \alpha^2 \Delta) \varphi(0, x) dx \end{aligned}$$

for any $\varphi \in C_0^\infty([0, T], \mathbb{R}^3)$ satisfying $\operatorname{div} \varphi = 0$. The proof of Theorem 2.2 is complete.

REFERENCES

- [1] C. Bardos, J. Linshiz and E.S. Titi. Global regularity for a Birkhoff-Rott- α approximation of the dynamics of vortex sheets of the 2D Euler equations. *Physica D*, 237:1905–1911, 2008.
- [2] C. Bardos, J. S. Linshiz, E. S. Titi. Global regularity and convergence of a Birkhoff-Rott- α approximation of the dynamics of vortex sheets of the 2D Euler equations. *Comm. Pure Appl. Math.*, (to appear).
- [3] C. Bardos and E.S. Titi. Euler equations of incompressible ideal fluids. *Uspekhi Matematicheskikh Nauk, UMN*, 62:3(375):5-6, 2007. Also in *Russian Mathematical Surveys*, 62(3):409–451, 2007.

- [4] C. Bardos and E.S. Titi. Loss of smoothness and energy conserving rough weak solutions for the 3d Euler equations. *Discrete and Continuous Dynamical Systems*, (submitted), 2009.
- [5] J.T. Beale, T. Kato and A. Majda. Remarks on the breakdown of smooth solutions for the 3D Euler equations. *Commun. Math. Phys.*, 94: 61–66, 1984.
- [6] A. L. Bertozzi, P. Constantin. Global regularity for vortex patches. *Comm. Math. Phys.*, 152(1):19–28, 1993.
- [7] A. V. Busuioc, T. S. Ratiu. Some remarks on a certain class of axisymmetric fluids of differential type. *Physical D*, 191:106–120, 2004.
- [8] R. Caffisch and O. Orellana. Singular solutions and ill-posedness for the evolution of vortex sheets. *SIAM J. Math. Anal.*, 20(2):293–307, 1989.
- [9] J.-Y. Chemin. Persistance de Structures geometriques dans les fluids incompressibles bidimensionnels. *Ann. Sci. Ecole Norm. Sup.*, 26(4):517–542, 1993.
- [10] J. Y. Chemin. Two-dimensional Euler system and the vortex patches problem. In *Handbook of mathematical fluid dynamics*, III:83-160, North-Holland, 2004.
- [11] P. Constantin and C. Foias. The Navier-Stokes Equations. *The University of Chicago Press*, Chicago, 1988.
- [12] J.-M. Delort. Existence de nappes de tourbillon en dimension deux. *J. Amer. Math. Soc.*, 4(3):553–586, 1991.
- [13] J. M. Delort. Une remarque sur le probleme des nappes de tourbillon axisymetriques sur R^3 . *J. Funct. Anal.*, 108:274–295, 1992.
- [14] J. Duchon and R. Robert. Global vortex sheet solutions of Euler equations in the plane. *J. Differential Equations*, 73(2):215–224, 1988.
- [15] J. E. Dunn and R. L. Fosdick. Thermodynamics, stability, and boundedness of fluids of complexity 2 and fluids of second grade. *Arch. Rat. Mech. Anal.*, 56:191–252, 1974.
- [16] J. E. Dunn and K.R. Rajagopal. Fluids of differential type: critical reviews and thermodynamic analysis. *Int. J. Engng. Sci.*, 33:689–729, 1995.
- [17] B. Ettinger and E.S. Titi. Global existence and uniqueness of weak solutions of 3-D Euler equations with helical symmetry in the absence of vorticity stretching. *SIAM J. Math. Anal.*, 41(1):269-296, 2009.
- [18] L. C. Evans and S. Müller. Hardy space and the two-dimensional Euler equations with non-negative vorticity. *J. Amer. Math. Soc.*, 7:199–219, 1994.
- [19] D. D. Holm, J. E. Marsden and T. Ratiu. Euler-Poincaré models of ideal fluids with nonlinear dispersion. *Phys. Rev. Lett.*, 80:4173–4176, 1998.
- [20] D. D. Holm, J. E. Marsden, and T.S. Ratiu. Euler-Poincaré equations and semidirect products with applications to continuum theories. *Adv. in Math.*, 137:1–81, 1998.
- [21] T. Y. Hou, C. Li. On global well-posedness of the lagrangian averaged Euler equations. *SIAM J. Math. Anal.*, 38(3):782–794, 2006.
- [22] Q. S. Jiu, D. J. Niu, E. S. Titi, Z. P. Xin. The Euler- α approximations to the 3D axisymmetric Euler equations with vortex-shhets initial data, preprint, 2009.
- [23] Q. S. Jiu, Z. P. Xin. Viscous approximation and decay rate of maximal vorticity function for 3-D axisymmetric Euler equations. *Acta Math. Sin. (Engl. Ser.)*, 20(3):385–404, 2004.
- [24] Q. S. Jiu and Z. P. Xin. On strong convergence to 3-D axisymmetric vortex sheets. *J. Differential Equations*, 223(1):33–50, 2006.
- [25] Q. S. Jiu and Z. P. Xin. Smooth Approximations and Exact Solutions of the 3D Steady Axisymmetric Euler Equations. *Comm. Math. Phys.*, 287:323–349, 2009.

- [26] R. P. Kanwal. Generalized Functions Theory and Technique. *Academic Press*, 1983.
- [27] S. Kouranbaeva, M. Oliver. Global well-posedness for the averaged Euler equations in two dimensions. *Physica D*, 138:197–209, 2000.
- [28] R. Krasny. Computation of vortex sheet roll-up in Trefftz plane. *J. Fluid Mech.*, 184:123–155, 1987.
- [29] G. Lebeau. Régularité du problème de Kelvin-Helmholtz pour l'équation d'Euler 2d. *ESAIM Control Optim. Calc. Var.*, 8:801–825, 2002 (electronic).
- [30] X. F. Liu and H. Y. Jia. Local existence and blowup criterion of the Langrangian averaged Euler equations in Besov spaces. *Commun. Pure Appl. Anal.*, 7(4):845–852, 2008.
- [31] J. G. Liu and Z. P. Xin. Convergence of vortex methods for weak solutions to the 2-D Euler equations with vortex sheet data. *Comm. Pure Appl. Math.*, 48:611–628, 1995.
- [32] M. C. Lopes Filho, H. J. Nussenzveig Lopes, S. Schochet. A criterion for the equivalence of the Birkhoff–Rott and Euler description of vortex sheet evolution. *Trans. Amer. Math. Soc.*, 359:4125–4142, 2007 (electronic).
- [33] A. Majda. Remarks on weak solutions for vortex sheets with a distinguished sign. *Indiana Univ. Math. J.*, 42:921–939, 1993.
- [34] A. Majda and A. Bertozzi. Vorticity and incompressible flow. Volume 27 of Cambridge Texts in Applied Mathematics. *Cambridge University Press*, Cambridge, 2002.
- [35] C. Marchioro, M. Pulvirenti. Mathematical Theory of Incompressible Nonviscous Fluids. *Springer-Verlag*, New York, 1994.
- [36] D. J. Niu, Q. S. Jiu and Z. P. Xin. Navier-Stokes approximations to 2D vortex sheets in half plane. *Methods and Applications of Analysis*, 14(3):263–272, 2007.
- [37] M. Oliver, S. Shkoller. The vortex blob method as a second-grade non-newtonian fluid. *Commun. in Partial Differential Equations*, 26(1&2):295–314, 2001.
- [38] E. Olson and E.S. Titi. Viscosity versus vorticity stretching: global well-posedness for a family of the Navier-Stokes alpha-like models. *Nonlinear Analysis*, 66(11):2427–2458, 2007.
- [39] S. Schochet. The weak vorticity formulation of the 2D Euler equations and concentration-cancellation. *Comm. P. D. E.*, 20:1077–1104, 1995.
- [40] R. Temam. Navier-Stokes Equations, Theory and Numerical Analysis. 3rd revised edition, *North-Holland*, 2001.
- [41] T. Shirota, T. Yanagisawa. Note on global existence for axially symmetric solutions of the Euler system. *Proc. Japan. Acad.*, 70(A):299–304, 1994.
- [42] V.I. Yudovich. Non-stationary flows of an ideal incompressible fluid. *Z. Vyčisl. Mat. i Mat. Fiz.*, 3:1032–1066, 1963.
- [43] S. Wu. Mathematical analysis of vortex sheets. *Comm. Pure Appl. Math.*, 59(8):1065–1206, 2006.