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# RELATIONSHIP BETWEEN A BUNCH CHARGE DISTRIBUTION AND THE TIME PROFILE OF A TRANSITION RADIATION FLASH*†

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**Relationship Between A Bunch Charge Distribution and The Time Profile of a Transition Radiation Flash**

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ABSTRACT

A relationship between a bunch charge distribution function and the time dependence of a transition radiation flash is determined. It is shown that in the optical frequency band this relationship allows one to determine the bunch longitudinal profile. The possibility of determination of the charge distribution in transverse section is considered as well.

In the present paper we have shown how the time profile of the transition radiation flash is related to the particle distribution along the bunch length. It is amply evident that the transition radiation intensity at a given instant of time and at a given point of space (with account of time delay) depends on the number of particles traversing the interface of two media at a given time moment. This circumstance, points out the possibility of determining the longitudinal component of the bunch particle distribution function by measuring the time dependence of transition radiation intensity.

For simplicity we will assume that the radiation takes place on a vacuum - ideally conductive surface boundary.

## 1. TRANSITION RADIATION OF A SINGLE CHARGE

Let a point-like charge q moving uniformly at a velocity $V = V_Z$ along the Z axis traverses normally the plane (x,y) which is a vacuum - ideal conductor interface (Fig. 1).

The "backward" radiation field - in the opposite direction to motion - can be written as a Fourier integral /1/:

$$\vec{E}(x,y,z;t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{E}^{0}_{\frac{\omega}{v},K_x,K_y}(z) e^{i\left(k_x x + K_y y - \frac{\omega}{v} \cdot vt\right)} dK_x dK_y d\frac{\omega}{v} / (2\pi)^3 \qquad (1.1)$$

where $\vec{E}^{0}_{\frac{\omega}{v},K_x,K_y}(Z)$ is a Fourier-transformed image of the radiation field with respect to time and transverse coordinates:

$$\vec{E}_{\frac{\omega}{v},K_x,K_y}(Z) = iea\left(K_x,K_y\right) \cdot \vec{e}\left(K_x,K_y\right) e^{-iz\sqrt{\omega^2/c^2 - K_x^2 - K_y^2}} \qquad (1.2)$$

In (1.2) by a $(K_x, K_y)$ we denote:

$$a(K_x, K_y) = \frac{4\pi\beta}{\omega}(\varepsilon - 1)S(K_x, K_y) \cdot V \tag{1.3}$$

where in turn

$$S(K_x, K_y) = \frac{æ\left(1 - \beta^2 - \beta\sqrt{\varepsilon - æ^2}\right)}{\left(1 - \beta^2 + \beta^2 æ^2\right)\left(1 + \beta\sqrt{\varepsilon - æ^2}\right)\left(\sqrt{\varepsilon - æ^2} + \varepsilon\sqrt{1 - æ^2}\right)} \tag{1.4}$$

$$æ = \frac{c}{\omega}\sqrt{K_x^2 + K_y^2}$$

Further in (1.2)

$$\vec{e}(K_x, K_y) = æ\vec{E} - \vec{q}\sqrt{1 - æ^2} \tag{1.5}$$

Vector $\vec{q}$ locates in the plane $(x, y)$ at angle $\varphi$ to the $x$ axis:

$$\vec{q} = \vec{x}\cos\varphi + \vec{y}\sin\varphi \tag{1.6}$$

Integral (1.1) is simplified in the "far zone" - for large values of $R = \sqrt{x^2 + y^2 + Z^2}$. In this case its asymptotic value can be obtained by the stationary phase method or by the saddle-point method /2/. With account of our assumed ideal conductor model and at $R >> 1$ for (1.1) we have

$$\vec{E}^o(x, y, z; t) = \vec{E}^o(R, \theta; L) = U(\theta; R)\delta(L)\vec{e}_\theta \tag{1.7}$$

where

$$U(\theta; R) = \frac{e\beta^2}{\pi R}\frac{\sin\theta}{1 - \beta^2\cos^2\theta} \tag{1.8}$$

defines spatial distribution of transition radiation field amplitude in the far zone. The angle between the negative direction of the $Z$ axis and the observation direction - $\theta$ together with the azimuthal angle φ and radius-vector $R$ characterize the observation point in the spherical coordinate system (Fig. 1). The unit vector $\bar{e}_\theta$ is the polar unit vector of this coordinate system.

Parameter $L$ introduced in (1.7) has a dimensionality of length:

$$L = \beta R - vt \tag{1.9}$$

At L = 0 the radiation, which according to (1.7) has a character of an instantaneous pulse emitted at the moment the charge traverses the interface, reaches the observation point.

Actually the measuring device that detects the transition radiation flash has a restricted frequency band $\omega_o - \Omega \le \omega \le \omega_o + \Omega$. In this case the field can be written as

$$\bar{E}^o(R,\theta;t) = U(\theta;R)\bar{e}_\theta \int\limits_{\frac{\omega_o-\Omega}{v}}^{\frac{\omega_o+\Omega}{v}} e^{i\frac{\omega}{v}L} d\frac{\omega}{v} = \bar{e}_\theta U(\theta;R)\frac{\sin\frac{\omega}{v}L}{L} e^{i\frac{\omega_o}{v}L} \tag{1.10}$$

In the case of oblique entrance of the charge the transition radiation field has a more complicated form /3-5/ - along with the component $E_\theta$ usually denoted by $E^{''}$ because it lies in the plane of the "bunch geometrical reflection" from the boundary, there also arises a component $E^1$ perpendicular to this plane.

When studying the properties of radiation in the oblique incidence case a spherical coordinate system $\theta^{''}$, $\varphi^{''}$ is used, the axis of which coincides with the direction of "geometro-optical" reflection (Fig. 2). It is shown /3,5/ that at low values of polar angle $(\theta^{''} << 1)$ and in the ultrarelativistic case($\gamma >> 1 - \gamma$ relativistic factor) the transition radiation field of the point-like charge in the far zone can be presented in the form:

$$E^{o''}(R,\theta^{''},\varphi^{''};t) = U'(\theta^{''},\varphi^{''};R)x(L),\ E^{o1}(R,\theta^{''},\varphi^{''};t) = U^1(\theta^{''},\varphi^{''};R)x \tag{1.11}$$

where

$$U'(\theta^{''},\varphi^{''};R) = U(\theta^{''};R)\cos\varphi^{''}, U^1(\theta^{''},\varphi^{''};R) = U(\theta^{''};R)\sin\gamma^{''} \tag{1.12}$$

Note that $U(\theta^{''};R)$ is defined according to (1.8),

$$x(L) = \delta(L) \tag{1.13}$$

at detection of the total emitted spectrum, and

$$x(L)=\frac{\sin\frac{\Omega}{v}L}{L}e^{i\frac{\omega_o}{v}L} \qquad (1.14)$$

at detection of radiation in the frequency band $\omega_o-\Omega\le\omega\le\omega_o+\Omega$.

## 2. THE BUNCH TRANSITION RADIATION FIELD FOR THE NORMAL ENTRANCE CASE

Now we assume that a bunch of charged particles strikes normally the boundary surface. We believe that velocities of all the particles are identical and equal to $V$. For definiteness, assume that the bunch is shaped as a cylinder with arbitrary cross-section. The cylinder generant is parallel to the Z axis (Fig. 1). Let $\alpha$ denote the cylinder length, and $a$ denote the maximum distance from the Z axis to the cylindrical surface generant. Following Ref. /3/ we will assume that the longitudinal Z and transverse $(z,\tilde{\varphi})$ coordinates of particles in the bunch are independent. In this case the bunch particle distribution function $f(r,z,\tilde{\varphi})$ can be presented as a product of the longitudinal $Z(z/d)$ and transverse $R(r,\tilde{\varphi})$ distribution functions

$$f(r,z,\tilde{\varphi})=Z(z/d)R(r,\tilde{\varphi}) \qquad (2.1)$$

which are normalized as follows

$$\int\limits_{-\infty}^{\infty}Z(z/d)dz=1,\ \int\limits_{o}^{\infty}\int\limits_{o}^{2\pi}R(r,\tilde{\varphi})rdrd\tilde{\varphi}=1 \qquad (2.2)$$

Let at instant $\tau=0$ the boundary is traversed by the bunch middle. Then the radiation from the particle being at a distance $Z$ from the boundary reaches the observation point with a delay $z/v$. Taking into account the fact that the distance from the point of intersection of the boundary by an arbitrary particle in the bunch cross-section to the observation point in the wave zone can be written in the form

$$\tilde{R}=R-\tilde{\mu},\qquad \mu=\tilde{\mu}\beta \qquad \text{where }\ \tilde{\mu}=r\sin\theta\cos(\varphi-\tilde{\varphi}) \qquad (2.3)$$

then the detected radiation field may be presented as

$$\vec{E}(z,r,\tilde{\varphi};R,\varphi,L)=\frac{q}{e}Z(z/d)R(r,\tilde{\varphi})x(L-\mu-z)U(\theta;R)\vec{e}_\theta \qquad (2.4)$$

where $q$ is total charge of the bunch. Function $x(L-\mu-z)$ is defined by relation (1.13) at detection of the total radiation spectrum, and by relation (1.14) at detection in frequency band $\omega_o-\Omega\leq\omega\leq\omega_o+\Omega$.

Having integrated (2.4) with respect to the bunch volume we obtain total radiation field

$$\vec{E}(R,\theta,\varphi;L)=\frac{q}{e}U(\theta,R)\int_{-\infty}^{\infty}\int_{o}^{a}\int_{o}^{2\pi}Z(z/d)R(r,\tilde{\varphi})X(L-\mu-z)rdrd\tilde{\varphi}dz\vec{e}_\theta \qquad (2.5)$$

In case the total radiation spectrum is detected, we obtain from (2.5) with account of (1.13):

$$\vec{E}(R,\theta,\varphi;L)=\frac{q}{e}U(\theta;R)\int_{o}^{a}\int_{o}^{2\pi}Z\left(\frac{L-\mu}{d}\right)R(r,\tilde{\varphi})rdrd\tilde{\varphi}\vec{e}_\theta \qquad (2.6)$$

If radiation is detected in frequency band $\omega_o-\Omega\leq\omega\leq\omega_o+\Omega$ we will have

$$\vec{E}(R,\theta,\varphi;L)=\frac{q}{e}U(\theta;R)\int_{-\infty}^{\infty}\int_{o}^{a}\int_{o}^{2\pi}Z\left(\frac{L-z-\mu}{d}\right)R(r,\tilde{\varphi})\frac{\sin\frac{\Omega}{v}z}{z}e^{i\frac{\omega_o}{v}z}rdrd\tilde{\varphi}dz\vec{e}_\theta \qquad (2.7)$$

As can be seen from (2.6) and (2.7), the radiation field in the general case represents a convolution containing longitudinal and transverse distribution functions. The longitudinal distribution function is feasible to extract from (2.6) and (2.7) if parameter $\mu$ in the argument of longitudinal distribution function is assumed small. It is evident that

$$\frac{|\mu|}{d}\leq\frac{a\beta\sin\theta}{d}\equiv æ \qquad (2.8)$$

i.e., parameter æ can be considered small at smallness of the ratio of transverse dimension to longitudinal one. In the ultra-relativistic case $(\beta\to 1)$ a peak of transition radiation intensity falls at direction $\theta\approx\sqrt{1-\beta^2}$ when angle $\theta$ in (2.8) can be considered small, too. Hence for the ultrarelativistic bunches the longitudinal distribution function also can be defined at commensurability of transverse and longitudinal dimensions.

Expand the longitudinal distribution function in integrals (2.7) and (2.8) into a series in parameter $\mu / d$ and preserve two first terms of the expansion. In the first case with respect to (2.2) we obtain

$$\bar{E}(R,\theta,\varphi;L) = \frac{q}{e} U(\theta;R) Z(L/d) \left\{ 1 - \frac{Z'(L/d)}{Z(L/d)} \right\} \beta \sin\theta \cdot \frac{G(\varphi)}{d} \bar{e}_\theta \tag{2.9}$$

where

$$G(\varphi) = \int_o^a \int_o^{2\pi} r^2 \cos(\varphi - \tilde{\varphi}) R(r,\tilde{\varphi}) r dr d\tilde{\varphi} \tag{2.10}$$

evidently

$$|G(\varphi)| \le a \int_o^a \int_o^{2\pi} R(r,\tilde{\varphi}) r dr d\tilde{\varphi} \tag{2.11}$$

from which for (2.9) we have

$$\bar{E}(R,\theta,\varphi;L) = \frac{q}{e} U(\theta;R) Z(L/d) \left\{ 1 - \frac{Z'(L/d)}{Z(L/d)} æ \right\} \bar{e}_\theta \tag{2.12}$$

where

$$æ = \frac{a\beta \sin\theta}{d} \approx \frac{a\beta}{\gamma d} << 1$$

In the second case we obtain

$$\bar{E}(R,\theta;\varphi;L) = \frac{q}{e} U(\theta;R) \tilde{Z}(L/d) \left\{ 1 - \frac{\tilde{Z}'(L/d)}{Z(L/d)} æ \right\} \bar{e}_\theta \tag{2.13}$$

and

$$\tilde{Z}(L/d) = \int_{-\infty}^{\infty} Z\left( \frac{L-Z}{d} \right) \frac{\sin \Omega Z}{Z} e^{i \frac{\omega_o}{v} Z} dZ \tag{2.14}$$

Thus, in the case of detection of the total radiation spectrum the transition radiation field reproduces longitudinal distribution function $Z(L/d)$ (at smallness of parameter æ). In the case of a signal detection in the band $\omega_o - \Omega \le \omega \le \omega_o + \Omega$ the function $\tilde{Z}(L/d)$ is reproduced.

Now we show that at large values of $\omega_o / v$ and $\frac{\Omega}{\omega_o} = const \ll 1$ function $\tilde{Z}(L/d)$ is identical to longitudinal distribution function $Z(L/d)$.

We present function $\tilde{Z}(L/d)$ in the form

$$\tilde{Z}(L/d) = \int_{-\infty}^{\infty} Z\left(\frac{L-Z}{d}\right) \frac{\sin\left\{\frac{\omega_o}{v}\left(1+\frac{\Omega}{\omega_o}\right)Z\right\} - \sin\left\{\frac{\omega_o}{v}\left(1-\frac{\Omega}{\omega_o}\right)Z\right\}}{Z} dZ$$
$$-i\int_{-\infty}^{\infty} Z\left(\frac{L/Z}{d}\right)\sin\frac{\Omega}{v}Z \cdot \frac{\sin\frac{\omega_o}{v}Z}{Z} dZ \tag{2.15}$$

With respect to asymptotic equality

$$\delta(x) = \frac{\sin \alpha x}{\tau / x} \qquad \text{at } \alpha \to \infty$$

we obtain for $\frac{\omega_o}{v} \gg 1$ and $\frac{\Omega}{\omega_o} \ll 1$:

$$\tilde{Z}(L/d) = \tau Z(L/d) \cdot \frac{2\Omega / \omega_o}{1-(\Omega/\omega_o)^2}. \tag{2.16}$$

Equality (2.16) corroborates the above statement.

Hence in the case with infinitely wide band detection as well as with detection of a signal in limited frequency band the transition radiation field reproduces the bunch charge longitudinal distribution function with an error not exceeding the quantity

$$\Delta(L) = \frac{Z'(L/d)}{Z(L/d)} æ \tag{2.17}$$

which along with parameter æ also depends on the degree of "smoothness" of the distribution function.

Now we'll illustrate what is stated above on a few examples. For simplicity, we will restrict ourselves to an axially symmetric bunch for which there is no dependence on $\tilde{\varphi}$ in transverse distribution function.

**Example 1.** The longitudinal and transverse distribution functions are uniform. Then the longitudinal distribution function has the form

$$Z\left(\frac{z}{d}\right)=\begin{cases}1/d & at\ |z|\leq d/2\\ 0 & at\ |z|\leq d/2\end{cases} \tag{2.18}$$

and the transverse one can be written as

$$R(r)=\begin{cases}\frac{1}{\pi a^2} & at\ r\leq a\\ 0 & at\ r>a\end{cases} \tag{2.19}$$

In the considered (axially symmetric) case parameter $a$ is a radius of the cylinder.

As can be seen from (2.18), function $Z(z/d)$ is constant on the section $|z|<d/2$; hence (see (2.17)) the distortions at its recovery are to arise in the edge region of the bunch $Z\sim d/2$. Indeed, when substituting (2.18) and (2.19) into (2.6), for the radiation field component $E_\theta$ we obtain

$$E_\theta(R,\theta;L)=\frac{q}{e}u(\theta;R)\cdot\frac{1}{d}F(L), \tag{2.20}$$

$$F(L)=\begin{cases}1 \quad at\ 0\leq L\leq d/2-æd\\ \frac{1}{\tau}\left\{\frac{\tau}{2}-\arcsin\left(\frac{L-d/2}{æd}\right)-\frac{L-d/2}{æd}\sqrt{1-\left(\frac{L-d/2}{æd}\right)^2}\right\} \quad at\ d/2-æd<L<d/2+æd\\ 0 \quad at\ L>d/2+æd\end{cases} \tag{2.20}$$

From (2.20) it follows (see Fig. 3) that at smallness of æ the detected signal reflects well the genuine longitudinal function of particle distribution in the bunch.

**Example 2.** The case of uniformly longitudinal and Gaussian radial distributions

Let the longitudinal distribution is expressed again by (2.18), and the radial one has the form:

$$R(r)=\frac{1}{\tau a^2}e^{-r^2/a^2} \qquad at\ \ 0\le r\le\infty \tag{2.21}$$

In this case the integral (2.6) reduces to the form

$$E_\theta(R,\theta;L)=\frac{q}{e}u(\theta;R)\cdot\frac{\sqrt{\pi}}{2}F(L),$$
$$F(L)=\frac{1}{2}erf\frac{L+d/2}{æd}-\frac{1}{2}erf\frac{L-d/2}{æd} \tag{2.22}$$

where |6|

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_o^x e^{-t^2}dt \tag{2.23}$$

and at $x\to\infty$ the asymptotic equality takes place |6|

$$erf(x)=1-\frac{e^{-x^2}}{2x} \tag{2.24}$$

Let us examine expression (2.22). Assuming parameter $1/2æ$ sufficiently large, using the asymptote (2.24) we obtain for at $0\le L<<d/2$

$$F(L=1) \tag{2.25}$$

Now we put $L\sim d/2-2æd$. In this case the first term in (2.22) again satisfies the conditions of asymptotic (2.24), while the argument of the second term is -2 at which the value of function $-erf(x)$ practically is 1|6|. At $L=d/2$ the second term in (2.22) vanishes, and the first one satisfies the asymptotic condition (2.24): at this point the function $F(L)$ is half the maximum value of $F\left(\frac{d}{2}\right)=1/2$. At $L=d/2+æd$ the first term in (2.22) is equal to $1/2$, while the second-term argument equals 2, i.e., the function $F(L)$ at this point practically vanishes. At $L>d/2+2æd$ the function of interest takes on positive values smaller than at the point $L=d/2+2æd$, i.e., vanishes again. So long as function $F(L)$ is even, the same regularities take place at $L<0$ too. Thus the behavior of function $F(L)$ differs from the initial distribution only in the intervals

$$\pm d/2 - 2æd < L < \pm d/2 + 2æd$$

which at $æ \ll 1$ means practically undistorted restoration of the initial distribution function (Fig. 4).

**Example 3**. The longitudinal and transverse distributions are Gaussian. The longitudinal distribution is described by the expression

$$Z(L/d) = e^{-4z^2/d^2} \qquad at\ -\infty < z < \infty \tag{2.26}$$

and the transverse one - by formula (2.21).

The substitution of (2.21) and (2.26) results in the following expression for the field

$$E_\theta(R,\theta;L) = \frac{q}{e} U(\theta;R) F(L) \tag{2.27}$$

where

$$F(L) = e^{-\frac{4L^2}{d^2}\left(1-\frac{4}{d^2\mu}\right)},\ \mu = \frac{4}{d^2} + \frac{1}{(æd)^2}. \tag{2.28}$$

Evidently, the resulting distribution also is Gaussian with the effective longitudinal dimension $D = d/\sqrt{1-4/d^2\mu}$. At $æ \ll 1$ parameter $\mu$ equals $1/(æd)^2$ which means that $D = d$.

Thus we have made sure on this example too that the detected signal reproduces the shape of longitudinal bunch particle distribution practically without distortion.

Summarizing what is stated above we can assert that the detection of transition radiation field or its intensity enables one to restore practically without distortion the shape of longitudinal bunch particle distribution by a detector with finite frequency band if our assumed conditions $æ \ll 1$, $\frac{\omega_o}{v} \gg 1$ and $\Omega/\omega_o \ll 1$ hold.

Now we refer again to the relation (2.7) that describes transition radiation field in frequency band $\omega_o - \Omega \le \omega \le \omega_o + \Omega$ and perform in it the change of variables $\xi = L - z - \mu$:

$$\bar{E}(R,\theta,\varphi;L)=\frac{q}{e}U(\theta;R)\int_{-\infty}^{\infty}\int_{0}^{a}\int_{0}^{2\pi}Z\left(\frac{\xi}{d}\right)R(r,\tilde{\varphi})\frac{\sin\frac{\Omega}{v}(L-\mu-\xi)}{L-\mu-\xi}e^{i\frac{\omega_o}{v}(L-\mu-\xi)}rdrd\tilde{\varphi}\xi\bar{e}_\theta \qquad (2.29)$$

Written in such a form expression (2.29) contains a part fast-oscillating over angular coordinates $\theta$ and $\varphi$ that enter into parameter $\mu$ (see (2.3)) and are stipulated by finite transverse dimensions of the bunch.

Now we expand function $\frac{\sin\frac{\Omega}{v}(L-\mu-\xi)}{L-\mu-\xi}$ in a Taylor series of parameter $\mu$ preserving the first two terms of the expansion. Then expression (2.29) will take the form

$$E_\theta(R,\theta,\varphi;L)=\frac{q}{e}U(\theta;R)\left\{\int_{-\infty}^{\infty}\int_{0}^{a}\int_{0}^{2\pi}Z\left(\frac{\xi}{d}\right)R(r,\tilde{\varphi})\frac{\sin\frac{\Omega}{v}(L-\xi)}{L-\xi}e^{i\frac{\omega_o}{v}(L-\mu-\xi)}rdrd\tilde{\varphi}d\xi\right.$$

$$\left.+\int_{-\infty}^{\infty}\int_{0}^{a}\int_{0}^{2\pi}Z\left(\frac{\xi}{d}\right)R(r,\tilde{\varphi})\frac{d}{d\xi}\left(\frac{\sin\frac{\Omega}{v}(L-\xi)}{L-\xi}\right)e^{i\frac{\omega_o}{v}(L-\mu-\xi)}\mu rdrd\tilde{\varphi}d\xi\right. \qquad (2.30)$$

The first integral in (2.30) can be written as a product of function $\tilde{Z}(L/d)$ (see (2.13)) and the integral containing the transverse distribution function. The second integral contains the function

$$\hat{Z}(L/d)=\int_{-\infty}^{\infty}Z(\xi/d)\frac{d}{d\xi}\left(\frac{\sin\frac{\Omega}{v}(L-\xi)}{L-\xi}\right)e^{-i\frac{\omega_o}{v}\xi}d\xi\cdot e^{i\frac{\omega_o}{v}L} \qquad (2.31)$$

which after integration by parts will take the form

$$\hat{Z}(L/d)=\left\{\frac{1}{d}\int_{-\infty}^{\infty}Z'(\xi/d)\frac{\sin\frac{\Omega}{v}(L-\xi)}{L-\xi}e^{-i\frac{\omega_o}{v}\xi}d\xi\right.$$

$$\left.-i\frac{\omega_o}{v}\int_{-\infty}^{\infty}Z(\xi/d)\frac{\sin\frac{\Omega}{v}(L-\xi)}{L-\xi}e^{-i\frac{\omega_o}{v}\xi}d\xi\right\}e^{i\frac{\omega_o}{v}L} \qquad (2.32)$$

At large values of parameter $\frac{\omega_o}{v}$, as was shown above (see (2.15) and (2.16)), expression (2.32) can be presented as

$$\hat{Z}(L/d) \sim \frac{1}{d} Z^{'}(L/d) - i\frac{\omega_o}{v} Z(L/d) = Z^{o}(L/d) \tag{2.33}$$

hence expression (2.30) under the same condition can be written as follows:

$$E_{\theta}(R,\theta,\varphi;L) = \frac{q}{e} U(\theta;R) \left\{ Z(L/d) G_1(\theta,\varphi) + Z^{o}(L/d) G_2(\theta,\varphi) \right\} \tag{2.34}$$

where

$$\begin{aligned} G_1(\theta,\varphi) &= \int_{o}^{a}\int_{o}^{2\pi} R(r,\tilde{\varphi}) e^{-i\frac{\omega_o}{v}\mu} r dr d\tilde{\varphi} \\ G_2(\theta,\varphi) &= \int_{o}^{a}\int_{o}^{2\pi} \mu R(r,\tilde{\varphi}) e^{-i\frac{\omega_o}{v}\mu} r dr d\tilde{\varphi} \end{aligned} \tag{2.35}$$

It is evident that

$$\left| G_2(\theta,\varphi) \right| \le a\beta \sin\theta \left| G_1(\theta,\varphi) \right| \tag{2.36}$$

With account of (2.36) expression (2.34) can be written in the form

$$E_{\theta}(R,\theta,\varphi;L) = \frac{q}{e} U(\theta;R) Z\left(\frac{L}{d}\right) G_1(\theta,\varphi) \left\{ 1 + \left( \frac{Z^{'}(L/d)}{Z(L/d)} - i\frac{\omega_o}{v} d \right) æ\, \mu(\varphi) \right\} \tag{2.37}$$

where

$$\mu(\varphi) = \frac{G_2(\theta,\varphi)}{G_1(\theta,\varphi)} \le 1$$

It follows from (2.37) that its second term can be neglected if

$$\left\{\left(\frac{Z'(L/d)}{Z(L/d)}\right)^2+\left(\frac{\omega_o d}{v}\right)^2\right\}^{1/2} æ<<1 \quad (2.38)$$

When

$$\frac{\omega_o d}{v}=\frac{2\pi d}{\lambda\beta}>>\frac{Z'(L/d)}{Z(L/d)} \quad (2.39)$$

which is the case if the bunch longitudinal dimensions multiply exceed the wavelength, and function $Z(L/d)$ is a slowly varying one as compared with $e^{-i\frac{\omega_o}{v}\xi}$, the condition (2.38) reduces to

$$\frac{2\pi a\sin\theta}{\lambda}<<1 \quad (2.40)$$

In other words, if the transverse dimensions of the bunch are of the order of the wavelength at which the radiation is detected, then at small observation angles the second terms in (2.34) and (2.35) can be neglected at spatial photometry of the transition radiation flash intensity spot: if the condition (2.40) is satisfied, then the diffraction picture will be observed by which one can obtain information about transverse dimensions of the bunch as well as about the character of the particle transverse distribution function.

It is evident that at $d>>\lambda$ the condition (2.40) is more rigorous than the condition $æ<<1$ from which the relation (2.14) results. Thus the diagnosis of transverse particle distribution is feasible not in all the cases the longitudinal profile can be determined.

## 3. TRANSITION RADIATION AT OBLIQUE ENTRANCE OF PARTICLES

In measurements of transition radiation intensity it is more convenient to use the monitors which are placed at an angle to the charge trajectory (see Fig. 2).

Let us obtain expressions for fields of the bunch at oblique entrance through the vacuum - ideal conductor boundary.

We introduce Cartesian coordinate systems x',y',z', x,y,z and x",y",z". The first of these systems is connected with the bunch trajectory - the z' axis is in the direction of velocity $\vec{V}$ and makes angle $\varphi_o$ with the Z axis. The x' axis is in the plane of z and z'; the y' axis is normal to this plane. z" is in the direction of "geometrooptical" reflection of particles from the boundary surface - it lies in the plane of zz' and makes angle $\varphi_o$ with Z. The x" axis also lies in the plane of zz', and y" is normal to it.

The expression for the field of the transition radiation bunch at oblique entrance can be obtained similarly as in the case with the point-like charge /4/.

A Fourier-transformed image in the considered case will be of the form

$$\bar{E}_{\frac{\omega}{v},K_x,K_y}(z)=\bar{E}^o_{\frac{\omega}{v},K_x,K_y}(z)\cdot\rho_{\frac{\omega}{v},K_x,K_y}(z=0) \tag{3.1}$$

here

$$\bar{E}^o_{\frac{\omega}{v},K_x,K_y}(z)=\bar{E}^o_{\frac{\omega}{v},K_x,K_y}e^{-iz\sqrt{\omega^2/c^2-K_x^2-K_y^2}} \tag{3.2}$$

is a Fourier-transformed image of single charge transition radiation field for the oblique entrance case; $\rho_{\frac{\omega}{v},K_x,K_y}(z)$ is a Fourier image of the bunch charge density distribution function. Writing the bunch charge density distribution in the coordinate system connected with the z' axis in the form of

$$\rho(x',y',z't)=qz(z'-vt)R(\tau,\tilde{\varphi}) \tag{3.3}$$

we'll arrive at the following expression for the Fourier image:

$$\rho_{\frac{\omega}{v},K_x',K_y'}(z)=qI_1\left(\frac{\omega}{v}\right)I_2\left(K_x',K_y'\right)e^{-i\frac{\omega}{v}z'} \tag{3.4}$$

where

$$I_1\left(\frac{\omega}{v}\right)=\int_{-\infty}^{\infty}Z\left(\frac{z}{d}\right)e^{-i\frac{\omega}{v}z}dz,I_2\left(K_z',K_y'\right)=\int_o^a\int_o^{2\pi}R(\tau,\tilde{\varphi})e^{-i\tau\left(K_x'\cos\tilde{\varphi}+K_y'\sin\tilde{\varphi}\right)}_{\tau d\tau d\tilde{\varphi}} \tag{3.5}$$

Then

$$\rho(x',y',z',t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}I_1\left(\frac{\omega}{v}\right)I_2\left(K_z',K_y'\right)e^{i\left(K_x'x+K_y'y-\frac{\omega}{v}vt\right)}_{dK_x'dK_y'd\frac{\omega}{v}} \tag{3.6}$$

Using the relations

$$X' = x\cos\varphi_o - \sin\varphi_o,\ \ y' = y,\ \ z' = x\sin\varphi_o + z\cos\varphi_o \tag{3.7}$$

we turn in (3.6) to the coordinate system $x, y, z$:

$$\rho(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_1\left(\frac{\omega}{v}\right) I_2\left(K_x', K_y'\right) e^{i\left\{x\left(\frac{\omega}{v}\sin\varphi_o + K_x'\cos\varphi_o + z\left(\frac{\omega}{v}\cos\varphi_o - K_x'\sin\varphi_o\right) + K_y'y - \frac{\omega}{v}vt\right)\right.} dK_x' dK_y' d\frac{\omega}{v} \tag{3.8}$$

and write the expression for the Fourier image in coordinates $x, y, vt$:

$$\rho_{\frac{\omega}{v}, K_x, K_y}(z) = I_1\left(\frac{\omega}{v}\right) I_2\left(\frac{K_x - \frac{\omega}{v}\sin\varphi_o}{\cos\varphi_o}, K_y\right) e^{i\left(\frac{\omega}{v}\cos\varphi - \frac{K_x - \frac{\omega}{v}\sin\varphi_o}{\cos\varphi_o}\sin\varphi_o\right)z} \tag{3.9}$$

Then using (3.1) and (3.9) we present the transition radiation field as a Fourier integral and passing with the help of the stationary phase or saddle-point method to the far zone we'll obtain

$$\vec{E}(R,\theta,\varphi,t) = \vec{U}^o(\theta,\varphi)\vec{F}(\theta,\varphi,L) \tag{3.10}$$

where $\vec{U}\cdot(\theta,\varphi)$ is the amplitude of the transition radiation field for the point-like particle, written in the coordinate system connected with the normal to the boundary. We do not present here the amplitude's explicit form /3-5/ because of its awkwardness, while the function - the bunch form-factor is defined as follows:

$$\tilde{F}(\theta,\varphi,L) = \int_{-\infty}^{\infty} I_1\left(\frac{\omega}{v}\right) I_2\left(K_x^o, K_y^o\right) e^{i\frac{\omega}{v}L} d\frac{\omega}{v} \tag{3.11}$$

where

$$K_x^o = \frac{\frac{\omega}{c}\sin\theta\cos\varphi - \frac{\omega}{v}\sin\varphi_o}{\cos\varphi_o},\ K_y^o = \frac{\omega}{c}\sin\theta\sin\varphi \tag{3.12}$$

When turning to the spherical coordinate system related to the direction of "geometrooptical" reflection, we use the relations that establish connection between $\theta,\varphi$ and $\theta^{"},\varphi^{"}$:

$$\begin{aligned} &\sin\theta\sin\varphi = \sin\theta^{"}\sin\varphi^{"} \\ &\sin\theta\cos\varphi = \cos\theta^{"}\sin\varphi_o + \sin\theta^{"}\cos\varphi_o\cos\varphi^{"} \end{aligned} \tag{3.13}$$

From the first relation (3.13) it follows that quantity $K_y^o$ when turning to the $\theta^{"},\varphi^{"}$ system remains unchanged:

$$K_y^o = \frac{\omega}{c}\sin\theta^{"}\sin\varphi^{"} \tag{3.14}$$

and quantity $K_x^o$ takes the form

$$K_x^o = \frac{1}{\cos\varphi_o}\left\{\frac{\omega}{c}\left(\cos\theta^{"}\sin\varphi_o + \sin\theta^{"}\cos\varphi_o\cos\varphi^{"}\right) - \frac{\omega}{v}\sin\varphi_o\right\} \tag{3.15}$$

Assuming for $\theta^{"} << 1$ in (3.15) $\cos\theta^{"} \approx 1$, $\sin\theta^{"} \approx \theta^{"}$ we have

$$K_x^o = \frac{1}{\cos\varphi_o}\left\{\left(\frac{\omega}{c} - \frac{\omega}{v}\right)\sin\varphi_o + \frac{\omega}{c}\theta^{"}\cos\varphi_o\cos\varphi^{"}\right\} \tag{3.16}$$

In the ultrarelativistic case (3.16) has the form

$$K_x^o = \frac{\omega}{c}\theta^{"}\cos\varphi^{"} \tag{3.17}$$

which coincides with the corresponding expression for normal entrance at replacements $\theta \to \theta^{"}$ and $\varphi \to \varphi^{"}$.

Thus the identity of form-factors for the both cases of normal and oblique entrances is found for the directions close to the direction of "geometrooptical" reflection. For these directions the amplitude components of field $\vec{U}^o(\theta,\varphi)$ are expressed in terms of (1.11) and (1.12), and the intensity in the same direction has the form identical to the normal entrance case.

APPENDIX

## 4. DETERMINATION OF LONGITUDINAL PARTICLE DISTRIBUTION FUNCTION BY A NARROW-BAND DETECTOR AT ARBITRARY FREQUENCY

1. Above we have shown that it is possible to determine the longitudinal bunch charge distribution function via the measurement of the time profile of transition radiation flash using a narrow-band detector only in the case when the condition $\omega_o >> 1$ is satisfied.

Now we'll investigate the possibility to determine this function at arbitrary $\omega_o \Omega$.

Recall that if parameter æ is sufficiently small, then the radiation field detected in the frequency band reproduces the function (2.13) which can be presented in the form:

$$\tilde{Z}(L/d) = \int_{\frac{\omega_o-\Omega}{v}}^{\frac{\omega_o+\Omega}{v}} I_1\left(\frac{\omega}{v}\right) e^{i\frac{\omega}{v}L} d\frac{\omega}{v} \sim E_\theta(R,\theta,L) \tag{4.1}$$

where $I_1\left(\frac{\omega}{v}\right)$ is a Fourier-transformed image of longitudinal distribution function. The Fourier image of the detected field reproduces the Fourier image of distribution function in the frequency band

$$A\left(\frac{\omega}{v}\right) = \int_{-\infty}^{\infty} E_\theta(R,\theta,L) e^{-i\frac{\omega}{v}L} dL = \begin{cases} I_1\left(\frac{\omega}{v}\right) & at \\ 0 & in\ other\ cases \end{cases} \tag{4.2}$$

So long as $A\left(\frac{\omega}{v}\right)$ like any function characterizing the object's image is analytical (see, e.g. /7/), it can be analytically continued outside the frequency band wherein measurements are performed:

$$\varphi\left(\frac{\omega}{v}\right) = \begin{cases} A\left(\frac{\omega}{v}\right) & at\ \omega_o - \Omega \le \omega \le \omega_o + \Omega \\ \tilde{A}\left(\frac{\omega}{v}\right) & at\ \omega < \omega_o - \Omega,\ \omega > \omega_o + \Omega \end{cases} \tag{4.3}$$

where $\tilde{A}\left(\frac{\omega}{v}\right)$ is analytical continuation of function $A\left(\frac{\omega}{v}\right)$ beyond the limits of determination (the detector band limits).

Thus, assuming

$$\varphi\left(\frac{\omega}{v}\right)=I_1\left(\frac{\omega}{v}\right) \quad at\ -\infty<\omega<\infty \tag{4.4}$$

we obtain the distribution function

$$Z(L/d)=\int_{-\infty}^{\infty}\varphi\left(\frac{\omega}{v}\right)e^{i\frac{\omega}{v}L}d\frac{\omega}{v}. \tag{4.5}$$

2. Now we consider the limiting case of absolutely incoherent radiation induced when the bunch crosses the boundary. In this case summation of intensities emitted by each particle of the bunch takes place. According to (1.10), the intensity emitted by the charge at the given instant of time in the given point of space is expressed as follows:

$$T_o(L,\theta)=|U(\theta,R)|^2\frac{\sin^2\frac{\omega}{v}L}{L^2} \tag{4.6}$$

Arguing in the same way as in Section 2, we can write the intensity emitted by the bunch in the form

$$T_o(L,\theta,\varphi)=|U(\theta,R)|^2\int_{-\infty}^{\infty}\int_{o}^{a}\int_{o}^{2\pi}Z\left(\frac{z}{d}\right)R(r,\tilde{\varphi})\frac{\sin^2\frac{\omega}{v}(L-\mu-Z)}{(L-\mu-Z)^2}rdrd\tilde{\varphi}dZ \tag{4.7}$$

where parameter $\mu$ is determined by (2.3). It can readily be shown that at $æ<<1$ for this case the estimate (2.14) is valid, where now

$$\tilde{Z}(L/d)=\int_{-\infty}^{\infty}Z(L/d)\frac{\sin^2\frac{\Omega}{v}(L-Z)}{(L-Z)^2}dZ \tag{4.8}$$

This function, not containing frequency $\omega_o$, is reproduced by the measured intensity:

$$T(L,\theta,\varphi) \sim \tilde{Z}(L/d) \tag{4.9}$$

The Fourier image of the measured intensity will be expressed via the Fourier image of longitudinal distribution function as follows:

$$B\left(\frac{\omega}{v}\right) = I_1\left(\frac{\omega}{v}\right) \wedge \left(\frac{\omega}{v}\right) \tag{4.10}$$

where

$$B\left(\frac{\omega}{v}\right) = \int_{-\infty}^{\infty} T(L,\theta,\varphi) e^{-i\frac{\omega}{v}L} dL, \quad I_1\left(\frac{\omega}{v}\right) = \int_{-\infty}^{\infty} Z(L/d) e^{-i\frac{\omega}{v}L} dL \tag{4.11}$$

$$\wedge\left(\frac{\omega}{v}\right) = \begin{cases} 0 & at \quad \left|\frac{\omega}{v}\right| > 2\Omega / v \\ \frac{2\Omega}{v} + \frac{\omega}{v} & at \quad -\frac{2\Omega}{v} \leq \frac{\omega}{v} \leq 0 \\ \frac{2\Omega}{v} - \frac{\omega}{v} & at \quad 0 \leq \frac{\omega}{v} \leq \frac{2\Omega}{v} \end{cases} \tag{4.12}$$

Thus in the frequency band $-2\Omega < \omega < 2\Omega$ the Fourier image of distribution function is to be determined by the relation

$$I_1\left(\frac{\omega}{v}\right) = B\left(\frac{\omega}{v}\right) / \wedge\left(\frac{\omega}{v}\right) \tag{4.13}$$

As far as function $B\left(\frac{\omega}{v}\right) / \wedge\left(\frac{\omega}{v}\right)$ is analytical in the inner region of some circle of radius $\Omega_o \leq 2\Omega$ on the complex plane, it can be continued outside this circle:

$$\varphi\left(\frac{\omega}{v}\right) = \begin{cases} B\left(\frac{\omega}{v}\right) / \wedge\left(\frac{\omega}{v}\right) & at \quad |\omega| \leq \Omega_o \\ \varphi\left(\frac{\omega}{v}\right) & at \quad |\omega| \geq \Omega_o \end{cases} \tag{4.14}$$

and the distribution function $Z(L/d)$ can be expressed through the function $\varphi\left(\frac{\omega}{v}\right)$ with the use of Fourier transformation:

$$Z(L/d)=\int_{-\infty}^{\infty}\varphi\left(\frac{\omega}{v}\right)e^{i\frac{\omega}{v}L}d\frac{\omega}{v} \tag{4.15}$$

As can be seen from (4.13)-(4.15), in the case of absolutely incoherent radiation the longitudinal distribution function is expressed through the transition radiation flash intensity; however this relationship is nonlinear as distinct from the coherent radiation whose field is linearly related to the bunch charge longitudinal distribution function.

SUMMARY

In the present work we have obtained relations on the basis of which one can realize the method of determination of the bunch charge longitudinal distribution using the shape of the transition radiation time profile the relations also allow one to estimate the occurring errors and minimize them.

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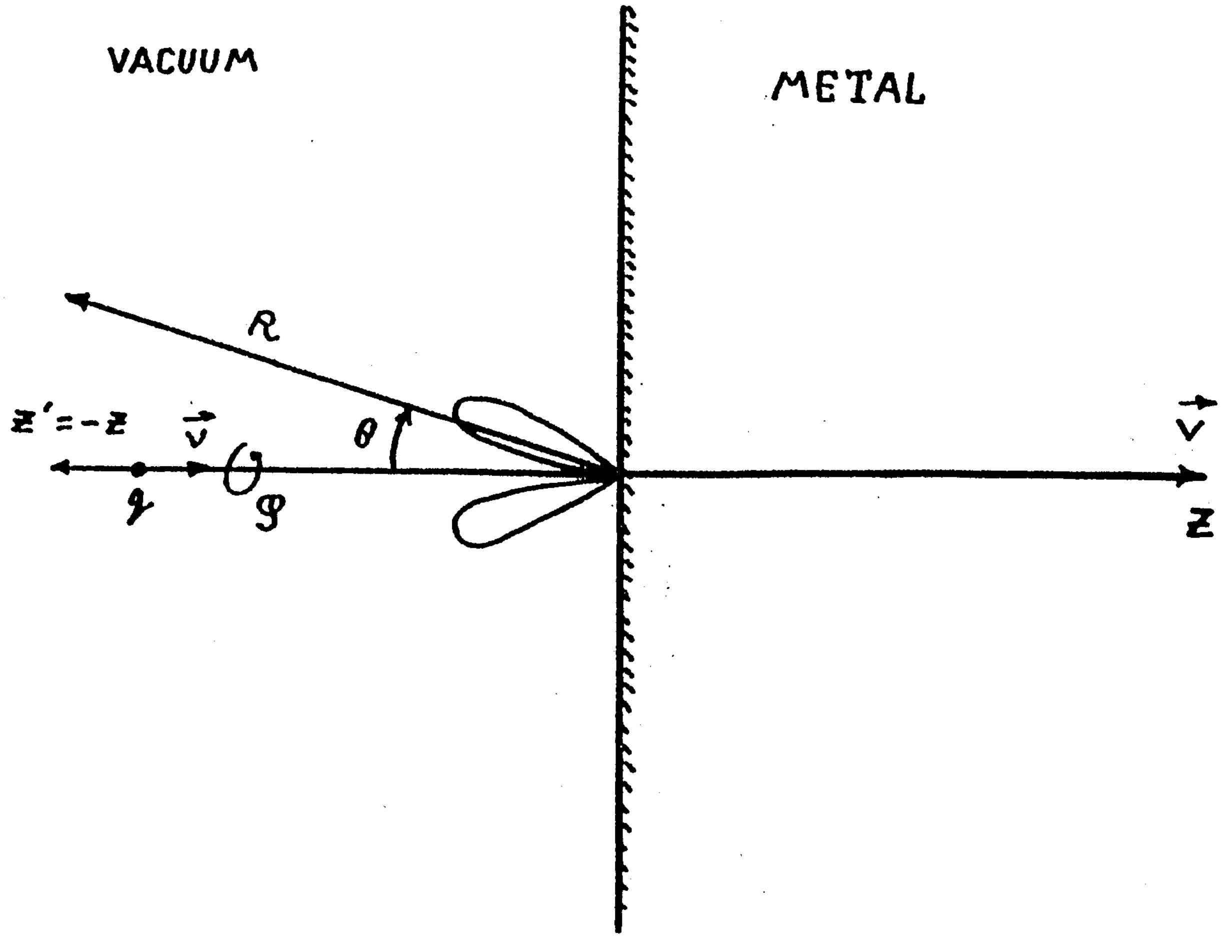


Fig.1. The normal entrance

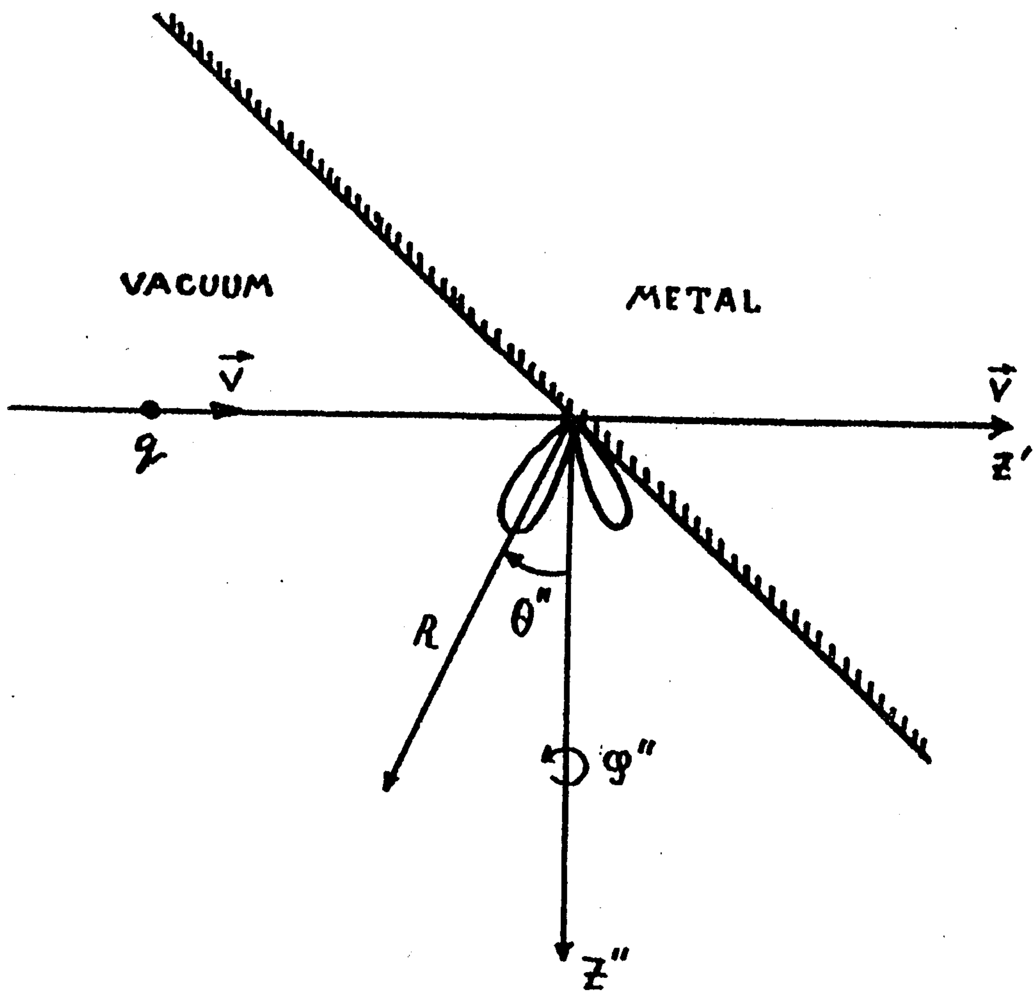


Fig.2. The oblique entrance

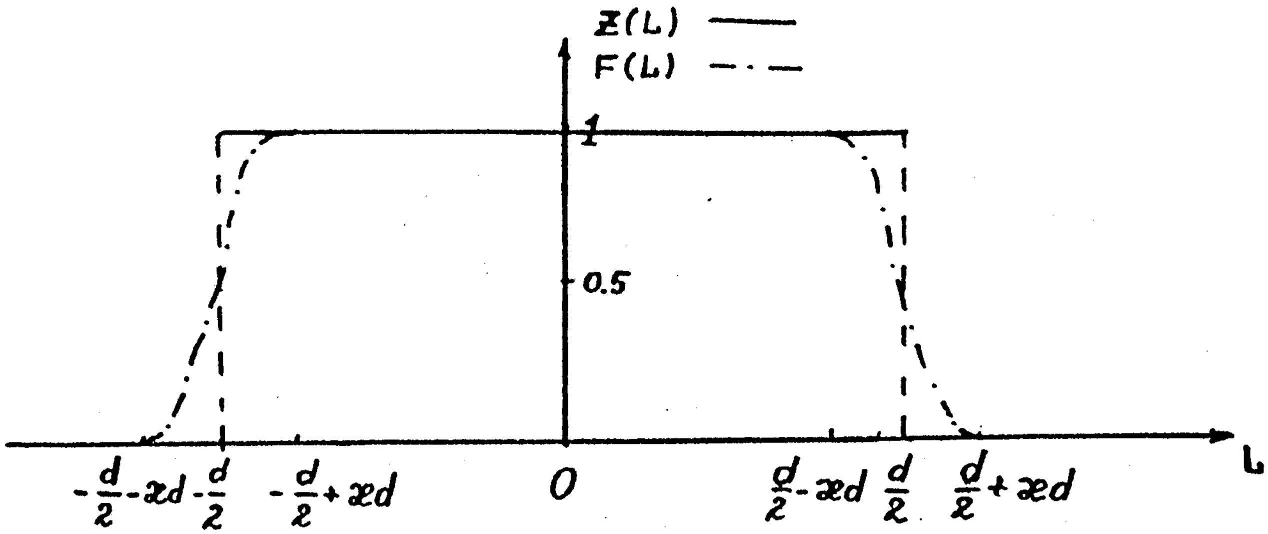


Fig.3. The time profile of light flash for the longitudinal charge distribution (Example 1)

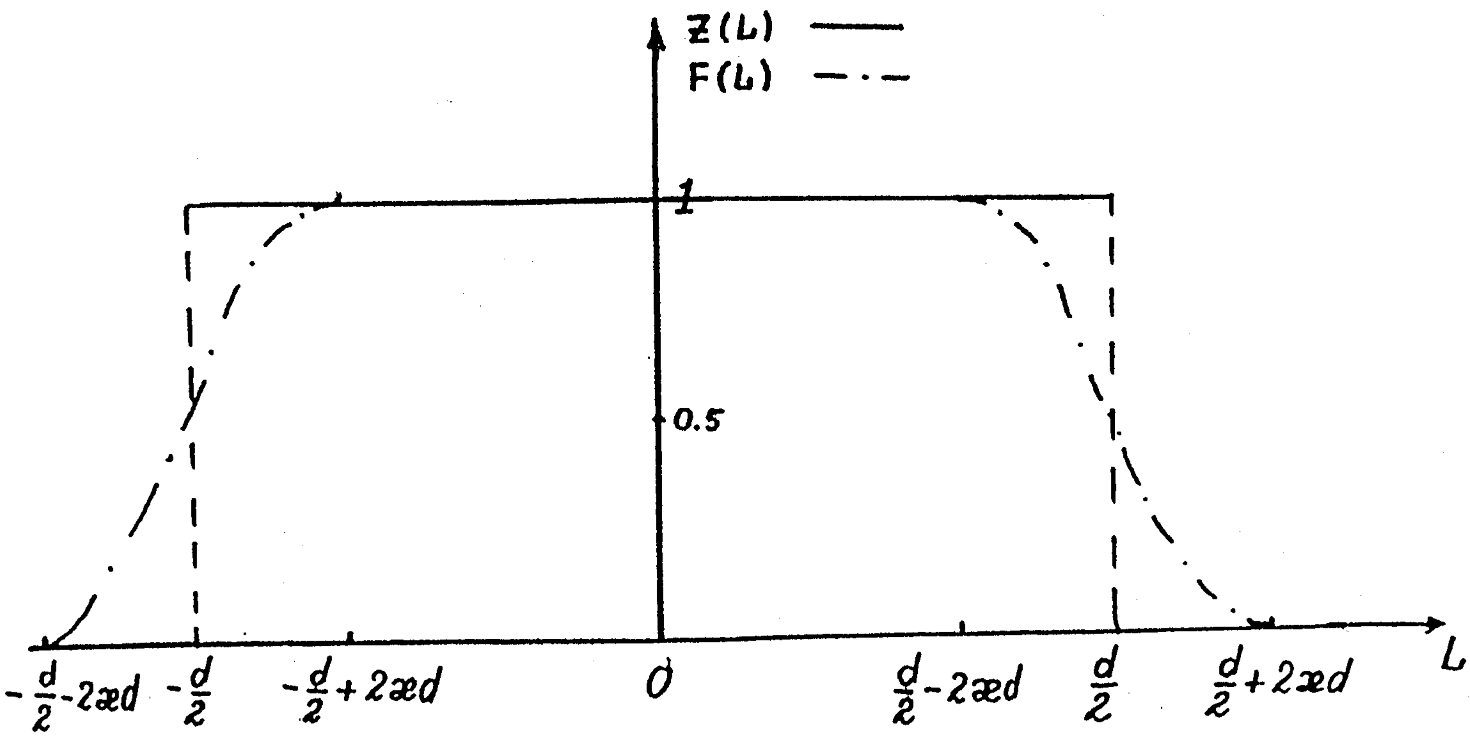


Fig.4. The time profile of light flash for the longitudinal charge distribution (Example 2)