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Extremum Seeking for Stabilization

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Engineering Sciences (Mechanical Engineering)

by

Alexander Scheinker

Committee in charge:

Professor Miroslav Krstić, Chair
Professor Jorge Cortés
Professor Bruce Driver
Professor Melvin Leok
Professor Maurício de Olíviera

2012

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University of California, San Diego

2012

DEDICATION

To Zina and Pima.

EPIGRAPH

There are moments in our lives, there are moments in a day, when we seem to see beyond the usual. Such are the moments of our greatest happiness. Such are the moments of our greatest wisdom. If one could but recall his vision by some sort of sign. It was in this hope that the arts were invented. Sign-posts on the way to what may be. Sign-posts toward greater knowledge. — Robert Henri

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A. Scheinker, M. Bland, M. Krstić, and J. Audia, “Rise-time optimization of accelerator high voltage converter modulator by extremum seeking," under review. (Chapter 4)

A. Scheinker and M. Krstić, “Extremum seeking-based tracking for unknown systems with unknowns control directions," *Proc. CDC*, Maui, HI, 2012. (Chapter 5)

The dissertation author was the primary author in these publications.

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A. Scheinker and M. Krstić, “Extremum seeking-based tracking for unknown systems with unknowns control directions," *Proc. CDC*, Maui, HI, 2012.

A. Scheinker and M. Krstić, “Maximum-seeking for CLFs: Universal semiglobally stabilizing feedback under unknown control directions," *IEEE Transactions on Automatic Control*, to appear.

A. Scheinker, M. Bland, M. Krstić, and J. Audia, “Rise-time optimization of accelerator high voltage converter modulator by extremum seeking," under review.

ABSTRACT OF THE DISSERTATION

Extremum Seeking for Stabilization

by

Alexander Scheinker

Doctor of Philosophy in Engineering Sciences (Mechanical Engineering)

University of California, San Diego, 2012

Professor Miroslav Krstić, Chair

This dissertation develops a new Extremum Seeking (ES) method, which has a long history as an optimization scheme, for stabilization of unknown possibly open-loop unstable dynamic systems. The main idea is in utilizing the ES method to minimize a known Lyapunov function candidate, such as the square sum of an unknown system's components with minimization becoming equivalent to stabilization. Several illustrative simulations, followed by a real world implementation of the technique are presented. The method is then extended to trajectory tracking for unknown and unstable systems. Finally non-smooth ES controllers are developed which have the advantage that their control efforts approach zero when the system's trajectory does, unlike the persistent oscillation present in typical ES schemes.

Chapter 1

Introduction

1.1 Background on Extremum Seeking

Perturbation driven extremum seeking, as a tool for finding and maintaining the extremum value of an unknown function, which may be the output of a known dynamic system, was first proposed in the 1922 paper of Leblanc [46]. In his paper, Leblanc reported on a new method for maintaining maximum power transfer from a transmission line to a tram car. Leblanc's paper was a description of an ingenious engineering design, but there was no mathematical analysis of the scheme's dynamics. Eventually, Leblanc's simple, powerful method became a popular tool for maximizing or minimizing unknown output functions for stable, known dynamic systems.

The intuitive explanation of why Leblanc's method works is very simple. Starting with some initial condition, $x(0)$ the system's input is perturbed in the form

$$x(t) = x(0) + \cos(\omega t),$$

which results in a perturbation of some unknown output function, $F(x(t)) = F(x(0) + \cos(\omega t))$. If $F(x)$ has an extremum point near $x(0)$ then $F(x(t))$ will be in or out of phase with the perturbing term $\cos(\omega t)$ as shown in Figure 1.1. The phase of the output function, relative to the perturbing term then gives the controller an idea of which way to move the parameter in order to approach the extremum point. By mixing and filtering this process is made into an automated feedback loop which drives a system's output

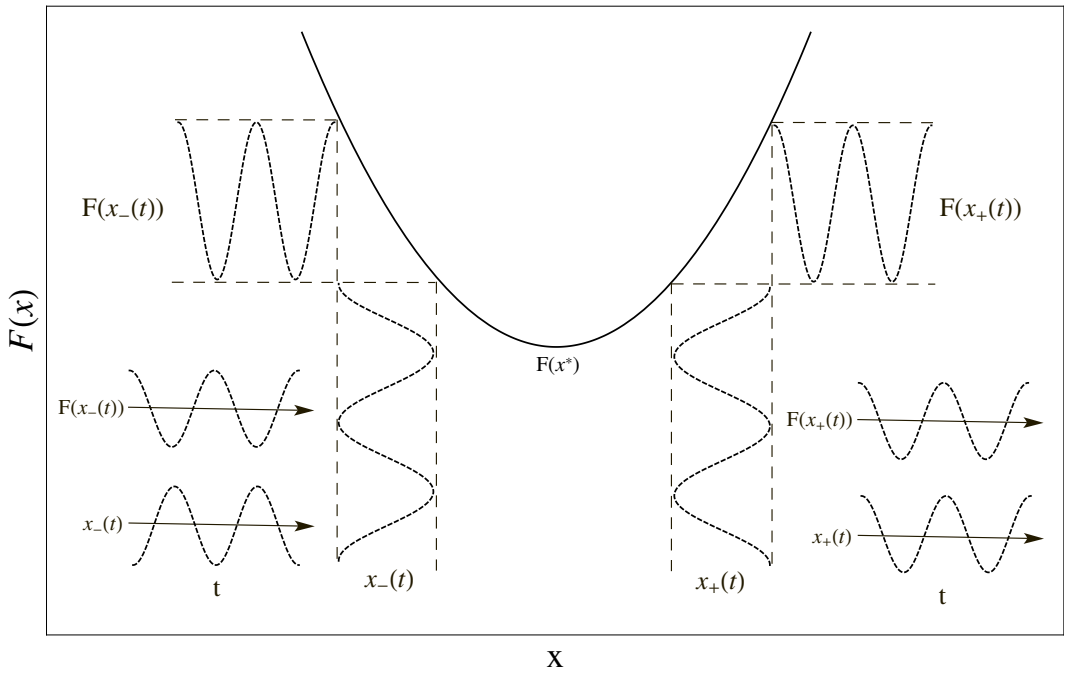


Figure 1.1: In searching for the minimum of $F(x) = x^2$, depending on whether $x(0)$ is greater than or less than $x^\star = 0$ the value of the perturbed function $F(x(t))$ will be in or out of phase with the perturbing signal, allowing for an estimate of the gradient for minimization.

towards its minimum.

This simple and effective optimization scheme gained further attention in Russia starting in the 1940s [35, 36]. In the U.S. the scheme was introduced in the 1950s [20] for optimization of the performance of an internal combustion engine. In the 1950s and 1960s the algorithm was modified and its performance for specific problems was analyzed [54, 55, 62, 63], but rigorous performance and design analysis had not taken place. In 1971 a Lyapunov based stability analysis of a particular extremum seeking system was performed [51].

In 2000 the first general stability analysis of the extremum seeking method was performed by Krstić and Wang [44], for stable dynamic systems, with unknown output functions as shown in Figure 1.2. What follows is an intuitive description of principle of operation of the ES scheme, for detailed analysis, which is an interesting combination of

standard averaging and singular perturbation theory, one should consult the reference.

In the setup shown in Figure 1.2, the dynamic system is given by $\dot{x} = f(x,u)$ for which a locally exponentially stabilizing family of controllers, parametrized by θ, in the form of $u = \alpha(x,\theta)$, is known and there exists a unique equilibrium map $l(\theta)$ such that $\dot{x} = 0$ if and only of $x = l(\theta)$. Furthermore, the unknown output function, $y = h(x)$ is assumed to have a minimum at some value of $\theta = \theta^\star$, such that $h(l(\theta))$ satisfies some usual minimization characteristics at $\theta = \theta^\star$ such as, for example $\frac{dh}{d\theta}\big|_{\theta^\star} = 0$.

In this setup a dithering term $a\sin(\omega t)$ is used to perturb the value of θ, in order to search for the optimal control which will minimize the output function $h(l(\theta))$. The value of the output function, $y = h(x)$ is first passed through the high pass filter, whose Laplace transform is represented as $\frac{s}{s+\omega_h}$ in order to remove any constant terms. The filtered signal is then mixed with the same term, $a\sin(\omega t)$, and again high pass filtered, a process which after some averaging analysis results in a few oscillating terms, one of which is of the form $\sin^2(\omega t)$ and proportional to $-\frac{dh}{d\theta}$, the gradient of the unknown output map. Upon integration, or low pass filtering through $\frac{k}{s}$ all oscillatory terms average to zero, except for the $-k\frac{dh}{d\theta}\sin^2(\omega t)$ term, resulting in θ following a gradient descent towards the minimizing value $\theta^\star$ of the unknown function $y = h(x) = h(l(\theta))$.

The work of Krstić and Wang sparked renewed interest in the field. Since 2000 the extremum seeking method, as a real-time non-model-based optimization approach, has seen significant theoretical advances [3, 13, 67], with extension to semi-global convergence [80], development of scalar Newton-like algorithms [56, 60], inclusion of measurement noise [77], extremum seeking with partial modeling information [1, 2, 18, 23, 28], and learning in noncooperative games [26, 76].

Extremum seeking has also been used in many diverse applications with unknown/uncertain systems, such as steering vehicles toward a source in GPS-denied environments [14, 15, 89], active flow control [6, 7, 8, 33, 39, 40], aeropropulsion [57, 85], cooling systems [48, 49], wind energy [17], photovoltaics [47], human exercise machines [90], optimizing the control of nonisothermal continuously stirred tank reactors [29], reducing the impact velocity of an electromechanical valve actuator [64], controlling Tokamak plasmas [11, 12], and enhancing mixing in magnetohydrodynamic channel flows [50].

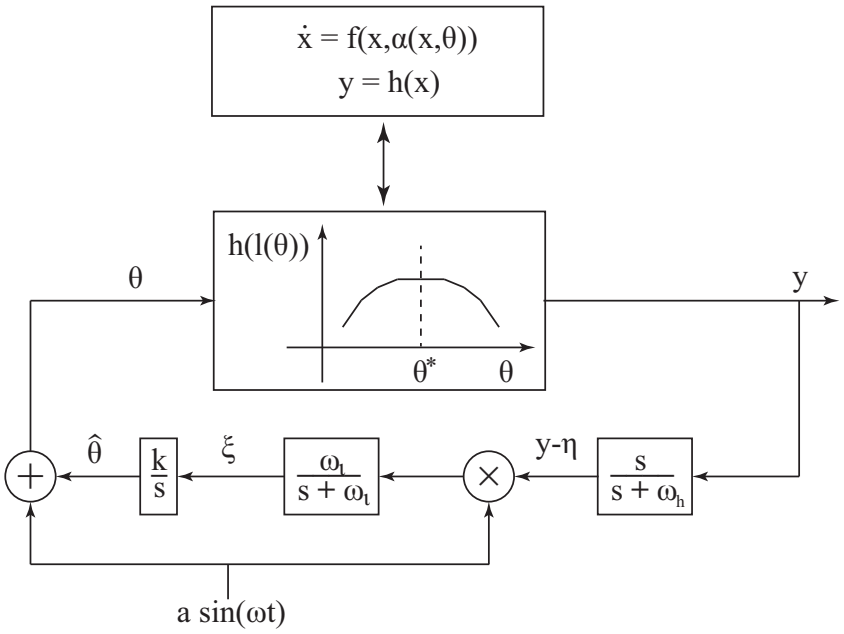


Figure 1.2: Extremum seeking for unknown output function minimization for a stable dynamic system.

As described, extremum seeking is a control framework that has traditionally been developed as a methodology for optimizing steady states for known stable systems. The ES method has seen many alterations, a major simplification and a first effort towards applying ES to unstable plants was presented by Zhang, Siranosian, Krstić [91], but only for simple linear examples and only for problems where instability is an obstacle to achieving optimization, rather than stabilization being the goal. The new scheme, as shown in Figure 1.3, was for velocity actuation of vehicles for minimization of the unknown output function $f(x,y)$. Because the position of a vehicle is the integral of its velocity, $x(t) = x(0) + \int_0^t v(\tau)d\tau$, the ES scheme was simplified by removal of the filter designated in Figure 1.2 as $\frac{k}{s}$, with this low pass filtering now being performed by the vehicles themselves. Furthermore, the addition of the ω as a gain of the dithering signal and the use of $\sin(\omega t)$ and $-\cos(\omega t)$ as mixing signals for dithering signals $\omega\cos(\omega t)$ and $\omega\sin(\omega t)$ respectively (which the mixing signals are the integrals of) resulted in the removal of the second high pass filter and an overall simplified averaging analysis.

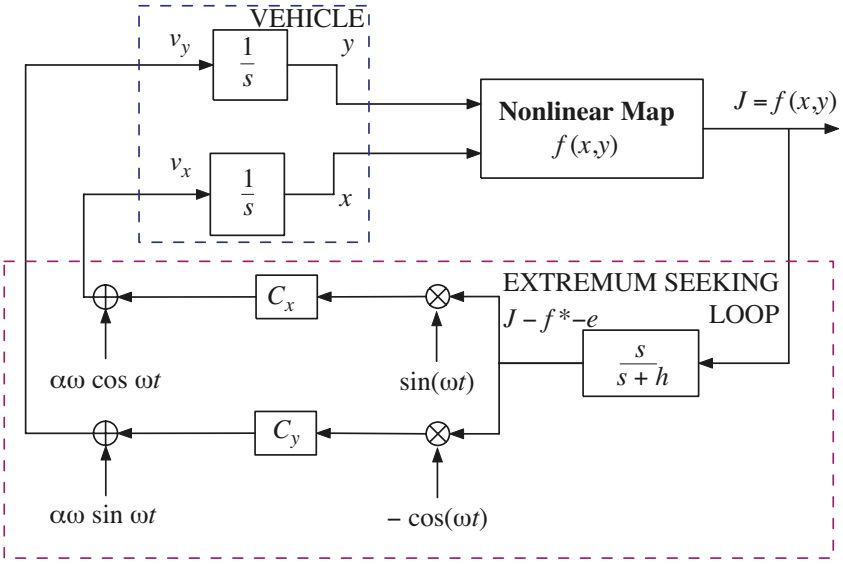


Figure 1.3: Simplified extremum seeking for unknown output function minimization, with integration performed by the system. Figure from [89].

1.2 Overview of Results

The main result of this thesis is the realization and proof that the minimum-seeking ability of the extremum seeking scheme can be utilized for stabilization of unknown open-loop unstable systems by employing the extremum seeker to minimize a system's Lyapunov function candidate. A basic example and the problem that started all of this work is the system shown in Figure 1.4, an ES scheme which is a further simplification of the scheme in Figure 1.3, whose dynamics are

$$\dot{x} = ax + bu \tag{1.1}$$

$$u = \alpha\omega\cos(\omega t) - k\sin(\omega t)x^2, \tag{1.2}$$

in which the perturbing term $\alpha\omega\cos(\omega t)$, after passing through the system's integrator, has the form $\alpha\sin(\omega t)$, whose influence on the cost function $V(x) = x^2(t)$ is then detected by mixing with the same frequency signal $-k\sin(\omega t)$, and then the product is again passed through the system's integrator in the closed loop, resulting in a minimum seeking scheme. In this case, the function being minimized is a Lyapunov function for our system, whose minimization is equivalent to stabilization.

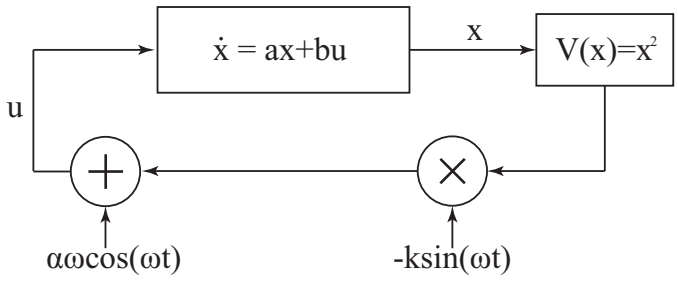


Figure 1.4: By minimizing $V(x) = x^2$, a Lyapunov function for $\dot{x} = ax + bu$, the ES controller stabilizes the system.

The stability of system (1.1), (1.2) may be established by standard averaging. By inspection, the solution of system (1.1), (1.2) must have a persistent term of the form $b\alpha\sin(\omega t)$. By guessing a solution of the form

$$x(t) = b\alpha\sin(\omega t) + \hat{x}(t)$$

and plugging back into (1.1), (1.2), one ends up with the system

$$\dot{\hat{x}} = ab\alpha\sin(\omega t) + a\hat{x} - b^3k\sin^3(\omega t) - 2b^2k\alpha\sin^2(\omega t)\hat{x} - bk\sin(\omega t)\hat{x}^2.$$

By considering the time scale $\tau = \omega t$ we have the system:

$$\frac{d\hat{x}}{d\tau} = \frac{1}{\omega}\left(ab\alpha\sin(\tau) + a\hat{x} - b^3k\sin^3(\tau) - 2b^2k\alpha\sin^2(\tau)\hat{x} - bk\sin(\tau)\hat{x}^2\right),$$

which, for large ω, after averaging the $\sin(\tau)$ and $\sin^3(\tau)$ terms to zero, looks like

$$\dot{\bar{\hat{x}}} = (a - b^2k\alpha)\bar{\hat{x}}. \tag{1.3}$$

An important feature of system (1.3) is that the control direction, b, has been squared, therefore stability is accomplished, regardless of the sign of b, by choosing $k\alpha > \frac{a}{b^2}$. Further analysis shows that b may even vary with time, passing through zero and changing sign, such as $b(t) = \sin(10t)$, and the controller will still stabilize the system as long as $b(t)$ is non-zero often enough.

The standard averaging analysis provides a local stability result and is possible due to the simplicity of linear systems, which allows for the possibility of guessing a

perturbation $\hat{x}$ of the actual solution. For nonlinear systems, or Lyapunov functions other than $V(x) = x^2$ the approach is based on the work of Dürr et al [21] in which a network of autonomous vehicles was self-organized by maximizing a global objective function. In their work, the scheme shown in Figure 1.3 was slightly modified, by splitting the gain ω on the dithering term, between the dithering and mixing terms, each now being multiplied by $\sqrt{\omega}$. This difference allowed for an innovative combination of certain Lie bracket-based averaging results of Kurzweil, Jarnik, Sussmann, Liu, Gurvits, and Li [30, 31, 32, 45, 78, 79] to be combined with results of Moreau and Aeyels [58] on semiglobal practical stability for nonlinear systems with small parametric perturbations.

For the scalar system (1.1), (1.2) we then use the new ES scheme, almost identical to that shown in Figure 1.3, with the modification that both the mixing and dithering terms have the gain $\sqrt{\omega}$, resulting in a controller of the form:

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)x^2, \tag{1.4}$$

which is in a form for which standard averaging does not apply, instead Lie bracket averaging must be used. For the linear system (1.1), the standard averaging and Lie bracket averaging techniques for controllers (1.2) and (1.4) respectively give the same average systems (1.3).

For nonlinear systems affine in control,

$$\dot{x} = f(x,t) + g(x,t)u, \tag{1.5}$$

we employ a nonlinear time-varying control law of the form

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)V(x), \tag{1.6}$$

as shown in Figure 1.5 where $V(x)$ is a Lyapunov function candidate for (1.5).

We now summarize the major differences between the controller (1.6) and typical ES schemes. One major difference, as already stated, is the $\sqrt{\omega}$ gain on the perturbing high-frequency terms and the fact that we are minimizing a known function $V(x)$ for an unknown system $f(x,t)$, $g(x,t)$, which contains its own integrator and is possibly open-loop unstable, rather than for a system that is open-loop stable and known, as in typical ES-schemes. The $\sqrt{\omega}$ term is a necessity for the application of Lie Bracket

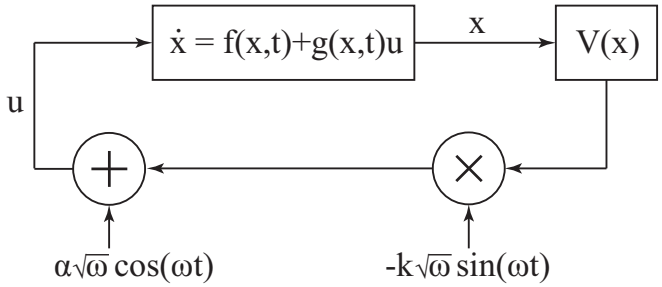


Figure 1.5: Extremum seeking for stabilization of unknown systems.

averaging and intuitively can be understood by the fact that a highly oscillatory term on average may have little or no influence on a system's overall dynamics unless its amplitude is appropriately increased. As in classical ES approaches we add a small perturbing term of the form (after integration) $\frac{\alpha}{\sqrt{\omega}}\sin(\omega t)$, which we then mix with a large signal of the same frequency $\sqrt{\omega}\sin(\omega t)$ in order to extract an estimate of the gradient of $V(x)$. One would expect that induced oscillations of any form such as square or triangular waves could successfully achieve similar ES results, as long as care is taken to match frequency and amplitude relationships, which may require more analysis when using non-smooth perturbations with infinite frequency spectrum.

As stated, the controller (1.6) is, in fact, a minimum-seeking controller for $V(x)$. Under this controller, the Lie bracket average system is

$$\dot{\bar{x}} = f(\bar{x},t) - k\alpha g(\bar{x},t)g^T(\bar{x},t)\left(\frac{\partial V(\bar{x})}{\partial \bar{x}}\right)^T. \tag{1.7}$$

This average system clearly displays that the ES algorithm introduces a gradient of the clf V with respect to the state $\bar{x}$, multiplied by the unknown square negative semidefinite matrix $g(\bar{x},t)g^T(\bar{x},t)$. Under suitable conditions, the choice of a sufficiently high gain product $k\alpha > 0$ makes the gradient term dominate the term $f(\bar{x},t)$ and globally asymptotically stabilizes the average system (1.7) and semi-globally practically stabilizes the original system (1.5) for sufficiently high ω.

For vector-valued linear time-varying plants,

$$\dot{x} = A(t)x + B(t)u,$$

the control law

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)|x|^2$$

results in the Lie bracket average LTV system

$$\dot{\bar{x}} = \left[A(t) - k\alpha B(t)B^T(t)\right]\bar{x},$$

which, under a persistency of excitation condition on the vector-valued function $B(t)$, is stabilized for sufficiently high $k\alpha > 0$.

Stabilization and control Lyapunov functions (clf) have been seminal accomplishments in the control field since the introduction of Sontag's formula in 1989 [75] and followed by constructive developments of clf's throughout the 1990s for systems with known models [41, 74], unknown parameters [42, 41, 81], uncertain nonlinearities [25], and stochastic and deterministic disturbances [42].

In the clf theory, a central place is occupied by "L_gV controllers" (damping controllers) [42, 74], which are capable of ensuring not only stability but also inverse optimality, and of which a representative example is Sontag's universal controller [75].

In developing stabilizing controllers for uncertain systems, the most challenging class of uncertainties is the unknown control direction, also referred to as the case with an unknown sign of the high frequency gain. This problem has a history that precedes clf's and goes to the early period of development of robust adaptive control. Posed in the early 1980s by Morse and first solved by Nussbaum [61], the problem of stabilization in the presence of unknown control direction has recently received increased attention in adaptive control of nonlinear systems but the classical parameter-adaptive solutions suffer from poor transient performance and fail to achieve exponential stability even in the absence of other uncertainties.

The work presented in this thesis builds on the connection between extremum seeking (ES), Lyapunov functions, and Lie bracket averaging, to develop a general framework for stabilization of systems with unknown models using clf's and L_gV-like controllers. Since our approach does not require the knowledge of the control direction (for systems affine in control, the input vector field g is allowed to be unknown), as a byproduct of our effort in designing ES-based L_gV-like controllers we provide a new solution to the problem of stabilization of systems with unknown direction. The new solution guarantees exponential convergence and does not suffer from poor transients that are characteristic of solutions that employ Nussbaum gain techniques.

In all our ES designs, the first step is the construction (or a guess) of a clf $V(x)$, the second step is a choice of sufficiently high gains k and α of the ES algorithm, such that the product $k\alpha$ is large enough to stabilize the Lie bracket average system, and the third step is the choice of a sufficiently high ω to satisfy the requirements of the Lie bracket averaging theorem of Kurzweil, Jarnik, Sussmann, Liu, Gurvits, and Li [30, 31, 32, 45, 78, 79] and of the semiglobal practical stability theorem of Moreau and Aeyels [58]. In the time-varying case, intuitively speaking, relative to the system dynamics, we must choose ω large enough such that the control oscillations are on a separate time-scale from the time-varying dynamics, $\omega \gg |\dot{A}(t)|, |\dot{B}(t)|$. The choice of ω is related to the region of attraction of our closed loop systems, with larger ω resulting in a smaller perturbing term $\frac{\alpha}{\sqrt{\omega}}\sin(\omega t)$ and providing stability for larger initial conditions.

Although the Lie bracket averaging analysis leads to a sufficiently condition for the product $k\alpha$ to be high enough, the individual terms k and α play significantly different roles in the ES algorithm and have different influences on the required choice of ω. The choice of ω is less sensitive to α, because the dithering term, after integration, is of the order of $\frac{\alpha}{\sqrt{\omega}}$. The choice of ω is much more sensitive to the value of k, which is a control gain that increases the convergence rate as well as decreases the size of the residual convergence set. For example, once a stabilizing choice of $k\alpha$ is made and a large enough ω is chosen, if one doubles k while halving α and thereby maintaining a fixed product $k\alpha$, if ω is not increased the system may possibly have large overshoot and may even become unstable. If, on the other hand, one halves the value of k while doubling α, the same choice of ω will usually maintain stability.

Clearly, our approach is of a high-gain type in requiring that both $k\alpha$ and ω be large, furthermore we introduce fast oscillations into the system, which may become impractical, due to actuator capabilities, for very large choices of ω. When considering problems with an unknown control direction, unlike the approach by Nussbaum [61] and Mudgett and Morse [59] (which we refer to as the "MMN approach"), our approach is neither global nor asymptotic—it guarantees semiglobal exponential practical stability. As such, our results are robust to disturbances. One of the major benefits of this approach is that noisy measurement of the cost function, $V(x)$, is not problematic. On average, as long as the noise is truly random and does not happen to perfectly match the

frequency of the controller's oscillations, it cannot destroy the overall system's stability. Furthermore, unlike the MMN approach, we can not only handle unknown signs of the high frequency gains, but also signs that change with time. In particular, we can allow $B(t)$ to go through zero. The price we pay, besides the lack of globality and of perfect regulation to the origin, is that our high-gain choice of $k\alpha$ requires that we know a lower bound on a mean value of $B(t)B^T(t)$ and upper bounds on mean-square values of $A(t)$ and $B(t)$.

The MMN controller is designed for the case of constant plant parameters. When it is applied to a system that is time-varying, and when the control coefficients are quickly varying with time, passing through zero, and alternating signs, such as when $B(t)$ contains sinusoids, then the MMN type controllers simply cannot keep up and repeatedly overshoot with greater and greater error magnitude. We illustrate this with an example. The controllers developed in this paper do not suffer from any such overshoots because the extremum seeking control scheme is by design operating on a faster time scale than the dynamics in the system's coefficients, and the system's behavior, as estimated by an averaged system, does not depend on the control coefficients' signs.

In Chapter 2 we state definitions of stability used throughout the thesis and provide a review of a Lie bracket averaging results of Kurzweil, Jarnik, Sussmann, Liu, Gurvits, and Li [30, 31, 32, 45, 78, 79] in order to derive an averaged system of our original system. In Chapter 3 we present the main results on which the remainder of the thesis is based. In Chapter 4 we demonstrate a real world application of the proposed control algorithm by utilizing it in hardware in order to optimize the output voltage waveform of a high voltage converter modulator (HVCM). In Chapter 5 we extend our stability results to trajectory tracking for unknown, unstable systems. In Chapter 6 we present a modified ES scheme which has the same Lie bracket average as the scheme in the material above, but the advantage that its control effort settles to zero if the system's trajectory ever reaches the origin.

This chapter contains material from

A. Scheinker and M. Krstić, "Maximum-seeking for CLFs: Universal semiglobally stabilizing feedback under unknown control directions," *IEEE Transactions on Automatic Control*, to appear.

The dissertation author was the primary author in this publication.

Chapter 2

Background on Lie Bracket Averaging and Semiglobal Practical Asymptotic Stability

In their approach, Dürr et al [21] make reference to and prove additional properties of the Lie bracket averaging results of Gurvits and Li [30, 31, 32], which focus on systems of the form:

$$\dot{x} = \sum_{i=1}^{m_1} b_i(x)\bar{u}_i(t) + \frac{1}{\sqrt{\varepsilon}} \sum_{i=1}^{m_2} \hat{b}_i(x)\hat{u}_i(t,\theta), \tag{2.1}$$

in which $\hat{u}_i(t,\theta)$ is T-periodic in $\theta = \frac{t}{\varepsilon}$, $T \in (0,\infty)$ and has zero average, $\int_0^T \hat{u}_i(t,\theta)d\theta = 0$. Relative to system (2.1), the following approximation model is considered:

$$\dot{\bar{x}} = \sum_{i}^{m_1} b_i(\bar{x})\bar{u}_i(t) + \frac{1}{T}\sum_{i<j}\left[\hat{b}_i,\hat{b}_j\right](\bar{x})\nu_{i,j}(t), \qquad \bar{x}(0) = x(0), \tag{2.2}$$

where

$$\begin{aligned} \nu_{i,j}(t) &= \int_0^T \int_0^\theta \hat{u}_i(t,\tau)\hat{u}_j(t,\theta)d\tau d\theta, \\ \left[\hat{b}_i,\hat{b}_j\right](\bar{x}) &= \frac{\partial \hat{b}_j(\bar{x})}{\partial \bar{x}}\hat{b}_i(\bar{x}) - \frac{\partial \hat{b}_i(\bar{x})}{\partial \bar{x}}\hat{b}_j(\bar{x}). \end{aligned} \tag{2.3}$$

Noting that a T-periodic function is also nT-periodic, If we replace T with nT, $n \in \mathbb{N}$ and apply the arguments of the original proof [30, 31, 32] as well as [22, Lemma 2], the systems (2.1) and (2.2) then satisfy the following approximation result.

Lemma 1 ([4, 22, 30, 31, 32])**.** *For period $T > 0$ and $n \in \mathbb{N}$, there exists $\varepsilon^\star$ such that for all $\varepsilon \in (0, \varepsilon^\star)$, the trajectory of system (2.1) is within a $\Delta(nT, \varepsilon)$-distance of the solution of system (2.2), namely,*

$$\max_{t \in [0, nT]} |x(t) - \bar{x}(t)| \leq \Delta(nT, \varepsilon), \tag{2.4}$$

where $\Delta(nT, \varepsilon) \to 0$ as $\varepsilon \to 0$.

The works of Gurvits and Li make reference to the work of Sussmann and Liu [78] in which highly oscillatory controls result in systems which are approximately described by Lie bracket averages. A proof of Sussmann and Liu's results was given by Sussmann (1992) [79, Section 5] and is a special case of Lemma 1. In fact systems of the form considered by Sussmann and Liu are exactly the types of systems that we consider in this work, therefore we present their proof as background. We note that Sussmann and Liu mentioned that their work is an extension of the results of Kurzweil and Jarnik [45].

Lemma 2. *Consider the vector valued system:*

$$\dot{x} = \sum_{i=1}^{m_1} b_i(x) u_i(t) + \sum_{i=1}^{m_2} \sqrt{\frac{n_i}{\varepsilon}} \left(b_{i,c}(x) u_{i,c}(t) \cos\left(\frac{n_i t}{\varepsilon}\right) + b_{i,s}(x) u_{i,s}(t) \sin\left(\frac{n_i t}{\varepsilon}\right) \right), \tag{2.5}$$

such that the trajectory $x(t)$ is always within some compact set $K \subset \mathbb{R}^n$, such that $b_i(x), b_{i,c}(x), b_{i,s}(x) \in C^1(K)$, where the $n_i \in \mathbb{R}$ are distinct and the functions $u_i(t)$, $u_{i,c}(t)$, $u_{i,s}(t) \in C^1(\mathbb{R})$. Consider relative to system (2.5) the following average system

$$\dot{\bar{x}} = \sum_{i=1}^{m_1} b_i(\bar{x}) u_i(t) + \frac{1}{2} \sum_{i=1}^{m_2} \left[b_{i,c}(\bar{x}), b_{i,s}(\bar{x}) \right] u_{i,c}(t) u_{i,s}(t), \quad \bar{x}(0) = x(0) \tag{2.6}$$

where

$$\left[b_{i,c}(\bar{x}), b_{i,s}(\bar{x}) \right] \;=\; \frac{\partial b_{i,s}(\bar{x})}{\partial \bar{x}} b_{i,c}(\bar{x}) - \frac{\partial b_{i,c}(\bar{x})}{\partial \bar{x}} b_{i,s}(\bar{x}).$$

By choosing small enough ε the two systems (2.5) and (2.6) may be forced to stay arbitrarily close together over any time interval $[0, T_2]$, in particular for any given $T_2 > 0$ and $\delta > 0$ there exists $\varepsilon^\star > 0$ such that for all $\varepsilon \in (0, \varepsilon^\star)$,

$$\max_{t \in [0, T_2]} |x(t) - \bar{x}(t)| < \delta. \tag{2.7}$$

Remark 1. *In the following proof we first fix $\delta > 0$, $T_2 > 0$, and a compact set $K \subset \mathbb{R}^n$ to which our analysis is confined. We perform several integrations by parts and show that many of the terms are bounded and multiplied by $\sqrt{\varepsilon}$ and therefore approach zero as ε does and for others apply the Riemann-Lebesgue Lemma to show the convergence of certain integrals to zero. Finally, the remaining terms are shown to uniformly converge, over the compact set $[0,T_2] \times K$ to within δ of the much simpler Lie bracket averaged system by application of the Arzel*à*-Ascoli Theorem, by choosing a sufficiently small value of ε. A background on the Functional Analysis techniques used throughout the proof can be found in any standard analysis text such as Folland [24].*

Proof. Consider the collection of solutions $\{x^\varepsilon(t)\}$ over the compact set $[0,T_2] \times K$, of system (2.5)

$$\begin{aligned} x^\varepsilon(t) &= x(0) + \sum_{i=1}^{m_1} \left(\int_0^t b_i(x^\varepsilon) u_i(\tau) d\tau \right) + \sum_{i=1}^{m_2} \left(\underbrace{\sqrt{\frac{n_i}{\varepsilon}} \int_0^t b_{i,c}(x^\varepsilon) u_{i,c}(\tau) \cos\left(\frac{n_i \tau}{\varepsilon}\right) d\tau}_{I_{i,c}} \right) \\ &\quad + \sum_{i=1}^{m_2} \left(\underbrace{\sqrt{\frac{n_i}{\varepsilon}} \int_0^t b_{i,s}(x^\varepsilon) u_{i,s}(\tau) \sin\left(\frac{n_i \tau}{\varepsilon}\right) d\tau}_{I_{i,s}} \right). \end{aligned}$$

Integrating by parts we get

$$\begin{aligned} I_{i,c} &= b_{i,c}(x^\varepsilon) \underbrace{\sqrt{\frac{n_i}{\varepsilon}} \int_0^t u_{i,c}(\tau) \cos\left(\frac{n_i \tau}{\varepsilon}\right) d\tau}_{J_{i,c}(t)} - \int_0^t J_{i,c}(\tau) \frac{\partial b_{i,c}(x^\varepsilon)}{\partial x^\varepsilon} \frac{\partial x^\varepsilon}{\partial \tau} d\tau \\ I_{i,s} &= b_{i,s}(x^\varepsilon) \underbrace{\sqrt{\frac{n_i}{\varepsilon}} \int_0^t u_{i,s}(\tau) \sin\left(\frac{n_i \tau}{\varepsilon}\right) d\tau}_{J_{i,s}(t)} - \int_0^t J_{i,s}(\tau) \frac{\partial b_{i,s}(x^\varepsilon)}{\partial x^\varepsilon} \frac{\partial x^\varepsilon}{\partial \tau} d\tau. \end{aligned}$$

Performing another integration by parts we get

$$\begin{aligned} J_{i,c}(t) &= \sqrt{\frac{\varepsilon}{n_i}} \left(u_{i,c}(t) \sin\left(\frac{n_i t}{\varepsilon}\right) \right) + \sqrt{\frac{\varepsilon}{n_i}} \int_0^t \frac{du_{i,c}(t)}{dt} \sin\left(\frac{n_i t}{\varepsilon}\right) dt \\ J_{i,s}(t) &= \sqrt{\frac{\varepsilon}{n_i}} \left(u_{i,s}(0) - u_{i,s}(t) \cos\left(\frac{n_i t}{\varepsilon}\right) \right) + \sqrt{\frac{\varepsilon}{n_i}} \int_0^t \frac{du_{i,s}(t)}{dt} \cos\left(\frac{n_i t}{\varepsilon}\right) dt. \end{aligned}$$

Because the functions $u_{i,c}(t)$ and $b_i(x^\varepsilon)$ are continuous they are bounded over the compact set $[0,T_2]\times K$, therefore the quantities

$$b_{i,c}(x^\varepsilon)\sqrt{\frac{\varepsilon}{n_i}}\left(u_{i,c}(t)\sin\left(\frac{n_i t}{\varepsilon}\right)\right)\to 0,\quad b_{i,s}(x^\varepsilon)\sqrt{\frac{\varepsilon}{n_i}}\left(u_{i,s}(0)-u_{i,s}(t)\cos\left(\frac{n_i t}{\varepsilon}\right)\right)\to 0$$

uniformly as $\varepsilon\to 0$. Furthermore, because $u_{i,c}(t),u_{i,s}(t)$ have continuous derivatives, $\frac{du_{i,c}(t)}{dt},\frac{du_{i,s}(t)}{dt}$ are clearly in $L^1[0,T_2]$, and therefore by the Riemann-Lebesgue Lemma the integrals

$$b_{i,c}(x^\varepsilon)\sqrt{\frac{\varepsilon}{n_i}}\int_0^t\frac{du_{i,s}(t)}{dt}\cos\left(\frac{n_i t}{\varepsilon}\right)dt\to 0,\quad b_{i,s}(x)\sqrt{\frac{\varepsilon}{n_i}}\int_0^t\frac{du_{i,c}(t)}{dt}\sin\left(\frac{n_i t}{\varepsilon}\right)dt\to 0$$

uniformly as $\varepsilon\to 0$. Substituting $\frac{\partial x}{\partial\tau}$ into $I_{i,c}$ and $I_{i,s}$ we get

$$\begin{aligned}
I_{i,c} &= -\int_0^t\sqrt{\frac{\varepsilon}{n_i}}\left(u_{i,c}(\tau)\sin\left(\frac{n_i\tau}{\varepsilon}\right)\right)\frac{\partial b_{i,c}(x^\varepsilon)}{\partial x^\varepsilon}b_{j,c}(x^\varepsilon)u_j(\tau)d\tau\\
&\quad-\sum_{j=1}^{m_2}\int_0^t\left(u_{i,c}(\tau)\sin\left(\frac{n_i\tau}{\varepsilon}\right)\right)\frac{\partial b_{i,c}(x^\varepsilon)}{\partial x}b_{j,c}j(x^\varepsilon)u_{j,c}(\tau)\cos\left(\frac{n_j\tau}{\varepsilon}\right)d\tau\\
&\quad-\sum_{j=1}^{m_2}\int_0^t\left(u_{i,c}(\tau)\sin\left(\frac{n_i\tau}{\varepsilon}\right)\right)\frac{\partial b_{i,c}(x^\varepsilon)}{\partial x}b_{j,s}(x^\varepsilon)u_{j,s}(\tau)\sin\left(\frac{n_j\tau}{\varepsilon}\right)d\tau\\
&\quad+\nu_{i,c}(x^\varepsilon,t,\varepsilon)\\
I_{i,s} &= \int_0^t\sqrt{\frac{\varepsilon}{n_i}}\left(u_{i,s}(\tau)\cos\left(\frac{n_i\tau}{\varepsilon}\right)-u_{i,s}(0)\right)\frac{\partial b_{i,s}(x^\varepsilon)}{\partial x^\varepsilon}b_{j,s}(x^\varepsilon)u_j(\tau)d\tau\\
&\quad-\sum_{j=1}^{m_2}\int_0^t\left(u_{i,s}(\tau)\cos\left(\frac{n_i\tau}{\varepsilon}\right)-u_{i,s}(0)\right)\frac{\partial b_{i,s}(x^\varepsilon)}{\partial x^\varepsilon}b_{j,c}(x^\varepsilon)u_{j,c}(\tau)\cos\left(\frac{n_j\tau}{\varepsilon}\right)d\tau\\
&\quad-\sum_{j=1}^{m_2}\int_0^t\left(u_{i,s}(\tau)\cos\left(\frac{n_i\tau}{\varepsilon}\right)-u_{i,s}(0)\right)\frac{\partial b_{i,s}(x^\varepsilon)}{\partial x^\varepsilon}b_{j,s}(x^\varepsilon)u_{j,s}(\tau)\sin\left(\frac{n_j\tau}{\varepsilon}\right)d\tau,\\
&\quad+\nu_{i,s}(x^\varepsilon,t,\varepsilon)
\end{aligned}$$

where the term $\nu_{i,c}(x^\varepsilon,t,\varepsilon)$ contains products of the terms $J_{i,c}\frac{\partial b_{i,c}(\partial x^\varepsilon)}{x^\varepsilon}$ and $\sum_{i=1}^{m_1}b_i(x)u_i(t)$ and the term $\nu_{i,s}(x^\varepsilon,t,\varepsilon)$ contains products of the terms $J_{i,s}\frac{\partial b_{i,s}(\partial x^\varepsilon)}{x^\varepsilon}$ and $\sum_{i=1}^{m_1}b_i(x)u_i(t)$ and therefore, by the above arguments uniformly converge to zero as $\varepsilon\to 0$ over the compact set $[0,T_2]\times K$. As $\varepsilon\to 0$, all mixed terms of the form $\cos\left(\frac{n_i\tau}{\varepsilon}\right)\sin\left(\frac{n_i\tau}{\varepsilon}\right)$ and terms of the form $\cos\left(\frac{n_i\tau}{\varepsilon}\right)\cos\left(\frac{n_j\tau}{\varepsilon}\right)$ and $\sin\left(\frac{n_i\tau}{\varepsilon}\right)\sin\left(\frac{n_j\tau}{\varepsilon}\right)$ such that $n_i\neq n_j$ weakly

converge to zero in $L^2[0,T_2]$. In the remaining terms:

$$-\int_0^t \frac{\partial b_{i,s}(x^\varepsilon)}{\partial x^\varepsilon} b_{i,c}(x^\varepsilon)u_{i,c}(\tau)u_{j,s}(\tau)\sin^2\left(\frac{n_i\tau}{\varepsilon}\right)d\tau$$
$$+\int_0^t \frac{\partial b_{i,c}(x^\varepsilon)}{\partial x^\varepsilon} b_{i,s}(x^\varepsilon)u_{i,s}(\tau)u_{j,c}(\tau)\cos^2\left(\frac{n_i\tau}{\varepsilon}\right)d\tau$$

the functions $\sin^2\left(\frac{n_i\tau}{\varepsilon}\right)$ and $\cos^2\left(\frac{n_i\tau}{\varepsilon}\right)$ converge weakly to $\frac{1}{2}$ in $L^2[0,T_2]$. Over the compact set $[0,T_2]\times K$ all the functions considered are uniformly bounded and equicontinuous, therefore for any countable sequence of values of $\{\varepsilon\}$, such that $\{\varepsilon_i\}\to 0$, by the Arzelà-Ascoli Theorem there exists a subsequence which uniformly converges to the limit

$$\begin{aligned} x^0(t) &= x(0)+\int_0^t \sum_{i=1}^m b_i(x^0)u_i(t)dt+\frac{1}{2}\int_0^t \sum_{i=1}^{m_2}\frac{\partial b_{i,s}(x^0)}{\partial x^0}b_{j,c}(x^0)u_{i,s}(t)u_{j,c}(t)dt \\ &\quad -\frac{1}{2}\int_0^t \sum_{i=1}^{m_2}\frac{\partial b_{i,c}(x^0)}{\partial x^0}b_{j,s}(x^0)u_{i,c}(t)u_{j,s}(t)dt, \end{aligned} \tag{2.8}$$

which is the solution of the differential equation

$$\begin{aligned} \dot{x}^0(t) &= \sum_{i=1}^m b_i(x^0)u_i(t)dt+\frac{1}{2}\sum_{i=1}^{m_2}\frac{\partial b_{i,s}(x^0)}{\partial x^0}b_{j,c}(x^0)u_{i,s}(t)u_{j,c}(t)dt \\ &\quad -\frac{1}{2}\sum_{i=1}^{m_2}\frac{\partial b_{i,c}(x^0)}{\partial x^0}b_{j,s}(x^0)u_{i,c}(t)u_{j,s}(t)dt, \quad x^0(0)=x(0) \end{aligned}$$

which we rewrite as

$$\dot{x}^0(t)=\sum_{i=1}^m b_i(x^0)u_i(t)+\frac{1}{2}\sum_{i=1}^{m_2}u_{i,c}(t)u_{j,s}(t)\left(\frac{\partial b_{i,s}(x^0)}{\partial x^0}b_{i,c}(x^0)-\frac{\partial b_{i,c}(x^0)}{\partial x^0}b_{i,s}(x^0)\right). \tag{2.9}$$

Because this solution is unique, every convergent subsequence must converge to this same limit and therefore so does the original uncountable collection of solutions $\{x^\varepsilon(t)\}$.

□

Before we can take advantage of these averaging results we make the following definitions as in Moreau and Aeyels [58]. In what follows, given a system

$$\dot{x}=f(t,x), \tag{2.10}$$

$\psi(t,t_0,x_0)$ denotes the solution of (2.10) which passes through the point x_0 at time t_0.

Definition 1. *Global Uniform Asymptotic Stability (GUAS): An equilibrium point of (2.10) is said to be GUAS if it satisfies the following three conditions:*

- *Uniform Stability: For every $c_2 \in (0,\infty)$ there exists $c_1 \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$,*

$$\|\psi(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0,\infty). \tag{2.11}$$

- *Uniform Boundedness: For every $c_1 \in (0,\infty)$ there exists $c_2 \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$,*

$$\|\psi(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0,\infty). \tag{2.12}$$

- *Global Uniform Attractivity: For all $c_1,c_2 \in (0,\infty)$ there exists $\bar{T} \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$,*

$$\|\psi(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0+\bar{T},\infty). \tag{2.13}$$

In conjunction with (2.10), we consider systems of the form

$$\dot{x} = f^{\varepsilon}(t,x) \tag{2.14}$$

whose trajectories are denoted as $\phi^{\varepsilon}(t,t_0,x_0)$.

Definition 2. *Converging Trajectories Property: The systems (2.10) and (2.14) are said to satisfy the converging trajectories property if for every $\hat{T} \in (0,\infty)$ and compact set $K \subset \mathbb{R}^n$ satisfying $\{(t,t_0,x_0) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n : t \in [t_0,t_0+\hat{T}], x_0 \in K\} \subset \mathrm{Dom}\psi$, for every $d \in (0,\infty)$ there exists $\varepsilon^\star$ such that for all $t_0 \in \mathbb{R}$, for all $x_0 \in K$ and for all $\varepsilon \in (0,\varepsilon^\star)$,*

$$\|\phi^{\varepsilon}(t,t_0,x_0) - \psi(t,t_0,x_0)\| < d, \ \ \forall t \in [t_0,t_0+\hat{T}]. \tag{2.15}$$

We then define the following form of stability for system (2.14).

Definition 3. *ε-Semiglobal Practical Uniform Asymptotic Stability (ε-SPUAS): An equilibrium point of (2.14) is said to be ε-SPUAS if it satisfies the following three conditions:*

- *Uniform Stability: For every $c_2 \in (0,\infty)$ there exists $c_1 \in (0,\infty)$ and $\hat{\varepsilon} \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\varepsilon \in (0,\hat{\varepsilon})$,*

$$\|\phi^{\varepsilon}(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0,\infty). \tag{2.16}$$

- *Uniform Boundedness: For every $c_1 \in (0,\infty)$ there exists $c_2 \in (0,\infty)$ and $\hat{\varepsilon} \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\varepsilon \in (0,\hat{\varepsilon})$,*

$$\|\phi^{\varepsilon}(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0,\infty). \tag{2.17}$$

- *Global Uniform Attractivity: For all $c_1,c_2 \in (0,\infty)$ there exists $\bar{T} \in (0,\infty)$ and $\hat{\varepsilon} \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\varepsilon \in (0,\hat{\varepsilon})$,*

$$\|\phi^{\varepsilon}(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0+\bar{T},\infty). \tag{2.18}$$

With these definitions the following result of Moreau and Aeyels [58] is used in the analysis that follows.

Theorem 1 ([58])**.** *If systems (2.14) and (2.10) satisfy the converging trajectories property and if the origin is a GUAS equilibrium point of (2.10), then the origin of (2.14) is ε-SPUAS.*

Finally, by combining the above results, we arrive at a relationship between the stability of the x and $\bar{x}$ systems, which allows us to perform stability analysis on and relate the stability of Lie bracket averaged systems of the form (2.1) to the stability of the original systems of the form (2.2).

Corollary 1. *If the origin of system (2.2) is GUAS, then the origin of system (2.1) is ε-SPUAS.*

Proof. Given any $\hat{T} > 0$, taking $n \in \mathbb{N}$ such that $nT > \hat{T}$. by Lemma 1 the solutions of (2.2) and (2.1) satisfy the converging trajectories property. Since the origin of (2.2) is GUAS, by Theorem 1, the origin of (2.1) is ε-SPUAS. □

This chapter contains material from A. Scheinker and M. Krstić, "Maximum-seeking for CLFs: Universal semiglobally stabilizing feedback under unknown control directions," *IEEE Transactions on Automatic Control*, to appear. Of which the dissertation author was the primary author.

Chapter 3

Main Results

In this chapter we present our main result, which is the utilization of a modified Extremum Seeking (ES) algorithm for stabilization of unknown and time-varying systems. In Section 3.1 we present a general framework for the design and analysis of stabilizing controllers for systems with unknown models by combining the ES approach with clf's. In Section 3.2 we introduce our stability results by considering a scalar system, we also compare the performance of our controller with the MMN controller for a scalar time-varying system. In Section 3.3 we present the first of our major results, for stabilization of unstable n-dimensional linear time-varying systems whose control vector coefficients may not only be of unknown sign but also of persistently changing sign. In Section 3.4 we consider unknown linear systems in strict-feedback form, as a representative of readily tractable but more notationally burdensome nonlinear systems in strict-feedback form, and design a stabilizing ES controller based on the backstepping approach [41, 81], which allows all the coefficients of the plant to be unknown, with only two mild conditions on bounds on the coefficients, which does not imply the knowledge of any of the coefficients' signs. In Section 3.5 we present results for MIMO nonlinear systems with matched uncertainties and illustrate how to achieve stabilization for uncertain nonlinearities of arbitrary growth, which allows us, for example, to stabilize systems with polynomial nonlinearities without requiring the knowledge of the nonlinearities' polynomial order, using exponential feedback of the state's norm.

Remark 2. *In all of the analysis that follows, in order to use the Lie bracket averaging*

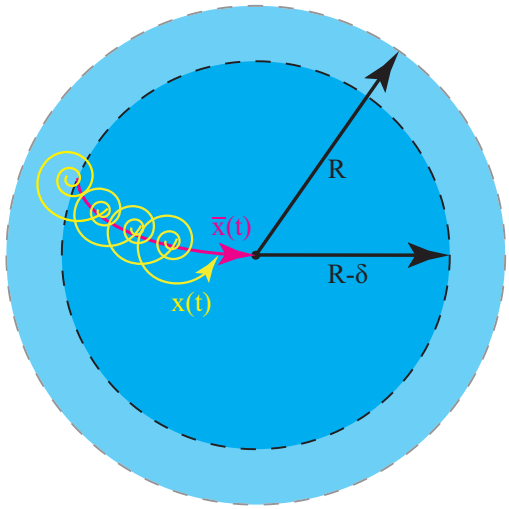


Figure 3.1: Trajectories with initial conditions $x(0) \in B(0, R-\delta)$ are guaranteed to remain within the compact set $\bar{B}(0,R)$.

results described above, the trajectories of our systems are required to be confined to compact sets. Although we do not state this explicitly in the analysis to follow, all of our trajectories are confined to compact sets, which is shown as follows. In each case to be considered in this and the following chapters we first choose $R > 0$, $\delta > 0$ and consider the compact set $\bar{B}(0,R) = \{x \in \mathbb{R}^n : |x| \leq R\}$. Over this compact set we can apply the above Lie bracket averaging results and therefore, for any fixed time length $\hat{T} > 0$, there exists $\varepsilon^\star$ such that for all $\varepsilon \in (0, \varepsilon^\star)$, for all $x(t)$, $\bar{x}(t) \in \bar{B}(0,R)$,

$$\max_{t \in [0,\hat{T}]} |x(t) - \bar{x}(t)| < \delta. \tag{3.1}$$

Now we restrict our analysis to the ball of initial conditions $B(0, R-\delta) = \{x \in \mathbb{R}^n : |x| < R-\delta\}$. We also take into account that in what follows, all of the Lie bracket average systems are shown to be exponentially stable, and therefore

$$\max_{t \in [0,\hat{T}]} |\bar{x}(t)| < |\bar{x}(0)| = |x(0)| < R - \delta. \tag{3.2}$$

Combining conditions (3.1) and (3.2), we then have, for all $x(0) \in B(0, R-\delta)$,

$$\begin{aligned} \max_{t \in [0,\hat{T}]} |x(t)| &= \max_{t \in [0,\hat{T}]} |x(t) - \bar{x}(t) + \bar{x}(t)| \\ &\leq \max_{t \in [0,\hat{T}]} |x(t) - \bar{x}(t)| + \max_{t \in [0,\hat{T}]} |\bar{x}(t)| < \delta + R - \delta = R. \end{aligned}$$

Therefore considering $x(0) \in B(0, R-\delta)$ we are guaranteed that all trajectories are confined to the compact set $\bar{B}(0,R)$, where the choices of R and δ can be made arbitrarily large and small respectively. A graphic representation of this simple result of the triangle inequality is shown in Figure 3.1.

3.1 Minimization of Lyapunov Functions

Our interest is in stabilization of the origin of systems of the form

$$\dot{x} = f(x) + g(x)u\,, \qquad f(0) = 0 \tag{3.3}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}$, and the vector fields f and g are unknown. Though our approach permits a time dependence in f and g, as long as they can be represented as sums of products of functions of x and functions of t, as required by the analysis methodology in Section 2, for clarity we concentrate in this section on time-invariant f and g.

Consider a controller in the form

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)V(x), \tag{3.4}$$

where $\alpha, k > 0$ and the function V is soon to be discussed. The Lie bracket average of the system (3.3), (3.4) is given by

$$\dot{\bar{x}} = f(\bar{x}) - k\alpha g(\bar{x})\left(L_g V(\bar{x})\right)^T\,, \tag{3.5}$$

where we use the standard Lie derivative notation $L_g V = \frac{\partial V}{\partial x} g$. The form of the system (3.5) motivates the following assumption.

Assumption 1 (strong L_gV-stabilizability)**.** *There exists a positive definite, radially unbounded, continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ and a constant $\beta > 0$ such that*

$$L_f V - \beta\left(L_g V\right)^2 < 0\,, \qquad \forall x \neq 0\,. \tag{3.6}$$

With Corollary 1 we establish the following result.

Theorem 2. *For given V and β, denote by $\mathscr{S}(V,\beta)$ the class of all systems (3.3) for which Assumption 1 is satisfied. Under the control law (3.4) all the systems in $\mathscr{S}(V,\beta)$ are $\frac{1}{\omega}$-SPUAS for all $k\alpha \geq \beta$.*

It is relevant to explore the special case of linear systems

$$\dot{x} = Ax + bu \tag{3.7}$$

with control

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)x^T Px, \tag{3.8}$$

where P is a positive definite and symmetric matrix. The Lie bracket average of the system (3.7), (3.8) is given by

$$\dot{\bar{x}} = \left(A - k\alpha bb^T P\right)\bar{x}\,. \tag{3.9}$$

Hence, the linear analog of Assumption 1 is that there exists a positive definite and symmetric *control Lyapunov matrix* (clm) P and a positive constant β such that

$$PA + A^T P - 2\beta Pbb^T P < 0\,. \tag{3.10}$$

Corollary 2. *For given P and β, denote by $\Sigma(P,\beta)$ the class of all pairs (A,b) for which (3.10) is satisfied. Under the control law (3.8) all the systems (3.7) in $\Sigma(P,\beta)$ are $\frac{1}{\omega}$-SPUAS for all $k\alpha \geq \beta$.*

3.1.1 Is Assumption 1 equivalent to stabilizability?

It is well known that a system (3.3) with smooth f and g is stabilizable by feedback continuous at the origin and smooth away from the origin if and only if there exists a control Lyapunov function (clf) with a suitable "small control property" (scp) [75], namely, a positive definite radially unbounded function W with the properties that $L_g W = 0 \Rightarrow L_f W < 0$ and $L_f W + L_g W \alpha_c < 0$ whenever $x \neq 0$, for some continuous function α_c.

Assumption 1 is somewhat stronger than mere stabilizability. For example, for the system

$$\dot{x} = x^3 + x^2 u\,, \tag{3.11}$$

which is stabilizable by simple smooth feedback $u = -2x$, no function V exists that satisfies (3.6) for some $\beta > 0$, and yet $W = x^2/2$ is a clf with an scp.

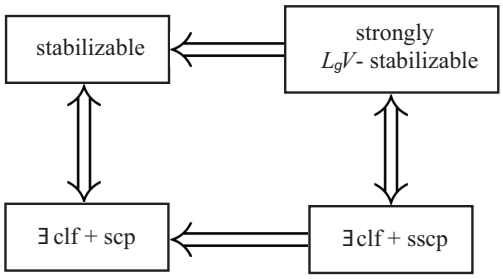


Figure 3.2: The existence of a clf with sscp is equivalent to the system being strongly L_gV-stabilizable, which implies that the system is stabilizable, which guarantees existence of a clf with scp [75], therefore, existence of a clf with sscp implies existence of a clf with scp.

However, Assumption 1 is satisfied for any stabilizable system whose clf W satisfies not only the clf condition $L_gW = 0 \Rightarrow L_fW < 0$ but also a *strong small control property* (sscp) that for $|x| = \varepsilon$

$$\lim_{\varepsilon \to 0} \max_{L_fW(x) > 0} \frac{L_fW(x)}{(L_gW(x))^2} < \infty. \tag{3.12}$$

Under condition (3.12), it can be shown, by slightly modifying the proof in [43, (75)–(80)], that Assumption 1 is satisfied for any $\beta \geq 1$ by a new clf V constructed as

$$V = \int_0^W \rho(r)dr, \tag{3.13}$$

where

$$\rho(r) = 1 + 2 \sup_{x:V(x) \leq r} \frac{L_fV + \sqrt{(L_fV)^2 + (L_gV)^4}}{(L_gV)^2}. \tag{3.14}$$

In simple terms, a system is strongly L_gV-stabilizable if it has a clf with a sscp. Though violated for the example (3.11), condition (3.12) is satisfied for many systems, including all systems in strict-feedback and strict-feedforward forms. Hence Assumption 1 is far from being overly restrictive, despite not being equivalent to stabilizability by continuous control.

Figure 3.2 shows relations between stabilizability and Assumption 1 by highlighting that both assumptions are equivalent to the existence of a clf, but with different small control properties.

In the linear case, the inequality (3.10) is simply a Riccati inequality and by no means appears to be a restrictive condition. However, when (A,b) are unknown, the designer can only guess a P, rather than solving (3.10) for a given matrix on the right-hand side of the inequality. As we shall see next, simple guesses will often violate (3.10). However, as we demonstrate in the rest of the paper, good guesses for a clm are available for some non-trivial classes of systems with unknown model parameters, including unknown control direction.

3.1.2 Is Assumption 1 reasonable for systems with unknown models?

Given how hard it is to find a clf when f and g are known, how can the designer have V and β that satisfy (3.6) when f and g are unknown?

For instance, for the scalar example $\dot{x} = f(x) + u$ with $f(x) = x^3$, the clf $V = x^2$ violates Assumption 1, though the clf $V = x^4$ verifies the assumption. In Section 3.5 we present an approach that allows the designer to construct a clf that verifies Assumption 1 despite not knowing f.

For the second-order linear example with $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is completely controllable, a simple clm $P = I$ violates (3.10) since $PA + A^T P - 2\beta P b b^T P = \begin{bmatrix} 2 & 1 \\ 1 & -\beta \end{bmatrix}$ cannot be made negative definite for any $\beta > 0$. Yet, as we shall see in Section 3.4, a more complicated, valid clm P that does not require exact knowledge of A and b can be constructed.

3.2 Scalar Linear Systems With Unknown Control Directions

Our main result for general n-th order LTV systems is given in Theorem 3. However, for clarity, we first present a simpler result for a scalar LTV case in Proposition 1, which is not a corollary to Theorem 3 but is proved under less restrictive conditions.

Proposition 1. *Consider the system*

$$\dot{x} = a(t)x + b(t)u \tag{3.15}$$

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)x^2, \tag{3.16}$$

and let there exist $\Delta > 0$, $\beta_0 > 0$, $\bar{a} > 0$, *and* $T > 0$ *such that* $a(t)$ *and* $b(t)$ *satisfy*

$$\frac{1}{\Delta}\int_s^{s+\Delta} b^2(\tau)d\tau \geq \beta_0, \ \forall\, s \geq T \tag{3.17}$$

$$\frac{1}{\Delta}\int_s^{s+\Delta} |a(\tau)|d\tau \leq \bar{a}, \ \forall\, s \geq T. \tag{3.18}$$

If

$$k\alpha > \frac{\bar{a}}{\beta_0}, \tag{3.19}$$

then the origin of (3.15), (3.16) is $\frac{1}{\omega}$*-SPUAS with a lower bound on the average decay rate given by*

$$\gamma_r = k\alpha\beta_0 - \bar{a} > 0. \tag{3.20}$$

Proof. System (3.15), (3.16) in closed loop form is

$$\dot{x} = a(t)x + b(t)\alpha\sqrt{\omega}\cos\omega t - b(t)k\sqrt{\omega}\sin(\omega t)x^2, \tag{3.21}$$

which has a Lie bracket average

$$\dot{\bar{x}} = \left[a(t) - k\alpha b^2(t)\right]\bar{x}. \tag{3.22}$$

If $k\alpha > \frac{\bar{a}}{\beta_0}$ we have from (3.17) that

$$k\alpha\int_s^{s+\Delta} b^2(\tau)d\tau > \Delta\bar{a}. \tag{3.23}$$

Therefore, for any $s \geq T$, $N \in \mathbb{N}$ the integral

$$\begin{aligned}
&\int_s^{s+N\Delta} \left[a(\tau) - k\alpha b^2(\tau)\right] d\tau \\
&= \sum_{j=0}^{N-1} \int_{s+j\Delta}^{s+(j+1)\Delta} \left[a(\tau) - k\alpha b^2(\tau)\right] d\tau \\
&= \sum_{j=0}^{N-1} \left[\int_{s+j\Delta}^{s+(j+1)\Delta} a(\tau)d\tau - \int_{s+j\Delta}^{s+(j+1)\Delta} k\alpha b^2(\tau)d\tau\right]
\end{aligned}$$

is, by application of (3.17), (3.18) and (3.19), bounded by

$$\int_s^{s+N\Delta} \left[a(\tau) - k\alpha b^2(\tau)\right] d\tau \leq \sum_{j=0}^{N-1} \left[\Delta\bar{a} - \Delta k\alpha\beta_0\right] = \sum_{j=0}^{N-1} (-\Delta\gamma_r) = -N\Delta\gamma_r < 0, \quad (3.24)$$

where $\gamma_r > 0$ is defined in (3.20). Hence, for any $s \geq T$, $N \in \mathbb{N}$ we have

$$|\bar{x}(s+N\Delta)| = |\bar{x}(s)| e^{\int_s^{s+N\Delta}\left[a(\tau)-k\alpha b^2(\tau)\right]d\tau} < |\bar{x}(s)| e^{-N\Delta\gamma_r}. \quad (3.25)$$

Because $\gamma_r > 0$ the state $\bar{x}(t)$ converges to zero. To study the convergence rate, for any $t \geq T$ we denote $N = \left\lfloor \frac{t-T}{\Delta} \right\rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. We then proceed to show that $|\bar{x}(t)| \leq M_0 e^{-\gamma_r t} |\bar{x}(0)|$, for all $t \geq 0$, for some $M_0 > 0$, and then, with the help of Corollary 1, complete the proof of the proposition. □

3.2.1 Comparison with Nussbaum type control

We now consider the scalar example

$$\dot{x} = x + \cos(10t)u \quad (3.26)$$

and compare our static time-varying feedback

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)x^2 \quad (3.27)$$

to the dynamic feedback scheme of Mudgett and Morse [59],

$$u = y^2\cos(y)x, \qquad \dot{y} = x^2, \quad (3.28)$$

which admittedly was designed only for constant input coefficients. We simulate the two closed loop systems starting from $x(0) = 5$, with $\omega = 100, k = 5, \alpha = 5$ for our controller and $y(0) = 10$ for the controller of Mudgett and Morse. As shown in Figure 3.3, the extremum-seeking method's performance is only slightly changed by the alternating sign of the input coefficient, at most kicking the system $\frac{\alpha}{\sqrt{\omega}}$ (the size of the perturbing signal) in the wrong direction. The MMN method on the other hand suffers from overshoot each time the sign change happens as $y(t)$ cannot change fast enough to maintain $\cos(10t)\cos(y(t)) < 0$. Worse yet, the growing size of $y(t)$ causes growth of the overshoot size as well.

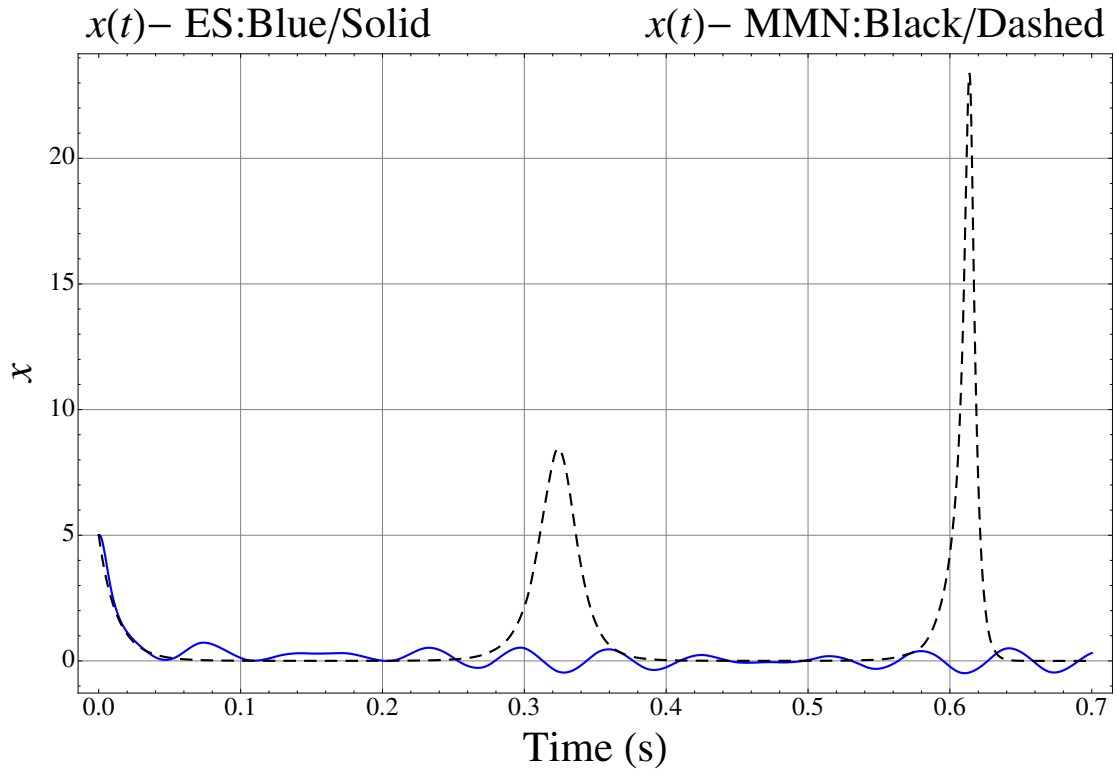


Figure 3.3: The repeated overshoot caused by the MMN-controller takes place whenever the value $\cos(10t)\cos(y(t))$ becomes positive, furthermore, because, when x is non-zero, y is always growing, the overshoots grow in severity as time goes on.

3.3 Vector Valued Linear Systems with Unknown Control Directions

Before we state our results we introduce the notation

$$\langle Z \rangle_{\Delta}(s) \triangleq \frac{1}{\Delta}\int_{s}^{s+\Delta} Z(\tau)d\tau \tag{3.29}$$

for $Z : \mathbb{R} \to \mathbb{R}$, and note that, for any column vector B, $BB^T \leq |B|^2 I$.

In what follows, the general n-dimensional case, is complicated by the possibility of cross talk between different components of vectors, a difficulty only possible in higher dimensions.

Theorem 3. *Consider the system*

$$\dot{x} = A(t)x + B(t)u, \tag{3.30}$$

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)|x|^2, \tag{3.31}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $u \in \mathbb{R}$, *and let there exist* $\Delta > 0$, $b_\star \geq \beta_0 > 0$, $a_\star \geq 0$, *and* $T \geq 0$ *such that* $A(t)$ *and* $B(t)$ *satisfy*

$$\frac{1}{\Delta}\int_s^{s+\Delta} B(\tau)B^T(\tau)d\tau \geq \beta_0 I,\ \forall s \geq T \tag{3.32}$$

$$\left\langle |B|^2 \right\rangle_\Delta (s) \leq b_\star,\ \forall s \geq 0, \tag{3.33}$$

$$\left\langle |A|^2 \right\rangle_\Delta (s) \leq a_\star,\ \forall s \geq 0. \tag{3.34}$$

The origin of system (3.30), (3.31) is $\frac{1}{\omega}$*-SPUAS with a lower bound on the average decay rate given by*

$$R = \frac{1}{2\Delta}\left[\ln\left(\frac{1}{\gamma}\right) - \gamma_2\right] - \sqrt{a_\star} > 0, \tag{3.35}$$

where

$$\gamma = 1 - \frac{k\alpha\Delta\beta_0}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} > 0 \tag{3.36}$$

$$\gamma_2 = \frac{4k\alpha\Delta^3 b_\star}{1 + 2k^2\alpha^2\Delta^2 b_\star^2}, \tag{3.37}$$

under either of the two conditions:

(i) Given $k\alpha > 0$ *and* $\Delta > 0$, $a_\star$ *is in the interval* $(0, \bar{a}_\star)$, *where*

$$\bar{a}_\star = \left(\frac{\ln\left(\frac{1}{\gamma}\right)}{\Delta + \sqrt{\Delta^2 + \gamma_2 \ln\left(\frac{1}{\gamma}\right)}}\right)^2. \tag{3.38}$$

(ii) For a given $a_\star$, *the window* Δ *satisfies* $\Delta \in (0, \bar{\Delta})$, *where*

$$\bar{\Delta} = \frac{1}{\sqrt{a_\star}}\min\left\{\bar{\Delta}_1,\ \bar{\Delta}_2\right\}, \tag{3.39}$$

where

$$\bar{\Delta}_1 = \frac{\frac{1}{2}\ln\left(\frac{2\sqrt{2}b_\star}{2\sqrt{2}b_\star - \beta_0}\right)}{1 + \frac{\sqrt{a_\star}}{b_\star}}, \tag{3.40}$$

$$\bar{\Delta}_2 = \frac{-1 + \sqrt{1 + \sqrt{2}\ln\left(\frac{2\sqrt{2}b_\star}{2\sqrt{2}b_\star - \beta_0}\right)}}{\sqrt{2}}, \tag{3.41}$$

and $k\alpha > 1$ is selected such that

$$k\alpha \in \left(\frac{1}{2\sqrt{2}\Delta b_\star}, \frac{1}{2\sqrt{2}\Delta b_\star} + M(a_\star, b_\star, \beta_0, \Delta) \right), \tag{3.42}$$

where

$$M = \frac{\sqrt{\beta_0^2 - 8b_\star^2 \left[1 - e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}\right]^2}}{8\Delta b_\star^2 \left[1 - e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}\right]} > 0. \tag{3.43}$$

Remark 3. *Theorem 3(i) is a robustness result. For any $k\alpha > 0$, the controller (3.31) allows some perturbation $A(t)x$ in the system (3.30), as long as the mean of $A(t)$ is sufficiently small, as quantified by (3.38). Theorem 3(ii) is a design result. If the window Δ is small enough, as quantified by (3.39) and known (it is reasonable to assume that Δ is known because otherwise the a priori bounds (3.32)–(3.34) would have no meaning for the user), then $k\alpha$ can be chosen in the interval (3.42) to guarantee stability. In summary, the controller (3.31) cannot dominate an arbitrarily large $A(t)$, but if $B(t)$ is persistently exciting (PE) over Δ that is sufficiently small in relation to the size of $A(t)$, then the controller (3.31) can stabilize the system (3.30). Furthermore the allowable $k\alpha$ is not arbitrarily large but is within an interval. Overly large $k\alpha$ results in instability despite $B(t)$ being PE because, for a given Δ, an overly large $k\alpha$ forces $x(t)$ to evolve within the time-varying null space of $B^T(t)$, rather than forcing $x(t)$ to converge to zero.*

Proof. The closed-loop system (3.30), (3.31) is given by

$$\dot{x} = A(t)x + B(t)\alpha\sqrt{\omega}\cos(\omega t) - B(t)k\sqrt{\omega}\sin(\omega t)|x|^2, \tag{3.44}$$

which we decompose as

$$\dot{x} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{a,i,j}(x)\bar{u}_{a,i,j}(t) + \sum_{j=1}^{n} b_{c,j}(x)\sqrt{\omega}\hat{u}_{c,j}(t,\theta) + \sum_{j=1}^{n} b_{s,j}(x)\sqrt{\omega}\hat{u}_{s,j}(t,\theta), \tag{3.45}$$

where

$$\begin{aligned} \bar{u}_{a,i,j}(t) &= a_{ji}(t), \quad \hat{u}_{c,j}(t,\theta) = b_j(t)\cos(\omega t) \\ \hat{u}_{s,j}(t,\theta) &= b_j(t)\sin(\omega t) \end{aligned}$$

and

$$b_{a,i,j}(x) = x_i e_j,\ b_{c,j}(x) = \alpha e_j,\ b_{s,j}(x) = -k|x|^2 e_j$$

where e_j is the standard j-th basis vector of $\mathbb{R}^n$. Applying Lie bracket averaging, we obtain the averaged system

$$\dot{\bar{x}} \quad = \quad A(t)\bar{x} - k\alpha B(t)B^T(t)\bar{x}. \tag{3.46}$$

Parts of this proof use steps developed in the proof of Theorem 4.3.2 (iii) in the second half of Section 4.8.3 in [34]. With the following Lyapunov function candidate

$$V(\bar{x}) = \frac{|\bar{x}|^2}{2} \tag{3.47}$$

we get

$$\dot{V}(\bar{x}) = \bar{x}^T\dot{\bar{x}} = \bar{x}^T A(t)\bar{x} - k\alpha\bar{x}^T B(t)B^T(t)\bar{x}. \tag{3.48}$$

Therefore, for any $s \geq T$ we have

$$V(s+\Delta) \quad = \quad V(s) - \underbrace{k\alpha\int_s^{s+\Delta}\left|\bar{x}^T(\tau)B(\tau)\right|^2 d\tau}_{I_1} + \underbrace{\int_s^{s+\Delta}\bar{x}^T(\tau)A(\tau)\bar{x}(\tau)d\tau}_{I_2}. \tag{3.49}$$

We first consider the term I_1 and rewrite

$$\bar{x}^T(\tau)B(\tau) \quad = \quad \bar{x}^T(s)B(\tau) + [\bar{x}(\tau) - \bar{x}(s)]^T B(\tau). \tag{3.50}$$

We can apply the inequality $(x+y)^2 \geq \frac{1}{2}x^2 - y^2$, obtaining

$$\left[\bar{x}^T(\tau)B(\tau)\right]^2 \quad \geq \quad \frac{1}{2}\left[\bar{x}^T(s)B(\tau)\right]^2 - \left[[\bar{x}(\tau) - \bar{x}(s)]^T B(\tau)\right]^2. \tag{3.51}$$

Thus, with (3.49) and (3.51) we get

$$I_1 \leq \underbrace{-k\alpha\frac{\bar{x}(s)^T}{2}\int_s^{s+\Delta}B(\tau)B^T(\tau)d\tau\bar{x}(s)}_{I_{11}} + \underbrace{k\alpha\int_s^{s+\Delta}\left[[\bar{x}(\tau) - \bar{x}(s)]^T B(\tau)\right]^2 d\tau}_{I_{12}}. \tag{3.52}$$

With (3.32) and (3.49) it readily follows that

$$I_{11} \leq -\frac{\bar{x}(s)^T}{2}k\alpha\Delta I\beta_0\bar{x}(s) = -k\alpha\Delta\beta_0 V(\bar{x}). \tag{3.53}$$

Next we address I_{12}. Using (3.44) we get

$$\bar{x}(\tau)-\bar{x}(s)=\int_s^\tau \dot{\bar{x}}(\sigma)d\sigma=\int_s^\tau A(\sigma)\bar{x}(\sigma)d\sigma-k\alpha\int_s^\tau B(\sigma)B^T(\sigma)\bar{x}(\sigma)d\sigma. \quad (3.54)$$

Transposing (3.54) and multiplying by $B(\tau)$ we get

$$[\bar{x}(\tau)-\bar{x}(s)]^T B(\tau)=\int_s^\tau \bar{x}^T(\sigma)A^T(\sigma)d\sigma B(\tau)-k\alpha\int_s^\tau \bar{x}^T(\sigma)B(\sigma)B^T(\sigma)B(\tau)d\sigma. \quad (3.55)$$

By using the representation in (3.55) together with the inequality $(x-y)^2\leq 2x^2+2y^2$ we get

$$\begin{aligned} I_{12}\leq & \underbrace{2k\alpha\int_s^{s+\Delta}\left[\int_s^\tau \bar{x}^T(\sigma)A^T(\sigma)d\sigma B(\tau)\right]^2 d\tau}_{I_{13}} \\ & +\underbrace{2k\alpha\int_s^{s+\Delta}\left[k\alpha\int_s^\tau \bar{x}^T(\sigma)B(\sigma)B^T(\sigma)B(\tau)d\sigma\right]^2 d\tau}_{I_{14}}. \end{aligned}$$

Next we consider the term I_{14}, to which we apply the Cauchy-Schwartz inequality followed by a change in the order of integration and obtain

$$I_{14}\leq 2k^3\alpha^3\Delta^2\left\langle|B|^2\right\rangle_\Delta^2\int_s^{s+\Delta}\left(\bar{x}^T(\sigma)B(\sigma)\right)^2 d\sigma. \quad (3.56)$$

Now we consider the term I_{13}, whose bound is given by

$$\begin{aligned} I_{13} &\leq 2k\alpha\int_s^{s+\Delta}|B(\tau)|^2\left[\int_s^\tau |A(\sigma)|\,|x(\sigma)|\,d\sigma\right]^2 d\tau \\ &\leq 4k\alpha\int_s^{s+\Delta}|B(\tau)|^2\int_s^\tau |A(\zeta)|^2 d\zeta\int_s^\tau V(\sigma)d\sigma d\tau \end{aligned} \quad (3.57)$$

and, changing the order of integration, we get

$$I_{13}\leq 4k\alpha\Delta^2\left\langle|B|^2\right\rangle_\Delta\left\langle|A|^2\right\rangle_\Delta\int_s^{s+\Delta}V(\sigma)d\sigma. \quad (3.58)$$

Combining results (3.52), (3.53), and the bounds on I_{13} and I_{14} we arrive at

$$I_1\leq -k\alpha\Delta\beta_0 V(\bar{x})+2k^3\alpha^3\Delta^2 b_\star^2\int_s^{s+\Delta}\left[\bar{x}^T(\sigma)B(\sigma)\right]^2 d\sigma+4k\alpha\Delta^2 b_\star a_\star\int_s^{s+\Delta}V(\sigma)d\sigma. \quad (3.59)$$

Moving the second term on the right hand side of (3.59) to the left, we obtain

$$I_1 \leq \frac{-k\alpha\Delta\beta_0 V(\bar{x}) + 4k\alpha\Delta^2 b_\star a_\star \int_s^{s+\Delta} V(\sigma)d\sigma}{1+2k^2\alpha^2\Delta^2 b_\star^2}. \tag{3.60}$$

Now we turn our attention to the term I_2 in (3.49). Noting that

$$\bar{x}^T(\tau)A(\tau)\bar{x}(\tau) \leq |A(\tau)|\,\bar{x}^T\bar{x} = 2\,|A(\tau)|\,V(\tau), \tag{3.61}$$

we get

$$I_2 \leq 2\int_s^{s+\Delta} |A(\tau)|\,V(\tau)d\tau. \tag{3.62}$$

Combining (3.49), (3.60) and (3.62) we obtain

$$V(s+\Delta) \leq \gamma V(s) + 2\int_s^{s+\Delta} |A(\tau)|\,V(\tau)d\tau + \frac{4k\alpha\Delta^2 b_\star a_\star \int_s^{s+\Delta} V(\sigma)d\sigma}{1+2k^2\alpha^2\Delta^2 b_\star^2}$$

which can be rewritten as

$$\gamma V(s) + \int_s^{s+\Delta} \left(2\,|A(\tau)| + \frac{4k\alpha\Delta^2 b_\star a_\star}{1+2k^2\alpha^2\Delta^2 b_\star^2}\right) V(\tau)d\tau, \tag{3.63}$$

where γ is defined in (3.36). Noting that

$$\frac{k\alpha\Delta\beta_0}{1+2k^2\alpha^2\Delta^2 b_\star^2} \leq \frac{\beta_0}{2\sqrt{2}b_\star} \tag{3.64}$$

and that $\beta_0 \leq b_\star$, we get that $\gamma \in \left(\frac{2\sqrt{2}-1}{2\sqrt{2}}, 1\right)$, which implies that γ is positive. We now apply the Bellman-Gronwall lemma, and get that for all $s \geq T$,

$$V(s+\Delta) \leq \gamma e^{\left(2\int_s^{s+\Delta}|A(\tau)|d\tau + \frac{4k\alpha\Delta^3 b_\star a_\star}{1+2k^2\alpha^2\Delta^2 b_\star^2}\right)} V(s). \tag{3.65}$$

We note that the Cauchy-Schwartz inequality yields $\int_s^{s+\Delta} |A(\tau)|\,d\tau \leq \Delta\sqrt{a_\star}$, so we get, for all $s \geq T$,

$$V(s+\Delta) \leq \gamma e^{\left(2\Delta\sqrt{a_\star} + \frac{4k\alpha\Delta^3 b_\star a_\star}{1+2k^2\alpha^2\Delta^2 b_\star^2}\right)} V(s). \tag{3.66}$$

Evidently for convergence we require that

$$\gamma e^{\left(2\Delta\sqrt{a_\star} + \frac{4k\alpha\Delta^3 b_\star a_\star}{1+2k^2\alpha^2\Delta^2 b_\star^2}\right)} = \left(1 - \frac{k\alpha\Delta\beta_0}{1+2k^2\alpha^2\Delta^2 b_\star^2}\right) e^{\left(2\Delta\sqrt{a_\star} + \frac{4k\alpha\Delta^3 b_\star a_\star}{1+2k^2\alpha^2\Delta^2 b_\star^2}\right)} < 1 \tag{3.67}$$

or equivalently

$$1-\frac{k\alpha\Delta\beta_0}{1+2k^2\alpha^2\Delta^2b_\star^2}<e^{-\left(2\Delta\sqrt{a_\star}+\frac{4k\alpha\Delta^3b_\star a_\star}{1+2k^2\alpha^2\Delta^2b_\star^2}\right)}. \tag{3.68}$$

To prove the theorem under condition (i), we now calculate the requirement on $a_\star$ for (3.68) to hold. We take ln of both sides of (3.68) which gives us

$$a_\star\frac{4k\alpha\Delta^3b_\star}{1+2k^2\alpha^2\Delta^2b_\star^2}+2\Delta\sqrt{a_\star}+\ln\left(1-\frac{k\alpha\Delta\beta_0}{1+2k^2\alpha^2\Delta^2b_\star^2}\right)<0. \tag{3.69}$$

We define γ_2 as in (3.37) and set the left side of (3.69) equal to zero, obtaining

$$a_\star\gamma_2+2\Delta\sqrt{a_\star}+\ln(\gamma)=0 \tag{3.70}$$

with roots

$$\sqrt{a_\star}=\frac{-2\Delta\pm\sqrt{4\Delta^2-4\gamma_2\ln(\gamma)}}{2\gamma_2}. \tag{3.71}$$

Since $\sqrt{a_\star}$ must be positive, we only consider the positive root

$$\sqrt{a_\star}=\frac{-\Delta+\sqrt{\Delta^2+\gamma_2\ln\left(\frac{1}{\gamma}\right)}}{\gamma_2}, \tag{3.72}$$

which is positive because $\gamma_2>0$ and $\gamma\in\left(\frac{2\sqrt{2}-1}{2\sqrt{2}}\right)$ implies that $\gamma_2\ln\left(\frac{1}{\gamma}\right)>0$. So we have

$$\bar{a}_\star=\left(\frac{\ln\left(\frac{1}{\gamma}\right)}{\sqrt{\Delta^2+\gamma_2\ln\left(\frac{1}{\gamma}\right)}+\Delta}\right)^2. \tag{3.73}$$

Since the left side of (3.69) is increasing as a function of $a_\star>0$, for all $a_\star\in(0,\bar{a}_\star)$ we satisfy (3.67). To study the convergence rate of our system we denote (3.67) as

$$\gamma_r=\gamma e^{\left(2\Delta\sqrt{a_\star}+\frac{4k\alpha\Delta^3b_\star a_\star}{1+2k^2\alpha^2\Delta^2b_\star^2}\right)}<1. \tag{3.74}$$

For any $t\geq T$ we denote $N=\left\lfloor\frac{t-T}{\Delta}\right\rfloor$, where $\lfloor\cdot\rfloor$ is the floor function. Then for $t\geq T$ we have

$$t=T+\Delta\left(\frac{t-T}{\Delta}-\left\lfloor\frac{t-T}{\Delta}\right\rfloor\right)+N\Delta \tag{3.75}$$

and from (3.66) we have the bound

$$V(t) \leq \gamma_r^N V\left(T+\Delta\left(\frac{t-T}{\Delta}-\left\lfloor\frac{t-T}{\Delta}\right\rfloor\right)\right). \tag{3.76}$$

This bound is obtained by noting from (3.66) and (3.74) that $V(s+N\Delta) \leq \gamma_r^N V(s)$ and by substituting $s = T+\Delta\left(\frac{t-T}{\Delta}-\left\lfloor\frac{t-T}{\Delta}\right\rfloor\right)$. Recalling that

$$\dot{V} = \bar{x}^T\left[A(t)-k\alpha B(t)B^T(t)\right]\bar{x} \leq 2\left|A(t)-k\alpha B(t)B^T(t)\right|V, \tag{3.77}$$

for

$$\Delta\left(\frac{t-T}{\Delta}-\left\lfloor\frac{t-T}{\Delta}\right\rfloor\right) \leq \Delta \tag{3.78}$$

we get the bound

$$V\left(T+\Delta\left(\frac{t-T}{\Delta}-\left\lfloor\frac{t-T}{\Delta}\right\rfloor\right)\right) \leq e^{2\int_0^{T+\Delta}\left|A(\tau)-k\alpha B(\tau)B^T(\tau)\right|d\tau}V(0), \tag{3.79}$$

and therefore

$$V(t) \leq e^{2\int_0^{T+\Delta}\left|A(\tau)-k\alpha B(\tau)B^T(\tau)\right|d\tau}\gamma_r^N V(0). \tag{3.80}$$

We now consider the term γ_r^N. Since $N = \frac{t-T-\Delta\left(\frac{t-T}{\Delta}-\left\lfloor\frac{t-T}{\Delta}\right\rfloor\right)}{\Delta} \geq \frac{t-T-\Delta}{\Delta}$, and $\gamma_r \in (0,1)$ it follows that

$$\gamma_r^N \leq \gamma_r^{\frac{t-T-\Delta}{\Delta}}. \tag{3.81}$$

With (3.80) and (3.81) we obtain

$$V(t) \leq e^{2\int_0^{T+\Delta}\left|A(\tau)-k\alpha B(\tau)B^T(\tau)\right|d\tau}\gamma_r^{-\frac{T+\Delta}{\Delta}}\gamma_r^{\frac{t}{\Delta}}V(0). \tag{3.82}$$

We now define

$$M_0 = \sqrt{e^{2\int_0^{T+\Delta}|A(\tau)-k\alpha B(\tau)B^T(\tau)|d\tau}\gamma_r^{-\frac{T+\Delta}{\Delta}}} \tag{3.83}$$

and rewrite

$$\gamma_r^{\frac{t}{\Delta}} = \left(\frac{1}{\gamma_r}\right)^{\frac{-t}{\Delta}} = e^{-\frac{\ln\left(\frac{1}{\gamma_r}\right)}{\Delta}t}. \tag{3.84}$$

Recalling that $\gamma_r \in (0,1)$ we define

$$R(k\alpha,\Delta,\beta_0,b_\star,a_\star) = \frac{\ln\left(\frac{1}{\gamma_r}\right)}{2\Delta} > 0, \tag{3.85}$$

and write the exponential decay of V as

$$V(t) \leq M_0^2 e^{-2Rt} V(0). \tag{3.86}$$

Substituting (3.74) into (3.85), we obtain (3.35). Finally recalling the definition of $V(t)$ we write the exponential decay of $|\bar{x}(t)|$ as

$$|\bar{x}(t)| \leq M_0 e^{-Rt} |\bar{x}(0)|. \tag{3.87}$$

Therefore, by Corollary 1, the origin of system (3.30), (3.31) is $\frac{1}{\omega}$-SPUAS, which proves the result under condition (i). Proceeding to the proof of the theorem under condition (ii), for any given $a_\star$ we want to find a range of stabilizing values of $k\alpha$ as a function of Δ. For a given β_0, $b_\star$, $a_\star$ we first consider over what range of $\Delta \in (0,\infty)$ it is possible to satisfy the convergence condition (3.68). We define the function

$$F(k\alpha,\Delta) = \frac{k\alpha\Delta\beta_0}{1+2k^2\alpha^2\Delta^2 b_\star^2} + e^{-\left(2\Delta\sqrt{a_\star}+\frac{4k\alpha\Delta^3 b_\star a_\star}{1+2k^2\alpha^2\Delta^2 b_\star^2}\right)} \tag{3.88}$$

which must achieve a value larger than 1 for (3.68) to be satisfied. In order to consider the maximum possible value of (3.88) we first fix Δ and set the derivative, with respect to $k\alpha$, of $F(k\alpha,\Delta)$ equal to zero, to find that $F(k\alpha,\Delta)$ has its maximum value at

$$(k\alpha)_{\mathrm{m}} = \frac{1}{\sqrt{2}\Delta b_\star} \tag{3.89}$$

and the maximum value is

$$F\left((k\alpha)_{\mathrm{m}},\Delta\right) = \frac{\beta_0}{2\sqrt{2}b_\star} + e^{-\left(2\Delta\sqrt{a_\star}+\sqrt{2}\Delta^2 a_\star\right)}. \tag{3.90}$$

The convergence condition requires this maximum value (3.90) to be greater than 1. We note that $F\left((k\alpha)_{\mathrm{m}},\Delta\right)$ is strictly decreasing as a function of $\Delta \in (0,\infty)$. Therefore if $F\left((k\alpha)_{\mathrm{m}},\Delta^\star\right) = 1$, it follows that $F\left((k\alpha)_{\mathrm{m}},\Delta\right) > 1$ for all $\Delta \in (0,\Delta^\star)$. The condition $F\left((k\alpha)_{\mathrm{m}},\Delta^\star\right) = 1$ implies that

$$2\Delta\sqrt{a_\star} + \sqrt{2}\Delta^2 a_\star - \ln\left(\frac{2\sqrt{2}b_\star}{2\sqrt{2}b_\star - \beta_0}\right) = 0 \tag{3.91}$$

from which we obtain the positive root

$$\Delta^\star = \frac{-1+\sqrt{1+\sqrt{2}\ln\left(\frac{2\sqrt{2}b_\star}{2\sqrt{2}b_\star-\beta_0}\right)}}{\sqrt{2a_\star}}. \tag{3.92}$$

Therefore it is possible to stabilize the system when $0<\Delta<\Delta^{\star}$ by choosing $k\alpha=(k\alpha)_{\mathrm{m}}$ as in (3.89). By continuity, for any $0<\Delta<\Delta^{\star}$ there must be an interval containing $(k\alpha)_{\mathrm{m}}$ such that all values of $k\alpha$ within that interval satisfy condition (3.68). For $\Delta\in(0,\Delta^{\star})$ we consider all values of $k\alpha$ that achieve $F(k\alpha,\Delta)>1$. Recalling the definition of $F(k\alpha,\Delta)$,

$$F(k\alpha,\Delta)=\frac{k\alpha\Delta\beta_0}{1+2k^2\alpha^2\Delta^2b_{\star}^2}+e^{-\left(2\Delta\sqrt{a_{\star}}+\frac{4k\alpha\Delta^3b_{\star}a_{\star}}{1+2k^2\alpha^2\Delta^2b_{\star}^2}\right)}, \tag{3.93}$$

to remove the $k\alpha$ dependence from the exponential in (3.93) we restrict our attention to $k\alpha>1$, in which case

$$e^{-\left(2\Delta\sqrt{a_{\star}}+\frac{4k\alpha\Delta^3b_{\star}a_{\star}}{1+2k^2\alpha^2\Delta^2b_{\star}^2}\right)}>e^{-2\Delta\left(\sqrt{a_{\star}}+\frac{a_{\star}}{b_{\star}}\right)}. \tag{3.94}$$

We satisfy (3.93) by restricting $k\alpha$ to satisfy

$$\frac{k\alpha\Delta\beta_0}{1+2k^2\alpha^2\Delta^2b_{\star}^2}+e^{-2\Delta\left(\sqrt{a_{\star}}+\frac{a_{\star}}{b_{\star}}\right)}>1. \tag{3.95}$$

Setting (3.95) equal to 1, we solve for $k\alpha$ as

$$k\alpha=\frac{\beta_0\pm\sqrt{\beta_0^2-8b_{\star}^2\left[1-e^{-2\Delta\left(\sqrt{a_{\star}}+\frac{a_{\star}}{b_{\star}}\right)}\right]^2}}{4\Delta b_{\star}^2\left[1-e^{-2\Delta\left(\sqrt{a_{\star}}+\frac{a_{\star}}{b_{\star}}\right)}\right]}. \tag{3.96}$$

To ensure $k\alpha$ is real valued we impose the condition

$$\beta_0^2\geq 8b_{\star}^2\left[1-e^{-2\Delta\left(\sqrt{a_{\star}}+\frac{a_{\star}}{b_{\star}}\right)}\right]^2 \tag{3.97}$$

which implies

$$e^{-2\Delta\left(\sqrt{a_{\star}}+\frac{a_{\star}}{b_{\star}}\right)}\geq 1-\frac{\beta_0}{2\sqrt{2}b_{\star}}. \tag{3.98}$$

Taking ln of each side of (3.98) we obtain the condition

$$-2\Delta\left(\sqrt{a_{\star}}+\frac{a_{\star}}{b_{\star}}\right)>\ln\left(1-\frac{\beta_0}{2\sqrt{2}b_{\star}}\right) \tag{3.99}$$

which implies that the new requirement on the possible values of Δ is

$$0<\Delta<\bar{\Delta}=\min\left\{\frac{\ln\left(\frac{1}{\sqrt{1-\frac{\beta_0}{2\sqrt{2}b_{\star}}}}\right)}{\left(\sqrt{a_{\star}}+\frac{a_{\star}}{b_{\star}}\right)},\ \Delta^{\star}\right\}. \tag{3.100}$$

With (3.92) and (3.100) we obtain (3.39). Returning to (3.96) and recalling the value $(k\alpha)_{\mathrm{m}} = \frac{1}{\sqrt{2}\Delta b_\star}$ we have the roots

$$k\alpha = \frac{(k\alpha)_{\mathrm{m}}}{\eta},\ k\alpha = (k\alpha)_{\mathrm{m}}\eta, \tag{3.101}$$

where

$$\eta = \frac{\beta_0 + \sqrt{\beta_0^2 - 8b_\star^2\left[1 - e^{-2\Delta\left(\sqrt{a_\star}+\frac{a_\star}{b_\star}\right)}\right]^2}}{2\sqrt{2}b_\star\left[1 - e^{-2\Delta\left(\sqrt{a_\star}+\frac{a_\star}{b_\star}\right)}\right]}. \tag{3.102}$$

Therefore the system is stable for

$$k\alpha \in \left(\frac{(k\alpha)_{\mathrm{m}}}{\eta},\ (k\alpha)_{\mathrm{m}}\eta\right). \tag{3.103}$$

We have thus derived sufficient conditions on Δ and $k\alpha$ to guarantee stability of our system. For each window Δ we have given an interval of stabilizing values of $k\alpha$, (5.66). However we now proceed to restrict our conditions on $k\alpha$ in order to give a more intuitive condition (3.42). We show that the interval (5.66) contains $(k\alpha)_{\mathrm{m}}$ by recalling (3.97) and calculating

$$\eta \geq 1 + \frac{\sqrt{\beta_0^2 - 8b_\star^2\left[1 - e^{-2\Delta\left(\sqrt{a_\star}+\frac{a_\star}{b_\star}\right)}\right]^2}}{2\sqrt{2}b_\star\left[1 - e^{-2\Delta\left(\sqrt{a_\star}+\frac{a_\star}{b_\star}\right)}\right]} \tag{3.104}$$

and

$$\frac{1}{\eta} \leq \frac{2\sqrt{2}b_\star\left[1 - e^{-2\Delta\left(\sqrt{a_\star}+\frac{a_\star}{b_\star}\right)}\right]}{\beta_0} < 1. \tag{3.105}$$

Therefore the interval (5.66) contains the more restrictive, but more illustrative interval (3.42), where we have explicitly written out the value $(k\alpha)_{\mathrm{m}} = \frac{1}{\sqrt{2}\Delta b_\star}$. From the presence of Δ in the denominator we see that this interval of stability grows unbounded in length as the window Δ decreases. □

Remark 4. *We recall from (3.39) that Δ must not exceed $\bar{\Delta}_1$. By recalling that $b_\star \geq \beta_0$, by using the fact that* $\ln\left(\frac{2\sqrt{2}}{2\sqrt{2}-1}\right) < \frac{1}{\sqrt{2}}$ *and by noting that*

$$\frac{\ln\left(\frac{2\sqrt{2}b_\star}{2\sqrt{2}b_\star - \beta_0}\right)}{2\left(\sqrt{a_\star}+\frac{a_\star}{b_\star}\right)} < \frac{\ln\left(\frac{2\sqrt{2}b_\star}{2\sqrt{2}b_\star - b_\star}\right)}{2\sqrt{a_\star}} = \frac{\ln\left(\frac{2\sqrt{2}}{2\sqrt{2}-1}\right)}{2\sqrt{a_\star}} < \frac{1}{2\sqrt{2a_\star}}, \tag{3.106}$$

we get that $\bar{\Delta}_1 < \frac{1}{2\sqrt{2a_\star}}$. *Hence, the stabilizing values of* $k\alpha$ *in the interval (3.42) must satisfy*

$$k\alpha > \frac{1}{2\sqrt{2}\Delta b_\star} > \frac{1}{2\sqrt{2}\bar{\Delta}_1 b_\star} > \frac{\sqrt{a_\star}}{b_\star}. \tag{3.107}$$

The condition (3.107) is very similar to the stability requirement that is established in the one-dimensional case, in Proposition 1. As $a_\star$ *increases, stability is ensured by increasing* $k\alpha$.

3.3.1 Vehicle Control 2-Dimensional Simulation Example

To demonstrate the extremum seeking controller's ability to handle unknown, quickly time varying control direction we consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2.1 & 4.9 \\ -7.5 & 3.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \cos(10t+.3) \\ \sin(10t+.3) \end{bmatrix} u\,. \tag{3.108}$$

A physical motivation for this example can be that $x = (x_1, x_2)$ is the planar coordinate of a mobile robot, with its angular velocity actuator failed and stuck at 10, and which has to be stabilized to the origin using the forward velocity input u only, in the presence of a position-dependent perturbation given by $\begin{bmatrix} 2.1 & 4.9 \\ -7.5 & 3.6 \end{bmatrix} x$. The uncontrolled system is unstable with poles at $2.85 \pm 10.7i$. We apply ES control

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)\left[x_1^2(t) + x_2^2(t)\right] \tag{3.109}$$

With $\omega = 100, k = 4, \alpha = 2$ and starting from $x_1(0) = 1, x_2(0) = -1$, Figure 3.4 shows the system's time response. Figure 3.5 shows a parametric view and a zoom in on the last second of the dynamics.

3.4 Linear Systems in Strict-Feedback Form

In this section we consider linear systems in strict-feedback form and design a controller based on the backstepping approach [41, 81].

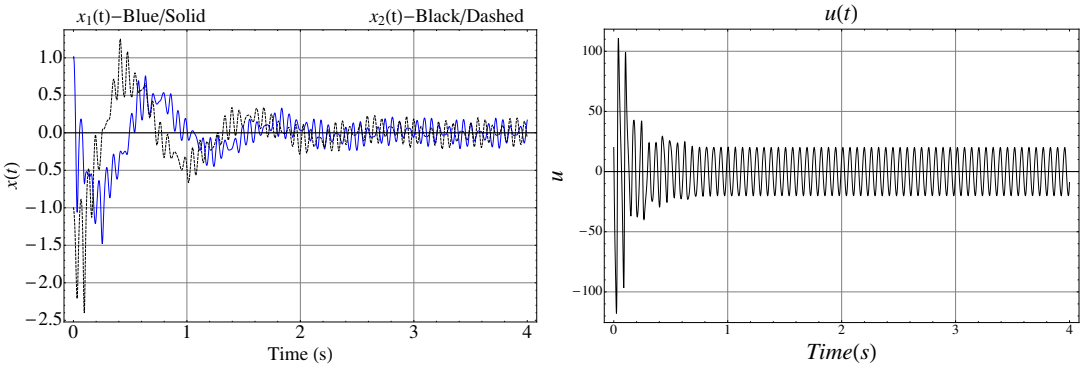


Figure 3.4: After a transient which, in the average sense, is underdamped, the solution of (3.108)–(3.109) settles to an $O\left(\frac{\alpha}{\sqrt{\omega}}\right)$ neighborhood of the origin.

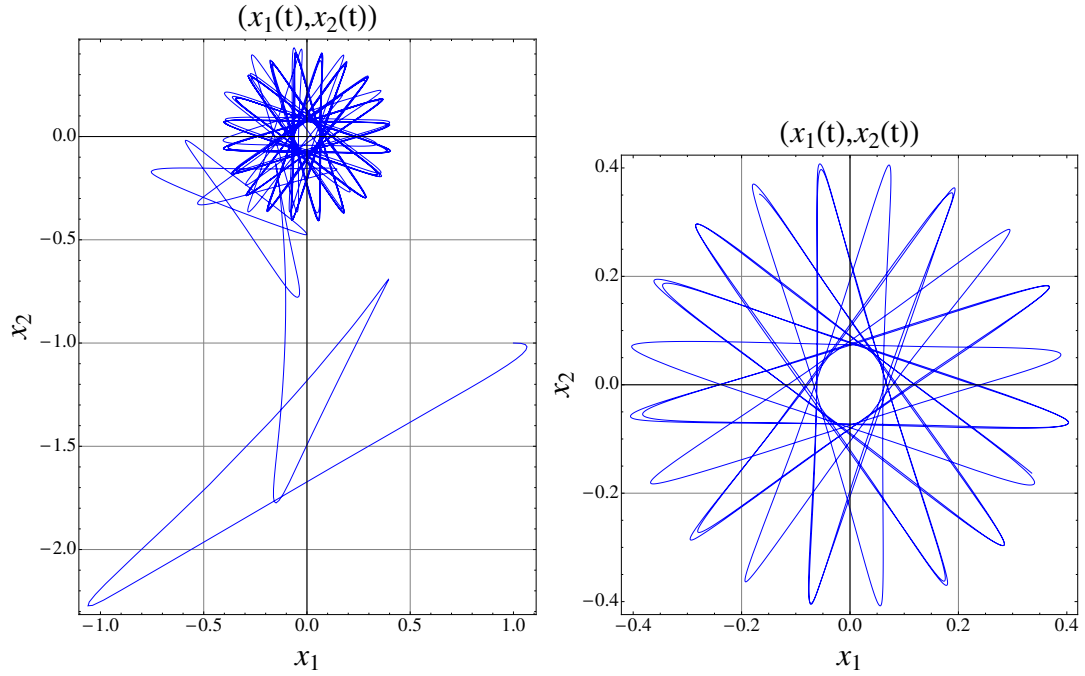


Figure 3.5: After a transient which, in the average sense, is underdamped, the solution of (3.108)–(3.109) settles to an $O\left(\frac{\alpha}{\sqrt{\omega}}\right)$ neighborhood of the origin. The figure on the right shows the last second of dynamics.

Theorem 4. *Consider the plant*

$$\dot{x}_i = \sum_{j=1}^{i} a_{ij}(t)x_j + x_{i+1},\ 1 \le i \le n-1 \tag{3.110}$$

$$\dot{x}_n = \sum_{j=1}^{n} a_{nj}(t)x_j + b(t)u, \tag{3.111}$$

with the control law

$$u=\alpha\sqrt{\omega}\cos(\omega t)-k\sqrt{\omega}\sin(\omega t)\left(\sum_{i=1}^{n-1}\left(\prod_{j=i}^{n-1}c_j\right)x_i+x_n\right)^2, \tag{3.112}$$

and let $\beta_0>0$ *and* $a_{\max}>0$ *be known such that for some* $T>0$ *and* $\Delta>0$*, for all* $s\geq T$*,*

$$\frac{1}{\Delta}\int_s^{s+\Delta}b^2(\tau)d\tau>\beta_0 \tag{3.113}$$

$$\frac{1}{\Delta}\int_s^{s+\Delta}|a_{ij}(\tau)|d\tau\leq a_{\max}, \qquad \forall i,j. \tag{3.114}$$

If $c_1,c_2,\ldots,c_n$ *are chosen recursively so that*

$$c_i>a_{\max}+\left\{C_{1i},\max_{2\leq j\leq i-2}C_{2ij},c_{i-1}\right\},1\leq i\leq n-1, \tag{3.115}$$

where $c_0=0$ *and*

$$C_{1i}=\frac{(n-1)^2(1+\bar{d}_{i,i-1})^2}{4\bar{d}_{i-1,i-1}}, \tag{3.116}$$

$$C_{2ij}=\frac{(n-1)^2\bar{d}_{ij}^2}{4\bar{d}_{jj}}, \tag{3.117}$$

and

$$\bar{d}_{ij}=a_{\max}+a_{\max}c_j+c_{i-1}\bar{d}_{i-1,j}, \quad 1\leq i\leq n, \quad 1\leq j\leq i-2 \tag{3.118}$$

$$\bar{d}_{ii}=c_i-c_{i-1}+a_{\max}, \quad 1\leq i\leq n-1, \tag{3.119}$$

$$\bar{d}_{nn}=b^2k\alpha-c_{n-1}+a_{\max}, \tag{3.120}$$

then if

$$k\alpha>\frac{c_{n-1}+a_{\max}}{\beta_0}, \tag{3.121}$$

the origin of system (3.110)-(3.112) is $\frac{1}{\omega}$*-SPUAS.*

Proof. We define

$$z_i=x_i+\sum_{k=1}^{i-1}\left(\prod_{j=k}^{i-1}c_j\right)x_k, 1\leq i\leq n \tag{3.122}$$

and rewrite the controller (3.112) as

$$u=\alpha\sqrt{\omega}\cos(\omega t)-k\sqrt{\omega}\sin(\omega t)z_n^2. \tag{3.123}$$

We get the Lie bracket averaged system (3.110)-(3.112) as

$$\dot{\bar{z}}=D\bar{z}, \tag{3.124}$$

where

$$D=\begin{pmatrix} -d_{11} & 1 & 0 & \dots & 0 & 0 \\ d_{21} & -d_{22} & 1 & \dots & 0 & 0 \\ d_{31} & d_{32} & -d_{33} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{n-1,1} & d_{n-1,2} & d_{n-1,3} & \dots & -d_{n-1,n-1} & 1 \\ d_{n1} & d_{n2} & d_{n3} & \dots & d_{n,n-1} & -d_{nn} \end{pmatrix}, \tag{3.125}$$

with the diagonal terms of (3.125) satisfying

$$\begin{aligned} d_{ii}&=c_i-c_{i-1}-a_{ii},\ 1\le i\le n-1 \\ d_{nn}&=b^2k\alpha-c_{n-1}-a_{nn}. \end{aligned}$$

The off-diagonal terms are defined as

$$d_{ij}=a_{ij}-a_{i,j+1}c_j+c_{i-1}d_{i-1,j},\quad 1\le i\le n,\quad 1\le j\le i-2.$$

Considering the Lyapunov function

$$V=\frac{1}{2}\sum_{i=1}^{n}\bar{z}_i^2, \tag{3.126}$$

we get

$$\dot{V}=-\sum_{i=1}^{n}d_{ii}\bar{z}_i^2+\sum_{i=2}^{n}(1+d_{i,i-1})\bar{z}_i\bar{z}_{i-1}+\sum_{i=3}^{n}\sum_{j=1}^{i-2}d_{ij}\bar{z}_i\bar{z}_j$$

which we rewrite as

$$\dot{V}=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[-\frac{d_{ii}}{n-1}\bar{z}_i^2+(1+d_{ji})\bar{z}_i\bar{z}_j-\frac{d_{jj}}{n-1}\bar{z}_j^2\right] \tag{3.127}$$

Note that $d_{ii} > 0 \; \forall i$ for c_i and $k\alpha$ that satisfy

$$c_i > c_{i-1} + a_{\max}, \; 1 \leq i \leq n-1, c_0 = 0. \tag{3.128}$$

We now rewrite (3.127) as

$$\dot{V} = \frac{-2}{n-1} \sum_{i=1}^{n-1} d_{ii} \sum_{j=i+1}^{n} \begin{bmatrix} z_i \\ z_j \end{bmatrix}^T \hat{D}_{ij} \begin{bmatrix} z_i \\ z_j \end{bmatrix}, \tag{3.129}$$

where

$$\hat{D}_{ij} = \frac{1}{2} \begin{bmatrix} 1 & \frac{(n-1)(1+d_{ij})}{d_{ii}} \\ \frac{(n-1)(1+d_{ij})}{d_{ii}} & \frac{d_{jj}}{d_{ii}} \end{bmatrix}. \tag{3.130}$$

To ensure that $\dot{V} < 0$, the matrices $\hat{D}_{ij} \neq D_{nn}$ are made positive definite by choosing

$$\sqrt{\frac{d_{ii}}{d_{i-1,i-1}}} > \frac{(n-1)(1+d_{i,i-1})}{2d_{i-1,i-1}}, 2 \leq i \leq n \tag{3.131}$$

and

$$\sqrt{\frac{d_{ii}}{d_{jj}}} > \frac{(n-1)d_{ij}}{2d_{jj}}, 3 \leq i \leq n, 2 \leq j \leq i-2, \tag{3.132}$$

which is accomplished by choosing c_i such that

$$c_i = a_{ii} + d_{ii} > a_{ii} + \frac{(n-1)^2(1+d_{i,i-1})^2}{4d_{i-1,i-1}}, \quad 2 \leq i \leq n$$
$$c_i = a_{ii} + d_{ii} > a_{ii} + \frac{(n-1)^2 d_{ij}^2}{4d_{jj}}, \quad 3 \leq i \leq n, \quad 2 \leq j \leq i-2.$$

Finally, by choosing

$$k\alpha > \frac{c_{n-1} + a_{\max}}{\beta_0} \tag{3.133}$$

we ensure that $\int_s^{s+\Delta} D_{nn}(\tau) d\tau < 0$, and proceeding as in the proof of Proposition 1, we ensure that $V(s+\Delta) < V(s)$ for all $s \geq T$, and as in Theorem 1 we guarantee that the origin is an exponentially stable equilibrium point of system (3.124). Therefore by Corollary 1, the origin of system (3.110)-(3.112) is $\frac{1}{\omega}$-SPUAS. □

A closer examination of the control law (3.112) and the clf (3.126), along with (3.122), shows that the control law is not exactly in the forms (3.4) and (3.8). The terms $z_1^2, \ldots, z_{n-1}^2$ are omitted because $L_g V = L_g \frac{z_n^2}{2} = z_n$.

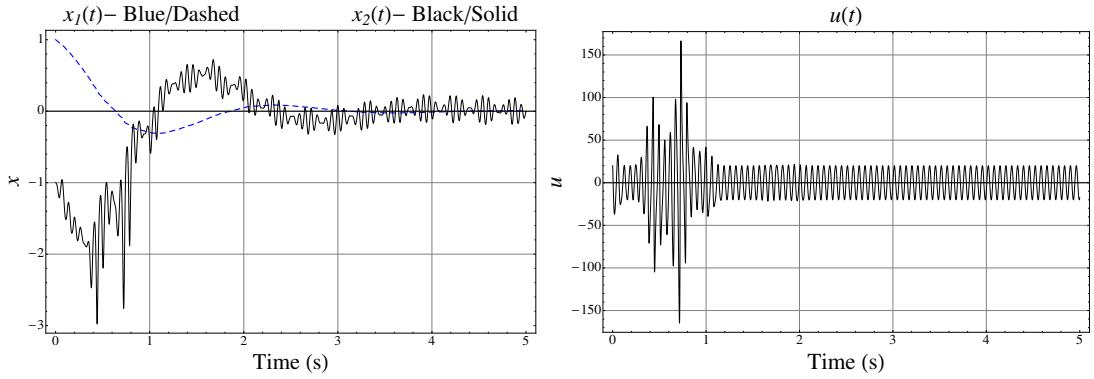


Figure 3.6: Although the sign of the applied force is unknown to the controller the position x_1 and velocity x_2 of system (3.134)–(3.135) quickly settle to $O\left(\frac{1}{\sqrt{\omega}}\right)$ neighborhoods of the origin.

3.4.1 Unknown Force Direction Simulation Example

Consider controlling the position and velocity of an object experiencing destabilizing forces proportional to its velocity and its distance from the origin, by applying a force u whose direction $b(t)$ is unknown. The dynamics are governed by Newton's law,

$$F_{\text{total}} = ma = m\ddot{x} = k_x x + k_v \dot{x} + b\sin(10t)u,$$

which may be written in strict-feedback form

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \frac{k_x}{m}x_1 + \frac{k_v}{m}x_2 + \frac{b}{m}\sin(10t)u. \tag{3.134}$$

We implement the feedback controller

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)\left(2x_1 + x_2\right)^2. \tag{3.135}$$

For the case $k_x = 1$, $k_v = 2$, $m = 1$, and $b = 1$, and with controller parameters $k = 4$, $\alpha = 2$, and $\omega = 100$, the simulation, with initial condition $x_1(0) = 1, x_2(0) = -1$, is shown in Figure 3.6.

3.5 Nonlinear MIMO Systems with Matched Uncertainties

While in Section 3.1 we presented a general approach for nonlinear systems based on an assumed availability of a clf V that satisfies the strong L_gV-stabilizability condition, in this section we turn our attention to a specific construction of such a clf for a limited but relevant class of systems that illustrates how to overcome the challenge of dealing with unknown nonlinearities.

In this section we study multi-input systems with the same number of controls and states. Admittedly, this is a class of "glorified first-order systems." However, we use this class to illustrate clearly how to deal with nonlinearities that are not only unknown but also have arbitrary growth (super-linear, exponential, or even faster than exponential). For systems with more states than controls, such as nth order systems in the strict-feedback form with one control and with only bounds on nonlinearities known, clfs satisfying Assumption 1 can be constructed using the approach introduced in [52, see Theorem 3.1, with (26) and (27) being the key steps], which we have actually used for linear strict-feedback systems in Section 3.4.

We consider only time-invariant nonlinear systems in this section. Time-varying systems, albeit linear, have already been dealt with in Section 3.3. The nonlinear systems studied in this section can be approached similarly but, for the sake of clarity, we choose not to pursue time-varying extensions here. Since the systems we consider here have the same number of controls and states, the input matrix is square. Given that the input matrix is not time-varying and thus persistency of excitation cannot be exploited in stabilization, we make an assumption that the input matrix multiplied by its transpose is positive definite for all x, which means that the system is completely controllable, though its control directions are unknown. Furthermore, the non-zero assumption on the input matrix $G(x)$ is motivated by the possible finite escape time of general nonlinear systems.

Theorem 5. *Consider the system*

$$\dot{x}=f(x)+G(x)u, \tag{3.136}$$

where $u,\ x\in\mathbb{R}^n$, *and* $f:\mathbb{R}^n\to\mathbb{R}^n$, $G:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ *and let there exist* $\beta_0>0$, *and*

$\eta \in \mathscr{K}_\infty$ such that $f(x)$ and $G(x)$ satisfy the following bounds for all $x \in \mathbb{R}^n$:

$$G(x)G^T(x) \geq \beta_0 I, \tag{3.137}$$

$$|f(x)| \leq \eta(|x|). \tag{3.138}$$

If k and α are chosen such that

$$k\alpha > \frac{1}{\beta_0} \tag{3.139}$$

then the controller

$$u_i = \alpha\sqrt{\omega\omega_i'}\cos(\omega\omega_i' t) - k\sqrt{\omega\omega_i'}\sin(\omega\omega_i' t)V(x), \tag{3.140}$$

where

$$V(x) = \int_0^{|x|} \eta(r)dr \tag{3.141}$$

and the frequencies ω_i' are distinct, renders the origin of (3.136), (3.140) $\frac{1}{\omega}$-SPUAS.

Proof. A common period for all of the controller components is given by $T = 2\pi\text{LCM}\left\{\frac{1}{\omega_i'}\right\}$. Therefore

$$\int_0^T \cos(\omega\omega_i' t)\cos(\omega\omega_j' t)dt = \int_0^T \sin(\omega\omega_i' t)\sin(\omega\omega_j' t)dt$$
$$= \int_0^T \sin(\omega\omega_i' t)\cos(\omega\omega_j' t)dt = 0, \ \ \forall\, i \neq j. \tag{3.142}$$

Consider the closed loop system

$$\dot{x} = f(x) + \sqrt{\omega}\sum_{i=1}^{n}\left[\alpha G(x)e_i\sqrt{\omega_i'}\cos(\omega_i'\theta) - kG(x)e_iV(x)\sqrt{\omega_i'}\sin(\omega_i'\theta)\right], \quad \theta = \omega t. \tag{3.143}$$

System (3.143) is in the form of system (2.1) to which we can apply Lie bracket averaging. Considering property (3.142), terms of different frequency combinations integrate to zero. Therefore the Lie bracket terms we are left with are

$$[G(\bar{x})e_i, G(\bar{x})e_iV(\bar{x})] = G(\bar{x})e_ie_i^TG^T(\bar{x})\left(\frac{\partial V(\bar{x})}{\partial \bar{x}}\right)^T. \tag{3.144}$$

Combining all terms of the form (3.144) we get

$$\sum_{i=1}^{n} Ge_ie_i^TG^T\left(\frac{\partial V}{\partial \bar{x}}\right)^T = GG^T\left(\frac{\partial V}{\partial \bar{x}}\right)^T, \tag{3.145}$$

resulting in the Lie bracket averaged system

$$\dot{\bar{x}} = f(\bar{x}) - \frac{k\alpha}{2} G(\bar{x}) G^T(\bar{x}) \eta(|\bar{x}|) \frac{\bar{x}}{|\bar{x}|}, \tag{3.146}$$

where we have used the fact that

$$\frac{\partial V(\bar{x})}{\partial \bar{x}} = \eta(|\bar{x}|) \frac{\bar{x}^T}{|\bar{x}|}. \tag{3.147}$$

With another Lyapunov function candidate

$$W(\bar{x}) = \frac{|\bar{x}|^2}{2}, \tag{3.148}$$

we get

$$\dot{W}(\bar{x}) = \bar{x}^T \dot{\bar{x}} = \bar{x}^T f(\bar{x}) - k\alpha \frac{\eta(|\bar{x}|)}{|\bar{x}|} \bar{x}^T G(\bar{x}) G^T(\bar{x}) \bar{x}. \tag{3.149}$$

From (3.138) we have

$$\left|\bar{x}^T f\right| \leq |\bar{x}|\,|f| \leq |\bar{x}| \eta(|\bar{x}|) \tag{3.150}$$

and from (3.137) we have that

$$k\alpha \frac{\eta(|\bar{x}|)}{|\bar{x}|} \bar{x}^T G(\bar{x}) G^T(\bar{x}) \bar{x} \geq k\alpha \frac{\eta(|\bar{x}|)}{|\bar{x}|} \beta_0 |\bar{x}|^2. \tag{3.151}$$

Plugging (3.150) and (3.151) into the equation for $\dot{W}(\bar{x})$ we get

$$\begin{aligned} \dot{W}(\bar{x}) \leq & |\bar{x}| \eta(|\bar{x}|) - k\alpha\beta_0 |\bar{x}| \eta(|\bar{x}|) \\ = & (1 - k\alpha\beta_0) |\bar{x}| \eta(|\bar{x}|), \end{aligned} \tag{3.152}$$

therefore by our choice of $k\alpha > \frac{1}{\beta_0}$, we guarantee that (3.152) is negative definite and therefore the Lie bracket averaged system (3.146) is globally uniformly asymptotically stable. By Corollary 1, system (3.136) is $\frac{1}{\omega}$-SPUAS. □

Remark 5. *Condition (3.137) can be relaxed to a functional lower bound $G(x)G^T(x) \geq \beta(|x|)I$ for some $\beta \in \mathscr{K}$. Then, for the average system, the Lyapunov inequality (3.152) is replaced by $\dot{W}(\bar{x}) \leq (1 - k\alpha\beta(|\bar{x}|))|\bar{x}|\eta(|\bar{x}|)$, which guarantees that, for $k\alpha > 1/\beta(\infty)$, the averaged system is globally ultimately bounded (GUUB) with an ultimate bound $\beta^{-1}\left(\frac{1}{k\alpha}\right)$. Though Theorem 6 only allows us to relate global asymptotic stability (GUAS) of the averaged system with $\frac{1}{\omega}$-SPUAS stability of the actual system, a similar*

relationship can be established between GUUB and what we refer to as $\frac{1}{\omega}$-Semiglobal Practical Uniform Ultimate Boundedness ($\frac{1}{\omega}$-SPUUB) of a system. The $\frac{1}{\omega}$-SPUUB property and its applications in tracking for unknown systems are presented in later chapters.

3.5.1 Nonlinear Simulation Example

We demonstrate the controller's ability to stabilize nonlinear systems with the following example:

$$\dot{x}=f(x)+\left(1-\frac{1}{2}\sin(x)\right)u\,,\qquad f(x)=x^2\,. \tag{3.153}$$

Assuming that we know that the nonlinearity $f(x)$ is polynomial, we know that $f(x)$ satisfies a bound of the form

$$|f(x)|<\gamma|x|e^{|x|}\,. \tag{3.154}$$

For $f(x)=x^2$, $\gamma=1$. Assuming γ to be known, and noting that $\int re^r dr=(r-1)e^r$, we choose the controller

$$u=\alpha\sqrt{\omega}\cos(\omega t)-k\sqrt{\omega}\sin(\omega t)\left[1+(|x|-1)e^{|x|}\right]\,. \tag{3.155}$$

With $k=7.5,\ \alpha=0.25$ and $\omega=70$, simulation results starting from $x(0)=2$ are shown in Figure 3.7.

This chapter contains material from
A. Scheinker and M. Krstić, "Maximum-seeking for CLFs: Universal semiglobally stabilizing feedback under unknown control directions," *IEEE Transactions on Automatic Control*, to appear.

The dissertation author was the primary author in this publication.

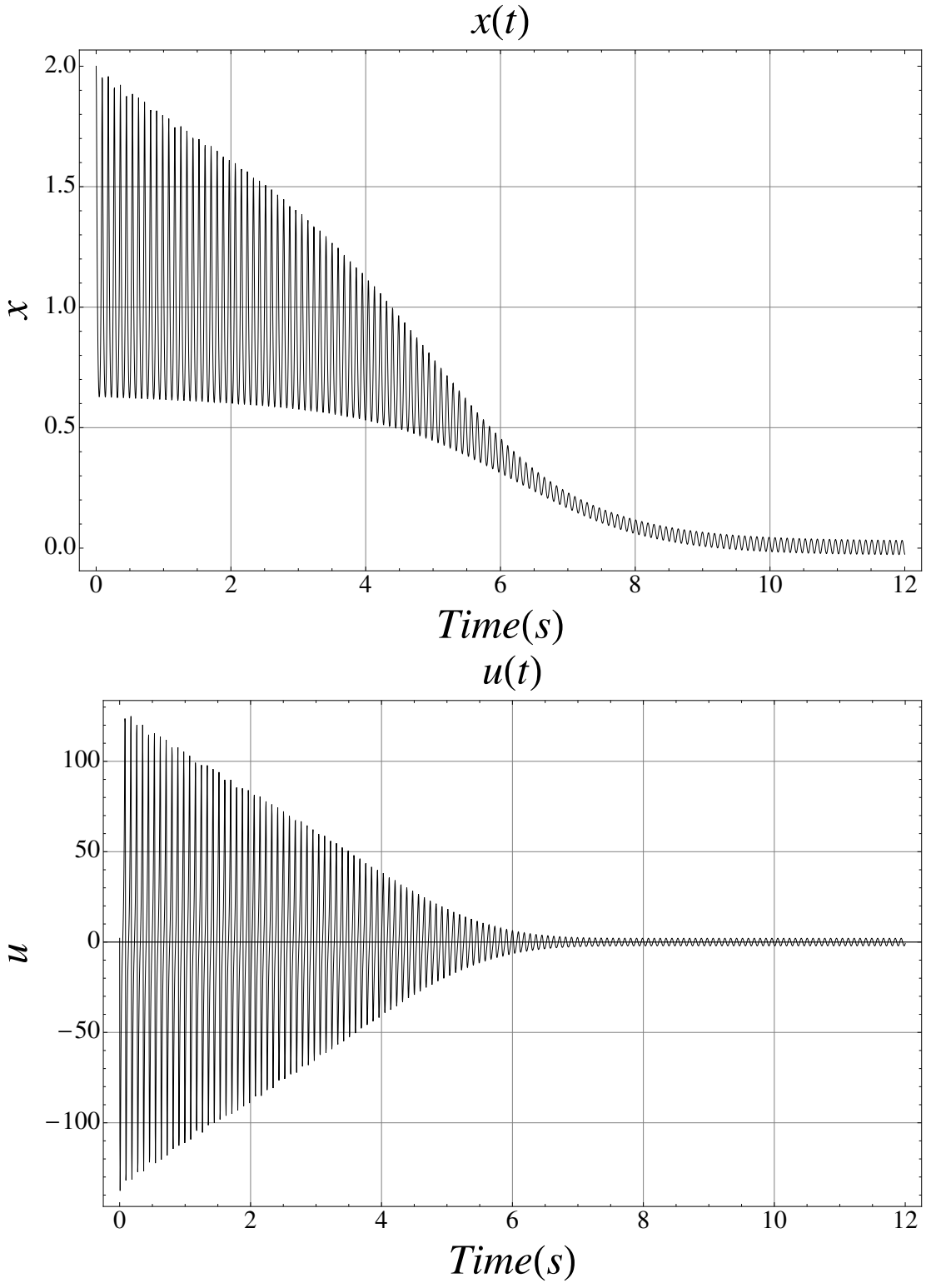


Figure 3.7: As system (3.153) settles to within a $O\left(\frac{1}{\sqrt{\omega}}\right)$ neighborhood of the origin. The control effort, initially large, settles to a steady state magnitude of $\alpha\sqrt{\omega}$.

Chapter 4

Iterative Application of Nonlinear MIMO ES for HVCM Voltage Output Optimization

In this chapter we demonstrate a real world application of the proposed control algorithm by utilizing it in hardware in order to optimize the output voltage waveform of a high voltage converter modulator (HVCM). In Section 4.1 we introduce HVCMs and their importance for the operation of modern high energy particle accelerators. In Section 4.2 we give a brief overview of the complex operation dynamics of an HVCM. In Section 4.3 we describe our iterative ES approach and in Section 4.4 we present the experimental results.

4.1 Introduction

Particle accelerators are essential for research in a wide range of fields, including everything from fundamental high-energy physics to the study of material properties and chemical reactions. As the required output particle energy of accelerators has increased to a range of MeV to TeV, modern particle accelerating structures have been able to limit their length to a range of one to tens of kilometers by operating with accelerating electric fields whose voltage gradients are on the MV/m scale. Because such high gradients are

not possible in DC systems, in which breakdown and arcing will destroy the fields and damage the accelerating structures, the use of resonant cavities, whose standing electromagnetic fields typically resonate at hundreds of megahertz, is required. To provide the energy needed to excite such accelerating cavities there is a requirement for high power radio frequency (RF) sources.

Modern particle accelerators utilize Klystrons, high frequency oscillators/amplifiers, as their RF sources [82]. The RF output of a Klystron must meet very stringent repeatability and stability constraints so that the cavities they power accelerate all particles by very precise amounts. Particles traveling too fast or too slow no longer match the very energy-dependent design of the accelerating structure. The Klystrons must therefore have very stable, very repeatable high voltage sources, such as High Voltage Converter Modulators (HVCM), which convert the energy stored in capacitor banks to large, flat voltage pulses. The growing popularity of HVCMs is due to the significant performance advantages that they offer over conventional modulator technologies for long pulse applications [9, 65]. These include high efficiency, low stored energy, and small size and cost. HVCMs are capable of providing extremely large output voltages (100kV) in extremely short periods of time (hundreds of μs), with very repeatable and stable, "flat-top" output waveforms ($< 1\%$ error).

While the steady state operation of HVCMs is well understood and there exist analytic results for flat top operation, including droop compensation [9, 10] the dynamics of the voltage rise time are complicated. Even if HVCM output voltage rise could be well described analytically, being a function of up to 24 parameters, its optimization (fast rise time, no overshoot) is a challenging task. For example, at ORNL, the HVCM pulses are "gently" initialized with one phase activated at a time, resulting in acceptable overshoot levels and a rise time of approximately 100μs. Therefore, in the experiment which is the subject of this paper, extremum seeking was utilized for output voltage optimization in the sense of decreasing rise time, while preventing dangerous overshoot.

We attempt to optimize the output voltage rise-time of the HVCM at Los Alamos National Laboratory to 50μs by three methods. In all methods we choose (experimentally found) very bad initial parameter settings, so that the system experiences unac-

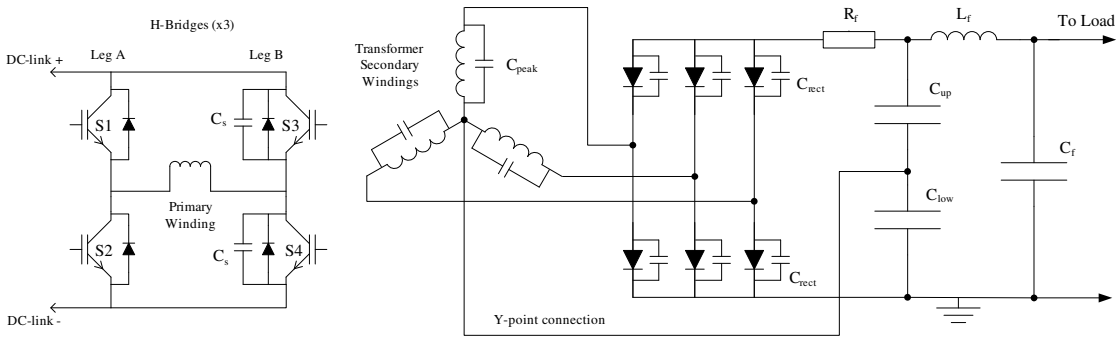


Figure 4.1: Simplified schematic of a typical HVCM. The primary windings of the three H-Bridges, shown on the left, are connected to the secondary windings in a 'Y' shown on the right.

ceptable overshoot, not settling to acceptable operating conditions until approximately 200μs into the pulse.

In the first approach we iteratively tune the first 6 switching edges of each of the three phase drive waveforms (18 variables total) of the HVCM by a simple update law based on a noisy cost function which is calculated from sampled raw data of the HVCM's output voltage. Despite the presence of random noise in the cost, because the noise is independent from the perturbing frequencies of the ES algorithm, the output voltage is brought within 2% error of the desired output level in approximately 10 minutes of run time.

In the second approach we again iteratively tune the first 6 switching edges of each of the three phase drive waveforms of the HVCM, but use an average of 5 output voltage waveforms which are created by firing the HVCM 5 times with fixed input parameters to calculate a smoother cost function. While averaging reduces random noise in the cost function it slows down the iterative process. With the cleaner cost function output voltage is brought within approximately 1.5% error of the desired output level, in approximately 30 minutes of run time.

In the third approach we increase the number of tuned parameters from the first 6 to the first 8 switching edges of each of the three phase drive waveforms (24 variables total) of the HVCM, and also increase averaging from 5 to 10 output voltage waveforms which are created by firing the HVCM 10 times with fixed input parameters to calculate

an even smoother cost function. Although the averaging used to reduce random noise in the cost function slows down the iterative process, an hour long tune up is more than acceptable because, once the input parameters have been optimized, they are set and maintained at the optimal settings. With the cleaner cost function the output voltage is successfully brought within 1% error of the desired output level, in approximately 60 minutes of run time.

The ES algorithm is digitally implemented in a Texas Instruments TMS320 F283335 Pulse Wave Modulator. Compared to the currently achievable rise time of 100μs for the HVCMs at the Spallation Neutron Source at Oak Ridge National Laboratory the improvement achieved is a $2\times$ reduction. Considering that HVCMs run at $100000kV$ with 60Hz repetition rates, the 50μs rise time reduction will result in very significant energy savings.

4.2 Background on High Voltage Converter Modulator Operation

The first generation of HVCMs were developed at Los Alamos National Laboratory for the Spallation Neutron Source (SNS) at Oak Ridge National Laboratory. High Voltage Converter Modulators (HVCM) offer significant performance advantages over conventional modulator technologies for long pulse applications [9, 65]. These include high efficiency, low stored energy, and small size and cost.

A simplified schematic of a typical HVCM is shown in Figure 4.1. Direct modulation of a switching power supply is used to produce the pulse. A high frequency transformer incorporated into the supply is used to step-up from voltages suitable for the semiconductors, to those required by the load. Three input H-Bridges (only one is shown in Figure 1) are connected to a common DC-link capacitor (not shown). The output of each H-Bridge is connected to the primary winding of a high voltage transformer. The transformer secondary windings are connected in 'Y' and to the output rectifier and filter.

A steady state analysis of the HVCM is presented in [10]. However, the interac-

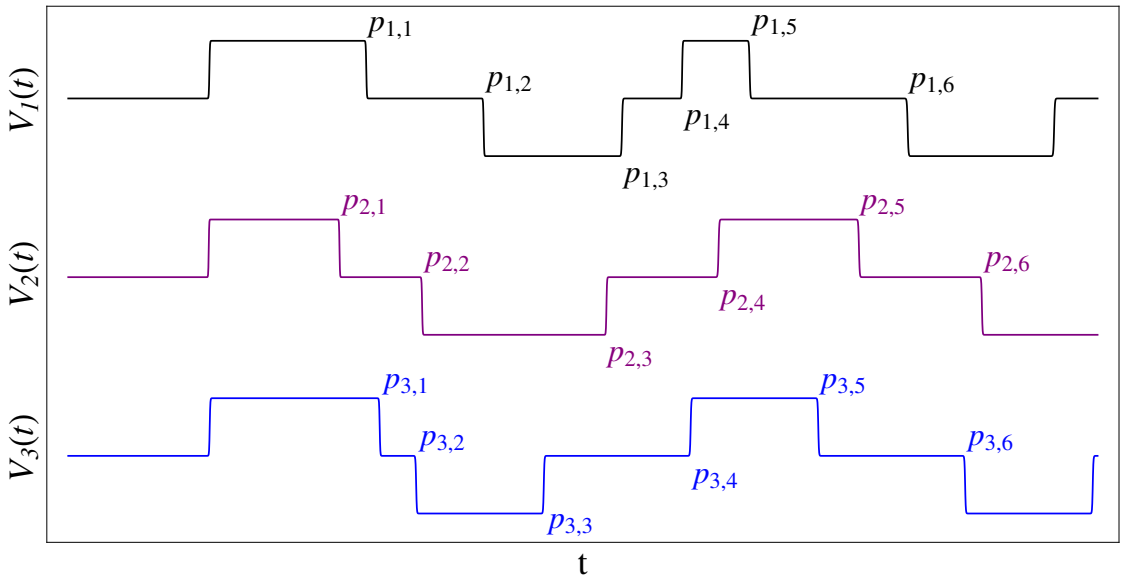


Figure 4.2: $p_{i,j}$ is the j^{th} switching edge of the i^{th} drive waveform V_i of each of the three input H-Bridges (as shown in Figure 4.1). The extremum seeker tunes only the first few switching edges of each drive waveform, which influence the rise time.

tion between the high frequency resonant components and the output filter make it very difficult to analyze the circuit under transient conditions. Therefore in order to optimize the rise time of the output voltage across the load the first few switching edges of the three phase drive waveforms of each of the three input H-Bridges (as shown in Figure 4.2) were perturbed according the extremum seeking scheme outlined in this paper.

4.3 The Extremum Seeking Algorithm for HVCM

In order to apply the ES method to the HVCM we must first define a cost function and input parameters which we perturb in order to minimize the given cost. Towards this end we first develop the following mathematical description of the dynamics and measurements of the HVCM's outputs that are of interest to us.

We consider the operation of the HVCM on two time scales, t and s. The time t is the actual time and s, which would typically be written as $n \in \mathbb{N}$, is iteration number, which in this case we consider to be an independent time scale from the viewpoint of the

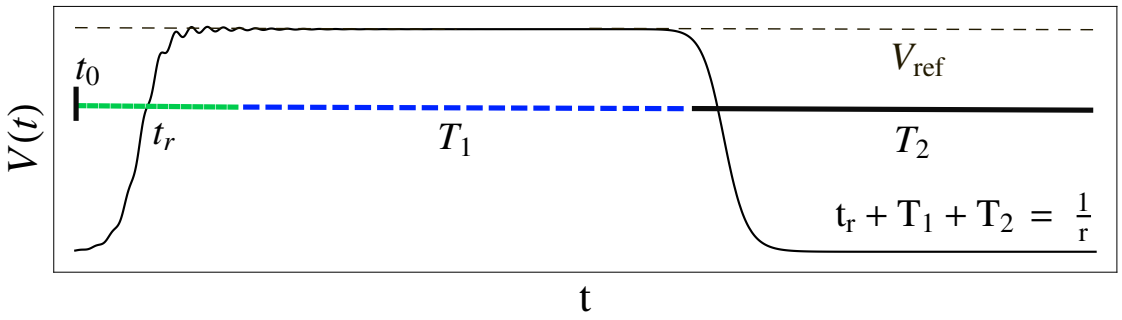


Figure 4.3: The HVCM is periodically activated with rise time t_r and pulse width T_1 defined such that $|V(t) - V_{\text{ref}}| < \frac{|V_{\text{ref}}|}{100}$ for all $t \in [t_r, t_r + T_1]$ and then turned off for T_2 seconds. The period T of one operation cycle is equal to $t_r + T_1 + T_2 = \frac{1}{r}$, resulting in r pulses per second.

digital controller, as described in more detail below. Relative to t, the HVCM is operated in a periodic fashion with an operator-prescribed repetition rate of r (typically 60Hz), as outlined in Figure 4.3. The operator defines a desired output voltage V_{ref} and pulse length T_1. At any t_0, with output voltage $V(t_0) = 0$, the HVCM is initialized and the output voltage begins to rise until $t = t_0 + t_{\text{r}}$ at which time 1% error is reached and maintained relative to the desired reference voltage for a time of T_1 so that $|V(t) - V_{\text{ref}}| < \frac{|V_{\text{ref}}|}{100}$ for all $t \in [t_0 + t_{\text{r}}, t_0 + t_{\text{r}} + T_1]$. At $t = t_0 + t_r + T_1$ the HVCM is turned off and the output voltage quickly drops to 0, and stays off for a time length of T_2, during which calculations and parameter updates may take place. T_2 is chosen such that $\frac{1}{r} = T = t_r + T_1 + T_2$, resulting in r pulses per second of pulse-width T_1. The HVCM's operation timing is linked to the accelerator's master timing system, so that the Klystrons fill the resonant accelerating cavities with electromagnetic fields for beam acceleration.

The dynamics of the output voltage V of the HVCM may therefore be described by a $\frac{1}{r} = T$-periodic system

$$\frac{\partial V(n,t)}{\partial t} = F\left(V, P(n), t\right) = F\left(V, P(n), t + T\right), \tag{4.1}$$

where $V(n, t_0) = V(n, t_0 + T) = 0$ and F is some nonlinear function describing the dynamics of the circuit in Figure 4.1.

Although it is possible to simulate the performance of the HVCM and to pro-

vide an analytic description of the steady-state behavior [9], there is no analytic model describing the dynamics of the HVCM and thereby a relationship between the positions of the switching edges of the three phase drive waveforms of each of the three input H-Bridges to the transient behavior of the output voltage. In this sense F is an unknown function of the 18 parameters:

$$\begin{bmatrix} p_{1,1}(n) & p_{1,2}(n) & p_{1,3}(n) & p_{1,4}(n) & p_{1,5}(n) & p_{1,6}(n) \\ p_{2,1}(n) & p_{2,2}(n) & p_{2,3}(n) & p_{2,4}(n) & p_{2,5}(n) & p_{2,6}(n) \\ p_{3,1}(n) & p_{3,2}(n) & p_{3,3}(n) & p_{3,4}(n) & p_{3,5}(n) & p_{3,6}(n) \end{bmatrix}$$

which we have grouped as $P(n)$, with each $p_{i,j}(n)$ being the j^{th} switching edge of the i^{th} drive waveform V_i of the three input H-Bridges as shown in Figure 4.2. $V(n,t)$ is the measurable HVCM output voltage at time t for parameter settings $p_{i,j}(n)$, where n is independent of t, as discussed in more detail below.

For any $t \in [t_0, t_0+t_r+T_1]$ the output voltage has dynamics as described by (4.1), and values

$$\begin{aligned} V(n,t) &= \int_{t_0}^{t_0+t} F\left(V(n,\tau), P(n), \tau\right) d\tau, \\ V(n,t) &= 0, \ t \in [t_0+t_r+T_1, T]. \end{aligned}$$

For $t \in [t_0+t_r+T_1, t_0+t_r+T_1+T_2]$ the system has T_2 seconds to perform calculations and update the parameter values $p_{i,j}(n)$.

Based on the results of [69], in order to minimize a cost function, as suggested by Lie bracket averaging results (discussed in more detail below) the iterative tuning law for the parameters $p_{i,j}(n)$ is chosen as:

$$\begin{aligned} p_{i,j}(n+1) = p_{i,j}(n) &+ \alpha\sqrt{\omega_{i,j}}\cos\left(\omega_{i,j} n dn\right) dn \\ &- k\sqrt{\omega_{i,j}}\sin\left(\omega_{i,j} n dn\right) C(n) dn, \end{aligned} \tag{4.2}$$

where we have defined the cost function

$$C(n) = \int_{t_1}^{t_2} \left(V(n,\tau) - V_{\text{ref}}\right)^2 d\tau, \tag{4.3}$$

where t_1 is the desired rise time and t_2 is chosen such that $t_1 < t_2 \leq T_1$ and is a long enough time so that the system has reached steady state.

Note 1. *The update law (4.2) also has rules which do not let the system do something non-physical, such as placing two switching edges on top of each other. A few of the many cases of such a rule are given as: If* $p_{i,j}(n+1) = p_{i,j+1}(n+1)$*, then* $p_{i,j}(n+1) = p_{i,j}(n)$*. If* $p_{i,1}(n+1) < 0$*, then* $p_{i,1}(n+1) = p_{i,1}(n)$*.*

The update law (4.2) is chosen with $dn \ll 1$ sufficiently small relative to the perturbation periods $T_{i,j} = \frac{2\pi}{\omega_{i,j}}$ so that based on the finite difference approximation of a derivative the system's n-dynamics approximately satisfy

$$\frac{\partial p_{i,j}}{\partial n} = \alpha\sqrt{\omega_{i,j}}\cos\left(\omega_{i,j}n\right) - k\sqrt{\omega_{i,j}}\sin\left(\omega_{i,j}n\right)C, \tag{4.4}$$

$$\frac{\partial C}{\partial n} = \sum_{i,j=1}^{m}\frac{\partial C}{\partial p_{i,j}}\sqrt{\omega_{i,j}}\left(\alpha\cos(\omega_{i,j}n) - k\sin(\omega_{i,j}n)C\right). \tag{4.5}$$

Therefore, by considering each $\frac{1}{\omega_{i,j}}$ as a small parameter $\varepsilon_{i,j}$, we can associate with system (4.4), (6.47) the Lie bracket averaged system:

$$\frac{\partial \bar{C}}{\partial n} = -\frac{k\alpha}{2}\sum_{i,j=1}^{m}\left(\frac{\partial \bar{C}}{\partial p_{i,j}}\right)^2, \quad \bar{C}(n_0) = C(n_0) \tag{4.6}$$

$$\frac{\partial \bar{p}_{i,j}}{\partial n} = -\frac{k\alpha}{2}\frac{\partial \bar{C}}{\partial \bar{p}_{i,j}}, \quad \bar{p}(n_0) = p(n_0), \tag{4.7}$$

which exponentially converges to a local minimum of the cost function $\bar{C}$. The relationship between C and $\bar{C}$ is such that for a given n-time length $\hat{N}$ and a desired accuracy $\delta > 0$ there exists $\omega^\star$ such that for all $\omega > \omega^\star$

$$\left|C(n) - \bar{C}(n)\right| < \delta, \quad \forall n \in \left[n_0, n_0 + \hat{N}\right]. \tag{4.8}$$

Remark 6. *On the choices of* ω*,* k*, and* α*.*

The term $\alpha\sqrt{\omega_{i,j}}\cos\left(\omega_{i,j}n\right)$ *in (4.4) can be thought of as the perturbing dither term in classical extremum seeking methods. Unlike the* $-k\sqrt{\omega_{i,j}}\sin\left(\omega_{i,j}n\right)C(n)$ *term, which decreases with decreased cost, the* α *term is always present, with an amplitude of (after integration)* $\frac{\alpha}{\omega_{i,j}}$*. Therefore, either a very small value of* α *or a very large value of the* $\omega_{i,j}$ *is required in order for the tuned parameters* $p_{i,j}$ *to settle near optimal set points as the cost function is minimized. Simply choosing a very small value of* α *is problematic because the overall system, on average, converges at a rate proportional to*

the product $k\alpha$. Therefore, in practice, a reasonable α is first chosen (by trial and error) and then a much larger value of k is also chosen, so that the product $k\alpha$ gives reasonably fast convergence, with the large k term vanishing as the cost function is decreased. The value of the $\omega_{i,j}$ are increased until stability and a desired level of convergence (relative to the size of α) is achieved.

Theoretically the choices of $\omega_{i,j}$ must be distinct, to satisfy the requirements of the Lie bracket averaging. In practice the choices of $\omega_{i,j}$ should further satisfy the property that for all distinct $\omega_{i,j}$, $\omega_{i_1,j_1} \neq \omega_{i_2,j_2} \pm \omega_{i_3,j_3}$, to prevent mixed signals in the nonlinear system from creating harmonics which cause cross talk between different components. In the experiment it was found that using perturbing frequencies of the form $\omega_{i_1,j_1} = 2\omega_{i_2,j_2}$ cause the system to periodically experience large disturbances. One simple method for choosing the perturbing frequencies was picking $\omega_{i,j} = \omega_0\sqrt{\Omega_{i,j}}$, where ω_0 was a fixed large scaling factor, and the $\Omega_{i,j}$ were numbers with irrational square roots. Although the numbers were truncated, removing all decimal components, the scheme successfully removed harmonic dependence among signals.

In order for the analysis and application presented here to be useful, to ensure that the averaged cost $\bar{C}(n)$ reaching a local minimum implies that the actual cost $C(n)$ also reaches a small neighborhood of a local minimum, the relationship in (4.8) must hold for a sufficiently small δ and sufficiently large $\hat{N}$. The frequencies $\omega_{i,j}$ must be large relative to the values of k and α and independent of the frequency components of any persistent noise present in the cost function measurements. Implementing the described controller with large values of $\omega_{i,j}$ is not problematic. The algorithm is implemented iteratively and the high frequency $\left(f_{i,j} = \frac{\omega_{i,j}}{2\pi}\right)$ perturbations are only of high frequency relative to the parameter n which is independent of actual time. The parameter n is only a digital entity, so the system itself does not oscillate, but is implemented at fixed settings of $p_{i,j}(n)$ for a given time period of T with only the digital update law (4.2) experiencing fast oscillations. Once $C(n)$ has reached a desired level at some iteration n_2 the extremum seeker is turned off, with parameter settings fixed at the constant values $p_{i,j}(n > n_2) \equiv p_{i,j}(n_2)$.

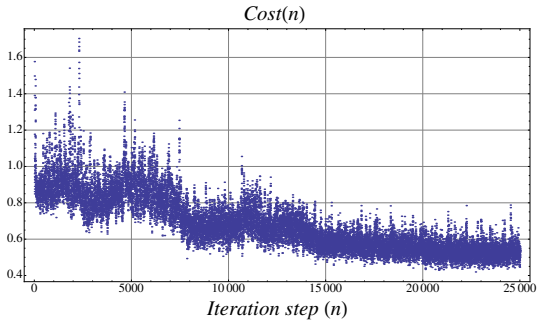


Figure 4.4: Noisy cost calculated in the digital signal processor due to noisy data during $n = 25,000$ iterative steps.

4.4 Experimental Results

Experimental results have confirmed the ability of the extremum seeking algorithm to optimize the output voltage rise-time of the HVCM at Los Alamos National Laboratory to 50μs, providing a reduction in half compared to the 100μs achieved at SNS. Furthermore, the results confirm the ability of the ES scheme to handle noisy measurements, as long as the noise does not have frequency components which match up exactly with the perturbing frequencies.

Throughout the experiments the desired output voltage was set at $V_{\text{ref}} = -10kV$, a voltage low enough to allow a 60Hz repetition rate with the available load. After optimization, with parameters fixed at their optimal settings, the voltage was turned up to approximately -50kV and the HVCM was fired to confirm the desired output voltage performance.

Following three separate experiments, which are discussed in detail below, the HVCM was fired at -90kV, a quantity limited by both the available power sources and current load setup, confirming $< 1\%$ error.

4.4.1 6 Edges Per Phase, Noisy Cost Function

In the first application of the ES algorithm we iteratively tune the first 6 switching edges of each of the three phase drive waveforms (18 variables total) of the HVCM with a rep rate of 60Hz, which was limited by the available load. Optimization took place

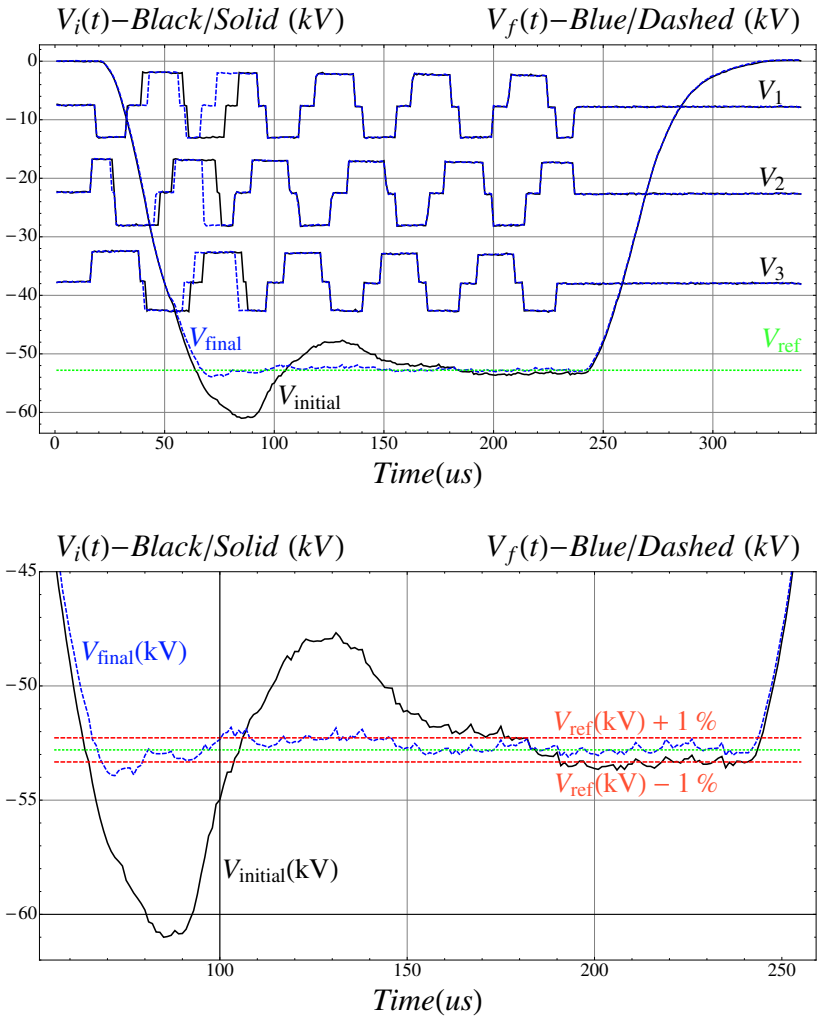


Figure 4.5: Without averaging and using only 6 edges per driving waveform the final output voltage was within 2% error. Waveform voltages are shown in units of kV, the driving waveforms V_1, V_2, and V_3 are shifted and dilated for comparison.

by the simple update law given by (4.2), with the cost calculated as given in (4.3), where $t_1 = 40\mu s$, $t_2 = 100\mu s$ is sufficiently large to capture voltage transients, $k = 500$, $\alpha = 0.1$, and the voltage output $V(t,n)$ depends on the parameter settings $p_{i,j}(n)$ at step

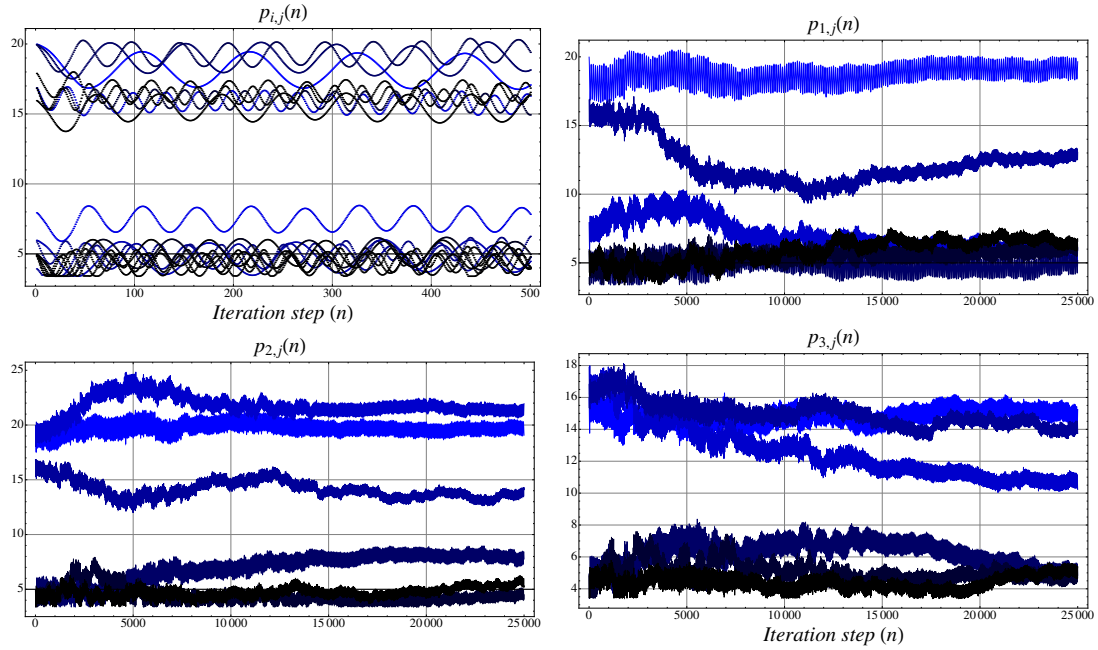


Figure 4.6: The motion of the switching edges of the driving waveforms V_1, V_2, and V_3 is shown during optimization. At the top we zoom in on the initial $n = 500$ steps, showing the smooth oscillation of all 18 parameters. Pulse widths are shown in microseconds.

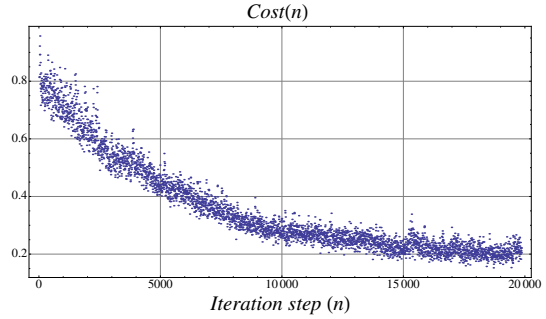


Figure 4.7: A less noisy cost was detected in the computer in $n = 20,000$ steps when each iterative extremum seeking step calculated the cost based on the average of 5 HVCM shots with fixed settings.

n. The perturbing angular frequencies $\omega_{i,j}$ of the parameters $p_{i,j}$ are chosen as

$$\begin{bmatrix} 115537 & 142643 & 164579 & 181076 & 199467 & 213282 \\ 229667 & 243898 & 256839 & 296917 & 319432 & 339395 \\ 378375 & 399167 & 413745 & 433573 & 455621 & 488106 \end{bmatrix}$$

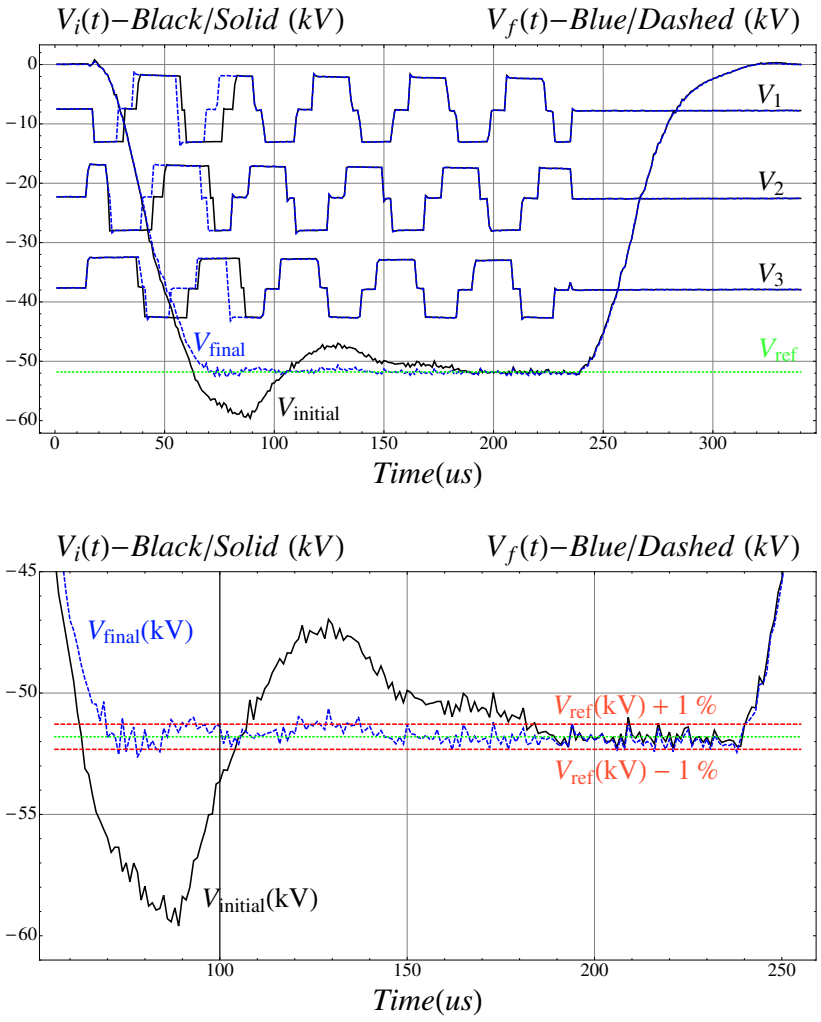


Figure 4.8: With averaging and 6 edges per driving waveform the final output voltage was closer, but still not within 1% error. Waveform voltages are shown in units of kV, the driving waveforms V_1, V_2, and V_3 are shifted and dilated for comparison.

and $\delta = 5 \times 10^{-7}$, so that $\frac{\max_{i,j}\{\omega_{i,j}\}}{2\pi}\delta < \frac{1}{30}$, guaranteeing that on the n-time scale the sines and cosines are smooth, with at least 30 points per 2π phase advance for the highest frequency. The evolution of the cost function $C(n)$ over $n = 25,000$ iterative steps, which is $25{,}000T \approx 5$ minutes of real time optimization is shown in Figure 4.4. While keeping the parameters $p_{i,j}$ fixed at the values optimized for -10kV, the voltage is then turned up to $-52.8kV$ and a comparison of the initial and final output voltage

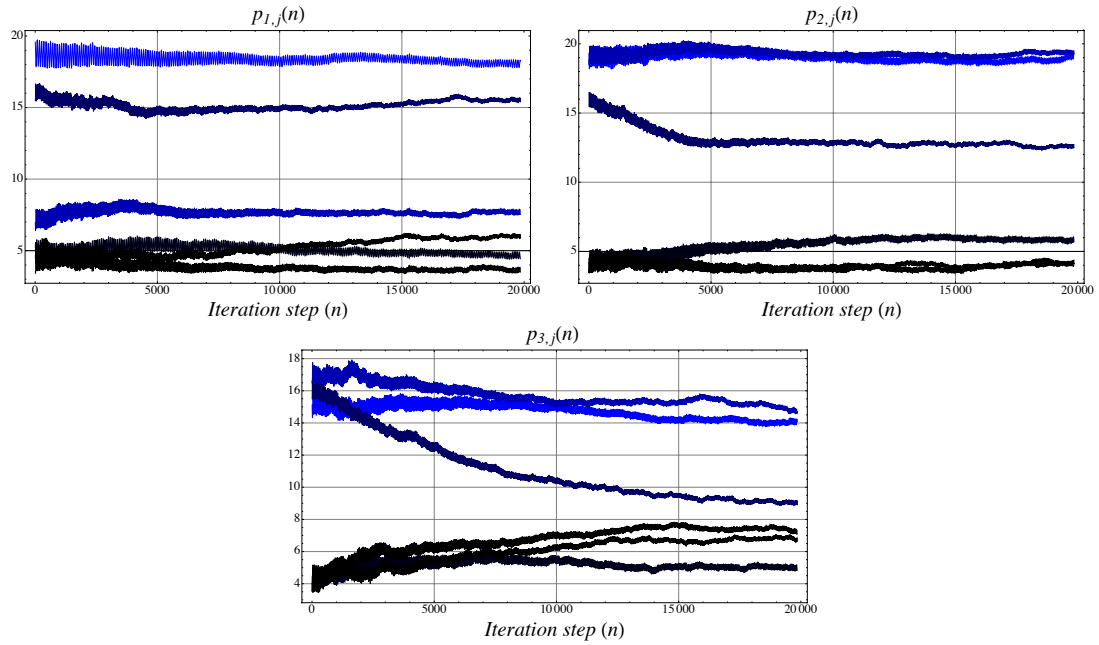


Figure 4.9: The evolution of the switching edges of the driving waveforms V_1, V_2, and V_3 during the extremum seeking optimization. Pulse widths are shown in microseconds.

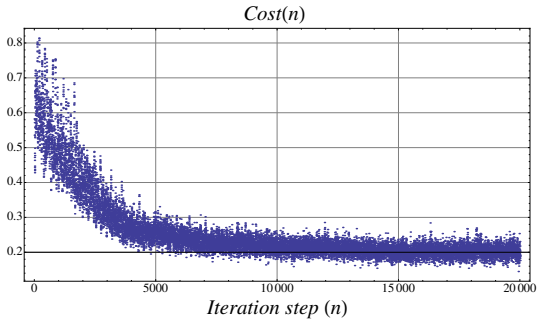


Figure 4.10: Cost function noise was significantly reduced when each iterative extremum seeking step calculated the cost based on the average of 10 HVCM shots. Combined with more perturbed edges resulted in the cost reaching a lower value than the previous two schemes.

waveforms is shown in Figure 4.5. Voltage output error was reduced to less than 2%. Figure 4.6 shows the evolution of the perturbed pulse widths (in microseconds) of the driving waveform V_1, V_2, and V_3 during the extremum seeking optimization.

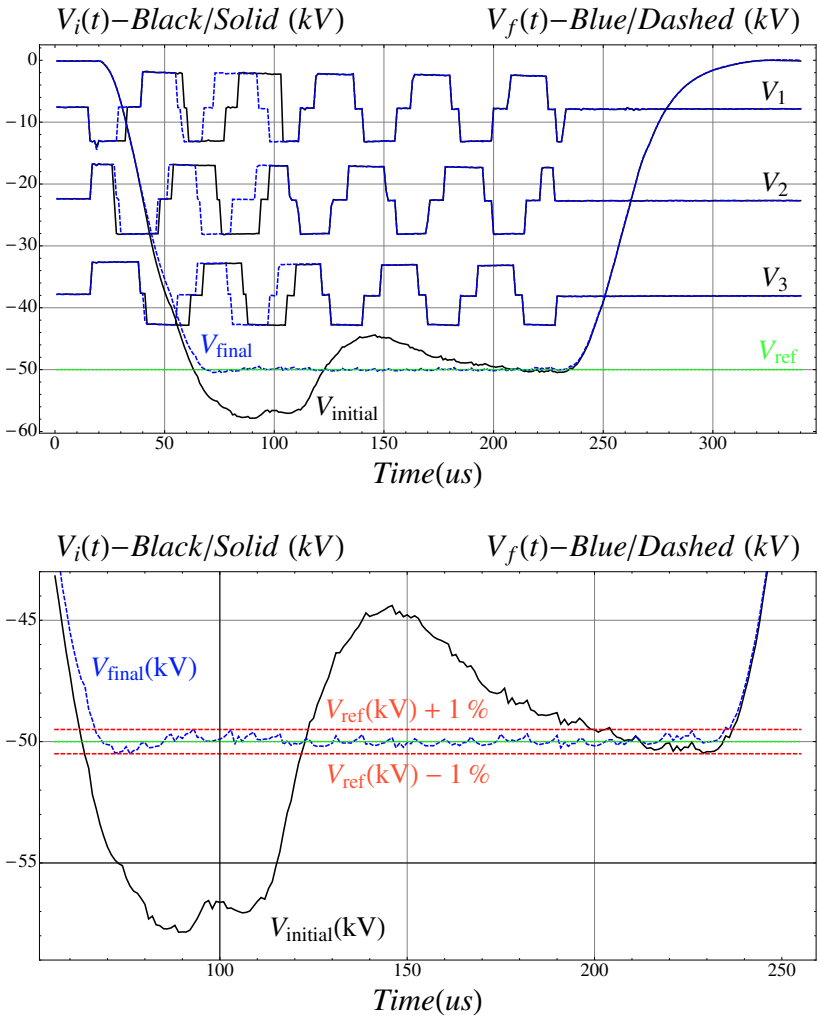


Figure 4.11: Using the averaging scheme and 8 edges per driving waveform the final output voltage was within 1% error. V_{initial}, V_{final}, and V_{ref} are shown in units of kV, the driving waveforms V_1, V_2, and V_3 are shifted and dilated for comparison.

4.4.2 6 Edges Per Phase, Averaged Cost Function

With the cost as noisy as it is shown in Figure 4.4 a slightly different extremum seeking scheme was attempted in which parameters were set and the HVCM was fired 5 times allowing the 5 output voltage waveforms to be averaged in order to calculate the cost, smoothing out random perturbations. In order to converge more closely to a minimum, at the cost of slower convergence, the value of α was decreased to 0.01.

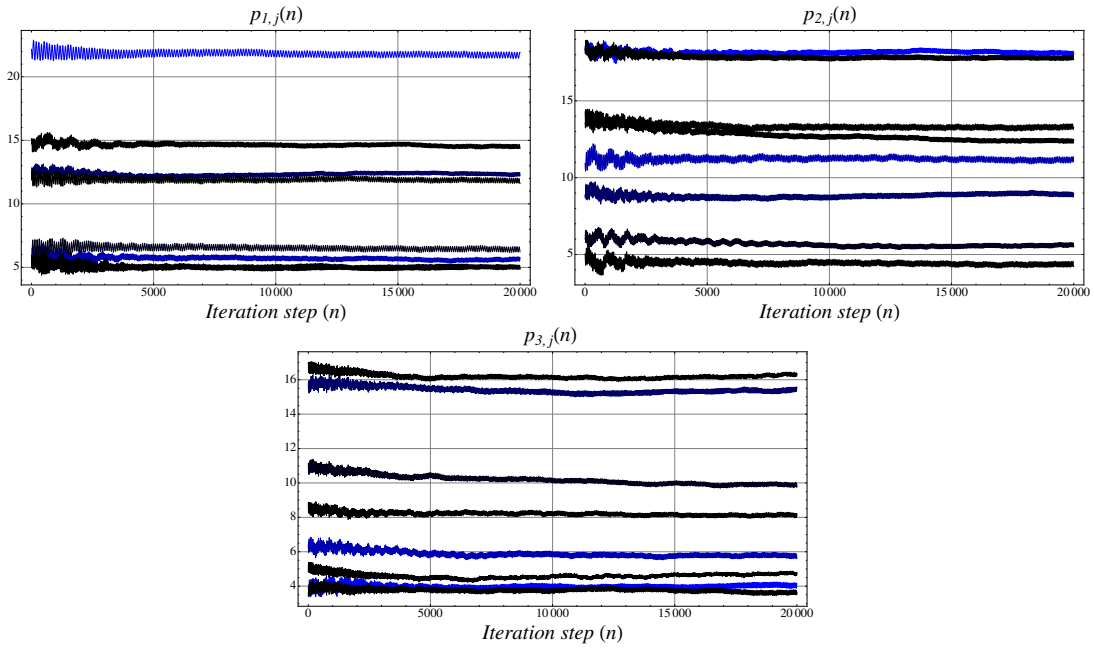


Figure 4.12: The evolution of the switching edges of the driving waveforms V_1, V_2, and V_3 during the extremum seeking optimization. Pulse widths are shown in microseconds.

Again, initially we set $V_{\text{ref}} = -10kV$ so that we can iteratively tune the first 6 switching edges of each of the three phase drive waveforms (18 variables total) of the HVCM with a rep rate of $60Hz$, which was limited by the available load. Optimization took place by the same update law (4.2), with the cost replaced by an averaged value $\bar{C}(n)$,which is calculated by averaging 5 cost measurements $C(n_1),\ldots,C(n_5)$, each calculated as:

$$C(n_i) = \int_{t_1}^{t_2} \left(V(n_i,\tau) - V_{\text{ref}}\right)^2 d\tau, \tag{4.9}$$

where the parameters $p_{i,j}(n_l)$ were kept constant during the five averaging shots $n_1 - n_5$, where again $t_1 = 50\mu s$, $t_2 = 100\mu s$ is sufficiently large to capture voltage transients, $k = 400$, $\alpha = 0.01$, and the voltage output $V(t,n)$ depends on the parameter settings $p_{i,j}(n)$ at step n. We again chose $\delta = 5 \times 10^{-7}$. The evolution of the cost function $C(n)$ over $n = 20,000$ averaged iterative steps (100,000 actual steps), which is $100{,}000T \approx$ 30 minutes of real time optimization is shown in Figure 4.7. The voltage is then turned up to $-51.8kV$ and a comparison of the initial and final output voltage waveforms is shown in Figure 4.8. The output voltage error, with the smoother cost the output voltage waveform was improved, but was still outside of the desired $< 1\%$ error range. Figure

4.9 shows the evolution of the perturbed pulse widths (in microseconds) of the driving waveform V_1, V_2, and V_3 during the extremum seeking optimization.

4.4.3 8 Edges Per Phase, Averaged Cost Function

Finally an extremum seeking scheme was attempted in which parameters were set and the HVCM was fired 10 times allowing the 10 output voltage waveforms to be averaged. A cost calculation based on more averaging helped in smoothing out random perturbations even further. Also, the number of switching edges per drive waveform was also increased to 8, giving the ES controller more influence over some of the bumps showing up later in the output voltage waveform. Again, initially we set $V_{\text{ref}} = -10kV$ so that we can iteratively tune the first 8 switching edges of each of the three phase drive waveforms (24 variables total) of the HVCM with a rep rate of $60Hz$, which was limited by the available load. Optimization again took place by the update law (4.2), with the cost replaced by an averaged value $\bar{C}(n)$,which is calculated by averaging 10 cost measurements $C(n_1),\ldots,C(n_{10})$, each calculated as in (4.9), where the parameters $p_{i,j}(n_l)$ were kept constant during the ten averaging shots $n_1,\ldots,n_{10}$, where again $t_1 = 50\mu s$, $t_2 = 100\mu s$ is sufficiently large to capture voltage transients, $k = 400$, $\alpha = 0.01$, and the voltage output $V(t,n)$ depends on the parameter settings $p_{i,j}(n)$ at step n. The perturbing angular frequencies $\omega_{i,j}$ of the 6 additional parameters $p_{i,j}$ were chosen as

$$\begin{bmatrix} 532841 & 576844 \\ 629285 & 664875 \\ 712039 & 754672 \end{bmatrix}$$

and $\delta = 5\times 10^{-7}$, so that $\frac{\max_{i,j}\{\omega_{i,j}\}}{2\pi}\delta < \frac{1}{16}$, guaranteeing that on the n-time scale the sines and cosines are smooth, with at least 16 points per 2π phase advance for the highest frequency. The evolution of the cost function $C(n)$ over $n = 20{,}000$ averaged iterative steps (200,000 actual steps), which is $200{,}000T \approx 1$ hour of real time optimization, is shown in Figure 4.10. The voltage is then turned up to $-50kV$ and a comparison of the initial and final output voltage waveforms is shown in Figure 4.11. The output voltage error, with the smoother cost and additional parameters has dropped within the desired $< 1\%$ range. Figure 4.12 shows the evolution of the perturbed pulse widths

(in microseconds) of the driving waveform V_1, V_2, and V_3 during the extremum seeking optimization.

This chapter contains material from

A. Scheinker, M. Bland, M. Krstić, and J. Audia, “Rise-time optimization of accelerator high voltage converter modulator by extremum seeking," under review.

The dissertation author was the primary author in this publication.

Chapter 5

Trajectory Tracking

In this chapter we extend our stability results to trajectory tracking for unknown, unstable systems. We introduce the problem in Section 5.1. In Section 5.2 we state definitions of stability used throughout the rest of the paper. We then use a slight modification of the result of Moreau and Aeyels [58] to show that semiglobal uniform ultimate boundedness of the averaged system implies stability of the original system to within the same δ-ball. In Section 5.3 we extend our stability results from Chapter 3 to that of trajectory tracking for linear and nonlinear systems. In Section 5.3.1 we demonstrate the controller's ability to track a trajectory with an unstable, 2-dimensional, linear time-varying system.

5.1 Introduction

Motivation A natural extension of the work develop so far is creating controllers for trajectory tracking, which is notoriously difficult in the case of unknown and changing control gain direction.

Overview of Results In this chapter we extend stability results to ultimately bounded trajectory tracking with a design based on extremum seeking [44, 80]. The focus of this chapter is on extending the stability results of Chapter 3 to trajectory tracking for LTV

and nonlinear systems of the form

$$\dot{x} = A(t)x + B(t)u \text{ and } \dot{x} = f(x,t) + G(x,t)u,$$

for a trajectory $r(t)$ having known magnitude and rate of change bounds. We define the error vector $e(t) = x(t) - r(t)$ and in the linear case, we use a nonlinear time-varying control law of the form

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)|e|^2, \tag{5.1}$$

and in the nonlinear case a control law of the form

$$u_i = \alpha\sqrt{\omega\omega_i'}\cos(\omega\omega_i' t) - k\sqrt{\omega\omega_i'}\sin(\omega\omega_i' t)V(e), \tag{5.2}$$

with distinct ω_i'. We then choose $V(e)$ and gain $k\alpha > 0$ sufficiently large to stabilize the linear Lie bracket average error system

$$\dot{\bar{e}} = \left[A(t) - k\alpha B(t)B^T(t)\right]\bar{e} + A(t)r - \dot{r}, \tag{5.3}$$

or the nonlinear Lie bracket average system

$$\dot{\bar{e}} = f(\bar{e},t) - k\alpha\gamma G(\bar{e},t)G^T(\bar{e},t)\left(\frac{\partial V}{\partial \bar{e}}\right)^T - \dot{r}, \tag{5.4}$$

to within a δ-ball of the origin. In the linear case we assume a persistency of excitation condition on the vector-valued function $B(t)$ and in the nonlinear case we assume that $G(e,t)G^T(e,t) > 0$. In both cases, following the stabilization of the averaged systems ω is chosen sufficiently large to satisfy the requirements of the Lie bracket averaging theorem of Kurzweil, Jarnik, Sussmann, Liu, Gurvits, and Li [30, 31, 32, 45, 78, 79] and of a semiglobal uniform ultimate boundedness theorem that is a slight modification of the results of Moreau and Aeyels [58].

In the linear case we can not only handle unknown signs of the high frequency gains, but also signs that change with time, allowing $B(t)$ to go through zero. The extremum seeking control scheme is by design operating on a faster time scale than the dynamics in the system's coefficients, and the system's behavior, as estimated by an averaged system, does not depend on the control coefficients' signs. In both cases, we accomplish tracking to within a δ-ball of the desired trajectory. Apart from the limited effort in [91], this paper provides the first results making extremum seeking to tracking for unstable plants.

5.2 Background on Semiglobal Practical Ultimate Boundedness

In order to discuss ultimate boundedness averaging results we make the following definitions similarly to Moreau and Aeyels [58]. In what follows, given a system

$$\dot{x} = f(t,x), \tag{5.5}$$

$\psi(t,t_0,x_0)$ denotes the solution of (5.5) which passes through the point x_0 at time t_0.

Definition 4. *Global Uniform Ultimate Boundedness with ultimate bound δ (δ-GUUB): For $\delta \geq 0$ an equilibrium point of (2.10) is said to be δ-GUUB if it satisfies the following three conditions:*

- *δ-Uniform Stability: For every $c_2 \in (\delta,\infty)$ there exists $c_1 \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$,*

$$\|\psi(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0,\infty). \tag{5.6}$$

- *δ-Uniform Ultimate Boundedness: For every $c_1 \in (0,\infty)$ there exists $c_2 \in (\delta,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$,*

$$\|\psi(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0,\infty). \tag{5.7}$$

- *δ-Global Uniform Attractivity: For all $c_1,c_2 \in (\delta,\infty)$ there exists $T \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$,*

$$\|\psi(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0+T,\infty). \tag{5.8}$$

In conjunction with (5.5), we consider systems of the form

$$\dot{x} = f^{\varepsilon}(t,x) \tag{5.9}$$

whose trajectories are denoted as $\phi^{\varepsilon}(t,t_0,x_0)$.

Definition 5. *Converging Trajectories Property: The systems (2.10) and (2.14) are said to satisfy the converging trajectories property if for every* $\hat{T} \in (0,\infty)$ *and compact set* $K \subset \mathbb{R}^n$ *satisfying* $\{(t,t_0,x_0) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n : t \in [t_0,t_0+\hat{T}], x_0 \in K\} \subset \mathrm{Dom}\psi$*, for every* $d \in (0,\infty)$ *there exists* $\varepsilon^\star$ *such that for all* $t_0 \in \mathbb{R}$*, for all* $x_0 \in K$ *and for all* $\varepsilon \in (0,\varepsilon^\star)$*,*

$$\|\phi^\varepsilon(t,t_0,x_0) - \psi(t,t_0,x_0)\| < d, \ \ \forall t \in [t_0,t_0+\hat{T}]. \tag{5.10}$$

We then define the following form of stability for system (5.9).

Definition 6. ε*-Semiglobal Practical Uniform Ultimate Boundedness with Ultimate Bound* δ *(*(ε,δ)*-SPUUB): The origin of (5.9) is said to be* (ε,δ)*-SPUUB if it satisfies the following three conditions:*

- (ε,δ)*-Uniform Stability: For every* $c_2 \in (\delta,\infty)$ *there exists* $c_1 \in (0,\infty)$ *and* $\hat{\varepsilon} \in (0,\infty)$ *such that for all* $t_0 \in \mathbb{R}$ *and for all* $x_0 \in \mathbb{R}^n$ *with* $\|x_0\| < c_1$ *and for all* $\varepsilon \in (0,\hat{\varepsilon})$*,*

$$\|\phi^\varepsilon(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0,\infty). \tag{5.11}$$

- (ε,δ)*-Uniform Ultimate Boundedness: For every* $c_1 \in (0,\infty)$ *there exists* $c_2 \in (\delta,\infty)$ *and* $\hat{\varepsilon} \in (0,\infty)$ *such that for all* $t_0 \in \mathbb{R}$ *and for all* $x_0 \in \mathbb{R}^n$ *with* $\|x_0\| < c_1$ *and for all* $\varepsilon \in (0,\hat{\varepsilon})$*,*

$$\|\phi^\varepsilon(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0,\infty). \tag{5.12}$$

- (ε,δ)*-Global Uniform Attractivity: For all* $c_1,c_2 \in (\delta,\infty)$ *there exists* $T \in (0,\infty)$ *and* $\hat{\varepsilon} \in (0,\infty)$ *such that for all* $t_0 \in \mathbb{R}$ *and for all* $x_0 \in \mathbb{R}^n$ *with* $\|x_0\| < c_1$ *and for all* $\varepsilon \in (0,\hat{\varepsilon})$*,*

$$\|\phi^\varepsilon(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0+T,\infty). \tag{5.13}$$

With these definitions the following modification of the result of Moreau and Aeyels [58] is used in the analysis that follows.

Theorem 6. *([58]) If systems (5.9) and (5.5) satisfy the converging trajectories property and if the origin is a* δ*-GUUB point of (5.5), then the origin of (5.9) is* (ε,δ)*-SPUUB.*

Proof. Given any $\hat{T} > 0$ we can always choose $nT > \hat{T}$ and therefore by Lemma 1 the solution of system (2.2) satisfies the converging trajectories property relative to the solution of system (2.1). The rest of the proof is a slight modification of the proof found in Moreau and Aeyels [58], with a lower bound of δ instead of 0 on the choices of c_2, b_2 and c_3, with details available from the author. □

5.3 Trajectory Tracking

We consider tracking for a trajectory $r(t)$ satisfying the bounds $|r(s)| < r_\star$ and $|\dot{r}(s)| < \rho_\star \; \forall \; s \geq T$, where we define $e(t) = x(t) - r(t)$, choose any $\delta > 0$ and perform our analysis for $|e| \geq \delta$.

Lemma 3. *Consider the system*

$$\dot{x}{=}a(t)x + b(t)u, \tag{5.14}$$

and let there exist $\triangle > 0$, $\beta_0 > 0$, $a_\star > 0$, *and* $T > 0$ *such that* $\forall \; s \geq T$, $a(t)$ *and* $b(t)$ *satisfy*

$$\frac{1}{\triangle}\int_s^{s+\triangle} b^2(\tau)d\tau \geq \beta_0, \; |a(s)| < a_\star. \tag{5.15}$$

Given the error system

$$\dot{e}{=}a(t)e + b(t)u + a(t)r - \dot{r} \tag{5.16}$$

$$u{=}\alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)e^2, \tag{5.17}$$

if

$$k\alpha > \frac{a_\star(r_\star + \delta) + \rho_\star}{\delta\beta_0} \tag{5.18}$$

then the origin of system (5.16), (5.17) is $\left(\frac{1}{\omega}, \delta\right)$*-SPUUB, with average convergence rate*

$$\gamma_r = 2\left[k\alpha\beta_0 - a_\star - \frac{a_\star r_\star + \rho_\star}{\delta}\right] > 0. \tag{5.19}$$

Proof. The Lie bracket averaged closed loop error system (5.16), (5.17) is

$$\dot{\bar{e}} = \left[a(t) - k\alpha b^2(t)\right]\bar{e} + a(t)r - \dot{r}. \tag{5.20}$$

Consider the Lyapunov function

$$V(t) = \frac{\bar{e}^2(t)}{2}. \tag{5.21}$$

For all $s \geq T$ and $|e| \geq \delta$ we get

$$\begin{aligned}\dot{V}(s) =& \left[a(s) - k\alpha b^2(s)\right]\bar{e}^2 + a(s)r\bar{e} - \dot{r}\bar{e} \\ \leq& 2\left[a(s) - k\alpha b^2(s) + \frac{a_\star r_\star + \rho_\star}{\delta}\right]V(s),\end{aligned} \tag{5.22}$$

which implies, by the Gronwall-Bellman lemma, that

$$V(s+\Delta) \leq V(s)\bar{e}^{\frac{2\Delta(a_\star r_\star + \rho_\star)}{\delta}}\bar{e}^{2\int_s^{s+\Delta} a(\tau) - k\alpha b^2(\tau)d\tau}. \tag{5.23}$$

From (5.15) and (5.18) we have

$$\begin{aligned}&\frac{2\Delta(a_\star r_\star + \rho_\star)}{\delta} + 2\int_s^{s+\Delta} a(\tau) - k\alpha b^2(\tau)d\tau \\ &\leq 2\Delta\left(\frac{a_\star r_\star + \rho_\star}{\delta} + a_\star - k\alpha\beta_0\right) = -\Delta\gamma_r < 0,\end{aligned} \tag{5.24}$$

therefore for $|\bar{e}(s)| \geq \delta$,

$$|\bar{e}(s+\Delta)| \leq |\bar{e}(s)|e^{-\Delta\gamma_r} < |\bar{e}(s)|. \tag{5.25}$$

Proceeding as in the proof of Theorem 7 we get the convergence rate. Therefore the averaged system is exponentially converging to within a δ-ball of the origin. Therefore the origin of system (5.16), (5.17) is $\left(\frac{1}{\omega}, \delta\right)$-SPUUB. □

Theorem 7. *Consider the system*

$$\dot{x} = A(t)x + B(t)u, \tag{5.26}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $u \in \mathbb{R}$, and let there exist $\Delta > 0$, $b_\star \geq \beta_0 > 0$, $a_\star \geq 0$, and $T \geq 0$ such that $A(t)$ and $B(t)$ satisfy

$$\frac{1}{\Delta}\int_s^{s+\Delta} B(\tau)B^T(\tau)d\tau \geq \beta_0 I,\ \forall s \geq T \tag{5.27}$$

$$\left\langle |B|^2 \right\rangle_\Delta (s) \leq b_\star,\ \forall s \geq 0, \tag{5.28}$$

$$\left\langle |A|^2 \right\rangle_\Delta (s) \leq a_\star,\ \forall s \geq 0. \tag{5.29}$$

We consider the error system

$$\dot{e}=A(t)e+B(t)u+A(t)r-\dot{r}, \tag{5.30}$$

$$u=\alpha\sqrt{\omega}\cos(\omega t)-k\sqrt{\omega}\sin(\omega t)|e|^2. \tag{5.31}$$

The origin of system (5.30), (5.31) is $\left(\frac{1}{\omega},\delta\right)$*-SPUUB with average decay rate*

$$R=\frac{1}{2\Delta}\left[\ln\left(\frac{1}{\gamma_3}\right)-a_\star\gamma_2\right]-\frac{\sqrt{a_\star}\delta+r_\star+\rho_\star}{\delta}>0, \tag{5.32}$$

where

$$\gamma=1-\frac{k\alpha\Delta\beta_0}{1+2k^2\alpha^2\Delta^2b_\star^2}>0, \tag{5.33}$$

$$\gamma_2=\frac{4k\alpha\Delta^3b_\star}{1+2k^2\alpha^2\Delta^2b_\star^2}, \tag{5.34}$$

$$\gamma_3=\frac{2\Delta b_\star(\sqrt{a_\star}r_\star+\rho_\star)(1-\gamma)}{\delta^2\beta_0}+\gamma \tag{5.35}$$

under either of the two conditions:

(i) Given $k\alpha>0$ *and* $\Delta>0$*,* $a_\star$ *is in the interval* $(0,\bar{a}_\star)$ *where* $\bar{a}_\star$ *satisfies*

$$\gamma_3(\bar{a}_\star)=e^{-\left(2\Delta\left[\sqrt{\bar{a}_\star}\left(1+\frac{r_\star}{\delta}\right)+\frac{\rho_\star}{\delta}\right]+\bar{a}_\star\gamma_2\right)}. \tag{5.36}$$

(ii) For a given $a_\star$*, the window* Δ *satisfies* $\Delta\in\left(0,\min\left\{\bar{\Delta}_1,\bar{\Delta}_2\right\}\right)$*, where* $\bar{\Delta}_1$ *and* $\bar{\Delta}_2$ *satisfy*

$$0=\bar{\Delta}_1^2\sqrt{2}a_\star+\bar{\Delta}_1 2\left(\sqrt{a_\star}\left[1+\frac{r_\star}{\delta}\right]+\frac{\rho_\star}{\delta}\right)+\ln\left(1+\frac{\beta_0}{2\sqrt{2}b_\star}\left[\frac{2\bar{\Delta}_1b_\star(\sqrt{a_\star}r_\star+\rho_\star)}{\delta^2\beta_0}-1\right]\right), \tag{5.37}$$

$$0=\beta_0\delta^2-2\bar{\Delta}_2b_\star(\sqrt{a_\star}r_\star+\rho_\star)-2\sqrt{2}b_\star\delta^2\left[1-e^{-2\bar{\Delta}_2\left(\frac{\sqrt{a_\star}\delta+r_\star+\rho_\star}{\delta}+\frac{a_\star}{b_\star}\right)}\right], \tag{5.38}$$

and $k\alpha>1$ *is selected such that*

$$k\alpha\in\left(\frac{1}{2\sqrt{2}\Delta b_\star},\frac{1}{2\sqrt{2}\Delta b_\star}+M_k(a_\star,b_\star,\beta_0,\Delta)\right), \tag{5.39}$$

where

$$M_k=\frac{\sqrt{\beta_0^2\left[\frac{1-\gamma_3}{1-\gamma}\right]^2-8b_\star^2\left[1-e^{-2\Delta\left(\frac{\sqrt{a_\star}\delta+r_\star+\rho_\star}{\delta}+\frac{a_\star}{b_\star}\right)}\right]^2}}{2\sqrt{2}b_\star\left[1-e^{-2\Delta\left(\frac{\sqrt{a_\star}\delta+r_\star+\rho_\star}{\delta}+\frac{a_\star}{b_\star}\right)}\right]}.$$

Remark 7. *Theorem 7(i) is a robustness result. For any $k\alpha > 0$, the controller (5.31) allows some perturbation $A(t)x$ in the system (5.26), as long as the mean of $A(t)$ is sufficiently small, as quantified by (5.36). Theorem 7(ii) is a design result. If an accuracy goal $\delta > 0$ is chosen such that the window Δ is small enough, as quantified by (5.37), (5.38) then it is possible to choose $k\alpha$ in some range, as quantified by (5.39) such that $x(t)$ exponentially converges to a trajectory that is within a δ-ball of the trajectory $r(t)$.*

Remark 8. *Parts of this proof use steps developed in the proof of Theorem 4.3.2 (iii) in the second half of Section 4.8.3 in [34].*

Proof. The Lie bracket averaged closed loop error system (5.30), (5.31) is

$$\dot{\bar{e}} = \left[A(t) - k\alpha B(t)B^T(t)\right]\bar{e} + A(t)r - \dot{r}. \tag{5.40}$$

With the Lyapunov function candidate $V(\bar{e}) = \frac{|\bar{e}|^2}{2}$ we get

$$\dot{V}(\bar{e}) = \bar{e}^T A(t)\bar{e} - k\alpha\bar{e}^T B(t)B^T(t)\bar{e} + \bar{e}^T A(t)r - \bar{e}^T\dot{r}.$$

Therefore, for any $s \geq T$ we have

$$\begin{aligned} V(s+\Delta) = V(s) &- \underbrace{k\alpha\int_s^{s+\Delta}\left|\bar{e}^T(\tau)B(\tau)\right|^2 d\tau}_{I_1} \\ &+ \underbrace{\int_s^{s+\Delta}\bar{e}^T(\tau)A(\tau)\bar{e}(\tau)d\tau}_{I_2} + \underbrace{\int_s^{s+\Delta}\bar{e}^T(\tau)A(\tau)r(\tau)d\tau}_{I_3} \\ &- \underbrace{\int_s^{s+\Delta}\bar{e}^T(\tau)\dot{r}(\tau)d\tau}_{I_4}. \end{aligned} \tag{5.41}$$

Applying analysis similar to [42] we get

$$I_1 \leq \frac{-k\alpha\Delta\beta_0 V(\bar{x})}{1+2k^2\alpha^2\Delta^2 b_\star^2} + \frac{k\alpha b_\star\Delta^2\left(\sqrt{a_\star}r_\star + \rho_\star + 4a_\star\int_s^{s+\Delta}V(\tau)d\tau\right)}{1+2k^2\alpha^2\Delta^2 b_\star^2} \tag{5.42}$$

and

$$I_2 \leq 2\int_s^{s+\Delta}|A(\tau)|\,V(\tau)d\tau. \tag{5.43}$$

The term I_3 satisfies the bound

$$|I_3| \leq \int_s^{s+\Delta} \left|\bar{e}^T(\tau)\right|^2 \frac{1}{|\bar{e}^T(\tau)|} |A(\tau)|\,|r(\tau)|\,d\tau, \tag{5.44}$$

which is, for all $|e| > \delta$ and taking into account the bound on $|r(t)|$, bounded by

$$|I_3| \leq \frac{r_\star}{\delta} \int_s^{s+\Delta} 2V(\tau)\,|A(\tau)|\,d\tau. \tag{5.45}$$

Similarly the term I_4 satisfies the bound

$$|I_4| \leq \int_s^{s+\Delta} \left|\bar{e}^T(\tau)\right|^2 \frac{1}{|\bar{e}^T(\tau)|} |\dot{r}(\tau)|\,d\tau, \tag{5.46}$$

which is, for all $|e| > \delta$ and taking into account the bound on $|\dot{r}(t)|$, bounded by

$$|I_4| \leq \frac{\rho_\star}{\delta} \int_s^{s+\Delta} 2V(\tau) d\tau. \tag{5.47}$$

Combining (5.41), (5.42), (5.43), (5.45) and (5.47), for all $|\bar{e}| > \delta$ we obtain

$$\begin{aligned} V(s+\Delta) \leq & \left(\frac{2\Delta b_\star (\sqrt{a_\star} r_\star + \rho_\star)(1-\gamma)}{\delta^2 \beta_0} + \gamma \right) V(s) \\ & + \int_s^{s+\Delta} \left(2|A(\tau)| \left(1 + \frac{r_\star}{\delta}\right) + \frac{2\rho_\star}{\delta} + \frac{a_\star}{\Delta}\gamma_2 \right) V(\tau) d\tau, \end{aligned}$$

where γ and γ_2 are defined in (5.33) and (5.34) respectively. Noting that $\frac{k\alpha\Delta\beta_0}{1+2k^2\alpha^2\Delta^2 b_\star^2} \leq \frac{\beta_0}{2\sqrt{2}b_\star}$ and that $\beta_0 \leq b_\star$, we get that $\gamma \in \left(\frac{2\sqrt{2}-1}{2\sqrt{2}}, 1\right)$, which implies that γ is positive. We now apply the Bellman-Gronwall lemma, and get that for all $s \geq T$, for all $|e| > \delta$,

$$V(s+\Delta) \leq \gamma_3 e^{\int_s^{s+\Delta} \left(2|A(\tau)|\left(1+\frac{r_\star}{\delta}\right) + \frac{2\rho_\star}{\delta} + \frac{a_\star}{\Delta}\gamma_2\right) d\tau} V(s),$$

Where γ_3 is as defined in (5.35). We note that the Cauchy-Schwartz inequality yields $\int_s^{s+\Delta} |A(\tau)|\,d\tau \leq \Delta\sqrt{a_\star}$, so we get, for all $s \geq T$,

$$V(s+\Delta) \leq \gamma_3 e^{\left(2\Delta\left[\sqrt{a_\star}\left(1+\frac{r_\star}{\delta}\right) + \frac{\rho_\star}{\delta}\right] + a_\star\gamma_2\right)} V(s), \tag{5.48}$$

where γ_2 is as defined in (5.34). Evidently for convergence we require that

$$\gamma_3 < e^{-\left(2\Delta\left[\sqrt{a_\star}\left(1+\frac{r_\star}{\delta}\right) + \frac{\rho_\star}{\delta}\right] + a_\star\gamma_2\right)}. \tag{5.49}$$

The left side of inequaity (5.49), γ_3, is strictly increasing as a function of $a_\star$. The exponential on the right side is strictly decreasing. Therefore if equality holds in (5.49) for some $\bar{a}_\star$ then inequality (5.49) is guaranteed to hold for all $a_\star \in (0, \bar{a}_\star)$, which by application of Theorem 6 proves that the origin of system (5.30), (5.31) is $\left(\frac{1}{\omega}, \delta\right)$-SPUUB, proving the theorem under condition (i). To study the convergence rate of our system we define

$$\gamma_r = \gamma_3 e^{\left(2\Delta\left[\sqrt{a_\star}\left(1+\frac{r_\star}{\delta}\right)+\frac{\rho_\star}{\delta}\right]+a_\star\gamma_2\right)} < 1. \tag{5.50}$$

For any $t \geq T$ we denote $N = \left\lfloor \frac{t-T}{\Delta} \right\rfloor$, where $\lfloor\cdot\rfloor$ is the floor function. Then for $t \geq T$ we have $t = T + \Delta\left(\frac{t-T}{\Delta} - \left\lfloor \frac{t-T}{\Delta} \right\rfloor\right) + N\Delta$ and from (5.48) we have the bound

$$V(t) \leq \gamma_r^N V\left(T + \Delta\left(\frac{t-T}{\Delta} - \left\lfloor \frac{t-T}{\Delta} \right\rfloor\right)\right). \tag{5.51}$$

This bound is obtained by noting from (5.48) and (5.50) that $V(s+N\Delta) \leq \gamma_r^N V(s)$ and by substituting $s = T + \Delta\left(\frac{t-T}{\Delta} - \left\lfloor \frac{t-T}{\Delta} \right\rfloor\right)$. Recalling that for $|e| > \delta$,

$$\begin{aligned}\dot{V} =& \bar{e}^T\left[A(t) - k\alpha B(t)B^T(t)\right]\bar{e} + \bar{e}^T A(t) r - \bar{e}^T\dot{r} \\ \leq& 2\left(\left|A(t) - k\alpha B(t)B^T(t)\right| + \frac{|A(t)r| + |\dot{r}|}{\delta}\right)V(t),\end{aligned}$$

we define

$$M_1 = e^{\int_0^{T+\Delta} 2\left(\left|A(\tau) - k\alpha B(\tau)B^T(\tau)\right| + \frac{|A(\tau)r(\tau)| + |\dot{r}(\tau)|}{\delta}\right)d\tau},$$

and because $\Delta\left(\frac{t-T}{\Delta} - \left\lfloor \frac{t-T}{\Delta} \right\rfloor\right) \leq \Delta$ we get the bound

$$V\left(T + \Delta\left(\frac{t-T}{\Delta} - \left\lfloor \frac{t-T}{\Delta} \right\rfloor\right)\right) \leq M_1 V(0).$$

Therefore

$$V(t) \leq M_1 \gamma_r^N V(0). \tag{5.52}$$

We now consider the term γ_r^N. Since $N = \frac{t-T-\Delta\left(\frac{t-T}{\Delta} - \left\lfloor \frac{t-T}{\Delta} \right\rfloor\right)}{\Delta} \geq \frac{t-T-\Delta}{\Delta}$, and $\gamma_r \in (0,1)$ it follows that

$$\gamma_r^N \leq \gamma_r^{\frac{t-T-\Delta}{\Delta}}. \tag{5.53}$$

With (5.52) and (5.53) we obtain

$$V(t) \leq M_1 \gamma_r^{-\frac{T+\Delta}{\Delta}} \gamma_r^{\frac{t}{\Delta}} V(0). \tag{5.54}$$

We now define $M = \sqrt{M_1 \gamma_r^{-\frac{T+\Delta}{\Delta}}}$ and rewrite $\gamma_r^{\frac{t}{\Delta}} = \left(\frac{1}{\gamma_r}\right)^{\frac{-t}{\Delta}} = e^{-\frac{\ln\left(\frac{1}{\gamma_r}\right)}{\Delta}t}$. Recalling that $\gamma_r \in (0,1)$ we define

$$R(k\alpha, \Delta, \beta_0, b_\star, a_\star) = \frac{\ln\left(\frac{1}{\gamma_r}\right)}{2\Delta} > 0, \tag{5.55}$$

and write the exponential decay of V as

$$V(t) \leq M^2 e^{-2Rt} V(0). \tag{5.56}$$

Substituting (5.50) into (5.55), we obtain (5.32). Finally recalling the definition of $V(t)$ we write the exponential decay of $|\bar{e}(t)|$ as

$$|\bar{e}(t)| \leq M e^{-Rt} |\bar{x}(0)|. \tag{5.57}$$

Therefore $\bar{e}$ exponentially converges to within a δ-ball of the origin with average convergence rate R as in (5.32). Proceeding to the proof of the theorem under condition (ii), for any given $a_\star$ we want to find a range of stabilizing values of $k\alpha$ as a function of Δ. For a given β_0, $b_\star$, $a_\star$ we first consider over what range of $\Delta \in (0,\infty)$ it is possible to satisfy the convergence condition (5.49). We define the function

$$F(k\alpha, \Delta) = \frac{k\alpha\Delta\beta_0}{1 + 2k^2\alpha^2\Delta^2 b_\star^2}\left[1 - \frac{2\Delta b_\star(\sqrt{a_\star}r_\star + \rho_\star)}{\delta^2\beta_0}\right] + e^{-\left(2\Delta\left[\sqrt{a_\star}\left(1+\frac{r_\star}{\delta}\right)+\frac{\rho_\star}{\delta}\right]+a_\star\gamma_2\right)} \tag{5.58}$$

which must have a value greater than 1 for (5.49) to be satisfied. In order to calculate the maximum possible value of (5.58) we fix Δ and set the derivative, with respect to $k\alpha$, of $F(k\alpha,\Delta)$ equal to zero, to find that $F(k\alpha,\Delta)$ has its maximum value at $(k\alpha)_{\mathrm{m}} = \frac{1}{\sqrt{2}\Delta b_\star}$ and the maximum value is

$$F\left((k\alpha)_{\mathrm{m}}, \Delta\right) = \frac{\beta_0}{2\sqrt{2}b_\star}\left[1 - \frac{2\Delta b_\star(\sqrt{a_\star}r_\star + \rho_\star)}{\delta^2\beta_0}\right] + e^{-\left(\sqrt{2}\Delta^2 a_\star + 2\Delta\sqrt{a_\star}\left[1+\frac{r_\star}{\delta}\right]+\frac{2\rho_\star\Delta}{\delta}\right)}. \tag{5.59}$$

The convergence condition requires this maximum value (5.59) to be greater than 1. We note that $F\left((k\alpha)_{\mathrm{m}}, \Delta\right)$ is strictly decreasing as a function of $\Delta \in (0,\infty)$. Therefore if $F\left((k\alpha)_{\mathrm{m}}, \bar{\Delta}_1\right) = 1$, it follows that $F\left((k\alpha)_{\mathrm{m}}, \Delta\right) > 1$ for all $\Delta \in \left(0, \bar{\Delta}_1\right)$. The condition $F\left((k\alpha)_{\mathrm{m}}, \bar{\Delta}_1\right) = 1$ implies that

$$0 = \bar{\Delta}_1^2\sqrt{2}a_\star + \bar{\Delta}_1 2\left(\sqrt{a_\star}\left[1 + \frac{r_\star}{\delta}\right] + \frac{\rho_\star}{\delta}\right) + \ln\left(1 + \frac{\beta_0}{2\sqrt{2}b_\star}\left[\frac{2\bar{\Delta}_1 b_\star(\sqrt{a_\star}r_\star + \rho_\star)}{\delta^2\beta_0} - 1\right]\right) \tag{5.60}$$

Therefore it is possible to stabilize the system when $0 < \Delta < \bar{\Delta}_1$ by choosing $k\alpha = (k\alpha)_{\mathrm{m}} = \frac{1}{\sqrt{2}\Delta b_\star}$. By continuity, for any $0 < \Delta < \bar{\Delta}_1$ there must be an interval containing $(k\alpha)_{\mathrm{m}}$ such that all values of $k\alpha$ within that interval satisfy condition (5.49). For $\Delta \in \left(0, \bar{\Delta}_1\right)$ we consider all values of $k\alpha$ that achieve $F(k\alpha,\Delta) > 1$. Recalling that

$$F(k\alpha,\Delta) = \frac{k\alpha\Delta\beta_0}{1+2k^2\alpha^2\Delta^2 b_\star^2}\left[1 - \frac{2\Delta b_\star(\sqrt{a_\star}r_\star + \rho_\star)}{\delta^2\beta_0}\right] + e^{-\left(2\Delta\left[\sqrt{a_\star}\left(1+\frac{r_\star}{\delta}\right)+\frac{\rho_\star}{\delta}\right]+a_\star\gamma_2\right)}, \tag{5.61}$$

to remove the $k\alpha$ dependence from the exponential in (5.61) we write out γ_2 and restrict our attention to $k\alpha > 1$, in which case

$$e^{-2\Delta\left(\frac{\sqrt{a_\star}\delta + r_\star + \rho_\star}{\delta} + \frac{2k\alpha\Delta^2 b_\star a_\star}{1+2k^2\alpha^2\Delta^2 b_\star^2}\right)} > e^{-2\Delta\left(\frac{\sqrt{a_\star}\delta + r_\star + \rho_\star}{\delta} + \frac{a_\star}{b_\star}\right)}. \tag{5.62}$$

We satisfy $F(k\alpha,\Delta) > 1$ by restricting $k\alpha$ to satisfy

$$\frac{k\alpha\Delta\beta_0}{1+2k^2\alpha^2\Delta^2 b_\star^2}\left[1 - \frac{2\Delta b_\star(\sqrt{a_\star}r_\star + \rho_\star)}{\delta^2\beta_0}\right] + e^{-2\Delta\left(\frac{\sqrt{a_\star}\delta + r_\star + \rho_\star}{\delta} + \frac{a_\star}{b_\star}\right)} > 1. \tag{5.63}$$

Setting (5.63) equal to 1, we solve for $k\alpha$ as:

$$\frac{\beta_0\left[\frac{1-\gamma_3}{1-\gamma}\right]}{4\Delta b_\star^2\left[1 - e^{-2\Delta\left(\frac{\sqrt{a_\star}\delta + r_\star + \rho_\star}{\delta} + \frac{a_\star}{b_\star}\right)}\right]} \pm \frac{\sqrt{\beta_0^2\left[\frac{1-\gamma_3}{1-\gamma}\right]^2 - 8b_\star^2\left[1 - e^{-2\Delta\left(\frac{\sqrt{a_\star}\delta + r_\star + \rho_\star}{\delta} + \frac{a_\star}{b_\star}\right)}\right]^2}}{4\Delta b_\star^2\left[1 - e^{-2\Delta\left(\frac{\sqrt{a_\star}\delta + r_\star + \rho_\star}{\delta} + \frac{a_\star}{b_\star}\right)}\right]} \tag{5.64}$$

To ensure $k\alpha$ is real valued we impose the condition

$$\beta_0^2\left[\frac{1-\gamma_3}{1-\gamma}\right]^2 = \beta_0^2\left[1 - \frac{2\Delta b_\star(\sqrt{a_\star}r_\star + \rho_\star)}{\delta^2\beta_0}\right]^2 \geq 8b_\star^2\left[1 - e^{-2\Delta\left(\frac{\sqrt{a_\star}\delta + r_\star + \rho_\star}{\delta} + \frac{a_\star}{b_\star}\right)}\right]^2. \tag{5.65}$$

Note that the left side of (5.65) is strictly increasing while the right side is strictly decreasing as Δ decreases. Therefore if equality in (5.65) holds for some $\bar{\Delta}_2$, then for all $\Delta < \bar{\Delta}_2$ the inequality will hold. Therefore (5.65) holds for all $0 < \Delta < \bar{\Delta}_2$ where $\bar{\Delta}_2$ satisfies

$$\beta_0\delta^2 - 2\bar{\Delta}_2 b_\star(\sqrt{a_\star}r_\star + \rho_\star) = 2\sqrt{2}b_\star\delta^2\left[1 - e^{-2\Delta\left(\frac{\sqrt{a_\star}\delta + r_\star + \rho_\star}{\delta} + \frac{a_\star}{b_\star}\right)}\right],$$

which implies that the new requirement on the possible values of Δ is $\Delta < \min\left\{\bar{\Delta}_1, \bar{\Delta}_2\right\}$, from which we obtain (5.37), (5.38). Returning to (5.64) and recalling the value $(k\alpha)_{\mathrm{m}} =$

$\frac{1}{\sqrt{2}\Delta b_\star}$ we have the roots $k\alpha = \frac{(k\alpha)_{\mathrm{m}}}{\eta}$ and $k\alpha = (k\alpha)_{\mathrm{m}}\eta$, where the value of η is given by:

$$\frac{\beta_0\left[\frac{1-\gamma_3}{1-\gamma}\right]}{2\sqrt{2}b_\star\left[1-e^{-2\Delta\left(\frac{\sqrt{a_\star}\delta+r_\star+\rho_\star}{\delta}+\frac{a_\star}{b_\star}\right)}\right]}+\frac{\sqrt{\beta_0^2\left[\frac{1-\gamma_3}{1-\gamma}\right]^2-8b_\star^2\left[1-e^{-2\Delta\left(\frac{\sqrt{a_\star}\delta+r_\star+\rho_\star}{\delta}+\frac{a_\star}{b_\star}\right)}\right]^2}}{2\sqrt{2}b_\star\left[1-e^{-2\Delta\left(\frac{\sqrt{a_\star}\delta+r_\star+\rho_\star}{\delta}+\frac{a_\star}{b_\star}\right)}\right]}.$$

Therefore the system is stable for

$$k\alpha \in \left(\frac{(k\alpha)_{\mathrm{m}}}{\eta},\ (k\alpha)_{\mathrm{m}}\eta\right). \tag{5.66}$$

We have thus derived sufficient conditions on Δ and $k\alpha$ to guarantee stability of our system. For each window Δ we have given an interval of stabilizing values of $k\alpha$, (5.66). However we now restrict our conditions on $k\alpha$ in order to give a more intuitive condition (5.39). We show that the interval (5.66) contains $(k\alpha)_{\mathrm{m}}$ by recalling (5.65) and verifying that $\eta \geq 1$ and therefore $\frac{1}{\eta} < 1$. Therefore the interval (5.66) contains the more restrictive, but more illustrative interval (5.39), where we have explicitly written out the value $(k\alpha)_{\mathrm{m}} = \frac{1}{\sqrt{2}\Delta b_\star}$. From the presence of $1-e^{-\Delta(\ldots)}$ in the denominator we see that this interval of stability grows unbounded in length as the window Δ decreases. □

The nonlinear extension of our tracking results, stated without proof is a trivial modification of Theorem 5.

Theorem 8. *Consider the nonlinear system*

$$\dot{x} = f(x,t) + G(x,t)u,$$

where $f, u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $G : \mathbb{R}^{n\times n} \times \mathbb{R} \to \mathbb{R}^{n\times n}$, with f and G each having separable dependence on x and t. Let there exist $\eta \in \mathscr{K}_\infty$ and $\beta_0 > 0$ such that f and G satisfy the following bounds for all $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$:

$$G(x,\tau)G^T(x,\tau) \geq \beta_0 I,\ |f(x,t)| \leq \eta(|x|). \tag{5.67}$$

Consider the error system

$$\dot{e} = f(e,t) + G(e,t)u - \dot{r}, \tag{5.68}$$

under the influence of the controller

$$u_i = \alpha\sqrt{\omega\omega_i^{'}}\cos(\omega\omega_i^{'}t) - k\sqrt{\omega\omega_i^{'}}\sin(\omega\omega_i^{'}t)\int_0^{|e|}\eta(r)dr, \tag{5.69}$$

where the frequencies $\omega_i^{'}$ are rational and distinct. If $k\alpha$ is chosen such that

$$k\alpha > \frac{1}{\beta_0} + \frac{\rho_\star}{\beta_0\eta(\delta)} \tag{5.70}$$

then origin of (5.68), (5.69) is $\left(\frac{1}{\omega},\delta\right)$-SPUUB.

5.3.1 Simulation Results

To demonstrate the extremum seeking controller's ability to handle unknown, quickly time varying control direction we consider the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1.1 & 1.2\\ -1.1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\cos(20t+.3)\\ \sin(20t+.3)\end{bmatrix}u, \tag{5.71}$$

which we want to follow the trajectory

$$r(t) = \left[\cos\left(t/4\right), \sin\left(t/2\right)\right]^T. \tag{5.72}$$

A physical motivation for this example can be that $x = (x_1, x_2)$ is the planar coordinate of a mobile robot, with its angular velocity actuator failed and stuck at 20, and which needs to follow the trajectory $r(t)$ using the forward velocity input u only, in the presence of a position-dependent perturbation given by $\begin{bmatrix}1.1 & 1.2\\ -1.1 & 1\end{bmatrix}x$. The uncontrolled system is unstable with poles at $1.05 \pm 1.15i$. We apply ES control

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)|e(t)|^2, \tag{5.73}$$

with parameters $\omega = 400$, $k = 10$, and $\alpha = 3$ and start from $x_1(0) = 1, x_2(0) = -1$. Figure 5.1 shows the system's trajectory over 26 seconds. Following an initial transient the control effort settles to a periodic slightly amplitude modulated waveform whose magnitude depends on $|e|$, also shown in Figure 5.1. This chapter contains material from

A. Scheinker and M. Krstić, "Extremum seeking-based tracking for unknown systems with unknowns control directions," *Proc. CDC*, Maui, HI, 2012.

The dissertation author was the primary author in this publication.

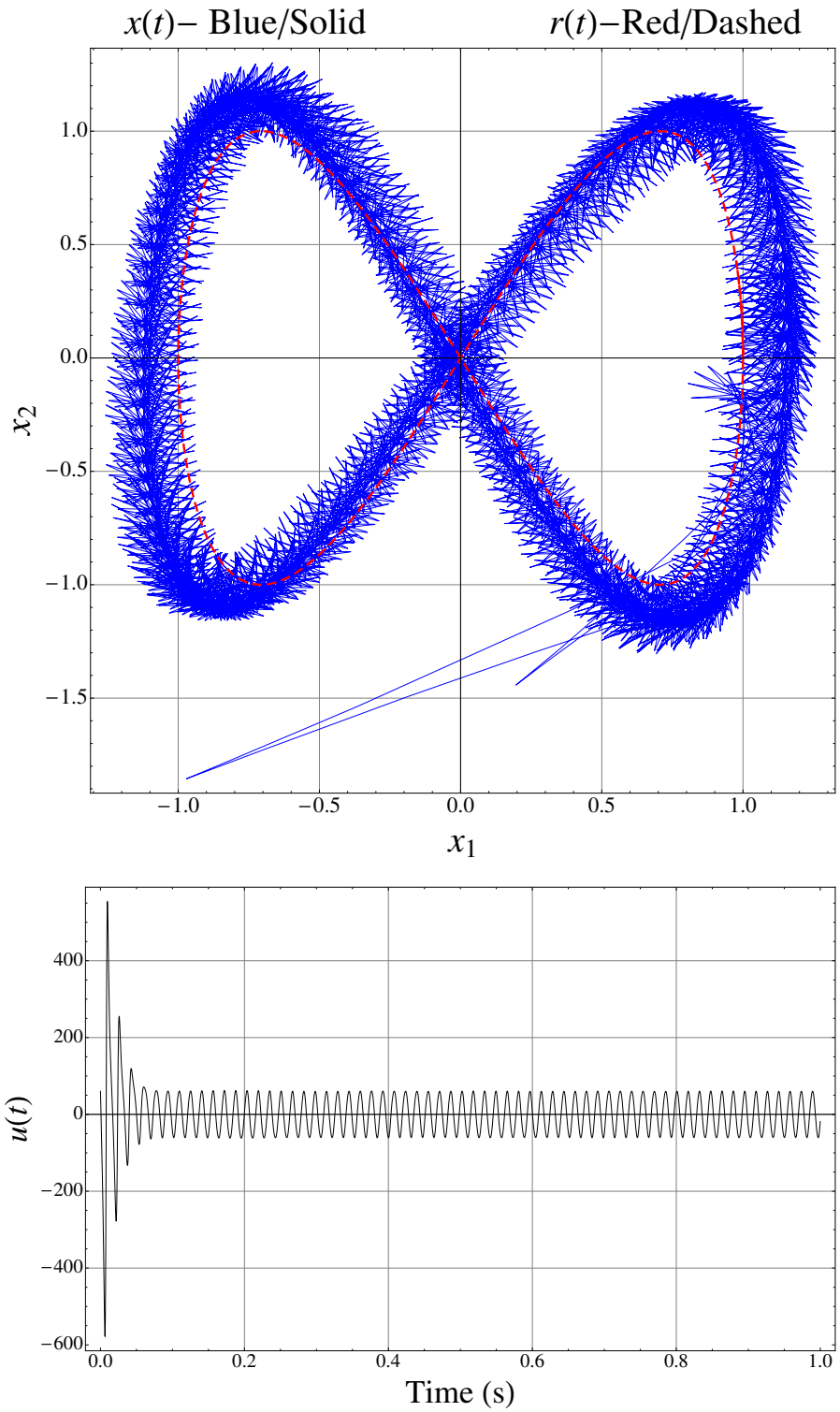


Figure 5.1: As the system trajectory approaches $r(t)$ the control effort quickly settles to an almost periodic waveform with amplitude modulation which is due to the fact that the disturbing term Ax has magnitude which depends on position.

Chapter 6

Non C^2 Controllers

In this chapter we present a modified ES scheme which has the same Lie bracket average as the scheme in the material above, but the advantage that its control effort settles to zero if the system's trajectory ever reaches the origin. In Section 6.1 we introduce the problem. In Section 6.2 we define forms of semiglobal practical boundedness inspired by the definitions of Moreau and Aeyels [58] which will be utilized in Section 6.5 for stability results. In Section 6.3 we prove our main averaging results for systems which are not differentiable at a point. In Section 6.4 we prove our main Lie bracket averaging results and apply them, together with the results of Section 6.3 to prove practical stability results for a class of Lie bracket averaged systems which are not differentiable at a point. In Section 6.5 we apply our results to a family of linear time varying systems and in Section 5.3.1 demonstrate, through simulation, the superiority of the non-C^2 controller over typical C^2 ES-based controllers. In Section 6.7 we extend our main averaging results for systems which are not well defined on entire sets and in Section 6.8 we utilize the results of Section 6.7 to prove practical stability results for a class of Lie bracket averaged systems which are not well defined on entire sets.

6.1 Introduction

One of the main limitations faced by all extremum seeking schemes is the presence of a persistent controller-induced perturbation term which prevents even stable

systems from settling at their equilibrium points. In this work, by introducing a non-smooth ES scheme we are able to reduce the introduced perturbations as the system settles towards equilibrium.

In extending the results of Moreau and Aeyels [58], in Theorem 9 we consider systems of the form:

$$\dot{x}=f(t,x), \tag{6.1}$$

$$\dot{x}=f^{\varepsilon}(t,x), \tag{6.2}$$

where the original system $f^{\varepsilon}(t,x):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, for a fixed time s is only assumed to satisfy $f^{\varepsilon}(s,x)\in C^2(\mathbb{R}^n\setminus B)$, for some subset $B\subset\mathbb{R}^n$, while the averaged system satisfies $f(s,x)\in C^2(\mathbb{R}^n)$. We first establish a practical stability result relating the stability of the averaged system (6.1) to that of system (6.2). Our stability result is based on the two systems satisfying a closeness of trajectories property, that is if for any given distance Δ and time length T there exists some $\hat{\varepsilon}$ such that for all $\varepsilon\in(0,\hat{\varepsilon})$ whenever the trajectory of system (6.2) is outside of the set B, it is within Δ of the trajectory of its averaged system (6.1). We show that if the two systems (6.1) and (6.2) satisfy such a closeness property then the convergence of (6.1) to the set B implies the practical convergence of (6.2) to the same set.

Having extended practical stability results for non-differentiable systems we consider, for $1\neq r\in\mathbb{R}$, systems and controllers of the form:

$$\dot{x}=A(t)x+B(t)u, \tag{6.3}$$

$$u=\alpha\sqrt{\omega}\cos(\omega t)|x|^r-\frac{k}{1-r}\sqrt{\omega}\sin(\omega t)|x|^{2-r}, \tag{6.4}$$

which give us the exact same C^2 Lie bracket averaged systems as in Chapters 1 - 5:

$$\dot{\bar{x}}=A(t)\bar{x}-k\alpha B(t)B^T(t)\bar{x},\qquad \bar{x}(t_0)=x(t_0), \tag{6.5}$$

for which we have proven practical stability results.

Remark 9. *For any choice of* $1\neq r\in\mathbb{R}$*, a controller of the form (6.4) results in the same Lie bracket averaged closed loop system (6.5). This is interesting considering that by choosing different values of r we get vastly different controllers, two of which (*$r=0$ *and*

$r = 2$) are smooth on $\mathbb{R}^n$, with all others non-smooth and even undefined at the origin. In general, we would like to choose $r \in (0,1)$ so that control effort is zero at the origin. In particular, we choose $r = \frac{1}{2}$ because through simulation it has shown the greatest balance in the magnitude of the perturbing term $\alpha\sqrt{\omega}\cos(\omega t)|x|^{\frac{1}{2}}$, the control gain term $2k\sqrt{\omega}\sin(\omega t)|x|^{\frac{3}{2}}$, and their decay as $|x|$ approaches zero. Therefore throughout the simulations we have chosen $r = 1 - \frac{1}{2} = \frac{1}{2}$ and used the controller

$$u = \alpha\sqrt{\omega}\cos(\omega t)|x|^{\frac{1}{2}} - 2k\sqrt{\omega}\sin(\omega t)|x|^{\frac{3}{2}}. \tag{6.6}$$

The controllers (6.4) obey a sort of symmetry about the forbidden choice $r = 1$, with the dither and control gain terms switching, depending which side of $r = 1$ we are on. We consider a few more values of r to demonstrate this symmetry.

If we choose $r = 1 + \frac{1}{2} = \frac{3}{2}$,

$$u = \alpha\sqrt{\omega}\cos(\omega t)|x|^{\frac{3}{2}} + 2k\sqrt{\omega}\sin(\omega t)|x|^{\frac{1}{2}},$$

which is very similar to the controller (6.6), but now the dominant term at large values of $|x|$ is the one with gain α and in this case we expect an increase in the value of α and decrease in k in order to achieve similar results.

When $r = 1 - 1 = 0$,

$$u = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)|x|^2,$$

resulting in the original smooth controllers used in Chapters 1 - 5.

When $r = 1 + 1 = 2$,

$$u = \alpha\sqrt{\omega}\cos(\omega t)|x|^2 + k\sqrt{\omega}\sin(\omega t),$$

again resulting in a smooth controller in which the gradient seeking and dithering terms have switched, which also faces the difficulty of persistent oscillation, even at the origin.

Finally, when $r = 1 - 2 = -1$,

$$u = \alpha\sqrt{\omega}\cos(\omega t)\frac{1}{|x|} - \frac{k}{2}\sqrt{\omega}\sin(\omega t)|x|^3,$$

and when $r = 1 + 2 = 3$,

$$u = \alpha\sqrt{\omega}\cos(\omega t)|x|^3 + \frac{k}{2}\sqrt{\omega}\sin(\omega t)\frac{1}{|x|},$$

giving us controllers which are not only non-smooth, but also undefined at the origin, which interestingly still achieve the same ultimate boundedness stability results on any region excluding the origin, which is topic of the remainder of this paper.

Recognizing that these new controllers, (6.4), for almost always all values of r, are not differentiable at the origin, we are motivated to extend the result of Gurvits and Li [30, 31, 32] to non-C^2 functions in order to prove a closeness of trajectories property, with which we establish the extended practical stability results of Moreau and Aeyels [58].

In what follows we consider systems of the form

$$\dot{x} = \sum_{i=1}^{m_1} b_i(x)\bar{u}_i(t) + \sum_{i=1}^{m_2} \hat{b}_i(x)\frac{1}{\sqrt{\varepsilon}}\hat{u}_i(t,\theta), \tag{6.7}$$

over compact sets $D \subset \mathbb{R}^n$, such that $b_i(x) \in C^2(\mathbb{R}^n)$ for all $i \in \{1,\ldots,m_1\}$ and $\hat{b}_i(x) \in C^2(D)$ for all $i \in \{1,\ldots,m_2\}$. We show that for $x(t), \bar{x}(t) \in D$ we can relate to (6.7) the Lie bracket averaged system

$$\dot{\bar{x}} = \sum_i b_i(\bar{x})\bar{u}_i(t) + \frac{1}{T}\sum_{i<j}\left[\hat{b}_i,\hat{b}_j\right](\bar{x})\nu_{i,j}(t), \quad \bar{x}(t_0) = x(t_0), \tag{6.8}$$

where

$$\nu_{i,j}(t) = \int_0^T \int_0^\theta \hat{u}_i(t,\tau)\hat{u}_j(t,\theta)d\tau d\theta$$
$$\left[\hat{b}_i,\hat{b}_j\right] = \frac{\partial \hat{b}_j}{\partial \bar{x}}\hat{b}_i - \frac{\partial \hat{b}_i}{\partial \bar{x}}\hat{b}_j,$$

and the function $\left[\hat{b}_i,\hat{b}_j\right]$ are in $C^2(\mathbb{R}^n)$. We show, as above, that the two systems satisfy the property that for any $\hat{T} > 0$ and $\delta > 0$, there exists $\varepsilon^\star$ such that for all $\varepsilon \in (0,\varepsilon^\star)$, $\forall t \in \left[t_0, t_0 + \hat{T}\right]$ for all $x(t) \in D$,

$$|x(t) - \bar{x}(t)| < \delta. \tag{6.9}$$

In Theorem 11, by utilizing the condition (6.9), we apply Theorem 9 to relate the stability of the origin of system (6.8) to the practical stability of system (6.7). In Proposition 2 and Theorem 12 we utilize Theorem 11 in order to establish practical stability results of scalar and vector LTV systems respectively, utilizing the non-smooth controllers.

In Theorem 13 we extend the practical stability results of Theorem 9 from points to sets. In Theorems 14 and 15 we then extend the results to practical stability of sets for systems well defined outside of sets to which their averaged systems are converging.

We point out that our contribution lies in the fact that we consider systems with high frequency $\left(\frac{1}{\varepsilon}\right)$ terms, which are not assumed to be globally C^2, but only in $C^2(D)$, for some $D \subset \mathbb{R}^n$. We show that even if a system's trajectory leaves D at some time $t \in \left[t_0, t_0 + \hat{T}\right]$ at which time it may no longer be well defined and therefore we can no longer associate with it an averaged system, the averaging relationship is re-established with the same choice of $\varepsilon^\star$ once the system's trajectory re-enters D. This allows us to develop non-C^2 controllers which achieve the same stability results as in [69, 70], but have the advantage that control effort settles to zero when the system reaches equilibrium.

In the case that the average system (6.8) is approaching a single point, $B = \{p\} \in D$ we re-establish the averaging result of Moreau and Aeyels, [58, Theorem 1] which state that if p is an asymptotically stable equilibrium point of (6.8) then in a sense p is also an equilibrium point of (6.7). In the case that the average system (6.8) is approaching a set $\bar{B} \subset D$, where $\bar{B}$ denotes the closure of the set B, we re-establish the averaging result of Dürr, Stanković, and Johansson, [22, Theorem 1] which states that if a compact invariant set $\bar{B}$ is a (globally) uniformly asymptotically stable set for system (6.8), then $\bar{B}$ is also, in a sense, a practically (globally) uniformly asymptotically stable for system (6.7).

6.2 Background on Semiglobal Practical Ultimate Boundedness

We start by making the following definitions similarly to Moreau and Aeyels [58]. In what follows, given a system

$$\dot{x} = f(t,x), \tag{6.10}$$

$\psi(t,t_0,x_0)$ denotes the solution of (6.10) which passes through the point x_0 at time t_0. In conjunction with (6.10), we consider systems of the form

$$\dot{x} = f^\varepsilon(t,x) \tag{6.11}$$

whose trajectories are denoted as $\psi^{\varepsilon}(t,t_0,x_0)$. We then define the following forms of stability for system (6.11).

Definition 7. *ε-Semiglobal Practical Uniform Ultimate Boundedness with ultimate bound δ ((ε,δ)-SPUUB): The origin of (6.11) is said to be (ε,δ)-SPUUB if it satisfies the following three conditions:*

- *(ε,δ)-Uniform Stability: For every $c_2 \in (\delta,\infty)$ there exists $c_1 \in (0,\infty)$ and $\hat{\varepsilon} \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\varepsilon \in (0,\hat{\varepsilon})$,*

$$\|\psi^{\varepsilon}(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0,\infty).$$

- *(ε,δ)-Uniform Ultimate Boundedness: For every $c_1 \in (0,\infty)$ there exists $c_2 \in (\delta,\infty)$ and $\hat{\varepsilon} \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\varepsilon \in (0,\hat{\varepsilon})$,*

$$\|\psi^{\varepsilon}(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0,\infty).$$

- *(ε,δ)-Global Uniform Attractivity: For all $c_1,c_2 \in (\delta,\infty)$ there exists $T \in (0,\infty)$ and $\hat{\varepsilon} \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\varepsilon \in (0,\hat{\varepsilon})$,*

$$\|\psi^{\varepsilon}(t,t_0,x_0)\| < c_2 \ \ \forall t \in [t_0+T,\infty).$$

6.3 Averaging for Systems not Differentiable at a Point

We start with a definition relating the trajectory of one system with another on a given set. The motivation for this definition is in order to perform analysis on a system which is only well defined on a subset of $\mathbb{R}^n$. We want to prove stability results about the original system by relating it to an averaged system, which is well defined on all of $\mathbb{R}^n$.

Definition 8. *Δ-Convergence of Trajectories on a set B (Δ-CT on B): For any $\Delta > 0$ and any set $B \subset \mathbb{R}^n$, the trajectory $\psi(t,t_0,x_0)$ of systems (2.10) is Δ-CT relative to the*

trajectory $\psi^{\varepsilon}(t,t_0,x_0)$ of system (2.14) on B if for any $\hat{T}>0$ there exists $\varepsilon^{\star}$ such that for all $\varepsilon\in(0,\varepsilon^{\star})$, for all $t\in\left[t_0,t_0+\hat{T}\right]$ for which $\psi^{\varepsilon}(t,t_0,x_0)\in B$,

$$|\psi(t,t_0,x_0)-\psi^{\varepsilon}(t,t_0,x_0)|<\Delta. \tag{6.12}$$

Theorem 9. *Consider the systems*

$$\dot{x}=f(t,x), \tag{6.13}$$

$$\dot{x}=f^{\varepsilon}(t,x), \tag{6.14}$$

whose trajectories, passing through the common point x_0 at time t_0 are denoted respectively as $\psi(t,t_0,x_0)$ and $\psi^{\varepsilon}(t,t_0,x_0)$. Consider functions f and f^{ε} such that for every fixed $s\in\mathbb{R}$, $f(s,x)\in C^2(\mathbb{R}^n)$ and $f^{\varepsilon}(s,x)\in C^2(\mathbb{R}^n\setminus\{0\})$. If for some $\delta>0$, $\psi(t)$ is Δ-CT relative to $\psi^{\varepsilon}(t)$ on $D=\mathbb{R}^n\setminus B(0,\delta)=\{x\in\mathbb{R}^n:|x|\geq\delta\}$, then if the origin is a GUAS equilibrium point of system (6.13) it is also a (ε,δ)-SPUUB equilibrium point of system (6.14).

6.3.1 Proof of (ε,δ)-Uniform Stability

For any $\delta>0$, for any $c_2\in(\delta,\infty)$, choose $\Delta\in(0,c_2)$. Because the origin of system (6.13) is Uniformly Stable, there exists $c_1\in(0,\infty)$ such that $\forall t_0\in\mathbb{R}$ and $|\psi(t_0)|<c_1$,

$$|\psi(t,t_0,x_0)|<c_2-\Delta,\ \forall t\in[t_0,\infty). \tag{6.15}$$

Note that by (6.15), $c_1<c_2$. Because the origin of system (6.13) is Globally Uniformly Attractive there exists $\bar{T}\in(0,\infty)$ such that

$$|\psi(t,t_0,x_0)|<c_1-\Delta\ \forall t\in[t_0+\bar{T},\infty). \tag{6.16}$$

We choose $\Delta_2\in(0,\min\{\Delta,c_1\})$, because the trajectory $\psi(t)$ is Δ_2-CT relative to $\psi^{\varepsilon}(t)$ on D, for $\hat{T}>\bar{T}$ there exists $\hat{\varepsilon}_1$ such that for all $\varepsilon\in(0,\hat{\varepsilon}_1)$, for all $t\in\left[t_0,t_0+\hat{T}\right]$ for which $|\psi^{\varepsilon}(t)|>\delta$, $|\psi(t)-\psi^{\varepsilon}(t)|<\Delta_2$. Therefore, for all $|\psi^{\varepsilon}(t_0)|<c_1$, $\forall t\in\left[t_0,t_0+\hat{T}\right)$, for which $|\psi^{\varepsilon}(t)|>\delta$, by (6.15)

$$|\psi^{\varepsilon}(t)|\leq|\psi^{\varepsilon}(t)-\psi(t)|+|\psi(t)|<\Delta_2+c_2-\Delta<c_2.$$

Otherwise, when $\psi^{\varepsilon}(t)$ is not in D, $|\psi^{\varepsilon}(t)| < \delta < c_2$. At $t = t_0 + \bar{T} < t_0 + \hat{T}$, by (6.16)

$$\begin{aligned}|\psi^{\varepsilon}(t_0+\bar{T})| &\leq |\psi^{\varepsilon}(t_0+\bar{T}) - \psi(t_0+\bar{T})| + |\psi(t_0+\bar{T})| \\ &< \Delta_2 + c_1 - \Delta_1 < c_1 < c_2,\end{aligned}$$

and therefore, by (6.15) we can repeat the above argument, extending the time interval indefinitely. Therefore for all $c_2 \in (0,\infty)$ there exists $\hat{\varepsilon}_1$ and c_1 such that for all $\varepsilon \in (0,\hat{\varepsilon}_1)$, for $|\psi^{\varepsilon}(t_0)| < c_1$, $\forall t \in [t_0,\infty)$,

$$|\psi^{\varepsilon}(t)| < c_2,$$

which proves that the origin of (6.14) is (ε,δ)-Uniformly Stable.

6.3.2 Proof of (ε,δ)-Uniform Ultimate Boundedness

Because the origin of system (6.13) is Uniformly Bounded, for every $c_1 \in (0,\infty)$ there exists $\hat{c}_2 \in (0,\infty)$ such that $\forall t_0 \in \mathbb{R}$ and $|x(t_0)| < c_1$,

$$|\psi(t)| < \hat{c}_2,\ \forall t \in [t_0,\infty). \tag{6.17}$$

Choose $\Delta_1 > 0$ and define c_2 as $c_2 = \hat{c}_2 + \Delta$. Again, because the origin of system (6.13) is Globally Uniformly Attractive there exists $\bar{T} \in (0,\infty)$ such that

$$|\psi(t)| < c_1 - \Delta_1\ \forall t \in [t_0+\bar{T},\infty). \tag{6.18}$$

We choose $\Delta_2 \in (0,\min\{\Delta_1,c_1,c_2\})$, $\psi(t)$ is Δ_2-CT relative to $\psi^{\varepsilon}(t)$ on D and therefore for any $\hat{T} > \bar{T}$ there exists $\hat{\varepsilon}_2$ such that for all $\varepsilon \in (0,\hat{\varepsilon}_2)$, for all $t \in \left[t_0, t_0+\hat{T}\right]$ for which $|\psi^{\varepsilon}(t)| > \delta$, $|\psi^{\varepsilon}(t) - \psi(t)| < \Delta_2$. Therefore, for all $|\psi(t_0)| < c_1$, $\forall t \in \left[t_0, t_0+\hat{T}\right)$, by (6.17)

$$|\psi^{\varepsilon}(t)| \leq |\psi^{\varepsilon}(t) - \psi(t)| + |\psi(t)| < \Delta_2 + \hat{c}_2 < c_2.$$

At $t = t_0 + \bar{T} < t_0 + \hat{T}$, by (6.18)

$$\begin{aligned}|\psi^{\varepsilon}(t_0+\bar{T})| &\leq |\psi^{\varepsilon}(t_0+\bar{T}) - \psi(t_0+\bar{T})| + |\psi(t_0+\bar{T})| \\ &< \Delta_2 + c_1 - \Delta_1 < c_1 < c_2,\end{aligned}$$

and therefore, by (6.17) we can repeat the above argument, extending the time interval indefinitely. Therefore for all $|\psi^{\varepsilon}(t_0)| < c_1$, there exists $\hat{\varepsilon}_2$ and c_2, such that for all $\varepsilon \in (0, \hat{\varepsilon}_2)$, $\forall t \in [t_0, \infty)$,

$$|\psi^{\varepsilon}(t)| < c_2,$$

which proves that the origin of (6.14) is (ε, δ)-Uniformly Ultimately Bounded.

6.3.3 Proof of (ε, δ)-Global Uniform Attractivity

Choose any $c_1 \in (0, \infty)$, $c_2 \in (\delta, \infty)$, and $\Delta_1 \in (0, c_2)$. Because the origin of system (6.13) is Uniformly Stable, there exists $\hat{c}_1 \in (0, \infty)$ such that $\forall t_0 \in \mathbb{R}$ and $|\psi(t_0)| < \hat{c}_1$,

$$|\psi(t)| < c_2 - \Delta_1, \ \forall t \in [t_0, \infty). \tag{6.19}$$

Because the origin of system (6.13) is Globally Uniformly Attractive there exists $\bar{T} \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and $|\psi(t_0)| < c_1$,

$$|\psi(t)| < \min\{c_1 - \Delta_1, \hat{c}_1 - \Delta_1\}, \ \forall t \in [t_0 + \bar{T}, \infty). \tag{6.20}$$

We choose $\Delta_2 \in (0, \min\{\Delta_1, c_1, \hat{c}_1\})$, $\psi(t)$ is Δ_2-CT relative to $\psi^{\varepsilon}(t)$ on D and therefore for any $\hat{T} > \bar{T}$ there exists $\hat{\varepsilon}_3$ such that for all $\varepsilon \in (0, \hat{\varepsilon}_3)$, for all $t \in \left[t_0, t_0 + \hat{T}\right]$ for which $|\psi^{\varepsilon}(t)| > \delta$, $|\psi^{\varepsilon}(t) - \psi(t)| < \Delta$.

Therefore, for all $|\psi^{\varepsilon}(t_0)| < c_1$, $\forall t \in \left[t_0, t_0 + \hat{T}\right)$, by (6.19)

$$|\psi^{\varepsilon}(t)| \leq |\psi^{\varepsilon}(t) - \psi(t)| + |\psi(t)| < \Delta_2 + c_2 - \Delta_1 < c_2.$$

At $t = t_0 + \bar{T} < t_0 + \hat{T}$, by (6.20)

$$\begin{aligned} |\psi^{\varepsilon}(t_0 + \bar{T})| &\leq |\psi^{\varepsilon}(t_0 + \bar{T}) - \psi(t_0 + \hat{T})| + |\psi(t_0 + \bar{T})| \\ &< \Delta_2 + \min\{c_1 - \Delta, \hat{c}_1 - \Delta\} \\ &\leq \min\{c_1, \hat{c}_1\}, \end{aligned}$$

and therefore, by (6.20) we can repeat the above argument, extending the time interval indefinitely. Therefore for all $|\psi^{\varepsilon}(t_0)| < c_1$, $\forall \varepsilon \in (0, \hat{\varepsilon}_3)$, $\forall t \in [t_0, \infty)$,

$$|\psi^{\varepsilon}(t)| < |\psi^{\varepsilon}(t) - \psi(t)| + |\psi(t)| < \Delta_2 + c_2 - \Delta_1 < c_2,$$

which proves that the origin of (6.14) is (ε,δ)-Globally Uniformly Attractive. By choosing $\hat{\varepsilon} = \min\{\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3\}$ the origin of (6.14) is (ε,δ)-SPUUB. ■

We now introduce a Lemma which we will use in the analysis that follows, which shows that if for a given system a point is (ε,δ)-SPUUB for all $\delta > 0$ then it is in fact ε-SPUAS.

Lemma 4. *If the origin of the system*

$$\dot{x} = f^{\varepsilon}(x,t), \tag{6.21}$$

is (ε,δ)-SPUUB for all $\delta > 0$, then the origin is ε-SPUAS.

Proof. If the origin of (6.21) is (ε,δ)-SPUUB for all $\delta > 0$, then for every $c_2 \in (0,\infty)$, for any $\delta \in (0,c_2)$, there exists $\hat{\varepsilon}$ such that the conditions of ε-SPUAS stability hold for all $\varepsilon \in (0,\hat{\varepsilon})$. □

Corollary 3. *Consider the systems*

$$\dot{x}=f(t,x), \tag{6.22}$$

$$\dot{x}=f^{\varepsilon}(t,x), \tag{6.23}$$

whose trajectories, passing through the common point x_0 at time t_0 are denoted respectively as $\psi(t,t_0,x_0)$ and $\psi^{\varepsilon}(t,t_0,x_0)$. Consider functions f and f^{ε} such that for every fixed $s \in \mathbb{R}$, $f(s,x) \in C^2(\mathbb{R}^n)$ and $f^{\varepsilon}(s,x) \in C^2(\mathbb{R}^n \setminus \{0\})$. If for all $\delta > 0$, $\psi(t)$ is Δ-CT relative to $\psi^{\varepsilon}(t)$ on $D = \mathbb{R}^n \setminus B(0,\delta) = \{x \in \mathbb{R}^n : |x| \geq \delta\}$, then if the origin is a GUAS equilibrium point of system (6.22) it is also a ε-SPUAS equilibrium point of system (6.23).

Proof. Because the origin of (6.23) is (ε,δ)-SPUUB for all $\delta > 0$, by Lemma 4 the origin of (6.23) is ε-SPUAS. □

Remark 10. *Corollary 3 extends to a non-differentiable point the results of Moreau and Aeyels, [58, Theorem 1] which state that if p is a GUAS equilibrium point of (6.23) then p is also an ε-SPUAS equilibrium point of (6.22).*

6.4 Lie bracket Averaging for Systems not Differentiable at a Point

We have established a stability relationship between two general systems (6.13) and (6.14), we now apply these results to a more specific class of systems.

We start by establishing the convergent trajectories property between a possibly not well defined system and its Lie bracket averaged version.

Theorem 10. *Given functions $b_i(x) \in C^2(\mathbb{R}^n)$ and $\hat{b}_i(x) \in C^2(\mathbb{R}^n \setminus \{0\})$, consider for $x \neq 0$ the system*

$$\dot{x} = \sum_{i=1}^{m_1} b_i(x)\bar{u}_i(t) + \sum_{i=1}^{m_2} \hat{b}_i(x)\frac{1}{\sqrt{\varepsilon}}\hat{u}_i(t,\theta), \tag{6.24}$$

where each $\hat{u}_i(t,\theta)$ is T-periodic in $\theta = \frac{t}{\varepsilon}$ and has zero average, $\int_\tau^{\tau+T} \hat{u}_i(t,\theta)d\theta = 0$. Consider also the Lie bracket averaged system

$$\dot{\bar{x}} = \sum_i b_i(\bar{x})\bar{u}_i(t) + \frac{1}{T}\sum_{i<j}\left[\hat{b}_i,\hat{b}_j\right](\bar{x})\nu_{i,j}(t), \quad \bar{x}(t_0) = x(t_0), \tag{6.25}$$

such that the average system is smooth, satisfying $\left[\hat{b}_i,\hat{b}_j\right](\bar{x}) \in C^2(\mathbb{R}^n)$. For any $R > r > 0$ and $\Delta > 0$, the trajectory of system (6.25) is Δ-CT relative to the trajectory of system (6.24) on the closed annulus $ann(0;r,R) = \{x \in \mathbb{R}^n : r \leq |x| \leq R\}$.

Proof. For given values of r and R let $D = ann(0;r,R)$. For $x(t_0) \in D$ we define the Lie bracket averaged system

$$\dot{\bar{x}}_1 = \sum_k b_i(\bar{x}_1)\bar{u}_i(t) + \frac{1}{T}\sum_{i<j}\left[\hat{b}_i,\hat{b}_j\right](\bar{x}_1)\nu_{i,j}(t), \quad \bar{x}_1(t_0) = x(t_0), \tag{6.26}$$

and exit time $t_{e_1} = \inf_{t\in\left(t_0,t_0+\hat{T}\right]}\{x(t) \text{ or } \bar{x}_1(t) \notin D\}$. Noting that a T-periodic function is also nT-periodic, we choose $n \in \mathbb{R}$ such that $nT > \hat{T} \geq t_{e_1}$ and apply the arguments of the original proof of [30, 31, 32] as well as [22, Lemma 2] and get the bound

$$\max_{t\in\left[t_0,t_0+t_{e_1}\right]} |x(t) - \bar{x}_1(t)| \leq \Delta_{1,\varepsilon} + \Delta_{2,\varepsilon}, \tag{6.27}$$

where

$$\Delta_{1,\varepsilon} \leq \sqrt{\varepsilon}\|\phi_1\| + O(\varepsilon) \tag{6.28}$$

$$\Delta_{2,\varepsilon} \leq \sqrt{\varepsilon}\left(\frac{e^{2\pi K}}{K}\right)\|f_2\|_{C\left[t_0,t_0+t_{e_1}\right]}, \tag{6.29}$$

where the functions f_2, ϕ_1 and the constant

$$K(D) = \sum_{i=1}^{m} \left\| \frac{\partial b_i}{\partial x} \right\| \|\bar{u}_i\| + \sum_{i<j} \left\| \frac{\partial \left[\hat{b}_i, \hat{b}_j\right]}{\partial x} \right\| \|\nu_{i,j}\|, \tag{6.30}$$

are as defined in [32]. Next we replace all of the norms in (6.28) - (6.30) by the maximum norm defined over the compact set $D \times \left[t_0, t_0 + \hat{T}\right]$, so for example we replace $\left\| \frac{\partial [b_i, b_j]}{\partial x} \right\| \|\nu_{i,j}\|$ with

$$\max_{x \in D} \left\{ \left\| \frac{\partial \left[\hat{b}_i(x), \hat{b}_j(x)\right]}{\partial x} \right\| \right\} \max_{t \in \left[t_0, t_0 + \hat{T}\right]} \left\{ \|\nu_{i,j}(t)\| \right\}.$$

By considering this maximum norm over the entire compact set $\left[t_0, t_0 + \hat{T}\right]$ we guarantee that (6.28) - (6.30) are independent of t_{e_1}. From (6.27) - (6.29) it is apparent that by choosing $\varepsilon^\star$ sufficiently small, for all $\varepsilon \in (0, \varepsilon^\star)$,

$$|x(t) - \bar{x}_1(t)| < \Delta, \ \forall t \in [t_0, t_{\mathrm{e}_1}). \tag{6.31}$$

If the trajectory of the system stays outside of D for all $t \in \left[t_{\mathrm{e}_1}, t_0 + \hat{T}\right]$ we can say nothing else and we are done. If on the other hand the trajectory of $x(t)$ re-enters D at time $t_{\mathrm{re}_1} \in \left(t_{\mathrm{e}_1}, t_0 + \hat{T}\right]$, then we define the second exit time, $t_{\mathrm{e}_2} = \inf_{t \in \left(t_{\mathrm{re}_1}, t_0 + \hat{T}\right]} \{x(t) \text{ or } \bar{x}(t) \notin D\}$ and the second Lie bracket averaged system $\bar{x}_2(t)$ such that $\bar{x}_2(t_{\mathrm{re}_1}) = x(t_{\mathrm{re}_1})$ and

$$\dot{\bar{x}}_2 = \sum_k b_k(\bar{x}_2)\bar{u}_i(t) + \frac{1}{T} \sum_{i<j} \left[\hat{b}_i, \hat{b}_j\right](\bar{x}_2)\nu_{i,j}(t). \tag{6.32}$$

Applying the same arguments, as above, noting that our previous choice of $\varepsilon^\star$ is independent of t_{re_1}, as it was chosen relative to the supremum over the entire compact set to which our analysis is confined, $D \times \left[t_0, t_0 + \hat{T}\right]$, we get that for all $\varepsilon \in (0, \varepsilon^\star)$,

$$|x(t) - \bar{x}_2(t)| < \Delta, \ \forall t \in [t_{\mathrm{re}_1}, t_{\mathrm{e}_2}). \tag{6.33}$$

Continuing in this way we define the ordered set of times $\{t_{\mathrm{e}_k}, t_{\mathrm{re}_k}\}_{k=1}^{N}$, $N \in \mathbb{N} \cup \{\infty\}$, at which either one of $x(t)$ or $\bar{x}_{\mathrm{e}_{k-1}}(t)$ exit and then $x(t)$ possibly re-enters the set D as:

$$\begin{aligned} t_{\mathrm{e}_k} &= \inf_{t \in \left(t_{\mathrm{re}_{k-1}}, t_0 + \hat{T}\right]} \left\{ x(t) \text{ or } \bar{x}_{\mathrm{e}_{k-1}}(t) \notin D \right\} \\ t_{\mathrm{re}_k} &= \min_{t \in \left(t_{\mathrm{e}_k}, t_0 + \hat{T}\right]} \{x(t) \in D\} \end{aligned}$$

and at each such re-entrant time, t_{re_k}, we define a Lie bracket averaged system $\bar{x}_k(t)$ such that $\bar{x}_k(t_{\mathrm{re}_{k-1}}) = x(t_{\mathrm{re}_{k-1}})$ and

$$\dot{\bar{x}}_k = \sum_k b_k(\bar{x}_k)\bar{u}_i(t) + \frac{1}{T}\sum_{i<j}\left[\hat{b}_i, \hat{b}_j\right](\bar{x}_k)v_{i,j}(t), \tag{6.34}$$

which satisfies that for all $\varepsilon \in (0, \varepsilon^\star)$,

$$|x(t) - \bar{x}_k(t)| < \Delta, \ \forall t \in \left[t_{\mathrm{re}_{k-1}}, t_{\mathrm{e}_k}\right). \tag{6.35}$$

Therefore, by definition, for any $t \in \left[t_0, t_0 + \hat{T}\right]$ such that $x(t) \in D$ there is some k such that $t \in \left[t_{\mathrm{re}_{k-1}}, t_{\mathrm{e}_k}\right)$ and there is an average system $\bar{x}_k(t)$ such that $|x(t) - \bar{x}_k(t)| < \Delta$. Therefore for each $t \in \left[t_0, t_0 + \hat{T}\right]$ such that $x(t) \in D$, we define the function

$$\bar{x}(t) = \bar{x}_k(t), \text{ s.t. } t \in \left[t_{\mathrm{re}_{k-1}}, t_{\mathrm{e}_k}\right), \tag{6.36}$$

and by definition, this $\bar{x}(t)$ satisfies the dynamics

$$\dot{\bar{x}} = \sum_k b_k(\bar{x})\bar{u}_i(t) + \frac{1}{T}\sum_{i<j}\left[\hat{b}_i, \hat{b}_j\right](\bar{x})v_{i,j}(t), \quad \bar{x}(t_0) = x(t_0), \tag{6.37}$$

and furthermore, for all $\varepsilon \in (0, \varepsilon^\star)$, for every $t \in \left[t_0, t_0 + \hat{T}\right]$ s.t. $x(t), \bar{x}(t) \in D$

$$|x(t) - \bar{x}(t)| = |x(t) - \bar{x}_k(t)| < \Delta, \tag{6.38}$$

and therefore the trajectory of system (6.24) is Δ-CT relative to the trajectory of (6.25) on $ann\,(0; r, R)$.

□

Having established the convergent trajectories property we now present our main result regarding Lie bracket averaged systems.

Theorem 11. *Given functions $b_i(x) \in C^2(\mathbb{R}^n)$ and $\hat{b}_i(x) \in C^2(\mathbb{R}^n \setminus \{0\})$, consider for $x \neq 0$ the system*

$$\dot{x} = \sum_{i=1}^{m_1} b_i(x)\bar{u}_i(t) + \frac{1}{\sqrt{\varepsilon}}\sum_{i=1}^{m_2}\hat{b}_i(x)\hat{u}_i(t, \theta), \tag{6.39}$$

where each $\bar{u}_i(t)$ and $\hat{u}_i(t, \theta)$ are bounded as functions of t and $\hat{u}_i(t, \theta)$ is T-periodic in $\theta = \frac{t}{\varepsilon}$ and has zero average, $\int_\tau^{\tau+T}\hat{u}_i(t, \theta)d\theta = 0$. Consider also the Lie bracket averaged system

$$\dot{\bar{x}} = \sum_i b_i(\bar{x})\bar{u}_i(t) + \frac{1}{T}\sum_{i<j}\left[\hat{b}_i, \hat{b}_j\right](\bar{x})v_{i,j}(t), \ \bar{x}(t_0) = x(t_0). \tag{6.40}$$

Assume that $\left[\hat{b}_i, \hat{b}_j\right](\bar{x}) \in C^2(\mathbb{R}^n)$*, namely, assume that the average system (6.40) is smooth despite* $\hat{b}_i(x)$ *being possibly non-differentiable at the origin. If the origin of 6.40 is GUAS then the origin of system (6.39) is* ε*-SPUAS.*

Proof. By Theorem 10, we have that for any $R, \delta > 0$ the trajectory of system (6.40) is Δ-CT relative to the trajectory of (6.39) on $\bar{B}_{R+\delta}(0) \setminus B_\delta(0) = \{x \in \mathbb{R}^n : \delta \leq |x| \leq R+\delta\}$. Therefore, for the given R and δ, for any $\hat{T} > 0$, there exists $\varepsilon^\star$ such that for all $\varepsilon \in (0, \varepsilon^\star)$ and any $x(0) \in \bar{B}_R(0) \setminus B_\delta(0)$,

$$\begin{aligned}\max_{t\in[0,\hat{T}]} |x(t)| &= \max_{t\in[0,\hat{T}]} |x(t) - \bar{x}(t) + \bar{x}(t)| \\ &\leq \max_{t\in[0,\hat{T}]} |x(t) - \bar{x}(t)| + \max_{t\in[0,\hat{T}]} |\bar{x}(t)| \\ &\leq \delta + R,\end{aligned}$$

where the last inequality holds due to the origin of (6.40) being GUAS. Therefore trajectories $x(t)$ starting in $\bar{B}_R(0) \setminus B_\delta(0)$ can only leave the set $\bar{B}_{R+\delta}(0) \setminus B_\delta(0)$ through the inner boundary $|x| = \delta$. Because the values R and δ may be chosen arbitrarily large and small respectively, by Theorem 9 the origin of system (6.39) is is (ε, δ)-SPUUB and by Corollary 3 it is in fact ε-SPUAS. □

6.5 Application

6.5.1 non-C^2 Control for Scalar Linear Time-Varying Systems

Proposition 2. *For* $1 \neq r \in \mathbb{R}$*, consider the system*

$$\dot{x} = a(t)x + b(t)u \tag{6.41}$$

$$u = \alpha\sqrt{\omega}\cos(\omega t)|x|^r - \frac{k\sqrt{\omega}}{1-r}\sin(\omega t)|x|^{2-r}, \tag{6.42}$$

and let there exist $\Delta > 0$*,* $\beta_0 > 0$*,* $a_\star > 0$*, and* $T > 0$ *such that* $a(t)$ *and* $b(t)$ *satisfy*

$$\frac{1}{\Delta}\int_s^{s+\Delta} b^2(\tau)d\tau \geq \beta_0, \ \forall\, s \geq T \tag{6.43}$$

$$\langle |a| \rangle_\Delta(s) \leq a_\star, \ \forall\, s \geq T. \tag{6.44}$$

If

$$k\alpha > \frac{a_\star}{\beta_0}, \tag{6.45}$$

then the origin of (6.41), (6.42) is $\frac{1}{\omega}$*-SPUAS with a lower bound on the average decay rate given by:*

$$\gamma_r = k\alpha\beta_0 - a_\star > 0. \tag{6.46}$$

Proof. System (6.41), (6.42) in closed loop form is

$$\dot{x} = a(t)x + b(t)\alpha\sqrt{\omega}\cos\omega t|x|^r - \frac{b(t)k\sqrt{\omega}}{1-r}\sin(\omega t)|x|^{2-r}, \tag{6.47}$$

which is C^2 on $\mathbb{R}^n \setminus \{0\}$ and has a Lie bracket average

$$\dot{\bar{x}} = \left[a(t) - k\alpha b^2(t)\right]\bar{x}, \tag{6.48}$$

which in Proposition 1 was shown to exponentially converge to the origin. Therefore, by Theorem 11 we conclude that the origin of (6.41), (6.42) is $\frac{1}{\omega}$-SPUAS. The decay rate is calculated as in [69]. □

6.5.2 non-C^2 Control for Vector Valued Linear Time-Varying Systems

Recall the notation

$$\langle Z\rangle_\Delta(s) \triangleq \frac{1}{\Delta}\int_s^{s+\Delta} Z(\tau)d\tau \tag{6.49}$$

for $Z : \mathbb{R} \to \mathbb{R}$, and note that, for any column vector B, $BB^T \leq |B|^2 I$.

Theorem 12. *For* $1 \neq r \in \mathbb{R}$*,*

$$\dot{x} = A(t)x + B(t)u, \tag{6.50}$$

$$u = \alpha\sqrt{\omega}\cos(\omega t)|x|^r - \frac{k\sqrt{\omega}}{1-r}\sin(\omega t)|x|^{2-r}, \tag{6.51}$$

where $x \in \mathbb{R}^n$*,* $A \in \mathbb{R}^{n\times n}$*,* $B \in \mathbb{R}^n$*,* $u \in \mathbb{R}$*, and let there exist* $\Delta > 0$*,* $b_\star \geq \beta_0 > 0$*,* $a_\star \geq 0$*, and* $T \geq 0$ *such that* $A(t)$ *and* $B(t)$ *satisfy*

$$\frac{1}{\Delta}\int_s^{s+\Delta} B(\tau)B^T(\tau)d\tau \geq \beta_0 I,\ \forall s \geq T \tag{6.52}$$

$$\left\langle |B|^2\right\rangle_\Delta(s) \leq b_\star,\ \forall s \geq 0, \tag{6.53}$$

$$\left\langle |A|^2\right\rangle_\Delta(s) \leq a_\star,\ \forall s \geq 0. \tag{6.54}$$

The origin of system (6.50), (6.51) is $\frac{1}{\omega}$-SPUAS with a lower bound on the average decay rate given by:

$$R = \frac{1}{2\Delta}\left[\ln\left(\frac{1}{\gamma}\right) - \gamma_2\right] - \sqrt{a_\star} > 0, \tag{6.55}$$

where

$$\gamma = 1 - \frac{k\alpha\Delta\beta_0}{1 + 2k^2\alpha^2\Delta^2 b_\star^2} > 0 \tag{6.56}$$

$$\gamma_2 = \frac{4k\alpha\Delta^3 b_\star}{1 + 2k^2\alpha^2\Delta^2 b_\star^2}, \tag{6.57}$$

under either of the two conditions:

(i) Given $k\alpha > 0$ and $\Delta > 0$, $a_\star$ is in the interval $(0, \bar{a}_\star)$, where

$$\bar{a}_\star = \left(\frac{\ln\left(\frac{1}{\gamma}\right)}{\Delta + \sqrt{\Delta^2 + \gamma_2 \ln\left(\frac{1}{\gamma}\right)}}\right)^2. \tag{6.58}$$

(ii) For a given $a_\star$, the window Δ satisfies $\Delta \in (0, \bar{\Delta})$, where

$$\bar{\Delta} = \frac{1}{\sqrt{a_\star}} \min\left\{\bar{\Delta}_1,\ \bar{\Delta}_2\right\}, \tag{6.59}$$

where

$$\bar{\Delta}_1 = \frac{\frac{1}{2}\ln\left(\frac{2\sqrt{2}b_\star}{2\sqrt{2}b_\star - \beta_0}\right)}{1 + \frac{\sqrt{a_\star}}{b_\star}}, \tag{6.60}$$

$$\bar{\Delta}_2 = \frac{-1 + \sqrt{1 + \sqrt{2}\ln\left(\frac{2\sqrt{2}b_\star}{2\sqrt{2}b_\star - \beta_0}\right)}}{\sqrt{2}}, \tag{6.61}$$

and $k\alpha > 1$ is selected such that

$$k\alpha \in \left(\frac{1}{2\sqrt{2}\Delta b_\star}, \frac{1}{2\sqrt{2}\Delta b_\star} + M(a_\star, b_\star, \beta_0, \Delta)\right), \tag{6.62}$$

where

$$M = \frac{\sqrt{\beta_0^2 - 8b_\star^2\left[1 - e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}\right]^2}}{8\Delta b_\star^2\left[1 - e^{-2\Delta\left(\sqrt{a_\star} + \frac{a_\star}{b_\star}\right)}\right]} > 0. \tag{6.63}$$

Proof. The closed-loop system (6.50), (6.51) is given by

$$\dot{x}=A(t)x+B(t)\alpha\sqrt{\omega}\cos(\omega t)|x|^r-\frac{B(t)k}{1-r}\sqrt{\omega}\sin(\omega t)|x|^{2-r}, \tag{6.64}$$

which is C^2 on $\mathbb{R}^n\setminus\{0\}$ and has a Lie bracket average

$$\dot{\bar{x}}=A(t)\bar{x}-k\alpha B(t)B^T(t)\bar{x}. \tag{6.65}$$

In Theorem 3 the exponential convergence of (6.65) to the origin was proven. Therefore, by the results of Theorem 11, the origin of (6.50), (6.51) is $\frac{1}{\omega}$-SPUAS. The decay rate is calculated as in Theorem 3. □

6.6 Comparison with C^2 controllers

6.6.1 Scalar Linear Time Varying System

We simulate and compare the performance of the system

$$\dot{x}=\sin(10t)x+\cos(10t)u,\ x(0)=1 \tag{6.66}$$

under the influence of the controllers

$$\begin{aligned}
u&=\alpha\sqrt{\omega}\cos(\omega t)|x|^{\frac{1}{2}}-2k\sqrt{\omega}\sin(\omega t)|x|^{\frac{3}{2}}\\
\tilde{u}&=\alpha\sqrt{\omega}\cos(\omega t)-k\sqrt{\omega}\sin(\omega t)|x|^2,\\
k&=10,\ \alpha=0.5,\ \omega=100.
\end{aligned}$$

We denote the trajectories of (6.66) with controllers $u(t)$ and $\tilde{u}(t)$ as $x(t)$ and $\tilde{x}(t)$ respectively, where we have chosen $r=\frac{1}{2}$ and $r=0$ respectively. The first 3 seconds of the simulation results are shown in Figure 6.1. In Figure 6.2 the last second of simulation is shown where both the state and control effort for system (6.66) with controller u have converged to zero.

Note 2. *In numerically simulating system (6.66) with controller u, as shown in Figures 6.1 and 6.2, the simulator was not able to reach the equilibrium point $x_1=0$, jumping through it repeatedly. In order to obtain this figure we had to artificially*

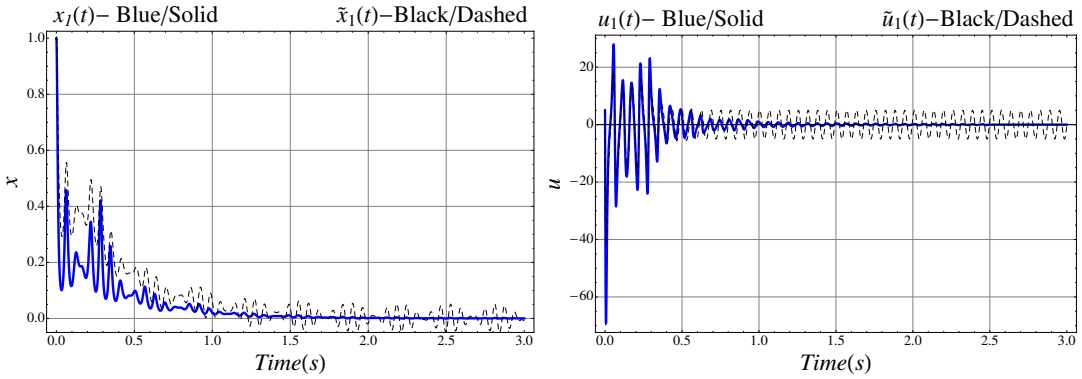


Figure 6.1: As the trajectories approach the origin the $\alpha\sqrt{\omega}\cos(\omega t)$ term of $\tilde{u}(t)$ persists which shows up in both the persistent control effort and the trajectory $\tilde{x}(t)$, which for $|\tilde{x}(t)| << 1$ settles to a steady state of $\tilde{x}(t) \approx \frac{\alpha}{\sqrt{\omega}}\sin(\omega t)$.

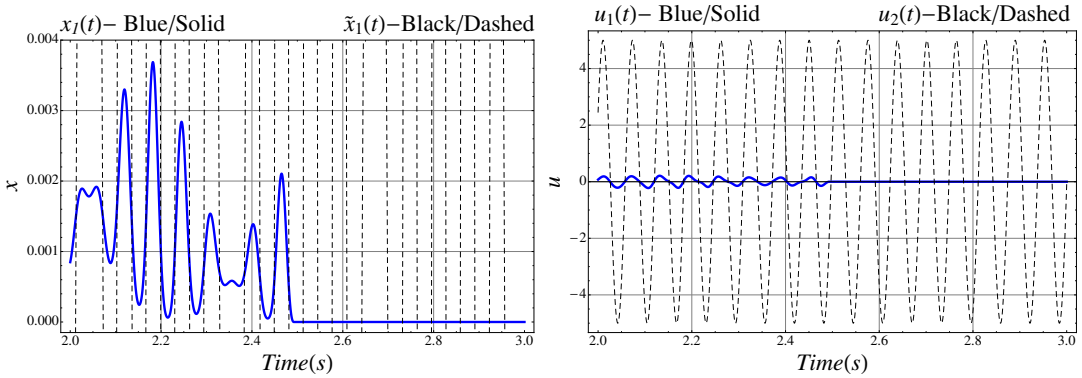


Figure 6.2: As the trajectory of system (6.66) with controller u approaches the origin the control effort also approaches zero. Once $x(t)$ reaches the origin, the system and control effort remain at that equilibrium point.

force the simulation of $x_1(t)$ *to reach zero by imposing the zero-forcing condition:* $\{\text{If } |x_1(t)| < 10^{-8}, x_1(t) = 0\}$. *This numerical problem was present in both Mathematica and Matlab simulations despite decreasing maximum simulation step size to* $dt = 10^{-7}$. *This numerical problem is due to the infinite derivative of u at* $x = 0$. *As* $x(t)$ *approaches* 0 *the digital simulation, working in finite steps, almost always jumps through zero and changes the sign of x causing a step change and change of sign in the derivative of u, which forces the control effort u to grow, resulting in an artificial*

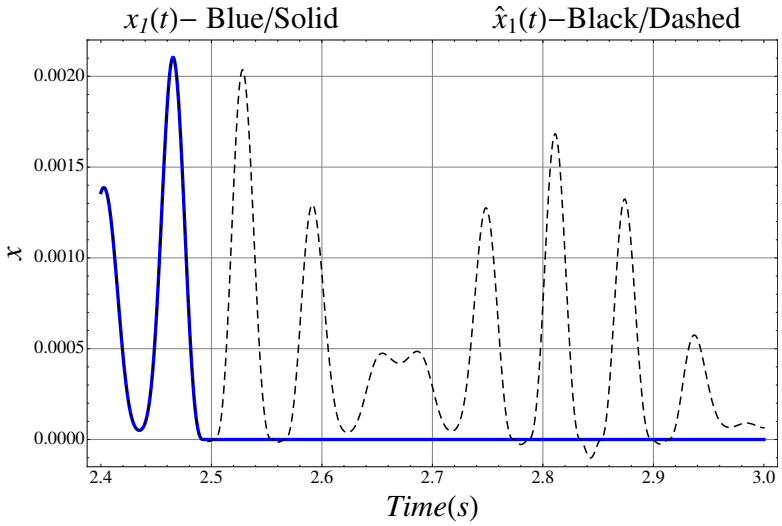


Figure 6.3: Without the zero condition: $\{\text{If } |x_1(t)| < 10^{-8}, x_1(t) = 0\}$, as in the dynamics of $\hat{x}(t)$, numerical limitations cause the simulation to pass through zero without settling, contradicting the dynamics as defined in (6.66) of which the origin is an equilibrium point.

persistent oscillation about zero. In Figure 6.3 we compare the results of performing the simulation with and without imposing the zero-forcing condition. In Figure 6.4 we focus on a zero crossing without the imposed zero-forcing condition to demonstrate the source of numerical error.

6.6.2 Two Dimensional Linear Time Varying System

To compare the C^2 and non-C^2 controllers' performances for a 2-dimensional system we recall the system (3.108):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2.1 & 4.9 \\ -7.5 & 3.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \cos(10t + .3) \\ \sin(10t + .3) \end{bmatrix} u\,.$$

A physical motivation for this example can be that $x = (x_1, x_2)$ is the planar coordinate of a mobile robot, with its angular velocity actuator failed and stuck at 10, and which has to be stabilized to the origin using the forward velocity input u only, in the presence

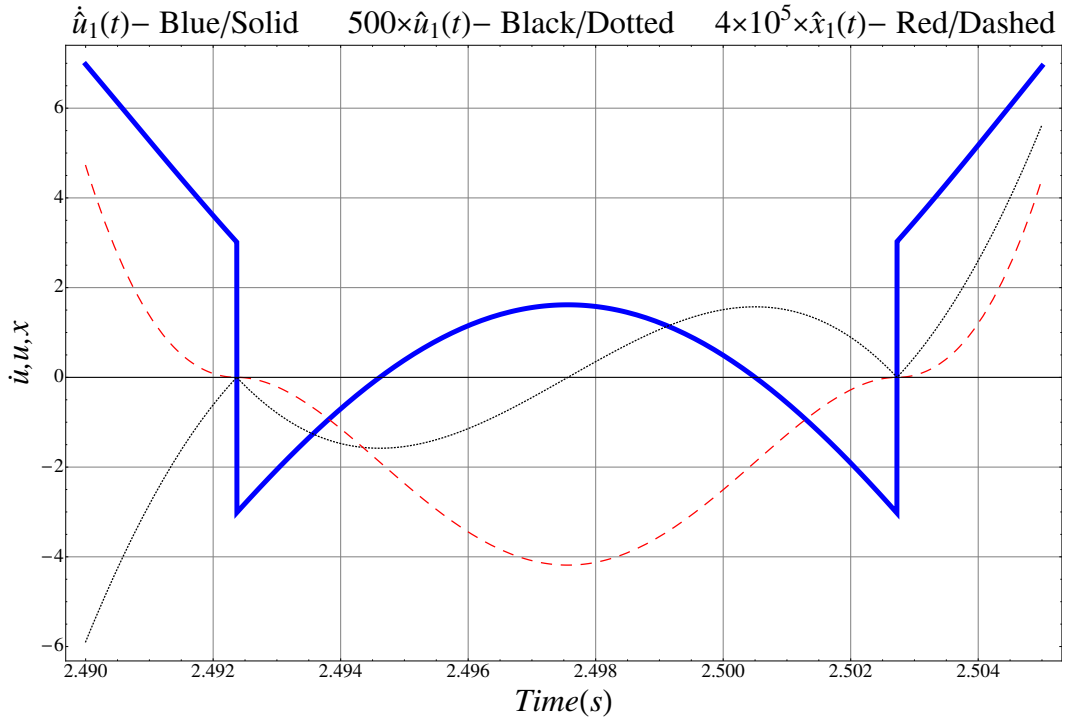


Figure 6.4: The numerical error in simulating system (6.66), due to the fact that the controller does not have a continuous derivative at zero, causes a step change in control derivative for values of $x(t)$ that jump past zero, is seen here in the plots of $\hat{x}(t)$, $\hat{u}(t)$ and $\dot{\hat{u}}(t)$.

of a position-dependent perturbation given by $\begin{bmatrix} 2.1 & 4.9 \\ -7.5 & 3.6 \end{bmatrix} x$. The uncontrolled system is unstable with poles at $2.85 \pm 10.7i$. Noting that $\beta_0 = 1$ and $a_\star = 8.325$, we apply ES controllers

$$u = \alpha\sqrt{\omega}\cos(\omega t)|x|^{\frac{1}{2}} - 2k\sqrt{\omega}\sin(\omega t)|x|^{\frac{3}{2}}$$
$$\tilde{u} = \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)|x|^2$$

with $\omega = 150, k = 4, \alpha = 2.5$, and start from $x_1(0) = 1, x_2(0) = -1$. We denote the trajectories of system (3.108) with controllers u and $\tilde{u}$ as $x = (x_1, x_2)^T$ and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)^T$ respectively, where we have chosen $r = \frac{1}{2}$ and $r = 0$ respectively. Figure 6.5 compares the systems' behavior over 3 seconds. Figure 6.7 zooms in on the last second, where the

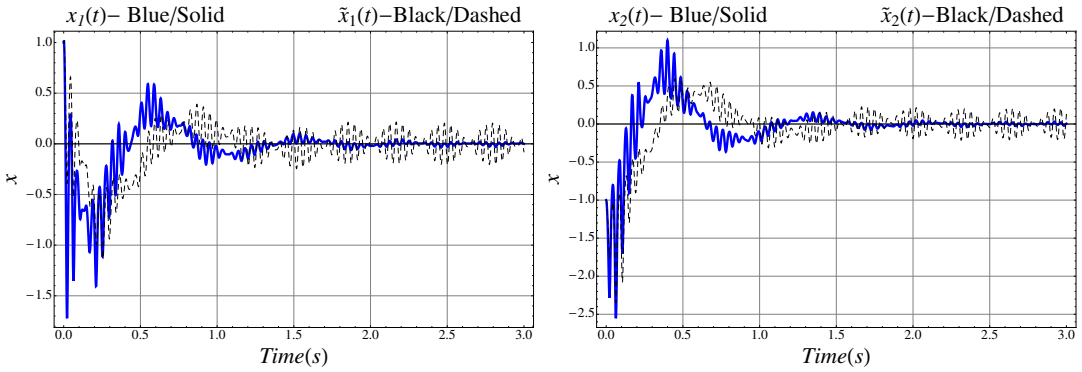


Figure 6.5: Near the origin the steady state oscillations of $x_i(t)$ due to the controller's input are greatly reduced relative to those of $\tilde{x}_i(t)$ which, for $|\tilde{x}| \ll 1$ reach a steady state proportional to $\frac{\alpha}{\sqrt{\omega}}$.

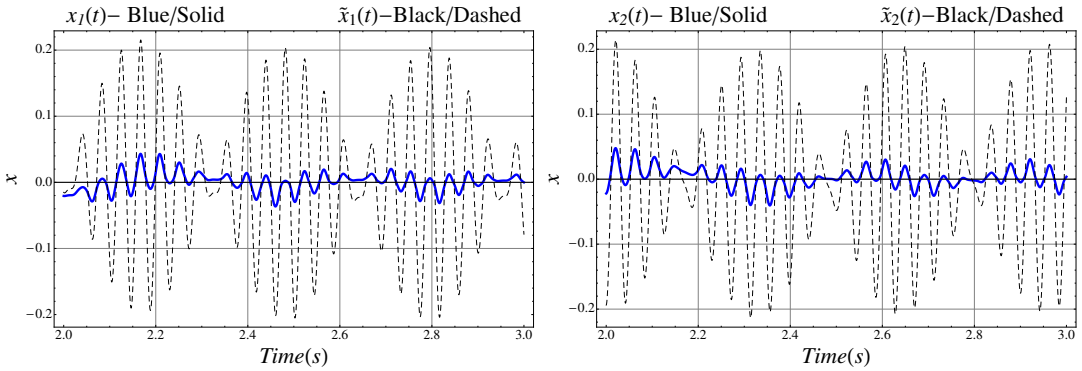


Figure 6.6: In the 2-dimensional case, even with the non-smooth controllers, the system may never settle to equilibrium at the origin, this would only be achieved if $x_1(t)$ and $x_2(t)$ reached the origin simultaneously.

system's dynamics are drastically different, due to the non-smooth controller's reduced dither near the origin. Figure 6.8 compares the systems' control efforts.

6.7 Averaging for Systems Undefined on a Set

Considering a system which is undefined or not continuously differentiable at many points which are known to be confined to some set, while their exact locations

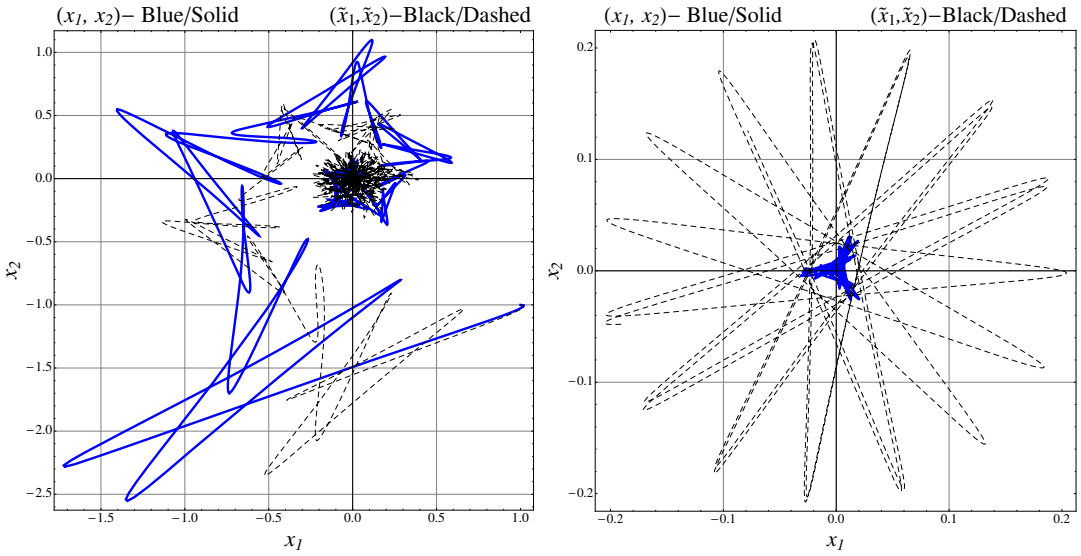


Figure 6.7: This figure shows the parametric view of the simulation of system 3.108 under both control laws. A zoom in on the last half second of the simulation clearly shows the benefit of the new controller.

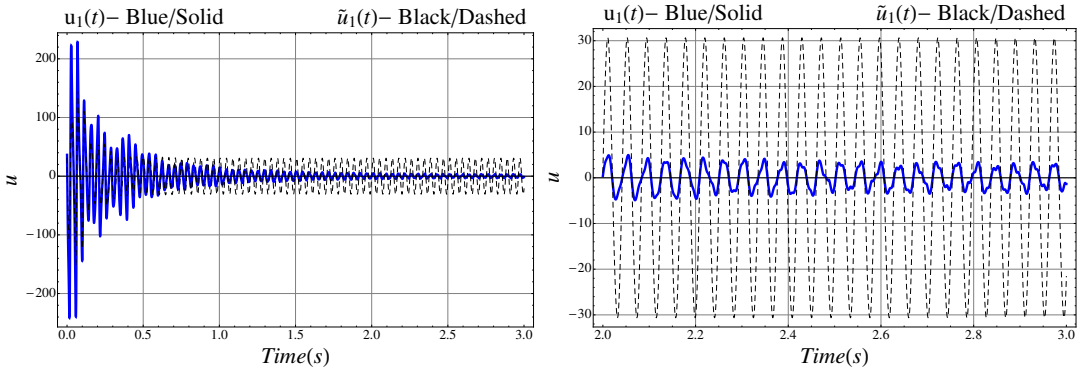


Figure 6.8: The initial effort of controller u is greater than that of $\tilde{u}$ due to the $2k$ term, which dominates for large values of $|x|$. The advantages of controller u are seen in the great reduction of control effort once the system has reached a neighborhood of the origin.

are unknown. For example, in the case of trajectory tracking, using the controllers described above, one may end up with terms of the form $|x(t) - r_i(t)|^{\frac{1}{2}}$, such that the multiple locations $r_i(t)$, of where the controller is non-differentiable are time-varying.

In such a case the analysis may be greatly complicated by having to consider collisions with these a priori unknown trajectories. However, if the trajectories are known to be confined to some set, and if our control goal is satisfied simply by entering this set, then the study of the system's convergence to the entire set, without regard of the details of what happens upon entrance of this set greatly simplifies the analysis. Therefore we now generalize the above results to compact sets, of which the set $\{0\}$ considered in the analysis of Section 6.3 is a special case.

For representing the distance between a point $p \in \mathbb{R}^n$ and a set $B \subset \mathbb{R}^n$ we introduce the notation

$$d(p,B) \triangleq \inf_{b \in B} \{|p-b|\}. \tag{6.67}$$

When convenient, for representing the distance $|p-b|$, between two points, p, $b \in \mathbb{R}^n$, we simply write $d(p,b)$.

We make the following definitions similarly to Moreau and Aeyels [58] and Dürr, Stanković, and Johansson, [22]. Again, in the following definitions, given a system

$$\dot{x} = f(t,x), \tag{6.68}$$

$\psi(t,t_0,x_0)$ denotes the solution of (6.68) which passes through the point x_0 at time t_0.

Definition 9. *Global Uniform Asymptotic Stability of a set (GUAS set) : A set $B \subset \mathbb{R}^n$ is said to be GUAS relative to (6.68) if it satisfies the following three conditions:*

- *Uniform Stability: For every $c_2 \in (0,\infty)$ there exists $c_1 \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $d(x_0,B) < c_1$,*

$$d\left(\psi(t,t_0,x_0),B\right) < c_2 \ \ \forall t \in [t_0,\infty).$$

- *Uniform Boundedness: For every $c_1 \in (0,\infty)$ there exists $c_2 \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $d(x_0,B) < c_1$,*

$$d\left(\psi(t,t_0,x_0),B\right) < c_2 < c_2 \ \ \forall t \in [t_0,\infty).$$

- *Global Uniform Attractivity: For all $c_1, c_2 \in (0,\infty)$ there exists $\bar{T} \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $d(x_0,B) < c_1$,*

$$d\left(\psi(t,t_0,x_0),B\right) < c_2 \ \ \forall t \in [t_0+\bar{T},\infty).$$

As before, in conjunction with (6.68), we consider systems of the form

$$\dot{x} = f^{\varepsilon}(t,x) \tag{6.69}$$

whose trajectories are denoted as $\psi^{\varepsilon}(t,t_0,x_0)$.

We then define the following forms of stability for system (6.69).

Definition 10. *ε-Semiglobal Practical Uniform Asymptotic Stability of a set(ε-SPUAS) set: A set $B \subset \mathbb{R}^n$ is said to be ε-SPUAS relative to (6.68) if it satisfies the following three conditions:*

- *Uniform Stability: For every $c_2 \in (0,\infty)$ there exists $c_1 \in (0,\infty)$ and $\hat{\varepsilon} \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $d(x_0,B) < c_1$ and for all $\varepsilon \in (0,\hat{\varepsilon})$,*

$$d\left(\psi^{\varepsilon}(t,t_0,x_0),B\right) < c_2 \ \ \forall t \in [t_0,\infty).$$

- *Uniform Boundedness: For every $c_1 \in (0,\infty)$ there exists $c_2 \in (0,\infty)$ and $\hat{\varepsilon} \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $d(x_0,B) < c_1$ and for all $\varepsilon \in (0,\hat{\varepsilon})$,*

$$d\left(\psi^{\varepsilon}(t,t_0,x_0),B\right) < c_2 \ \ \forall t \in [t_0,\infty).$$

- *Global Uniform Attractivity: For all $c_1,c_2 \in (0,\infty)$ there exists $\bar{T} \in (0,\infty)$ and $\hat{\varepsilon} \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $d(x_0,B) < c_1$ and for all $\varepsilon \in (0,\hat{\varepsilon})$,*

$$d\left(\psi^{\varepsilon}(t,t_0,x_0),B\right) < c_2 \ \ \forall t \in [t_0+\bar{T},\infty).$$

Definition 11. *ε-Semiglobal Practical Uniform Ultimate Boundedness of a set with ultimate bound δ ((ε,δ)-SPUUB set): A set $B \subset \mathbb{R}^n$ is said to be (ε,δ)-SPUUB relative to (6.68) if it satisfies the following three conditions:*

- *(ε,δ)-Uniform Stability: For every $c_2 \in (\delta,\infty)$ there exists $c_1 \in (0,\infty)$ and $\hat{\varepsilon} \in (0,\infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $d(x_0,B) < c_1$ and for all $\varepsilon \in (0,\hat{\varepsilon})$,*

$$d\left(\psi^{\varepsilon}(t,t_0,x_0),B\right) < c_2 \ \ \forall t \in [t_0,\infty).$$

- (ε,δ)-*Uniform Ultimate Boundedness: For every* $c_1 \in (0,\infty)$ *there exists* $c_2 \in (\delta,\infty)$ *and* $\hat{\varepsilon} \in (0,\infty)$ *such that for all* $t_0 \in \mathbb{R}$ *and for all* $x_0 \in \mathbb{R}^n$ *with* $d(x_0,B) < c_1$ *and for all* $\varepsilon \in (0,\hat{\varepsilon})$*,*

$$d\left(\psi^{\varepsilon}(t,t_0,x_0),B\right) < c_2 \ \ \forall t \in [t_0,\infty).$$

- (ε,δ)-*Global Uniform Attractivity: For all* $c_1,c_2 \in (\delta,\infty)$ *there exists* $T \in (0,\infty)$ *and* $\hat{\varepsilon} \in (0,\infty)$ *such that for all* $t_0 \in \mathbb{R}$ *and for all* $x_0 \in \mathbb{R}^n$ *with* $d(x_0,B) < c_1$ *and for all* $\varepsilon \in (0,\hat{\varepsilon})$*,*

$$d\left(\psi^{\varepsilon}(t,t_0,x_0),B\right) < c_2 \ \ \forall t \in [t_0+T,\infty).$$

Theorem 13. *Consider the systems*

$$\dot{x}=f(t,x), \tag{6.70}$$

$$\dot{x}=f^{\varepsilon}(t,x), \tag{6.71}$$

whose trajectories, passing through the common point x_0 *at time* t_0 *are denoted respectively as* $\psi(t,t_0,x_0)$ *and* $\psi^{\varepsilon}(t,t_0,x_0)$*. Consider functions* f *and* f^{ε} *such that for some set* $B \subset \mathbb{R}^n$*, for every fixed* $s \in \mathbb{R}$*,* $f(s,x) \in C^2(\mathbb{R}^n)$ *and* $f^{\varepsilon}(s,x) \in C^2(\mathbb{R}^n/B)$*. If for some* $\delta > 0$*,* $\psi(t)$ *is* Δ*-CT relative to* $\psi^{\varepsilon}(t)$ *on* $D = \mathbb{R}^n/B$*, then if the set B is a GUAS relative to system (6.70) it is also* (ε,δ)*-SPUUB relative to system (6.71).*

Proof. The proof follows exactly as in the proof of Theorem 9 with $|\cdot|$ replaced by $d(\cdot,B)$ throughout. □

Exactly as in Lemma 4 we then have the result:

Lemma 5. *If a set B, relative to the system*

$$\dot{x} = f^{\varepsilon}(x,t), \tag{6.72}$$

whose trajectories are denoted as $\psi^{\varepsilon}(t,t_0,x_0)$*, is* (ε,δ)*-SPUUB for all* $\delta > 0$*, then B is* ε*-SPUAS relative to system (6.72).*

6.8 Lie bracket Averaging for Systems Undefined on a Set

In order to extend the above results to Lie bracket averaging we now prove a converging trajectories property which gives us a bound on the distance between systems of the form (6.39) and (6.40) which have common initial conditions, for times when each system is within a compact set D on which f is C^2, allowing the trajectory of either system to leave and then re-enter D.

Theorem 14. *Given a compact set $D \subset \mathbb{R}^n$, and the functions $b_i(x) \in C^2(\mathbb{R}^n)$ and $\hat{b}_i(x) \in C^2(D)$, for $x \in D$ consider the system*

$$\dot{x} = \sum_{i=1}^{m_1} b_i(x)\bar{u}_i(t) + \sum_{i=1}^{m_2} \hat{b}_i(x)\frac{1}{\sqrt{\varepsilon}}\hat{u}_i(t,\theta), \tag{6.73}$$

where each $\hat{u}_i(t,\theta)$ is T-periodic in $\theta = \frac{t}{\varepsilon}$ and has zero average, $\int_\tau^{\tau+T} \hat{u}_i(t,\theta)d\theta = 0$. Consider also the Lie bracket averaged system

$$\dot{\bar{x}} = \sum_i b_i(\bar{x})\bar{u}_i(t) + \frac{1}{T}\sum_{i<j}\left[\hat{b}_i,\hat{b}_j\right](\bar{x})\nu_{i,j}(t), \quad \bar{x}(t_0) = x(t_0), \tag{6.74}$$

such that the average system is smooth, satisfying $\left[\hat{b}_i,\hat{b}_j\right] \in C^2(\mathbb{R}^n)$. For any $\Delta > 0$, the trajectory of system (6.73) is Δ-CT relative to the trajectory of system (6.74) on D.

Proof. For a given $x(t_0) \in D$ we define the Lie bracket averaged system

$$\dot{\bar{x}}_1 = \sum_i b_i(\bar{x}_1)\bar{u}_i(t) + \frac{1}{T}\sum_{i<j}\left[\hat{b}_i,\hat{b}_j\right](\bar{x}_1)\nu_{i,j}(t), \quad \bar{x}_1(t_0) = x(t_0), \tag{6.75}$$

and exit time $t_{e_1} = \inf_{t\in\left(t_0,t_0+\hat{T}\right]}\{x(t) \text{ or } \bar{x}_1(t) \notin D\}$. Noting that a T-periodic function is also nT-periodic, we choose $n \in \mathbb{R}$ such that $nT > \hat{T} \geq t_{e_1}$ and apply the arguments of the original proof of [30, 31, 32] as well as [22, Lemma 2] and get the bound

$$\max_{t\in\left[t_0,t_0+t_{e_1}\right]} |x(t) - \bar{x}_1(t)| \leq \Delta_{1,\varepsilon} + \Delta_{2,\varepsilon}, \tag{6.76}$$

where

$$\Delta_{1,\varepsilon} \leq \sqrt{\varepsilon}\|\phi_1\| + O(\varepsilon) \tag{6.77}$$

$$\Delta_{2,\varepsilon} \leq \sqrt{\varepsilon}\left(\frac{e^{2\pi K}}{K}\right)\|f_2\|_{C\left[t_0,t_0+t_{e_1}\right]}, \tag{6.78}$$

where the functions f_2, ϕ_1 and the constant

$$K(D) = \sum_{i=1}^{m} \left\| \frac{\partial b_i}{\partial x} \right\| \|\bar{u}_i\| + \sum_{i<j} \left\| \frac{\partial \left[\hat{b}_i, \hat{b}_j\right]}{\partial x} \right\| \|\nu_{i,j}\|, \tag{6.79}$$

are as defined in [32]. Next we replace all of the norms in (6.77) - (6.79) by the maximum norm defined over the compact set $D \times \left[t_0, t_0 + \hat{T}\right]$, so for example we replace $\left\| \frac{\partial \left[\hat{b}_i, \hat{b}_j\right]}{\partial x} \right\| \|\nu_{i,j}\|$ with

$$\max_{x \in D} \left\{ \left\| \frac{\partial \left[\hat{b}_i(x), \hat{b}_j(x)\right]}{\partial x} \right\| \right\} \max_{t \in \left[t_0, t_0 + \hat{T}\right]} \left\{ \|\nu_{i,j}(t)\| \right\}.$$

By considering this maximum norm over the entire compact set $\left[t_0, t_0 + \hat{T}\right]$ we guarantee that (6.77) - (6.79) are independent of t_{e_1}. From (6.76) - (6.78) it is apparent that by choosing $\varepsilon^\star$ sufficiently small, for all $\varepsilon \in (0, \varepsilon^\star)$,

$$|x(t) - \bar{x}_1(t)| < \delta, \ \forall t \in [t_0, t_{\mathrm{e}_1}). \tag{6.80}$$

If the trajectory of the system stays outside of D for all $t \in \left[t_{\mathrm{e}_1}, t_0 + \hat{T}\right]$ we can say nothing else and we are done. If on the other hand the trajectory of $x(t)$ re-enters D at time $t_{\mathrm{re}_1} \in \left(t_{\mathrm{e}_1}, t_0 + \hat{T}\right]$, then we define the second exit time, $t_{\mathrm{e}_2} = \inf_{t \in \left(t_{\mathrm{re}_1}, t_0 + \hat{T}\right]} \{x(t) \text{ or } \bar{x}_1(t) \notin D\}$ and the second Lie bracket averaged system $\bar{x}_2(t)$ such that $\bar{x}_2(t_{\mathrm{re}_1}) = x(t_{\mathrm{re}_1})$ and

$$\dot{\bar{x}}_2 = \sum_i b_i(\bar{x}_2)\bar{u}_i(t) + \frac{1}{T} \sum_{i<j} \left[\hat{b}_i, \hat{b}_j\right](\bar{x}_2)\nu_{i,j}(t). \tag{6.81}$$

Applying the same arguments, as above, noting that our previous choice of $\varepsilon^\star$ is independent of t_{re_1}, as it was chosen relative to the supremum over the entire compact set to which our analysis is confined, $D \times \left[t_0, t_0 + \hat{T}\right]$, we get that for all $\varepsilon \in (0, \varepsilon^\star)$,

$$|x(t) - \bar{x}_2(t)| < \delta, \ \forall t \in [t_{\mathrm{re}_1}, t_{\mathrm{e}_2}). \tag{6.82}$$

Continuing in this way we define the ordered set of times $\{t_{\mathrm{e}_k}, t_{\mathrm{re}_k}\}_{k=1}^{N}$, $N \in \mathbb{N} \cup \{\infty\}$, at which either one of $x(t)$ or $\bar{x}_2(t)$ exit and then $x(t)$ possibly re-enters the set D as:

$$\begin{aligned} t_{\mathrm{e}_k} &= \inf_{t \in \left(t_{\mathrm{re}_{k-1}}, t_0 + \hat{T}\right]} \left\{ x(t) \text{ or } \bar{x}_{\mathrm{e}_{k-1}}(t) \notin D \right\} \\ t_{\mathrm{re}_k} &= \min_{t \in \left(t_{\mathrm{e}_k}, t_0 + \hat{T}\right]} \{x(t) \in D\} \end{aligned}$$

and at each such re-entrant time, t_{re_k}, we define a Lie bracket averaged system $\bar{x}_k(t)$ such that $\bar{x}_k(t_{\mathrm{re}_{k-1}}) = x(t_{\mathrm{re}_{k-1}})$ and

$$\dot{\bar{x}}_k = \sum_i b_i(\bar{x}_k)\bar{u}_i(t) + \frac{1}{T}\sum_{i<j}\left[\hat{b}_i, \hat{b}_j\right](\bar{x}_k)\nu_{i,j}(t), \tag{6.83}$$

which satisfies that for all $\varepsilon \in (0, \varepsilon^\star)$,

$$|x(t) - \bar{x}_k(t)| < \delta, \ \forall t \in \left[t_{\mathrm{re}_{k-1}}, t_{\mathrm{e}_k}\right). \tag{6.84}$$

Therefore, by definition, for any $t \in \left[t_0, t_0 + \hat{T}\right]$ such that $x(t) \in D$ there is some k such that $t \in \left[t_{\mathrm{re}_{k-1}}, t_{\mathrm{e}_k}\right)$ and there is an average system $\bar{x}_k(t)$ such that $|x(t) - \bar{x}_k(t)| < \delta$. Therefore for each $t \in \left[t_0, t_0 + \hat{T}\right]$ such that $x(t) \in D$, we define the function

$$\bar{x}(t) = \bar{x}_k(t), \text{ s.t. } t \in \left[t_{\mathrm{re}_{k-1}}, t_{\mathrm{e}_k}\right), \tag{6.85}$$

and by definition, this $\bar{x}(t)$ satisfies the dynamics

$$\dot{\bar{x}} = \sum_i b_i(\bar{x})\bar{u}_i(t) + \frac{1}{T}\sum_{i<j}\left[\hat{b}_i, \hat{b}_j\right](\bar{x})\nu_{i,j}(t), \quad \bar{x}(t_0) = x(t_0), \tag{6.86}$$

and furthermore, for all $\varepsilon \in (0, \varepsilon^\star)$, for every $t \in \left[t_0, t_0 + \hat{T}\right]$ s.t. $x(t), \bar{x}(t) \in D$

$$|x(t) - \bar{x}(t)| = |x(t) - \bar{x}_k(t)| < \delta. \tag{6.87}$$

□

In the following Theorem, for representing points that are at least a distance $\delta > 0$ within the interior of a set $D \subset \mathbb{R}^n$ we introduce the notation

$$\mathrm{int}_\delta(D) = \{x \in \mathbb{R}^n : x \in D \text{ and } d(x, \partial D) \geq \delta\}, \tag{6.88}$$

where ∂D denotes the boundary of D.

Theorem 15. *Given a compact set $D \subset \mathbb{R}^n$, and the functions $b_i(x) \in C^2(\mathbb{R}^n)$ and $\hat{b}_i(x) \in C^2(D)$, for $x \in D$ consider the system*

$$\dot{x} = \sum_{i=1}^{m_1} b_i(x)\bar{u}_i(t) + \sum_{i=1}^{m_2}\hat{b}_i(x)\frac{1}{\sqrt{\varepsilon}}\hat{u}_i(t, \theta), \tag{6.89}$$

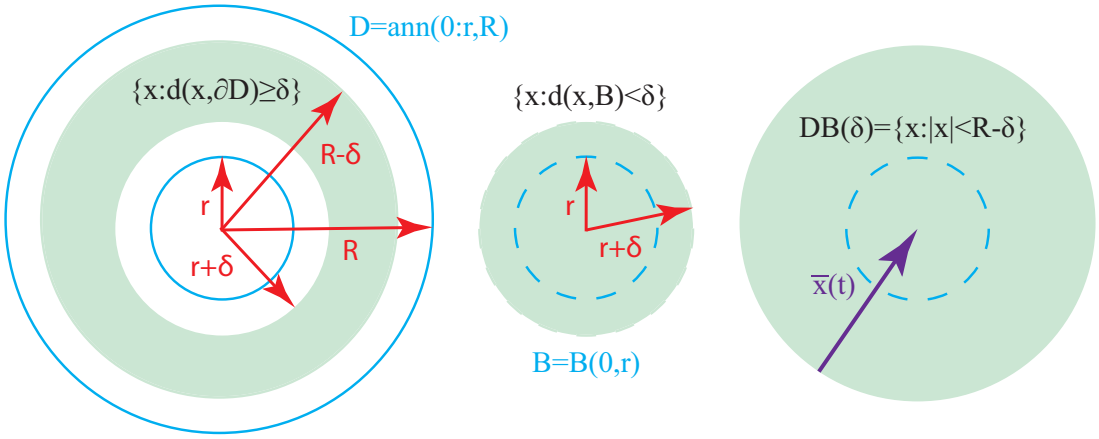


Figure 6.9: This figure shows the set $DB(\delta)$ for D the closure of an annulus of inner radius r and outer radius R, and B being a ball of radius r.

where each $\hat{u}_i(t,\theta)$ *is* T*-periodic in* $\theta = \frac{t}{\varepsilon}$ *and has zero average,* $\int_\tau^{\tau+T} \hat{u}_i(t,\theta)d\theta = 0$. *Consider also the Lie bracket averaged system*

$$\dot{\bar{x}} = \sum_i b_i(\bar{x})\bar{u}_i(t) + \frac{1}{T}\sum_{i<j}\left[\hat{b}_i,\hat{b}_j\right](\bar{x})v_{i,j}(t), \quad \bar{x}(t_0) = x(t_0), \tag{6.90}$$

such that the average system is smooth, satisfying $\left[\hat{b}_i,\hat{b}_j\right](\bar{x}) \in C^2(\mathbb{R}^n)$. *If a set* $B \subset \mathbb{R}^n$ *satisfies the property that* B *is GUAS relative to system (6.90), and for any* $\delta > 0$, *any trajectory of (6.90) starting within the set* $\text{int}_\delta(D)$ *stays within a distance* δ *from the boundary of* D, *unless it enters the set* B, *that is any trajectory* $\bar{x}(t)$ *such that* $\bar{x}(0) \in \text{int}_\delta(D)$ *stays within the set*

$$DB(\delta) = \{x \in \mathbb{R}^n : d(x,\partial D) \geq \delta\} \cup \{x \in \mathbb{R}^n : d(x,B) \leq \delta\},$$

then B *is* ε*-SPUAS relative to system (6.89).*

Remark 11. *A simple example of a trajectory satisfying the above mentioned property for a set of the type* $DB(\delta)$ *as defined above is when* D *is the closure of the annulus of inner radius* r *and outer radius* R, B *is a ball of radius* r, *and* $\bar{x}(t)$ *is a trajectory starting within* $\text{int}_\delta(D)$ *and moving in a straight line towards the origin. The set* $DB(\delta)$ *is then just the ball of radius* $R-\delta$, *as shown in Figure 6.9.*

Proof. By Theorem 14, we have that the trajectory of system (6.90) is Δ-CT relative to the trajectory of (6.89) on B. Therefore, for any given $\delta > 0$, for any $\hat{T} > 0$, there exists $\varepsilon^\star$ such that for all $\varepsilon \in (0, \varepsilon^\star)$ and any $x(0) \in D$, for all t such that $x(t) \in D$,

$$\max_{t \in [0, \hat{T}]} |x(t) - \bar{x}(t)| < \frac{\delta}{2}. \tag{6.91}$$

In particular, consider $x(t)$ such that $x(0) \in \text{int}_{\frac{\delta}{2}}(D)$, we then get, for all t such that $x(t) \in D$,

$$\begin{aligned} \max_{t \in [0, \hat{T}]} d(x(t), \partial D) &\geq \max_{t \in [0, \hat{T}]} d(\bar{x}(t), \partial D) - \max_{t \in [0, \hat{T}]} d(x(t), \bar{x}(t)) \\ &\geq \max_{t \in [0, \hat{T}]} d(\bar{x}(t), \partial D) - \frac{\delta}{2}. \end{aligned}$$

Because the $\bar{x}(0) = x(0) \in \text{int}_{\frac{\delta}{2}}(D)$, by the assumption in the statement of the theorem $\bar{x}(t) \in BD\left(\frac{\delta}{2}\right)$ and therefore either

$$\max_{t \in [0, \hat{T}]} d(\bar{x}(t), \partial D) > \delta \Longrightarrow \max_{t \in [0, \hat{T}]} d(x(t), \partial D) > 0$$

or else

$$\begin{aligned} \max_{t \in [0, \hat{T}]} d(\bar{x}(t), B) < \delta \Longrightarrow \max_{t \in [0, \hat{T}]} d(x(t), B) &\leq \max_{t \in [0, \hat{T}]} d(x(t), \bar{x}(t)) + \max_{t \in [0, \hat{T}]} d(\bar{x}(t), B) \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

Therefore trajectories $x(t)$ starting in $\text{int}_{\frac{\delta}{2}}(D)$ always remain inside the set D or otherwise reach within a distance δ of the set B. Therefore by Theorem 13 the origin of system (6.39) is (ε, δ)-SPUUB. Because the choice of δ may be made arbitrarily small, by Lemma 5 the origin of system (6.39) is ε-SPUAS. □

Chapter 7

Conclusion

The work presented here offers a new application of Extremum Seeking, as a method for stabilization of unknown systems as well as trajectory tracking and optimization. The stabilization of unknown systems is possible due to the controller's ability to minimize Lyapunov functions.

The extremum seeking algorithm creates a closed loop system that is independent of the control vector's direction. This is a useful property which allows us to stabilize and perform trajectory tracking with *unknown*, unstable, control direction-varying systems using a particular form of time-varying nonlinear high-gain feedback. In the LTV case the only restriction to the applicability of the control law (5.1) is that, for a given bound on $A(t)$, the vector $B(t)$ be persistently exciting over a sufficiently short window Δ, namely, that the variations of $B(t)$ are sufficiently fast. In the nonlinear case we require that the control vector $G(x,t)$ is non-zero. In both cases we achieve semiglobal uniform ultimate boundedness of the error system, with ultimate bound $\delta > 0$.

We also extend the averaging results of Kurzweil, Jarnik, Sussmann, Liu, Gurvits, and Li to non-C^2 systems which allows us to design non-C^2 extremum-seeking based controllers. The advantage of these non-C^2 controllers is that their efforts settle to zero if the system ever reaches its equilibrium and when the system does not reach, but is within some small neighborhood of equilibrium the control effort is reduced. We are able to stabilize *unknown*, unstable, control direction-varying systems using time-

varying nonlinear high-gain feedback, which in the scalar case does not suffer from the persistent oscillations present in typical ES control schemes, once the system's trajectory has reached the origin. In the multi-dimensional case, control effort is greatly reduced near the origin.

In a real world demonstration, the proposed algorithm has been implemented to simultaneously tune 24 parameters of an unknown, non-linear dynamic system, successfully decreasing the output voltage rise time of a high voltage converter modulator by a factor of 2 to within a 1% error. Furthermore, as shown in the results of Section 4.4.1, the ES algorithm is successful despite a possibly noisy cost, as long as the noise doesn't have frequency components which match up with any of the ES algorithm's perturbing frequencies.

Although Lyapunov function minimization is extremely useful in providing stability results, the controller designed here may work to maximize or minimize arbitrary functions. This happens because the system's trajectory is pushed along the functions' gradients. Any system of the form

$$\begin{aligned} \dot{x} &= f(x,t) + g(x,t)u \\ u &= \alpha\sqrt{\omega}\cos(\omega t) - k\sqrt{\omega}\sin(\omega t)V(x,t), \end{aligned}$$

where $f(x,t)$, $g(x,t)$, and $V(x,t)$ have separable x and t dependence has Lie bracket average

$$\dot{\bar{x}} = f(\bar{x},t) - k\alpha g(\bar{x})g^T(\bar{x},t)\left(\frac{\partial V(\bar{x},t)}{\partial \bar{x}}\right)^T, \tag{7.1}$$

which approaches a minimum or maximum of $V(\bar{x},t)$ depending on the sign of $k\alpha$ with a bound on how close to the minimum/maximum we can get depending on the amplitude of $\left|k\alpha g(\bar{x})g^T(\bar{x},t)\left(\frac{\partial V(\bar{x},t)}{\partial \bar{x}}\right)^T\right|$ relative to $|f(\bar{x},t)|$. One topic of future work will be on such an implementation for minimization of unknown output functions for unknown systems.

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