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On log del Pezzo surfaces in characteristic different from two and three

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Justin Lacini

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2020

The dissertation of Justin Lacini is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California San Diego

2020

DEDICATION

To my family

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Finally, I would like to conclude by thanking Prof. Michael McQuillan. His visits at UC San Diego have always brought new life to the department and our conversations about mathematics have been the source of months of interesting research.

The present work is being prepared for submission for publication. The dissertation author was the primary investigator and author of this material.

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ABSTRACT OF THE DISSERTATION

On log del Pezzo surfaces in characteristic different from two and three

by

Justin Lacini

Doctor of Philosophy in Mathematics

University of California San Diego, 2020

Professor James McKernan, Chair

A log del Pezzo surface is a klt projective surface whose canonical divisor is anti-ample. We classify all log del Pezzo surfaces of Picard number one defined over algebraically closed fields of characteristic different from two and three. We also discuss some consequences of the classification. For example, we show that log del Pezzo surfaces defined over algebraically closed fields of characteristic higher than five have at most four singular points and admit a log resolution that lifts to characteristic zero over a smooth base.

Chapter 1

Preliminaries

1.1   Introduction

The canonical divisor plays a central role in the classification of algebraic varieties. For example, the Minimal Model Program predicts that every smooth projective variety can be "built" from the following three families:

1. Fano varieties, on which the canonical class is anti-ample.

2. Varieties with canonical class that is trivial (for example Calabi-Yau or abelian varieties).

3. Canonical models, on which the canonical class is ample.

The aim of this dissertation is to classify all klt Fano surfaces (called *log del Pezzo surfaces*) of Picard number one over algebraically closed fields of characteristic different

from two and three. More precisely:

Theorem 1.1.1 (Classification of rank one log del Pezzo surfaces, Theorem 4.1.20)**.** *Let S be a rank one log del Pezzo surface defined over an algebraically closed field of characteristic different from two and three. If S is smooth then $S = \mathbb{P}^2$; if S is Gorenstein, then S is one of the surfaces described in Theorem B.2.2; otherwise, it belongs to one of the families LDP1 to LDP19 described in Section 4.*

Theorem 1.1.1 has several immediate consequences. For instance, it is well-known that rank one log del Pezzo surfaces defined over the complex numbers have at most four singular points (see [KM99, Corollary 1.8.1] and [Bel09]). This result, known as Bogomolov bound, fails in characteristic two, as Keel and McKernan exhibited examples of rank one log del Pezzo surfaces with arbitrarily many singular points ([KM99, Chapter 9]). Nevertheless, in light of Theorem 1.1.1 we have the following:

Corollary 1.1.2 (Bogomolov bound)**.** *Let S be a rank one log del Pezzo surface defined over an algebraically closed field of characteristic different from two, three and five. Then S has at most four singular points.*

By slightly modifying Keel and McKernan's example, in characteristic three one easily gets a rank one log del Pezzo surface with seven singularities (see Example 5.2.1). By carefully examining the classification of extremal rational elliptic surfaces, we were also able to find a counter-example to Bogomolov's bound in characteristic five.

Theorem 1.1.3. *(Example 5.2.3) There exists a rank one log del Pezzo surface S defined over any algebraically closed field of characteristic five and a curve $C \subseteq S$ such that*

1. *$-(K_S + C)$ is ample.*

2. *$K_S + C$ is dlt.*

3. *S has five singular points.*

There has been recently a great deal of interest in extending the classical results of the Minimal Model Program to algebraically closed fields of positive characteristic (see for instance [HX15]). It is therefore natural to ask which properties of complex log del Pezzo surfaces carry to positive characteristic. Cascini, Tanaka and Witaszek [CTW17] have proved that in large characteristic all log del Pezzo surfaces either lift to characteristic zero over a smooth base, or are globally F-regular. Our desire to determine the exact characteristic has been one of the main motivations behind Theorem 1.1.1.

Theorem 1.1.4 (Lifting to characteristic zero, Theorem 5.1.2)**.** *Let S be a rank one log del Pezzo surface defined over an algebraically closed field of characteristic $p > 5$. Let $\pi : \tilde{S} \to S$ be the minimal resolution of S. Then $(\tilde{S}, \mathrm{Ex}(\pi))$ lifts to characteristic zero over a smooth base.*

Notice that Theorem 1.1.4 is sharp, in light of Theorem 1.1.3 and of the Bogomolov bound. Furthermore, Theorem 1.1.4 implies that Kodaira's vanishing theorem holds for rank one log del Pezzo surfaces in characteristic $p > 5$ (see Theorem 5.3.1). This behavior

in dimension two should be contrasted with the failure of Kodaira's vanishing theorem in higher dimension, even for smooth Fano varieties (see [Tot17]).

We give now a brief sketch of the proof of Theorem 1.1.1. Let S be a log del Pezzo surface of Picard number one. A natural approach is to try to "simplify" the singularities of S by extracting an exceptional divisor E_1 of its minimal resolution. Let $f : T_1 \to S$ be the extraction of E_1. Since S has Picard number one, T_1 has Picard number two, and therefore the closed cone of curves of T_1 is generated by two rays. One of the two rays is generated by the class of E_1. The main idea is to realize that we may then play the two-ray game on T_1 by contracting the other ray. Let $\pi : T_1 \to S_1$ be the contraction. There is a priori no reason why S_1 would be any "simpler" that S, and in fact this is not the case for most choices of E_1. The truly remarkable fact, which makes the classification possible at all, is that by choosing E_1 to be the divisor with the worst singularity (as measured by the discrepancy), then S_1 is indeed simpler than S, and one may even classify the possibilities for the contraction π_1.

The idea is then to continue this sequence of extractions and contractions, which produces a sequence of increasingly simpler surfaces S_1, S_2 and so on. This process, which was first introduced by Keel and M$^\text{c}$Kernan [KM99], is called the *hunt*. The hunt is very efficient and usually terminates within three steps, yielding either a Gorenstein log del Pezzo surface of Picard number one, a cone over the rational normal curve of degree n, or a Mori fiber space. One may therefore recover S by classifying all such surfaces and the contractions π_i that appear during the hunt.

Keel and McKernan introduced the hunt in order to prove that the smooth locus of log del Pezzo surfaces of rank one is uniruled. Their proof is divided in two cases, based on the following notion:

Definition 1.1.5. Let (X, Δ) be a $\mathbb{Q}$-factorial projective log pair. A *special tiger* for $K_X + \Delta$ is an effective $\mathbb{Q}$-divisor α such that $K_X + \Delta + \alpha$ is numerically trivial, but not klt. If there is a special tiger, a *tiger* is any divisor E with discrepancy at most -1. ◄

For surfaces that admit a tiger they presented a short proof by using deformation theory. To complete their argument, however, they classified all complex log del Pezzo surfaces with no tigers (more precisely, they constructed a family of surfaces that contains all those that have no tigers). By pushing these methods a bit further, they actually classified all simply connected rank one log del Pezzo surfaces, *with the exception of a bounded family.*

In this paper we use Keel and McKernan's ideas to complete the classification over the complex numbers, and to extend it to algebraically closed fields of characteristic $p > 3$. In positive characteristic a whole set of additional difficulties appear. For starters, one cannot use topological arguments in order to simplify the classification, such as reducing to the simply connected case. Furthermore, one cannot use the Bogomolov bound as we do not *a priori* know whether it holds in characteristic $p > 5$ or not. Other issues are that the classification of rank one Gorenstein log del Pezzo surfaces was not available in positive characteristic (to the best of author's knowledge), and that the proof of [KM99, Lemma 22.2] on the existence of complements does not *a priori* carry through in positive

characteristic, as it uses the Kawamata-Viehweg vanishing theorem.

The rest of the paper is organized as follows. In Appendix A we describe in detail klt, dlt and lc surfaces singularities and techniques to deal with them efficiently. In Appendix B we classify Gorenstein log del Pezzo surfaces in characteristic different from two and three. In Section 1.3 we recall the main results concerning the hunt from [KM99], in an effort to make the presentation as self-contained as possible. We also develop the hunt in the level case, which will play a role in classifying log del Pezzo surfaces with tigers. In Section 2 we start carrying the hunt for log del Pezzo surfaces that do not have tigers, and in Section 3 we deal instead with the case in which there are tigers. As a byproduct, we classify all pairs (S, C) such that S is a rank one log del Pezzo surface and $C \subseteq S$ is a curve such that $K_S + C$ is anti-nef. In Section 4 we summarize our findings and list all the families of rank one log del Pezzo surfaces. Finally, we conclude by considering liftability to characteristic zero and other applications in Section 5.

1.2   Notation

A **pair** (X, Δ) is given by a normal projective variety X and a Weil $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. If Δ is effective, (X, Δ) is a **log pair**. We say that Δ is a **boundary** if $\Delta = \sum_i c_i D_i$ where D_i are irreducible distinct Weil divisors and $0 < a_i \leqslant 1$. We say that a birational morphism $f : Y \to X$ is a **log resolution** of (X, Δ) if Y is smooth and $f_*^{-1}(\Delta) + \mathrm{Ex}(f)$ is a simple normal crossings divisor.

Let $f : Y \to X$ be any birational morphism with Y normal. We can write $K_Y + f_*^{-1}\Delta = f^*(K_X + \Delta) + \sum_i a_i E_i$ where E_i are f-exceptional divisors. The numbers $a_i = a_i(E_i; X, \Delta)$ are called **the discrepancy** of E_i with respect to the pair (X, Δ). We define **the coefficient** of E_i with respect to the pair (X, Δ) to be $e(E_i; X, \Delta) = -a_i$. This is just the coefficient with which E_i appears in the divisor Γ defined by $K_Y + \Gamma = f^*(K_X + \Delta)$. We call Γ the **log pullback** of Δ.

We say that (X, Δ) is **log canonical** (or simply lc) if $a_i \geqslant -1$ for every E_i and every f. We say that (X, Δ) is **Kawamata log terminal** (or simply klt) if $\lfloor \Delta \rfloor = 0$ and $a_i > -1$ for every E_i and every f. It is in fact sufficient to check the above definitions on any given log resolution of (X, Δ). Finally, we say that (X, Δ) is **divisorially log terminal** (or simply dlt) if there is a log resolution $f : Y \to (X, \Delta)$ such that $a(E_i; X, \Delta) > -1$ for all f-exceptional divisors. This is equivalent to requiring that there is a closed subset $Z \subseteq X$ such that $(X \setminus Z, \Delta_{|X \setminus Z})$ has simple normal crossings and, if E is an irreducible divisor over X with center contained in Z, then $a(E; X, \Delta) > -1$.

In this paper we will mainly use the above definitions in the case of surfaces. If S is

a normal surface we indicate by $\tilde{S}$ its minimal resolution. If $C \subseteq S$ is an effective divisor, then we indicate by $\tilde{C} \subseteq \tilde{S}$ its strict transform. We indicate by $\mathbb{F}_n$ the Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ over $\mathbb{P}_k^1$. We denote by $\overline{\mathbb{F}}_n$ the surface obtained by contracting the unique $(-n)$ curve of $\mathbb{F}_n$. A **log del Pezzo surface** is a surface S with only klt singularities such that $-K_S$ is ample. We often call the Picard number of a log del Pezzo surface simply as **rank**. For example, $\overline{\mathbb{F}}_n$ is a log del Pezzo surface of rank one.

1.3   The hunt

Let S be a rank one log del Pezzo surface defined over an algebraically closed field k of any characteristic. In this section we describe how to produce a sequence of progressively simpler log del Pezzo surfaces. This process is called the "hunt", and will be the main tool of our analysis. A hunt step consists of a K-positive extraction and a K-negative contraction. The reason that the hunt is so useful is that it is possible to classify its birational transformations. Therefore, if we reach a log del Pezzo surface that we fully understand, we can get useful information about S by reversing the process.

The hunt and its properties are described in [KM99, Chapter 8]. In addition to the hunt for flush pairs, which is described in [KM99], we will also need the hunt for level pairs (see Definition 1.3.2 for the definition of flush and level). Throughout this section we adopt the following practice: we will state all the relevant results for which a proof can be found in [KM99] and provide the reference within that paper, whereas we will prove all the further results that we need, even if this only consists in making a minor change to a proof from [KM99]. We hope in this way to keep the exposition as self-contained and detailed as possible, while avoiding repetitions with [KM99].

1.3.1   Flush and level

Here we develop the general theory that will be used during the hunt. In this sub-section Δ is a boundary with support D.

Definition 1.3.1. Let (X, Δ) be a log pair and suppose that $\Delta = \sum_i a_i D_i$ is a boundary (we assume that D_i are all irreducible and distinct). We define $m = m(\Delta)$ be the minimum of the non-zero a_i, with the convention that if Δ is empty then $m(\Delta) = 0$. ◀

Definition 1.3.2. Let (X, Δ) be a log pair and suppose that Δ is a boundary. Let E be an exceptional divisor over X. We say that the pair (X, Δ) is:

1. flush (respectively level) at E if $e(E; X, \Delta) < m$ (respectively $e(E; X, \Delta) \leqslant m$).

2. flush (respectively level) if $e(E; X, \Delta) < m$ (respectively $e(E; X, \Delta) \leqslant m$) for all exceptional divisors E.

◀

Lemma 1.3.3. *Let X be a $\mathbb{Q}$-factorial variety and let Δ be a boundary. Suppose that $f : Y \to X$ is a birational morphism with irreducible divisorial exceptional locus E. Assume that $e = e(E; X, \Delta) \geqslant 0$. Let $\Gamma = eE + f_*^{-1}\Delta$ be the log pullback of Δ. Assume Γ is a boundary. Then:*

1. *If (X, Δ) is level and E has maximal coefficient for (X, Δ) then $K_Y + \Gamma$ is level.*

2. *If (Y, Γ) is flush (respectively level), then (X, Δ) is not flush (respectively level) for a divisor F if and only $F = E$, and the coefficient of E in Γ is at least as large as (respectively strictly larger than) the coefficient of some non exceptional component of Γ.*

Proof. Both statements immediately follow from the definitions. □

The following lemma provides a useful criterion to check the flush and level properties.

Lemma 1.3.4. *Let S be the germ of a klt surface at p in the étale topology.*

1. *If (S, D) has normal crossings (in particular, p is smooth) then (S, Δ) is level. If furthermore $\lfloor \Delta \rfloor = 0$ then (S, Δ) is flush.*

2. *Assume (S, D) does not have normal crossings (that is either p is singular, or D has worse than a simple node). Let $f : T \to S$ be a log resolution of (S, Δ) and assume that $e(F; S, \Delta) \leqslant 1$ for every f-exceptional divisor F. Then for any exceptional divisor V there is an f-exceptional divisor F such that $e(V; S, \Delta) \leqslant e(F; S, \Delta)$. In particular, (S, Δ) is flush (respectively level) if and only if it is flush (respectively level) at all exceptional divisors of f.*

Proof. This is [KM99, Lemma 8.3.2]. We only remark that it is sufficient to check the property of being flush and level in an étale neighborhood, so that the same proof as in [KM99] applies. $\square$

Now we remark that in the case of surfaces it suffices to check the property of being divisorially log terminal or log canonical for divisors in the minimal resolution. This will play an important role throughout the paper.

Lemma 1.3.5. *Let S be the germ of a klt surface at p in the étale topology. Assume that p is singular.*

1. If the coefficient of every exceptional divisor of the minimal resolution $\pi : \tilde{S} \to S$ for (S, D) is strictly less than one, then (S, D) is dlt.

2. If the coefficient of every exceptional divisor of the minimal resolution $\pi : \tilde{S} \to S$ for (S, D) is less or equal than one, then (S, D) is lc, except the following cases:

    (a) There is exactly one irreducible component of $\tilde{S}$ over p, and D is simply tangent to it.

    (b) There is exactly one irreducible component of $\tilde{S}$ over p, and D has two branches meeting transversally on it.

    (c) There are exactly two irreducible components E_1 and E_2 of $\tilde{S}$ over p, and D has one branch meeting each component transversally at $E_1 \cap E_2$.

Proof. Part (1) of the statement is [KM99, Lemma 8.3.3]. We will go over its proof to get part (2) of the statement.

Suppose (S, D) is not log canonical. D is not empty, since S is klt by assumption. Let E be the reduced exceptional locus of π. By Lemma 1.3.4, π is not a log resolution for (S, D). By the classification of surface singularities in Appendix A (see in particular Lemma A.0.5, Lemma A.0.6 and Lemma A.0.7), E has simple normal crossings and $\tilde{D}$ does not have simple normal crossings with E by Lemma 1.3.4. As the computation of the coefficients is purely numerical (see the remark after Lemma A.0.7), we may replace $\tilde{D}$ by a disjoint union of irreducible curves, each meeting E transversally and such that at least two curves meet the same irreducible component of E. Let D' be the pushforward

of the new configuration. Clearly π is a log resolution of (S, D') by construction.

Suppose for the moment that (S, D') is not log canonical at p. Then, by Lemma 1.3.4, there exists a component of E with coefficient strictly larger than one for (S, D'), contradiction. Therefore assume that (S, D') is log canonical at p. By Lemma A.0.6, (S, D') is not dlt. Also, by Lemma A.0.7, E has at most two components over p. The only ways this can happen are listed in part (2) of the statement. $\qquad \square$

In order to capture the cases in Lemma 1.3.5 we introduce the following definition.

Definition 1.3.6. We say a pair (S, D) is almost log canonical if the coefficient of every exceptional divisor of the minimal resolution with respect to (S, D) is less or equal than one. Equivalently, either (S, D) is log canonical at singular points p or there is an étale neighborhood of p such that one of the cases (a)-(c) in the previous lemma holds. $\qquad \blacktriangleleft$

The next results describe the geometric consequences of being flush and level. These geometric consequences are the main reason why the hunt works. In fact, the hunt (almost) preserves the property of being flush or level, which in turn controls the geometry of the curves that we extract and contract.

Lemma 1.3.7. Let $f : T \to S$ extract the irreducible divisor E. Assume that $e(E; S, \Delta) \geqslant 0$. Let Γ be the log pullback of Δ and assume Γ is also a boundary. Let $\pi : \tilde{S} \to S$ be the minimal resolution.

1. If p is singular, and (S, Δ) is flush at every π-exceptional divisor F then (S, D) is dlt and (S, Δ) is flush.

2. *If (S, Δ) is dlt and (S, Δ) is level at every π-exceptional divisor F, then $(S, s\Delta)$ is flush for every $s > 1$ such that $s\Delta$ is a boundary.*

3. *Suppose p is singular. If f is a K_T-negative contraction and (T, Γ) is flush at every exceptional divisor of the minimal resolution of T, then (S, Δ) is flush.*

4. *Suppose p is singular and (S, D) is log canonical. Then for any exceptional divisor V there is some π-exceptional divisor F with $e(F; S, \Delta) \geqslant e(V; S, \Delta)$.*

Proof. This is [KM99, Lemma 8.3.5]. $\qquad\square$

We will also need a slightly different version of the previous lemma.

Lemma 1.3.8. *Let $f : T \to S$ extract the irreducible divisor E. Assume that $e(E; S, \Delta) \geqslant 0$. Let Γ be the log pullback of Δ and assume Γ is also a boundary. Let $\pi : \tilde{S} \to S$ be the minimal resolution.*

1. *If p is a singular point, and (S, Δ) is level at every π-exceptional divisor F then (S, D) is almost log canonical at p. If furthermore (S, D) is log canonical then (S, Δ) is level.*

2. *Suppose $(S, D + D')$ is log canonical. If there exists $0 < c \leqslant 1$ such that $(S, D + cD')$ is level at every π-exceptional divisor F, then $(S, D + sD')$ is level for any $c \leqslant s \leqslant 1$.*

3. *Suppose $p = f(E)$ is singular. If f is a K_T-negative contraction and (T, Γ) is level at every exceptional divisor of the minimal resolution of T, then (S, D) is almost log canonical at p.*

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Proof. As usual, we go through the proof of [KM99] and make the necessary changes. We start with (1). Let λ be the smallest coefficient of Δ. Let F be a π-exceptional irreducible divisor and consider the function $f(t) = e(F; S, tD)$. Now $f(0) \geqslant 0$ since F is π-exceptional. We have that $f(\lambda) = e(F; S, \lambda D) \leqslant e(F; S, \Delta)$ since $\lambda D \leqslant \Delta$. Also, since (S, Δ) is level at F, we have that $e(F; S, \Delta) \leqslant \lambda$. Putting all this together we have

$$f(\lambda) \leqslant e(F; S, \Delta) \leqslant \lambda$$

As f is an affine function, $f(t) \leqslant t$ for all $t \geqslant \lambda$. In particular $e(F; S, D) \leqslant 1$, so that (S, D) is almost log canonical. Suppose now that (X, D) is log canonical. Then π is étale locally a log resolution of (X, D), so (X, Δ) is level by Lemma 1.3.4. This proves (1).

Now we prove (2). Since $(S, D + D')$ is log canonical, by Lemma 1.3.4 it suffices to check that $(S, D + sD')$ is level at every π-exceptional divisor F. Consider $f(t) = e(F; S, D + tD')$. Clearly $f(1) \leqslant 1$ since $(S, D + D')$ is log canonical. Also, $f(c) \leqslant c$ since $(S, D + cD')$ is level at F. Hence $f(t) \leqslant t$ for every $c \leqslant t \leqslant 1$, giving the result.

Finally we prove (3). Since (T, Γ) is level at every exceptional divisor of the minimal resolution of T, then $(T, \lceil \Gamma \rceil)$ is almost log canonical by (1). The pair (S, Δ) is level at each exceptional divisor of the minimal resolution of S above p, since every such divisor also appears in the minimal resolution of T. Therefore (S, D) is almost log canonical at p. $\qquad \square$

The next lemma shows that being flush or level gives strong control on the singularities of the boundary divisor at smooth points of S.

Lemma 1.3.9. *Suppose p is smooth and the pair (S, Δ) is flush. Let $\Delta = \sum_i a_i D_i$ and $m = m(\Delta)$. Then*

1. *If M_i is the multiplicity of D_i, then $\sum_i a_i M_i - 1 < m$. In particular $m < 1/(M-1)$ where M is the multiplicity of D.*

2. *If D has a node of genus at least two at p, and the coefficient of the two branches of D at p are $a \geqslant b$, then $2a + b < 2$.*

3. *If p is a cusp of D and a is the coefficient of the branch of D at p, then $a < 4/5$.*

4. *If $m \geqslant 4/5$ then D has normal crossings.*

   If (S, Δ) is level at p instead then $(1) - (4)$ hold by switching "$<$" and "$\leqslant$".

Proof. This is [KM99, Lemma 8.3.7]. □

1.3.2 The first two hunt steps in the flush case

In this sub-section we describe the setting and the outcome of the hunt in the flush case. This is the main type of hunt we run in this paper, and it is especially useful in classifying log del Pezzo surfaces without tigers (see below for the definition). The underlying idea is that this hunt preserves flushness, which in turn controls the geometry involved. Let's start with the following fundamental definition.

Definition 1.3.10. Let (X, Δ) be a log pair. Let $f : Y \to X$ be a birational morphism. We say X has a tiger in Y if there exists an effective $\mathbb{Q}$-Cartier divisor α such that

1. $K_X + \Delta + \alpha$ is numerically trivial.

2. If Γ is the log pullback of $\Delta + \alpha$ in Y, there is a divisor E of coefficient at least one in Γ.

Any such divisor E is called tiger. ◀

The following geometric situations will be common, and we name them according to [KM99].

Definition 1.3.11. Let A and B be two rational curves on a klt surface S such that $K_S + A + B$ is divisorially log terminal at any singular point of S. We say that $(S, A + B)$ is a

1. banana, if A and B meet in exactly two points, and there normally.

2. fence, if A and B meet at exactly one point, and there normally.

3. tacnode, if A and B meet at most at two points, there is one point $q \in A \cap B$ such that $A + B$ has a node of genus $g \geqslant 2$ at q, and if there is a second point of intersection, then A and B meet there transversally.

◀

Now we describe the scaling of the hunt in the flush case.

Lemma 1.3.12. *Let (S, Δ) be a log pair such that S is a rank one log del Pezzo surface. Let $f : T \to S$ be an extraction of relative Picard number one of an irreducible divisor E of the minimal resolution. The cone of curves of T has two edges, one generated by the class of E, and let the other be generated by the class of R. Let $x = f(E)$, Γ such that $K_T + \Gamma = f^*(K_S + \Delta)$ and $\Gamma_\epsilon = \Gamma + \epsilon E$, where $0 < \epsilon \ll 1$. Assume that $-(K_S + \Delta)$ is ample, then*

1. *R is K_T-negative and contractible, hence there is a rational curve Σ that generates the same ray. Let π be the contraction morphism.*

2. *$K_T + \Gamma_\epsilon$ is anti-ample.*

3. *Γ_ϵ is E negative.*

4. *There is a unique rational number λ such that with $\Gamma' = \lambda \Gamma_\epsilon$, $K_T + \Gamma'$ is R trivial. $\lambda > 1$.*

5. *$K_T + \Gamma'$ is E negative.*

6. *π is either birational or a $\mathbb{P}^1$-fibration (called "net").*

7. *If $\pi : T \to S_1$ is birational, and $\Delta_1 = \pi(\Gamma')$, then $K_{S_1} + \Delta_1$ is anti-ample and S_1 is a rank one log del Pezzo.*

8. *If (S, Δ) does not have a tiger in a surface Y that dominates T, then neither do (T, Γ') and (S_1, Δ_1).*

Proof. See [KM99, Definition-Lemma 8.2.5]. □

Definition 1.3.13. We call the above transformations (f, π) a hunt step (in the flush case) for (S, Δ) if $e(E; S, \Delta)$ is maximal among exceptional divisors of the minimal resolution of S. There might be multiple choices. If x is a chain singularity we allow any choice E that is not a (-2) curve (this is always possible). If x is a non chain singularity we require E to be the central curve (which has maximal coefficient by Lemma A.0.14). If two points have the same coefficient, we can pick one at our choice, but unless stated otherwise our choice will be a chain singularity that allows us to extract the curve with lowest self-intersection. ◄

We fix now the notation that we will always use when running the hunt in the flush case.

Notation. We always start from a surface without boundary, so $\Delta_0 = \emptyset$. We index by (f_i, π_{i+1}) the next hunt step for (S_i, Δ_i). Define:

- $x_i = f(E_{i+1}) \in S_i$

- $q_{i+1} = \pi_{i+1}(\Sigma_{i+1}) \in S_{i+1}$.

- Γ_{i+1} to be the log pullback of Δ_i: $K_{T_{i+1}} + \Gamma_{i+1} = f_i^*(K_{S_i} + \Delta_i)$.

- $\Delta_{i+1} = \pi_{i+1}(\Gamma'_{i+1})$; it satisfies $K_{T_{i+1}} + \Gamma'_{i+1} = \pi_{i+1}^*(K_{S_{i+1}} + \Delta_{i+1})$.

- $A_1 = \pi_1(E_1) \subset S_1$ and $B_2 = \pi_2(E_2) \subset S_2$.

- A_2 the strict transform of A_1 on S_2.

Let a_1, b_2 be the coefficients of A_1, B_2 in Δ_1, Δ_2 (which are also the coefficients of E_1, E_2 in Γ_1', Γ_2') and a_2 the coefficient of A_2 in Δ_2. We remark that a_1, $b_2 < a_2$ by the flush condition and the previous scaling. Let e_i be the coefficient of E_{i+1} in (S_{i+1}, Γ_{i+1}). This is also the coefficient of the pair (S_i, Δ_i). Finally, we indicate by $\overline{\Sigma}_i$ the image of Σ_i in S_0 or S_1, depending on the context.

We are ready to describe the first two hunt steps in the flush case.

Proposition 1.3.14. *Suppose that S is a rank one log del Pezzo surface that has no tigers in $\tilde{S}$. For the first hunt step: $K_{T_1} + E_1$ is divisorially log terminal, $K_{T_1} + \Gamma_1'$ is flush and one of the following holds.*

1. *T_1 is a net.*

Otherwise $K_{S_1} + a_1 A_1$ is flush and one of the following holds

2. *$g(A_1) > 1$.*

3. *$g(A_1) = 1$ and A_1 has an ordinary node at $q = q_1$.*

4. *$g(A_1) = 1$ and A_1 has ordinary cusp at $q = q_1$.*

5. *$g(A_1) = 0$ and $K_{S_1} + A_1$ is divisorially log terminal.*

For the second hunt step one of the following holds.

6. *T_2 is a net.*

7. A_1 is contracted by π_2, $K_{T_2}+\Gamma_2'$ is flush, $K_{S_1}+A_1$ is divisorially log terminal, q_2 is a smooth point of S_2, B_2 is singular at q_2 with a unibranch singularity, and $K_{S_2}+\Delta_2$ is flush away from q_2, but is not level at q_2. Σ_2 is the only exceptional divisor at which $K_{S_2}+\Delta_2$ fails to be flush.

8. Δ_2 has two components.

Suppose that in this last case that $a_2+b_2 \geqslant 1$. Then $\overline{\Sigma}_2 \cap \mathrm{Sing}(A_1) = \emptyset$, $K_{T_2}+\Gamma_2'$ is flush away from $\mathrm{Sing}(A_1)$. $K_{S_2}+\Delta_2$ is flush away from $\pi_2(\mathrm{Sing}(A_1))$, and at least one of $-(K_{S_2}+A_2)$ or $-(K_{S_2}+B_2)$ is ample. Also, one of the following holds:

9. (S_2, A_2+B_2) is a fence.

10. (S_2, A_2+B_2) is a banana, $K_{S_2}+B_2$ is plt, and $x_1 \in A$.

11. (S_2, A_2+B_2) is a tacnode, with tacnode at q_2. $K_{S_2}+B_2$ is plt. If $x_1 \in A_1$, $A_2 \cap B_2 = \{x_1, q_2\}$. If $x_1 \notin A_1$ then $A_2 \cap B_2 = \{q_2\}$.

Proof. See [KM99, Proposition 8.4.7]. $\qquad\square$

Remark 1.3.1. In Proposition 1.3.14 we only need the hypothesis that S has no tigers in $\tilde{S}$ to ensure that the Δ_i are always boundaries. Therefore the proposition still holds under any assumptions that guarantees the same.

1.3.3 The first two hunt steps in the level case

In classifying log del Pezzo surfaces with tigers we will use a slightly different version of the hunt. In particular, we change the scaling convention so that the Δ_i remain boundary divisors.

Let (S, C) be a pair such that S is a rank one log del Pezzo surface and C is a reduced curve in S. Suppose that $K_S + C$ is anti-nef and that (S, C) is log canonical. In particular, C is a tiger of S and (S, C) is level. Let $T_1 \to S$ be the extraction of a divisor of coefficient maximal coefficient for (S, C). This divisor lies in the minimal resolution by Lemma A.0.6, Lemma A.0.7 and Lemma 1.3.4. We write $K_{T_1} + C + e_1 E_1 = f_0^*(K_S + C)$.

Lemma 1.3.12 (1) gives us a K_T-negative contraction π_1. Assume that $\pi_1 : T_1 \to S_1$ is a birational morphism and let $A_1 = \pi_1(E_1)$. Assume also that π_1 does not contract the strict transform of C in T_1 and call C_1 its image in S_1. We define $a_1 \geqslant e_1$ be such that $(K_{T_1} + C + a_1 E_1)$ is π_1-trivial. If $a_1 > 1$, then $K_{S_1} + C_1 + A_1$ is log canonical, since $K_{T_1} + C + E_1$ is log canonical and Σ_1 has negative coefficient. In particular, $K_{S_1} + C_1 + A_1$ is anti-nef and level. In this case we re-define $a_1 = 1$ and go to the second hunt step. If instead $a_1 \leqslant 1$, $K_{S_1} + C_1 + a_1 A_1$ is anti-nef and $K_{T_1} + C + E_1 = \pi_1^*(K_{S_1} + C_1 + a_1 A_1)$. The pair $(S_1, C_1 + a_1 A_1)$ is level by Lemma 1.3.3. In particular, $(S_1, C_1 + A_1)$ is almost log canonical at singular points by Lemma 1.3.8. This concludes the first hunt step in the level case. For the second hunt step, let $f_1 : T_2 \to S_1$ be the extraction of the exceptional

divisor of maximal coefficient E_2 in $\tilde{S}_1$ relative to the pair $(S_1, C_1 + a_1 A_1)$. We write

$$K_{T_2} + C_1 + a_1 A_1 + e_2 E_2 = f^*(K_{S_1} + C_1 + a_1 A_1)$$

Suppose that $\pi_2 : T_2 \to S_1$ is a birational transformation and that neither C_1 or A_1 get contracted. Call C_2 and A_2 the strict transforms of C and A_1 in S_2. Let $B_2 = \pi_2(E_2)$. Define again $b_2 \geqslant e_2$ so that $K_{T_1} + C_1 + b_2 E_2$ is π_2-trivial. As above, if $b_2 > 1$ then $K_{S_2} + C_2 + A_2 + B_2$ is log canonical and anti-nef. We may therefore assume that $b_2 \leqslant 1$. Then $K_{S_2} + C_2 + A_2 + b_2 B_2$ is anti-nef, and level by Lemma 1.3.3. It follows that $(S_2, C_2 + A_2 + B_2)$ is almost log canonical at singular points by Lemma 1.3.8. At smooth points, we can control singularities by Lemma 1.3.9.

1.3.4 Classification of the hunt contractions in the flush case

In this sub-section we describe the geometry of the contractions that appear during a hunt in the flush case. The description we give is étale local. First, we explain our setting, following [KM99, Chapter 11]. Let T be a $\mathbb{Q}$-factorial klt projective surface and let $\pi : T \to S$ be a proper birational contraction of a K_T extremal ray. Denote by Σ the exceptional divisor and let q be the image of Σ. We will be concerned with the étale local description of T around Σ. Let $W \subset T$ be a curve with smooth components crossing normally. Assume W has at most two irreducible components X, Y and that $K_T + cX + dY$ is π-trivial and flush, with $0 < c, d < 1$. Assume also that $\pi|_W$ is finite

(which is to say that Σ is not a component of W). Thus $K_S + cX + dY$ is flush by Lemma 1.3.3. Let D be the image of W in S and let $h : \tilde{T} \to \tilde{S}$ be the induced map between the minimal resolutions.

Before stating the classification, we need to give a name to some particular geometric configurations.

Definition 1.3.15. Let S be a surface, and D be a curve in S. Choose a point $q \in D$. If D has two smooth branches X and Y meeting to order g at q we say that D has a node of order g. If $g \geqslant 1$ we have to blow $g - 1$ times to make $X + Y$ have normal crossings. We call this configuration 0. If we further blow up once at $X \cap Y$ we reach configuration I. Now blow up at $X \cap \Sigma$, where Σ is the unique (-1) curve. This is configuration II. We denote by (II) and a string of x and y (for example (II, x, x, y)) a configuration that is reached by starting with configuration II and then blowing up the intersection of the unique (-1) curve with the branch corresponding to the letter of the string. ◀

Definition 1.3.16. Let S be a surface, and D be a curve in S. Choose a point $q \in D$. If D has a unibranch double point at q we say D has a cusp of order g. We call X the branch of D at q. We need g blowups along X to remove its singularity. We call this configuration I. Next, we blow up at $X \cap \Sigma$, where as usual Σ is the unique (-1) curve. This is configuration II. Next, we blow up again at $X \cap \Sigma$, reaching configuration III. We have now three choices:

1. If we blow up at the intersection with the unique exceptional (-3) curve, we reach configuration U.

2. If we blow up at $X \cap \Sigma$ then we reach configuration V.

3. If we blow up at the intersection with the unique (-2) curve, we reach configuration W.

After this, at each step we may only choose to blow up one of two points on the last (-1) curve: either the nearest point to X, in which case we add an n to the string of letters we have formed so far, or the farthest point from X, in which case we add an f. ◀

Notation. If p is a chain singularity, we mark with a prime the component touched by Σ and we underline the component touched by D.

Going back to our original setting, if D has multiplicity two at q, then q is smooth and we have the following classification.

Lemma 1.3.17. *Suppose D has a node of order g. One of the following holds:*

1. *T has type I or 0, and $c + d = 1$.*

2. *$g \geqslant 2$, T has type II and $gd + (g+1)c = g+1$.*

   $g = 1$, $K_T + \Sigma + X + Y$ is log canonical and either

3. *T has type (II, x^{r-1}) with $r \geqslant 1$, there is a unique singularity, an A_r point, X meets Σ at a smooth point, Y meets Σ at the A_r point, which has type $(\underline{2}, ..., 2')$ and $c + \frac{d}{r+1} = 1$.*

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4. Σ meets X and Y each at a singular point of T, and those are the only singularities along Σ.

Proof. See [KM99, Lemma 11.1.1]. □

Lemma 1.3.18. *Suppose D has a cusp of order g. Then either*

1. *T has type I and $c = 1/2$.*

2. *T has type II and $c = (g+1)/(2g+1)$.*

3. *T has type III and $c = (g+1)/(2g+1)$.*

4. *$g = 1$, T has type u and $c = 3/4$ or $g = 2$ and $c = 9/14$.*

5. *$g = 1$, T has type v and $c = 5/7$ or $g = 2$ and $c = 7/11$.*

6. *$g = 1$, T has type w and $c = 7/9$.*

7. *$g = 1$, T has type $(u; n)$ and $c = 11/14$.*

8. *$g = 1$, T has type $(v; f)$ and $c = 10/13$.*

9. *$g = 1$, T has type $(v; f^2)$ and $c = 15/19$.*

10. *$g = 1$, T has type $(v; n)$ and $c = 3/4$.*

11. *$g = 1$, T has type $(v; n^2)$ and $c = 7/9$.*

Proof. See [KM99, Lemma 11.2.1]. □

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In case D has multiplicity three, we only have the following two cases.

Lemma 1.3.19. *Suppose that D has multiplicity three. If D has two branches at q, then one branch has a double point, a simple cusp, and the other is smooth. If X is the branch with the cusp, then Σ meets X normally at one smooth point, Σ contains two singularities (2) and (3), and Y meets the -2 curve, and is disjoint from Σ on $\tilde{T}$.*

Proof. See [KM99, Lemma 11.3.2]. $\square$

Lemma 1.3.20. *Suppose that D has multiplicity three. If D is unibranch, then either*

1. *Σ has singularities $(3, 2')$, (3) and meets X normally at a smooth point*

2. *Σ has singularities (3), (2) and on the minimal resolution X meets Σ normally at the intersection of Σ and the -2 curve.*

Proof. See [KM99, Lemma 11.3.3]. $\square$

The last type of contraction we are interested in are fibrations (which we call "nets", following [KM99]). Let $\pi : T \to C$ be a $\mathbb{P}^1$-fibration of relative Picard number one, with T a normal surface. Let $\tilde{\pi}$ be the composition $\tilde{\pi} : \tilde{T} \to T \to C$, where $\tilde{T}$ is the minimal resolution of T. We describe the fiber F of $\tilde{T}$ above $p \in C$ as the sequence $k(-a) + l(-b) + \cdots + m(-c)$, by which we mean a chain of curves of self intersection $-a$, $-b$ and so on, with multiplicities k, l and so on.

We recall here the following definition from Appendix A.

Definition 1.3.21. A non Du Val klt singularity with coefficient strictly less than $1/2$ is called almost Du Val. They all are of the form $(3, A_k)$ for some k. ◀

Lemma 1.3.22. *Assume T is klt, G is a multiple fiber of π of multiplicity m, and G contains a cyclic singularity, either Du Val or almost Du Val. If $e(T) < 2/3$ then G is one of the following:*

If (T, G) is not dlt at any singular point:

1. $(2, 2', 2), m = 2$.

2. $(3, 2', 2, 2), m = 3$.

If (T, G) is dlt at one singular point, but not dlt:

3. $(2', z), m = 4$. z *is a non chain singularity, with center -2 and branches* (2), (2) *and* $(2, \cdots, 3')$ *(or* $(3')$*).*

4. $(2, 3', 2; 2'), m = 4$.

If (T, G) is dlt:

5. $(A_k; (k+1)'), k \leqslant 4, m = k+1$. *The fiber is* $-(k+1) + [k+1](-1) + k(-2) + [k-1](-2) + \cdots + (-2)$.

6. $(2, 3'; 2', 3), m = 5$. *The fiber is* $(-2) + 2(-3) + 5(-1) + 3(-2) + (-3)$.

7. $(3, 2, 2'; 4', 2), m = 7$. *The fiber is* $(-3) + 3(-2) + 5(-2) + 7(-1) + 2(-4) + (-2)$.

8. $(4, 2'; 3', 2, 2), m = 7$. *The fiber is* $(-4) + 4(-2) + 3(-1) + 7(-3) + 2(-2) + (-2)$.

Proof. See [KM99, Lemma 11.5.9]. □

Lemma 1.3.23. *If G is a multiple fiber of multiplicity three and the coefficient $e(T) <$ 2/3, then G is one of the fibers of the above classification.*

Proof. See [KM99, Lemma 11.5.13]. □

1.3.5 Useful facts

Here we collect some results that are somehow unrelated to the previous discussion, but that will be useful later on.

Lemma 1.3.24. *Let S be a rank one log del Pezzo surface and $\tilde{S}$ its minimal resolution. Then $K_{\tilde{S}}^2 + \rho(\tilde{S}) = 10$.*

Proof. Run the minimal model program on $\tilde{S}$. The end result S_{min} is a Mori fiber space because $\tilde{S}$ is birational to S. Then S_{min} is either $\mathbb{P}^2$ or a ruled surface. In this last cast it's a Hirzebruch surface because it's rational. In any case, it follows that $K_{S_{min}}^2 + \rho(S_{min}) = 10$ and a sequence of smooth blow ups does not change the equality. □

Lemma 1.3.25. *Let S be a rank one log del Pezzo surface. Let n be the number of exceptional components in the minimal resolution. Then*

$$K_S^2 = 9 - n + \sum_i e_i(-2 - E_i^2)$$

where e_i is the coefficient of the divisor E_i. If $e(S) < 1/2$, u is the number of exceptional components coming from Du Val singualrities, and n_r is the number of points

of type $(3, A_r)$, *then the above formula takes the form*

$$K_S^2 = 9 - u - \sum_r n_r(r+1)\Big(1 - \frac{1}{2r+3}\Big)$$

Proof. Obvious. $\qquad\square$

Lemma 1.3.26. *Let S be a rank one log del Pezzo surface and $\tilde{S}$ its minimal resolution. Then $\rho(\tilde{S}) \geqslant 11$ if S has no tigers in $\tilde{S}$.*

Proof. This follows from [KM99, (10.3)]. $\qquad\square$

Lemma 1.3.27. *Let p be a klt singularity with n components and coefficient $e < 3/5$. Then $n \geqslant \sum e_i(-2 - E_i^2)$ and equality holds if and only if p is of type* (4).

Proof. Obvious by Proposition A.0.15. $\qquad\square$

Lemma 1.3.28. *Suppose C is a smooth rational curve in the smooth locus of a rank one log del Pezzo surface S. Then either $S = \mathbb{P}^2$ or $S = \overline{\overline{\mathbb{F}}}_n$, in which case $C \in |\sigma_n|$.*

Proof. See [KM99, Lemma 13.7]. $\qquad\square$

Chapter 2

Log del Pezzo surfaces without tigers

We will assume throughout this chapter that S_0 is a rank one log del Pezzo surface with no tigers in $\tilde{S}_0$. Our goal is to explicitly classify all such surfaces. To this end we will apply the hunt in the flush case (see Subsection 1.3.2) and carefully analyze each step (all notation is fixed as in ()). We start by showing that T_1 is not a net, in analogy with [KM99, Chapter 14].

2.1 T_1 a net

Here we assume that T_1 is a net and, if $\text{char}(k) \neq 3$, we derive a contradiction (see Proposition 2.1.3).

Recall that $K_{T_1} + a_1 E_1$ is flush, klt and anti-nef by Proposition 1.3.14. We start with the following easy observation.

Lemma 2.1.1. *The f_0-exceptional divisor E_1 is not contained in a fiber of π_1. Let F be the general fiber of π_1 and consider $d = F \cdot E_1$. Then $d \geqslant 3$ and $e_0 < a_1 = 2/d$.*

Proof. The proof of [KM99, Lemma 14.2] applies independently of the characteristic, and we briefly recall it here. Since E_1 is f_0-exceptional, $E_1^2 < 0$ and thus E_1 cannot be a fiber of π_1. It follows then that the two edges of the cone of curves of T_1 are generated by F and E_1. Now notice that $E_1^2 < 0$ implies that $(K_{T_1} + E_1) \cdot E_1 < (K_{T_1} + e_0 E_1) \cdot E_1$. The right hand side of this inequality is equal to $f_0^*(K_{S_0}) \cdot E_1 = 0$. Therefore $(K_{T_1} + E_1) \cdot E_1 < 0$.

On the other hand, $(K_{T_1} + E_1) \cdot F = -2 + d$. The divisor $-(K_{T_1} + E_1)$ is not nef since S_0 has no tigers in $\tilde{S}_0$, and therefore $-2 + d > 0$. Since d is clearly an integer, we get that $d \geqslant 3$. Finally, in order to find a_1 we simply solve $(K_{T_1} + a_1 E_1) \cdot F = 0$. $\square$

Lemma 2.1.2. *If $\mathrm{char}(k) \neq 3$, then $e(S_0) \geqslant 1/2$.*

Proof. Suppose that we have $e_0 < 1/2$ instead. By the classification of klt singularities of low coefficient (see Proposition A.0.15), all singularities are either Du Val or of the form $(3, A_r)$, for some $r \geqslant 0$. Suppose there are n non Du Val singularities. If $n \leqslant 2$ then S_0 has a tiger by [KM99, Lemma 10.4], so $n \geqslant 3$. In particular there are at least two non Du Val points on T_1, S_0 is not Gorenstein and $e_0 \geqslant 1/3$. Also, by Lemma 2.1.1, $d \leqslant 5$. The first hunt step extracts the (-3) curve E_1 from a point with maximal r.

Case 1: Suppose for the moment that $r = 0$. On T_1 we then only have singularities of type (3) or Du Val. By Lemma 1.3.22, any fiber through a (3) point has multiplicity three and contains only one other singularity, an A_2 point. Since E_1 is in the smooth

locus of T_1, m divides d and $d = 3$. There are at least two (3) points on T_1, so that if we apply Riemann-Hurwitz to the morphism $\pi_1|_{E_1} \to \mathbb{P}^1$, we see that there are exactly two multiple fibers. This implies that the Picard number of $\tilde{T}_1$ is eight. But then $\tilde{S}_0$ has Picard number eight too and hence S_0 has a tiger by Lemma 1.3.26.

Case 2: Hence $r \geqslant 1$ and on the exceptional divisor E_1 we only have one singularity, an A_r point. Also, $e_0 \geqslant 2/5$, which in turn implies that $d \leqslant 4$ by Lemma 2.1.1. Note that a fiber can contain at most two singular points by Lemma 1.3.22, so there is at least one non Du Val point which is not in the same fiber as the A_r point. Call this point p and the respective fiber F. The fiber F meets E_1 only at smooth points because of the way we defined p. This tells us that the multiplicity of F can't be more than four. The only non Du Val fibers with multiplicity at most four have multiplicity exactly three by Lemma 1.3.22 and the fact that $e_0 < 1/2$. Hence d is a multiple of three, and so $d = 3$.

In summary, F has multiplicity exactly three and passes through either just one singularity of type $(3, 2, 2, 2)$, or two singularities, an A_2 point and a (3) point. This means that p is either of type $(3, 2, 2, 2)$ or (3), and is the only non Du Val point on F. By the above there is at least another non Du Val point on T_1, which we call q. Finally, let G be the fiber through the A_r point of E_1.

Suppose first that q lies on G. By Lemma 1.3.22, the singularities on G are A_2 and (3), G has multiplicity three and $K_{T_1} + G$ is dlt. This implies that the intersection of E_1 with G at q is either one or two, depending on whether they meet at opposite ends of the A_2 chain or at the same end. This means that they necessarily intersect at another

point, but since all other points on E_1 are smooth, we get that $G \cdot E_1 \geqslant 4$, contradiction.

Hence q lies on a fiber distinct from F and G, which we call H. Clearly H has multiplicity three. Notice now that one of the points of intersection of E_1 and G is ramified for $\pi_1|_{E_1}$, for either they only meet at the A_r point, which is then ramified, or they meet at another point, which is ramified since it is a smooth point and G is a multiple fiber. But then applying Riemann-Hurwitz we get a contradiction, since there would be at least three ramified points, two of which ramified of order three. $\qquad\square$

Proposition 2.1.3. *If* $\operatorname{char}(k) \neq 3$ *then* T_1 *is not a net.*

Proof. Suppose T_1 is a net. By Lemma 2.1.2, $e(S_0) \geqslant 1/2$. By Lemma 2.1.1, $d = 3$ and $e(S_0) < 2/3$. The spectral value of each singularity along E_1 is at most one by Lemma A.0.11. The classification of Lemma 1.3.22 therefore applies: either the multiple fiber meets E_1 at smooth points, in which case we can use Lemma 1.3.23 because $m = 3$, or it meets E_1 in singular points of spectral value less or equal to one, which are cyclic Du Val or almost Du Val by [KM99, Lemma 8.0.8]. There are four possible cases: T_1 has zero, one, two or three singularities on E_1.

Case 1: T_1 is smooth along E_1. Every multiple fiber has multiplicity three and either contains just one singularity, a $(3, 2, 2, 2)$ point, or two singularities, a (3) point and an A_2 point. In any case, the singular points on a multiple fiber contribute at most four to the Picard number of the minimal resolution. By Riemann-Hurwitz applied to $\pi_1|_{E_1}$, there are exactly two singular fibers, and hence the Picard number of the minimal resolution of S_0 is at most ten, which contradicts Lemma 1.3.26.

Case 2: There is just one singular point, which we call p, on E_1. Let F be the the fiber through it.

Case 2a: Let us first make the further assumption that the multiplicity of F is at least three. Then p is ramified of order three and there can be at most one other singular fiber, necessarily of multiplicity three. As we saw above, the singular points of this other fiber contribute to the Picard number of the minimal resolution by at most four. Therefore, the minimal resolution of T_1 must have at least five exceptional components above the singularities in F by Lemma 1.3.26. By Lemma 1.3.22, the list of possibilities is $(A_4; 5)$ and $m = 5$ or $(3, 2, 2; 4, 2)$ and $m = 7$ or $(2; z)$ with z a non chain singularity and $m = 4$. The first case does not occur: since the spectral value of p is at most one, p must be the A_4 point. Since in this case both $K_{T_1} + F$ and $K_{T_1} + E_1$ are dlt, an easy computation shows that the intersection $F \cdot E_1$ can't be three, which contradicts the fact that $d = 3$. In the second case p must be the $(3, 2, 2)$ singularity. Again $K_{T_1} + F$ is dlt and F can touch either end of the singularity. A computation shows that $E_1 \cdot F = 3$ is only achieved when both the strict transforms of E_1 and F on the minimal resolution touch the (-3) curve, and do not meet on it. But then the singularity of x_0 has to be of the form $(k, 3, 2, 2)$. Since E_1 must be the $(-k)$ curve, and since its coefficient is the highest due to the definition of the hunt, we must have $k \geqslant 5$. However this gives $e_0 > 2/3$, contradiction. In the third case p is the (2) point, $K_{T_1} + F$ is dlt at p and F can't meet E_1 on any other point since $m = 4$. Under these hypotheses however $F \cdot E_1 \neq 3$, contradiction.

Case 2b: We now assume that the multiplicity of F is two. By Lemma 1.3.22, F contains either only one A_3 singularity or two A_1 singularities. Notice that F can't meet E_1 on three different points, so there is at least one ramification point on it. Any other fiber has multiplicity three and contributes by at most four to the Picard number of the minimal resolution. There must be at least two of them by Lemma 1.3.26, but then this contradicts Riemann-Hurwitz applied to $\pi_1|_{E_1}$.

Case 3: Suppose there are exactly two singular points p and q on E_1. If p and q lie on the same fiber, they must be the only points of E_1 on the fiber since $d = 3$ and the fiber is multiple. In particular, exactly one of them is a ramification point for $\pi_1|_{E_1}$. Since both p and q are Du Val or almost Du Val, their fiber contributes by at most four to the Picard number of the minimal resolution by Lemma 1.3.22. There are then at least two more singular fibers, each of multiplicity three, contradicting Riemann-Hurwitz. Hence p and q lie on distinct fibers F and G respectively.

Case 3a: We further suppose there is another multiple fiber, which we call H, necessarily of multiplicity three. Then F and G have multiplicity two by Riemann-Hurwitz. Again by Lemma 1.3.26 and Lemma 1.3.22, at least one of the singularities on E_1, say p, is an A_3 point. Notice F meets transversally the middle component of p and that $K_{T_1} + E_1$ is dlt. This implies that $F \cdot E_1 = 1$. Therefore F must intersect E_1 on another point, which is necessarily a smooth point of T_1 and which is ramification point of order two for $\pi_1|_{E_1}$. Note that x_0 can't have type $(2, 2, 2, k, 2, 2, 2)$ with $k \geqslant 3$ because $e(S_0) < 2/3$, hence G contains two A_1 singularities. H passes through a $(3, 2, 2, 2)$ point by considerations on

the Picard number. Notice that G is disjoint from E_1 on the minimal resolution above the singular point, since otherwise the A_1 point would be a ramification point of order three, contradicting Riemann-Hurwitz. It follows that G also meets E_1 on a point different from q, which is necessarily a smooth point of ramification of order two. Finally E_1 is a (-3) curve because $e_0 < 2/3$. Contracting all the (-1) curves from the minimal resolution of T_1 we perform nine blowdowns on E_1. This means that we reach a Hirzebruch surface, the image of E_1 has self intersection six and is a triple section. But there are no such surfaces, hence this case is impossible too.

Case 3b: Now assume there are only two singular fibers. One of them contributes by at least five to the Picard number of the minimal resolution by Lemma 1.3.26. But this kind of fiber was ruled out in Case 2a.

Case 4: As the last case, suppose there are three singularities p, q and r along E_1. Since x_0 is a klt singularity with coefficient less than $2/3$, the singularities on E_1 are respectively $(2), (2)$ and $(A_j, 3)$ for some j. The two A_1 points p and q can't lie on the same fiber F, otherwise $K_{T_1} + F$ is dlt and there is no configuration in which $E_1 \cdot F = 3$. So F and G contain each two A_1 points and $K_{T_1} + F + G$ is dlt. By Lemma 1.3.22 an A_1 point and $(A_j, 3)$ can't lie on the same fiber either. Hence p, q and r lie on different fibers, say F, G and H. The fibers through the A_1 points can't contain a non chain singularity, for otherwise $m = 4$ and there is no way to get $E_1 \cdot F = 3$. Since the $(A_j, 3)$ point can lie only in fibers of multiplicity at least three, $\pi_1|_{E_1}$ is ramified of order three at r. The contribution of G to the Picard number of the minimal resolution of T_1 must be at least

five, since there are no more multiple fibers by Riemann-Hurwitz. Then $j = 2$ and G contains a $(4, 2)$ point by Lemma 1.3.22. The coefficient of $(4, 2)$ is $4/7 > 1/2$, which contradicts the choice of x_0 in the hunt. $\hspace{1em}\square$

2.2 $g(A_1) > 1$

In the previous section we saw that T_1 can't be a net, so π_1 is a birational contraction. Here we prove:

Proposition 2.2.1. *If $g(A_1) > 1$ and $\operatorname{char}(k) \neq 2$, then S_0 is the surface of [KM99, Lemma 15.2].*

We will compare the local information we get by the geometry of the hunt contraction, and the global information we get from Lemma 1.3.25 to obtain numerical obstructions. First note that S_0 is not Gorenstein, for otherwise it would have a tiger by Lemma [KM99, Lemma 10.4]. Hence $1/3 \leqslant e_0 < a_1$. Therefore A_1 has either a point of multiplicity three or a double point by Lemma 1.3.9.

Lemma 2.2.2. *A_1 cannot have multiplicity three.*

Proof. Suppose that A_1 has multiplicity three. Then $a_1 < 1/2$ by Lemma 1.3.9 and all the singularities in S_0 are either Du Val or almost Du Val. The possible configurations for the contraction are given by Lemma 1.3.19 and Lemma 1.3.20.

Assume that π_1 is given by Lemma 1.3.19. Then x_0 has type $(3, 2)$. Say that on S_0 we have n singularities of type (3), m of type $(3, 2)$ and possibly some Du Val ones. We compute

$$K_{S_0}^2 = \frac{(K_{S_0} \cdot \overline{\Sigma}_1)^2}{\overline{\Sigma}_1^2} = \frac{1}{15 \cdot 11}$$

From the push-pull formula we get that $K_{S_0}^2 = K_{\tilde{S}}^2 + n/3 + 2m/5$, but this can't happen because there is a factor eleven in the denominator of $K_{S_0}^2$ and $K_{\tilde{S}}^2$ is an integer.

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Now assume that π_1 is given by the first case of Lemma 1.3.20 (the second case is numerically the same as above). Say x_0 has type $(3, A_r)$, with $r \geqslant 1$. We have that $K_{S_0} \cdot \overline{\Sigma}_1 = -7/15 + (r+1)/(2r+3)$. By adjunction $\overline{\Sigma}_1^2 = -1/15 + (r+1)/(2r+3)$. Then, as above,

$$K_{S_0}^2 = \frac{(r-6)^2}{15(2r+3)(13r+12)}$$

Since $K_{S_0} \cdot \overline{\Sigma}_1 < 0$, we have that $r \leqslant 5$, hence the non Du Val singularities can only be of type $(3, A_j)$ with $j \leqslant r \leqslant 5$. One checks that for any $r \leqslant 5$ Lemma 1.3.25 cannot hold. Suppose for example that $r = 1$. Then $K_{S_0}^2 = 1/75$, hence there is a factor twenty five in the denominator, contradiction by Lemma A.0.15. All the other cases are similar, except when $r = 5$. In this case $K_{S_0}^2 = 1/(3 \cdot 5 \cdot 7 \cdot 11 \cdot 13)$. For this to be possible, there should be a singularity of type $(3, A_j)$ for $j = 0, 1, 2, 4, 5$. But then, again by Lemma 1.3.25 and Lemma 1.3.27 we get $K_{S_0}^2 < 0$, contradiction. $\qquad \square$

Now we consider double points of genus $g \geqslant 2$. We analyze them via Lemma 1.3.17 and Lemma 1.3.18. Note that configuration 0 of Lemma 1.3.17 does not arise since E_1 is smooth. Also note that configurations I and II are numerically the same in the node and cusp cases. In these cases we will only discuss cusps since we only use the numerical properties of the configuration. Let's start with configuration I, where we allow also $g = 1$ because it simplifies some work later on.

Lemma 2.2.3. *Suppose A_1 has a double point and $g(A_1) \geqslant 1$. Then configuration I does not arise.*

Proof. By Lemma 1.3.17 $a_1 = 1/2$, so that $1/3 \leqslant e_0 < 1/2$. Therefore x_0 is of type $(3, A_r)$ for some $r \geqslant 0$. The local configuration tells us that

$$K_{S_0}^2 = \frac{(K_{S_0} \cdot \overline{\Sigma}_1)^2}{\overline{\Sigma}_1^2} = \frac{(-1 + 2e_0)^2}{4e_0 - 1/g}$$

Lemma 1.3.25 has finitely many solutions thanks to Lemma 1.3.27. By running a computer program (or by a direct and tedious computation) one sees that the only solutions for Lemma 1.3.25 are $r = 0$, $g = 1$ and $r = 1$, $g = 1$.

In the case $r = 0$, $g = 1$, A_1 is in the smooth locus and is a tiger by adjunction. So assume $r = 1$, $g = 1$. There are three solutions compatible with these numbers: in the notation of Lemma 1.3.25, we have $u = 6$, $n_0 = 2$, $n_1 = 1$ or $u = 4$, $n_0 = 5$, $n_1 = 1$ or $u = 2$, $n_0 = 8$, $n_1 = 1$. Now we will apply the results of [KM99, Chapter 12] to our situation. By [KM99, Lemma 12.1] we have that $F = \emptyset$ and $M \in |K_{S_1} + A_1|$ is an irreducible rational curve that passes through the only singular point on A_1, a (2) point. M also passes through all the non Du Val points of S_1, which are all (3) points because $n_1 = 1$ and $n_i = 0$ for $i \geqslant 2$. Let $\overline{M}$ be the reduction of M. Since $(K_{S_1} + M) \cdot M < 0$ and M passes through a (2) point and at least two (3) points, we have by adjunction that $K_{S_1} + \overline{M}$ is dlt and there are exactly two (3) points on M. Hence we are in the case $u = 6$, $n_0 = 2$, $n_1 = 1$. Now consider the transformation $S_1 \dashrightarrow W$ of [KM99, Lemma-Definition 12.4]. W is a Gorenstein log del Pezzo of Picard number one. Let C be the exceptional (-2) curve over the singular point contained in A_1. Then, after contracting

M, C becomes a (-1) curve passing through the intersection of the exceptional curves of an A_2 point. Notice that $K_W^2 = 1$ by Lemma 1.3.24 since $u = 6$ and there is a A_2 point. By the discussion in Appendix B.2, after blowing up the unique base point p of $|-K_{\tilde{W}}|$ we get an extremal rational elliptic surface. Notice that we must have $p = A_1 \cap C$ since $A_1 \equiv C \equiv -K_W$ and the Picard group of $\tilde{W}$ has no torsion. By blowing up p, we get an elliptic fibration which has I_1 and IV among its singular fibers, but this does not appear in the classification of Theorem B.1.5, contradiction. $\qquad\square$

Lemma 2.2.4. *Suppose A_1 has a double point and $g(A_1) \geqslant 2$. If $\mathrm{char}(k) \neq 2$ then configuration II does not arise, unless S_0 is the surface described in [KM99, Lemma 15.2].*

Proof. By Lemma 1.3.18 we get that $e_0 < a_1 = (g+1)/(2g+1) \leqslant 3/5$. By the classification of klt singularities of low coefficient (see Appendix A and in particular Lemma A.0.15), either $x_0 = (2,3,2,2)$ and $g = 2$, or $x_0 = (3, A_g)$.

Case 1: Suppose that $x_0 = (3, A_g)$ and hence $e_0 < 1/2$. The local configuration tells us that

$$K_{S_0}^2 = \frac{(K_{S_0} \cdot \overline{\Sigma}_1)^2}{\overline{\Sigma}_1^2} = \frac{(-1 + \frac{2g+1}{2g+3})^2}{\frac{6g+1}{2g+3} - 1}$$

By running a computer program we see that the only solutions to Lemma 1.3.25 are $g = 2$, $u = 3$, $n_0 = 5$, $n_2 = 1$ and $g = 2$, $u = 5$, $n_0 = 2$, $n_2 = 1$. In any case, we have $g = 2$ and $u > 0$. Again, since $a_1 = 3/5 > 1/2$, let F and M be as in [KM99, Chapter 12]. A_1 is in the smooth locus of S_1 by the description of configuration II and

by the assumption that $x_0 = (3, A_g)$. Also, $A_1^2 = -3 + 9 = 6$. It follows by [KM99, Lemma 12.1.7] that $M = \emptyset$. Since $(K_{S_1} + F) \cdot F < 0$ and since F passes through all the non Du Val points, we must be in the case $u = 5$, $n_0 = 2$. After extracting the two (-3) curves E_2 and E_3, $|F|$ is base point free and gives a fibration $T \to \mathbb{P}^1$ by [KM99, 12.3]. Since the relative Picard number of T is two, there is exactly one reducible fiber with two irreducible components F_1 and F_2. All other fibers lie in the smooth locus, since E_2 are E_3 are sections. Since all singularities on T are Du Val, F_1 and F_2 must meet at an A_5 point at the opposite ends of the chain. The component F_1 cannot meet both E_1 and E_2 since otherwise F_2 would be contractible on S_1. Therefore assume $F_i \cdot E_i = 1$ for $i = 1, 2$. By contracting F_2, we reach the Hirzebruch surface $\mathbb{F}_3$. Let us denote by a bar the image of the curves in $\mathbb{F}_3$. With this convention, $\overline{A_1}$ is a double section, $\overline{E_1}$ is the negative section and $\overline{E_2}$ is a positive section disjoint from E_1. Depending on whether $F_1 \cap A_1 = 2$ or $F_1 \cap A_1 = 1$, we have that $\overline{A_1}^2 = 6$ or $\overline{A_1}^2 = 0$ respectively. The first case cannot happen since numerically $\overline{A_1} \equiv aE_1 + 2F$ and the equation $3a^2 - 4a + 6 = 0$ has no solutions. The second case cannot happen either since A_2 cannot be a fiber.

Case 2: Suppose now that $x_0 = (2, 3, 2, 2)$, which gives $e_0 = 6/11$. For this case, the argument given in [KM99, Chapter 15] goes through without any changes, but we'll show it here too for sake of clarity. Since $K_{S_0}^2 = 1/(11 \cdot 13)$, Lemma 1.3.25 implies that there must be a $(3, A_5)$ point to compensate for the factor thirteen in the denominator. The equality in Lemma 1.3.25 is already satisfied with these two singularities, hence the additional singular points can only be (4) points by Lemma 1.3.27. By [KM99, Lemma

12.3] all these additional points need to be on M, and since $K_{S_1} + M$ is negative, there can be at most one such point. If M passed through a (4) point, M would be smooth by adjunction. One could then contract M on S_1, contradiction. This means that $(2, 3, 2, 2)$ and $(3, A_5)$ are the only singularities in S_0, so now one can proceed as in [KM99, Definition-Lemma 15.2]. □

Lemma 2.2.5. *Suppose A_1 has a double point and $g(A_1) \geqslant 2$. Then configuration III does not arise.*

Proof. We will only sketch the proof, as the idea behind it is very simple but the computations are rather tedious. By Lemma 1.3.17, $a_1 = (g + 1)/(2g + 1)$. We explicitly know the geometry of configuration III, so that if we were able to determine the type of singularity at x_0 as well, we could compute

$$K_{S_0}^2 = \frac{(K_{S_0} \cdot \overline{\Sigma}_1)^2}{\overline{\Sigma}_1^2}$$

One could then use a computer program to see if there are any combinations of singularities that satisfy Lemma 1.3.25. Note however that the possibilities for x_0 are limited by the fact that $e_0 < 3/5$ and by Lemma A.0.15. One can therefore repeat the above procedure for each case of the singularity at x_0 and conclude. In the following we only mention the cases in which Lemma 1.3.25 has a solution and the case in which $e_0 < 1/2$ (which does not yield a solution to Lemma 1.3.25) as an example of the strategy we just described.

The only cases that give solutions to Lemma 1.3.25 are $x_0 = (4)$ and $x_0 = (4, 2)$, both for $g = 2$. Suppose first that $x_0 = (4)$. By [KM99, Lemma 12.1] we have that $M = \emptyset$. By [KM99, 12.3] F only passes through almost Du Val points. The only solution to Lemma 1.3.25 that is compatible with $(K_{S_1} + F) \cdot F < 0$, is $u = 6$, $n_0 = 2$ and $n_1 = 1$. We may run therefore the same argument of Lemma 2.2.4, Case 1. After contracting F_2, however, we reach immediately a contradiction: $\overline{E_1}$ and $\overline{E_2}$ are disjoint sections with self intersections -3 and 4 respectively.

Suppose then that $x_0 = (4, 2)$. Then $K_{S_0}^2 = \frac{2}{13 \cdot 35}$. By Lemma 1.3.25 and Lemma 1.3.27 the singular points of S_0 are $(4, 2)$, $(3, 2)$, (2), $(3, A_5)$ and possibly some (4) points.

Case 1: Suppose there is at least one (4) point. The second hunt step extracts an exceptional (-4) curve E_2, which lies in the smooth locus of T_2. We also have that $a_2 > b_2 > e_1 = 1/2$. In particular, Proposition 1.3.14 applies.

Case 1a: Suppose furthermore that T_2 is a net. E_2 is then a multi-section. If E_2 is not a section, then $\Sigma_2 \cdot E_2 \geqslant 2$. This in turn implies that $(K_T + (3/5)A_1 + (1/2)E_2) \cdot \Sigma_2 > 0$, contradiction. If E_2 is instead a section, then T_2 is smooth, also a contradiction.

Case 1b: Assume that $(S_2, A_2 + B_2)$ is a fence. Since E_2 is in the smooth locus, and since $B_2^2 > 0$, we must have that Σ_2 meets A_1 at an A_r point with $r \geqslant 4$, contradicting the fact that A_1 does not contain any such point.

Case 1c: Let $(S_2, A_2 + B_2)$ be a tacnode of genus g. Since E_2 is in the smooth locus and since there is only a (2) point in A_1, the genus of the node is exactly one and we are in either configuration I or II. In any case, $B_2^2 < 0$ since $E_2^2 = -4$, contradiction.

Since A_1 can't be contracted, and since $x_1 \notin A_1$, the above cases exhaust all the possibilities in Proposition 1.3.14. Therefore there cannot be a (4) point in S_1. This means that the second hunt step extracts the (-3) curve of the $(3, A_5)$ singularity. We again go over all cases.

Case 2a: Suppose that T_2 is a net. Just as in case $(1a)$ we can deduce that E_2 is a section. This however contradicts Lemma 1.3.22.

Case 2b: Assume that $(S_2, A_2 + B_2)$ is a fence. Just as in case $(1b)$ this implies that $B_2^2 < 0$, contradiction.

Case 2c: Let $(S_2, A_2 + B_2)$ be a tacnode of genus g. The only configuration compatible with the geometry at hand is II, where Σ_2 meets E_2 at the A_5 point. Therefore $g = 5$. This implies that B_2 is in the smooth locus of S_2, $B_2^2 = 6$, and the only singular point of S_2 is a (2) point. This contradicts Lemma 1.3.28.

The above cases show that x_0 is not a (4) point or a $(4, 2)$ point. We show now, just as an example, how one can exclude simple based on numerical computations the case in which $e_0 < 1/2$ (the other cases are involve entirely analogous computations). Let x_0 be of type $(3, A_r)$. Then

$$K_{S_0}^2 = \frac{2(g + r + 2)^2}{(2g + 1)(2r + 3)(4gr + 4g - 1)}$$

Since there is (2) point and since $g - 1 = r$, in the sum of the singularities of Lemma 1.3.25 we already have a term which is at least $1 + 2g(1 - \frac{1}{2g+1})$. By Lemma 1.3.27 we get that $8 > 2g(1 - \frac{1}{2g+1})$ and hence $g \leqslant 4$. For the case $g = 4$ note that by Lemma 1.3.25

we have $r = 3$, otherwise the right hand side would be negative. By the above we get $K_{S_0}^2 = 2/(7 \cdot 9)$, which means that there would be an $(3, A_2)$ point. This again contradicts Lemma 1.3.25. Suppose $g = 3$. If $3 \leqslant r \leqslant 5$, the $12r + 11$ part of the denominator has big primes in its factorization, contradicting Lemma 1.3.25. If $r = 2$, then $K_{S_0}^2 = 2/35$, hence the only non Du Val singularities are of type $(3, 2)$ and $(3, 2, 2)$. However it's easy to see that $2/35 = 9 - u - (8/5)n_1 - (18/7)n_2$ has no solutions in integers.

Finally, for the case $g = 2$, running a computer program shows there are no solutions either. $\qquad\square$

Lemma 2.2.6. *If* char$(k) \neq 2$ *and* π_1 *has type U or V, then S_0 has a tiger.*

Proof. The proof goes almost exactly along the lines of [KM99, Lemma 15.4], with only minor modifications. In particular, we have that F passes through at least two singular points (we need char$(k) \neq 2$ in order to apply [KM99, Lemma 15.3]), M is empty, and A_1 is in the Du Val locus of S_1, with at most one singular point on it.

Suppose first that A_1 is in the in the smooth locus of S_1. By [KM99, Lemma 12.3] all the non Du Val points on S_1 are almost Du Val and F does not pass through Du Val points. By [KM99, 15.4.1], there can't be two non Du Val points on F, contradiction.

So there is exactly one Du Val point on A_1 (necessarily in $A \cap F$), and it must be a (2) point by [KM99, 15.4.1]. Again by [KM99, 15.4.1], there is exactly one other singular point on F, which is almost Du Val. We can't be in configuration U, since otherwise $x_0 = (2, 3, 2, 2)$ or $x_0 = (2, 4, 2, 2)$. In the first case, the point $(4, 2, 2)$ in configuration U has higher coefficient, contradicting the choice of x_0 in the hunt. In the second case, x_0

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has coefficient $e(x_0) > 2/3$, contradicting Lemma 1.3.18. So we are in configuration V and $x_0 = (4, 2)$. Now we compute $(K_{S_1} + A_1) \cdot A_1 = 5/2$ by adjunction. Also, $A_1^2 = 15/2$ by the description of the configuration. This implies that $K_{S_1} \equiv -(2/3)A_1$ and that $K_{S_1}^2 = 10/3$. In particular, the only non Du Val point in S_2 is a (3) point. By Lemma 1.3.25 applied to S_1 we see that there must be at least two singular components coming from Du Val points outside A_1. Now extract the Du val singularity on A_1 and the only non Du Val singularity. Following the argument in Lemma 2.2.4, Case 1, get a contradiction. In fact, say $E_1^2 = -2$, $E_2^2 = -3$ and $F_i \cdot E_i = 1$. By contracting F_2 we go to $\mathbb{F}_2$, but the image of E_2 is a section of self intersection zero, contradiction. $\qquad \square$

Proof of Proposition 2.2.1. This follows from Lemma 1.3.17, Lemma 1.3.18 and the lemmas above. $\qquad \square$

2.3 A_1 has a simple cusp

Here we prove:

Proposition 2.3.1. *If* $\mathrm{char}(k) \neq 2, 3$ *and* A_1 *has a cusp of genus one then* $\tilde{S}_0$ *is isomorphic to one of the following:*

1. *Take the cubic* C *given by* $Z^2 X = Y^3$ *in* $\mathbb{P}^2$, *the line* L *given by* $Y = 0$ *and a line* E *meeting* C *at* $L \cap C$ *and two other distinct points* p *and* q. *Blow up four times above* $[0, 0, 1]$ *along* C. *This gives the minimal resolution of the Gorenstein log del Pezzo* $S(A_4)$. *Now blow up twice on* p *along* E. *This gives the minimal resolution of the Gorenstein log del Pezzo surface* $S(A_1 + A_5)$. *Next, blow up on the cusp of* C *four times along* C.

2. *Take the cubic* C *given by* $Z^2 X = Y^3$ *in* $\mathbb{P}^2$, *the line* L *given by* $Y = 0$ *and a line* E *meeting* C *at two points* p *and* q *with order two and one respectively. Blow up three times above* $[0, 0, 1]$ *along* C. *This gives the minimal resolution of the Gorenstein log del Pezzo* $S(A_1 + A_2)$. *Now blow up twice above* p *along* E. *This gives the minimal resolution of the Gorenstein log del Pezzo surface* $S(3A_2)$. *Next, blow up on the cusp of* C *four times along* C.

In the following discussion we always assume that $\mathrm{char}(k) \neq 2, 3$ and we discuss the case $\mathrm{char}(k) = 5$ separately when needed. First notice that $K_{T_1} + a_1 \Sigma_1$ and $K_{S_1} + a_1 A_1$ are flush by Proposition 1.3.14. The geometric configurations are described in Lemma 1.3.18. As case I has been ruled out in Lemma 2.2.3, we may assume that $a_1 \geqslant 2/3$. Clearly

A_1 is not the in smooth locus of S_1, for otherwise it would be a tiger by adjunction. By Lemma 1.3.18 we have that $2/3 \leqslant a_1 < 4/5$. Note that $A_1 \neq \Sigma_2$ because Σ_2 is smooth. By Lemma A.0.13 we have that $e_1 \geqslant 1/3$, so that $a_1 + e_1 \geqslant 1$ and the second hunt step is classified in Proposition 1.3.14. Clearly A_2 is still a rational curve with a single cusp, of genus one, by Proposition 1.3.14. Also, at least one of $-(K_{S_2} + A_2)$ and $-(K_{S_2} + B_2)$ is ample, hence $-(K_{S_2} + B_2)$ is ample and B_2 is smooth.

Lemma 2.3.2. *The pair $(S_2, A_2 + B_2)$ is not a fence.*

Proof. Assume $(S_2, A_2 + B_2)$ is a fence.

Case 1: Suppose first that A_2 is not in the smooth locus of S_2. Then, since $K_{S_2} + (2/3)A_2$ is negative, we have a birational map to a Gorenstein log del Pezzo surface W by [KM99, Lemma-Definition 12.4]. As usual, we let M be as in [KM99, Chapter 12] (and $F = \emptyset$). The rank of W is one more than the difference between the number of singular points on A_2 and the number of irreducible components of M. By [KM99, Lemma 12.5], no irreducible component of M passes through more than two singular points. Furthermore $0 < (K_{S_2} + A_2) \cdot B_2 < 1$, hence M meets B_2 at least once, and only at singular points of S_2. If the rank of W were more than one, there would be a component of M which passes through two singular points of A_2. This is however absurd, since it would also meet B_2 at a singular point, and therefore it would pass through at least three singular points. So the rank of W is one. Let G be the exceptional divisor adjacent to A_2. We have that A_2 is a cuspidal rational curve in the smooth locus of W, $A_2 \in |-K_W|$ and $K_W \cdot G = K_W \cdot B_2 = -1$. If $\mathrm{char}(k) \neq 5$ this contradicts Lemma B.2.4.

Suppose then that $\mathrm{char}(k) = 5$. In this case W is the Gorenstein log del Pezzo surface described in Example B.3.1. Therefore W has exactly two singularities, p and q, each of type A_4. Notice that G and B_2 meet only once in W and are both nodal, with each a node at one of p and q. This, however, is impossible by the description of the transformation $S_2 \dashrightarrow W$.

Case 2: Suppose now that A_2 is in the smooth locus of S_2. This clearly implies that S_2 is Gorenstein. Notice that since $K_{S_2} + B_2$ is dlt we can immediately rule out the case $\mathrm{char}(k) = 5$ by the explicit description of $S(2A_4)$ in this case (Example B.3.1). Suppose therefore that $\mathrm{char}(k) \neq 5$. Again using the fact that $K_{S_2} + B_2$ is dlt we deduce that $0 < B_2^2 = -2 + \deg(\mathrm{Diff}_{B_2}(0)) - K_{S_2} \cdot B_2$ by adjunction. However $K_{S_2} \equiv -A_2$, hence $\deg(\mathrm{Diff}_{B_2}(0)) > 1$, and there are at least two singular points on B_2. By Lemma B.2.4, S_2 is $S(A_1 + A_2)$, and is obtained by taking a flex cubic in $\mathbb{P}^2$ and the tangent line to its flex, blowing up three times to separate them and the blowing down the (-2) curves. However then we have tigers: if $a_2 \geqslant 5/6$ there is a tiger over the singular point of A_2, the (-1) curve of the resolution of the cusp, and otherwise B_2 is a tiger as $K_{S_2} + (5/6)A_2 + B_2$ is numerically trivial. $\qquad \square$

Lemma 2.3.3. *The Gorenstein log del Pezzo surface W associated to S_1 as in [KM99, Lemma-Definition 12.4] has rank at least two.*

Proof. Suppose W has rank one.

Case 1: Suppose also that A_1 contains at least two singular points of S_1. By [KM99, Lemma-Definition 12.4], the image of A_1 in W has a cusp and meets two (-1)

curves. Furthermore these two (-1) curves must meet the image of A_1 in W at different points by [KM99, Lemma 12.1 (7)]. But then we obtain a contradiction by Lemma B.2.4 as in Lemma 2.3.2 Case 1.

Case 2: Suppose now that A_1 has just one singularity, which implies that M of [KM99, Definition-Lemma 12.0] is irreducible by [KM99, Lemma-Definition 12.4 (1)]. Consider the morphism $f : Y \to S_1$ extracting the exceptional divisor G adjacent to A_1 and the morphism $\pi : Y \to W$ contracting M. Define Γ by $K_Y + \Gamma = f^*(K_{S_1} + a_1 A_1)$ and $\Gamma' = \lambda(\Gamma + \epsilon G)$ such that $K_Y + \Gamma'$ is π-trivial. Let $\Delta' = \pi(\Gamma')$. Clearly $K_W + \Delta'$ is negative by Lemma 1.3.12. Now A_1 is in the smooth locus of W and $K_W \cdot G = -A_1 \cdot G = -1$. If $K_W^2 \geqslant 2$, then $K_W + G$ is anti-ample because $(K_W + G) \cdot A_1 < 0$, and hence G is a smooth (-1) curve. If $K_W^2 = 1$, we consider the associated extremal rational elliptic surface. For the moment we would like to show that G is smooth. There are two cases.

Case 2a: Suppose $\text{char}(k) \neq 5$. Then $W = S(E_8)$ by Lemma B.2.4. Since the pullback of G is in $|-K_{\tilde{W}}|$, it follows by the description of the fibers of Theorem B.1.5 that G is smooth.

Case 2b: Suppose $\text{char}(k) = 5$. Since $K_{S_1} + M$ is negative, M is a smooth rational curve. Again by Theorem B.1.5 we deduce that either $W = S(E_8)$, in which case we conclude as above, or $W = S(2A_4)$. Suppose we have the latter. The fact that the pullback of G is in $|-K_{\tilde{W}}|$ implies that G is a nodal rational curve, with the node passing through an A_4 point. It is easy to see however that since $K_{S_1} + A_1$ is dlt, and since M is smooth, there is no way to get such a geometric configuration. Therefore this case does

not occur either.

In conclusion, G is a smooth (-1) curve. Now, if $K_W + G$ is dlt, $A_1 + G$ is a fence and we can proceed as in Lemma 2.3.2.

Otherwise, G meets a unique curve V of the minimal resolution since it's a fiber of the associated extremal rational elliptic surface. Let $h : Q \to W$ extract V. G is a (-1) curve in the smooth locus of Q, so we can contract it with $r : Q \to W_1$. Notice that $K_{W_1}^2 = K_W^2 - 1$. Scaling again as in Lemma 1.3.12 and repeating the process with A_1 and $r(V)$, we can induct on K_W^2, by Lemma 1.3.12. Eventually we therefore we reach the case in which $K_W + G$ dlt, which we can discard as above via Lemma 2.3.2. $\square$

Lemma 2.3.4. *S_1 is singular along A_1 in at least two points and is Du Val outside A_1.*

Proof. The first part is obvious from the above and second part follows from [KM99, Lemma 12.5] (compare with [KM99, Lemma 16.3]). $\square$

Lemma 2.3.5. *S_1 is not Gorenstein unless S_0 is one of the surfaces described in Proposition 2.3.1.*

Proof. From now on we suppose that S_1 is Gorenstein. Let's start by noticing that $(K_{S_1} + A_1) \cdot A_1 \geqslant 1$ by adjunction. Since $(K_{S_1} + 2/3A_1) \cdot A_1 < 0$, we have that $A_1^2 > 3$. Note also that $K_{S_1}^2 > (4/9)A_1^2$, hence $K_{S_1}^2 \geqslant 2$. Also, recall that $a_1 < 4/5$ and $e_0 \geqslant e_1 > 0$, so that $E_1^2 < -2$.

Case 1: Suppose that A_1 passes through three singularities. By the classification of klt singularities, these would be either $2A_1 + A_n$ or $A_1 + A_2 + A_k$, with $k \leqslant 4$. By

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Theorem B.2.2 the only possibilities for S_1 are $S_1 = S(2A_1 + A_3)$ or $S_1 = S(3A_1 + D_4)$. In the first case, $4/5 \leqslant e_0 < a_1$, contradicting the fact that $a_1 < 4/5$. In the second case $a_1 > e_0 \geqslant 2/3$. Since $(K_{S_1} + A_1) \cdot A_1 = 3/2$ we get that $K_{S_1}^2 \geqslant 3$ following the same reasoning at the beginning of the proof of this lemma, contradicting the fact that $S_1 = S(3A_1 + D_4)$ implies that $K_{S_1}^2 = 2$.

Case 2: So A_1 passes through just two singularities. We start by proving that they can't be both A_1 points. Suppose by contradiction that both of the singular points are A_1 points. It immediately follows by adjunction that $(K_{S_1} + A_1) \cdot A_1 = 1$. Therefore $M \cdot A_1 = 1$, which implies that $K_{S_1} + M$ is dlt. Notice that M does not contain other singular points by [KM99, Lemma 12.5]. Extract the (-2) curves of the A_1 points and contract M and one of the extracted curves. The other curve becomes a curve with self-intersection zero, contradicting the fact that the Picard number is one.

Now we prove that the two singular points on A_1 can't be an A_1 and one A_2 point respectively. In fact, by the classification of Theorem B.2.2 and the fact that $K_{S_1}^2 \geqslant 2$, we see that $S_1 = S(A_1 + A_2)$ and $K_{S_1}^2 = 6$. But then $\tilde{A_1}^2 = K_W^2 > K_Y^2 = K_{S_1}^2 = 6$, and we obtain a contradiction thanks to [KM99, Lemma 16.5].

In summary, so far we have we have shown that if A_1 has a (2) point on it, then it also has either an A_3 or an A_5, again by the classification of Theorem B.2.2. Configurations II, u, w, $(U;n)$, $(V;f)$, $(V;f^2)$ of Lemma 1.3.18 do not occur, for otherwise x_0 would be a non chain singularity and $e_0 \geqslant 4/5$, contradiction. Now we are left with configurations V, $(V;n)$, $(V;n^2)$. We have an improved estimate on a_1, namely $a_1 \geqslant 5/7$. If $K_{S_1}^2 \geqslant 6$,

we are done as in the case $K_{S_1}^2 = 6$ described above. Also, there are no Gorenstein log del Pezzo surfaces with $K_S^2 = 5$ and at least two singularities, by the list in Theorem B.2.2. On the other hand, combining $K_{S_1} + 5/7A_1 \leqslant 0$ and $(K_{S_1} + A_1) \cdot A_1 \geqslant 5/4$, we get that $K_{S_1}^2 \geqslant 3$. Hence $K_{S_1}^2$ is either three or four, and the only possibilities for S_1 are $S(2A_1 + A_3)$, $S(A_1 + A_5)$ and $S(3A_2)$.

Consider M as in [KM99, Chapter 12]. Every component of M contains at least two singularities by [KM99, Lemma 12.2] and the number of components of M is at most two by [KM99, Lemma 12.1]. Suppose that M has two components M_1 and M_2. Consider W of [KM99, Lemma - Definition 12.4]. Since $K_W^2 = K_{S_1}^2 + 2 \leqslant 6$, W is either $S(A_4)$ or $S(A_1 + A_2)$ by Lemma B.2.4. Now $M_1 + M_2$ is log canonical at $b = M_1 \cap M_2$ and dlt away from b by adjunction. It is easy now to see that there are no compatible configurations with this geometric description. We may therefore assume that M has only one component. Now we analyze the possible contractions π_1.

Configurations V and $(V; n)$: We have that $E_1^2 = -3$ since $e_0 < a_1$. Recalling that $M \equiv K_{S_1} + A_1$ one then computes $(K_{S_1} + M) \cdot M = -2/3$ if S_1 is either $S(A_1 + A_5)$ or $S(3A_2)$, and $(K_{S_1} + M) \cdot M = -55/64$. Adjunction implies that only the first two cases are possible, and in both of them M passes only through the two singular points in A_1. Furthermore, since $M \cdot A_1 = (K_{S_1} + A_1) \cdot A_1 = 4/3$ we have that M meets the same exceptional curves as A_1 in $\tilde{S}_1$. In the case in which $S_1 = S(3A_2)$ extract one of the exceptional divisors E of the A_2 points touching A_1. By contracting M we get that $W = S(A_1 + A_2)$ and the image of E is a curve of self intersection one which touches the

strict transform of A_1 in two points with multiplicities one and two. Working backwards, one gets the description in Lemma 2.3.1 (a).

Suppose now $S_1 = S(A_1 + A_5)$ and extract the (-2) curve E above the A_5 point which is adjacent to A_1. Let W be the surface obtained by contracting M. We have that $W = S(A_4)$ and the image of E is a zero curve. Working backwards, one may see that S is the surface described in Lemma 2.3.1 (b).

Configuration $(V; n^2)$: Since there is a point of coefficient $2/3$ we necessarily have that $E_1^2 = -4$. If $S_1 = S(A_1 + A_5)$ or $S_1 = S(3A_2)$, one computes $K_{S_0}^2 = \frac{1}{18 \cdot 19}$, which contradicts Lemma 1.3.25. Finally, if $S = S_1(2A_1 + A_3)$, then one computes $(K_{S_1} + M) \cdot M = -15/16$, which is impossible by adjunction. $\qquad\square$

Lemma 2.3.6. *If* $\mathrm{char}(k) \neq 2, 3$ *then* S_2 *is not a banana or a net.*

Proof. The proofs in [KM99, Lemma 16.5, Lemma 16.6] carry through. We just point out that the argument in [KM99, Lemma 16.5] does not actually use the Bogomolov bound, and that the reduction to $S(A_1)$, $S(A_1 + A_2)$ and $S(A_4)$ still holds without the simply connected hypothesis thanks to Lemma B.2.2 and Theorem B.1.5. $\qquad\square$

Proof of Proposition 2.3.1. This follows from Lemmas 2.3.2 - 2.3.6 and Proposition 1.3.14.

$\qquad\square$

2.4 A_1 has a simple node

Throughout this section we suppose that $\mathrm{char}(k) \neq 2, 3$, that A_1 has a simple node and that S_0 doesn't have tigers in $\tilde{S}_0$.

Remark 2.4.1. The proof of [KM99, Proposition 13.5] still works in our setting.

We fix notation as in [KM99, Chapter 17].

Notation. Let C and D be the two branches of A_1 at the node, and c, d be the points of T_1 where the branches meet Σ_1. We can assume that the first two blow ups of $h : \tilde{T}_1 \to S_1$ are along C. Let $r + 1$ be the initial number of blow ups along C, $r \geqslant 1$. Note that d in necessarily singular.

Lemma 2.4.1. *Notation as above.*

1. $K_{T_1} + \Sigma_1 + E_1$ *is log canonical.*

2. $\overline{\Sigma}_1$ *has two smooth branches through x_0 and meets no other singularities.*

3. *If c is smooth, then d in an A_r point, $r \geqslant 1$ and $a_1 = (r+1)/(r+2)$.*

4. T_1 *is singular at some point of $E_1 \setminus E_1 \cap \Sigma_1$.*

5. A_1 *contains exactly one singularity.*

6. S_0 *has exactly two non Du Val points and $e_0 > 1/2$.*

7. $a_1 \geqslant 2/3$, *and $a_1 \geqslant 4/5$, unless we have (3) with $r \leqslant 2$.*

Proof. This is [KM99, Lemma 17.2]. The same proof applies, using Lemma B.2.5. $\qquad \square$

Let x_0 and y be the non Du Val points on S_0, and let z be the singular point of S_1 contained in A_1, of index s.

Proposition 2.4.2. $(S_2, A_2 + B_2)$ *is a fence.* $g(A_2) = 1$ *and* B_2 *is smooth.*

1. *If* $x_1 \in A_1$ *then* Σ_2 *meets* E_2 *at a smooth point, and contains a unique singular point* $(A_t, 3, A_{j-2})$ *for some t.* $K_T + \Sigma_2$ *is dlt and* Σ_2 *meets the end of the* A_{j-2} *chain.* $(S_2, A_2 + B_2)$ *is given by [KM99, 13.5] and* q_2 *is an* A_{t+1} *point.*

2. *If* $x_1 \notin A_1$ *and* A_2 *is not in the smooth locus of* S_2*, then* S_2 *is obtained by starting from* $S(2A_1 + A_3)$ *(see Lemma B.2.5 for a detailed description of this surface), picking a* (-1) *curve* B*, blowing up on the* (-1) *curve* $C \neq B$ *at the* A_3 *point once and then contracting* C*. Let* A *be the nodal curve contained in the smooth locus of* S_2*. To obtain* S_1 *blow up on the intersection of* A B *twice along* A*, then contract* B*. To obtain* S_0 *blow up twice along one of the branches of* A_1 *and then contract* A*. In particular* x_0 *is a chain singularity.*

3. *If* $x_1 \notin A_1$ *and* A_2 *is in the smooth locus of* S_2*, then one of the following holds:*

   (a) $a_2 < 6/7$ *and* $(S_2, A_2 + B_2)$ *is given by [KM99, 13.5.1], or*

   (b) S_0 *is obtained as follows: start with* $S(A_1 + A_5)$ *or* $S(3A_2)$*, consider a* (-1) *curve* B *passing through two of the singularities, consider a rational nodal curve* A *in the smooth locus, blow up twice on* $A \cap B$ *along* B*, three times on the node*

of A along the same branch, and then contract all the negative curves with self

intersection less than (-1); or

(c) *S_0 is obtained as follows: start with $S(2A_1 + A_3)$, consider a (-1) curve B*

passing through two singularities, blow up on $A \cap B$ twice along B, blow up on

the node either four or five times along the same branch, the contract as above;

or

(d) *S_0 is obtained as follows: start with $S(2A_1 + A_3)$, consider a (-1) curve B*

passing through two singularities, blow up three times on $A \cap B$ along B, blow

up four times on the node along the same branch and then contract as above.

In particular, if $S_2 \neq S(A_1 + A_2)$ then x_0 is a chain singularity.

Proof. Part (1) follows as in [KM99, Proposition 17.3], therefore from now on we assume

that $x_1 \notin A_1$.

Case 1: Suppose also that $A_2 + B_2$ has a node of genus $g \geqslant 2$. Then we have

that $z \in \Sigma_2$, $a_1 = 2/3$, $g = 2$, $r = 1$, $E_1^2 \leqslant -3$ exactly as in [KM99, Proposition 17.3].

The point x_0 is a chain singularity of type $(2, 2, -E_1^2, 2)$, hence $E_1^2 = -3$, for otherwise

the coefficient would be too high. Thus $K_{S_2}^2 = A_2^2 = r + 4 + g + E_1^2 = 4$. Now notice that

$(K_{S_2} + 2/3 A_2 + B_2) \cdot B_2 > 0$ because there are no tigers, hence $(K_{S_2} + B_2) \cdot B_2 > -2/3$.

By adjunction there are at least two singular points of S_2 on B_2. By checking the list

in Theorem B.2.2, we see that $S_2 = S(2A_1 + A_3)$ because it's the only surface with

$K_S^2 = 4$ and at least two singularities. But then, applying adjunction again we see that

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B_2 has to contain all of these singularities. This implies that x_1 is a non chain singularity. Furthermore, since it's not Du Val by Lemma 2.4.1, $E_2^2 \leqslant -3$ and $e_1 > 2/3$, contradiction.

Case 2: So we must go to a fence. If A_2 is not in the smooth locus of S_2, the proof of [KM99, Proposition 17.3] gives the result. Suppose therefore that A_2 is in the smooth locus. If $S_2 = S(A_1 + A_2)$, the result follows again by the proof of [KM99, Proposition 17.3]. Assume then that $S_2 \neq S(A_1 + A_2)$.

Case 2a: Let's start with the case in which x_1 is a non chain singularity. Since S_2 has no tigers we must have $(K_{S_2} + A_2 + B_2) \cdot B_2 > 0$. It then follows that B_2 contains at least two singularities and they can't be both (2) points. Looking at the possibilities of [KM99, Lemma 13.5] and comparing them with the classification of non chain klt singularities, we see that Σ_2 meets E_2 at a point of index two, or three if $S_2 = S(2A_1 + A_3)$. If Σ_2 meets E_2 at a (2) point, then $z = (A_k, 3)$ and $\tilde{E}_2^2 = -3 - k$, with $k \geqslant 0$. Since $S_2 \neq S(A_1 + A_2)$ we have that $A_2^2 = K_{S_2}^2 \leqslant 4$, which means that $K_{\tilde{S}_1} \cdot \tilde{A}_1 \geqslant -3$ by adjunction and the description of the configuration. Hence $K_{S_1} \cdot A_1 \geqslant -3 + 1/(2k + 3)$. We also have that $(K_{S_1} + A_1) \cdot A_1 = (2k + 2)/(2k + 3)$ by adjunction, and therefore by linearity of the intersection product we get

$$a_1 < \frac{3 - 1/(2k + 3)}{3 - 1/(2k + 3) + (2k + 2)/(2k + 3)} = \frac{3k + 4}{4k + 5}$$

One checks that the singularity at x_1 has always greater coefficient, contradiction.

Let's consider now the case where Σ_2 meets E_2 at a (3) point. If it meets A_1 at a (2) point, then E_1 is a (-2) curve, $\tilde{A}_1^2 = 2$, $K_{S_1} \cdot A_1 = -2$ and $(K_{S_1} + A_1) \cdot A_1 = 1/2$. That

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gives us that $a_1 < 4/5 = e_1$, contradiction. Otherwise $\overline{\Sigma}_2$ meets A_1 at an $(A_k, -3, -2)$ point. By making the same computations as above, we get that $\tilde{E}_2^2 = -3 - k$ and

$$a_1 < \frac{3 - 2/(3k + 5)}{3 - 2/(3k + 5) + (3k + 4)/(3k + 5)} = \frac{9k + 13}{12k + 17}$$

Again, the singularity at x_1 has always greater coefficient, contradiction. The only remaining case is that Σ_2 meets E_2 at an A_2 point and A_1 at a $(A_k, -4)$ point. We do not treat this case, as it can be ruled out by analogous computations as above.

Case 2b: Let's consider now the case in which x_1 is a chain singularity. Then the configuration is given by (3) of Lemma 1.3.17, Σ_2 meets A_1 at an A_k point and $b_2 + a_2/(k + 1) = 1$. Since $(K_{S_2} + a_2 A_2 + b_2 B_2) \cdot A_2 < 0$, we get that

$$a_2 < \frac{K_{S_2}^2 - 1}{K_{S_2}^2 - 1/(k + 1)}$$

We must have $K_{S_2}^2 \geqslant 3$, for otherwise $a_1 < 2/3$, contradicting Lemma 2.4.1 (7).

If $K_{S_2}^2 = 3$, then $S_2 = S(A_1 + A_5)$ or $S(3A_2)$ by Lemma B.2.2. Also, $a_1 \leqslant 3/4$, again by the above inequality and Lemma 2.4.1. Since B_2 is a (-1) curve, we get that $\tilde{E}_2^2 = -2 - k$, and hence $k = 1$ because $e_1 < a_2$. One computes $e_1 = 3/5$, hence $\tilde{E}_1^2 = -4$ and $r = 2$. This is case (b) in Proposition 2.4.2 (3).

Suppose now that $K_{S_2}^2 = 4$, which implies that $S_2 = S(2A_1 + A_3)$. One computes $e_1 = \frac{4k}{4k+3}$. Setting $e_1 < a_2$ we get that $k = 1$ or $k = 2$, and that $a_2 < 6/7$ or $a_2 < 9/11$ respectively. Suppose that x_0 is a chain singularity. It follows from Lemma 2.4.1 that

$a_1 = (r+1)/(r+2)$. Let's start with $k = 1$. Then $\tilde{E}_1^2 = -1 - r$, and $e_0 = \frac{2r^2-2}{2r^2+r+1}$. The only solutions to $e_1 < e_0 < a_2$ are $r = 3$ or $r = 4$. This leads to case (c) of Proposition 2.4.2 (3). In the case with $k = 2$, one finds again $r = 3$ exactly as above. This is case (d) of Proposition 2.4.2 (3).

If x_0 is a non chain singularity instead there are no solutions, for one can check that if there are many blow ups at the node, the coefficient is at least $6/7$, and if there are few then the coefficient the coefficient is lower than e_1. $\qquad\square$

Lemma 2.4.3. *Suppose that x_0 is a non chain singularity. Then $x_1 \in A_1$ and S_0 is obtained by blowing up the end of the A_5 point in $S(A_1 + A_5)$ along the (-1) curve Y, then blowing up the node twice along one branch and then once along the nearest point of the other branch, and finally contracting down all the K-positive curves.*

Proof. The proof is the same as in [KM99, Lemma 17.4], by using Proposition 2.4.2 instead of [KM99, Proposition 17.3]. $\qquad\square$

We conclude this section by noting now that the same classification as in [KM99, 17.7 - 17.14] carries through, by the previous results.

2.5   Smooth fences

In this section we collect some useful facts about abstract smooth fences. We will then apply these results during the hunt in the next sections. Throughout the subsection $(S, X + Y)$ will be a smooth fence. By this we mean that S is a rank one log del Pezzo surface and that $X + Y$ is a fence, with X and Y both smooth rational curves. No assumption about the existence of tigers and about the characteristic of the ground field are made. We define $\alpha = -(K_S + X) \cdot X$ and $\beta = -(K_S + Y) \cdot Y$. Let's begin with the following elementary but fundamental lemma.

Lemma 2.5.1. *The following hold:*

1. *$\alpha = 1$ if and only if $\beta = 1$.*

2. *$\alpha < 1$ if and only if $\beta < 1$.*

3. *If $\beta \neq 1$ then $X^2 = \frac{1-\alpha}{1-\beta}$.*

4. *If $\beta \neq 1$ then $(K_S + tX) \cdot X = (t - \beta)\frac{1-\alpha}{1-\beta} - 1$.*

5. *If $K_S + aX + bY$ is anti-ample and $\alpha, \beta < 1$, then $a(1 - \alpha) + b(1 - \beta) < 1 - \alpha\beta$.*

Proof. Since S is rank one and $X + Y$ is a fence, $K_S + X \equiv -\alpha Y$ and $K_S + Y \equiv -\beta X$. Hence $K_S + X + \alpha Y \equiv K_S + Y + \beta X$, from which $(1 - \beta)X \equiv (1 - \alpha)Y$. Now the results follows easily. $\qquad\square$

Lemma 2.5.2. *If $\alpha > 0$ then $\tilde{Y}^2 \neq -1$.*

Proof. Suppose that Y is a (-1) curve. Then $K_S \cdot Y \geqslant -1$, hence $(K_S + X) \cdot Y \geqslant 0$. It follows then that $K_S + X$ is nef, contradicting the fact that $\alpha > 0$. $\qquad\square$

Lemma 2.5.3. *Suppose Y passes through exactly two singular points and X passes through at least two singular points. Then $\tilde{Y}^2 \neq 0$.*

Proof. Suppose by contradiction that $\tilde{Y}^2 = 0$. Let $f : T \to S$ be the extraction of the two adjacent curves L and M to Y. Since $\tilde{Y}^2 = 0$, then we have a contraction $\pi : T \to \mathbb{P}^1$. Clearly Y is a fiber and X and the two extracted curves are sections. Since the relative Picard number of the contraction is two, there is exactly one reducible fiber. However, since X contains at least two singularities of T, there is a multiple and irreducible fiber F. This fiber can contain at most two singularities, hence it touches either L or M at a smooth point, contradiction. $\qquad\square$

Lemma 2.5.4. *X and Y can't both contain exactly two singularities.*

Proof. Suppose they do. Then $\alpha, \beta > 0$. Without loss of generality we may assume that $Y^2 \leqslant 1$. However, this contradicts either Lemma 2.5.2 or Lemma 2.5.3. $\qquad\square$

Corollary 2.5.5. *Let $S(X + Y)$ be a fence such that X contains at least two singularities of S and Y contains exactly two singularities, one Du Val and the other a $(-n)$ point. Then $\tilde{Y}^2 \neq -1$.*

Proof. Suppose $\tilde{Y}^2 = -1$. Let $f : T \to S$ be the extraction of the $(-n)$ curve E on Y. Then the strict transform of Y is the curve that gets contracted in Lemma 1.3.12. Clearly

we get a birational morphism $\pi : T \to S_1$, and the image of E is a smooth rational curve in the smooth locus of S_1. However, since there are at least two singularities along X, we get a contradiction by Lemma 1.3.28. $\qquad\square$

Corollary 2.5.6. *Let $S(X+Y)$ be a fence such that X contains exactly three singularities and Y contains exactly two singularities. Suppose also that one of the two singularities on Y, p, is an A_1 point. Then $\tilde{Y}^2 \neq -1$.*

Proof. Suppose $\tilde{Y}^2 = -1$. Let $f_0 : T_1 \to S$ be the extraction of the divisor E adjacent to Y which does not lie over p. As above, let $\pi_1 : T_1 \to S_1$ be the contraction of Y. We have that X and E meet once with order two on S_1. If E is in the smooth locus of S_1, we are done by Lemma 1.3.28. Otherwise there is a singular point on E. Notice that E is either a (-1)-curve or a zero curve on S_1 since is has self intersection at most (-2) on T_1 and $K_{S_1} \cdot E < 0$. It can't be a (-1) curve, for otherwise it would contract on S_1, contradiction. Therefore it is a zero curve. Let $f_1 : T_2 \to S_1$ be the extraction of the adjacent divisor G to E. Since E is a zero curve, it is contractible and therefore T_2 is a net. Clearly G is a section. There is at most one singular fiber on T_2 since G has at most one singular point. However there are two singular points in X, which are not contained in G. Since a multiple fiber can contain at most two singular points, we have a contradiction in this case too. In conclusion, $\tilde{Y}^2 \neq -1$. $\qquad\square$

Lemma 2.5.7. *Suppose that X contains at least three singularities and that Y contains exactly two singularities. Then $\beta \leqslant 2/3$.*

Proof. Suppose $\beta > 2/3$. Then there is an A_1 singularity on Y by adjunction. Since X has at least three singularities $\alpha \leqslant 1/2$, so that $Y^2 = \frac{1-\beta}{1-\alpha} < 1$. Therefore $\tilde{Y}^2$ is either zero or negative one and we can conclude by Lemmas 2.5.6 and 2.5.3. $\qquad\square$

Lemma 2.5.8. *Suppose that X contains exactly three singularities and that Y contains exactly two singularities. Then $\alpha \leqslant 1/6$, and if $\alpha > 0$ then $\beta < \alpha$.*

Proof. Let's start by noticing that if $\alpha \leqslant \beta$, then $Y^2 \leqslant 1$ by Lemma 2.5.1. Therefore Y must be a (-1) curve by Lemma 2.5.3. In that case we must have $\alpha \leqslant 0$ by Lemma 2.5.2.

It remains to see that $\alpha \leqslant 1/6$. Suppose that $\alpha > 1/6$. We must have $\beta < \alpha$ for otherwise $\alpha \leqslant 0$ by the previous remark. By adjunction we see that the singularities on X are two A_1 points and a point of index k, with $2 \leqslant k \leqslant 5$. A simple local computation shows that $X^2 = l/k$, with k and l coprime. Since $\alpha = 1/k$ by adjunction, we also have that $X^2 = \frac{(k-1)/k}{1-\beta}$ by Lemma 2.5.1. This leads to $\beta = 1 - \frac{k-1}{l}$. By the above $\beta < \alpha$, therefore $\frac{k-1}{l} > \frac{k-1}{k}$ and $l < k$. But then $\beta \leqslant 0$, contradiction by adjunction. $\qquad\square$

Lemma 2.5.9. *Suppose that X contains exactly three singularities and that Y contains exactly two singularities. Then $K_S + (4/5)X + (2/3)Y$ is nef.*

Proof. Clearly $\alpha < 1$ by adjunction. Suppose by contradiction that $K_S + (4/5)X + (2/3)Y$ is anti-ample. By Lemma 2.5.1, we have that $(4/5)\alpha + (2/3)\beta - \alpha\beta > 7/15$. Notice that $\alpha \leqslant 1/6$ by Lemma 2.5.8 and that $\beta \leqslant 2/3$ by Lemma 2.5.7. But then

$$7/15 = (4/5) \cdot (1/6) + (2/3) \cdot (2/3) - (1/6) \cdot (2/3) \geqslant (4/5)\alpha + (2/3)\beta - \alpha\beta > 7/15$$

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which is a contradiction. □

Lemma 2.5.10. *If X contains at most one singularity, so does Y.*

Proof. Suppose Y contains at least two singularities. Then $\beta \leqslant 1 < \alpha$, but this can't happen by Lemma 2.5.1. □

Now we characterize smooth fences with that contain exactly one singular points on each branch.

Lemma 2.5.11. *Suppose that X and Y contain both exactly one singular point and assume $Y^2 \leqslant X^2$. Then $\tilde{S}$ is obtained as follows: on the Hirzebruch surface $\mathbb{F}_n$ pick the section with negative self intersection C, choose a disjoint section D (which has necessarily positive self intersection) and choose a fiber F; now blow up once at $F \cap C$ or $F \cap D$, then blow up once more at the intersection of the exceptional divisor and F and keep blowing up at either end of the (-1) curve. To get S we contract all the K-positive curves. X will be the strict transform of D and Y will be the strict transform of a fiber different from F.*

Proof. Notice that Y is a curve with zero self intersection on the minimal resolution since $Y^2 \leqslant 1$ and Y passes exactly through one singularity. Therefore, after extracting it's adjacent divisor E, we get a net T. It's clear that Y is a fiber, X and E are sections. The divisor E does not lie in the smooth locus of T, for otherwise T would be smooth. Therefore E passes through a unique singularity, which lies on the same fiber as the

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singularity contained in X. Now the result easily follows from running a relative MMP on the minimal resolution of T, just as in [KM99, 11.5.4]. □

2.6 A_1 is smooth

In this section we classify all cases in which A_1 is smooth. Recall that $K_{S_1} + a_1 A_1$ is flush and that $K_{S_1} + A_1$ is dlt by Proposition 1.3.14. Since S_0 has no tigers in $\tilde{S}_0$, we must have $(K_{S_1} + A_1) \cdot A_1 > 0$. We deduce then by adjunction that A_1 contains at least three singularities. On the other hand, by the description of klt singularities and by the description of the hunt contractions, A_1 contains at most four singularities. Most of this subsection is devoted to proving the following proposition.

Proposition 2.6.1. *Suppose* $\mathrm{char}(k) \neq 2, 3$. *If* A_1 *is smooth, then* A_1 *contains exactly three singularities and* $(S_2, A_2 + B_2)$ *is a smooth banana.*

Let's start with a preliminary lemma.

Lemma 2.6.2. $\tilde{A}_1^2 \geqslant -1$, $a_1 > 2/3$, $e_1 > 1/2$.

Proof. $\tilde{A}_1^2 \geqslant -1$ is clear, since $K_{S_1} \cdot A_1 < 0$.

Notice that the lemma follows from [KM99, Lemma 18.2.4] in the case in which A_1 contains three singularities of S_1. Suppose then that A_1 contains four singularities. This implies that x_0 is a non chain singularity and that Σ_1 meets E_1 at a smooth point. Let's start by showing that $a_1 > 2/3$. If $e_0 \neq 1/2$ then $a_1 > e_0 \geqslant 2/3$ and we are done. Suppose that $e_0 = 1/2$. If there was a chain singularity with maximal coefficient we would have chosen that singularity in the hunt by Definition 1.3.13. Therefore all chain singularities are either Du Val or almost Du Val. Let the branches of x_0 be (2), (2) and $(A_r, 3)$. In order to create the fourth singular point, there is a $(3, A_k)$ point (with possibly $k = 0$)

in S_0. If $k \geqslant 1$ and Σ_1 meets the end corresponding to a (-2)-curve, then $a_1 > 2/3$.

Suppose therefore that Σ_1 meets the (-3) curve. Then $K_{S_0} \cdot \overline{\Sigma}_1 = -1/2 + (k+1)/(2k+3)$

and $\overline{\Sigma}_1^2 = -(k+2)/(2k+3) + r + 3/2$. By Lemma 1.3.25 and the estimate $K_{S_0}^2 > 0$ we

immediately get $r + k \leqslant 4$. To rule out each one of these cases, one computes:

$$K_{S_0}^2 = \frac{(K_{S_0} \cdot \overline{\Sigma}_1)^2}{\overline{\Sigma}_1^2} = \frac{1}{2(2k+3)(4k+4kr+6r+5)}$$

The only solutions to Lemma 1.3.25 are given by $k = r = 0$, in which case there

must be three extra $(3, 2)$ points and no other singularities. In this situation, $a_1 = 2/3$

and the (3) point on A_1 has coefficient $5/9$. The second hunt step extracts this divisor,

which we call E_2.

Case 1: Suppose that T_2 is a net. Then E_2 is a multi-section. A_1 is not a fiber

since it contains three singular points. Since $(K_{T_2} + (2/3)A_1 + (5/9)E_2) \cdot F < 0$ for the

general fiber, and since E_2 cannot be a section, we see that A_1 must be a section. There

are therefore three triple fibers, each containing a $(3, 2)$ and an A_1 point. This implies

that E_2 is a triple section, which is ramified at least in three points with order three.

This, however, contradicts Riemann-Hurwitz.

Case 2: Suppose that $\pi_2 : T_2 \to S_2$ is a birational contraction. The curve A_1 is

not contracted since it contains three (2) points. $(S_2, A_2 + B_2)$ cannot be a banana, for

otherwise B_2 would be in the smooth locus of S_2, contradicting Lemma 1.3.28. $(S_2, A_2 +$

$B_2)$ cannot be a tacnode because the curve A_1 only contains A_1 singularities and E_2 is

in the smooth locus of T_2. $(S_2, A_2 + B_2)$ cannot be a smooth fence by Lemma 2.5.10 and cannot be a singular fence either by the description of the singularities in S_2. Therefore we get a contradiction by Proposition 1.3.14.

Let's now show that $e_1 > 1/2$. If there is either a non Du Val point or a point of index at least four on A_1, we conclude by Lemma A.0.13. If not, then E_1 is not a (-2) curve, for otherwise $e_0 = 0$. If there is a branch of index three in x_0, we get that $a_1 > e_0 \geqslant 3/4$ and we conclude again by Lemma A.0.13. Suppose therefore that the three branches of x_0 are all (2) points. If $E_1^2 \leqslant -4$ we have that $a_1 > e_0 \geqslant 4/5$, so that $e_1 > 1/2$ by Lemma A.0.13. If not, $E_1^2 = -3$ and $e_0 = 2/3$. The fourth singular point on A_1 can't be (2), for otherwise there would be a tiger, hence it's $(2, 2)$. That means that Σ_1 meets the first curve of an $(A_k, 3, 2)$ singularity, with k possibly zero. Now we can compute $K_{S_1} \cdot A_1 = -k$ and $A_1^2 = k + 1/6$, hence $K_{S_1}^2 = 6k^2/(6k + 1)$. If we had $e_1 \leqslant 1/2$, using Lemma 1.3.25 one obtains immediately a contradiction. $\qquad\square$

Lemma 2.6.3. $(S_2, A_2 + B_2)$ *is not a tacnode.*

Proof. Suppose that S_2 is a tacnode and define l so that A_1 is a $(l - 2)$ curve. As in [KM99, Lemma 18.3 - 18.5], one shows that

1. S_1 is Du Val along A_1.

2. $(K_{S_1} + A_1) \cdot A_1 \geqslant l/3 \geqslant 1/3$.

3. $(K_{S_1} + (3/4)A_1) \cdot A_1 > 0$.

By Lemma 1.3.17 we have that $c + dg/(g+1) = 1$, where d is the coefficient of the branch containing the A_g point. Since $a_2 > 2/3$ and $b_2 > 1/2$, the only possibility is that $g = 2$ and the singularity lies on the curve A_1 (in other words, $c = b_2$ and $d = a_2$).

Case 1: Suppose that A_1 contains exactly three singular points. Let the two singular points that are not contained in Σ_1 be an A_t and an A_m point respectively, with $t \geqslant m$. By adjunction $(K_{S_1} + A_1) \cdot A_1 < 2/3$, hence $l = 1$. Since S_1 is Du Val along A_1, we have that $K_{S_1} \cdot A_1 = -1$ and $A_1^2 = -1/3 + t/(t+1) + m/(m+1)$. However $(K_{S_1} + 3/4 A_1) \cdot A_1 > 0$, hence $-1 - 1/4 + 3/4(t/(t+1) + m/(m+1)) > 0$. This implies that $t/(t+1) + m/(m+1) > 5/3$. Now recall that $b_2 + (2/3)a_2 = 1$. Since $b_2 > a_2 t/(t+1)$ by Lemma A.0.13 and since $a_2 > 2/3$, we have that $t/t(+1) < 5/6$, which contradicts the previous inequality.

Case 2: Let's consider the case in which A_1 contains four singular points. Then x_0 is necessarily a non chain singularity. It follows by adjunction that $(K_{S_1} + A_1) \cdot A_1 < 2/3$ and therefore $l = 1$. Also, $e_0 = 2/3$ since $a_1 < 3/4$. This implies that $E_1^2 = -2$ by inspecting all non chain singularities of coefficient $2/3$. The fourth singularity on A_1 is necessarily created by contracting Σ_1, which meets E_1 at a smooth point and meets a $(3, A_k)$ point at its (-3) curve. But then $K_{S_0} \cdot \overline{\Sigma}_1 \geqslant -1 + 1/3 + 2/3 = 0$, contradiction. $\qquad \square$

The next step towards the proof of Proposition 2.6.1 is to prove that $S_2(A_2 + B_2)$ can't be a fence. The main tools for this will be the results in Section 2.5. Let's start with the following.

Lemma 2.6.4. *Suppose that $S_2(A_2 + B_2)$ is a fence and $x_1 \notin A_1$. Then Σ_2 can't meet E_2*

at an A_r point and A_1 at a smooth point.

Proof. Suppose it does. Then $a_2 + b_2/(r+1) = 1$, hence $b_2 < (r+1)/(r+2)$. Since S_0 has no tigers in $\tilde{S}_0$, B_2 contains at least two singular points. This implies that B_2 contains exactly two singular points and that x_1 is a non chain singularity. We know from Lemma 2.6.2 that $e_1 > 1/2$, which implies that $e_1 \geqslant 2/3$ since x_0 is a non chain singularity. Therefore $b_2 > e_1 \geqslant 2/3$. The two singular points on B_2 can't be both A_1 points because S_0 would have a tiger, hence $r \leqslant 4$ by the classification of klt singularities. Note also that $r \neq 1$ because $b_2 > 2/3$, so that the two singular points on B_2 are a (2) point and a point of index d, with $3 \leqslant d \leqslant 5$. But then we conclude by Lemma 2.5.7. $\qquad\square$

Lemma 2.6.5. *Suppose that $S_2(A_2 + B_2)$ is a fence and $x_1 \notin A_1$. Then Σ_2 can't meet E_2 at a smooth point and A_1 at an A_r point.*

Proof. Suppose it does. Then $a_2/(r+1) + b_2 = 1$, hence $a_2 > (r+1)/(r+2) > b_2$. On the other hand $(K_{S_1} + A_1) \cdot \overline{\Sigma}_2 = -1 + e_1 + 1/(r+1) > 0$, hence $e_1 > r/(r+1)$. If x_1 is a non chain singularity, its coefficient is a rational number of the form $m/(m+1)$ for some m by [KM99, Lemma 8.3.9], contradicting the inequality $(r+1)/(r+2) > e_1 > r/(r+1)$. So x_1 is a chain singularity, and A_1 must contain four singular points by Lemma 2.5.4. Let's proceed by cases on r.

Case 1: Suppose $r = 1$. Then $1/2 < e_1 < 2/3$. Since $B_2^2 > 0$, we necessarily have that $E_2^2 = -3$, hence $\tilde{B}_2$ is a (-1) curve. It follows by Lemma 2.5.2 that $\alpha \leqslant 0$. This, on the other hand, implies that the (2) point that gets contracted by Σ_2 is a branch of the non chain singularity x_0. If $b_2 < 3/5$, then x_1 is of the form $(2, 3, A_k)$ for $2 \leqslant k \leqslant 4$, but

this is impossible by Lemma 2.5.6. So $b_2 \geqslant 3/5$, and we also get that $e_0 < a_2 \leqslant 4/5$. This implies that e_0 is either $2/3$ or $3/4$.

Case 1a: Suppose $e_0 = 3/4$. Then by $e_0/2 + b_2 < 1$ we get that $b_2 < 5/8$. By inspecting all non chain singularities of coefficient $3/4$ we may conclude that the branches of x_0 are (2), (2) and $(2, 2)$, for otherwise one of them would have been chosen in the second hunt step. Since $\alpha \leqslant 0$, the fourth singularity on A_1 has index at least six. But then it would be chosen by the second hunt step, contradiction.

Case 1b: Suppose $e_0 = 2/3$. Then x_0 can't have a branch which is either $(A_k, 4)$ or $(A_k, 3, 2)$, because in both cases they would be chosen by the second hunt step. Hence $E_1^2 = -2$ and x_0 has branches (2), $(2, 2)$ and (3). To rule out this case, some extra work is needed. First of all notice that by Lemma 2.5.6 and by the inequality $1/2 < e_1 < 2/3$ the singularities on B_2 are $(2, 2)$ and either $(2, 2)$ or $(2, 2, 2)$. These cases correspond to $B_2^2 = 1/3$ or $5/12$. In each case we must have $\alpha = 0$, hence the third singularity on A_2 is either $(2, 2)$ or (3). Suppose this singularity is a (3) point, so that it is obtained by contracting Σ_1 which passes through an $(A_k, 4)$ point. Since $K_{S_0} \cdot \overline{\Sigma}_1 < 0$ we must have that $k \geqslant 1$. If $k \geqslant 2$ we have that $A_2^2 > 2$ and if $k = 1$ we have that $A_2^2 = 7/3$. In each case $A_2^2 \cdot B_2^2 \neq 1$, contradiction. One similarly rules out the case in which the third singularity on A_2 is an $(2, 2)$ point and it is obtained by contracting a curve passing through an $(A_k, 3, 2)$ point.

Case 2: Suppose now that $r = 2$. This implies that $e_1 > 2/3$ and $e_0 \geqslant 3/4$. We divide our analysis in cases once again, depending on whether the A_2 point belongs to E_1

or not.

Case 2a: Suppose that the A_2 point is constructed by contracting Σ_1 which passes through an $(A_k, 3, 2)$ point. Then $a_1 = (3k+3)/(3k+5)$, with $k \geqslant 1$ since $a_1 > 2/3$. Hence $k \geqslant 2$ and $a_1 \geqslant 9/11 > 4/5$. Now we conclude by Lemma 2.5.9.

Case 2b: Suppose that the A_2 point belongs to E_1 instead. If $e_0 \geqslant 4/5$ we are done again by Lemma 2.5.9. Suppose that $e_0 = 3/4$. Then the two other singularities on E_1 must be both A_1 points. This implies that $\alpha > 0$ and therefore $\beta < \alpha \leqslant 1/6$ by Lemma 2.5.8. We can finally conclude by applying Lemma 2.5.1.

Case 3: As the last case, suppose that $r \geqslant 3$. In this instance, we can directly conclude by Lemma 2.5.9 since $a_2 > 4/5$ and $b_2 > 3/4$. $\qquad\square$

Lemma 2.6.6. *Suppose that $S_2(A_2 + B_2)$ is a fence and $x_1 \notin A_1$. Then Σ_2 can't meet both A_1 and E_2 at singular points.*

Proof. Suppose it does. The point x_1 must by a non chain singularity by Lemma 2.5.10. Also, A_1 must have four singular points by Lemma 2.5.4, so that x_0 is a non chain singularity as well. The singular points on B_2 can't be both (2) points, for otherwise B_2 would be a tiger. If Σ_2 meets E_2 at a point of index bigger than two, then there is a (2) point on B_2 and the other point has index at most five by the classification of non-chain klt singularities. Therefore we get a contradiction by Lemma 2.5.7.

If Σ_2 meets E_2 on a (2) point instead, then it meets A_1 on an $(3, A_k)$, with Σ_2 touching the (-3) curve and A_1 touching the (-2) curve on the minimal resolution. By imposing $K_{S_1} \cdot \overline{\Sigma}_2 < 0$ and $(K_{S_1} + A_1) \cdot \overline{\Sigma}_2 > 0$, we get that $(2k+2)/(2k+3) < e_1 <$

$(2k + 4)/(2k + 5) < a_2$, hence $e_1 = (2k + 3)/(2k + 4)$. Now we conclude by Lemma 2.5.9. $\qquad\square$

Lemma 2.6.7. *Suppose that $S_2(A_2 + B_2)$ is a fence. Then B_2 is singular.*

Proof. By the previous lemmas, $x_1 \in A_1$. Suppose by contradiction that B_2 is smooth. A_2 contains either two or three singular points, while B_2 contains either one or two singular points. We can then see by Lemma 2.5.10 and Lemma 2.5.4 that the only possibility is that A_2 contains three singular points and B_2 two. In particular this implies that x_0 is a non-chain singularity and therefore $e_0 \geqslant 2/3$. Let $x \in A_1$ be the fourth singularity. Suppose that E_2 lies over x. Then by explicit computation one can see that there exists an integer n such that $\alpha = 1/n > 0$. By Lemma 2.5.8 we have that $0 < \beta < \alpha \leqslant 1/6$. In particular $n \geqslant 6$ and $0 < A_2^2 < 1$. It's also easy to see by an explicit computation that this implies that $A_2^2 \leqslant (n-1)/n$, since different possible singularities on A_2 give rise to values of A_2^2 which differ by multiples of $1/n$. But then by Lemma 2.5.1 we get that $\frac{n-1}{n(1-\beta)} \leqslant \frac{n-1}{n}$, which implies that $\beta \leqslant 0$, contradiction. Therefore E_2 is extracted from one of the singularities of E_1.

If E_1 contains two (2) points, then we get again that $\alpha = 1/n > 0$ for some integer n and the above argument still holds. The same happens if E_1 does not contain two (2) points, but the index of singularity at x is at most five. Therefore we may assume that the index of x is at least six. Also, E_1 must contain a singularity of index at least four which is not Du Val, for otherwise the hunt would extract E_2 from x. In particular, $e_0 \geqslant 4/5$. We see then by Lemma A.0.13 that $e_1 > 2/3$. This contradicts Lemma 2.5.9. $\qquad\square$

Lemma 2.6.8. *If $(S_2, A_2 + B_2)$ is a fence. Then*

1. $x_1 \in A_1$.

2. B_2 *is singular and contains at most one singular point of S_2.*

3. $\tilde{A}_2^2 = -1$.

4. A_2 *contains at least one non Du Val point.*

5. *If* $\mathrm{char}(k) \neq 2, 3$ *then B_2 has genus at least two.*

Proof. Suppose that $(S_2, A_2 + B_2)$ is a fence. By the previous lemmas, $x_1 \in A_1$, B_2 is singular and contains at most one singular point of S_2. The argument in [KM99, Lemma 18.6] then proves that A_2 is a (-1) curve. The argument presented there uses deformation theory, and we give here a more elementary proof using Lemma 2.5.1. First of all notice that since B_2 is singular $\beta \leqslant 0$. If $\beta = 0$ then B_2 is a genus one rational curve lying in the smooth locus of S_2. This implies that S_2 is Gorenstein, and since $K_{S_2} \cdot A_2 = -B_2 \cdot A_2 = -1$, we get that A_2 is a (-1) curve. If $\beta \neq 0$, on the other hand, we must have $\beta \leqslant -1/2$. It's easy to see then that $\alpha > \beta$. In fact if A_2 contains at most two singularities, then $\alpha > 0$. If A_2 contains instead three singularities, then at least two of them are already present on E_1. This implies that their indexes are at most two and three respectively, which yields $\alpha > -(-2 + 1/2 + 2/3 + 1) = -1/6 \geqslant \beta$. By Lemma 2.5.1 and the fact that $\alpha > \beta$, we get that $A_2^2 < 1$. Clearly $\tilde{A}_2^2 \neq 0$ since otherwise we would get a net and B_2 would be a section, contradicting the fact that B_2 is singular. Therefore $\tilde{A}_2^2 = -1$.

Next, we notice that A_2 is in the Du Val locus if and only if B_2 has genus one and is in the smooth locus. In fact, if A_2 is in the Du Val locus, then $(K_{S_2} + B_2) \cdot A_2 = 0$, which implies that $(K_{S_2} + B_2) \cdot B_2 = 0$ and B_2 is in the smooth locus. Vice versa, if B_2 has genus one and is in the smooth locus, then S_2 is Gorenstein and obviously A_2 is in the Du Val locus.

If B_2 has genus one and lies in the smooth locus, however, S_2 is Gorenstein and, in the notation of Lemma 2.5.1, $\beta = 0$. Since B_2^2 is an integer, we must have that $1/(1 - \alpha)$ is an integer. This implies that $\alpha = 0$ and therefore $A_2^2 = 1$. It's now easy to see that this is impossible since $\tilde{A}_2^2 = -1$ and A_2 contains at most three singularities, just as above. This proves (4).

Finally, suppose that B_2 has genus one. If B_2 has a cusp, then consider the Gorenstein log del Pezzo surface W of [KM99, Lemma-Definition 12.4]. Since there are at least two (-1) curve touching the strict transform of B_2, the Mordell-Weil group of the corresponding elliptic surface has at least two elements. By Lemma B.2.4 we must have that $\mathrm{char}(k) = 5$ and $W = S(2A_4)$. In the notation of [KM99, Lemma 12.4], the image of G in W is a smooth (-1) curve. By the description in Example B.3.1, the image of G must pass through both the two A_4 singularities. Therefore M must meet G at a smooth point and contract to an A_4 singularity. Since there cannot be any other singularities, M must pass through all the singular points of A_2. In particular the image of G will meet the A_4 point created by the contraction of M at the intersection of two exceptional components. This contradicts the description in Example B.3.1, contradiction.

Therefore if B_2 has genus one, then B_2 is nodal. Also, since both E_2 and B_2 contain exactly one singular point, Σ_2 must meet E_2 at smooth points. By Lemma 1.3.17, $b_2 = 1/2$. However by Lemma 2.6.2, $e_1 > 1/2$, contradiction. $\qquad\square$

Lemma 2.6.9. *If* $\operatorname{char}(k) \neq 2,3$ *then* $(S_2, A_2 + B_2)$ *is not a fence.*

Proof. Suppose it is. By Lemma 2.6.8 the genus of B_2 is at least two. By Lemma 1.3.17 and Lemma 1.3.18 we have that $b_2 < 2/3$, so that the spectral value of Δ_1 is at most one by [KM99, Lemma 8.0.7]. Therefore, by Lemma A.0.11, A_1 has Du Val or almost Du Val singularities. By Lemma 2.6.2 we have that $e_1 > 1/2$ so that configuration I of Lemma 1.3.17 and Lemma 1.3.18 does not occur.

Case 1: Suppose we have a contraction of type II (recall that the nodal case and the cuspidal case are numerically the same in this configuration). Then we argue as in [KM99, Lemma 18.6]. We have:

$$e_1 < b_2 < (g+1)/(2g+1) \leqslant 2/3$$

Notice now that x_1 is an A_{g+1} point and Σ_2 passes through the A_g point of E_2. In fact, if x_1 was an $(3, A_g)$ point, then its spectral value would be at least two. Recall there is at least one non Du Val point on A_2. Therefore, A_1 does have a point of spectral value at least one. Since we still chose the Du Val point x_1 in the hunt, we must have that

$$1/3 + a_1/3 \leqslant a_1(g+1)/(g+2)$$

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Thus $a_1 \geqslant (g+2)/(2g+1)$ and $e_1 \geqslant a_1(g+1)/(g+2) = (g+1)/(2g+1)$, contradiction.

As a consequence, we have shown that B_2 must have a cusp since there are no more configurations for Lemma 1.3.17. Now we go over the remaining configurations of Lemma 1.3.18.

Case 2: Suppose we have a contraction of type III. Let r be the index of a non Du Val point on A_2. By Lemma A.0.13 and Lemma 2.6.2 we get that $b_2 > 2/3 - 1/(3r)$. However $b_2 \leqslant 3/5$ by Lemma 1.3.18, hence $r \leqslant 4$.

Case 2a: Suppose x_1 is non Du Val, hence a (3) point. We see from the configuration and adjunction that $(K_{S_2} + B_2) \cdot B_2 = 2g - 2$ and $B_2^2 = 4g - 1$. By Lemma 2.5.1 we see that $4g - 1 = (2g-1)/(1-\alpha)$, hence $\alpha = 2g/(4g-1)$. Since $\alpha > 1/2$, A_2 contains only two singular points of S_2. The idea now is to apply Lemma 1.3.25 on S_2, since the configurations are greatly simplified by the inequality $b_2 \leqslant 3/5$ and by the shape of α. If we call x and y the indexes of the singularities in A_2, we get $1/x + 1/y = \alpha$ by the definition of α. Let's start with $g = 2$, which gives $\alpha = 4/7$. This implies that $x = 2$ and $y = 14$. Therefore there is an A_{14} point on A_2, which of course contradicts Lemma 1.3.25. One can similarly check all cases up to $g = 7$ by hand. If $g \geqslant 8$, however, one concludes immediately by Lemma 1.3.25 since $K_{S_2}^2 = (K_{S_2} \cdot B_2)^2/(B_2)^2 \geqslant g + 1$.

Case 2b: Suppose then that x_1 is Du Val, say an A_{j+1} point. Since there is a non Du Val point on A_1, but x_1 got extracted anyway, we must have $j \geqslant 3$. Now, applying adjunction on B_2 we get $(K_{S_2} + B_2) \cdot B_2 = -2 + 2g + j/(j+1)$. By the description of

configuration III we get that $B_2^2 = 4g + j/(j+1)$. But then

$$K_{S_2}^2 \geqslant \frac{4(1+g)^2}{4g+1} \geqslant 4$$

By Lemma 1.3.25 there must be just two singularities on A_2, either two (3) points or a (2) point and a (3) point. In each case, we get $A_2^2 > 2/3$. Since $B_2^2 > 2$, this contradicts Lemma 2.5.1.

Case 3: Suppose now that we are in configuration U. In particular, $g = 2$ and $b_2 = 9/14$. By the configuration, $x_1 = (j, 2, 2)$. The spectral value is at most one so $j = 2$. But then $B_2^2 = 8$, again by the configuration. By Lemma 2.5.1 we also have that $B_2^2 = 3/(1 - \alpha)$, hence $\alpha = 5/8$. This implies as above that A_2 has two singularities, one of which with index eight. But this clearly contradicts the choice of x_1 in the hunt.

Case 4: Finally, suppose we are in configuration V. In particular, $g = 2$ and $b_2 = 7/11$.

Case 4a: Let x_1 be Du Val, say an A_{j+1} point. Just as above in Case 2b, we get that $j \geqslant 3$. By adjunction by the description of the configuration, one gets $(K_{S_2} + B_2) \cdot B_2 = 2 + j/(j+1)$ and $B_2^2 = 9 + j/(j+1)$. But then $\alpha = (6j+6)/(10j+9)$. Since $\alpha > 1/2$ one immediately sees that A_2 has only two singularities on it. It is easy to see however that the equation $1/x + 1/y = (6j+6)/(10j+9)$ does not have a solutions in positive integers.

Case 4b: Suppose that x_1 is a (3) point. The same computation as in Case 3

gives $\alpha = 5/8$. One the concludes exactly as in Case 3.

Case 4c: Suppose that x_1 is almost Du Val, say of type $(3, A_j)$ with $j \geqslant 1$. Since the spectral value of Δ_1 is at most one, A_1 meets the last (-2) curve. Again by adjunction we get that $(K_{S_2} + B_2) \cdot B_2 = 2 + 2j/(2j + 1)$. Also, $B_2^2 = 9 + (2j - 1)/(2j + 1)$, so that $\alpha = (12j + 5)/(20j + 8)$. The equation $1/x + 1/y = (12j + 5)/(20j + 8)$ has no solutions in positive integers. $\qquad \square$

Lemma 2.6.10. *Let $S(X + Y)$ be a smooth banana. Then X and Y can't have contain exactly two and one singular points respectively.*

Proof. Suppose they do. Let $f : T \to S$ extract the adjacent divisor E to Y. We get a contraction π from Lemma 1.3.12.

Case 1: Suppose T is a net. If Y is a fiber, then E is a section with at most one singular point, hence there is exactly one singular fiber. However X has two singular points, contradicting the fact that singular fibers contain at most two singular point. If Y is a multisection, then $Y \cdot F \geqslant 2$ because T is not smooth. But then $(K_S + Y) \cdot f_* F \geqslant 0$. However $K_S + Y$ is negative by adjunction, contradiction.

Case 2: Suppose $\pi : T \to S_1$ is a birational contraction. Clearly π contracts a curve Σ that doesn't touch Y. If S_1 is not smooth, then $Y^2 \geqslant 2$ by Lemma 1.3.28. On S we have that $X^2 > Y^2$ by adjunction (and the analogue of Lemma 2.5.1). Hence $Y^2 < 2$, contradiction. On the other hand, if S_1 is smooth, then it's $\mathbb{P}^2$. Since $E \cdot Y = 1$, they must both be lines and X is a smooth conic. However Σ must pass through both it's singularities, hence X has a node, contradiction. $\qquad \square$

Proof of 2.6.1. By Lemma 2.6.9, $S_2(A_2 + B_2)$ is not a fence. Lemmas [KM99, Lemma 18.7 - 18.8] show that A_1 can't be contracted and T_2 is not a net. The only option left in Proposition 1.3.14 is that $(S_2, A_2 + B_2)$ is a smooth banana. By Lemma 2.6.10 we may also deduce that A_1 contains exactly three singularities. $\qquad \square$

Thanks to Lemma 2.6.10 S_2 is one of the smooth bananas described in [KM99, Lemma 13.2] and we can get the classification of [KM99, Chapter 19].

Chapter 3

Log del Pezzo surfaces with tigers

In this chapter we classify all log canonical pairs (S, C) such that S is a rank one log del Pezzo surface, C is a reduced integral Weil divisor and $K_S + C$ is anti-nef. This classification will allow us to complete the classification of log del Pezzo surfaces of rank one in the next section.

3.1 (S, C) is divisorially log terminal

In this section we furthermore assume that $K_S + C$ is dlt. In the following, we shall slightly change the scaling convention of Lemma 1.3.12 to avoid getting coefficients higher than one.

Case 1: If C contains at least three singular points, we run the hunt for (K_S, aC) with $1 - \epsilon < a < 1$ and we don't rescale by multiplying by λ. Instead, we write $K_{T_1} + \Gamma_1 = f^*(K_S + aC)$, $\Gamma'_1 = \Gamma + b_1 E_1$ with $R \cdot \Gamma'_1 = 0$, and $\Delta_1 = (\pi_1)_*(\Gamma'_1)$. It's easy to see that

if π_1 is a birational contraction, then $b_1 < a$ and the statement of Proposition 1.3.14 remains true with these conventions. In fact, multiplication by λ was only needed to ensure flushness with respect to the divisor aC in Γ'_1, but in our setting this is immediate since $a > 1 - \epsilon$ (compare with the claim on page 141 of [KM99]). Furthermore, since we can pass to the limit $a \to 1$ by continuity, we may even assume $a = 1$.

Case 2: Suppose instead that C contains at most two singular points. Then we run the hunt in the level case for (S, C). Notice that the discussion in Section 1.3.3 shows that $(S_1, C_1 + A_1)$ is dlt since (S, C) is dlt.

We start our analysis by extending [KM99, Proposition 23.5].

Proposition 3.1.1. *Suppose* $\mathrm{char}(k) \neq 2, 3$ *and that* C *contains at least three singularities.*

If $-(K_S + C)$ *is ample then:*

1. *If* $\mathrm{char}(k) \neq 5$ *then* $S \setminus C$ *has exactly one singular point, a non cyclic singularity, z. If* $Z \to S$ *extracts the central divisor* E *of* z*, then* Z *is a* $\mathbb{P}^1$ *fibration and* E *and* C *are sections. The pair* (S, C) *is uniquely determined by* z*, and all non cyclic singularities* z *occur in this way for some pair* (S, C)*. These surfaces are classified in [KM99, Lemma 23.5.1.1].*

2. *If* $\mathrm{char}(k) = 5$*, in addition to the above,* (S, C) *could also be obtained by resolving the singularity of the unique cuspidal rational curve in the smooth locus of* $S(2A_4)$ *and taking* C *to be the last* (-1) *curve. This is example 5.2.3.*

If $K_S + C$ is numerically trivial then S is Du Val, and (S, C) is one of five families:

3. *$S = S(A_1 + 2A_3)$. (S, C) is given by [KM99, Lemma 19.2], Case 1, $s = 3$ and $r = 1$.*

4. *$S = S(3A_2)$. (S, C) is given by [KM99, Lemma 19.2], Case 2, $s = 2$ and $r = 1$.*

5. *$K_S^2 = 1$, C is a (-1) curve and S is one of $S(A_1 + A_2 + A_5)$, $S(2A_1 + A_3)$, $S(4A_1)$. The pairs are obtained from [KM99, Lemma 13.5] by blowing up the node of the nodal curve always along the same branch.*

Proof. The proof of [KM99, Proposition 23.5] goes through with only minor modifications, which we shall point out.

Notice that $e_1 \geqslant 1/2$ by Lemma A.0.13 since $a = 1$ and since there is a singular point on C. Furthermore, $e_1 \geqslant 2/3$ unless all the singularities on C are A_1 points. If follows that $2a + b_1 > 2$. Hence we can't have a tacnode or a triple point by Lemma 1.3.9.

Case 1: Suppose $x \notin C$. Notice that Σ_1 passes through a singular point of C, for otherwise $(K_{T_1} + C + b_1 E_1) \cdot \Sigma_1 > 0$.

Case 1a: Suppose we have a birational contraction to S_1 (notice that our convention has a different indexing with respect to [KM99]). This must be a smooth fence by Proposition 1.3.14. If C contains four singular points, then its corresponding branch in the fence contains three (2) points. The branch corresponding to E can't contain two singular points by Lemma 2.5.8 or one singular point by Lemma 2.5.10. But then x is a non chain singularity, and its coefficient is $1/2$ since $(K_{T_1} + C + b_1 E_1) \cdot \Sigma_1 = 0$. From this we deduce that $\tilde{E}_1^2 = -2$. The description of the configuration then tells us that the

corresponding branch of the fence will be a zero curve. Extracting its singularities we get a net on which C is a section. Also, there is a smooth section corresponding to the (-2) branch of x. Now one proceeds as in the proof of Lemma 2.5.3: there is an irreducible multiple fiber since there are three singularities on C, and this contradicts the existence of a smooth section.

Suppose then that C has exactly three singular points. Notice that Σ_1 can't meet E_1 at a singular point because of Lemma 2.5.4 and Lemma 2.5.10. From this argument it also follows that x is a non chain singularity. We want to apply the results of Section 2.5 with $X = A_1$ and $Y = C_1$. Now, since $K_{S_1} + C_1 + b_1 A_1$ is anti-nef, in the notation of Lemma 2.5.1 we get that $b_1 \leqslant -(K_{S_1} + C_1) \cdot C_1 = \beta$. Since $\beta \geqslant b_1 \geqslant 1/2 \geqslant \alpha > 0$ we get a contradiction by Lemma 2.5.8.

Case 1b: Suppose instead we get a net. Then we get description (1) by the argument in [KM99, Proposition 23.5].

Case 2: Suppose now that $x \in C$. If we get a banana in the hunt, then we get descriptions (3) and (4) by using Lemma 2.6.10 and [KM99, Lemma 13.2].

We can't get a smooth fence in the hunt because of Lemma 2.5.4 and Lemma 2.5.10. Hence if we get a fence, it will be a singular one. Notice that C_1 is smooth and $\tilde{C}_1^2 = -1$ by the proof of Lemma 2.6.8 (3). C_1 is contained in the Du Val locus if and only if A_1 is contained in the smooth locus by adjunction. Now we can again use the argument in [KM99, Proposition 23.5] to get description (5). A little bit of care is needed in applying [KM99, Proposition 13.4]: if $\mathrm{char}(k) \neq 5$ then one uses Lemma B.2.4 as usual

to replace arguments about triviality of the fundamental group; if $\text{char}(k) = 5$ one has to further discard the case in which $W = S(2A_4)$. To do so, notice that C_1 is a (-1) curve which contains two singular points and such that $K_{S_1} + C_1$ is dlt. This contradicts the explicit description in Example B.3.1.

If we get a net, then the proof given [KM99, Proposition 23.5] carries through with no modifications. In the case in which C gets contracted, however, there is a difference if $\text{char}(k) = 5$. In fact, if A_1 is in the smooth locus, then S_1 can also be $S(2A_4)$, leading to description (2). All the rest proceeds in the same way as in [KM99]. $\square$

Proposition 3.1.2. *Assume C contains exactly two singular points. Then $(\tilde{S}, \tilde{C})$ is one of the following*

1. *Start with $\mathbb{F}_n$, pick the negative section E and a positive section C disjoint from E. Blow up along C or E to create two multiple fibers, while keeping C and E disjoint, E K-positive and the K-positive curves contractible to klt singularities.*

2. *Start with a smooth fence containing one singular point in each branch. These are classified in Lemma 2.5.11. Blow up $X \cap Y$ and then keep blowing up at the intersection of the last exceptional divisor with either branch of the fence, while keeping the K-positive curves contractible to klt singularities. $\tilde{C}$ will be the strict transform of X.*

3. *Stat with $\mathbb{F}_n$, pick the negative section E and a positive section C meeting E exactly once. Blow up along C or E to create one multiple fiber, while keeping E K-positive*

and the K-positive curves contractible to klt singularities.

4. *Start with either $\mathbb{F}_2$ or $\mathbb{P}^2$ and pick a banana. Blow up repeatedly above one intersection point to make one of the branches K-positive, while keeping the other branch K-negative and the K-positive curves contractible to klt singularities. The K-negative branch is $\tilde{C}$.*

5. *Start with either a fence as described in Lemma 2.5.11, or by the fence in $\mathbb{F}_n$ given by a fiber C and a positive section E. Then blow up repeatedly over E in order to make E contractile to a klt chain singularity.*

6. *Start with a dlt pair (S', C'), where C' is a smooth rational curve containing either one or two singular points. Blow up one of the singular points (or a smooth point if C' has only one singularity) always along C' to make C' K-positive and contractible to a klt singularity. $\tilde{C}$ is the last (-1) curve.*

Proof. We run the hunt in the level case for (S, C) and analyze all possible cases.

Case 1: Suppose $x \notin C$.

Case 1a: Suppose furthermore that T_1 is a net. Clearly C is not contractible on S and therefore it's not contractible on T_1 either since $x \notin C$. On the other hand, E_1 has negative self intersection because it's exceptional over S. Therefore C and E_1 are are both multi-sections on T_1. Let F be a general fiber of the net and recall that $(K_{T_1} + C + b_1 E_1) \cdot F = 0$ by definition. Since $b_1 \geqslant 1/2$, either C and E_1 are both sections or C is a section and E_1 is a double section. The latter case does not occur: since in this

case $b_1 = 1/2$, the two singular points on C are A_1 points; by Lemma 1.3.22 then $K_{T_1} + F$ is dlt and there is exactly one other A_1 point on F, contradicting $F \cdot E_1 = 2$.

Suppose therefore that C and E are both sections. Running a relative MMP for the morphism $\tilde{T}_1 \to \mathbb{P}^1$ one gets a Hirzebruch surface $M = \mathbb{F}_n$. On M the curves C and E_1 are disjoint, C is a section of positive self intersection, and E_1 is the section of negative self intersection. This process can be reversed to obtain $\tilde{S}$. More precisely, one can start from $\mathbb{F}_n$, pick the negative section E_1 and a disjoint positive section C, and then blow up either on E_1 or C so that to create disjoint two multiple fibers. This is case (1).

Case 1b: If we get a birational contraction instead, we must have a fence. By using Lemma 2.5.10 and Lemma 2.5.4 we see that x must be a chain singularity, that Σ_1 meets C at a singular point, and E_1 either at a smooth point (if E_1 has just one singular point on it) or at a singular point (if E_1 has two singular points on it). Smooth fences with one singular point on each branch are classified in Lemma 2.5.11. Then one can obtain $\tilde{S}$ by blowing up at the intersection of the branches of the fence. This is case (2).

Case 2: Suppose now that $x \in C$.

Case 2a: Suppose T_1 is a net. C can't be a fiber because it's dlt and has only one singularity. Hence C and E_1 are both sections as above, since $(K_{T_1} + C + b_1 E_1) \cdot F = 0$. Since C contains only one singular point, there is exactly one multiple fiber. Therefore E_1 must also contain exactly one singular point on that multiple fiber. Running a relative MMP over the base we go to $\mathbb{F}_n$. E_1 must be the negative section of $\mathbb{F}_n$. C is a positive section meeting E_1 exactly once in $\mathbb{F}_n$. This gives case (3).

Case 2b: Suppose then that we get a birational contraction down to S_1. If we go to a banana, Σ_1 must meet C. Since $(K_{T_1} + C + b_1 E_1) \cdot \Sigma_1 = 0$, Σ_1 meets C at a singular point and therefore C_1 is in the smooth locus of S_1. By adjunction, A_1 also is in the smooth locus. Since $C_1 \cdot A_1 = 2$ we must have $S_1 = \mathbb{F}_2$ or $\mathbb{P}^2$ by Lemma 1.3.28. To get back to S it suffices to reverse the process: pick a smooth banana in $\mathbb{F}_2$ or $\mathbb{P}^2$, blow up repeatedly one of the intersection points, then contract one of the two branches of the banana. This gives case (4).

If we go to a smooth fence, we have at most one singularity on the branch corresponding to E_1 by Lemma 2.5.10. This is case (5).

Suppose we go to a singular fence. If A_1 is in the smooth locus of S_1 then S_1 is Gorenstein. But then C_1 is contracible, since it contains only one singular point, contradiction. If instead A_1 contains a singular point on S_1, we must have that $b_1 = 1/2$ by $(K_{T_1} + C + b_1 E_1) \cdot \Sigma_1 = 0$. This however implies that in S the singularities on C are both A_1 points, contradiction.

Finally, if C gets contracted we can induct on (S_1, A_1), giving description (6). $\square$

Proposition 3.1.3. *Assume C contains exactly one singular point. Then $(\tilde{S}, \tilde{C})$ is one of the following:*

1. *Start with $\mathbb{F}_n$. Pick the negative section E and a positive section C, which is disjoint from E. Blow up repeatedly points on a fixed fiber so that C remains a positive curve, there are K-positive curves lying both over E and C, the K-positive curves are contractible to klt chain singularities and C and E are disjoint.*

2. *Start with $\mathbb{P}^2$. Pick two lines C and E and blow up at the intersection three times along E. Then continue blowing up while keeping the K-positive curves contractible to klt chain singularities and C and E disjoint.*

3. *Start with $\mathbb{F}_n$. Pick a positive section C and a fiber F. Blow up twice at $F \cap C$ along F. Then continue blowing up while keeping the K-positive curves contractible to klt chain singularities and C and F disjoint.*

4. *$(\mathbb{F}_n, C)$, where $n \geqslant 2$ and C is any positive section meeting the negative section exactly once.*

5. *Start with $\mathbb{F}_n$. Pick the negative section E, a point p and a fiber C not passing through p. Blow up at p to obtain at most one multiple fiber, while keeping the K-positive curves contractible to klt chain singularities.*

6. *Start with $\mathbb{P}^2$. Pick two lines C and E. Pick a point $p \in E \setminus C$, blow up p along E, while keeping the K-positive curves contractible to klt chain singularities.*

7. *Start with $\mathbb{F}_n$. Pick the negative section E, a fiber F and a positive section C. Blow up at a point in $F \setminus (C \cup E)$ to make F contractible. Continue blowing up while keeping the K-positive curves contractible to klt chain singularities.*

Proof. We run the hunt in the level case for (S, C) and analyze all possible cases.

Case 1: Suppose $x \notin C$.

Case 1a: Suppose that T_1 is a net. With usual arguments as in Lemma 3.1.1 and Lemma 3.1.2, we see that C and E_1 are both sections. Since there is one singular point

on C, there must be one singular point on E_1 as well, and they must lie on the same multiple fiber. By running a relative MMP we get $\mathbb{F}_n$. C must be a positive section and E_1 must be the negative section. This is case (1).

Case 1b: If we get a birational contraction, we must go to a fence. Since $a = 1$, Σ_1 must pass through the only singular point on C. Therefore the strict transform of C is a smooth rational curve in the smooth locus of S_1. By Lemma 2.5.10, x is a chain singularity. Therefore E_1 contains either one or two singular points of T_1. In the first case, S_1 is $\mathbb{P}^2$, and C_1 and A_1 are lines in S_1. This is case (2). If E contains two singularities instead, we get case (3) by Lemma 1.3.28.

Case 2: Suppose now that $x \in C$.

Case 2a: Suppose T_1 is a net. If C is a section, then T_1 is smooth. Therefore $T_1 = \mathbb{F}_n$ for some n, and S is just $\overline{\overline{\mathbb{F}}}_n$. This is case (4). Suppose then that C is a fiber. Clearly this implies that E_1 is a section. Running a relative MMP one gets case (5).

Case 2b: Suppose we get a birational contraction. We can't get a banana because C has only one singular point of S and $a + b_1 > 1$. We can't get a singular fence either, for otherwise C would be a (-1) curve, hence contractible on S. Notice furthermore that C can't get contracted on T_1 for otherwise $C^2 < 0$ in S.

Therefore we get a smooth fence. If A_1 is in the smooth locus of S_1, then S_1 is $\mathbb{P}^2$, C_1 and A_1 are lines, and we obtain T_1 by blowing up a point of A_1 multiple times. This is case (6). If A_1 is not in the smooth locus of S_1 instead, we get case (7). $\qquad\square$

We conclude with an almost trivial case.

Lemma 3.1.4. *Assume that C is in the smooth locus of S. Then $(\tilde{S}, \tilde{C})$ is one of the following:*

1. *$(\mathbb{P}^2, C)$, where C is a line.*

2. *$(\mathbb{P}_2, C)$, where C is a smooth conic.*

3. *$(\mathbb{F}_n, C)$, where C is a positive section not meeting the negative section.*

Proof. This follows immediately from Lemma 1.3.28. $\square$

3.2 (S, C) is log canonical but not divisorially log terminal

In this section we classify pairs (S, C) of a log del Pezzo surface S and a curve C such that $K_S + C$ is anti-nef and log canonical, but not divisorially log terminal. Throughout this subsection we run the hunt in the level case as in Section 1.3.3.

Lemma 3.2.1. *Let (S, C) be a pair of a rank one log del Pezzo S and an irreducible curve C such that $K_S + C$ is anti-nef, log canonical but not dlt. Suppose also that C contains three singular points of S. Then $(\tilde{S}, \tilde{C})$ is described as follows. Start with $\mathbb{F}_n$, pick the negative section E and a positive section C such that $C \cap E$ contains at most one point. Now create two dlt multiple fibers of multiplicity two. If $C \cap E$ contains exactly one point stop. If $C \cap E = \emptyset$, create a third multiple fiber such that it contains exactly one chain singularity.*

Proof. Two of the singular points on C must be A_1 points, at which $K_S + C$ is dlt by Lemma A.1.4. For the last point, p, there are two cases:

1. p is a non chain singularity, with two (2) branches. C meets the opposite end of the third branch, or

2. p is a $(2, n, 2)$ point, and $\tilde{C}$ meets the central curve.

   Case 1: Let the hunt extract the central curve of the singularity at p.

Case 1a: Suppose we get a net. Clearly C can't be a fiber since it contains three singularities. Hence C and E_1 are multisections, and they are both sections since $K_{T_1} + C + E_1$ is numerically trivial. There are two multiple fibers with multiplicity two passing through the (2) points. Finally there is a third multiple fiber passing through $C \cap E_1$. This is precisely one of the two situations described in the statement.

Case 1b: Suppose we get a birational contraction. Clearly C does not get contracted as it contains three singular points. $K_{S_1} + C + A_1$ is log canonical at singular points and doesn't have triple points because it's level. Σ_1 meets C and E_1 only at singular points, hence the only possibility is that it passes through $C \cap E_1$, by contractibility considerations. In this case, however, we get that $C_1^2 = A_1^2 = 1$ by symmetry (proceeding as in Lemma 2.5.1). Extract the two (-2) curves on C_1. Since $C_1^2 = 1$, C_1 becomes a curve of zero self-intersection after the extraction. Therefore we get a net of relative Picard number two. This is however impossible, since C_1 would be an irreducible dlt multiple fiber containing only one singularity, contradiction.

Case 2: This case can be thought of as a "degenerate" version of Case 1. The first hunt step extracts the middle curve above p. It's easy to see that the contraction of Σ_1 does not yield a birational contraction. In fact, there are only A_1 points lying on C and E_1, and there is no configuration that allows a birational contraction. Suppose therefore that T_1 is a net. If C is a fiber, then it has multiplicity two and E_1 is a double section, which is impossible since $\tilde{E}_1^2 \leqslant -2$ (see also [Har77, Chapter V, Proposition 2.20]). Therefore C and E_1 are both sections, meeting at a smooth point. This situation

is described in the statement. □

Lemma 3.2.2. *Suppose (S, C) is a pair of a rank one log del Pezzo S and a curve C such that $K_S + C$ is anti-nef, log canonical but not divisorially log terminal. If C passes through two singular points, and is dlt at one of them, say p, then $(\tilde{S}, \tilde{C})$ is one of the following:*

1. *Pick a pair (S, E) such that E passes through a $(2, n, 2)$, and E is lc but not dlt at this point (in other words E crosses transversally the $(-n)$ curve). Suppose that E contains at most one other singular point, and E is dlt along it. These are described in parts (2) and (3) of the statement. Blow up above the dlt point (or at a smooth point if E has just one singularity), always along E so that it becomes contractible to a klt singularity. $\tilde{C}$ is the final (-1) curve.*

2. *Start with a dlt pair (S, E) such that E contains two A_1 points, and possibly a third singular point. If there are just two singular points, blow up above a smooth point of E always along E, otherwise blow up above the third point, always along E. $\tilde{C}$ is the last (-1) curve.*

3. *Start with a fence as described in Lemma 2.5.11 with a branch, E, containing an A_1 point. $\tilde{C}$ is the other branch. Now blow up a smooth point on E to create another A_1 point and make E contractible.*

Proof. The configurations for the non dlt point are classified by Lemma A.1.4 just as in Lemma 3.2.1. We continue to use the same division in cases as in Lemma 3.2.1.

Case 1: This time we choose to extract the adjacent curve E_1 of the non chain singularity on C.

Case 1a: Suppose T_1 is a net. C and E_1 can't be sections since they don't touch on the multiple fiber, but one of the singularities is not dlt for this fiber. C can't be a fiber either because it's dlt and has only one singular point on it.

Case 1b: Hence we get a birational contraction. If C gets contracted, then we are in the same situation as in the hypothesis of the lemma, but with a a log canonical point of smaller index, therefore we can apply induction. This is description (1). If Σ_1 passes through both the singularities on C and E_1, then we would get two smooth rational curves meeting at a smooth point and at a chain singularity at the opposite ends of it. Notice that one may argue as in Lemma 2.5.1 in the case of bananas as well. Since α, $\beta \neq -2$, we get $C_1^2 = A_1^2 = 2$ by symmetry. Adjunction, however, would then give a non integer value for $\tilde{C}^2$, contradiction. Also, Σ_1 can't meet E_1 at a smooth point since $K_S + C$ is anti-ample and E_1 has coefficient one. The only option left is that Σ_1 meets one of the two (2) ends of the non chain singularity in E_1, and therefore S_1 is a smooth fence. However then E_1 would have positive self-intersection in T_1, contradiction.

Case 2: As above one can show that T_1 is not a net. If C gets contracted, E is dlt with two A_1 points and possibly another dlt point on it. We can then we can use Lemma 3.1.1 and Lemma 3.1.2 to classify such configurations and get back to the original pair (S, C). This is description (2).

If Σ_1 meets both C and E_1 at singular points, we must go to a banana. It is easy

to see however this cannot happen since the singular points on E are A_1 points. The only remaining option is that Σ_1 passes through an A_1 point of E_1 and another singular point on T_1. Since we then go to a fence, we see that Σ_1 contracts to a smooth point on A_1 by Lemma 2.5.10. We get therefore a fence with one singular point on each branch, one of which is an A_1 point. It is possible to get then S by starting with such a fence, as classified in Lemma 2.5.11, and then blowing up a point in along E_1 multiple times, and finally blowing up once away, to create Σ_1. Finally one can contract E_1. This is case (3). $\qquad\square$

Lemma 3.2.3. *Suppose (S, C) is a pair of a rank one log del Pezzo S and a curve C such that $K_S + C$ is anti-nef, log canonical but not divisorially log terminal. If C passes through two singularities, and is not dlt at either, then $(\tilde{S}, \tilde{C})$ is one of the following:*

1. *Start with $\mathbb{F}_n$, pick a double section E and let the two fibers tangent to E be F and G. Blow up on the intersection $F \cap E$ any number of times, always along E. C is the last (-1) curve. Now blow up at $E \cap G$ twice to separate them and call H the (-1) curve. Finally blow up on H at a point not contained in any of the other components, always along H.*

2. *Start with $\mathbb{F}_n$, pick a double section E, a tangent fiber F and a transverse fiber G. Blow up on the intersection $F \cap E$ any number of times along E; $\tilde{C}$ is the last (-1) curve. Then blow on one of the points in $G \cap E$, and once at the intersection of the two (-1) curves.*

3. *Start from $\mathbb{F}_n$, pick a positive section E and a negative section F. Blow up to create two dlt double fibers. Now blow up any point on E always along E, and if the point was singular then make a last blow up away from E to create an A_1 point on E. $\tilde{C}$ is given by the choice of any smooth fiber.*

Proof. Let p and q be the two singularities. There are three cases.

Case 1: Suppose first that both of them are non chain singularities and extract the adjacent divisor E_1 above q.

Case 1a: Suppose T_1 is a net. C and E_1 can't be both sections because otherwise they would meet on the singular fiber. If C is a fiber instead, it must have multiplicity two because for a general fiber F we have $(K_{T_1} + C + E_1) \cdot F = 0$, so that $F \cdot E_1 = 2 = 2C \cdot E_1$. This is description (1).

Case 1b: Suppose we get a birational contraction instead. If C gets contracted, then it contracts to a smooth point since $K_{S_1} + A_1$ is anti-nef and log canonical. This is however not possible since C contains a non chain singular point. Thus $\Sigma_1 \neq C$. Therefore Σ_1 can meet C and E_1 only at singular points. If it meets both curves we must go to a smooth banana since $K_{S_1} + C + A_1$ is anti-nef and log canonical. It's easy to see that this is again impossible since C contains a non chain singular point. Therefore Σ_1 only meets E_1 and we go to a smooth fence. By Lemma 2.5.1 we must have a non chain singularity on each branch. By symmetry $C_1^2 = A_1^2 = 1$, hence after extracting the adjacent curve D of p at C_1, we get that $C_1^2 = 0$, E_1 and D are sections. But then we would have a multiple fiber with two non dlt sections, which is impossible.

Case 2: Suppose p is a non chain singularity, and q is a $(2, n, 2)$ point. Extract the $(-n)$ curve E_1. If T_1 is a net, then C must be a double fiber and E_1 a double section. This is description (2). If we get a birational contraction, we must go to a fence, where C has a non chain point and A_1 has two (2) points. After extracting the adjacent divisor D of the non chain singularity we would have a net, with A_1 and D being a sections, which is impossible.

Case 3: Finally, suppose p is a $(2, n, 2)$ point and q is a $(2, m, 2)$ point. Extract the $(-m)$ curve E. If T is a net, we get again description (2). Suppose therefore that we have a birational contraction to S_1. Using the usual arguments we deduce that $C_1 + A_1$ is a fence. Now, after extracting the $(-n)$ curve D from C_1, we clearly get a net where C_1 is a fiber, D and E_1 are sections. This is description (3). $\qquad\square$

Lemma 3.2.4. *Suppose (S, C) is a pair of a rank one log del Pezzo S and a curve C such that $K_S + C$ is anti-nef, log canonical but not divisorially log terminal. If C passes through only one singular point p of S then $(\tilde{S}, \tilde{C})$ is obtained as follows.*

1. *Start with $\mathbb{F}_n$, pick the negative section E, and a positive section C meeting E just once and transversally. Pick a point in $(C \cup E) \setminus (C \cap E)$ and blow up any number of times to make a divisorially log terminal fiber. Then perform a blow up on an interior point of the last (-1) curve so that it creates a chain singularity connecting C and E, and finally continue blowing up on the (-1) curve at the nearest point to C and E.*

2. *Start with $\mathbb{F}_n$, pick the negative section E, and a positive section C meeting E exactly twice and transversally. Blow up at one of intersections once, then blow up at the intersection of the exceptional component and the strict transform of the fiber. Now keep blowing up at the intersection of the last (-1) curve with the exceptional component joining C and E any number of times.*

3. *Start with a log canonical pair (S, E), where S is a rank one log del Pezzo surface, $K_S + E \equiv 0$, E has exactly one node, which is a singular point of S as well. Blow up repeatedly at one of the branches of the node, so that E becomes K-positive, and the K-positive curves are contractible to klt singularities. $\tilde{C}$ is the last (-1) curve.*

4. *Start with a Gorenstein log del Pezzo surface of rank one, with a nodal rational curve E in its smooth locus. Then blow up at the node of E always along the same branch, finally contract. C is the last (-1) curve.*

5. *Start with a smooth banana in either $\mathbb{P}^2$ or $\mathbb{F}_2$. Then blow up on one intersection until one curve is negative enough, continue blowing up to make a chain singularity connecting the two branches. $\tilde{C}$ is the other branch of the banana.*

6. *Start with $\mathbb{F}_n$, pick the negative section E and a fiber C. Then create two dlt double fibers away from C.*

7. *Start from $\mathbb{F}_2$, pick a fiber E and a positive section C. Blow up at a smooth point of E so that E is negative and has an A_1 point on it.*

Proof. We divide once again our analysis on the type of log canonical singularity at p. Since C contains exactly one singular point of S, in addition to the cases of the previous lemmas, C could also be nodal at p (see Lemma A.1.4).

Case 1: Suppose p is a non chain singular point. Then after extracting the divisor E_1 adjacent to C, the curve C is in the smooth locus of T_1.

Case 1a: Suppose T_1 is a net. C can't be a section as T_1 is singular. If C is a fiber, E_1 is a section. But then $K_{T_1} + E_1$ must be dlt, contradiction.

Case 1b: Suppose we get a birational contraction to S_1. C can't get contracted because $\tilde{C}^2 > -1$. Σ_1 can only meet E_1 since C is in the smooth locus of T_1, so that we get a smooth fence. Notice that C_1 is in the smooth locus of S_1. Therefore $S_1 = \overline{\mathbb{F}}_n$ for some n by Lemma 1.3.28. However there is no configuration in which Σ_1 contracts to a $(-n)$ singularity, given the fact that $K_{T_1} + E_1$ is not divisorially log terminal.

Case 2: Suppose now that p is a node of C and extract one of the two divisors adjacent to C. Notice that $K_S + C \equiv 0$.

Case 2a: Suppose T_1 is a net and C contains a singular point of T_1. C can't be a fiber since $K_{T_1} + C$ is dlt and C contains only one singular point. If C and E_1 are sections, we get description (1).

Case 2b: Suppose T_1 is a net and C lies in the smooth locus of T_1. If C is a section, we get description (2). If C is a fiber instead, then E_1 is a double section with negative self-intersection, contradiction.

Case 2c: Assume we get a birational transformation. If C contracts, then (S_1, A_1)

satisfies again the hypothesis of the lemma, unless S_1 is Gorenstein. This way we get descriptions (3) and (4). If we go to a smooth banana instead, $C_1 \cap A_1$ must consist of smooth points (since otherwise $C_1^2 = 2$ and $A_1^2 = 2$, which would contradict the fact that $\tilde{C}_1$ is an integer) and we get description (5).

Case 3: The last case is the one in which C passes through a $(2, n, 2)$ point. Extract the $(-n)$ curve. If T_1 is a net, then C must be a fiber, E_1 a section. This is description (6). If we have a birational contraction instead, C can't contract and we must have a fence. Clearly there must be only one (2) point on A_1 since C_1 is in the smooth locus of S_1, therefore Σ_1 contracts the other A_1 point down to a smooth point. This is description (7). $\qquad\square$

We conclude with the following easy observation.

Lemma 3.2.5. *Suppose (S, C) is a pair of a rank one log del Pezzo S and a curve C such that $K_S + C$ is anti-nef, log canonical but not divisorially log terminal. If C is in the smooth locus of S then C is nodal and S is Gorenstein.*

Proof. Immediate from adjunction. $\qquad\square$

Chapter 4

Classification

Let k be an algebraically closed field of characteristic different from two and three. In this section we classify all rank one log del Pezzo surfaces defined over k. We start with a preliminary definition.

Definition 4.0.1. A tiger E for S is called exceptional if E does not lie in S. ◄

Theorem 4.0.2. *Let S be a rank one log del Pezzo surface.*

1. *If S has no tigers in $\tilde{S}$ then S is one of the surfaces described in Section 2.*

2. *If S has an exceptional tiger in $\tilde{S}$ then S is obtained by starting with a log canonical pair (S_1, C) such that $K_{S_1} + C$ is anti-nef (these are described in Section 3), by choosing a point in $\tilde{C}$ and blowing up on that point so that $\tilde{C}$ becomes contractible, and the K-positive curves are contractible to klt singularities.*

3. *If S has tigers in $\tilde{S}$, but none of which is exceptional, then S is one of the surfaces described in Section 3.*

Proof. Part (1) of the statement is clear. Suppose S has an exceptional tiger E in $\tilde{S}$. By definition, this means that there is an effective divisor α such that $K_S + \alpha$ is anti-nef and an extraction $f : T \to S$ of an exceptional divisor E in $\tilde{S}$ such that $K_T + E + f_*^{-1}(\alpha) = f^*(K_S + \alpha)$. Choose E and α such that the coefficient of E for $(S, \emptyset)$ is maximal. Let $\pi : T \to S_1$ be the associated K_T-negative contraction as Lemma 1.3.12 and let $C = \pi(E)$. Choose $a > e$ such that $K_T + aE$ is π_1-trivial. Notice that $K_{S_1} + C$ is clearly anti-nef since

$$\pi^*(K_{S_1} + C) \cdot E \leqslant (K_T + E + f_*^{-1}(\alpha)) \cdot E \leqslant 0$$

If $a \geqslant 1$, the pair (S_1, C) is log canonical, since so is $K_{T_1} + E_1$. Suppose $a < 1$. We claim that $(S_1 + C)$ is still log canonical. If that was not the case, let c be the log canonical threshold. Clearly $c \geqslant a$ since $K_{S_1} + aC$ is log canonical. By Lemma A.1.3, C and the exceptional curves of the minimal resolution mapping to C form the graph of a log canonical singularity. The proof also tells us that c is exactly the coefficient of C in this log canonical singularity, and that there is a curve of coefficient one for $K_{S_1} + cC$ lying in the minimal resolution of S_1. Let E_1 be this curve and let $\tilde{E}_1$ be its strict transform in $\tilde{S}$. Let e_1 be the coefficient of E_1 for $(S_1, \emptyset)$. Denote by $f(t) = e(\tilde{E}_1; S_1, tC)$. Clearly $f(0) \geqslant 0$ since E_1 is exceptional. Also, $f(c) = 1$ by definition. Since $f(t)$ is an affine function, we must have $f(a) > a$. But then $e(\tilde{E}_1; T, aE) > a$. Consider $g(t) = e(\tilde{E}_1; T, tE)$. We have

just seen that $g(a) > a$. Also, $g(0) \geqslant 0$. Therefore $g(e) > e$. However this contradics the maximality of e since $e(\tilde{E}_1; S, \alpha) = e(\tilde{E}_1; T, E + f_*^{-1}(\alpha)) \geqslant e(\tilde{E}_1; S_1, cC) = 1$. Therefore (S_1, C) is log canonical and $K_{S_1} + C$ is anti-nef, so that we can again use Section 3, proving part (2).

Finally, we prove part (3). Let C be a non-exceptional tiger, so that $K_S + C$ is anti-nef. If (S, C) is log canonical, we are done. If (S, C) is not log canonical let c be the log canonical threshold. By Lemma A.1.3, there is a divisor of coefficient one with respect to (S, cC) in the minimal resolution of S. Therefore there is an exceptional tiger, contradiction. $\qquad \square$

4.1 The list

Definition 4.1.1 (LDP 1)**.** Let A and B be two smooth conics meeting to order four in $\mathbb{P}^2$ at a point c. Pick a point $b \neq c$ of B. Pick a point a in the intersection of A with the tangent line M_b to B at b. Let b' be the other point of the the intersection of B with the line passing through a and c. Let a' be a point of the intersection of A with the line through b and b'.

Let $\tilde{S} \to \mathbb{P}^2$ blow up once at a, a', b', twice along M_b at b, and five times along A at c. $\qquad \blacktriangleleft$

Definition 4.1.2 (LDP 2)**.** Take the cubic C given by $Z^2 X = Y^3$ in $\mathbb{P}^2$, the line L given by $Y = 0$ and a line E meeting C at $L \cap C$ and two other distinct points p and q. Blow

up four times above $[0, 0, 1]$ along C. This gives the minimal resolution of the Gorenstein log del Pezzo $S(A_4)$. Now blow up twice on p along E. This gives the minimal resolution of the Gorenstein log del Pezzo surface $S(A_1 + A_5)$. Next, blow up on the cusp of C four times along C. ◄

Definition 4.1.3 (LDP 3). Take the cubic C given by $Z^2 X = Y^3$ in $\mathbb{P}^2$, the line L given by $Y = 0$ and a line E meeting C at two points p and q with order two and one respectively. Blow up three times above $[0, 0, 1]$ along C. This gives the minimal resolution of the Gorenstein log del Pezzo $S(A_1 + A_2)$. Now blow up twice above p along E. This gives the minimal resolution of the Gorenstein log del Pezzo surface $S(3A_2)$. Next, blow up on the cusp of C four times along C. ◄

Definition 4.1.4 (LDP 4). Let B and C be two dlt (-1) curves meeting a nodal curve of $S(2A_1 + A_3)$ at smooth points, each passing through an A_1 point, and meeting at opposite ends of the A_3 point (as in Lemma B.2.5).

Let $\tilde{S} \to \tilde{S}(2A_1 + A_3)$ blow up once at the intersection of C with the (-2) curve at the A_3 singularity, twice at $A \cap B$ along A and twice along on the branches of the node of A. ◄

Definition 4.1.5 (LDP 5). Let B be a (-1) curve in either $S_2 = S(A_1 + A_5)$ or $S_2 = S(3A_2)$ passing through two singularities, B dlt. Let A be a nodal rational curve in the smooth locus of S_2.

Let $\tilde{S} \to \tilde{S}_2$ blow up twice at $A \cap B$ along B and three times on the node of A along the same branch. ◄

Definition 4.1.6 (LDP 6). Let A be a nodal curve in the smooth locus of $S(2A_1 + A_3)$, B a dlt (-1) curve through A_1 and A_3.

Let $\tilde{S} \to \tilde{S}(2A_1 + A_3)$ blow up on $A \cap B$ twice along B, blow up the node of A four or five times along the same branch. ◀

Definition 4.1.7 (LDP 7). Let A be a nodal curve in the smooth locus of $S(2A_1 + A_3)$, B a dlt (-1) curve through A_1 and A_3.

Let $\tilde{S} \to \tilde{S}(2A_1 + A_3)$ blow up three times on $A \cap B$ along B, blow up four times on the node along the same branch. ◀

Definition 4.1.8 (LDP 8). Let A be a nodal curve in the smooth locus of $S(A_1 + A_5)$ and B a log terminal (-1) curve.

Let $\tilde{S} \to \tilde{S}(A_1 + A_5)$ blow up at the intersection of B with the (-2) curve in the A_5 singularity, blow up the node of A twice along one branch and then once along the nearest point of the other branch. ◀

Definition 4.1.9 (LDP 9). Let A be a nodal curve in the smooth locus of the Gorenstein log del Pezzo S_2 and B a dlt (-1) curve that passes through two singular points. Let $\tilde{S} \to \tilde{S}_2$ blow up t times on the intersection of B with the (-2) curve relative to the specified point p, always along B, then blow up s times at the node of A, always along the same branch, for p, t and s as follows.

1. $S_2 = S(A_1 + A_2)$, $p = A_1$ and $(t, s) = (2, 6), (1, 6), (1, 7)$.

2. $S_2 = S(A_1 + A_2)$, $p = A_2$ and $(t, s) = (3, 6), (2, 6), (1, 6), (1, 7), (1, 8)$.

3. $S_2 = S(A_1 + A_5)$, $p = A_1$ and $(t, s) = (1, 3)$.

4. $S_2 = S(A_1 + A_5)$, $p = A_5$ and $(t, s) = (2, 3), (1, 3), (1, 4)$.

5. $S_2 = S(3A_2)$, $p = A_2$ and $(t, s) = (1, 3)$.

6. $S_2 = S(A_2 + A_5)$, $p = A_5$ and $(t, s) = (1, 2)$.

7. $S_2 = S(A_2 + A_5)$, $p = A_2$ and $(t, s) = (1, 2)$.

8. $S_2 = S(A_1 + 2A_3)$, $p = A_3$ and $(t, s) = (1, 2)$.

9. $S_2 = S(2A_1 + A_3)$, $p = A_3$ and $(t, s) = (1, 5), (1, 4), (2, 4)$.

10. $S_2 = S(2A_1 + A_3)$, $p = A_1$ and $(t, s) = (1, 4)$.

◀

Definition 4.1.10 (LDP 10)**.** Let A be a nodal rational curve in the smooth locus of $S(A_1 + A_2)$, and B a dlt (-1) curve passing through the two singularities.

Let $\tilde{S} \to \tilde{S}(A_1 + A_2)$ blow up t times $A \cap B$ along B and s times at the node of A, always along the same branch for $(t, s) = (2, 5), (2, 6), (3, 6), (4, 6), (5, 6), (2, 7), (3, 7)$, $(2, 8),(2, 9)$. ◀

Definition 4.1.11 (LDP 11)**.** Let A be a nodal rational curve in the smooth locus of $S(A_1 + A_2)$, and B a dlt (-1) curve passing through the two singularities.

Let $\tilde{S} \to \tilde{S}(A_1 + A_2)$ blow up $A \cap B$ twice along A, and once near B, then blow up the node at A five, six or seven times along the same branch. ◀

Definition 4.1.12 (LDP 12)**.** Let A be the nodal rational curve in the smooth locus of $S(A_1 + A_2)$, and B the dlt (-1) curve passing through the two singularities.

Let $\tilde{S} \to \tilde{S}(A_1 + A_2)$ blow up $A \cap B$ twice along B, and once near A, then blow up the node at A six times. ◀

Definition 4.1.13 (LDP 13)**.** Let A and B be two positive sections disjoint from the negative section in $\mathbb{F}_2$. Suppose A and B intersect at p and q. Let F be a fiber, not passing through p and q. Blow up $r + 1$ times at $u = A \cap F$ along F and $s + 1$ times at p along B. Now continue in one of the following manners.

If $(s, r) = (3, 2)$ then

1. Blow up above q along A four or five times, or

2. Blow up above p along A three or four times, or

3. Blow up on u along A three times.

If $(s, r) = (4, 1)$ then

1. Blow up above q along A five, six, seven or eight times, or

2. Blow up above p along A k times with $3 \leqslant k \leqslant 7$, or

3. Blow up on u along A three or four times.

◀

Definition 4.1.14 (LDP 14)**.** Let A and B be two positive sections not intersecting the negative section in f $\mathbb{F}_2$ intersecting at p and q. Let F be a fiber, not passing through p and q. Blow up $r + 1$ times at $u = B \cap F$ along F and $s + 1$ times at p along B. Now continue in one of the following manners.

If $(s, r) = (3, 2)$ then blow up $A \cap F$ along A four times.

If $(s, r) = (4, 1)$ then

1. Blow up q along A six times, or

2. Blow up above $A \cap F$ along A four times, or

3. Blow up on p along A five times.

If $(s, r) = (3, 1)$ then

1. Blow up q along A k times, with $5 \leqslant 9$ times, or

2. Blow up $A \cap F$ k times, with $4 \leqslant 7$, or

3. Blow up p along A k times, with $4 \leqslant k \leqslant 8$.

◀

Definition 4.1.15 (LDP 15)**.** Let A and B be two positive sections disjoint from the negative section in $\mathbb{F}_2$ intersecting at p and q. Let F be a fiber, not passing through p and q. Blow up twice at $u = A \cap F$ along A, and five times at p along B, twice along A and once away from A. ◀

Definition 4.1.16 (LDP 16). Let A be a positive section on $\mathbb{F}_n$ and E the negative section. Pick three points on E or A all lying in distinct fibers, blow them up once, then blow up at the intersection of the (-1) curves and continue blowing up to get log terminal fibers such that E and the singularities on it form a log terminal non chain singularity. Now to define $\tilde{S}$ pick any point on A intersecting a K-positive curve, and blow up that point in any fashion so that A becomes K-positive and the K-positive curves are contractible to klt singularities. ◄

Definition 4.1.17 (LDP 17). Suppose $\operatorname{char}(k) = 5$. Start with the surface in Definition B.3.1. Blow up its cusp three times, then contract the resulting rational curve and two of the exceptional divisors. Therefore we obtain a log del Pezzo surface with singularities $2A_4 + (5) + (3) + (2)$. ◄

Definition 4.1.18 (LDP 18). Let (S, C) be one of the surfaces in Proposition 3.1.1 (3)-(5), Lemma 3.1.2, Lemma 3.1.3, Lemma 3.2.1, Lemma 3.2.2, Lemma 3.2.3 or Lemma 3.2.4. Define $\tilde{S} = S_1$. ◄

Definition 4.1.19 (LDP 19). Let (S, C) be one of the surfaces in Proposition 3.1.1, Lemma 3.1.2, Lemma 3.1.3, Lemma 3.1.4, Lemma 3.2.1, Lemma 3.2.2, Lemma 3.2.3, Lemma 3.2.4 or Lemma 3.2.5. Then define $\tilde{S}$ by a sequence of blow ups on C such that C becomes K-positive, and K-positive curves are contractible to klt singularities. ◄

We are now ready to state the classification theorem.

Theorem 4.1.20 (Classification of rank one log del Pezzo surfaces)**.** *Let S be a rank one log del Pezzo surface over an algebraically closed field of characteristic different from two and three. If S is smooth then $S = \mathbb{P}^2$; if S is Gorenstein, then S is one of the surfaces in Theorem B.2.2, otherwise it is one of the log del Pezzo surfaces in the families LDP1 to LDP19.*

Proof. This is the content of Theorem 4.0.2. $\square$

Chapter 5

Applications

In this section we highlight some rather immediate applications of Theorem 4.1.20.

5.1   Liftability to characteristic zero

One of the main applications is that every log del Pezzo surface defined over an algebraically closed field of characteristic strictly higher than five lifts to characteristic zero over a smooth base.

This answers the question after [CTW17, Theorem 1.1]. Let's start first with the definition of liftability as in [CTW17].

Definition 5.1.1. Let X be a smooth variety over a perfect field k of characteristic $p > 0$, and let D be a simple normal crossing divisor on X. Write $D = \sum_i D_i$, where D_i are the irreducible components of D. We say that a pair (X, D) is liftable to characteristic zero

over a smooth base if there exists

1. A scheme T smooth and separated over Spec $\mathbb{Z}$.

2. A smooth and separated morphism $\mathcal{X} \to T$.

3. Effective Cartier divisors $\mathcal{D}_1..., \mathcal{D}_r$ on $\mathcal{X}$ such that the scheme-theoretic intersection $\bigcap_i \mathcal{D}_i$ for any subset $J \subseteq \{1, ..., r\}$ is smooth over T.

4. Amorphism $\alpha : \text{Spec } k \to T$ such that the base changes of the schemes $\mathcal{X}, \mathcal{D}_i$ over T by α are isomorphic to X, D_i respectively.

◀

Theorem 5.1.2. *Let S be a rank one log del Pezzo surface over an algebraically closed field of characteristic $p > 5$. Then there is a log resolution $\nu : V \to S$ such that $(V, Exc(\nu))$ is liftable to characteristic zero over a smooth base.*

Proof. If S is Gorenstein, the result follows by writing the integral Weierstrass model of the corresponding elliptic surface (see [JLR12]). Suppose therefore that S is not Gorenstein. Notice that in the classification of Theorem 4.1.20 each S is determined by the geometry of the first surface from which we start the construction of S. Since these clearly lift to characteristic zero, and the smooth blow ups also do, we deduce that S has a log resolution that lifts as well. □

As a byproduct of Section 3 we also get the following:

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Theorem 5.1.3. *Let S be a rank one log del Pezzo surface over an algebraically closed field of characteristic $p > 5$ and C a curve in S. Suppose that $K_S + C$ is anti-nef and log canonical. Then (S, C) lifts to characteristic zero over a smooth base.*

Proof. Immediate from the classification in Section 3. □

5.2   Non liftable examples in low characteristic

In this subsection we will see that in characteristic two, three and five, there are log del Pezzo surfaces of rank one that do not lift to characteristic zero over a smooth base. Therefore the conclusion of Theorem 5.1.2 is sharp. The first such example was shown in [KM99, Chapter 9] in characteristic two. The following examples are not liftable to characteristic zero because they do not satisfy the Bogomolov bound of [KM99, Chapter 9] (see the proof of [CTW17, Theorem 1.3]).

We start with an example in characteristic three due to Fabio Bernasconi.

Example 5.2.1 (characteristic 3)**.** Pick C to be $y = x^3$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Choose three points on C and blow up three times each along C. This gives a log del Pezzo surface with singularities $4(3) + 3A_2$.

The following example in characteristic three is new.

Example 5.2.2 (characteristic 3)**.** Pick C to be $y = x^3$ in $\mathbb{P}^2$. Choose three flex lines L_1, L_2, L_3 such that their intersection with C lie on a line L_4. Blow up three times two flex points, blow up twice the third one, always along C and then blow up $L_1 \cap L_2 \cap L_3$.

Finally blow up the cusp of C. This gives a log del Pezzo surface with singularities $2(3) + E_6 + (2, 3)$.

Finally let's conclude with an example in characteristic five.

Example 5.2.3 (characteristic 5)**.** Start with the surface in Definition B.3.1. Blow up its cusp three times, then contract the resulting rational curve and two of the exceptional divisors. Therefore we obtain a log del Pezzo surface with singularities $2A_4 + (5) + (3) + (2)$.

5.3   Kodaira's vanishing theorem

Kodaira's vanishing theorem is known to fail for log del Pezzo surfaces in characteristic two and three. However, a consequence of Theorem 5.1.2 is that it holds for rank one log del Pezzo surfaces in characteristic strictly higher than five. More precisely:

Theorem 5.3.1. *Let k be an algebraically closed field of characteristic $\mathrm{char}(k) > 5$. Let S be a rank one log del Pezzo surface defined over k and let A be an ample Cartier divisor on S. Then $H^i(S, K_S + A) = 0$ for all $i > 1$.*

Proof. See [CTW17, Lemma 6.1]. □

Appendix A

Appendix: Surface singularities

In this Appendix we give a detailed description of klt, lc and dlt surface singularities. We also provide techniques that are useful in computations. The main references for the discussion are [Kol97, Chapter 3] and [KM99]. In what follows, unless specified otherwise, S indicates the germ (in the Zariski topology) of a normal surface at a point $p \in S$. We start with some definitions.

Definition A.0.1. Let $p \in S$ be a singular point and let $\pi : \tilde{S} \to S$ be the minimal resolution. The point p is a chain singularity if the dual graph of the minimal resolutions above p is a chain. If the dual graph is not a chain, we will say that p is a non chain singularity.

If p is a chain singularity, we say that p has type $(-E_1^2, -E_2^2, \cdots, -E_n^2)$, where $E_1, \cdots, E_n$ are the exceptional curves over p. ◀

Definition A.0.2. A point $p \in S$ is called Du Val if the coefficient of any exceptional

irreducible divisor of the minimal resolution over p is zero. By adjunction, this is equivalent to saying that for every such divisor E is a smooth rational curve and that we have $E^2 = -2$. ◀

Remark A.0.1. In the case of surfaces, Du Val singularities are the same as klt Gorenstein singularities.

Definition A.0.3. We denote the singularity $(2, \cdots, 2)$ by A_j, where j is the number of components of the dual graph. ◀

Definition A.0.4. Let $p \in S$ be a singular point and let Γ be the dual graph of its minimal resolution. The index $\Delta(\Gamma)$ is the absolute value of the determinant of the intersection matrix of Γ. ◀

Lemma A.0.5. *Let $p \in S$ be a klt singularity. Then every curve of the dual graph of its minimal resolution is a smooth rational curve. If p is a non chain singularity, then p has a fork with three chain branches. The indexes $(\Delta_1, \Delta_2, \Delta_3)$ of the branches are one of the following: $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$. We will refer to the fork vertex as the central curve of p.*

There is an analogous description for lc singularities, for which we refer to [Kol97, Chapter 3, page 58].

Lemma A.0.6. *Let $p \in S$ and let C be an irreducible and reduced curve through p. If (S, C) is dlt at p then either:*

1. p is smooth and C has simple normal crossings at p or

2. p is singular, in which case it must be a chain singularity, C is smooth at p and touches normally one of the ends of the chain.

Lemma A.0.7. *Let $p \in S$ and let C be an irreducible and reduced curve through p. If (S, C) is lc but not dlt at p then either:*

1. p is smooth and C has a simple node at p or

2. p is a chain singularity and C touches both ends of the singularity normally or

3. p is a chain singularity with exactly three components and C touches the central component normally or

4. p is a non chain singularity, in which case two of the branches have index two, and C touches the opposite end of the third branch normally.

Once we know the dual graph of the minimal resolution over a klt singularity p and all the self-intersections E_i^2, we can compute the coefficients of the E_i. In fact it's sufficient to solve the n equations $(K_{\tilde{S}} + \sum_i e_i E_i) \cdot E_i = 0$ by using adjunction. The following lemmas can be useful in speeding up computations, especially when p is a non chain singularity.

Lemma A.0.8. *Let Γ be the dual graph of the minimal resolution at p and let v be a*

vertex of Γ. Let $v_1, \cdots, v_s$ be the vertices adjacent to v. Then

$$\Gamma(\Delta) = n \cdot \Delta(\Gamma - v) - \sum_i \Delta(\Gamma - v - v_i)$$

Lemma A.0.9. *The coefficients of the exceptional divisors E_j over p are given by*

$$1 - e_i = \frac{1}{\Delta(\Gamma)} \sum_k \Delta(\Gamma - (path\ from\ v_j\ to\ v_k)) \cdot c_k$$

where c_k is given by $c_k = 2 - E_k(\sum_{l \neq k} E_l)$. It is useful to notice that $c_k = 0$ if and only if v_k has exactly two neighbors, $c_k = 1$ if and only if it has one neighbor and $c_k = -1$ if and only if it has three neighbors.

Definition A.0.10. Let p be a klt cyclic singularity $(-E_1^2, \cdots, -E_n^2)$. We say that p has spectral value k if the coefficient of E_1 has the form k/r where r is the index of the singularity at p. ◀

Lemma A.0.11. *If $\beta = (j, \alpha)$ then the difference of the spectral value of β and α is $(j-2)r$, where r is the index of α. In particular:*

1. *α has spectral value 0 if and only if $\alpha = A_j$.*

2. *α has spectral value 1 if and only if $\alpha = (A_j, 3)$ or (3).*

3. *α has spectral value 2 if and only if $\alpha = (4)$ or $\alpha = (3, 2)$.*

Definition A.0.12. We say that a point $p \in S$ of spectral value 1 is almost Du Val. ◀

While the notion of spectral value is important for the "hunt", the main relevance in the present discussion comes from the following lemmas.

Lemma A.0.13. *Suppose that $p \in S$ is a klt chain singularity and that D is an irreducible divisor through p, touching E_1 normally (i.e. (S, D) is dlt at p). Then*

$$e(E_1, K_S + \lambda D) = (k/r) + \lambda(r - 1 - k)/r = \lambda(r - 1)/r + (1 - \lambda)(k/r)$$

Lemma A.0.14. *Let p be a klt non chain singularity with branches $\beta_1, \beta_2, \beta_3$. Assume that the central curve is a $(-l)$ curve. Then the coefficient of e of p is the same as the coefficient of the central curve, and it is a rational number of the form $k/(k+1)$ for some positive integer k.*

Moreover, if $\beta_1 = \beta_2 = (2)$ and $\alpha = (l, \beta_3)$ then k is the spectral value of α.

For the convenience of the reader, we list below all klt singularities with small coefficient. This can be done with a straightforward computation and we refer to [KM99, Proposition 10.1].

Proposition A.0.15. *Let $p \in S$ be a klt singularity with coefficient $0 < e < 3/5$. Then the possibilities for p are as follows.*

1. *$e < 1/2$: $(3, A_j)$. $e = (j + 1)/(2j + 1)$.*

2. *$e = 1/2$:*

   (a) *(4)*

(b) $(3, A_j, 3)$

(c) $(2, 3, 2)$

(d) p is a non chain singularity, with center (2) and branches (2), (2) and $(A_j, 3)$, with the central curve and the unique (-3) curve meeting opposite ends of the A_j chain. This is the only non chain singularity with $e < 2/3$.

3. *$1/2 < e < 3/5$:*

   (a) $(2, 3, A_j)$ *with* $2 \leqslant j \leqslant 4$. *$e = (2j + 2)/(3j_5)$.*

   (b) $(4, 2)$. *$e = 4/7$.*

A.1   Adjunction

A recurring topic in our discussion is the fact that "negativity" of the canonical divisor should control the behavior of singularities. In the following we collect some useful lemmas that deal with the case in which $K_S + \Delta$ is non-positive and Δ is an integral divisor. We start first by recalling the results in [KM99, Appendix L]

Lemma A.1.1. *Let $C \subseteq S$ be a reduced irreducible curve on a $\mathbb{Q}$-factorial projective surface. If $(K_S + C) \cdot C < 0$ then C is rational.*

Proof. See [KM99, Lemma L.2.2] $\square$

Lemma A.1.2. *Let S be a $\mathbb{Q}$-factorial projective surface, and C, $D \subseteq S$ two curves, with C integral. If $(K_S + C + D) \cdot C \leqslant 0$, and $C \cap D$ contains at least two points, then*

K_S+C+D is log canonical, $C \cap D$ consists exactly of two points, and $(K_S+C+D) \cdot C = 0$.

We will also need the following stronger results.

Lemma A.1.3. *Suppose that S is a surface with only klt singularities and that C is a curve such that $(K_S + C) \cdot C \leqslant 0$. Then the dual graph of the singularities on C and the curve C is the dual graph of a log canonical singularity.*

Proof. If C has any singularity in the smooth locus of S there is nothing to prove. Suppose therefore that all singularities of C are contained in $\mathrm{Sing}(S)$. We can furthermore assume that C is not contained in the smooth locus of S. Therefore C is rational curve. If K_S+C is log canonical the statement follows from the classification of surface log canonical singularities (see, for example [Kol97, Chapter 3]). Suppose then that $K_S + C$ is not log canonical and consider its log canonical threshold $e < 1$. Let $\tilde{S} \to S$ be the minimal resolution of S, with exceptional divisors E_i. We let $a_i = \tilde{C} \cdot E_i$ and we define the numbers b_i by $f^*(K_S + C) = K_T + \tilde{C} + \sum_i b_i E_i$. Notice that the numbers b_i are solutions to the equations $f^*(K_S + C) \cdot E_i = 0$, and they are therefore determined by the numbers a_i and by the self-intersections E_i^2. The hypothesis of the lemma is also determined by the same data since $(K_S + C) \cdot C \leqslant 0$ if and only if $-2 + \sum_i a_i b_i \leqslant 0$. Therefore the datum of $\tilde{C}^2$ does not influence neither the hypothesis of the lemma, nor the conclusion. Define now the numbers d_i by $f^*(C) = \tilde{C} + \sum_i d_i E_i$. The coefficients e_i of the divisors E_i are given by $e_i = b_i - d_i$. Clearly, $0 \leqslant e_i < 1$. The datum of $C^2 = \tilde{C}^2 + \sum_i a_i d_i E_i^2$ is determined by the numbers a_i, e_i, E_i^2 and $\tilde{C}^2$. Since we have the freedom of changing $\tilde{C}^2$ as we want

while keeping a_i, e_i and E_i^2 constant, we also have the freedom of changing the value of C^2 in order to prove the lemma.

In particular, we can declare C^2 to be such that $(K_S + eC) \cdot C = 0$. From this and the fact that $(K_S + C) \cdot C \leqslant 0$, we get that $C^2 < 0$. Define $D = f^*(C)$ and consider the intersection matrix of the divisors E_i and D. It is clearly negative definite on the space generated by the E_i. Also, D is orthogonal to that space. We may therefore conclude that the whole intersection matrix is negative definite since $D^2 = C^2 < 0$. It follows that the intersection matrix of E_i and $\tilde{C}$ is also negative definite since D is obtained by applying the Gram-Schmidt process to $\tilde{C}$. Now we can conclude by using [Kol97, Lemma 2.19.3] and the argument in [Kol97, Chapter 3]. □

Lemma A.1.4. *Let S be a log del Pezzo surface and C an irreducible curve in S. Suppose that (S, C) is log canonical but not divisorially log terminal.*

1. *If $K_S + C$ is anti-ample, then C contains at most two singular points. If furthermore C contains exactly two singular points, then $K_X + C$ is dlt at one of them.*

2. *If $K_S + C$ is numerically trivial, C contains at most three singular points. If furthermore C contains exactly three singular points, then two of them are A_1 points and (S, C) is dlt along them, and (S, C) is not dlt at the third point.*

3. *If $K_S + C$ is anti-nef and C contains at least two singular points, then C is smooth.*

Proof. Let start by proving (1). Suppose by contradiction that C contains at least three singular points p, q, r. We may assume that $K_S + C$ is not dlt at p. By inversion of

adjunction, the coefficient of p in $\mathrm{Diff}_C(0)$ is at least one. Since the coefficient of any singular point is at least $1/2$, however, we have that $(K_S + C) \cdot C = \deg(\mathrm{Diff}_C(0)) \geqslant 0$, contradiction. Therefore C contains at most two singular points. Suppose now that C contains exactly two singular points. If (S, C) is not dlt at either point, then both of their coefficients in $\mathrm{Diff}_C(0)$ are one. This implies again that $(K_S + C) \cdot C = \deg(\mathrm{Diff}_C(0)) = 0$, contradiction.

This proves (1). The same type of reasoning gives (2). For (3) notice that if C is not smooth then by adjunction $(K_S + C) \cdot C \geqslant 0$, with equality exactly when C has one node and contains no singularities of S outside the node. Equivalently, one may just use Lemma A.1.3. $\qquad\square$

Appendix B

Appendix: Gorenstein log del Pezzo surfaces

Recall that a normal projective variety X is called Gorenstein if the canonical divisor K_X is Cartier. In this appendix we classify Gorenstein log del Pezzo surfaces of Picard number one over algebraically closed fields of characteristic different from two and three. Their singularities were first classified over the complex numbers by Furushima [Fur86]. The analogous result in positive characteristic can easily be derived from existing papers, but we were unable to find it explicitly stated in the literature. In the following treatment, we take the approach of Ye [Ye02], who reduces the study of Gorenstein log del Pezzo surfaces to the study of extremal rational elliptic surfaces. These were classified over the complex numbers by Miranda and Persson in [MP86], and over algebraically closed fields of positive characteristic by Lang in [Lan91] and [Lan94].

B.1 Extremal rational elliptic surfaces

Definition B.1.1. An elliptic surface is a smooth relatively minimal surface X over a curve C, such that the general fiber is a smooth curve of genus one. ◀

Let X be a smooth projective variety and let $f : X \to C$ be a fibration such that the general fiber is irreducible. Over the complex numbers we can deduce that the general fiber is in fact smooth by generic smoothness. In positive characteristic, however, generic smoothness no longer holds and some care is needed. The following lemma, largely taken from [Mir89], shows that we recover smoothness of the general fiber in a special case.

Lemma B.1.2. *Let $f : X \to \mathbb{P}^1$ be a smooth relatively minimal surface with section such the general fiber is irreducible and of arithmetic genus one. Suppose also that X is rational, the image of the section is a (-1) curve and $\mathrm{char}(k) \neq 2, 3$. Then $f : X \to \mathbb{P}^1$ is obtained by resolving the rational map induced by a pencil of cubic curves in $\mathbb{P}^2$ whose general member is smooth. In particular X is an elliptic surface.*

Proof. For the general fiber F we have that $K_X \cdot F = 0$ by adjunction, hence $K_X \equiv nF$ for some integer n. Consider the image S of the section of f, which is a (-1) curve by assumption. Again by adjunction we get that $K_X \cdot S = -1$, hence $K_X \equiv -F$. It follows that $-K_X$ is nef and that every rational curve has self intersection at least -2. Let $g : X \to M$ be a blowdown to a Mori fiber space M. From the above considerations, M can only be $\mathbb{F}_0, \mathbb{F}_2$ or $\mathbb{P}^2$. In each of these cases X dominates $\mathbb{P}^2$ and therefore we get a contraction $h : X \to \mathbb{P}^2$. Pushing forward $|F|$ in $\mathbb{P}^2$ we obtain a pencil of curves that are

numerically equivalent to the push forward elements in $|-K_X|$. It follows then that $f_*|F|$ is a pencil in $|-K_{\mathbb{P}^2}|$, and it is therefore a pencil of cubics. Finally note that the general member of a pencil of cubics in characteristic different from two or three is smooth by [Mir89, Lemma I.5.2] and the comment after it. $\qquad\square$

Definition B.1.3. Let $f : X \to C$ be an elliptic surface with section σ. The section σ naturally gives all the fibers the structure of elliptic curves. The set of all sections of f is then a group, where the multiplication is done fiber by fiber and σ is the identity element. This group is called the Mordell-Weil group of X and is denoted by $\mathrm{MW}(X)$. $\qquad\blacktriangleleft$

Definition B.1.4. Let $f : X \to C$ be a rational elliptic surface with section σ. We say that X is extremal if the Mordell-Weil group of X is finite and $\mathrm{NS}(X)_{\mathbb{Q}}$ is generated by the classes of $\sigma(C)$ and of the vertical components. $\qquad\blacktriangleleft$

Theorem B.1.5. *The classification of the singular fibers of extremal rational elliptic surfaces over algebraically closed fields with* $\mathrm{char}(k) \neq 2, 3, 5$ *is the same as over* $\mathbb{C}$*. The configurations are listed in the following table using Kodaira's notation:*

Singular fibers	$|\operatorname{MW}(X)|$	Singular fibers	$|\operatorname{MW}(X)|$
II, II^*	1	I_1^*, I_4, I_1	2
III, III^*	2	I_2^*, I_2, I_2	4
IV, IV^*	3	I_9, I_1, I_1, I_1	3
I_0^*, I_0^*	4	I_8, I_2, I_1, I_1	4
II^*, I_1, I_1	1	I_6, I_3, I_2, I_1	6
III^*, I_2, I_1	2	I_5, I_5, I_1, I_1	5
IV^*, I_3, I_1	3	I_4, I_4, I_2, I_2	8
I_4^*, I_1, I_1	2	I_3, I_3, I_3, I_3	9

If $\operatorname{char}(k) = 5$ the classification is the same, except that in the above table one replaces I_5, I_5, I_1, I_1 with I_5, I_5, II. Furthermore, if $\operatorname{char}(k) \neq 2, 3$, for every configuration of possible singular fibers in the above table there is a unique extremal rational elliptic surface with section with that configuration of singular fibers, except for the case (I_0^*, I_0^*). In this case there are infinitely many extremal rational elliptic surfaces with that configuration of fibers.

Proof. See [Lan91, Theorem 2.1], [Lan94, Theorem 4.1], [MP86, Theorem 4.1] and [Mir89, Theorem VIII.1.5]. □

B.2 Reduction to elliptic surfaces

We recall here the relation between Gorenstein log del Pezzo surfaces and extremal rational elliptic surfaces. Throughout this discussion we assume $\text{char}(k) \neq 2, 3$. Let V be a Gorenstein log del Pezzo surface of rank one such that $K_V^2 = 1$. Let U be its minimal resolution, with exceptional locus D, consisting of eight (-2) curves by Lemma 1.3.24. The general member of the pencil $|-K_U|$ is reduced and irreducible by [Dem80, Chapter III, Théorème 1, (b')] (see the remark in the introduction to Chapter IV), and of arithmetic genus one by adjunction. This pencil has only one base point p by [Dem80, Chapter III, Proposition 2]. After blowing it up, we are in the hypothesis of Lemma B.1.2, where we may take the exceptional curve of the blow up as a section. Therefore $X = \text{Bl}_p(U)$ is an elliptic surface. Notice that p in not contained in the support of D because $K_U \cdot D = 0$. Hence the strict transform of D in X is contained in the union of the singular fibers and does not meet the section. By Lemma 1.3.24 we have that $\rho(U) = 10 - K_U^2 = 9$, and therefore $\rho(X) = 10$. By the Shioda-Tate formula ([SS10, Corollary 6.13]), however, we have that $\rho(X) = \sharp D + \text{rank}(\text{MW}(X)) + 2$, where $\sharp D$ indicates the number of irreducible components of the support of D. It follows then that the Mordell-Weil group of X is finite and therefore X is an extremal rational elliptic surface.

From this discussion we see that we may obtain every rank one Gorenstein log del Pezzo surface V with $K_V^2 = 1$ by starting with an extremal rational elliptic surface of Picard number ten, selecting a section, contracting all the (-2) curves not meeting the section and then blowing down the section. If $K_V^2 = 8$ then V is obviously isomorphic to

$\overline{\mathbb{F}}_2$.

Suppose now that V is a Gorenstein log del Pezzo surface such that $2 \leqslant K_V^2 \leqslant 7$ and let U be its minimal resolution. Let C be any (-1) curve on U (such a curve exists because the Picard number of U is at least three). Every irreducible divisor A in the linear system $|-K_U|$ meets C exactly once, and by taking A to be general, we may assume that there are no other (-1) curves passing through $p = A \cap C$. Blowing up p and then contracting all the (-2) curves, we get a rank one log del Pezzo surface V' such that $K_{V'}^2 = K_V^2 - 1$. By repeating this process we may reduce our analysis to the case when $K_V^2 = 1$, which is the content of Lemma [Ye02, Lemma 3.2].

We summarize our conclusions in the following result:

Theorem B.2.1. *Let V be a rank one Gorenstein log del Pezzo surface with $K_V^2 \leqslant 7$ and let $U \to V$ be the minimal resolution. If $\mathrm{char}(k) \neq 2, 3$, there exists an extremal rational elliptic surface Y and a morphism $f : Y \to U$ that is a composition of blow downs of some (-1)-curves.*

Proof. Immediate from the previous description. For more detail see [Ye02, Theorem 3.4]. $\square$

Thanks to Theorem B.2.1 and Theorem B.1.5, one can now classify all Gorenstein log del Pezzo surfaces. In fact, the only (-1) curves in an extremal rational elliptic surface W are the members of the Mordell-Weil group. In order to get the minimal resolution of any Gorenstein log del Pezzo surface it is therefore enough to consider an extremal

rational elliptic surface X, contract a section, and then keep contracting the (-1) curves that are formed in the process along the singular fibers. The result is the following:

Theorem B.2.2. *Let V be a rank one Gorenstein log del Pezzo surface over an algebraically closed field such that $\text{char}(k) \neq 2, 3$. The singularity types on V are listed in the following table. Furthermore, V is uniquely determined by its singularities, with the exception of the cases $E_8, A_1 + E_7, A_2 + E_6$, which have two classes of isomorphism each, and the case $2D_4$, which has infinitely many classes of isomorphism.*

A_1	$A_1 + A_2$	A_4	$2A_1 + A_3$	D_5	$A_1 + A_5$
$3A_2$	E_6	$3A_1 + D_4$	A_7	$A_1 + D_6$	E_7
$A_1 + 2A_3$	$A_2 + A_5$	D_8	$2A_1 + D_6$	E_8	$A_1 + E_7$
$A_1 + A_7$	$2A_4$	A_8	$A_1 + A_2 + A_5$	$A_2 + E_6$	$A_3 + D_5$
$4A_2$	$2A_1 + 2A_3$	$2D_4$			

Proof. The same proof as in [Ye02, Theorem 1.2] applies, using Theorem B.1.5. $\qquad\square$

Notation. We will denote any Gorenstien log del Pezzo surface by the corresponding singularity type. For example, $S(A_1)$ denotes the unique Gorenstein log del Pezzo surface with only one singularity, of type A_1. Obviously, $S(A_1)$ is isomorphic to $\overline{\overline{\mathbb{F}}}_2$.

Lemma B.2.3. *Let $\text{char}(k) \neq 2, 3$ and let S be a rank one Gorenstein log del Pezzo surface. Suppose that $\tilde{S}$ is obtained by contracting (-1) curves of the extremal rational elliptic surface X as in Theorem B.2.1. Then the (-1) curves of $\tilde{S}$ are either images of elements of $\text{MW}(X)$ or images of the rational curves contained in the fibers of X.*

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Proof. Obvious by Theorem B.2.1 and the discussion after it. □

Lemma B.2.4. *Let* $\mathrm{char}(k) \neq 2, 3$ *and let* S *be a rank one log del Pezzo surface. Suppose that there is a rational cuspidal curve* C *in the smooth locus of* S. *Then* S *is Gorenstein. Furthermore:*

1. *if* $\mathrm{char}(k) \neq 5$, S *is obtained by blowing down* (-1) *curves on the extremal rational elliptic surface with singular fibers* II, II^* *and is one of the following:* $S(E_8)$, $S(E_7)$, $S(E_6)$, $S(D_5)$, $S(A_4)$ *or* $S(A_1 + A_2)$. *On* S *there is one and only one* (-1) *curve.*

2. *if* $\mathrm{char}(k) = 5$, *there is one more case:* S *is obtained by blowing down one of the sections of the extremal rational elliptic surface with singular fibers* I_5, I_5, II. *In this case* $S = S(2A_4)$, *which is fully described in Example B.3.1.*

Proof. Notice that $C \equiv -K_S$ by adjunction, so S is Gorenstein. Every rank one Gorenstein log Del Pezzo surface is obtained by blowing down (-1) curves in an extremal rational elliptic surface X by Theorem B.2.1. Consider the strict transform $\tilde{C}$ of C in X and let F be a general fiber. If $\tilde{C} \cdot F > 1$ then by looking at the description of the contractions we see that there is a (-1) curve D in S such that $C \cdot D > 1$, contradicting the fact that $K_S \cdot D = -1$. If $\tilde{C} \cdot F = 1$ then $\tilde{C}$ is a section of the fibration $X \to \mathbb{P}^1$, contradicting the fact that $\tilde{C}$ is cuspidal. Therefore we see that $\tilde{C}$ is a fiber of X. This concludes the proof by Theorem B.1.5 and Lemma B.2.3. □

Lemma B.2.5. *Let* $\mathrm{char}(k) \neq 2, 3$ *and let* S *be a rank one log del Pezzo surface. Suppose that:*

1. *There is a rational nodal curve A in its smooth locus*

2. *There are two rational curves C and D such that $K_S \cdot C = K_S \cdot D = -1$*

3. *$K_S^2 \geqslant 4$.*

Then S is Gorenstein and $S = S(2A_1 + A_3)$. Furthermore, $C \cap D \cap A = \emptyset$, C and D each pass through one of the A_1 points and meet at opposite ends of the A_3 point. Finally, C and D are the only two (-1) curves on S.

Proof. We have again that $A \equiv -K_S$ by adjunction, so S is Gorenstein. Notice that $(K_S + C + D) \cdot A \leqslant -2$, hence C and D are smooth by adjunction. Let X be an extremal rational elliptic surface that dominates $\tilde{S}$ as in Theorem B.2.1. By Lemma B.2.3 we have that $|\operatorname{MW}(X)| \geqslant 2$. Also, by the same reasoning as in Lemma B.2.4, we have that the strict transform of A in X is a fiber. By looking at the classification in Theorem B.1.5, we see that starting with an extremal rational elliptic surface X with non trivial Mordell-Weil group and a fiber of type I_1, the only Gorenstein log del Pezzo surface with $K_S^2 \geqslant 4$ that we obtain is $S(2A_1 + A_3)$. This case is achieved by taking X to be the extremal rational elliptic surface with singular fibers III^*, I_2, I_1. Now one can get the result by either looking at the description of the elliptic surface, or by following the argument in [KM99, Lemma 3.9.2]. $\qquad\square$

B.3 Special examples

Here we study more in detail some specific Gorenstein log del Pezzo surfaces that turn out to be especially important in the present paper.

Example B.3.1. Let $\mathrm{char}(k) = 5$ and consider $S = S(2A_4)$. This surface is obtained by taking W to be the unique extremal rational elliptic surface with singular fibers of type I_5, I_5, II, contracting one of the five sections and the contracting all the (-2) curves. It is clear from this description that there is a unique rational cuspidal curve D in its smooth locus. We would like to present in the following a slightly more explicit approach, and explicitly carry on the computations.

Consider the following four points in $\mathbb{P}^2_k$: $a = [-1, 1, 1]$, $b = [-1, -1, 1]$, $c = [1, -1, 1]$ and $d = [1, 1, 1]$. Let L_{ab}, L_{ac}, L_{ad}, L_{bc}, L_{bd} and L_{cd} be the six lines between them. Consider the conics $C_0 = L_{ab} + L_{ac} + L_{cd}$ and $C_\infty = L_{ad} + L_{bd} + L_{bc}$. The equation of C_0 is $(Y^2 - Z^2)(X + Y) = 0$ and the equation of C_∞ is $(X^2 - Z^2)(Y - X) = 0$. Consider the pencil of cubics C_t generated by C_0 and C_∞, namely the curves C_t whose equations are

$$(Y^2 - Z^2)(X + Y) + t(X^2 - Z^2)(Y - X) = 0$$

Clearly the case locus of this pencil is given by the points a, b, c, d counted with multiplicity two, and by the point $f = [0, 0, 1]$, counted with multiplicity one. We claim that the curve C_2 is a rational cuspidal curve, with cusp at $[1, 3, 0]$. First, it's immediate

to check that C_2 passes through $[1, 3, 0]$. Now we study C_2 in the chart $X \neq 1$, with local coordinates $y = Y/X$ and $z = Z/X$. Its equation is

$$(y^2 - z^2)(y + 1) + 2(1 - z^2)(y - 1) = 0$$

The Jacobian and the Hessian of C_2 in these coordinates are

$$J = (3y^2 + 2z^2 + 2y + 2, -yz + 2z)$$

and

$$H = \begin{bmatrix} y + 2 & 4z \\ 4z & -y + 2 \end{bmatrix}$$

respectively. Evaluating J and H in $(3, 0)$ shows that C_2 has a cusp.

Now we resolve the base locus of the pencil at a, b, c, d. Since the multiplicity at these points is two, we need to blow ups at each point. We denote by E_1^a the first blow up over a, and we use the analogous notation for the other points. We also slightly abuse notation by denoting the strict transforms of the lines and of the cubics considered so far with their original names. Consider the blow up $\tilde{S} \to \mathbb{P}^2$ on $E_1^a \cap L_{ad}$, $E_1^b \cap L_{ab}$, $E_1^c \cap L_{bc}$ and $E_1^d \cap L_{cd}$ and call E_2^a, E_2^b, E_2^c and E_2^d the respective exceptional divisors. Notice that the curves E_1^a, L_{ab}, L_{cd} and E_1^c form an ordered chain of four (-2) curves. Similarly the curves E_1^d, L_{ad}, L_{bc} and E_1^b form another ordered chain of (-2) curves. Our

surface S is therefore obtained by contracting these two chains to two points, p and q respectively. The strict transform of C_2 in S is contained in the smooth locus and is a member of $|-K_S|$ by adjunction. There strict transforms of L_{ac} and L_{bd} are the only two (-1) curves in $|-K_S|$. There are exactly four other (-1) curves on S by Lemma B.2.3: the images of E_2^a, E_2^b, E_2^c and E_2^d.

Next we study the geometry of these curves with respect to the singular points on S. The curve L_{ac} is nodal, with a node in p. The two branches of the node touch the opposite curves of the A_4 chain singularity at p. Similarly L_{bd} is nodal with a node at q. Notice that $L_{ac} \cap L_{bd}$ is the only point in the base locus of $|-K_S|$, which is the image of $[0, 0, 1]$. For the remaining (-1) curves we notice that they are smooth, pass through both p and q, are dlt at one but not lc at the other. Furthermore, given any two such curves, p and q are the only points where they intersect.

Example B.3.2. Let $\operatorname{char}(k) \neq 2, 3, 5$ and $S = S(2A_4)$. Most of the analysis in Example B.3.1 carries through. The only difference is that S contains exactly two singular rational curves in its smooth locus, both with nodal singularities.

Example B.3.3. Let $\operatorname{char}(k) \neq 2, 3$ and $S = S(A_1 + A_2)$. Then S is obtained as follows. Consider the flex cubic C given by $XZ^2 = Y^3$ in $\mathbb{P}_k^2$ and the line L given by $Y = 0$. Blow up three times over the intersection $L \cap C$ to separate them, then contract the three (-2) curves obtained in the process. Notice that the strict transform of C in S is a cuspidal rational curve contained in the smooth locus. The image of the last exceptional curve created by blowing up at $L \cap C$ is a (-1) curve. This is also the unique (-1) curve, for

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example by Lemma B.2.4.

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