UNIVERSITY OF CALIFORNIA
RIVERSIDE

Koszulity of Directed Graded k-linear Categories and Their Quadratic Dual

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Jordan Christopher Tousignant

June 2018

Dissertation Committee:
Dr. Wee Liang Gan, Chairperson
Dr. Vyjayanthi Chari
Dr. Jacob Greenstein

The Dissertation of Jordan Christopher Tousignant is approved:

Committee Chairperson

University of California, Riverside

ABSTRACT OF THE DISSERTATION

Koszulity of Directed Graded k-linear Categories
and Their Quadratic Dual

by

Jordan Christopher Tousignant

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2018
Dr. Wee Liang Gan, Chairperson

We define a family of categories $\mathcal{FI}^n_{t,A}$ related to the category $\mathcal{FI}$ of finite sets and injective functions. We show that the k-linearizations of these categories are Koszul, where k is a field of characteristic 0, using the language of directed graded k-linear categories. We also describe their quadratic dual categories in special cases.

Table of Contents

1. Introduction

1.1. Motivation.

Before we introduce examples of the categories under consideration, we need to introduce some notation from [SS].

Let

$$GL = GL(\infty) = \lim_{\rightarrow} GL(n) = \coprod_{n \geq 1} GL(n)/\sim$$

be the direct limit of the groups $GL(n)$ of invertible $n \times n$ invertible matrices under upper left corner inclusion. Let $V = \mathbb{C}^\infty$ be a countable dimensional complex vector space. Then GL acts on V by left multiplication. Let $GA = GA(\infty)$ be the subgroup of GL stabilizing a nonzero linear map $t : V \to \mathbb{C}$ which annihilates all but finitely many basis vectors. Let $O = O(\infty)$ be the subgroup of GL stabilizing a nondegenerate symmetric bilinear form $\omega : V \times V \to \mathbb{C}$ such that e_i is orthogonal to all but finitely many e_j. Given a category $\mathcal{C}$, let $\mathrm{Mod}^{\mathrm{f}}_{\mathcal{C}}$ be the category of functors of finite length from $\mathcal{C}$ to the category of complex vector spaces.

Example 1.1.1. Let $\mathcal{FI}$ be the category of finite sets and injective functions. S. Sam and A. Snowden call this the "upwards subset" category and denote it by (us); the opposite category is called (ds) for "downwards subset". They define a category $\mathrm{Rep}^{\mathrm{pol}}(GA)$ of polynomial representations of GA, i.e. those appearing as a subquotient of a finite direct sum of various tensor powers of V. They describe a functor $K : (\mathrm{ds}) \to \mathrm{Rep}^{\mathrm{pol}}(GA)$, which gives rise to an equivalence of categories $\mathrm{Mod}^{\mathrm{f}}_{(\mathrm{us})} \to \mathrm{Rep}^{\mathrm{pol}}(GA)$ [SS, Corollary 5.2.4].

Example 1.1.2. Let $\mathcal{FIM}$ be the category of finite sets and injective functions that are equipped with a perfect matching on the complement of the image. S. Sam and A. Snowden call this the "upwards Brauer" category and denote it by (ub); the opposite category is called (db) for "downwards Brauer". They define a category $\mathrm{Rep}(O)$ of algebraic representations of O, i.e. those appearing as a subquotient of a finite direct sum of various tensor powers of V. They describe a functor $K : (\mathrm{db}) \to \mathrm{Rep}(O)$, which gives rise to an equivalence of categories $\mathrm{Mod}^{\mathrm{f}}_{(\mathrm{ub})} \to \mathrm{Rep}(O)$ [SS, Corollary 4.2.7].

Example 1.1.3. Let $\mathcal{FIM}_w$ be the category whose objects are pairs of finite sets and whose morphisms are injective functions that are equipped with a perfect bipartite matching on the complement of the image. S. Sam and A. Snowden call this the "upwards walled Brauer" category and denote it by (uwb); the opposite category is called (dwb) for "downwards walled Brauer". They define a category $\mathrm{Rep}(GL)$ of algebraic representations of GL, i.e. those appearing as a subquotient of a finite direct sum of various tensor powers of V and its restricted dual V_*. They describe a functor $K : (\mathrm{dwb}) \to \mathrm{Rep}(GL)$, which gives rise to an equivalence of categories $\mathrm{Mod}^{\mathrm{f}}_{(\mathrm{uwb})} \to \mathrm{Rep}(GL)$ [SS, Corollary 3.2.12].

Example 1.1.4. Let $\mathcal{FIM}'_w$ be the category whose objects are pairs of finite sets and whose morphisms are injective functions that are equipped with a perfect bipartite matching on the complement of the image, such that each element of the domain and of the matching is either "marked" or "unmarked". The opposite category $(\mathcal{FIM}'_w)^{op}$ is used by D. Grantcharov and V. Serganova to diagrammatically describe $\mathrm{Hom}_{\mathfrak{q}(\infty)}(T^{p,q}, T^{r,s})$ [GS, Section 5], where $\mathfrak{q}(\infty)$ is a certain Lie superalgebra of linear operators in $\mathrm{End}(V) \oplus \mathrm{End}(W)$, with V and W being countable dimensional complex super vector spaces equipped with a certain bilinear form $W \times V \to \mathbb{C}$, and $T^{m,n} = V^{\otimes m} \otimes W^{\otimes n}$.

Thus, each of these four "diagram" categories are used to describe morphisms between tensor powers of certain countable dimensional representations of GA, O, GL, and $\mathfrak{q}(\infty)$, over $\mathbb{C}$.

E. Dan-Cohen, I. Penkov, and V. Serganova proved that the $\mathbb{C}$-linearization of $\mathcal{FIM}$ is Koszul using the language of tensor representations of the infinite dimensional orthogonal Lie algebra $\mathfrak{o}(\infty)$ [DPS, Theorem 5.5].

W. L. Gan and L. Li proved that the k-linearizations of $\mathcal{FI}$ and several other related categories are Koszul, when k is a field of characteristic 0 [GL, Corollary 5.12]. They use the framework of Koszul theory for directed graded k-linear categories, which we will utilize to establish our results here.

In this paper, we shall define a 3-parameter family of categories $\mathcal{FI}^n_{t,A}$, which include the above four categories as special cases. We give a direct proof that the k-linearization of $\mathcal{FI}^n_{t,A}$ is Koszul when k is a field of characteristic 0. We also describe the quadratic dual category of the k-linearization of $\mathcal{FI}^n_{t,A}$ under a certain restriction on the parameter t.

1.2. Notation and conventions.

Let $\mathbb{N}$ be the set of positive integers, and $\mathbb{N}_0$ be the set of non-negative integers. For any $n \in \mathbb{N}_0$, let $[n] = \{1, ..., n\}$; in particular, $[0] = \varnothing$. We use $\amalg$ to denote the disjoint union of sets, and $\sqcup$ to indicate the union of sets that happen to be disjoint. We write $\subset$ to signify $\subsetneq$. For us, k will always denote a field. For any k-vector space V, we write V^* for its dual space $\mathrm{Hom}_k(V, k)$. For any finite set $S = \{s_1, ..., s_m\}$, we write kS for the m-dimensional k-vector space with basis S (if $S = \{s\}$ has only one element, we will write ks instead of $k\{s\}$). Also, we denote by $\det(S)$ the 1 dimensional k-vector space $\bigwedge^m kS$; in particular, $\det(\varnothing) = k$. We denote by k-Mod the category of k-vector spaces, and by k-gMod the category of $\mathbb{N}_0$-graded k-vector spaces whose morphisms are homogeneous of some fixed degree. By a category $\mathcal{C}$ we mean a small category. We write $X \in \mathcal{C}$ to mean $X \in \mathrm{Ob}(\mathcal{C})$. Given $X, Y \in \mathcal{C}$, we write $\mathcal{C}(X, Y)$ for the set of morphisms in $\mathcal{C}$ from X to Y. The composite of two morphisms $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ is written as $gf \in \mathcal{C}(X, Z)$. We denote by 1_X the identity morphism of $X \in \mathcal{C}$.

2. Preliminaries

2.1. Directed graded k-linear categories.

Given a category $\mathcal{C}$ and a field k, the *k-linearization* of $\mathcal{C}$ is the category $\underline{\mathcal{C}}$ having $\mathrm{Ob}(\underline{\mathcal{C}}) = \mathrm{Ob}(\mathcal{C})$ and $\underline{\mathcal{C}}(X, Y) = k\mathcal{C}(X, Y)$ for any $X, Y \in \mathcal{C}$. A *k-linear category* is a category $\mathcal{C}$ enriched over k-Mod. Thus, the k-linearization $\underline{\mathcal{C}}$ of a category $\mathcal{C}$ is a k-linear category. We shall denote k-linear categories by $\underline{\mathcal{C}}$ rather than $\mathcal{C}$. A *graded k-linear category* is a category $\underline{\mathcal{C}}$ enriched over k-gMod. In particular, $\underline{\mathcal{C}}(X, Y) = \bigoplus_{i \geq 0} \underline{\mathcal{C}}(X, Y)_i$ for any $X, Y \in \underline{\mathcal{C}}$. We shall refer to $f \in \underline{\mathcal{C}}(X, Y)_i$ as a morphism of degree i. By letting $\underline{\mathcal{C}}_i = \bigoplus_{X, Y \in \mathcal{C}} \underline{\mathcal{C}}(X, Y)_i$ for each $i \geq 0$, we get a graded k-algebra $\bigoplus_{i \geq 0} \underline{\mathcal{C}}_i$ whose multiplication is given by composition of morphisms. A k-linear category $\underline{\mathcal{C}}$ is *directed* if there is a partial order $\leq$ on $\mathrm{Ob}(\underline{\mathcal{C}})$ such that whenever $\underline{\mathcal{C}}(X, Y) \neq 0$, we have $X \leq Y$. A full subcategory $\underline{\mathcal{D}}$ of a directed k-linear category $\underline{\mathcal{C}}$ is *convex* if for any $X, Y, Z \in \underline{\mathcal{C}}$ satisfying $X \leq Y \leq Z$, we have $Y \in \underline{\mathcal{D}}$ whenever $X, Z \in \underline{\mathcal{D}}$. The *convex hull* of a given set $S \subseteq \mathrm{Ob}(\underline{\mathcal{C}})$ is the smallest convex full subcategory of $\underline{\mathcal{C}}$ containing S.

Definition 2.1.1. We say that $\underline{\mathcal{C}}$ is a *directed graded k-linear category* if $\underline{\mathcal{C}}$ is a graded k-linear category that is directed, and which satisfies the following additional conditions:

(A1) $\underline{\mathcal{C}}(X, Y)$ is finite dimensional as a k-vector space for every $X, Y \in \underline{\mathcal{C}}$;

(A2) $\underline{\mathcal{C}}(X, X)$ is semisimple as a k-algebra for every $X \in \underline{\mathcal{C}}$;

(A3) if $X \neq Y$, then $\underline{\mathcal{C}}(X, Y)_0 = 0$;

(A4) for every $X \in \underline{\mathcal{C}}$ and $i > 0$, we have $\underline{\mathcal{C}}(X, X)_i = 0$;

(A5) for each $X \in \underline{\mathcal{C}}$, there are only finitely many $Y \in \underline{\mathcal{C}}$ such that $\underline{\mathcal{C}}(X, Y)_1 \neq 0$ or $\underline{\mathcal{C}}(Y, X)_1 \neq 0$;

(A6) $\underline{\mathcal{C}}_1 \cdot \underline{\mathcal{C}}_i = \underline{\mathcal{C}}_{i+1}$ for every $i \geq 0$;

(A7) the convex hull of any finite set $S \subseteq \mathrm{Ob}(\underline{\mathcal{C}})$ contains only finitely many objects.

Note 2.1.2. By conditions (A3) and (A4), a directed graded k-linear category $\underline{\mathcal{C}}$ is skeletal. To see this, let $f \in \underline{\mathcal{C}}(X, Y)$ be an isomorphism. Then there exists $f^{-1} \in \underline{\mathcal{C}}(Y, X)$ such that $f^{-1} f = 1_X \in \underline{\mathcal{C}}(X, X)_0$. Since morphisms in $\underline{\mathcal{C}}$ are graded, this forces $f \in \underline{\mathcal{C}}(X, Y)_0$. Hence, $\underline{\mathcal{C}}(X, Y)_0 \neq 0$ implies $X = Y$.

2.2. Graded $\underline{\mathcal{C}}$-modules.

Let $\underline{\mathcal{C}}$ be a k-linear category. A *(left) $\underline{\mathcal{C}}$-module* is a (covariant) k-linear functor $M : \underline{\mathcal{C}} \to k$-Mod. By k-linear we mean that $M(cf + g) = cM(f) + M(g)$ for all $f, g \in \mathcal{C}(X, Y)$ and $c \in k$. Given $\underline{\mathcal{C}}$-modules M, N, a *$\underline{\mathcal{C}}$-module homomorphism* $T : M \to N$ is a natural transformation of functors. We denote by $\underline{\mathcal{C}}$-Mod the category of $\underline{\mathcal{C}}$-modules.

Note 2.2.1. A *right $\underline{\mathcal{C}}$-module* is a contravariant k-linear functor $M : \underline{\mathcal{C}} \to k$-Mod. It is understood that all definitions and results stated for $\underline{\mathcal{C}}$-modules are to hold analogously for right $\underline{\mathcal{C}}$-modules.

Let $\underline{\mathcal{C}}$ be a graded k-linear category. A *graded $\underline{\mathcal{C}}$-module* is a degree-preserving k-linear functor $M : \underline{\mathcal{C}} \to k\text{-gMod}$. In particular, $M(X) = \bigoplus_{i \geq 0} M(X)_i$ for any $X \in \underline{\mathcal{C}}$. By degree-preserving we mean that if $f \in \underline{\mathcal{C}}(X, Y)_j$ is a morphism of degree j, then $M(f)(M(X)_i) \subseteq M(Y)_{i+j}$ for all $i \geq 0$; i.e. $M(f) : \bigoplus_{i \geq 0} M(X)_i \to \bigoplus_{i \geq 0} M(Y)_i$ is homogeneous of degree j. Given graded $\underline{\mathcal{C}}$-modules M, N, a *graded $\underline{\mathcal{C}}$-module homomorphism* $T : M \to N$ is a degree-preserving natural transformation of functors. By degree-preserving we mean that $T_X(M(X)_i) \subseteq N(X)_i$ for all $X \in \underline{\mathcal{C}}$ and $i \geq 0$; i.e. $T_X : \bigoplus_{i \geq 0} M(X)_i \to \bigoplus_{i \geq 0} N(X)_i$ is homogeneous of degree 0. Let $\underline{\mathcal{C}}$-gMod be the category of graded $\underline{\mathcal{C}}$-modules.

Example 2.2.2. Let $\underline{\mathcal{C}}$ be a directed graded k-linear category, and $X \in \underline{\mathcal{C}}$.

(a) The (covariant) Hom functor $\underline{\mathcal{C}}(X, -) : \underline{\mathcal{C}} \to k\text{-gMod}$ is a graded $\underline{\mathcal{C}}$-module. The right $\underline{\mathcal{C}}$-module version of this is the contravariant Hom functor $\underline{\mathcal{C}}(-, X) : \underline{\mathcal{C}} \to k\text{-gMod}$.

(b) $\underline{\mathcal{C}}(X, X) : \underline{\mathcal{C}} \to k\text{-gMod}$ is a graded $\underline{\mathcal{C}}$-module in the following way. We define $\underline{\mathcal{C}}(X, X)$ on objects $Y \in \underline{\mathcal{C}}$ by

$$Y \mapsto \begin{cases} \underline{\mathcal{C}}(X, X) & \text{if } Y = X \\ 0 & \text{if } Y \neq X, \end{cases}$$

and on morphisms $f \in \underline{\mathcal{C}}(Y, Z)$ by $f \mapsto \{g \mapsto fg\}$. The right $\underline{\mathcal{C}}$-module version of this is defined the same way on objects and on morphisms $f \in \underline{\mathcal{C}}(Y, Z)$ by $f \mapsto \{g \mapsto gf\}$.

Note 2.2.3. The categories $k-$Mod and $k-$gMod are abelian, hence so are $\underline{\mathcal{C}}$-Mod and $\underline{\mathcal{C}}$-gMod. When a statement is made about a (graded) $\underline{\mathcal{C}}$-module homomorphism $T : M \to N$, we mean that statement is true for all of its components $T_X : M(X) \to N(X)$. For example, by saying that $T : M \to N$ is injective (resp. surjective) we mean that $T_X : M(X) \to N(X)$ is injective (resp. surjective) for every object X. Also, by saying that a sequence of (graded) $\underline{\mathcal{C}}$-module homomorphisms $L \xrightarrow{S} M \xrightarrow{T} N$ is exact at M we mean that $\text{im}(S_X) = \ker(T_X)$ for every object X.

Proposition 2.2.4. Let $\underline{\mathcal{C}}$ be a directed graded k-linear category, and $X \in \underline{\mathcal{C}}$. Then $\underline{\mathcal{C}}(X, -)$ is a projective object in $\underline{\mathcal{C}}$-gMod.

Proof. Consider a diagram of graded $\underline{\mathcal{C}}$-modules with bottom row exact:

$$\begin{array}{ccccc} & & \underline{\mathcal{C}}(X,-) & & \\ & & \Big\downarrow{\scriptstyle T} & & \\ M & \xrightarrow[S]{} & N & \longrightarrow & 0 \end{array}$$

Evaluation at $X \in \underline{\mathcal{C}}$ gives a diagram of graded k-vector spaces with bottom row exact:

$$\begin{array}{ccccc} & & \underline{\mathcal{C}}(X,X) & & \\ & & \Big\downarrow{\scriptstyle T_X} & & \\ M(X) & \xrightarrow[S_X]{} & N(X) & \longrightarrow & 0 \end{array}$$

Take the identity morphism $1_X \in \underline{\mathcal{C}}(X,X)_0$ and apply T_X to get $T_X(1_X) \in N(X)_0$. Since S_X is surjective, there exists $m_X = \sum_{i\geq 0} m_{X,i} \in \bigoplus_{i\geq 0} M(X)_i = M(X)$ such that $S_X(m_X) = \sum_{i\geq 0} S_X(m_{X,i}) = T_X(1_X)$. Because S_X is degree-preserving, we must have $S_X(m_{X,i}) = 0$ for all $i > 0$, so $S_X(m_{X,0}) = T_X(1_X)$. Define $R : \underline{\mathcal{C}}(X,-) \to M$ as follows: for any $Y \in \underline{\mathcal{C}}$, let $R_Y : \underline{\mathcal{C}}(X,Y) \to M(Y)$ be given by $R_Y(f) = M(f)(m_{X,0})$. If $g \in \underline{\mathcal{C}}(Y,Z)$, then we get a commutative diagram

$$\begin{array}{ccc} \underline{\mathcal{C}}(X,Y) & \xrightarrow{R_Y} & M(Y) \\ \Big\downarrow & & \Big\downarrow \\ \underline{\mathcal{C}}(X,Z) & \xrightarrow[R_Z]{} & M(Z) \end{array}$$

since $f \mapsto M(f)(m_{X,0}) \mapsto M(g)M(f)(m_{X,0})$ around the top right corner and $f \mapsto gf \mapsto M(gf)(m_{X,0})$ around the bottom left corner. Thus, R is a natural transformation. If $f \in \underline{\mathcal{C}}(X,Y)_i$, then $M(f)$ is homogeneous of degree i, so $R_Y(f) = M(f)(m_{X,0}) \in M(Y)_i$. Hence, $R_Y(\underline{\mathcal{C}}(X,Y)_i) \subseteq M(Y)_i$ and so R is degree-preserving. Therefore, R is a graded $\underline{\mathcal{C}}$-module homomorphism.

Now we check that $S_Y R_Y = T_Y$ for any $Y \in \underline{\mathcal{C}}$. Let $Y \in \underline{\mathcal{C}}$ and $f \in \underline{\mathcal{C}}(X,Y)$. Then the naturality of S and T yield commutative diagrams:

$$\begin{CD} M(X) @>{S_X}>> N(X) \\ @V{M(f)}VV @VV{N(f)}V \\ M(Y) @>>{S_Y}> N(Y) \end{CD} \qquad \begin{CD} \underline{\mathcal{C}}(X,X) @>{T_X}>> N(X) \\ @VVV @VV{N(f)}V \\ \underline{\mathcal{C}}(X,Y) @>>{T_Y}> N(Y) \end{CD}$$

Therefore,

$$S_Y R_Y(f) = S_Y(M(f)(m_{X,0})) = N(f)(S_X(m_{X,0})) = N(f)(T_X(1_X)) = T_Y(f1_X) = T_Y(f).$$

It follows that $SR = T$ and so $\underline{\mathcal{C}}(X,-)$ is projective as a graded $\underline{\mathcal{C}}$-module. $\square$

Remark 2.2.5.

(a) If V is a 1-dimensional k-vector space, then $\underline{\mathcal{C}}(X,-) \otimes_k V$ is a graded $\underline{\mathcal{C}}$-module isomorphic to $\underline{\mathcal{C}}(X,-)$ as a graded $\underline{\mathcal{C}}$-module.

(b) A direct sum of projective graded $\underline{\mathcal{C}}$-modules is a projective graded $\underline{\mathcal{C}}$-module.

Let $\underline{\mathcal{C}}$ be a directed graded k-linear category. Given a $\underline{\mathcal{C}}$-module M, a *$\underline{\mathcal{C}}$-submodule* of M is a $\underline{\mathcal{C}}$-module $N : \underline{\mathcal{C}} \to k - \text{Mod}$ satisfying $N(X) \subseteq M(X)$ for all $X \in \underline{\mathcal{C}}$ and $N(f) = M(f)|_{N(X)}$ for all $f \in \underline{\mathcal{C}}(X,Y)$. Also, we say that M *contains* a set S if $S \subseteq \bigcup_{X \in \underline{\mathcal{C}}} M(X)$. A graded $\underline{\mathcal{C}}$-module M is *generated in degree* $i \geq 0$ if the only $\underline{\mathcal{C}}$-submodule of M containing $\bigcup_{X \in \underline{\mathcal{C}}} M(X)_i$ is M itself.

Definition 2.2.6. A graded $\underline{\mathcal{C}}$-module M is *Koszul* if it has a linear projective resolution

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

in $\underline{\mathcal{C}}$-gMod. By linear we mean that each P_i is generated in degree i. The category $\underline{\mathcal{C}}$ is *Koszul* if for every $X \in \underline{\mathcal{C}}$, the graded $\underline{\mathcal{C}}$-module $\underline{\mathcal{C}}(X,X)$ is Koszul.

3. The category $\mathcal{FI}^n_{t,A}$

3.1. Partition types.

Let $\mathcal{FI}$ be the category of finite sets and injective functions. Let $n \in \mathbb{N}$, and $\mathcal{FI}^n$ be the n-fold product category $\mathcal{FI} \times \cdots \times \mathcal{FI}$. Then an arbitrary object $X \in \mathcal{FI}^n$ is of the form $X = (X_1, ..., X_n)$ for some finite sets $X_i \in \mathcal{FI}$. If $X = (X_1, ..., X_n) \in \mathcal{FI}^n$, then by $x \in X$ we mean $x \in X_i$ for some i. By a *partition* P_X of an object $X = (X_1, ..., X_n) \in \mathcal{FI}^n$, we mean a partition of $X_1 \amalg \cdots \amalg X_n$. We allow $P_\varnothing = \varnothing$ to be a partition of $\varnothing \in \mathcal{FI}^n$. Let $\mathcal{P}$ be the set of all partitions of every $X \in \mathcal{FI}^n$. By a *property* t on $\mathcal{P}$ we mean a map t from $\mathcal{P}$ to a 2-element set {yes, no}. We say that a partition P_X of $X \in \mathcal{FI}^n$ *has property* t if $t(P_X) =$ yes; otherwise we say P_X does not have property t.

Definition 3.1.1. We say that a property t is a *partition type* if the following conditions are satisfied:

(P0) $P_\varnothing$ has property t;

(P1) if P_X has property t and $f \in \mathcal{FI}^n(X, Y)$, then $f(P_X)$ has property t;

(P2) if V, W are disjoint subsets of $X \in \mathcal{FI}^n$ and P_V, P_W have property t, then $P_V \sqcup P_W$ has property t;

(P3) if P_X has property t and $S \in P_X$, then $P_X \backslash \{S\}$ has property t;

(P4) there exists $M \in \mathbb{N}$ such that if P_X has property t, then $|S| \leq M$ for all $S \in P_X$.

If t is a partition type and P_X is a partition of $X \in \mathcal{FI}^n$ which has property t, then we say that P_X is a *partition of type* t.

Example 3.1.2.

(a) Let $m \in \mathbb{N}$. Define property m by declaring that $m(P_X) =$ yes for partitions P_X of $X \in \mathcal{FI}^n$ satisfying $|S| = m$ for all $S \in P_X$; $m(P_X) =$ no otherwise. Then m is a partition type.

(b) Let $m \in \mathbb{N}$. Define property $\leq m$ by requiring that $\leq m(P_X) =$ yes for partitions P_X of $X \in \mathcal{FI}^n$ satisfying $|S| \leq m$ for all $S \in P_X$; $\leq m(P_X) =$ no otherwise. Then $\leq m$ is a partition type.

(c) Define property n^* by declaring that $n^*(P_X) =$ yes for partitions P_X of $X = (X_1, ..., X_n) \in \mathcal{FI}^n$ satisfying $|S \cap X_i| = 1$ for all $S \in P_X$, $1 \leq i \leq n$; $n^*(P_X) =$ no otherwise. Then n^* is a partition type. Note that if P_X is a partition of type n^*, then $|S| = n$ for all $S \in P_X$.

(d) Let $(m_1, ..., m_n) \in \mathbb{N}^n$. Define property $(m_1, ..., m_n)$ by requiring that $(m_1, ..., m_n)(P_X) =$ yes for partitions P_X of $X = (X_1, ..., X_n) \in \mathcal{FI}^n$ satisfying $|S \cap X_i| = m_i$ for all $S \in P_X$, $1 \leq i \leq n$; $(m_1, ..., m_n)(P_X) =$ no otherwise. Then $(m_1, ..., m_n)$ is a partition type.

3.2. The category $\mathcal{FI}^n_{t,A}$.

Let $n \in \mathbb{N}$, t be a partition type, and A be an abelian group. We define a category $\mathcal{FI}^n_{t,A}$ having the same objects as $\mathcal{FI}^n$, and morphisms defined by the following data: if $X, Y \in \mathcal{FI}^n_{t,A}$, then a morphism $(f, P_f, \alpha_f) : X \to Y$ in $\mathcal{FI}^n_{t,A}$ consists of a morphism $f \in \mathcal{FI}^n(X, Y)$, a partition P_f of $Y \backslash f(X) \in \mathcal{FI}^n$ of type t, and a function $\alpha_f : X \amalg P_f \to A$. If all 3 of these items do not exist for a certain $X, Y \in \mathcal{FI}^n$, then $\mathcal{FI}^n_{t,A}(X, Y) = \varnothing$. The composite of two morphisms $(f, P_f, \alpha_f) : X \to Y$ and $(g, P_g, \alpha_g) : Y \to Z$ in $\mathcal{FI}^n_{t,A}$ is given by the morphism $(g, P_g, \alpha_g)(f, P_f, \alpha_f) = (gf, P_{gf}, \alpha_{gf}) :$ $X \to Z$ in $\mathcal{FI}^n_{t,A}$, where gf is the composite of f followed by g in $\mathcal{FI}^n$, P_{gf} is the partition $g(P_f) \sqcup P_g$ of $Z \backslash gf(X) \in \mathcal{FI}^n$ of type t, and $\alpha_{gf} : X \amalg P_{gf} \to A$ is the function defined by

$$\begin{aligned} \alpha_{gf}(x) &= \alpha_f(x) + \alpha_g(f(x)) \text{ for } x \in X, \\ \alpha_{gf}(g(S)) &= \alpha_f(S) + \sum_{y \in S} \alpha_g(y) \text{ for } S \in P_f, \\ \alpha_{gf}(T) &= \alpha_g(T) \text{ for } T \in P_g. \end{aligned}$$

To show that composition in $\mathcal{FI}^n_{t,A}$ is associative, let $(f, P_f, \alpha_f) : W \to X$, $(g, P_g, \alpha_g) : X \to Y$, and $(h, P_h, \alpha_h) : Y \to Z$ be morphisms in $\mathcal{FI}^n_{t,A}$. Then $h(gf) = (hg)f$,

$$\begin{aligned} P_{h(gf)} &= h(P_{gf}) \sqcup P_h \\ &= h(g(P_f) \sqcup P_g) \sqcup P_h \\ &= hg(P_f) \sqcup h(P_g) \sqcup P_h \\ &= hg(P_f) \sqcup P_{hg} \\ &= P_{(hg)f}, \end{aligned}$$

and

$$\begin{aligned}\alpha_{h(gf)}(w) &= \alpha_{gf}(w) + \alpha_h(gf(w)) \\ &= \alpha_f(w) + \alpha_g(f(w)) + \alpha_h(gf(w)) \\ &= \alpha_f(w) + \alpha_{hg}(f(w)) \\ &= \alpha_{(hg)f}(w) \text{ for } w \in W,\end{aligned}$$

$$\begin{aligned}\alpha_{h(gf)}(h(g(R))) &= \alpha_{gf}(g(R)) + \sum_{y\in g(R)} \alpha_h(y) \\ &= \alpha_f(R) + \sum_{x\in R} \alpha_g(x) + \sum_{y\in g(R)} \alpha_h(y) \\ &= \alpha_f(R) + \sum_{x\in R} (\alpha_g(x) + \alpha_h(g(x))) \\ &= \alpha_f(R) + \sum_{x\in R} \alpha_{hg}(x) \\ &= \alpha_{(hg)f}((hg)(R)) \text{ for } R \in P_f,\end{aligned}$$

$$\begin{aligned}\alpha_{h(gf)}(h(S)) &= \alpha_{gf}(S) + \sum_{y\in S} \alpha_h(y) \\ &= \alpha_g(S) + \sum_{y\in S} \alpha_h(y) \\ &= \alpha_{hg}(h(S)) \\ &= \alpha_{(hg)f}(h(S)) \text{ for } S \in P_g,\end{aligned}$$

$$\begin{aligned}\alpha_{h(gf)}(T) &= \alpha_h(T) \\ &= \alpha_{hg}(T) \\ &= \alpha_{(hg)f}(T) \text{ for } T \in P_h.\end{aligned}$$

Thus, $(h(gf), P_{h(gf)}, \alpha_{h(gf)}) = ((hg)f, P_{(hg)f}, \alpha_{(hg)f})$. The identity morphism of $X \in \mathcal{FI}^n_{t,A}$ is $(1_X, \varnothing, 0) : X \to X$, where $0 : X \to A$ is the zero map.

Example 3.2.1.

(a) If $n = 1$, t is partition type 1, and A is the trivial abelian group 0, then $\mathcal{FI}^1_{1,0}$ is the category $\mathcal{FI}$ from Example 1.1.1.

(b) If $n = 1$, t is partition type 2, and A is the trivial abelian group 0, $\mathcal{FI}^1_{2,0}$ is the category $\mathcal{FIM}$ from Example 1.1.2.

(c) If $n = 2$, t is partition type 2^*, and A is the trivial abelian group 0, then $\mathcal{FI}^1_{2^*,0}$ is the category $\mathcal{FIM}_w$ from Example 1.1.3.

(d) If $n = 2$, t is partition type 2^*, and A is the abelian group $\mathbb{Z}/2\mathbb{Z}$, then $\mathcal{FI}^1_{2^*,\mathbb{Z}/2\mathbb{Z}}$ is the category $\mathcal{FIM}'_w$ from Example 1.1.4.

Thus, the family of categories $\mathcal{FI}^n_{t,A}$ unify the four categories from the Introduction in a general setup.

Remark 3.2.2. If $n \in \mathbb{N}$, t is a partition type, and A is an abelian group, then the full subcategory of $\mathcal{FI}^n_{t,A}$ on objects of the form $X = ([x_1], ..., [x_n])$ for $x_i \in \mathbb{N}_0$ $(1 \leq i \leq n)$, is skeletal, and hence equivalent to $\mathcal{FI}^n_{t,A}$.

Let $n \in \mathbb{N}$, t be a partition type, and A be a finite abelian group. Let $\mathcal{C}$ be the skeletal subcategory of $\mathcal{FI}^n_{t,A}$ on objects of the form $X = ([x_1], ..., [x_n])$ for $x_i \in \mathbb{N}_0$ $(1 \leq i \leq n)$. Then $\mathcal{C}$ is equivalent to $\mathcal{FI}^n_{t,A}$. Let k be a field of characteristic 0, and $\underline{\mathcal{C}}$ be the k-linearization of $\mathcal{C}$.

Proposition 3.2.3. $\underline{\mathcal{C}}$ is a directed graded k-linear category.

Proof. For any $X, Y \in \mathcal{C}$ and $i \geq 0$, define the degree i component of $\underline{\mathcal{C}}(X,Y)$ to be

$$\underline{\mathcal{C}}(X,Y)_i = \bigoplus_{\substack{(f,P_f,\alpha_f)\in\mathcal{C}(X,Y)\\ |P_f|=i}} k(f, P_f, \alpha_f).$$

Then $\underline{\mathcal{C}}(X,Y) = \bigoplus_{i\geq 0} \underline{\mathcal{C}}(X,Y)_i$ is a graded k-vector space. So $(f, P_f, \alpha_f) \in \underline{\mathcal{C}}(X,Y)_i$ is a morphism of degree i if and only if $|P_f| = i$. For any $X, Y, Z \in \mathcal{C}$, the composition map $\underline{\mathcal{C}}(Y,Z) \otimes_k \underline{\mathcal{C}}(X,Y) \to \underline{\mathcal{C}}(X,Z)$ preserves the grading, and hence is a morphism in k-gMod. It follows that $\underline{\mathcal{C}}$ is a graded k-linear category. The objects of $\underline{\mathcal{C}}$ are partially ordered by inclusion $\subseteq$, such that $\underline{\mathcal{C}}(X,Y) \neq 0$ implies $X \subseteq Y$. So $\underline{\mathcal{C}}$ is a directed k-linear category.

We now check that $\underline{\mathcal{C}}$ meets additional conditions (A1)-(A7).

(A1) For any $X, Y \in \mathcal{C}$, $\mathcal{C}(X,Y)$ is a finite set, so $\underline{\mathcal{C}}(X,Y)$ is a finite dimensional k-vector space.

(A2) For any $X = ([x_1], ..., [x_n]) \in \mathcal{C}$, a morphism $(f, P_f, \alpha_f) \in \mathcal{C}(X,X)$ consists of a bijection $f \in \mathcal{FI}^n(X,X)$ and a function $\alpha_f : X \to A$, because $P_f = \varnothing$. Note that $f \in S_{x_1} \times \cdots \times S_{x_n}$, where S_{x_i} is the symmetric group on $[x_i]$, and $\alpha_f \in A^X$, where A^X is the group of all functions from X to A. Therefore, $\mathcal{C}(X,X)$ is the finite group $S_{x_1} \times \cdots \times S_{x_n} \times A^X$. Since char $k = 0$, the group algebra $\underline{\mathcal{C}}(X,X)$ is semisimple, by Maschke's theorem.

(A3) Let $X, Y \in \mathcal{C}$. If $(f, P_f, \alpha_f) \in \underline{\mathcal{C}}(X,Y)_0$ is a basis element, then $P_f = \varnothing$, which forces $X = Y$. So if $X \neq Y$, then $\underline{\mathcal{C}}(X,Y)_0 = 0$.

(A4) Let $X \in \mathcal{C}$ and $i > 0$. If $(f, P_f, \alpha_f) \in \underline{\mathcal{C}}(X,X)_i$ is a basis element, then $P_f = \varnothing$ since $f \in \mathcal{FI}^n(X,X)$ is a bijection. But $|P_f| = i > 0$ since (f, P_f, α_f) is a morphism of degree i. So for all $X \in \mathcal{C}$ and $i > 0$, we must have $\underline{\mathcal{C}}(X,X)_i = 0$.

(A5) Let $X \in \mathcal{C}$. Then there are only finitely many $Y \in \mathcal{C}$ such that $\underline{\mathcal{C}}(Y,X)_1 \neq 0$. By condition (P4) of the partition type t, there are only finitely many $Y \in \mathcal{C}$ such that $\underline{\mathcal{C}}(X,Y)_1 \neq 0$.

(A6) Let $i \geq 0$. In order to prove $\underline{\mathcal{C}}_1 \cdot \underline{\mathcal{C}}_i = \underline{\mathcal{C}}_{i+1}$, it is enough to show that any morphism $(f, P_f, \alpha_f) \in \mathcal{C}(X,Y)$ of degree $i+1$ can be factored as a composite of a degree i morphism followed by a degree 1 morphism. Let $(f, P_f, \alpha_f) \in \mathcal{C}(X,Y)$ be a morphism of degree $i+1$. Then $|P_f| = i+1 \geq 1$. Pick $T \in P_f$. Since $\mathcal{C}$ is skeletal, there is a unique $Y' \in \mathcal{C}$ and a bijection $g \in \mathcal{FI}^n(Y', Y \backslash T)$. Let $f' \in \mathcal{FI}^n(X, Y \backslash T)$ be the morphism obtained by restricting the codomain of f from Y to $Y \backslash T$. Define $f_1 = g^{-1} f' \in \mathcal{FI}^n(X, Y')$, $P_{f_1} = \{g^{-1}(P_f \backslash \{T\})\}$, and $\alpha_{f_1} : X \amalg P_{f_1} \to A$ by $\alpha_{f_1}(x) = \alpha_f(x)$ for all $x \in X$ and $\alpha_{f_1}(S) = \alpha_f(g(S))$ for all $S \in P_{f_1}$. Then $(f_1, P_{f_1}, \alpha_{f_1}) \in \mathcal{C}(X, Y')$ is a morphism of degree i. Let $\iota \in \mathcal{FI}^n(Y \backslash T, Y)$ be the inclusion map. Define $g_1 = \iota g \in \mathcal{FI}^n(Y', Y)$, $P_{g_1} = \{T\}$, and $\alpha_{g_1} : Y' \amalg P_{g_1} \to A$ by $\alpha_{g_1}(y) = 0$ for all $y \in Y'$ and $\alpha_{g_1}(T) = \alpha_f(T)$. Then $(g_1, P_{g_1}, \alpha_{g_1}) \in \mathcal{C}(Y', Y)$ is a morphism of degree 1. Now $g_1 f_1 = \iota g g^{-1} f' = f$, $P_{g_1 f_1} = g_1(P_{f_1}) \sqcup P_{g_1} = (P_f \backslash \{T\}) \sqcup \{T\} = P_f$, and

$$\alpha_{g_1 f_1}(x) = \alpha_{f_1}(x) + \alpha_{g_1}(f_1(x)) = \alpha_f(x) \text{ for } x \in X,$$
$$\alpha_{g_1 f_1}(g_1(S)) = \alpha_{f_1}(S) + \sum_{y \in S} \alpha_{g_1}(y) = \alpha_f(g(S)) \text{ for } S \in P_{f_1},$$
$$\alpha_{g_1 f_1}(T) = \alpha_{g_1}(T) = \alpha_f(T).$$

Hence, $(f, P_f, \alpha_f) = (g_1, P_{g_1}, \alpha_{g_1})(f_1, P_{f_1}, \alpha_{f_1})$ is the composite of a degree i morphism followed by a degree 1 morphism, as desired.

(A7) Let $X, Z \in \mathcal{C}$. Because $\mathcal{C}$ is skeletal, it is totally ordered by $\subseteq$. So without loss of generality, suppose $X \subseteq Z$. Again since $\mathcal{C}$ is skeletal, there are only finitely many $Y \in \mathcal{C}$ such that $X \subseteq Y \subseteq Z$. It follows that the convex hull of any finite set $S \subseteq \mathrm{Ob}(\underline{\mathcal{C}})$ contains only finitely many objects.

Therefore, $\underline{\mathcal{C}}$ is a directed graded k-linear category. $\square$

Notation 3.2.4. For any $X, Y \in \mathcal{C}$, we shall write degree 1 morphisms in $\mathcal{C}(X, Y)$ simply as (f, R, α_f) for $P_f = \{R\}$, where $R \subseteq Y$.

4. Koszulity

Fix $n \in \mathbb{N}$, a partition type t, and a finite abelian group A. Let $\mathcal{C} = \mathcal{FI}^n_{t,A}$, and $\underline{\mathcal{C}}$ be the k-linearization of $\mathcal{C}$, where k is a field of characteristic 0. By Proposition 3.2.3, $\underline{\mathcal{C}}$ is a directed graded k-linear category. In this section, we will prove that $\underline{\mathcal{C}}$ is Koszul (Corollary 4.3.2). To do this, we will construct a linear projective resolution $C_\bullet(-)(Y) \to \underline{\mathcal{C}}(Y, Y)$ of graded right $\underline{\mathcal{C}}$-modules for arbitrary $Y \in \mathcal{C}$.

4.1. The complex $C_\bullet(-)(Y)$.

Fix $Y \in \mathcal{C}$ for the remainder of this section. For any $m \in \mathbb{N}_0$, define a functor $C_m(-)(Y) : \underline{\mathcal{C}} \to$ k-gMod as follows. For any object $X \in \mathcal{C}$, let

$$C_m(X)(Y) = \bigoplus_{(I,\alpha)} \underline{\mathcal{C}}(X, Y \backslash I) \otimes_k \det(I),$$

the direct sum being over all pairs (I, α), where $I = I_1 \sqcup \cdots \sqcup I_m$ is the union of m mutually disjoint nonempty subsets $I_j \subseteq Y$ such that $\{I_j\}$ is a partition of type t, and $\alpha : \{I_1, ..., I_m\} \to A$ is a function. By $\det(I)$ we mean the 1-dimensional k-vector space $\bigwedge^m k\{I_1, ..., I_m\}$.

To see that $C_m(X)(Y)$ is a graded k-vector space, let

$$C_m(X)(Y)_i = \bigoplus_{(I,\alpha)} \bigoplus_{\substack{(f,P_f,\alpha_f)\in\mathcal{C}(X,Y\backslash I)\\ |P_f|=i-m}} k(f, P_f, \alpha_f) \otimes_k \det(I)$$

for $i \geq m$ and $C_m(X)(Y)_i = 0$ for $i < m$. Then

$$\begin{aligned} C_m(X)(Y) &= \bigoplus_{(I,\alpha)} \underline{\mathcal{C}}(X, Y\backslash I) \otimes_k \det(I) \\ &= \bigoplus_{(I,\alpha)} \bigoplus_{i \geq m} \bigoplus_{\substack{(f,P_f,\alpha_f) \in \mathcal{C}(X,Y\backslash I) \\ |P_f| = i-m}} k(f, P_f, \alpha_f) \otimes_k \det(I) \\ &= \bigoplus_{i \geq m} C_m(X)(Y)_i. \end{aligned}$$

So $C_m(X)(Y)$ is a graded k-vector space living in degrees $\geq m$. This completes the definition of $C_m(-)(Y)$ on objects.

For any morphism $(g, P_g, \alpha_g) \in \underline{\mathcal{C}}(X, X')$, we define a map $C_m(X')(Y) \to C_m(X)(Y)$ on direct summands corresponding to (I, α) by k-linear extension of the assignment

$$(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^{m} I_j \mapsto (f, P_f, \alpha_f)(g, P_g, \alpha_g) \otimes \bigwedge_{j=1}^{m} I_j.$$

In other words, a basis element $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^{m} I_j$ in a direct summand $\underline{\mathcal{C}}(X', Y\backslash I) \otimes_k \det(I)$ of $C_m(X')(Y)$ corresponding to (I, α) gets sent to the basis element $(f, P_f, \alpha_f)(g, P_g, \alpha_g) \otimes \bigwedge_{j=1}^{m} I_j$ in the direct summand $\underline{\mathcal{C}}(X, Y\backslash I) \otimes_k \det(I)$ of $C_m(X)(Y)$ corresponding to (I, α).

To see that $C_m(X')(Y) \to C_m(X)(Y)$ is a morphism of graded k-vector spaces, let $(g, P_g, \alpha_g) \in \underline{\mathcal{C}}(X, X')_j$ be a morphism of degree $j \geq 0$, and $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^{m} I_j$ be in the degree $i \geq m$ component of $C_m(X')(Y)$. Then $|P_g| = j$ and $|P_f| = i - m$, which implies $|P_{fg}| = |f(P_g)| + |P_f| = (i+j) - m$. So $(f, P_f, \alpha_f)(g, P_g, \alpha_g) \otimes \bigwedge_{j=1}^{m} I_j$ is in the degree $i + j$ component of $C_m(X)(Y)$, hence $C_m(X')(Y) \to C_m(X)(Y)$ is homogeneous of degree j. This completes the definition of $C_m(-)(Y) : \underline{\mathcal{C}} \to k$-gMod on morphisms.

So $C_m(-)(Y) : \underline{\mathcal{C}} \to k$-gMod is a degree-preserving k-linear functor that is contravariant. Thus, we have a graded right $\underline{\mathcal{C}}$-module $C_m(-)(Y)$ for each $m \in \mathbb{N}_0$. In particular, $C_0(-)(Y) = \underline{\mathcal{C}}(-, Y)$.

Before we define a differential $\partial : C_m(-)(Y) \to C_{m-1}(-)(Y)$, we need the notion of an inclusion morphism in $\mathcal{C}$.

Definition 4.1.1. Let $X, Y \in \mathcal{C}$ with $X \subseteq Y$ such that $\{Y \backslash X\}$ is a partition of type t. We say that $(\iota, P_\iota, \alpha_\iota) \in \mathcal{C}(X, Y)$ is an *inclusion morphism* if $\iota(x) = x$ for all $x \in X$, $P_\iota = \{Y \backslash X\}$, and $\alpha_\iota(x) = 0$ for all $x \in X$.

Note 4.1.2. In order to completely describe an inclusion morphism, one still needs to specify $\alpha_\iota(Y \backslash X) \in A$. Thus, there may be many inclusion morphisms from X to Y. Also, since inclusion morphisms are of degree 1, we shall write them as $(\iota, Y \backslash X, \alpha_\iota)$.

Remark 4.1.3. Inclusion morphisms "commute" in the following sense. Let $Y \in \mathcal{C}$ and $I_1, I_2 \subset Y$ be two disjoint nonempty subsets such that $\{I_1\}, \{I_2\}$ are partitions of type t. Suppose we have inclusion morphisms

$$(\iota_1, I_1, \alpha_{\iota_1}) \in \mathcal{C}(Y \backslash I_1, Y),\ (\iota_2, I_2, \alpha_{\iota_2}) \in \mathcal{C}(Y \backslash I_2, Y)$$

and

$$(\jmath_1, I_1, \alpha_{\jmath_1}) \in \mathcal{C}(Y \backslash (I_1 \sqcup I_2), Y \backslash I_2),\ (\jmath_2, I_2, \alpha_{\jmath_2}) \in \mathcal{C}(Y \backslash (I_1 \sqcup I_2), Y \backslash I_1)$$

such that $\alpha_{\iota_1}(I_1) = \alpha_{\jmath_1}(I_1)$ and $\alpha_{\iota_2}(I_2) = \alpha_{\jmath_2}(I_2)$. Then

$$(\iota_1, I_1, \alpha_{\iota_1})(\jmath_2, I_2, \alpha_{\jmath_2}) = (\iota_2, I_2, \alpha_{\iota_2})(\jmath_1, I_1, \alpha_{\jmath_1})$$

because

$$\iota_1 \jmath_2(y) = y = \iota_2 \jmath_1(y) \text{ for } y \in Y \backslash (I_1 \sqcup I_2),$$

$$P_{\iota_1 \jmath_2} = \iota_1(\{I_2\}) \sqcup \{I_1\} = \{I_1, I_2\} = \iota_2(\{I_1\}) \sqcup \{I_2\} = P_{\iota_2 \jmath_1},$$

and

$$\begin{aligned} \alpha_{\iota_1 \jmath_2}(y) &= 0 = \alpha_{\iota_2 \jmath_1}(y) \text{ for } y \in Y \backslash (I_1 \sqcup I_2), \\ \alpha_{\iota_1 \jmath_2}(I_1) &= \alpha_{\iota_1}(I_1) = \alpha_{\jmath_1}(I_1) = \alpha_{\iota_2 \jmath_1}(I_1), \\ \alpha_{\iota_1 \jmath_2}(I_2) &= \alpha_{\jmath_2}(I_2) = \alpha_{\iota_2}(I_2) = \alpha_{\iota_2 \jmath_1}(I_2). \end{aligned}$$

For every $m \in \mathbb{N}$, we define a graded $\underline{\mathcal{C}}$-module homomorphism $\partial : C_m(-)(Y) \to C_{m-1}(-)(Y)$ as follows. For any $X \in \mathcal{C}$, let $\partial_X : C_m(X)(Y) \to C_{m-1}(X)(Y)$ be defined on each direct summand corresponding to (I, α) by k-linear extension of the assignment

$$(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^{m} I_j \mapsto \sum_{j=1}^{m} (-1)^{j-1} (\iota_j, I_j, \alpha_j)(f, P_f, \alpha_f) \otimes I_1 \wedge \cdots \widehat{I_j} \cdots \wedge I_m,$$

where $(\iota_j, I_j, \alpha_j) \in \mathcal{C}(Y \backslash I, (Y \backslash I) \sqcup \{I_j\})$ is the inclusion morphism defined by $\alpha_j(I_j) = \alpha(I_j) \in A$. Note that for each $j = 1, ..., m$, $(\iota_j, I_j, \alpha_j)(f, P_f, \alpha_f) \otimes I_1 \wedge \cdots \widehat{I_j} \cdots \wedge I_m$ is in the direct summand of $C_{m-1}(X)(Y)$ corresponding to $(I \backslash I_j, \alpha|_{\{I_1, ..., I_m\} \backslash \{I_j\}})$.

To see that ∂ is a natural transformation, let $(g, P_g, \alpha_g) \in \mathcal{C}(X, X')$. Then we get a commutative diagram:

$$\begin{array}{ccc} C_m(X')(Y) & \longrightarrow & C_m(X)(Y) \\ \Big\downarrow \partial_{X'} & & \Big\downarrow \partial_X \\ C_{m-1}(X')(Y) & \longrightarrow & C_{m-1}(X)(Y) \end{array}$$

because

$$\begin{aligned} (f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^{m} I_j &\mapsto (f, P_f, \alpha_f)(g, P_g, \alpha_g) \otimes \bigwedge_{j=1}^{m} I_j \\ &\mapsto \sum_{j=1}^{m} (-1)^{j-1} (\iota_j, I_j, \alpha_j)(f, P_f, \alpha_f)(g, P_g, \alpha_g) \otimes I_1 \wedge \cdots \widehat{I_j} \cdots \wedge I_m \end{aligned}$$

around the top right corner, while

$$\begin{aligned} (f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^{m} I_j &\mapsto \sum_{j=1}^{m} (-1)^{j-1} (\iota_j, I_j, \alpha_j)(f, P_f, \alpha_f) \otimes I_1 \wedge \cdots \widehat{I_j} \cdots \wedge I_m \\ &\mapsto \sum_{j=1}^{m} (-1)^{j-1} (\iota_j, I_j, \alpha_j)(f, P_f, \alpha_f)(g, P_g, \alpha_g) \otimes I_1 \wedge \cdots \widehat{I_j} \cdots \wedge I_m \end{aligned}$$

around the bottom left corner.

To see that ∂ is degree-preserving, let $X \in \mathcal{C}$ and $i \geq m$. We must show that $\partial_X(C_m(X)(Y)_i) \subseteq C_{m-1}(X)(Y)_i$. If $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j$ is a basis element in a direct summand $\underline{\mathcal{C}}(X, Y\backslash I) \otimes_k \det(I)$ of $C_m(X)(Y)_i$ corresponding to (I, α), then $(f, P_f, \alpha_f) \in \mathcal{C}(X, Y\backslash I)$ with $|P_f| = i - m$. For each $j = 1, ..., m$, we have $(\iota_j, I_j, \alpha_j)(f, P_f, \alpha_f) \in \mathcal{C}(X, (Y\backslash I) \sqcup \{I_j\})$ with

$$|P_{\iota_j f}| = |\iota_j(P_f) \sqcup \{I_j\}| = |P_f| + 1 = i - (m-1).$$

So

$$\partial_X((f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j) = \sum_{j=1}^m (-1)^{j-1} (\iota_j, I_j, \alpha_j)(f, P_f, \alpha_f) \otimes I_1 \wedge \cdots \widehat{I_j} \cdots \wedge I_m$$

belongs to $C_{m-1}(X)(Y)_i$. Thus, $\partial : C_m(-)(Y) \to C_{m-1}(-)(Y)$ is a graded $\underline{\mathcal{C}}$-module homomorphism for each $m \in \mathbb{N}$.

For any $X \in \mathcal{C}$, observe that

$$\begin{aligned}
(\partial_X)^2((f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j) &= \sum_{j=1}^m (-1)^{j-1} \partial_X((\iota_j, I_j, \alpha_j)(f, P_f, \alpha_f) \otimes I_1 \wedge \cdots \widehat{I_j} \cdots \wedge I_m) \\
&= \sum_{i<j} (-1)^{i+j-2} (\iota_i, I_i, \alpha_i)(\iota_j, I_j, \alpha_j)(f, P_f, \alpha_f) \otimes I_1 \wedge \cdots \widehat{I_i} \cdots \widehat{I_j} \cdots \wedge I_m \\
&\quad + \sum_{i>j} (-1)^{i+j-1} (\iota_i, I_i, \alpha_i)(\iota_j, I_j, \alpha_j)(f, P_f, \alpha_f) \otimes I_1 \wedge \cdots \widehat{I_j} \cdots \widehat{I_i} \cdots \wedge I_m \\
&= 0
\end{aligned}$$

after switching i and j in the third line and using the fact that the inclusion morphisms (ι_i, I_i, α_i) and (ι_j, I_j, α_j) commute, by Remark 4.1.5. So $\partial^2 = 0$ and hence ∂ is a differential.

By putting $C_m(-)(Y) = 0$ for all $m < 0$, we obtain a complex $C_\bullet(-)(Y)$ of graded right $\underline{\mathcal{C}}$-modules.

Remark 4.1.4. For any $m \in \mathbb{N}_0$, $C_m(-)(Y)$ can be viewed as a direct sum $\bigoplus_{(I,\alpha)} \underline{\mathcal{C}}(-, Y\backslash I) \otimes_k \det(I)$ of graded right $\underline{\mathcal{C}}$-modules $\underline{\mathcal{C}}(-, Y\backslash I) \otimes_k \det(I)$. By Remark 2.2.5, $C_m(-)(Y)$ is projective as a graded right $\underline{\mathcal{C}}$-module.

Proposition 4.1.5. For any $m \in \mathbb{N}_0$, $C_m(-)(Y)$ is generated in degree m.

Proof. Let M be a right $\underline{\mathcal{C}}$-submodule of $C_m(-)(Y)$ containing $\bigcup_{X\in\mathcal{C}} C_m(X)(Y)_m$. We must show $M(X) = C_m(X)(Y)$ for all $X \in \mathcal{C}$. Let $X \in \mathcal{C}$ and $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j$ be a basis element in a direct summand of $C_m(X)(Y)$ corresponding to (I, α). We will show $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j \in M(X)$ by factorizing $(f, P_f, \alpha_f) \in \mathcal{C}(X, Y\backslash I)$ as follows. Choose $Y' \in \mathcal{C}$ and an isomorphism $(g, \varnothing, 0) \in \mathcal{C}(Y', Y\backslash I)$ (in which $g \in \mathcal{FI}^n(Y', Y\backslash I)$ is a bijection, $P_g = \varnothing$, and $\alpha_g : Y' \to A$ is the zero map). Then $(g, \varnothing, 0) \otimes \bigwedge_{j=1}^m I_j$ belongs to the direct summand of $C_m(Y')(Y)_m$ corresponding to (I, α). Define $f' = g^{-1}f$, $P_{f'} = g^{-1}(P_f)$, and $\alpha_{f'} : X \amalg P_{f'} \to A$ by $\alpha_{f'}(x) = \alpha_f(x)$ for all $x \in X$, $\alpha_{f'}(S) = \alpha_f(g(S))$ for all $S \in P_{f'}$. This defines a morphism $(f', P_{f'}, \alpha_{f'}) \in \mathcal{C}(X, Y')$ such that $gf' = f$, $P_{gf'} = g(P_{f'}) = P_f$, and

$$\alpha_{gf'}(x) = \alpha_{f'}(x) + \alpha_g(f'(x)) = \alpha_f(x) \text{ for } x \in X,$$
$$\alpha_{gf'}(g(S)) = \alpha_{f'}(S) + \sum_{y'\in S} \alpha_g(y') = \alpha_f(g(S)) \text{ for } S \in P_{f'}.$$

Hence, $(g, \varnothing, 0)(f', P_{f'}, \alpha_{f'}) = (f, P_f, \alpha_f)$. Since M is a right $\underline{\mathcal{C}}$-submodule of $C_m(-)(Y)$,

$$M((f', P_{f'}, \alpha_{f'})) : M(Y') \to M(X)$$

is the restriction of $C_m(Y')(Y) \to C_m(X)(Y)$ to $M(Y')$. Because M contains $\bigcup_{X\in\mathcal{C}} C_m(X)(Y)_m$, we have $(g, \varnothing, 0) \otimes \bigwedge_{j=1}^m I_j \in C_m(Y')(Y)_m \subseteq M(Y')$. So

$$(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j = (g, \varnothing, 0)(f', P_{f'}, \alpha_{f'}) \otimes \bigwedge_{j=1}^m I_j = M((f', P_{f'}, \alpha_{f'}))((g, \varnothing, 0) \otimes \bigwedge_{j=1}^m I_j) \in M(X).$$

Hence, $C_m(-)(Y)$ is generated in degree m. □

We therefore have a complex $C_\bullet(-)(Y)$ of graded right $\underline{\mathcal{C}}$-modules

$$\cdots \to C_m(-)(Y) \to C_{m-1}(-)(Y) \to \cdots \to C_1(-)(Y) \to \underline{\mathcal{C}}(-, Y) \to 0$$

in which $C_m(-)(Y)$ is projective and generated in degree m for all $m \in \mathbb{N}_0$.

4.2. Exactness of $C_\bullet(-)(Y)$ in positive degrees.

In this subsection, we will show that $H_m(C_\bullet(-)(Y)) = 0$ for all $m \in \mathbb{N}$. First, we will prove that $H_1(C_\bullet(-)(Y)) = 0$ with the aid of the following lemma.

Lemma 4.2.1. Let $X, Y \in \mathcal{C}$ and $I_1, I_2 \subset Y$ be distinct nonempty subsets such that $\{I_1\}, \{I_2\}$ are partitions of type t. For $i = 1, 2$, let $(f_i, P_{f_i}, \alpha_{f_i}) \in \mathcal{C}(X, Y \backslash I_i)$ be morphisms and $(\iota_i, I_i, \alpha_{\iota_i}) \in \mathcal{C}(Y \backslash I_i, Y)$ be inclusion morphisms. If

$$(\iota_1, I_1, \alpha_{\iota_1})(f_1, P_{f_1}, \alpha_{f_1}) = (\iota_2, I_2, \alpha_{\iota_2})(f_2, P_{f_2}, \alpha_{f_2}),$$

then $I_1 \cap I_2 = \varnothing$ and there exists a morphism $(f, P_f, \alpha_f) \in \mathcal{C}(X, Y \backslash (I_1 \sqcup I_2))$ and inclusion morphisms $(\jmath_1, I_1, \alpha_{\jmath_1}) \in \mathcal{C}(Y \backslash (I_1 \sqcup I_2), Y \backslash I_2)$, $(\jmath_2, I_2, \alpha_{\jmath_2}) \in \mathcal{C}(Y \backslash (I_1 \sqcup I_2), Y \backslash I_1)$ such that

$$(\jmath_1, I_1, \alpha_{\jmath_1})(f, P_f, \alpha_f) = (f_2, P_{f_2}, \alpha_{f_2})$$

and

$$(\jmath_2, I_2, \alpha_{\jmath_2})(f, P_f, \alpha_f) = (f_1, P_{f_1}, \alpha_{f_1}).$$

Proof. From $(\iota_1, I_1, \alpha_{\iota_1})(f_1, P_{f_1}, \alpha_{f_1}) = (\iota_2, I_2, \alpha_{\iota_2})(f_2, P_{f_2}, \alpha_{f_2})$, we get $\iota_1 f_1 = \iota_2 f_2$, $P_{f_1} \sqcup \{I_1\} = P_{f_2} \sqcup \{I_2\}$, and $\alpha_{\iota_1 f_1} = \alpha_{\iota_2 f_2}$. These respectively imply that $f_1(x) = f_2(x)$ for all $x \in X$, $P_{f_1} \backslash \{I_2\} = P_{f_2} \backslash \{I_1\}$, and

$$\alpha_{f_1}(x) = \alpha_{\iota_1 f_1}(x) = \alpha_{\iota_2 f_2}(x) = \alpha_{f_2}(x) \text{ for } x \in X,$$
$$\alpha_{f_1}(S) = \alpha_{\iota_1 f_1}(S) = \alpha_{\iota_2 f_2}(S) = \alpha_{f_2}(S) \text{ for } S \in P_{f_1} \backslash \{I_2\} = P_{f_2} \backslash \{I_1\}.$$

In particular, $I_1 \in P_{f_2}$ and $I_2 \in P_{f_1}$ are disjoint because they are distinct elements of the same partition $P_{f_1} \sqcup \{I_1\} = P_{f_2} \sqcup \{I_2\}$. Define $(f, P_f, \alpha_f) \in \mathcal{C}(X, Y \backslash (I_1 \sqcup I_2))$ by setting $f(x) = f_1(x) = f_2(x)$ for all $x \in X$, $P_f = P_{f_1} \backslash \{I_2\} = P_{f_2} \backslash \{I_1\}$, and $\alpha_f = \alpha_{f_1}|_{X \amalg P_f} = \alpha_{f_2}|_{X \amalg P_f}$. Let $(\jmath_1, I_1, \alpha_{\jmath_1}) \in \mathcal{C}(Y \backslash (I_1 \sqcup I_2), Y \backslash I_2)$ and $(\jmath_2, I_2, \alpha_{\jmath_2}) \in \mathcal{C}(Y \backslash (I_1 \sqcup I_2), Y \backslash I_1)$ be the inclusion morphisms defined by $\alpha_{\jmath_1}(I_1) = \alpha_{f_2}(I_1)$ and $\alpha_{\jmath_2}(I_2) = \alpha_{f_1}(I_2)$. Then $\jmath_1 f(x) = f_2(x)$ for all $x \in X$, $P_{\jmath_1 f} = P_f \sqcup \{I_1\} = P_{f_2}$,

and

$$\begin{aligned}\alpha_{\jmath_1 f}(x) &= \alpha_f(x) = \alpha_{f_2}(x) \text{ for } x \in X,\\ \alpha_{\jmath_1 f}(S) &= \alpha_f(S) = \alpha_{f_2}(S) \text{ for } S \in P_f = P_{f_2} \backslash \{I_1\},\\ \alpha_{\jmath_1 f}(I_1) &= \alpha_{\jmath_1}(I_1) = \alpha_{f_2}(I_1).\end{aligned}$$

So $(\jmath_1, I_1, \alpha_{\jmath_1})(f, P_f, \alpha_f) = (f_2, P_{f_2}, \alpha_{f_2})$. Similarly, $\jmath_2 f(x) = f_1(x)$ for all $x \in X$, $P_{\jmath_2 f} = P_f \sqcup \{I_2\} = P_{f_1}$, and

$$\begin{aligned}\alpha_{\jmath_2 f}(x) &= \alpha_f(x) = \alpha_{f_1}(x) \text{ for } x \in X,\\ \alpha_{\jmath_2 f}(S) &= \alpha_f(S) = \alpha_{f_1}(S) \text{ for } S \in P_f = P_{f_1} \backslash \{I_2\},\\ \alpha_{\jmath_2 f}(I_2) &= \alpha_{\jmath_2}(I_2) = \alpha_{f_1}(I_2).\end{aligned}$$

So $(\jmath_2, I_2, \alpha_{\jmath_2})(f, P_f, \alpha_f) = (f_1, P_{f_1}, \alpha_{f_1})$. □

Remark 4.2.2. Let $X, Y \in \mathcal{C}$ and $I \subseteq Y$ be a subset such that $\{I\}$ is a partition of type t. For $i = 1, 2$, let $(f_i, P_{f_i}, \alpha_{f_i}) \in \mathcal{C}(X, Y \backslash I)$ be morphisms and $(\iota_i, I, \alpha_{\iota_i}) \in \mathcal{C}(Y \backslash I, Y)$ be inclusion morphisms. If

$$(\iota_1, I, \alpha_{\iota_1})(f_1, P_{f_1}, \alpha_{f_1}) = (\iota_2, I, \alpha_{\iota_2})(f_2, P_{f_2}, \alpha_{f_2}),$$

then $f_1(x) = f_2(x)$ for all $x \in X$, $P_{f_1} = P_{f_2}$, $\alpha_{f_1}(x) = \alpha_{f_2}(x)$ for all $x \in X$, and $\alpha_{f_1}(S) = \alpha_{f_2}(S)$ for all $S \in P_{f_1} = P_{f_2}$. So $(f_1, P_{f_1}, \alpha_{f_1}) = (f_2, P_{f_2}, \alpha_{f_2})$.

Proposition 4.2.3. $H_1(C_\bullet(-)(Y)) = 0$.

Proof. Let us abbreviate morphisms (f, P_f, α_f) in $\mathcal{C}$ simply as f, while keeping in mind the remaining data P_f and α_f that define them. Let $X \in \mathcal{C}$ and consider the tail of the complex $C_\bullet(X)(Y)$:

$$\cdots \to C_2(X)(Y) \stackrel{\partial_2}{\to} C_1(X)(Y) \stackrel{\partial_1}{\to} \underline{\mathcal{C}}(X, Y) \to 0.$$

Recall that

$$C_1(X)(Y) = \bigoplus_{(I,\alpha)} \underline{\mathcal{C}}(X, Y \backslash I) \otimes_k \det(I),$$

where $I \subseteq Y$ such that $\{I\}$ is a partition of type t, $\alpha(I) \in A$, and $\det(I) = kI$. Thus, an arbitrary element of $C_1(X)(Y)$ is of the form

$$u = (\sum_{i_1} c_{i_1} f_{i_1}) \otimes I_1 + \cdots + (\sum_{i_s} c_{i_s} f_{i_s}) \otimes I_s,$$

where the $I_j \subseteq Y$ are distinct and equipped with an element $\alpha_j(I_j) = a_j \in A$ $(j = 1, ..., s)$, the $f_{i_j} \in \mathcal{C}(X, Y \backslash I_j)$ are all distinct, and $c_{i_j} \in k$. For simplicity, we reindex this sum as

$$u = \sum_i c_i f_i \otimes I_i,$$

in which the I_i are no longer distinct, yet the f_i remain distinct. Suppose $u \in \ker(\partial_1)$. Then

$$\partial_1(u) = \sum_i c_i \iota_i f_i = 0,$$

where the ι_i are the inclusion morphisms $(\iota_i, I_i, \alpha_i) \in \mathcal{C}(Y \backslash I_i, Y)$ defined by $\alpha_i(I_i) = a_i \in A$. Now, some of the $\iota_i f_i$ may have composed to the same element in $\mathcal{C}(X, Y)$. By grouping together all such terms in the sum $\partial_1(u)$, we see that the sum of the corresponding c_i is zero in each group. By reindexing if necessary, we get

$$c_1 + \cdots + c_{i_1} = c_{i_1+1} + \cdots + c_{i_2} = \cdots = c_{i_{r-1}+1} + \cdots + c_{i_r} = 0$$

for some r, and so

$$c_1 = -c_2 - \cdots - c_{i_1}, c_{i_1+1} = -c_{i_1+2} - \cdots - c_{i_2}, ..., c_{i_{r-1}+1} = -c_{i_{r-1}+2} - \cdots - c_{i_r}.$$

Hence, the sum u can be rearranged to

$$\begin{aligned} &c_2(f_2 \otimes I_2 - f_1 \otimes I_1) + \cdots + c_{i_1}(f_{i_1} \otimes I_{i_1} - f_1 \otimes I_1) + \\ &c_{i_1+2}(f_{i_1+2} \otimes I_{i_1+2} - f_{i_1+1} \otimes I_{i_1+1}) + \cdots + c_{i_2}(f_{i_2} \otimes I_{i_2} - f_{i_1+1} \otimes I_{i_1+1}) + \cdots \\ &+c_{i_{r-1}+2}(f_{i_{r-1}+2} \otimes I_{i_{r-1}+2} - f_{i_{r-1}+1} \otimes I_{i_{r-1}+1}) + \cdots + c_{i_r}(f_{i_r} \otimes I_{i_r} - f_{i_{r-1}+1} \otimes I_{i_{r-1}+1}). \end{aligned}$$

Each of the terms $f_i \otimes I_i - f_j \otimes I_j$ in this sum result from morphisms for which $\iota_i f_i = \iota_j f_j$ in the sum $\partial_1(u)$. If $I_i = I_j$, then by Remark 4.2.2, we must have $f_i = f_j$, contradicting that the f_i are distinct. Thus, $I_i \neq I_j$ for each of the terms $f_i \otimes I_i - f_j \otimes I_j$. In particular, $\iota_1 f_1 = \iota_2 f_2$ and $I_1 \neq I_2$, so by Lemma 4.2.1, there exists a morphism $f \in \mathcal{C}(X, Y \backslash (I_1 \sqcup I_2))$ and inclusion morphisms $\jmath_1 \in \mathcal{C}(Y \backslash (I_1 \sqcup I_2), Y \backslash I_2)$, $\jmath_2 \in \mathcal{C}(Y \backslash (I_1 \sqcup I_2), Y \backslash I_1)$ such that $\jmath_1 f = f_2$ and $\jmath_2 f = f_1$. Let $I = I_1 \sqcup I_2$ and define $\alpha : \{I_1, I_2\} \to A$ by $\alpha(I_1) = \alpha_{\jmath_1}(I_1)$, $\alpha(I_2) = \alpha_{\jmath_2}(I_2)$. Then $f \otimes I_1 \wedge I_2$ belongs to the direct summand of $C_2(X)(Y)$ corresponding to (I, α), and

$$\partial_2(f \otimes I_1 \wedge I_2) = \jmath_1 f \otimes I_2 - \jmath_2 f \otimes I_1 = f_2 \otimes I_2 - f_1 \otimes I_1.$$

Hence, $f_2 \otimes I_2 - f_1 \otimes I_1 \in \operatorname{im}(\partial_2)$. Likewise, every term in the above sum u belongs to $\operatorname{im}(\partial_2)$. Therefore, $\ker(\partial_1) \subseteq \operatorname{im}(\partial_2)$ and so $H_1(C_\bullet(X)(Y)) = 0$. □

Theorem 4.2.4. $H_m(C_\bullet(-)(Y)) = 0$ for all $m \geq 2$.

Proof. We proceed by induction on $|Y|$. If $|Y| = 0$, then $Y = \varnothing$ implies $C_m(-)(\varnothing) = 0$ for all $m \geq 1$, so $H_m(C_\bullet(-)(Y)) = 0$ for all $m \geq 2$ in the base case. Let $N > 0$ and assume $H_m(C_\bullet(X)(Y)) = 0$ for all $m \geq 2$ and $X, Y \in \mathcal{C}$ for which $|Y| < N$. Let $X, Y \in \mathcal{C}$ with $|Y| = N$. We must show $H_m(C_\bullet(X)(Y)) = 0$ for all $m \geq 2$. We may assume $X = ([x_1], ..., [x_n])$ and $Y = ([y_1], ..., [y_n])$, where $\sum_{i=1}^n y_i = N$. Since $N > 0$, we must have $y_i > 0$ for some i. Fix the element $y_i \in Y$.

Define a subcomplex $S_\bullet(X)(Y)$ of $C_\bullet(X)(Y)$ as follows. For each $m \in \mathbb{N}_0$, let $S_m(X)(Y)$ be the k-submodule of $C_m(X)(Y)$ spanned by the direct summands $\underline{\mathcal{C}}(X, Y \backslash I) \otimes_k \det(I)$ such that $y_i \notin I$. Then for a typical basis element $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j \in S_m(X)(Y)$ in a direct summand corresponding to (I, α), we have $(f, P_f, \alpha_f) \in \mathcal{C}(X, Y \backslash I)$, $I = I_1 \sqcup \cdots \sqcup I_m$, $\alpha : \{I_1, ..., I_m\} \to A$, and $y_i \notin I_j$ for any j. Because $y_i \notin I$ implies $y_i \notin I \backslash I_j$ for every $j = 1, ..., m$, we see that ∂_X maps $S_m(X)(Y)$ into $S_{m-1}(X)(Y)$ for each $m \in \mathbb{N}$. Hence, $S_\bullet(X)(Y)$ is a subcomplex of $C_\bullet(X)(Y)$.

For any $m \in \mathbb{N}_0$, given a basis element $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j$ in a direct summand of $S_m(X)(Y)$ corresponding to (I, α), we have either $y_i \in f(X)$ or $y_i \notin f(X)$. If $y_i \in f(X)$, then $f(x) = y_i$ for some unique $x \in [x_i]$ and $\alpha_f(x) \in A$. By restricting the domain of f to $X \backslash \{x\}$ and the domain of α_f to $X \backslash \{x\} \amalg P_f$, we can regard $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j$ as an element of $\bigoplus_{x \in [x_i]} C_m(X \backslash \{x\})(Y \backslash \{y_i\})^{\oplus |A|}$, by identifying it with $(f \mid_{X \backslash \{x\}}, P_f, \alpha_f \mid_{(X \backslash \{x\}) \amalg P_f}) \otimes \bigwedge_{j=1}^m I_j$. On the other hand, if $y_i \notin f(X)$, then $y_i \in S$ for some unique $S \in P_f$ and $\alpha_f(S) \in A$. By restricting the codomain of f to $Y \backslash S$ and the domain of α_f to $X \amalg (P_f \backslash \{S\})$, we can regard $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j$ as an element of

$\bigoplus_S C_m(X)(Y\backslash S)^{\oplus|A|}$, where the direct sum is over all $S \subseteq Y$ such that $\{S\}$ is a partition of type t and $y_i \in S$, by identifying it with $(f, P_f\backslash\{S\}, \alpha_f\mid_{X\amalg(P_f\backslash\{S\})}) \otimes \bigwedge_{j=1}^m I_j$. This gives a map

$$S_m(X)(Y) \to \bigoplus_{x\in[x_i]} C_m(X\backslash\{x\})(Y\backslash\{y_i\})^{\oplus|A|} \oplus \bigoplus_S C_m(X)(Y\backslash S)^{\oplus|A|}.$$

The inverse map

$$\bigoplus_{x\in[x_i]} C_m(X\backslash\{x\})(Y\backslash\{y_i\})^{\oplus|A|} \oplus \bigoplus_S C_m(X)(Y\backslash S)^{\oplus|A|} \to S_m(X)(Y)$$

is defined as follows. Given a basis element $(f, P_f, \alpha_f)\otimes\bigwedge_{j=1}^m I_j$ of a direct summand $C_m(X\backslash\{x\})(Y\backslash\{y_i\})$ corresponding to both $x \in [x_i]$ and $a \in A$, we identify it with $(\overline{f}, P_f, \overline{\alpha_f}) \otimes \bigwedge_{j=1}^m I_j$ in $S_m(X)(Y)$, where $\overline{f}$ extends f by $\overline{f}(x) = y_i$, and $\overline{\alpha_f}$ extends α_f by $\overline{\alpha_f}(x) = a$. On the other hand, given a basis element $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j$ of a direct summand of $C_m(X)(Y\backslash S)$ corresponding to both $S \subseteq Y$ (where $\{S\}$ is a partition of type t and $y_i \in S$) and $a \in A$, we identify it with $(\iota_S, S, \alpha_S)(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j$ in $S_m(X)(Y)$, where $(\iota_S, S, \alpha_S) \in \mathcal{C}((Y\backslash S)\backslash I, Y\backslash I)$ is the inclusion morphism defined by $\alpha_S(S) = a$.

Therefore, we identify the subcomplex $S_\bullet(X)(Y)$ with

$$\bigoplus_{x\in[x_i]} C_\bullet(X\backslash\{x\})(Y\backslash\{y_i\})^{\oplus|A|} \oplus \bigoplus_S C_\bullet(X)(Y\backslash S)^{\oplus|A|}.$$

Consider the quotient complex $C_\bullet(X)(Y)/S_\bullet(X)(Y)$. Since $S_0(X)(Y) = \underline{\mathcal{C}}(X,Y) = C_0(X)(Y)$, we have $C_0(X)(Y)/S_0(X)(Y) = 0$. If $m \in \mathbb{N}$, then a typical basis element in the quotient $C_m(X)(Y)/S_m(X)(Y)$ is represented by $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j$ in $\underline{\mathcal{C}}(X, Y\backslash I) \otimes \det(I)$, with $y_i \in I$. So if

$$(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j + S_m(X)(Y) \in C_m(X)(Y)/S_m(X)(Y),$$

then $y_i \in I_j$ for exactly one j. By removing I_j from I, along with its associated element $\alpha(I_j) \in A$, we can regard $(f, P_f, \alpha_f)\otimes\bigwedge_{j=1}^m I_j + S_m(X)(Y)$ as an element of $\bigoplus_S C_{m-1}(X)(Y\backslash S)^{\oplus|A|}$, the direct sum being over all $S \subseteq Y$ such that $\{S\}$ is a partition of type t and $y_i \in S$, by identifying it with

$$\operatorname{sgn}(\sigma)(f, P_f, \alpha_f) \otimes I_1 \wedge \cdots \widehat{I_j} \cdots \wedge I_m \in \underline{\mathcal{C}}(X, (Y\backslash I_j)\backslash(I\backslash I_j)) \otimes_k \det(I\backslash I_j),$$

where $\sigma = (1, 2, ..., j-1, j)$ is the permutation that moves I_j to the leftmost position in the wedge product and preserves the order of the remaining wedge factors. This gives a map

$$C_m(X)(Y)/S_m(X)(Y) \to \bigoplus_S C_{m-1}(X)(Y \backslash S)^{\oplus |A|}.$$

The inverse map

$$\bigoplus_S C_{m-1}(X)(Y \backslash S)^{\oplus |A|} \to C_m(X)(Y)/S_m(X)(Y)$$

is defined as follows. Given a basis element $(f, P_f, \alpha_f) \otimes I_1 \wedge \cdots \wedge I_{m-1}$ of a direct summand $C_{m-1}(X)(Y \backslash S)$ corresponding to both $S \subseteq Y$ (where $\{S\}$ is a partition of type t and $y_i \in S$) and $a \in A$, we identify it with

$$(f, P_f, \alpha_f) \otimes S \wedge I_1 \wedge \cdots \wedge I_{m-1} + S_m(X)(Y) \in C_m(X)(Y)/S_m(X)(Y).$$

Here, if $(f, P_f, \alpha_f) \otimes I_1 \wedge \cdots \wedge I_{m-1}$ belongs a direct summand of $C_{m-1}(X)(Y \backslash S)$ corresponding to $(I_1 \sqcup \cdots \sqcup I_{m-1}, \alpha)$, then the representative $(f, P_f, \alpha_f) \otimes S \wedge I_1 \wedge \cdots \wedge I_{m-1}$ belongs to the direct summand of $C_m(X)(Y)$ corresponding to $(I_1 \sqcup \cdots \sqcup I_{m-1} \sqcup S, \overline{\alpha})$, where $\overline{\alpha}$ extends α by $\overline{\alpha}(S) = a$. Therefore, we identify the quotient complex $C_\bullet(X)(Y)/S_\bullet(X)(Y)$ with

$$\left(\bigoplus_S C_\bullet(X)(Y \backslash S)^{\oplus |A|} \right) [-1],$$

where $[-1]$ is the shift functor on complexes defined by $(K[-1])_m = K_{m-1}$ for any complex $K_\bullet$.

Thus, we have a short exact sequence of complexes

$$0 \to S_\bullet(X)(Y) \to C_\bullet(X)(Y) \to C_\bullet(X)(Y)/S_\bullet(X)(Y) \to 0,$$

which gives a long exact sequence in homology:

$$\begin{aligned} \cdots \quad \to \quad & H_m(S_\bullet(X)(Y)) \to H_m(C_\bullet(X)(Y)) \to H_m(C_\bullet(X)(Y)/S_\bullet(X)(Y)) \to \cdots \\ \to \quad & H_2(S_\bullet(X)(Y)) \to H_2(C_\bullet(X)(Y)) \to H_2(C_\bullet(X)(Y)/S_\bullet(X)(Y)) \to \cdots . \end{aligned}$$

By the subcomplex identification and the induction hypothesis, we get $H_m(S_\bullet(X)(Y)) = 0$ for all $m \geq 2$. By the quotient complex identification, the induction hypothesis, and since $H_1(C_\bullet(X)(Y)) =$

0 for all $X, Y \in \mathcal{C}$, we obtain $H_m(C_\bullet(X)(Y)/S_\bullet(X)(Y)) = 0$ for all $m \geq 2$. By exactness of the homology sequence, we arrive at $H_m(C_\bullet(X)(Y)) = 0$ for all $m \geq 2$. □

Remark 4.2.5. We will show that $C_\bullet(X)(Y)$ is the mapping cone of a certain morphism of chain complexes, with the same notation of Theorem 4.2.4. For any $m \in \mathbb{N}_0$, consider the direct summand $\bigoplus_S C_m(X)(Y \backslash S)^{\oplus|A|}$ in the degree m part of the subcomplex identification:

$$S_m(X)(Y) = \bigoplus_{x \in [x_i]} C_m(X\backslash\{x\})(Y\backslash\{y_i\})^{\oplus|A|} \oplus \bigoplus_S C_m(X)(Y\backslash S)^{\oplus|A|}.$$

Under this identification, $\bigoplus_S C_m(X)(Y\backslash S)^{\oplus|A|}$ is the k-subspace of $S_m(X)(Y)$ spanned by the elements $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j$ for which $y_i \notin f(X)$. Since $y_i \notin f(X)$ implies $y_i \notin \iota_j f(X)$ for every inclusion morphism $\iota_j \in \mathcal{FI}^n(Y\backslash I, (Y\backslash I) \sqcup \{I_j\})$ $(1 \leq j \leq m)$, $\bigoplus_S C_\bullet(X)(Y\backslash S)^{\oplus|A|}$ is a subcomplex of $S_\bullet(X)(Y)$. For any $m \in \mathbb{N}_0$, let

$$i_m : \bigoplus_S C_m(X)(Y\backslash S)^{\oplus|A|} \hookrightarrow S_m(X)(Y)$$

be the inclusion map. Recall that a basis element $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j$ in a direct summand $C_m(X)(Y\backslash S)$ corresponding to both $S \subseteq Y$ (where $\{S\}$ is a partition of type t and $y_i \in S$) and $a \in A$, gets identified with $(\iota_S, S, \alpha_S)(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j$ in $S_m(X)(Y)$ (where $(\iota, S, \alpha_S) \in \mathcal{C}((Y\backslash S)\backslash I, Y\backslash I)$ is the inclusion morphism defined by $\alpha_S(S) = a$). So

$$i_m((f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j) = (\iota_S, S, \alpha_S)(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^m I_j$$

for all $m \in \mathbb{N}_0$. The diagram

$$\begin{array}{ccc}
\bigoplus_S C_m(X)(Y\backslash S)^{\oplus|A|} & \xrightarrow{\partial} & \bigoplus_S C_{m-1}(X)(Y\backslash S)^{\oplus|A|} \\
\Big\downarrow i_m & & \Big\downarrow i_{m-1} \\
S_m(X)(Y) & \xrightarrow[\partial]{} & S_m(X)(Y)
\end{array}$$

commutes since

$$
\begin{aligned}
(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^{m} I_j &\mapsto \sum_{j=1}^{m} (-1)^{j-1} (\iota_j, I_j, \alpha_j)(f, P_f, \alpha_f) \otimes I_1 \wedge \cdots \widehat{I_j} \cdots \wedge I_m \\
&\mapsto \sum_{j=1}^{m} (-1)^{j-1} (\iota_S, S, \alpha_S)(\iota_j, I_j, \alpha_j)(f, P_f, \alpha_f) \otimes I_1 \wedge \cdots \widehat{I_j} \cdots \wedge I_m
\end{aligned}
$$

around the top right corner, while

$$
\begin{aligned}
(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^{m} I_j &\mapsto (\iota_S, S, \alpha_S)(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^{m} I_j \\
&\mapsto \sum_{j=1}^{m} (-1)^{j-1} (\iota_j, I_j, \alpha_j)(\iota_S, S, \alpha_S)(f, P_f, \alpha_f) \otimes I_1 \wedge \cdots \widehat{I_j} \cdots \wedge I_m
\end{aligned}
$$

around the bottom left corner, and the inclusion morphisms (ι_S, S, α_S), (ι_j, I_j, α_j) commute by Remark 4.1.3. Thus,

$$
i : \bigoplus_S C_\bullet(X)(Y \backslash S)^{\oplus |A|} \hookrightarrow S_\bullet(X)(Y)
$$

is a morphism of complexes.

The mapping cone of $-i$ is the complex

$$
\left(\bigoplus_S C_\bullet(X)(Y \backslash S)^{\oplus |A|} \right) [-1] \oplus S_\bullet(X)(Y)
$$

whose differential d in degree m is given by

$$
d(u', u) = (-\partial(u'), \partial(u) + i_{m-1}(u'))
$$

for $u' \in \bigoplus_S C_{m-1}(X)(Y \backslash S)^{\oplus |A|}$ and $u \in S_m(X)(Y)$. By the quotient complex identification, we have an isomorphism of k-vector spaces

$$
\alpha_m : \bigoplus_S C_{m-1}(X)(Y \backslash S)^{\oplus |A|} \cong C_m(X)(Y)/S_m(X)(Y),
$$

in which a basis element $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^{m-1} I_j$ in a direct summand $C_{m-1}(X)(Y \backslash S)$ corresponding to $S \subseteq Y$ and $a \in A$ gets identified with

$$(f, P_f, \alpha_f) \otimes S \wedge I_1 \wedge \cdots \wedge I_m + S_m(X)(Y)$$

in $C_m(X)(Y)/S_m(X)(Y)$. Also, there is a k-vector space isomorphism

$$\beta_m : (C_m(X)(Y)/S_m(X)(Y)) \oplus S_m(X)(Y) \cong C_m(X)(Y)$$

given by

$$\beta_m(v + S_m(X)(Y), u) = v + u$$

for $v \in C_m(X)(Y)$ and $u \in S_m(X)(Y)$. For any $m \in \mathbb{N}_0$, we define the k-vector space isomorphism

$$e_m : \bigoplus_S C_{m-1}(X)(Y \backslash S)^{\oplus |A|} \oplus S_m(X)(Y) \xrightarrow{\sim} C_m(X)(Y)$$

to be

$$e_m(u', u) = \rho_m \alpha_m(u') + u$$

for $u' \in \bigoplus_S C_{m-1}(X)(Y \backslash S)^{\oplus |A|}$ and $u \in S_m(X)(Y)$, where $\rho_m : C_m(X)(Y)/S_m(X)(Y) \to C_m(X)(Y)$ is the splitting map $\rho_m(v + S_m(X)(Y)) = v$.

Observe that for any $m \in \mathbb{N}_0$, we have

$$e_{m-1}d(u', u) = e_{m-1}(-\partial(u'), \partial(u) + i_{m-1}(u')) = -\rho_m \alpha_m \partial(u') + \partial(u) + i_{m-1}(u')$$

and

$$\partial e_m(u', u) = \partial(\rho_m \alpha_m(u') + u) = \partial \rho_m \alpha_m(u') + \partial(u)$$

for $u' \in \bigoplus_S C_{m-1}(X)(Y \backslash S)^{\oplus |A|}$ and $u \in S_m(X)(Y)$. So $e_{m-1}d = \partial e_m$ provided that

$$\partial \rho_m \alpha_m(u') = -\rho_m \alpha_m \partial(u') + i_{m-1}(u')$$

for $u' \in \bigoplus_S C_{m-1}(X)(Y \backslash S)^{\oplus |A|}$. Take a basis element $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^{m-1} I_j$ in a direct summand of $C_{m-1}(X)(Y\backslash S)$ corresponding to $S \subseteq Y$ and $a \in A$. Then

$$\begin{aligned}\partial\rho_m\alpha_m((f,P_f,\alpha_f)\otimes\bigwedge_{j=1}^{m-1}I_j) &= \partial((f,P_f,\alpha_f)\otimes S\wedge I_1\wedge\cdots\wedge I_{m-1})\\ &= (\iota_S,S,\alpha_S)(f,P_f,\alpha_f)\otimes\bigwedge_{j=1}^{m-1}I_j\\ &\quad+\sum_{j=1}^{m-1}(-1)^j(\iota_j,I_j,\alpha_j)(f,P_f,\alpha_f)\otimes S\wedge I_1\wedge\cdots\widehat{I_j}\cdots\wedge I_{m-1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}-\rho_m\alpha_m\partial((f,P_f,\alpha_f)\otimes\bigwedge_{j=1}^{m-1}I_j) &= -\rho_m\alpha_m(\sum_{j=1}^{m-1}(-1)^{j-1}(\iota_j,I_j,\alpha_j)(f,P_f,\alpha_f)\otimes I_1\wedge\cdots\widehat{I_j}\cdots\wedge I_{m-1})\\ &= \sum_{j=1}^{m-1}(-1)^j(\iota_j,I_j,\alpha_j)(f,P_f,\alpha_f)\otimes S\wedge I_1\wedge\cdots\widehat{I_j}\cdots\wedge I_{m-1}\end{aligned}$$

and

$$i_{m-1}((f,P_f,\alpha_f)\otimes\bigwedge_{j=1}^{m-1}I_j) = (\iota_S,S,\alpha_S)(f,P_f,\alpha_f)\otimes\bigwedge_{j=1}^{m-1}I_j.$$

It follows that $e_{m-1}d = \partial e_m$ for all $m \in \mathbb{N}_0$, which means the diagram

$$\begin{array}{ccc}\bigoplus_S C_{m-1}(X)(Y\backslash S)^{\oplus|A|}\oplus S_m(X)(Y) & \xrightarrow{d} & \bigoplus_S C_{m-2}(X)(Y\backslash S)^{\oplus|A|}\oplus S_{m-1}(X)(Y)\\ \downarrow e_m & & \downarrow e_{m-1}\\ C_m(X)(Y) & \xrightarrow[\partial]{} & C_{m-1}(X)(Y)\end{array}$$

commutes. Hence, $\left(\bigoplus_S C_\bullet(X)(Y\backslash S)^{\oplus|A|}\right)[-1]\oplus S_\bullet(X)(Y)$ and $C_\bullet(X)(Y)$ are isomorphic as complexes of k-vector spaces. Therefore, $C_\bullet(X)(Y)$ is the mapping cone of $-i$.

4.3. $C_\bullet(-)(Y)$ is a resolution of $\underline{\mathcal{C}}(Y,Y)$.

We have a complex of graded right $\underline{\mathcal{C}}$-modules

$$\cdots\to C_m(-)(Y)\to C_{m-1}(-)(Y)\to\cdots\to C_1(-)(Y)\to\underline{\mathcal{C}}(-,Y)\to 0$$

which is exact at each $m \in \mathbb{N}$, and in which $C_m(-)(Y)$ is projective and generated in degree m for all $m \in \mathbb{N}_0$.

Consider the augmented sequence

$$\cdots \to C_m(-)(Y) \to C_{m-1}(-)(Y) \to \cdots \to C_1(-)(Y) \xrightarrow{\partial} \underline{\mathcal{C}}(-,Y) \xrightarrow{\varepsilon} \underline{\mathcal{C}}(Y,Y) \to 0$$

in which $\varepsilon : \underline{\mathcal{C}}(-,Y) \to \underline{\mathcal{C}}(Y,Y)$ is the graded right $\underline{\mathcal{C}}$-module homomorphism defined for each $X \in \mathcal{C}$ by

$$\varepsilon_X = \begin{cases} 1_{\underline{\mathcal{C}}(Y,Y)} & \text{if } X = Y \\ 0 & \text{if } X \neq Y. \end{cases}$$

The augmented sequence is exact at $\underline{\mathcal{C}}(Y,Y)$ because ε_X is surjective for each $X \in \mathcal{C}$.

Proposition 4.3.1. The augmented sequence is exact at $\underline{\mathcal{C}}(-,Y)$.

Proof. Let $X \in \mathcal{C}$. We must show that $\mathrm{im}(\partial_X) = \ker(\varepsilon_X)$.

For $(\subseteq)$, take a basis element $(f, P_f, \alpha_f) \otimes I$ in a direct summand of $C_1(X)(Y)$ corresponding to (I, α), with $\varnothing \neq I \subseteq Y$ and $\alpha(I) \in A$. Then $\partial_X((f, P_f, \alpha_f) \otimes I) = (\iota, I, \alpha_\iota)(f, P_f, \alpha_f)$, where $(\iota, I, \alpha_\iota) \in \mathcal{C}(Y \backslash I, Y)$ is the inclusion morphism defined by $\alpha_\iota(I) = \alpha(I)$. Since $\underline{\mathcal{C}}$ is directed, we have $X \subseteq Y \backslash I \subseteq Y$. But then $X \neq Y$, because otherwise $Y \backslash I = Y$ contradicts $I \neq \varnothing$. So $(\iota, I, \alpha_\iota)(f, P_f, \alpha_f) \in \mathcal{C}(X,Y)$ with $X \neq Y$, hence $\varepsilon_X((\iota, I, \alpha_\iota)(f, P_f, \alpha_f)) = 0$. Thus, $\mathrm{im}(\partial_X) \subseteq \ker(\varepsilon_X)$.

For $(\supseteq)$, let $(f, P_f, \alpha_f) \in \mathcal{C}(X,Y)$ be a basis element such that $\varepsilon_X((f, P_f, \alpha_f)) = 0$. If $X = Y$, then ε_X is the identity map on $\underline{\mathcal{C}}(Y,Y)$, which forces $(f, P_f, \alpha_f) = 0$ and contradicts that (f, P_f, α_f) was a basis element. So $X \neq Y$ and hence $X \subset Y$ because $\underline{\mathcal{C}}$ is directed. We claim that (f, P_f, α_f) has a preimage in $C_1(X)(Y)$ under ∂_X. Since $X \subset Y$, there exists $I \in P_f$ and $\alpha_f(I) \in A$. By restricting the codomain of $f : X \to Y$ to $Y \backslash I$ and the domain of $\alpha_f : X \amalg P_f \to A$ to $X \amalg (P_f \backslash \{I\})$, we get a morphism $(f, P_f \backslash \{I\}, \alpha_f \mid_{X \amalg (P_f \backslash \{I\})}) \in \mathcal{C}(X, Y \backslash I)$. Then $(f, P_f \backslash \{I\}, \alpha_f \mid_{X \amalg (P_f \backslash \{I\})}) \otimes I$ belongs to the direct summand of $C_1(X)(Y)$ corresponding to $(I, \alpha_f|_{\{I\}})$ and

$$\partial_X((f, P_f \backslash \{I\}, \alpha_f \mid_{X \amalg (P_f \backslash \{I\})}) \otimes I) = (\iota, I, \alpha_\iota)(f, P_f \backslash \{I\}, \alpha_f \mid_{X \amalg (P_f \backslash \{I\})}) = (f, P_f, \alpha_f),$$

where $(\iota, I, \alpha_\iota) \in \mathcal{C}(Y\backslash I, Y)$ is the inclusion morphism defined by $\alpha_\iota(I) = \alpha_f(I)$. So $(f, P_f, \alpha_f) \in \mathrm{im}(\partial_X)$, and hence $\mathrm{im}(\partial_X) \supseteq \ker(\varepsilon_X)$.

Therefore, $\mathrm{im}(\partial_X) = \ker(\varepsilon_X)$ and the augmented complex is exact at $\underline{\mathcal{C}}(-, Y)$. $\square$

This completes the construction of our linear projective resolution

$$\cdots \to C_m(-)(Y) \to C_{m-1}(-)(Y) \to \cdots \to C_1(-)(Y) \to \underline{\mathcal{C}}(-, Y) \to \underline{\mathcal{C}}(Y, Y) \to 0$$

of $\underline{\mathcal{C}}(Y, Y)$ in $\underline{\mathcal{C}}$-gMod. We conclude that the category $\underline{\mathcal{C}}$ is Koszul, summarized in the Corollary below.

Corollary 4.3.2. Let $n \in \mathbb{N}$, t be a partition type, A be a finite abelian group, and k be a field of characteristic 0. Then the k-linearization of $\mathcal{FI}^n_{t,A}$ is Koszul.

Remark 4.3.3. If we only allow left modules in the definition of Koszulity, then what we have proven is that the k-linearization of $(\mathcal{FI}^n_{t,A})^{op}$ is Koszul. By [GL, Proposition 3.5], the k-linearization of $\mathcal{FI}^n_{t,A}$ is Koszul.

5. The Quadratic Dual

5.1. Preliminaries.

Let $\mathcal{C}$ be a category such that $\mathcal{C}(X, X)$ is a group for any $X \in \mathcal{C}$ (in other words, $\mathcal{C}$ is an EI category, one whose endomorphisms are isomorphisms). Let k be a field, and $\underline{\mathcal{C}}$ be the k-linearization of $\mathcal{C}$. Assume that $\underline{\mathcal{C}}$ is a directed graded k-linear category.

Remark 5.1.1. Let $X, Y \in \mathcal{C}$. If V is a $(\underline{\mathcal{C}}(Y, Y), \underline{\mathcal{C}}(X, X))$-bimodule, then V^* is a $(\underline{\mathcal{C}}(X, X), \underline{\mathcal{C}}(Y, Y))$-bimodule with left action given by k-linear extension of $(\sigma \cdot v^*)(v') = v^*(v'\sigma)$ for $\sigma \in \mathcal{C}(X, X)$, and right action given by k-linear extension of $(v^* \cdot \tau)(v') = v^*(\tau v')$ for $\tau \in \mathcal{C}(Y, Y)$, where $v^* \in V^*$ and $v' \in V$. If $\phi : V \to W$ is a $(\underline{\mathcal{C}}(Y, Y), \underline{\mathcal{C}}(X, X))$-bimodule homomorphism, then $\phi^* : W^* \to V^*$ defined by $\phi^*(w^*) = w^*\phi$ is a $(\underline{\mathcal{C}}(X, X), \underline{\mathcal{C}}(Y, Y))$-bimodule homomorphism.

In particular, $\underline{\mathcal{C}}(X,Y)_1$ is a $(\underline{\mathcal{C}}(Y,Y),\underline{\mathcal{C}}(X,X))$-bimodule by composition of morphisms, so $\underline{\mathcal{C}}(X,Y)_1^*$ is a $(\underline{\mathcal{C}}(X,X),\underline{\mathcal{C}}(Y,Y))$-bimodule. Also, for any $X,Y,Z \in \mathcal{C}$, the composition map

$$\gamma_{XYZ} : \underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1 \to \underline{\mathcal{C}}(X,Z)_2$$

is a $(\underline{\mathcal{C}}(Z,Z),\underline{\mathcal{C}}(X,X))$-bimodule homomorphism, thus

$$\gamma_{XYZ}^* : \underline{\mathcal{C}}(X,Z)_2^* \to (\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1)^*$$

is a $(\underline{\mathcal{C}}(X,X),\underline{\mathcal{C}}(Z,Z))$-bimodule homomorphism.

Remark 5.1.2. Let $X,Y \in \mathcal{C}$, and V be a $(\underline{\mathcal{C}}(Y,Y),\underline{\mathcal{C}}(X,X))$-bimodule that is finite dimensional as a k-vector space. Let $\{e_1,...,e_n\}$ be a basis of V, and $\{e_1^*,...,e_n^*\}$ be the dual basis of V^*, so that $V \cong V^*$ as k-vector spaces by the correspondence $e_i \leftrightarrow e_i^*$. Then V^* is a $(\underline{\mathcal{C}}(X,X),\underline{\mathcal{C}}(Y,Y))$-bimodule in the following way. Define the left action of $\underline{\mathcal{C}}(X,X)$ on V^* by k-linear extension of $\sigma \cdot v^* = (v\sigma^{-1})^*$ for $\sigma \in \mathcal{C}(X,X)$, and the right action of $\underline{\mathcal{C}}(Y,Y)$ on V^* by k-linear extension of $v^* \cdot \tau = (\tau^{-1}v)^*$ for $\tau \in \mathcal{C}(Y,Y)$, where $v \leftrightarrow v^*$ correspond under the isomorphism $V \cong V^*$. These actions are compatible since

$$(\sigma \cdot v^*) \cdot \tau = (v\sigma^{-1})^* \cdot \tau = (\tau^{-1}(v\sigma^{-1}))^* = ((\tau^{-1}v)\sigma^{-1})^* = \sigma \cdot (\tau^{-1}v)^* = \sigma \cdot (v^* \cdot \tau).$$

Note 5.1.3. If $X,Y \in \mathcal{C}$ and V is a $(\underline{\mathcal{C}}(Y,Y),\underline{\mathcal{C}}(X,X))$-bimodule that is finite dimensional over k, then in order to distinguish between the two ways in which V^* is a $(\underline{\mathcal{C}}(X,X),\underline{\mathcal{C}}(Y,Y))$-bimodule, let us call the bimodule action of Remark 5.1.1 the “natural” action, and the bimodule action of Remark 5.1.2 the “inverse” action.

Proposition 5.1.4. Let $X,Y,Z \in \mathcal{C}$. Suppose V is a $(\underline{\mathcal{C}}(Y,Y),\underline{\mathcal{C}}(X,X))$-bimodule and W is a $(\underline{\mathcal{C}}(Z,Z),\underline{\mathcal{C}}(Y,Y))$-bimodule, both of which are finite dimensional over k. Then $(W \otimes_{\underline{\mathcal{C}}(Y,Y)} V)^* \cong V^* \otimes_{\underline{\mathcal{C}}(Y,Y)} W^*$ as $(\underline{\mathcal{C}}(X,X),\underline{\mathcal{C}}(Z,Z))$-bimodules under the inverse action.

Proof. Define

$$\varphi : (W \otimes_{\underline{\mathcal{C}}(Y,Y)} V)^* \to V^* \otimes_{\underline{\mathcal{C}}(Y,Y)} W^*$$

by k-linear extension of the assignment $(w \otimes v)^* \mapsto v^* \otimes w^*$. To see that φ is well defined, note that

$$\varphi((w\alpha \otimes v)^*) = v^* \otimes (w\alpha)^* = v^* \otimes \alpha^{-1} \cdot w^* = v^* \cdot \alpha^{-1} \otimes w^* = (\alpha v)^* \otimes w^* = \varphi((w \otimes \alpha v)^*)$$

for any $\alpha \in \mathcal{C}(Y,Y)$. To see that φ is a $(\underline{\mathcal{C}}(X,X), \underline{\mathcal{C}}(Z,Z))$-bimodule homomorphism, observe that

$$\sigma \cdot (w \otimes v)^* = ((w \otimes v)\sigma^{-1})^* = (w \otimes v\sigma^{-1})^* \mapsto (v\sigma^{-1})^* \otimes w^* = (\sigma \cdot v^*) \otimes w^* = \sigma \cdot (v^* \otimes w^*)$$

for any $\sigma \in \mathcal{C}(X,X)$, and

$$(w \otimes v)^* \cdot \tau = (\tau^{-1}(w \otimes v))^* = (\tau^{-1} w \otimes v)^* \mapsto v^* \otimes (\tau^{-1} w)^* = v^* \otimes (w^* \cdot \tau) = (v^* \otimes w^*) \cdot \tau$$

for any $\tau \in \mathcal{C}(Z,Z)$. Next, define

$$\psi : V^* \otimes_{\underline{\mathcal{C}}(Y,Y)} W^* \to (W \otimes_{\underline{\mathcal{C}}(Y,Y)} V)^*$$

by k-linear extension of the assignment $v^* \otimes w^* \mapsto (w \otimes v)^*$. Then ψ well defined since

$$\psi((v^* \cdot \alpha) \otimes w^*) = \psi((\alpha^{-1} v)^* \otimes w^*) = (w \otimes \alpha^{-1} v)^* = (w\alpha^{-1} \otimes v)^* = \psi(v^* \otimes (w\alpha^{-1})^*) = \psi(v^* \otimes (\alpha \cdot w^*))$$

for any $\alpha \in \mathcal{C}(Y,Y)$. Also, ψ is a $(\underline{\mathcal{C}}(X,X), \underline{\mathcal{C}}(Z,Z))$-bimodule homomorphism because

$$\sigma \cdot (v^* \otimes w^*) = (\sigma \cdot v^*) \otimes w^* = (v\sigma^{-1})^* \otimes w^* \mapsto (w \otimes v\sigma^{-1})^* = ((w \otimes v)\sigma^{-1})^* = \sigma \cdot (w \otimes v)^*$$

for any $\sigma \in \mathcal{C}(X,X)$, and

$$(v^* \otimes w^*) \cdot \tau = v^* \otimes (w^* \cdot \tau) = v^* \otimes (\tau^{-1} w)^* \mapsto (\tau^{-1} w \otimes v)^* = (\tau^{-1}(w \otimes v))^* = (w \otimes v)^* \cdot \tau$$

for any $\tau \in \mathcal{C}(Z,Z)$. Thus, $(W \otimes_{\underline{\mathcal{C}}(Y,Y)} V)^* \cong V^* \otimes_{\underline{\mathcal{C}}(Y,Y)} W^*$ as $(\underline{\mathcal{C}}(X,X), \underline{\mathcal{C}}(Z,Z))$-bimodules under the inverse action. $\square$

Remark 5.1.5. In particular, for any $X, Y, Z \in \mathcal{C}$, $(\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1)^* \cong \underline{\mathcal{C}}(X,Y)_1^* \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)_1^*$ as $(\underline{\mathcal{C}}(X,X), \underline{\mathcal{C}}(Z,Z))$-bimodules under the inverse action. Moreover, it follows by induction that

$$(\underline{\mathcal{C}}(Y_i,Z)_1 \otimes_{\underline{\mathcal{C}}(Y_i,Y_i)} \cdots \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \underline{\mathcal{C}}(X,Y_1)_1)^* \cong \underline{\mathcal{C}}(X,Y_1)_1^* \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_i,Y_i)} \underline{\mathcal{C}}(Y_i,Z)_1^*$$

as $(\underline{\mathcal{C}}(X,X), \underline{\mathcal{C}}(Z,Z))$-bimodules under the inverse action, for any $X, Y_1, ..., Y_i, Z \in \mathcal{C}$, $i \geq 1$.

Proposition 5.1.6. Let $X, Y, Z \in \mathcal{C}$. Then $\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1$ is a $(\underline{\mathcal{C}}(Z,Z), \underline{\mathcal{C}}(X,X))$-bimodule that is finite dimensional over k, and the natural and inverse actions on $(\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1)^*$ coincide.

Proof. $\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1$ is spanned by the finite set

$$S = \{g \otimes f \,:\, g \in \mathcal{C}(Y,Z),\ f \in \mathcal{C}(X,Y) \text{ are morphisms of degree } 1\},$$

so S contains a basis of $\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1$. To show the natural and inverse actions on $(\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1)^*$ coincide, it is enough to check this on basis elements. Let $g, g' \in \mathcal{C}(Y,Z)$, $f, f' \in \mathcal{C}(X,Y)$ be morphisms of degree 1 and let $\sigma \in \mathcal{C}(X,X)$, $\tau \in \mathcal{C}(Y,Y)$. Under the natural action, we have

$$(\sigma \cdot (g \otimes f)^*)(g' \otimes f') = (g \otimes f)^*(g' \otimes f'\sigma) = \delta_{g\otimes f, g'\otimes f'\sigma}$$

and

$$((g \otimes f)^* \cdot \tau)(g' \otimes f') = (g \otimes f)^*(\tau g' \otimes f') = \delta_{g\otimes f, \tau g'\otimes f'}.$$

On the other hand, under the inverse action, we have

$$(\sigma \cdot (g \otimes f)^*)(g' \otimes f') = (g \otimes f\sigma^{-1})^*(g' \otimes f') = \delta_{g\otimes f\sigma^{-1}, g'\otimes f'}$$

and

$$((g \otimes f)^* \cdot \tau)(g' \otimes f') = (\tau^{-1} g \otimes f)^*(g' \otimes f') = \delta_{\tau^{-1}g\otimes f, g'\otimes f'}.$$

But $\delta_{g\otimes f, g'\otimes f'\sigma} = \delta_{g\otimes f\sigma^{-1}, g'\otimes f'}$ and $\delta_{g\otimes f, \tau g'\otimes f'} = \delta_{\tau^{-1}g\otimes f, g'\otimes f'}$, so the natural and inverse actions coincide. □

Therefore, we may unambiguously identify $\operatorname{im}(\gamma^*_{XYZ})$ as a $(\underline{\mathcal{C}}(X,X),\underline{\mathcal{C}}(Z,Z))$-subbimodule of $\underline{\mathcal{C}}(X,Y)^*_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)^*_1$, where

$$\gamma^*_{XYZ} : \underline{\mathcal{C}}(X,Z)^*_2 \to (\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1)^* \cong \underline{\mathcal{C}}(X,Y)^*_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)^*_1.$$

5.2. The category $\underline{\mathcal{C}}^!$.

Definition 5.2.1. Let $\underline{\mathcal{C}}$ be a directed graded k-linear category. Recall that $A = \bigoplus_{i\geq 0} \underline{\mathcal{C}}_i$ is a graded k-algebra, where $\underline{\mathcal{C}}_i = \bigoplus_{X,Y\in\underline{\mathcal{C}}} \underline{\mathcal{C}}(X,Y)_i$. Note that $\underline{\mathcal{C}}_0$ is a ring, with multiplicative identity $\sum_{X\in\underline{\mathcal{C}}} 1_X$ if and only if $|\mathrm{Ob}(\underline{\mathcal{C}})|$ is finite. Since $\underline{\mathcal{C}}_1$ is a $(\underline{\mathcal{C}}_0,\underline{\mathcal{C}}_0)$-bimodule, we can form the graded k-algebra

$$T = \underline{\mathcal{C}}_0 \oplus \underline{\mathcal{C}}_1 \oplus (\underline{\mathcal{C}}_1 \otimes_{\underline{\mathcal{C}}_0} \underline{\mathcal{C}}_1) \oplus \cdots .$$

Let $\gamma : T \to A$ be the map induced by composition of morphisms. It is a graded k-algebra homomorphism, and is surjective by condition (A6). Let $K = \ker(\gamma)$, which is a graded ideal of T. If K is generated by its degree 2 component $K_2 = K \cap (\underline{\mathcal{C}}_1 \otimes_{\underline{\mathcal{C}}_0} \underline{\mathcal{C}}_1)$, then we call $\underline{\mathcal{C}}$ a *quadratic* category.

Remark 5.2.2. If a directed graded k-linear category $\underline{\mathcal{C}}$ is Koszul, then it is quadratic by [GL, Proposition 3.10].

Remark 5.2.3. Let $R = \bigoplus_{i\geq 0} R_i$ be a graded ring, and $a, b \in R_0$. Then $aRb = \bigoplus_{i\geq 0} aR_ib$ is a graded subring of R. Let $S \subseteq R_i$ for some $i \geq 0$, and I be the ideal of R generated by S. Then $I = \bigoplus_{i\geq 0}(I \cap R_i)$ is a graded ideal of R. Put $I_i = I \cap R_i$ for each $i \geq 0$. Then $aIb = \bigoplus_{i\geq 0} aI_ib$ is a graded ideal of aRb, and hence admits a graded quotient ring $aRb/aIb = \bigoplus_{i\geq 0}(aR_ib/aI_ib)$.

Let $\mathcal{C}$ be an EI category, and $\underline{\mathcal{C}}$ be the k-linearization of $\mathcal{C}$. Assume that $\underline{\mathcal{C}}$ is a directed graded k-linear category. Recall that $\underline{\mathcal{C}}$ is skeletal, by Note 2.1.2.

Definition 5.2.4. Let $\underline{\mathcal{C}}_1^* = \bigoplus_{X,Y\in\mathcal{C}} \underline{\mathcal{C}}(X,Y)_1^*$. Since $\underline{\mathcal{C}}(X,Y)_1^*$ is a $(\underline{\mathcal{C}}(X,X), \underline{\mathcal{C}}(Y,Y))$-bimodule for any $X, Y \in \mathcal{C}$, it follows that $\underline{\mathcal{C}}_1^*$ is a $(\underline{\mathcal{C}}_0, \underline{\mathcal{C}}_0)$-bimodule. So we can form the graded k-algebra

$$R = \underline{\mathcal{C}}_0 \oplus \underline{\mathcal{C}}_1^* \oplus (\underline{\mathcal{C}}_1^* \otimes_{\underline{\mathcal{C}}_0} \underline{\mathcal{C}}_1^*) \oplus \cdots .$$

Put $R_0 = \underline{\mathcal{C}}_0$ and $R_i = \underline{\mathcal{C}}_1^* \otimes_{\underline{\mathcal{C}}_0} \cdots \otimes_{\underline{\mathcal{C}}_0} \underline{\mathcal{C}}_1^*$ (i factors) for $i > 0$, so that $R = \bigoplus_{i\geq 0} R_i$. Note that $R_0 = \bigoplus_{X\in\mathcal{C}} \underline{\mathcal{C}}(X,X)_0$ by condition (A3), and $\underline{\mathcal{C}}(X,X) = \underline{\mathcal{C}}(X,X)_0$ for any $X \in \mathcal{C}$ by condition (A4). So for any $X, Z \in \mathcal{C}$, we have $1_X, 1_Z \in R_0$. Hence, $1_X R 1_Z = \bigoplus_{i\geq 0} 1_X R_i 1_Z$ is a graded k-subalgebra of R. Notice that

$$1_X R_0 1_Z = \begin{cases} \underline{\mathcal{C}}(X,X) & \text{if } X = Z \\ 0 & \text{if } X \neq Z, \end{cases}$$

$$1_X R_1 1_Z = \underline{\mathcal{C}}(X,Z)_1^*,$$

$$1_X R_2 1_Z = \bigoplus_{Y\in\mathcal{C}} \underline{\mathcal{C}}(X,Y)_1^* \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)_1^*,$$

and for $i > 2$, we have

$$1_X R_i 1_Z = \bigoplus_{Y_1,\ldots,Y_{i-1}\in\mathcal{C}} \underline{\mathcal{C}}(X,Y_1)_1^* \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \underline{\mathcal{C}}(Y_{i-1},Z)_1^*.$$

Let $S = \bigcup_{X,Y,Z\in\mathcal{C}} \operatorname{im}(\gamma_{XYZ}^*)$. Since

$$R_2 = \bigoplus_{X,Y,Z\in\mathcal{C}} \underline{\mathcal{C}}(X,Y)_1^* \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)_1^*,$$

we have $S \subseteq R_2$. Let I be the ideal of R generated by S. Then $I = \bigoplus_{i\geq 0} I_i$ is a graded ideal of R, where $I_i = I \cap R_i$ for all $i \geq 0$; in particular, $I_0 = I_1 = 0$. So $1_X I 1_Z = \bigoplus_{i\geq 0} 1_X I_i 1_Z$ is a graded ideal of $1_X R 1_Z$, and hence admits a graded quotient k-algebra $1_X R 1_Z / 1_X I 1_Z = \bigoplus_{i\geq 0} (1_X R_i 1_Z / 1_X I_i 1_Z)$.

The *quadratic dual* of $\underline{\mathcal{C}}$ is the k-linear category $\underline{\mathcal{C}}^!$ having the same objects as $\underline{\mathcal{C}}$ and morphisms defined by

$$\underline{\mathcal{C}}^!(Z,X) = 1_X R 1_Z / 1_X I 1_Z = \bigoplus_{i\geq 0} (1_X R_i 1_Z / 1_X I_i 1_Z)$$

for any $X, Z \in \mathcal{C}$. For any $X, Y, Z \in \mathcal{C}$, the composition map $\underline{\mathcal{C}}^!(Y,X) \otimes_k \underline{\mathcal{C}}^!(Z,Y) \to \underline{\mathcal{C}}^!(Z,X)$ is defined by

$$(1_X r 1_Y + 1_X I 1_Y)(1_Y s 1_Z + 1_Y I 1_Z) = 1_X rs 1_Z + 1_X I 1_Z.$$

To see this is well-defined, suppose

$$1_X r 1_Y + 1_X I 1_Y = 1_X r' 1_Y + 1_X I 1_Y \text{ and } 1_Y s 1_Z + 1_Y I 1_Z = 1_Y s' 1_Z + 1_Y I 1_Z.$$

Then

$$1_X r 1_Y - 1_X r' 1_Y = 1_X (r - r') 1_Y \in 1_X I 1_Y \text{ and } 1_Y s 1_Z - 1_Y s' 1_Z = 1_Y (s - s') 1_Z \in 1_Y I 1_Z$$

implies $r - r', s - s' \in I$. So

$$\begin{aligned} 1_X rs 1_Z - 1_X r's' 1_Z &= 1_X rs 1_Z - 1_X rs' 1_Z + 1_X rs' 1_Z - 1_X ss' 1_Z \\ &= 1_X r(s - s') 1_Z + 1_X (r - r') s' 1_Z \end{aligned}$$

belongs to $1_X I 1_Z$, and hence $1_X rs 1_Z + 1_X I 1_Z = 1_X r's' 1_Z + 1_X I 1_Z$. For any $X \in \mathcal{C}$, the identity morphism in $\underline{\mathcal{C}}^!(X, X)$ is $1_X + 1_X I 1_X$.

Proposition 5.2.5. $\underline{\mathcal{C}}^!$ is a directed graded k-linear category.

Proof. Let $X, Z \in \mathcal{C}$. Recall that

$$\underline{\mathcal{C}}^!(Z, X) = 1_X R 1_Z / 1_X I 1_Z = \bigoplus_{i \geq 0} (1_X R_i 1_Z / 1_X I_i 1_Z).$$

Put $\underline{\mathcal{C}}^!(Z, X)_i = 1_X R_i 1_Z / 1_X I_i 1_Z$ for each $i \geq 0$. Then

$$\underline{\mathcal{C}}^!(Z, X) = \bigoplus_{i \geq 0} \underline{\mathcal{C}}^!(Z, X)_i$$

is a graded k-vector space. Notice that

$$\underline{\mathcal{C}}^!(Z, X)_0 = \begin{cases} \underline{\mathcal{C}}(X, X) & \text{if } X = Z \\ 0 & \text{if } X \neq Z, \end{cases}$$

$$\underline{\mathcal{C}}^!(Z,X)_1 = \underline{\mathcal{C}}(X,Z)_1^*,$$

$$\underline{\mathcal{C}}^!(Z,X)_2 = \bigoplus_{Y\in\mathcal{C}}(\underline{\mathcal{C}}(X,Y)_1^* \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)_1^*/\mathrm{im}(\gamma_{XYZ}^*))$$

and for $i > 2$,

$$\underline{\mathcal{C}}^!(Z,X)_i = \bigoplus_{Y_1,\ldots,Y_{i-1}\in\mathcal{C}}((\underline{\mathcal{C}}(X,Y_1)_1^* \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \underline{\mathcal{C}}(Y_{i-1},Z)_1^*)/W_i),$$

with

$$W_i = \sum_{j=1}^{i-1} \underline{\mathcal{C}}(X,Y_1)_1^* \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_{j-1},Y_{j-1})} \mathrm{im}(\gamma^*_{Y_{j-1}Y_jY_{j+1}}) \otimes_{\underline{\mathcal{C}}(Y_{j+1},Y_{j+1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \underline{\mathcal{C}}(Y_{i-1},Z)_1^*$$

in which $X = Y_0$ and $Z = Y_i$. For any $X, Y, Z \in \mathcal{C}$, the composite of a degree i morphism $1_X r_i 1_Y + 1_X I_i 1_Y$ in $\underline{\mathcal{C}}^!(Y,X)_i$ and a degree j morphism $1_Y r_j 1_Z + 1_Y I_j 1_Z$ in $\underline{\mathcal{C}}^!(Z,Y)_j$ is the degree $i+j$ morphism $1_X r_i r_j 1_Z + 1_X I_{i+j} 1_Z$ in $\underline{\mathcal{C}}^!(Z,X)_{i+j}$. Hence, the composition map $\underline{\mathcal{C}}^!(Y,X) \otimes_k \underline{\mathcal{C}}^!(Z,Y) \to \underline{\mathcal{C}}^!(Z,X)$ is a morphism in k-gMod. It follows that $\underline{\mathcal{C}}^!$ is a graded k-linear category.

Since $\underline{\mathcal{C}}$ is directed, Ob($\underline{\mathcal{C}}$) is partially ordered by $\subseteq$ such that whenever $\underline{\mathcal{C}}(X,Y) \neq 0$, we have $X \subseteq Y$. We partially order Ob($\underline{\mathcal{C}}^!$) by $\supseteq$, and claim that whenever $\underline{\mathcal{C}}^!(Z,X) \neq 0$, we have $Z \supseteq X$. To show this, let $X, Z \in \mathcal{C}$ and suppose $\underline{\mathcal{C}}^!(Z,X) \neq 0$. Then we must have a nonzero morphism $u \in \underline{\mathcal{C}}^!(Z,X)_i$ for some $i \geq 0$. If $i = 0$, then $\underline{\mathcal{C}}^!(Z,X)_0 \neq 0$ forces $X = Z$, so $Z \supseteq X$. If $i = 1$, consider $\underline{\mathcal{C}}^!(Z,X)_1 = \underline{\mathcal{C}}(X,Z)_1^*$. Since $\underline{\mathcal{C}}(X,Z)_1$ is finite dimensional as a k-vector space by condition (A1), $\underline{\mathcal{C}}(X,Z)_1^* \cong \underline{\mathcal{C}}(X,Z)_1$ as k-vector spaces. Hence, $\underline{\mathcal{C}}(X,Z) \neq 0$ implies $X \subseteq Z$, so $Z \supseteq X$. If $i \geq 2$, then there exist $Y_1, \ldots, Y_{i-1} \in \mathcal{C}$ and nonzero morphisms $u_1 \in \underline{\mathcal{C}}(X,Y_1)_1^* \cong \underline{\mathcal{C}}(X,Y_1)_1$,...,$u_i \in \underline{\mathcal{C}}(Y_{i-1},Z)_1^* \cong \underline{\mathcal{C}}(Y_{i-1},Z)_1$. So the composite $u_i \cdots u_1 \in \underline{\mathcal{C}}(X,Z)_i$ is nonzero. Hence, $\underline{\mathcal{C}}(X,Z) \neq 0$ implies $X \subseteq Z$, so $Z \supseteq X$. Thus, $\underline{\mathcal{C}}^!$ is directed.

It remains to verify conditions (A1)-(A7) for $\underline{\mathcal{C}}^!$.

(A1) Let $X, Z \in \mathcal{C}$. We must show that $\underline{\mathcal{C}}^!(Z,X)$ is finite dimensional as a k-vector space. Define the graded k-linear category $\underline{\mathcal{C}}^*$ by setting Ob($\underline{\mathcal{C}}^*$) = Ob($\underline{\mathcal{C}}$) and $\underline{\mathcal{C}}^*(Y,W) = 1_W R 1_Y$ for any $W, Y \in \mathcal{C}$, where

$$R = \underline{\mathcal{C}}_0 \oplus \underline{\mathcal{C}}_1^* \oplus (\underline{\mathcal{C}}_1^* \otimes_{\underline{\mathcal{C}}_0} \underline{\mathcal{C}}_1^*) \oplus \cdots.$$

It suffices to show that $\underline{\mathcal{C}}^*(Z,X)$ is finite dimensional. $\underline{\mathcal{C}}^*$ is directed by $\supseteq$ in the same way as $\underline{\mathcal{C}}^!$.

Let $\underline{\mathcal{D}}$ be the convex hull of $\{X, Z\}$ in $\underline{\mathcal{C}}^*$. Then $\underline{\mathcal{D}}(Z,X) = \bigoplus_{i \geq 0} \underline{\mathcal{D}}(Z,X)_i$, where

$$\underline{\mathcal{D}}(Z,X)_0 = \begin{cases} \underline{\mathcal{C}}(X,X) & \text{if } X = Z \\ 0 & \text{if } X \neq Z, \end{cases}$$

$$\underline{\mathcal{D}}(Z,X)_1 = \underline{\mathcal{C}}(X,Z)_1^*,$$

and for $i \geq 2$,

$$\underline{\mathcal{D}}(Z,X)_i = \bigoplus_{Y_1,\dots,Y_{i-1} \in \underline{\mathcal{D}}} ((\underline{\mathcal{C}}(X,Y_1)_1^* \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \underline{\mathcal{C}}(Y_{i-1},Z)_1^*).$$

For any finite set $S \subseteq \mathrm{Ob}(\mathcal{C})$, the convex hull of S in $\underline{\mathcal{C}}^*$ is the same as the convex hull of S in $\underline{\mathcal{C}}$. By condition (A7) for $\underline{\mathcal{C}}$, $\underline{\mathcal{D}}$ has only finitely many objects. Hence, each $\underline{\mathcal{D}}(Z,X)_i$ is finite dimensional and $\underline{\mathcal{D}}(Z,X)_i = 0$ for all $i > |\mathrm{Ob}(\underline{\mathcal{D}})| + 1$. Thus, $\underline{\mathcal{C}}^*(Z,X) = \underline{\mathcal{D}}(Z,X)$ is finite dimensional.

(A2) Let $X \in \mathcal{C}$. By condition (A4) for $\underline{\mathcal{C}}^!$ below, $\underline{\mathcal{C}}^!(X,X)_i = 0$ for all $i > 0$. Thus, $\underline{\mathcal{C}}^!(X,X) = \underline{\mathcal{C}}^!(X,X)_0 = \underline{\mathcal{C}}(X,X)$. Hence, $\underline{\mathcal{C}}^!(X,X)$ is semisimple as a k-algebra by condition (A2) for $\underline{\mathcal{C}}$.

(A3) $\underline{\mathcal{C}}^!(Z,X)_0 = 0$ if $X \neq Z$ by definition.

(A4) Let $X \in \mathcal{C}$. We will show that $\underline{\mathcal{C}}^!(X,X)_i = 0$ for all $i > 0$. For $i = 1$, we have $\underline{\mathcal{C}}^!(X,X)_1 = \underline{\mathcal{C}}(X,X)_1^* \cong \underline{\mathcal{C}}(X,X)_1$ as k-vector spaces. Hence, $\underline{\mathcal{C}}^!(X,X)_1 = 0$ by condition (A4) for $\underline{\mathcal{C}}$. Next suppose $\underline{\mathcal{C}}^!(X,X)_i \neq 0$ for some $i \geq 2$, for sake of contradiction. Then there exist $Y_1, \dots, Y_{i-1} \in \mathcal{C}$ such that $\underline{\mathcal{C}}(X,Y_1)_1^* \cong \underline{\mathcal{C}}(X,Y_1)_1, \dots, \underline{\mathcal{C}}(Y_{i-1},X)_1^* \cong \underline{\mathcal{C}}(Y_{i-1},X)_1$ are nonzero. Since $\underline{\mathcal{C}}$ is directed, this implies $X \subseteq Y_1 \subseteq \cdots \subseteq Y_{i-1} \subseteq X$, which gives $X = Y_1 = \cdots = Y_{i-1}$. But then $\underline{\mathcal{C}}(X,Y_1)_1^*, \dots, \underline{\mathcal{C}}(Y_{i-1},X)_1^*$ are all isomorphic to $\underline{\mathcal{C}}(X,X)_1$ as k-vector spaces. Hence, $\underline{\mathcal{C}}(X,Y_1)_1^*, \dots, \underline{\mathcal{C}}(Y_{i-1},X)_1^* = 0$ by condition (A4) for $\underline{\mathcal{C}}$. This is a contradiction, so we must have $\underline{\mathcal{C}}^!(X,X)_i = 0$ for all $i \geq 2$. Therefore, $\underline{\mathcal{C}}^!(X,X)_i = 0$ for every $X \in \mathcal{C}$ and $i > 0$.

(A5) Let $X \in \mathcal{C}$. By condition (A5) for $\underline{\mathcal{C}}$, there are only finitely many $Y \in \mathcal{C}$ such that $\underline{\mathcal{C}}^!(Y,X)_1 \cong \underline{\mathcal{C}}(X,Y)_1 \neq 0$ or $\underline{\mathcal{C}}^!(X,Y)_1 \cong \underline{\mathcal{C}}(Y,X)_1 \neq 0$.

(A6) We must show that $\underline{\mathcal{C}}_1^! \cdot \underline{\mathcal{C}}_i^! = \underline{\mathcal{C}}_{i+1}^!$ for every $i \geq 0$, where $\underline{\mathcal{C}}_i^! = \bigoplus_{X,Z \in \mathcal{C}} \underline{\mathcal{C}}^!(Z,X)_i$ for each $i \geq 0$. Note that $\underline{\mathcal{C}}_1^! = \bigoplus_{X,Z \in \mathcal{C}} \underline{\mathcal{C}}(X,Z)_1^*$ by definition, and $\underline{\mathcal{C}}_0^! = \bigoplus_{Z \in \mathcal{C}} \underline{\mathcal{C}}(Z,Z)_0$ by condition (A3) for $\underline{\mathcal{C}}^!$. Let $X, Z \in \mathcal{C}$. If $f^* \in \underline{\mathcal{C}}^!(Z,X)_1 = \underline{\mathcal{C}}(X,Z)_1^*$, then $1_Z \in \underline{\mathcal{C}}^!(Z,Z)_0 = \underline{\mathcal{C}}(Z,Z)$ and $f^* = f^* \cdot 1_Z$. It follows that $\underline{\mathcal{C}}_1^! \cdot \underline{\mathcal{C}}_0^! = \underline{\mathcal{C}}_1^!$. If $f_1^* \otimes \cdots \otimes f_{i+1}^* + W_{i+1}$ belongs to a direct summand of $\underline{\mathcal{C}}^!(Z,X)_{i+1}$ corresponding to some $Y_1, \dots, Y_i \in \mathcal{C}$, $i \geq 1$, then $f_1^* \otimes \cdots \otimes f_i^* + W_i$ is in the direct summand of

$\underline{\mathcal{C}}^!(Z,X)_i$ corresponding to $Y_1,...,Y_{i-1} \in \mathcal{C}$ and $f_{i+1}^* \in \underline{\mathcal{C}}(Y_i,Z)_1^*$. So

$$f_1^* \otimes \cdots \otimes f_{i+1}^* + W_{i+1} = (f_1^* \otimes \cdots \otimes f_i^* + W_i) \cdot f_{i+1}^*,$$

and it follows that $\underline{\mathcal{C}}_{i+1}^! = \underline{\mathcal{C}}_1^! \cdot \underline{\mathcal{C}}_i^!$ for $i \geq 1$.

(A7) Since $\mathrm{Ob}(\underline{\mathcal{C}}^!) = \mathrm{Ob}(\underline{\mathcal{C}})$ and $Z \supseteq Y \supseteq X$ if and only if $X \subseteq Y \subseteq Z$, the convex hull of any finite set $S \subseteq \mathrm{Ob}(\underline{\mathcal{C}}^!)$ contains only finitely many objects, by condition (A7) for $\underline{\mathcal{C}}$.

Therefore, $\underline{\mathcal{C}}^!$ is a directed graded k-linear category. □

Remark 5.2.6. Let $A^!$ be the graded k-algebra $\bigoplus_{i\geq 0} \underline{\mathcal{C}}_i^!$, where $\underline{\mathcal{C}}_i^! = \bigoplus_{X,Z\in\mathcal{C}} \underline{\mathcal{C}}^!(Z,X)_i$ for each $i \geq 0$. Because $\underline{\mathcal{C}}_0^! = \bigoplus_{X\in\mathcal{C}} \underline{\mathcal{C}}(X,X)$ and $\underline{\mathcal{C}}_1^! = \bigoplus_{X,Z\in\mathcal{C}} \underline{\mathcal{C}}(X,Z)_1^*$, $\underline{\mathcal{C}}_1^!$ is a $(\underline{\mathcal{C}}_0^!, \underline{\mathcal{C}}_0^!)$-bimodule. So we can form the graded k-algebra

$$T^! = \underline{\mathcal{C}}_0^! \oplus \underline{\mathcal{C}}_1^! \oplus (\underline{\mathcal{C}}_1^! \otimes_{\underline{\mathcal{C}}_0^!} \underline{\mathcal{C}}_1^!) \oplus \cdots$$

and let $\gamma^! : T^! \to A^!$ be the graded k-algebra homomorphism induced by composition of morphisms in $\underline{\mathcal{C}}^!$. Then $\gamma^!$ is surjective by condition (A6) for $\underline{\mathcal{C}}^!$, and $K^! = \ker(\gamma^!)$ is generated by its degree 2 component $K_2^! = K^! \cap (\underline{\mathcal{C}}_1^! \otimes_{\underline{\mathcal{C}}_0^!} \underline{\mathcal{C}}_1^!)$ by construction of morphisms in $\underline{\mathcal{C}}^!$. Thus, $\underline{\mathcal{C}}^!$ is quadratic.

5.3 The category $\underline{\mathcal{C}}^{tw}$.

Let $n \in \mathbb{N}$, t be a partition type, and A be a finite abelian group. Let $\mathcal{C}$ be the skeletal subcategory of $\mathcal{FI}_{t,A}^n$ on objects $X = ([x_1],...,[x_n])$. Let $\underline{\mathcal{C}}$ be the k-linearization of $\mathcal{C}$, where k is a field of characteristic 0. Then $\mathcal{C}$ is an EI category and $\underline{\mathcal{C}}$ is a directed graded k-linear category, by Proposition 3.2.3.

Definition 5.3.1. The *twist* of $\underline{\mathcal{C}}$ is the k-linear category $\underline{\mathcal{C}}^{tw}$ having the same objects as $\underline{\mathcal{C}}$, and morphisms defined by

$$\underline{\mathcal{C}}^{tw}(X,Y) = \bigoplus_{(f,P_f,\alpha_f)\in\mathcal{C}(X,Y)} k(f,P_f,\alpha_f) \otimes_k \det(P_f)$$

for any $X, Y \in \mathcal{C}$, where $\det(P_f) = \bigwedge^{|P_f|} kP_f$. The composite of two morphisms $(g,P_g,\alpha_g) \otimes \bigwedge_{j=1}^m S_j$

in $\underline{\mathcal{C}}^{tw}(Y,Z)$ and $(f,P_f,\alpha_f)\otimes\bigwedge_{j=1}^{l}R_j$ in $\underline{\mathcal{C}}^{tw}(X,Y)$ is defined to be

$$(gf,P_{gf},\alpha_{gf})\otimes g(R_1)\wedge\cdots\wedge g(R_l)\wedge S_1\wedge\cdots\wedge S_m$$

in $\underline{\mathcal{C}}^{tw}(X,Z)$. Composition in $\underline{\mathcal{C}}^{tw}$ is associative because if

$$(h,P_h,\alpha_h)\otimes\bigwedge_{j=1}^{p}T_j\in\underline{\mathcal{C}}^{tw}(Y,Z),\ (g,P_g,\alpha_g)\otimes\bigwedge_{j=1}^{m}S_j\in\underline{\mathcal{C}}^{tw}(X,Y),\ (f,P_f,\alpha_f)\otimes\bigwedge_{j=1}^{l}R_j\in\underline{\mathcal{C}}^{tw}(W,X),$$

then

$$((h,P_h,\alpha_h)\otimes\bigwedge_{j=1}^{p}T_j)((gf,P_{gf},\alpha_{gf})\otimes\bigwedge_{j=1}^{l}g(R_j)\wedge\bigwedge_{j=1}^{m}S_j)$$

and

$$((hg,P_{hg},\alpha_{hg})\otimes\bigwedge_{j=1}^{m}h(S_j)\wedge\bigwedge_{j=1}^{p}T_j)((f,P_f,\alpha_f)\otimes\bigwedge_{j=1}^{l}R_j)$$

both equal

$$(hgf,P_{hgf},\alpha_{hgf})\otimes\bigwedge_{j=1}^{l}hg(R_j)\wedge\bigwedge_{j=1}^{m}h(S_j)\wedge\bigwedge_{j=1}^{p}T_j.$$

The identity morphism in $\underline{\mathcal{C}}^{tw}(X,X)$ is $(1_X,\varnothing,0)\otimes 1$, where $0:X\to A$ is the zero map.

Proposition 5.3.2. $\underline{\mathcal{C}}^{tw}$ is a directed graded k-linear category.

Proof. For any $X,Y\in\mathcal{C}$, define the degree $i\geq 0$ part of $\underline{\mathcal{C}}^{tw}(X,Y)$ by

$$\underline{\mathcal{C}}^{tw}(X,Y)_i=\bigoplus_{\substack{(f,P_f,\alpha_f)\in\mathcal{C}(X,Y)\\|P_f|=i}}k(f,P_f,\alpha_f)\otimes_k\det(P_f).$$

Then $\underline{\mathcal{C}}^{tw}(X,Y)=\bigoplus_{i\geq 0}\underline{\mathcal{C}}^{tw}(X,Y)_i$ is a graded k-vector space, such that $\underline{\mathcal{C}}^{tw}(Y,Z)_j\otimes_k\underline{\mathcal{C}}^{tw}(X,Y)_i\to\underline{\mathcal{C}}^{tw}(X,Z)_{i+j}$ for every $i,j\geq 0$. It follows that $\underline{\mathcal{C}}^{tw}$ is a graded k-linear category. Also, $\underline{\mathcal{C}}^{tw}$ is directed by the same partial order $\subseteq$ on $\underline{\mathcal{C}}$. We now verify conditions (A1)-(A7) for $\underline{\mathcal{C}}^{tw}$ below.

(A1) Let $X,Y\in\mathcal{C}$. Since $\det(P_f)$ is 1-dimensional for any $(f,P_f,\alpha_f)\in\mathcal{C}(X,Y)$, $\underline{\mathcal{C}}^{tw}(X,Y)\cong\underline{\mathcal{C}}(X,Y)$ as k-vector spaces. By condition (A1) for $\underline{\mathcal{C}}$, $\underline{\mathcal{C}}^{tw}(X,Y)$ is finite dimensional as a k-vector space.

(A2) Let $X \in \mathcal{C}$. Then

$$\underline{\mathcal{C}}^{tw}(X,X) = \bigoplus_{(f,\varnothing,\alpha_f)\in\mathcal{C}(X,X)} k(f,\varnothing,\alpha_f) \otimes_k k$$

is isomorphic to $\underline{\mathcal{C}}(X,X)$ as a k-algebra. So $\underline{\mathcal{C}}^{tw}(X,X)$ is semisimple as a k-algebra for any $X \in \mathcal{C}$, by condition (A2) for $\underline{\mathcal{C}}$.

(A3) If $X \neq Y$, then there is no $(f, P_f, \alpha_f) \in \mathcal{C}(X,Y)$ for which $|P_f| = 0$. Hence, $\underline{\mathcal{C}}^{tw}(X,Y)_0 = 0$ if $X \neq Y$.

(A4) If $X \in \mathcal{C}$ and $i > 0$, then there is no $(f, P_f, \alpha_f) \in \mathcal{C}(X,X)$ for which $|P_f| = i$. Thus, $\underline{\mathcal{C}}^{tw}(X,X)_i = 0$ for every $X \in \mathcal{C}$ and $i > 0$.

(A5) For any $X, Y \in \mathcal{C}$,

$$\underline{\mathcal{C}}^{tw}(X,Y)_1 = \bigoplus_{\substack{(f,P_f,\alpha_f)\in\mathcal{C}(X,Y)\\ |P_f|=1}} k(f,P_f,\alpha_f) \otimes_k kP_f$$

is isomorphic to $\underline{\mathcal{C}}(X,Y)_1$ as a k-vector space. So for any $X \in \mathcal{C}$, there are only finitely many $Y \in \mathcal{C}$ such that $\underline{\mathcal{C}}^{tw}(X,Y)_1 \neq 0$ or $\underline{\mathcal{C}}^{tw}(Y,X)_1 \neq 0$, by condition (A5) for $\underline{\mathcal{C}}$.

(A6) Let $X, Y \in \mathcal{C}$. If $(f, P_f, \alpha_f) \otimes R \in \underline{\mathcal{C}}^{tw}(X,Y)_1$ is a morphism of degree 1, where $P_f = \{R\}$, then the identity morphism $(1_X, \varnothing, 0) \otimes 1 \in \underline{\mathcal{C}}^{tw}(X,X)_0$ is a morphism of degree 0, and

$$(f, P_f, \alpha_f) \otimes R = ((f, P_f, \alpha_f) \otimes R) \circ ((1_X, \varnothing, 0) \otimes 1).$$

It follows that $\underline{\mathcal{C}}_1 \cdot \underline{\mathcal{C}}_0 = \underline{\mathcal{C}}_1$. Next, let $(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^{i} T_j \in \underline{\mathcal{C}}^{tw}(X,Y)_i$ be a morphism of degree $i > 1$. By condition (A6) for $\underline{\mathcal{C}}$, $(f, P_f, \alpha_f) = (g', P_{g'}, \alpha_{g'})(f', P_{f'}, \alpha_{f'})$ for some morphisms $(f', P_{f'}, \alpha_{f'}) \in \mathcal{C}(X, Y')$, $(g', P_{g'}, \alpha_{g'}) \in \mathcal{C}(Y', Y)$, of degrees $i-1$ and 1, respectively, for some $Y' \in \mathcal{C}$. Let $P_{f'} = \{R_1, ..., R_{i-1}\}$ and $P_{g'} = \{S\}$. Then $\{T_1, ..., T_i\} = P_f = P_{g'f'} = \{g'(R_1), ..., g'(R_{i-1}), S\}$. So $(f', P_{f'}, \alpha_{f'}) \otimes \bigwedge_{j=1}^{i-1} R_j \in \underline{\mathcal{C}}^{tw}(X, Y')_{i-1}$ and $(g', P_{g'}, \alpha_{g'}) \otimes S \in \underline{\mathcal{C}}^{tw}(X, Y')_1$ with

$$((g', P_{g'}, \alpha_{g'}) \otimes S) \circ ((f', P_{f'}, \alpha_{f'}) \otimes \bigwedge_{j=1}^{i-1} R_j) = (f, P_f, \alpha_f) \otimes g'(R_1) \wedge ... \wedge g'(R_{i-1}) \wedge S,$$

which equals $\pm(f, P_f, \alpha_f) \otimes \bigwedge_{j=1}^{l} T_j$. It follows that $\underline{\mathcal{C}}_1 \cdot \underline{\mathcal{C}}_i = \underline{\mathcal{C}}_{i+1}$ for every $i \geq 0$.

(A7) The convex hull of any finite set $S \subseteq \mathrm{Ob}(\underline{\mathcal{C}}^{tw})$ contains only finitely many objects, by condition (A7) for $\underline{\mathcal{C}}$.

Therefore, $\underline{\mathcal{C}}^{tw}$ is a directed graded k-linear category. $\square$

Note 5.3.3. For any $X \in \mathcal{C}$, we have $\underline{\mathcal{C}}(X,X) = \underline{\mathcal{C}}(X,X)_0$ and $\underline{\mathcal{C}}^{tw}(X,X) = \underline{\mathcal{C}}^{tw}(X,X)_0$ by condition (A4) for $\underline{\mathcal{C}}$ and $\underline{\mathcal{C}}^{tw}$, respectively. Let us identify $\underline{\mathcal{C}}(X,X)_0 \cong \underline{\mathcal{C}}^{tw}(X,X)_0$ as k-algebras under the correspondence $(f,\varnothing,\alpha_f) \leftrightarrow (f,\varnothing,\alpha_f)\otimes 1$. Since $\underline{\mathcal{C}}_0 = \bigoplus_{X\in\mathcal{C}} \underline{\mathcal{C}}(X,X)_0$ and $\underline{\mathcal{C}}_0^{tw} = \bigoplus_{X\in\mathcal{C}} \underline{\mathcal{C}}^{tw}(X,X)_0$ by condition (A3) for $\underline{\mathcal{C}}$ and $\underline{\mathcal{C}}^{tw}$, respectively, we shall identify $\underline{\mathcal{C}}_0^{tw} \cong \underline{\mathcal{C}}_0$ as k-algebras. Then for any $X, Y \in \mathcal{C}$, $\underline{\mathcal{C}}^{tw}(X,Y)_1$ is $(\underline{\mathcal{C}}(Y,Y), \underline{\mathcal{C}}(X,X))$-bimodule with left action given by k-linear extension of

$$(\tau,\varnothing,\alpha_\tau)\cdot((f,P_f,\alpha_f)\otimes R) = (\tau f, P_{\tau f}, \alpha_{\tau f})\otimes \tau(R)$$

for $(\tau,\varnothing,\alpha_\tau) \in \mathcal{C}(Y,Y)$, and right action given by k-linear extension of

$$((f,P_f,\alpha_f)\otimes R)\cdot(\sigma,\varnothing,\alpha_\sigma) = (f\sigma, P_{f\sigma}, \alpha_{f\sigma})\otimes R$$

for $(\sigma,\varnothing,\alpha_\sigma) \in \mathcal{C}(X,X)$, where $P_f = \{R\}$.

Notation 5.3.4. Given a group G, let X be a right G-set and Y be a left G-set. Define an equivalence relation $\sim$ on $X \times Y$ be declaring that $(x,y) \sim (x',y')$ if and only if there exists $g \in G$ such that $x = x'g$ and $gy = y'$. Let $X \times_G Y = X \times Y/\sim$ denote the set of equivalence classes. For any $X, Y \in \mathcal{C}$, let $\mathcal{C}(X,Y)_1$ be the set of morphisms in $\mathcal{C}(X,Y)$ of degree 1. Then for any $X, Y, Z \in \mathcal{C}$, $\mathcal{C}(Y,Z)_1$ is a right $\mathcal{C}(Y,Y)$-set and $\mathcal{C}(X,Y)_1$ is a left $\mathcal{C}(Y,Y)$-set, so we can form the set $\mathcal{C}(Y,Z)_1 \times_{\mathcal{C}(Y,Y)} \mathcal{C}(X,Y)_1$.

Proposition 5.3.5. $\underline{\mathcal{C}}^{tw}$ is quadratic.

Proof. Let $A^{tw} = \bigoplus_{i\geq 0} \underline{\mathcal{C}}_i^{tw}$, where $\underline{\mathcal{C}}_i^{tw} = \bigoplus_{X,Y\in\mathcal{C}} \underline{\mathcal{C}}^{tw}(X,Y)_i$. Then A^{tw} is a graded k-algebra with multiplication given by composition of morphisms in $\underline{\mathcal{C}}^{tw}$. Since $\underline{\mathcal{C}}_1^{tw}$ is a $(\underline{\mathcal{C}}_0, \underline{\mathcal{C}}_0)$-bimodule, we can form the graded k-algebra

$$T^{tw} = \underline{\mathcal{C}}_0^{tw} \oplus \underline{\mathcal{C}}_1^{tw} \oplus (\underline{\mathcal{C}}_1^{tw} \otimes_{\underline{\mathcal{C}}_0} \underline{\mathcal{C}}_1^{tw}) \oplus \cdots .$$

Let $\gamma^{tw} : T^{tw} \to A^{tw}$ be the graded k-algebra homomorphism induced by composition of morphisms in $\underline{\mathcal{C}}^{tw}$. It is surjective by condition (A6) for $\underline{\mathcal{C}}^{tw}$. Let $K^{tw} = \ker(\gamma^{tw})$, which is a graded ideal of T^{tw}.We must show that K^{tw} is generated by its degree 2 component $K_2^{tw} = \ker(\gamma^{tw}) \cap (\underline{\mathcal{C}}_1^{tw} \otimes_{\underline{\mathcal{C}}_0} \underline{\mathcal{C}}_1^{tw})$.

Let us abbreviate morphisms (f, P_f, α_f) in $\mathcal{C}$ simply by f. Also, for any morphism f, pick an ordering of the elements of P_f and write $\wedge_f$ for the basis element of $\det(P_f)$.

We define a graded k-algebra $\tilde{T}^{tw} = \bigoplus_{i \geq 0} \tilde{T}_i^{tw}$ as follows. Let $\tilde{T}_0^{tw} = \underline{\mathcal{C}}_0$ and for $i \geq 1$, let

$$\tilde{T}_i^{tw} = \bigoplus_{X, Y_1, ..., Y_{i-1}, Z \in \mathcal{C}} \bigoplus_{(f_i, ..., f_1)} k(f_i, ..., f_1) \otimes_k \det(P_{f_i \cdots f_1}),$$

where the inner direct sum is over all

$$(f_i, ..., f_1) \in \mathcal{C}(Y_{i-1}, Z)_1 \times_{\mathcal{C}(Y_{i-1}, Y_{i-1})} \cdots \times_{\mathcal{C}(Y_1, Y_1)} \mathcal{C}(X, Y_1)_1.$$

The product of two basis elements

$$(g_j, ..., g_1) \otimes \wedge_{g_j \cdots g_1} \in \tilde{T}_j^{tw} \text{ and } (f_i, ..., f_1) \otimes \wedge_{f_i \cdots f_1} \in \tilde{T}_i^{tw}$$

is defined to be

$$(g_j, ..., g_1, f_i, ..., f_1) \otimes \wedge_{g_j \cdots g_1 f_i \cdots f_1} \in \tilde{T}_{i+j}^{tw}.$$

Let $i \geq 2$ and $X, Y_1, ..., Y_{i-1}, Z \in \mathcal{C}$. For any

$$(f_i, ..., f_1) \in \mathcal{C}(Y_{i-1}, Z)_1 \times_{\mathcal{C}(Y_{i-1}, Y_{i-1})} \cdots \times_{\mathcal{C}(Y_1, Y_1)} \mathcal{C}(X, Y_1)_1,$$

choose the ordering $(f_i \cdots f_2(R_1), ..., R_i)$ of the elements of $P_{f_i \cdots f_1}$, where $P_{f_l} = \{R_l\}$ for each $1 \leq l \leq i$.

Define $\tilde{\gamma}^{tw} : \tilde{T}^{tw} \to A^{tw}$ in degree $i \geq 1$ by k-linear extension of the assignment

$$(f_i, ..., f_1) \otimes \wedge_{f_i \cdots f_1} \mapsto f_i \cdots f_1 \otimes \wedge_{f_i \cdots f_1},$$

where $\wedge_{f_i \cdots f_1} = f_i \cdots f_2(R_1) \wedge \cdots \wedge R_i$. To see that $\tilde{\gamma}^{tw}$ is a k-algebra homomorphism, take basis elements

$$(g_j, ..., g_1) \otimes \wedge_{g_j \cdots g_1} \in \tilde{T}_j^{tw} \text{ and } (f_i, ..., f_1) \otimes \wedge_{f_i \cdots f_1} \in \tilde{T}_i^{tw}.$$

Then

$$\begin{aligned}\tilde{\gamma}^{tw}(((g_j,...,g_1)\otimes\wedge_{g_j\cdots g_1})((f_i,...,f_1)\otimes\wedge_{f_i\cdots f_1})) &= \tilde{\gamma}^{tw}((g_j,...,g_1,f_i,...,f_1)\otimes\wedge_{g_j\cdots g_1 f_i\cdots f_1}) \\ &= g_j\cdots g_1 f_i\cdots f_1\otimes\wedge_{g_j\cdots g_1 f_i\cdots f_1} \\ &= ((g_j\cdots g_1)\otimes\wedge_{g_j\cdots g_1})((f_i\cdots f_1)\otimes\wedge_{f_i\cdots f_1}) \\ &= \tilde{\gamma}^{tw}((g_j,...,g_1)\otimes\wedge_{g_j\cdots g_1})\tilde{\gamma}^{tw}((f_i,...,f_1)\otimes\wedge_{f_i\cdots f_1})\end{aligned}$$

where

$$\wedge_{g_j\cdots g_1 f_i\cdots f_1} = g_j\cdots g_1 f_i\cdots f_2(R_1)\wedge\cdots\wedge g_j\cdots g_1(R_i)\wedge g_j\cdots g_2(S_1)\wedge\cdots\wedge S_j$$

for $P_{f_l}=\{R_l\}$ $(1\le l\le i)$ and $P_{g_p}=\{S_p\}$ $(1\le p\le j)$.

Recall that $\gamma: T\to A$ is a graded k-algebra homomorphism given by composition of morphisms in $\underline{\mathcal{C}}$, where $A=\bigoplus_{i\ge 0}\underline{\mathcal{C}}_i$ and $T=\underline{\mathcal{C}}_0\oplus\underline{\mathcal{C}}_1\oplus(\underline{\mathcal{C}}_1\otimes_{\underline{\mathcal{C}}_0}\underline{\mathcal{C}}_1)\oplus\cdots$. Note that $\tilde{T}^{tw}\cong T$ as graded k-algebras by k-linear extension of the correspondence

$$(f_i,...,f_1)\otimes\wedge_{f_i\cdots f_1}\leftrightarrow f_i\otimes\cdots\otimes f_1$$

in degree $i\ge 1$.

For any $i\ge 2$, an arbitrary element in a direct summand of $\tilde{T}^{tw}_i$ corresponding to $X,Y_1,...,Y_{i-1},Z\in\mathcal{C}$ is of the form

$$\tilde{x}=\sum_j c_{i_j,...,1_j}(f_{i_j},...,f_{1_j})\otimes\wedge_{f_{i_j}\cdots f_{1_j}}.$$

Then

$$\tilde{\gamma}^{tw}(\tilde{x})=\sum_j c_{i_j,...,1_j}f_{i_j}\cdots f_{1_j}\otimes\wedge_{f_{i_j}\cdots f_{1_j}}$$

belongs to the direct summand of A^{tw}_i corresponding to $X,Z\in\mathcal{C}$. Let

$$x=\sum_j c_{i_j,...,1_j}f_{i_j}\otimes\cdots\otimes f_{1_j}$$

be the image of $\tilde{x}$ under the isomorphism $\tilde{T}^{tw} \cong T$. Then

$$\gamma(x) = \sum_j c_{i_j,\ldots,1_j} f_{i_j} \cdots f_{1_j}$$

belongs to the direct summand of A_i corresponding to $X, Z \in \mathcal{C}$. Group together like terms in both of the sums $\tilde{\gamma}^{tw}(\tilde{x})$ and $\gamma(x)$, i.e. those for which the composite $f_{i_j} \cdots f_{1_j}$ is the same element in $\mathcal{C}(X, Z)$. If $\tilde{x} \in \ker(\tilde{\gamma}^{tw})$, then

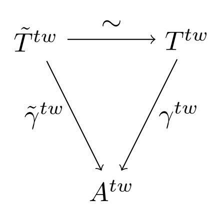
$$\tilde{\gamma}^{tw}(\tilde{x}) = \sum_j c_{i_j,\ldots,1_j} f_{i_j} \cdots f_{1_j} \otimes \wedge_{f_{i_j} \cdots f_{1_j}} = 0$$

implies that the sum of the corresponding scalars is 0 in each group, and hence $x \in \ker(\gamma)$. Since $\underline{\mathcal{C}}$ is quadratic by Remark 5.2.2, x is generated by the degree 2 component of $\ker(\gamma)$. Because $\tilde{T}^{tw} \cong T$ as graded k-algebras, $\tilde{x}$ is generated by the degree 2 component of $\ker(\tilde{\gamma}^{tw})$. It follows that $\ker(\tilde{\gamma}^{tw})$ by its degree 2 component.

Observe that $\tilde{T}^{tw} \cong T^{tw}$ as graded k-algebras by k-linear extension of the correspondence

$$(f_i, \ldots, f_1) \otimes \wedge_{f_i \cdots f_1} \leftrightarrow (f_i \otimes R_i) \otimes \cdots \otimes (f_1 \otimes R_1)$$

in degree $i \geq 1$, where $P_{f_l} = \{R_l\}$ for each $1 \leq l \leq i$. Hence, we get a commutative diagram



because

$$\tilde{\gamma}^{tw}((f_i, \ldots, f_1) \otimes \wedge_{f_i \cdots f_1}) = f_i \cdots f_1 \otimes \wedge_{f_i \cdots f_1} = \gamma^{tw}((f_i \otimes R_i) \otimes \cdots \otimes (f_1 \otimes R_1)).$$

Since $\ker(\tilde{\gamma}^{tw})$ is generated by its degree 2 component and $\tilde{T}^{tw} \cong T^{tw}$ as graded k-algebras, $\ker(\gamma^{tw})$ is generated by its degree 2 component. Therefore, $\underline{\mathcal{C}}^{tw}$ is quadratic. $\square$

5.4. Description of $\underline{\mathcal{C}}^!$.

Let $n \in \mathbb{N}$, t be a partition type, and A be a finite abelian group. Let $\mathcal{C}$ be the skeletal subcategory of $\mathcal{FI}^n_{t,A}$ on objects of the form $X = ([x_1], ..., [x_n])$, where $x_i \in \mathbb{N}_0$, $1 \leq i \leq n$. Let k be a field of characteristic 0, and $\underline{\mathcal{C}}$ be the k-linearization of $\mathcal{C}$. Then $\underline{\mathcal{C}}$ is Koszul by Corollary 4.3.2 and hence quadratic by [GL, Proposition 3.10].

In this section we will show that $\underline{\mathcal{C}}^!$ and $(\underline{\mathcal{C}}^{tw})^{op}$ are isomorphic categories (Corollary 5.4.10), provided that we make a further assumption on the partition type t (Remark 5.4.2). To prove $\underline{\mathcal{C}}^! \cong (\underline{\mathcal{C}}^{tw})^{op}$, we need to show that $\underline{\mathcal{C}}^!(Z, X)_i \cong \underline{\mathcal{C}}^{tw}(X, Z)_i$ as k-vector spaces for any $X, Z \in \mathcal{C}$ and $i \geq 0$.

Note 5.4.1. For $i = 0, 1$, this is immediate. For any $X, Z \in \mathcal{C}$, recall that

$$\underline{\mathcal{C}}^!(Z, X)_0 = \begin{cases} \underline{\mathcal{C}}(X, X) & \text{if } X = Z \\ 0 & \text{if } X \neq Z \end{cases}$$

and

$$\underline{\mathcal{C}}^!(X, Z)_1 = \underline{\mathcal{C}}(X, Z)_1^*.$$

On the other hand,

$$\underline{\mathcal{C}}^{tw}(X, Z)_0 = \begin{cases} \underline{\mathcal{C}}(X, X) \otimes_k k & \text{if } X = Z \\ 0 & \text{if } X \neq Z \end{cases}$$

and

$$\underline{\mathcal{C}}^{tw}(X, Z)_1 = \bigoplus_{\substack{(f, P_f, \alpha_f) \in \mathcal{C}(X,Y) \\ |P_f| = 1}} k(f, P_f, \alpha_f) \otimes_k \det(P_f).$$

Thus, $\underline{\mathcal{C}}^!(Z, X)_i \cong \underline{\mathcal{C}}^{tw}(X, Z)_i$ as k-vector spaces for $i = 0, 1$.

Remark 5.4.2. For $i \geq 2$, we must make a further assumption on the partition type t. There are two cases depending on n. If $n = 1$, then fix $m \in \mathbb{N}$ and assume t is partition type m; if $n > 1$, then assume t is partition type n^*.

Proposition 5.4.3. Let $X, Z \in \mathcal{C}$ and $i \geq 2$. If $\underline{\mathcal{C}}(X, Z)_i \neq 0$, then there is a unique sequence of objects $Y_1, ..., Y_{i-1} \in \mathcal{C}$ such that $\underline{\mathcal{C}}(X, Y_1)_1$,...,$\underline{\mathcal{C}}(Y_{i-1}, Z)_1$ are nonzero.

Proof. Recall that any morphism of degree $i \geq 2$ factors into a composite of degree 1 morphisms by condition (A6) for $\underline{\mathcal{C}}$. Let $X = ([x_1], ..., [x_n])$ and $Z = ([z_1], ..., [z_n])$. If $n = 1$, then we must have $[z_1] = [x_1 + im]$, so $Y_1 = [x_1 + m]$,...,$Y_{i-1} = [x_1 + (i-1)m]$ are the only objects in $\mathcal{C}$ for which $\underline{\mathcal{C}}(X, Y_1)_1$,...,$\underline{\mathcal{C}}(Y_{i-1}, Z)_1$ are nonzero. On the other hand, if $n > 1$, then we must have $([z_1], ..., [z_n]) = ([x_1 + i], ..., [x_n + i])$, so $Y_1 = ([x_1 + 1], ..., [x_n + 1])$,...,$Y_{i-1} = ([x_1 + (i-1)], ..., [x_n + (i-1)])$ are the only objects in $\mathcal{C}$ for which $\underline{\mathcal{C}}(X, Y_1)_1$,...,$\underline{\mathcal{C}}(Y_{i-1}, Z)_1$ are nonzero. □

Corollary 5.4.4. Let $X, Z \in \mathcal{C}$ and $i \geq 2$. Suppose $\underline{\mathcal{C}}(X, Z)_i \neq 0$, and let $Y_1, ..., Y_{i-1} \in \mathcal{C}$ be the unique objects such that $\underline{\mathcal{C}}(X, Y_1)_1$,...,$\underline{\mathcal{C}}(Y_{i-1}, Z)_1$ are nonzero. Then

$$\underline{\mathcal{C}}^!(Z, X)_i = (\underline{\mathcal{C}}(X, Y_1)_1^* \otimes_{\underline{\mathcal{C}}(Y_1, Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_{i-1}, Y_{i-1})} \underline{\mathcal{C}}(Y_{i-1}, Z)_1^*)/W_i,$$

where

$$W_i = \sum_{j=1}^{i-1} \underline{\mathcal{C}}(X, Y_1)_1^* \otimes_{\underline{\mathcal{C}}(Y_1, Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_{j-1}, Y_{j-1})} \mathrm{im}(\gamma^*_{Y_{j-1} Y_j Y_{j+1}}) \otimes_{\underline{\mathcal{C}}(Y_{j+1}, Y_{j+1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_{i-1}, Y_{i-1})} \underline{\mathcal{C}}(Y_{i-1}, Z)_1^*,$$

in which $X = Y_0$ and $Z = Y_i$.

To show $\underline{\mathcal{C}}^!(Z, X)_i \cong \underline{\mathcal{C}}^{tw}(X, Z)_i$ as k-vector spaces for any $X, Z \in \mathcal{C}$ and $i \geq 2$, we will exhibit a surjection

$$\phi : \underline{\mathcal{C}}(X, Y_1)_1^* \otimes_{\underline{\mathcal{C}}(Y_1, Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_{i-1}, Y_{i-1})} \underline{\mathcal{C}}(Y_{i-1}, Z)_1^* \twoheadrightarrow \underline{\mathcal{C}}^{tw}(X, Z)_i$$

whose kernel is W_i. The map ϕ is constructed as follows.

By Remark 5.1.5, we have an isomorphism of $(\underline{\mathcal{C}}(X, X), \underline{\mathcal{C}}(Z, Z))$-bimodules

$$\underline{\mathcal{C}}(X, Y_1)_1^* \otimes_{\underline{\mathcal{C}}(Y_1, Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_{i-1}, Y_{i-1})} \underline{\mathcal{C}}(Y_{i-1}, Z)_1^* \cong (\underline{\mathcal{C}}(Y_{i-1}, Z)_1 \otimes_{\underline{\mathcal{C}}(Y_{i-1}, Y_{i-1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_1, Y_1)} \underline{\mathcal{C}}(X, Y_1)_1)^*.$$

Since

$$\underline{\mathcal{C}}(Y_{i-1}, Z)_1 \otimes_{\underline{\mathcal{C}}(Y_{i-1}, Y_{i-1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_1, Y_1)} \underline{\mathcal{C}}(X, Y_1)_1$$

is finite dimensional as a k-vector space, we have an isomorphism of k-vector spaces

$$(\underline{\mathcal{C}}(Y_{i-1}, Z)_1 \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \underline{\mathcal{C}}(X, Y_1)_1)^* \cong \underline{\mathcal{C}}(Y_{i-1}, Z)_1 \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \underline{\mathcal{C}}(X, Y_1)_1.$$

Let

$$\alpha : \underline{\mathcal{C}}(X, Y_1)_1^* \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \underline{\mathcal{C}}(Y_{i-1}, Z)_1^* \xrightarrow{\sim} \underline{\mathcal{C}}(Y_{i-1}, Z)_1 \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \underline{\mathcal{C}}(X, Y_1)_1$$

be the composite of the above two k-vector space isomorphisms.

Recall that $\underline{\mathcal{C}}^{tw}(X, Y)_1 \cong \underline{\mathcal{C}}(X, Y)_1$ as $(\underline{\mathcal{C}}(Y, Y), \underline{\mathcal{C}}(X, X))$-bimodules for any $X, Y \in \mathcal{C}$. It follows by induction that

$$\underline{\mathcal{C}}^{tw}(Y_{i-1}, Z)_1 \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \underline{\mathcal{C}}^{tw}(X, Y_1)_1 \cong \underline{\mathcal{C}}(Y_{i-1}, Z)_1 \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \underline{\mathcal{C}}(X, Y_1)_1$$

as $(\underline{\mathcal{C}}(Z, Z), \underline{\mathcal{C}}(X, X))$-bimodules. Let

$$\beta : \underline{\mathcal{C}}(Y_{i-1}, Z)_1 \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \underline{\mathcal{C}}(X, Y_1)_1 \xrightarrow{\sim} \underline{\mathcal{C}}^{tw}(Y_{i-1}, Z)_1 \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \underline{\mathcal{C}}^{tw}(X, Y_1)_1$$

be this isomorphism as k-vector spaces.

Let

$$\gamma^{tw} : \underline{\mathcal{C}}^{tw}(Y_{i-1}, Z)_1 \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \underline{\mathcal{C}}^{tw}(X, Y_1)_1 \twoheadrightarrow \underline{\mathcal{C}}^{tw}(X, Z)_i$$

be the composition map in $\underline{\mathcal{C}}^{tw}$, which is surjective by condition (A6) for $\underline{\mathcal{C}}^{tw}$.

Thus, we define

$$\phi : \underline{\mathcal{C}}(X, Y_1)_1^* \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \underline{\mathcal{C}}(Y_{i-1}, Z)_1^* \twoheadrightarrow \underline{\mathcal{C}}^{tw}(X, Z)_i$$

as the composite $\gamma^{tw}\beta\alpha$.

Notation 5.4.5. For arbitrary $X, Y, Z \in \mathcal{C}$, let us denote by $\phi_{XYZ} : \underline{\mathcal{C}}(X,Y)_1^* \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)_1^* \twoheadrightarrow \underline{\mathcal{C}}^{tw}(X,Z)_2$ the composite $\gamma_{XYZ}^{tw}\beta_{XYZ}\alpha_{XYZ}$, where

$$\begin{aligned} \alpha_{XYZ} : \underline{\mathcal{C}}(X,Y)_1^* \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)_1^* &\xrightarrow{\sim} \underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1, \\ (f, P_f, \alpha_f)^* \otimes (g, P_g, \alpha_g)^* &\mapsto (g, P_g, \alpha_g) \otimes (f, P_f, \alpha_f), \end{aligned}$$

$$\begin{aligned} \beta_{XYZ} : \underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1 &\xrightarrow{\sim} \underline{\mathcal{C}}^{tw}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}^{tw}(X,Y)_1 \\ (g, P_g, \alpha_g) \otimes (f, P_f, \alpha_f) &\mapsto ((g, P_g, \alpha_g) \otimes S) \otimes ((f, P_f, \alpha_f) \otimes R), \end{aligned}$$

$$\begin{aligned} \gamma_{XYZ}^{tw} : \underline{\mathcal{C}}^{tw}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}^{tw}(X,Y)_1 &\twoheadrightarrow \underline{\mathcal{C}}^{tw}(X,Z)_2 \\ ((g, P_g, \alpha_g) \otimes S) \otimes ((f, P_f, \alpha_f) \otimes R) &\mapsto (gf, P_{gf}, \alpha_{gf}) \otimes g(R) \wedge S, \end{aligned}$$

with $P_g = \{S\}$, $P_f = \{R\}$.

Since $\underline{\mathcal{C}}^{tw}$ is quadratic by Proposition 5.3.5, $\ker(\gamma^{tw})$ equals

$$\sum_{j=1}^{i-1} \underline{\mathcal{C}}^{tw}(Y_{i-1}, Z)_1 \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_{j+1},Y_{j+1})} \ker(\gamma^{tw}_{Y_{j-1}Y_jY_{j+1}}) \otimes_{\underline{\mathcal{C}}(Y_{j-1},Y_{j-1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \underline{\mathcal{C}}^{tw}(X, Y_1)_1,$$

in which $X = Y_0$ and $Z = Y_i$. Because α, β are k-vector space isomorphisms, we get $\ker(\phi) = \alpha^{-1}\beta^{-1}(\ker(\gamma^{tw}))$. Hence, $\ker(\phi)$ equals

$$\sum_{j=1}^{i-1} \underline{\mathcal{C}}(X, Y_1)_1^* \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_{j-1},Y_{j-1})} \ker(\phi_{Y_{j-1}Y_jY_{j+1}}) \otimes_{\underline{\mathcal{C}}(Y_{j+1},Y_{j+1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \underline{\mathcal{C}}(Y_{i-1}, Z)_1^*.$$

Thus, it suffices to show that $\ker(\phi_{XYZ}) = \mathrm{im}(\gamma_{XYZ}^*)$, where $X, Z \in \mathcal{C}$ are arbitrary and Y is the unique object in $\mathcal{C}$ such that $\underline{\mathcal{C}}(X,Y)_1, \underline{\mathcal{C}}(Y,Z)_1 \neq 0$.

Definition 5.4.6. Let $X, Z \in \mathcal{C}$ and $i \geq 0$. Given a morphism $(g, P_g, \alpha_g) \in \underline{\mathcal{C}}(X,Z)_i$, there is a unique sequence of objects $Y_1, ..., Y_{i-1} \in \mathcal{C}$ such that $\underline{\mathcal{C}}(X, Y_1)_1, ..., \underline{\mathcal{C}}(Y_{i-1}, Z)_1$ are nonzero. We call an element

$$(f_i, P_{f_i}, \alpha_{f_i}) \otimes \cdots \otimes (f_1, P_{f_1}, \alpha_{f_1}) \in \underline{\mathcal{C}}(Y_{i-1}, Z)_1 \otimes_{\underline{\mathcal{C}}(Y_{i-1}, Y_{i-1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_1, Y_1)} \underline{\mathcal{C}}(X, Y_1)_1$$

a *factorization* of (g, P_g, α_g) if

$$(f_i, P_{f_i}, \alpha_{f_i}) \cdots (f_1, P_{f_1}, \alpha_{f_1}) = (g, P_g, \alpha_g).$$

Lemma 5.4.7. If $X, Z \in \mathcal{C}$, then any basis element $(h, P_h, \alpha_h) \in \underline{\mathcal{C}}(X, Z)_2$ has exactly two factorizations in $\underline{\mathcal{C}}(Y, Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X, Y)_1$, where Y is the unique object in $\mathcal{C}$ such that $\underline{\mathcal{C}}(X, Y)_1, \underline{\mathcal{C}}(Y, Z)_1 \neq 0$.

Proof. By the proof of condition (A6) for $\underline{\mathcal{C}}$, we know that (h, P_h, α_h) factors into a composite of two degree 1 morphisms $(f, P_f, \alpha_f) \in \underline{\mathcal{C}}(X, Y)_1$ and $(g, P_g, \alpha_g) \in \underline{\mathcal{C}}(Y, Z)_1$. We shall reproduce that argument here for future reference.

Since (h, P_h, α_h) is a degree 2 morphism, we have $P_h = \{T_1, T_2\}$ for some disjoint nonempty subsets $T_1, T_2 \subset Z$. Then there is a bijection $\tau_1 \in \mathcal{FI}^n(Y, Z \backslash \{T_1\})$. Let $\iota_1 \in \mathcal{FI}^n(Z \backslash \{T_1\}, Z)$ be the inclusion map. Define $g_1 = \iota_1 \tau_1 \in \mathcal{FI}^n(Y, Z)$, $P_{g_1} = \{T_1\}$, and $\alpha_{g_1}(y) = 0$ for $y \in Y$, and $\alpha_{g_1}(T_1) = \alpha_h(T_1)$. Then $(g_1, P_{g_1}, \alpha_{g_1}) \in \mathcal{C}(Y, Z)$ is a morphism of degree 1. Let $h' \in \mathcal{FI}^n(X, Z \backslash \{T_1\})$ be the morphism obtained by restricting the codomain of $h \in \mathcal{FI}^n(X, Z)$ to $Z \backslash \{T_1\}$. Define $f_1 = \tau_1^{-1} h' \in \mathcal{FI}^n(X, Y)$, $P_{f_1} = \{\tau_1^{-1}(T_2)\}$, and $\alpha_{f_1}(x) = \alpha_h(x)$ for $x \in X$, and $\alpha_{f_1}(\tau_1^{-1}(T_2)) = \alpha_h(T_2)$. Then $(f_1, P_{f_1}, \alpha_{f_1}) \in \mathcal{C}(X, Y)$ is a morphism of degree 1.

So

$$g_1 f_1 = \iota_1 \tau_1 \tau_1^{-1} h' = \iota_1 h' = h,$$

$$P_{g_1 f_1} = g_1(P_{f_1}) \sqcup P_{g_1} = \{\iota_1 \tau_1 \tau_1^{-1}(T_2)\} \sqcup \{T_1\} = \{T_1, T_2\} = P_h,$$

and

$$\alpha_{g_1f_1}(x) = \alpha_{f_1}(x) + \alpha_{g_1}(f(x)) = \alpha_h(x) \text{ for } x \in X,$$
$$\alpha_{g_1f_1}(T_2) = \alpha_{g_1f_1}(g_1(\tau_1^{-1}(T_2))) = \alpha_{f_1}(\tau_1^{-1}(T_2)) + \sum_{y\in\tau_1^{-1}(T_2)} \alpha_{g_1}(y) = \alpha_h(T_2),$$
$$\alpha_{g_1f_1}(T_1) = \alpha_{g_1}(T_1) = \alpha_h(T_1).$$

Thus, $(g_1, P_{g_1}, \alpha_{g_1})(f_1, P_{f_1}, \alpha_{f_1}) = (h, P_h, \alpha_h)$. So

$$(g_1, P_{g_1}, \alpha_{g_1}) \otimes (f_1, P_{f_1}, \alpha_{f_1}) \in \underline{\mathcal{C}}(Y, Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X, Y)_1$$

is a factorization of (h, P_h, α_h) such that $P_{g_1} = \{T_1\}$ and $g_1(P_{f_1}) = \{T_2\}$.

Similarly, we can find a bijection $\tau_2 \in \mathcal{FI}^n(Y, Z\backslash\{T_2\})$ and construct two degree 1 morphisms $(f_2, P_{f_2}, \alpha_{f_2}) \in \mathcal{C}(X, Y)$, $(g_2, P_{g_2}, \alpha_{g_2}) \in \mathcal{C}(Y, Z)$ such that $(h, P_h, \alpha_h) = (g_2, P_{g_2}, \alpha_{g_2})(f_2, P_{f_2}, \alpha_{f_2})$, so that

$$(g_2, P_{g_2}, \alpha_{g_2}) \otimes (f_2, P_{f_2}, \alpha_{f_2}) \in \underline{\mathcal{C}}(Y, Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X, Y)_1$$

is a factorization of (h, P_h, α_h) such that $P_{g_2} = \{T_2\}$ and $g_2(P_{f_2}) = \{T_1\}$. Hence, $(h, P_h, \alpha_h) \in \mathcal{C}(X, Z)$ has at least two factorizations $(g_i, P_{g_i}, \alpha_{g_i}) \otimes (f_i, P_{f_i}, \alpha_{f_i})$ $(i = 1, 2)$ in $\underline{\mathcal{C}}(Y, Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X, Y)_1$.

We now show that these are the only two factorizations of (h, P_h, α_h) in $\underline{\mathcal{C}}(Y, Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X, Y)_1$. Suppose $(h, P_h, \alpha_h) = (g, P_g, \alpha_g)(f, P_f, \alpha_f)$ for some degree 1 morphisms $(f, P_f, \alpha_f) \in \mathcal{C}(X, Y)$ and $(g, P_g, \alpha_g) \in \mathcal{C}(Y, Z)$. Say $P_g = \{S\}$ and $P_f = \{R\}$. Then $\{T_1, T_2\} = P_h = g(P_f) \sqcup P_g = \{g(R), S\}$ implies that either $T_1 = g(R)$ and $T_2 = S$, or $T_1 = S$ and $T_2 = g(R)$. Without loss of generality, assume $T_1 = S$ and $T_2 = g(R)$. We claim that $(g, P_g, \alpha_g) \otimes (f, P_f, \alpha_f) = (g_1, P_{g_1}, \alpha_{g_1}) \otimes (f_1, P_{f_1}, \alpha_{f_1})$ in $\underline{\mathcal{C}}(Y, Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X, Y)_1$. To prove this, we must find $(\sigma, \varnothing, \alpha_\sigma) \in \mathcal{C}(Y, Y)$ such that $(g, P_g, \alpha_g) = (g_1, P_{g_1}, \alpha_{g_1})(\sigma, \varnothing, \alpha_\sigma)$ and $(\sigma, \varnothing, \alpha_\sigma)(f, P_f, \alpha_f) = (f_1, P_{f_1}, \alpha_{f_1})$.

Let $g' \in \mathcal{FI}^n(Y, Z\backslash\{T_1\})$ be the morphism obtained by restricting the codomain of $g \in \mathcal{FI}^n(Y, Z)$ to $Z\backslash\{T_1\}$. Recall that $(g_1, P_{g_1}, \alpha_{g_1}) \in \mathcal{C}(Y, Z)$ was defined by $g_1 = \iota_1\tau_1 \in \mathcal{FI}^n(Y, Z)$, $P_{g_1} = \{T_1\}$, and $\alpha_{g_1}(y) = 0$ for $y \in Y$, $\alpha_{g_1}(T_1) = \alpha_h(T_1)$, where $\iota_1 \in \mathcal{FI}^n(Z\backslash\{T_1\}, Z)$ is inclusion and $\tau_1 \in \mathcal{FI}^n(Y, Z\backslash\{T_1\})$ is a bijection. Define $(\sigma, \varnothing, \alpha_\sigma) \in \mathcal{C}(Y, Y)$ by $\sigma = \tau_1^{-1}g' \in \mathcal{FI}^n(Y, Y)$ and

$\alpha_\sigma(y) = \alpha_g(y)$ for $y \in Y$. Then

$$g_1\sigma = \iota_1\tau_1\tau_1^{-1}g' = g$$

and

$$\alpha_{g_1\sigma}(T_1) = \alpha_{g_1}(T_1) = \alpha_h(T_1) = \alpha_{gf}(T_1) = \alpha_g(T_1),$$

so $(g_1, P_{g_1}, \alpha_{g_1})(\sigma, \varnothing, \alpha_\sigma) = (g, P_g, \alpha_g)$. On the other hand, recall that $(f_1, P_{f_1}, \alpha_{f_1}) \in \mathcal{C}(X, Y)$ was defined by $f_1 = \tau_1^{-1}h' \in \mathcal{FI}^n(X, Y)$, $P_{f_1} = \{\tau_1^{-1}(T_2)\}$, $\alpha_{f_1}(x) = \alpha_h(x)$ for $x \in X$ and $\alpha_{f_1}(\tau_1^{-1}(T_2)) = \alpha_h(T_2)$, where $h' \in \mathcal{FI}^n(X, Z\backslash\{T_1\})$ is the morphism obtained by restricting the codomain of $h \in \mathcal{FI}^n(X, Z)$ to $Z\backslash\{T_1\}$. Then

$$\sigma f = \tau_1^{-1}g'f = \tau_1^{-1}h' = f_1,$$

$$\sigma(P_f) = \tau_1^{-1}g'(\{R\}) = \{\tau_1^{-1}(T_2)\} = P_{f_1},$$

and

$$\alpha_{\sigma f}(x) = \alpha_f(x) + \alpha_\sigma(f(x)) = \alpha_f(x) + \alpha_g(f(x)) = \alpha_h(x) = \alpha_{f_1}(x) \text{ for } x \in X,$$

$$\alpha_{\sigma f}(\sigma(R)) = \alpha_f(R) + \sum_{y\in R}\alpha_\sigma(y) = \alpha_f(R) + \sum_{y\in R}\alpha_g(y) = \alpha_h(g(R)) = \alpha_h(T_2) = \alpha_{f_1}(\tau_1^{-1}(T_2)).$$

So $(\sigma, \varnothing, \alpha_\sigma)(f, P_f, \alpha_f) = (f_1, P_{f_1}, \alpha_{f_1})$, and hence

$$(g, P_g, \alpha_g) \otimes (f, P_f, \alpha_f) = (g_1, P_{g_1}, \alpha_{g_1}) \otimes (f_1, P_{f_1}, \alpha_{f_1})$$

in $\underline{\mathcal{C}}(Y, Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X, Y)_1$.

Similarly, if we had assumed $T_1 = g(R)$ and $T_2 = S$, we would get

$$(g, P_g, \alpha_g) \otimes (f, P_f, \alpha_f) = (g_2, P_{g_2}, \alpha_{g_2}) \otimes (f_2, P_{f_2}, \alpha_{f_2})$$

in $\underline{\mathcal{C}}(Y, Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X, Y)_1$.

Therefore, $(h, P_h, \alpha_h) \in \underline{\mathcal{C}}(X, Z)_2$ has exactly two factorizations $(g_1, P_{g_1}, \alpha_{g_1}) \otimes (f_1, P_{f_1}, \alpha_{f_1})$ and $(g_2, P_{g_2}, \alpha_{g_2}) \otimes (f_2, P_{f_2}, \alpha_{f_2})$ in $\underline{\mathcal{C}}(Y, Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X, Y)_1$, such that $P_{g_1} = \{T_1\}$, $g_1(P_{f_1}) = \{T_2\}$ and $P_{g_2} = \{T_2\}$, $g_2(P_{f_2}) = \{T_1\}$. □

Proposition 5.4.8. $\ker(\phi_{XYZ})$ is the span of all elements of the form

$$f_1^* \otimes g_1^* + f_2^* \otimes g_2^* \in \underline{\mathcal{C}}(X,Y)_1^* \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)_1^*$$

for which $f_1^* \otimes g_1^* \neq f_2^* \otimes g_2^*$ and $g_1 f_1 = g_2 f_2$.

Proof. Recall that

$$\phi_{XYZ} : \underline{\mathcal{C}}(X,Y)_1^* \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)_1^* \to \underline{\mathcal{C}}^{tw}(X,Z)_2$$

is defined as the composite $\gamma^{tw}_{XYZ}\beta_{XYZ}\alpha_{XYZ}$, where

$$\alpha_{XYZ}((f,P_f,\alpha_f)^* \otimes (g,P_g,\alpha_g)^*) = (g,P_g,\alpha_g) \otimes (f,P_f,\alpha_f),$$

$$\beta_{XYZ}((g,P_g,\alpha_g) \otimes (f,P_f,\alpha_f)) = ((g,P_g,\alpha_g) \otimes S) \otimes ((f,P_f,\alpha_f) \otimes R),$$

with $P_g = \{S\}$, $P_f = \{R\}$, and

$$\gamma^{tw}_{XYZ}(((g,P_g,\alpha_g) \otimes S) \otimes ((f,P_f,\alpha_f) \otimes R)) = (gf, P_{gf}, \alpha_{gf}) \otimes g(R) \wedge S.$$

So

$$\phi_{XYZ}((f,P_f,\alpha_f)^* \otimes (g,P_g,\alpha_g)^*) = (gf, P_{gf}, \alpha_{gf}) \otimes g(R) \wedge S,$$

where $P_f = \{R\}$ and $P_g = \{S\}$.

Consider the composite of the second and third map:

$$\gamma^{tw}_{XYZ}\beta_{XYZ} : \underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1 \xrightarrow{\sim} \underline{\mathcal{C}}^{tw}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}^{tw}(X,Y)_1 \twoheadrightarrow \underline{\mathcal{C}}^{tw}(X,Z)_2.$$

For simplicity, we shall abbreviate degree 1 morphisms (f, P_f, α_f) in $\mathcal{C}$ as f. Then an arbitrary element of $\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1$ is of the form $x = \sum a_{ij}(g_j \otimes f_i)$, where $a_{ij} \in k$ and $f_i \in \mathcal{C}(X,Y)$, $g_j \in \mathcal{C}(Y,Z)$ are degree 1 morphisms, say with $P_{f_i} = \{R_i\}$, $P_{g_j} = \{S_j\}$. We may assume that all $g_j \otimes f_i$ in x are distinct. Thus, x is a k-linear combination of distinct factorizations in $\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1$ of various degree two morphisms $h \in \mathcal{C}(X,Z)$. By Lemma 5.4.7, each degree two morphism $h \in \mathcal{C}(X,Z)$ has exactly two factorizations in $\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1$. So for each $h \in \mathcal{C}(X,Z)$, at most 2 of its factorizations are present in x. Suppose $x \in \ker(\gamma^{tw}_{XYZ}\beta_{XYZ})$. Then

$$\gamma^{tw}_{XYZ}\beta_{XYZ}(x) = \sum a_{ij}(g_j f_i \otimes g_j(R_i) \wedge S_j) = 0$$

in

$$\underline{\mathcal{C}}^{tw}(X,Z)_2 = \bigoplus_{\substack{h \in \mathcal{C}(X,Z) \\ |P_h|=2}} kh \otimes_k \det(P_h).$$

Fix a direct summand corresponding to $h \in \mathcal{C}(X,Z)$, $|P_h| = 2$. Then there are at most two terms in $\gamma^{tw}_{XYZ}\beta_{XYZ}(x)$ that belong to $kh \otimes_k \det(P_h)$, and the sum of those terms is 0. If there is only one such term $a(gf \otimes g(R) \wedge S)$, then $a(gf \otimes g(R) \wedge S) = 0$ implies $a = 0$. If there are two such terms $a_1(g_1 f_1 \otimes g_1(R_1) \wedge S_1)$ and $a_2(g_2 f_2 \otimes g_2(R_2) \wedge S_2)$, then

$$0 = a_1(g_1 f_1 \otimes g_1(R_1) \wedge S_1) + a_2(g_2 f_2 \otimes g_2(R_2) \wedge S_2) = (a_1 - a_2)(g_1 f_1 \otimes g_1(R_1) \wedge S_1),$$

since $g_1 f_1 = g_2 f_2$ and $g_1(R_1) = S_2$, $g_2(R_2) = S_1$. Hence, $a_1 - a_2 = 0$ implies $a_1 = a_2$. Because this holds for each direct summand, it follows that $\ker(\gamma^{tw}_{XYZ}\beta_{XYZ}) \subseteq \text{span}(U)$, where

$$U = \{g_1 \otimes f_1 + g_2 \otimes f_2 \in \underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1 \,:\, g_1 \otimes f_1 \neq g_2 \otimes f_2 \text{ and } g_1 f_1 = g_2 f_2\}.$$

Conversely, if $g_1 \otimes f_1, g_2 \otimes f_2 \in \underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1$ are distinct elements such that $g_1 f_1 = g_2 f_2$, then

$$\gamma^{tw}_{XYZ}\beta_{XYZ}(g_1 \otimes f_1 + g_2 \otimes f_2) = g_1 f_1 \otimes g_1(R_1) \wedge S_1 + g_2 f_2 \otimes g_2(R_2) \wedge S_2 = 0,$$

again since $g_1 f_1 = g_2 f_2$ and $g_1(R_1) = S_2$, $g_2(R_2) = S_1$. Thus, $\ker(\gamma^{tw}_{XYZ}\beta_{XYZ}) = \text{span}(U)$.

Looking back at our isomorphism

$$\alpha_{XYZ} : \underline{\mathcal{C}}(X,Y)_1^* \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)_1^* \xrightarrow{\sim} \underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1,$$

we conclude that $\ker(\phi_{XYZ})$ is the span of all elements of the form

$$f_1^* \otimes g_1^* + f_2^* \otimes g_2^* \in \underline{\mathcal{C}}(X,Y)_1^* \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)_1^*$$

for which $f_1^* \otimes g_1^* \neq f_2^* \otimes g_2^*$ and $g_1 f_1 = g_2 f_2$. □

Proposition 5.4.9. $\mathrm{im}(\gamma_{XYZ}^*)$ is the span of all elements of the form

$$f_1^* \otimes g_1^* + f_2^* \otimes g_2^* \in \underline{\mathcal{C}}(X,Y)_1^* \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)_1^*,$$

where $f_1^* \otimes g_1^* \neq f_2^* \otimes g_2^*$ and $g_1 f_1 = g_2 f_2$.

Proof. Recall that

$$\gamma_{XYZ}^* : \underline{\mathcal{C}}(X,Z)_2^* \to (\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1)^*$$

is the $(\underline{\mathcal{C}}(X,X), \underline{\mathcal{C}}(Z,Z))$-bimodule homomorphism obtained by dualizing the composition map

$$\gamma_{XYZ} : \underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1 \to \underline{\mathcal{C}}(X,Z)_2.$$

For simplicity, we again write morphisms (f, P_f, α_f) in $\mathcal{C}$ as f. Then for any dual basis element $h^* \in \underline{\mathcal{C}}(X,Z)_2^*$ and for any basis elements $g \in \underline{\mathcal{C}}(Y,Z)_1$, $f \in \underline{\mathcal{C}}(X,Y)_1$, we have

$$\gamma_{XYZ}^*(h^*)(g \otimes f) = h^*(gf) = \begin{cases} 1 & \text{if } h = gf \\ 0 & \text{else.} \end{cases}$$

Fix a dual basis element $h^* \in \underline{\mathcal{C}}(X,Z)_2^*$. Let $\gamma_{XYZ}^*(h^*) = \sum a_{ij}(g_j \otimes f_i)^*$ in $(\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1)^*$, where $a_{ij} \in k$ and the $g_j \otimes f_i$ are distinct and range over all basis elements in

$\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1$. If $g_j f_i = h$, then

$$a_{ij} = \gamma^*_{XYZ}(h^*)(g_j \otimes f_i) = 1.$$

If $g_j f_i \neq h$, then

$$a_{ij} = \gamma^*_{XYZ}(h^*)(g_j \otimes f_i) = 0.$$

By Lemma 5.4.7, h has exactly two factorizations in $\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1$, both of which are present within the sum $\gamma^*_{XYZ}(h^*)$. If $g_1 \otimes f_1$ and $g_2 \otimes f_2$ are these factorizations, then

$$\gamma^*_{XYZ}(h^*) = (g_1 \otimes f_1)^* + (g_2 \otimes f_2)^*.$$

Since this holds for each $h^* \in \underline{\mathcal{C}}(X,Z)^*_2$, it follows that $\mathrm{im}(\gamma^*_{XYZ}) \subseteq \mathrm{span}(V)$, where

$$V = \{(g_1 \otimes f_1)^* + (g_2 \otimes f_2)^* \in (\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1)^* : (g_1 \otimes f_1)^* \neq (g_2 \otimes f_2)^* \text{ and } g_1 f_1 = g_2 f_2\}.$$

Conversely, if $(g_1 \otimes f_1)^*$, $(g_2 \otimes f_2)^*$ are distinct elements of $(\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1)^*$ for which $g_1 f_1 = g_2 f_2$, let $h = g_1 f_1$. Then $h^* \in \underline{\mathcal{C}}(X,Z)^*_2$ and

$$\gamma^*_{XYZ}(h^*) = (g_1 \otimes f_1)^* + (g_2 \otimes f_2)^*.$$

Thus, $\mathrm{im}(\gamma^*_{XYZ}) = \mathrm{span}(V)$. By identifying $\mathrm{im}(\gamma^*_{XYZ})$ under the $(\underline{\mathcal{C}}(X,X), \underline{\mathcal{C}}(Z,Z))$-bimodule isomorphism $(\underline{\mathcal{C}}(Y,Z)_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(X,Y)_1)^* \cong \underline{\mathcal{C}}(X,Y)^*_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)^*_1$, we conclude that $\mathrm{im}(\gamma^*_{XYZ})$ is the span of all elements of the form

$$f_1^* \otimes g_1^* + f_2^* \otimes g_2^* \in \underline{\mathcal{C}}(X,Y)^*_1 \otimes_{\underline{\mathcal{C}}(Y,Y)} \underline{\mathcal{C}}(Y,Z)^*_1,$$

where $f_1^* \otimes g_1^* \neq f_2^* \otimes g_2^*$ and $g_1 f_1 = g_2 f_2$. $\square$

By Propositions 5.4.8 and 5.4.9, $\ker(\phi_{XYZ}) = \mathrm{im}(\gamma^*_{XYZ})$ for any $X, Z \in \mathcal{C}$ and Y being the unique object in $\mathcal{C}$ for which $\underline{\mathcal{C}}(X,Y)_1, \underline{\mathcal{C}}(Y,Z)_1 \neq 0$. So $\ker(\phi)$ equals

$$\sum_{j=1}^{i-1} \underline{\mathcal{C}}(X,Y_1)^*_1 \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_{j-1},Y_{j-1})} \mathrm{im}(\gamma^*_{Y_{j-1}Y_jY_{j+1}}) \otimes_{\underline{\mathcal{C}}(Y_{j+1},Y_{j+1})} \cdots \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \underline{\mathcal{C}}(Y_{i-1},Z)^*_1,$$

in which $X = Y_0$ and $Z = Y_i$. Hence, $\ker(\phi)$ is precisely W_i. Therefore,

$$\underline{\mathcal{C}}^{tw}(X,Z)_i \cong (\underline{\mathcal{C}}(X,Y_1)_1^* \otimes_{\underline{\mathcal{C}}(Y_1,Y_1)} \cdots \otimes_{\underline{\mathcal{C}}(Y_{i-1},Y_{i-1})} \underline{\mathcal{C}}(Y_{i-1},Z)_1^*)/W_i = \underline{\mathcal{C}}^!(Z,X)_i$$

as k-vector spaces for any $X, Z \in \mathcal{C}$ and $i \geq 0$. This proves that $\underline{\mathcal{C}}^!(Z,X) \cong \underline{\mathcal{C}}^{tw}(X,Z)$ for all $X, Z \in \mathcal{C}$. It follows that $\underline{\mathcal{C}}^!$ and $(\underline{\mathcal{C}}^{tw})^{op}$ are isomorphic categories, summarized in the Corollary below.

Corollary 5.4.10. Let $n \in \mathbb{N}$, t be a partition type, A be a finite abelian group, and k be a field of characteristic 0. Let $\mathcal{C}$ be the skeletal subcategory of $\mathcal{FI}^n_{t,A}$ on objects of the form $X = ([x_1], ..., [x_n])$ for $x_i \in \mathbb{N}_0$ $(1 \leq i \leq n)$. Let $\underline{\mathcal{C}}$ be the k-linearization of $\mathcal{C}$. If t is partition type m for some $m \in \mathbb{N}$ when $n = 1$, or if t is partition type n^* when $n > 1$, then $\underline{\mathcal{C}}^! \cong (\underline{\mathcal{C}}^{tw})^{op}$.

References

[BGS] Alexander Beilinson, Victor Ginzburg, Wolfgang Soergel; Koszul Duality Patterns in Representation Theory; J. Amer. Math. Soc. Vol. 9 Num. 2, April 1996, 473-527.

[DPS] Elizabeth Dan-Cohen, Ivan Penkov, Vera Serganova; A Koszul Category of Representations of Finitary Lie Algebras; arXiv: 1105.3407v2, 25 May 2012.

[GL] Wee Liang Gan, Liping Li; Koszulity of Directed Categories in Representation Stability Theory; arXiv: 1411.5308v2, 21 Nov 2014.

[GL2] Wee Liang Gan, Liping Li; A Remark on FI-Module Homology; arXiv: 1505.01777v4, 11 May 2016.

[GS] Dimitar Grantcharov, Vera Serganova; Tensor Representations of $\mathfrak{q}(\infty)$; arXiv: 1605.02389v1, 8 May 2016.

[SS] Steven V. Sam, Andrew Snowden; Stability Patterns in Representation Theory; arXiv:1302.5859v1, 24 Feb 2013.