Positively multiplicative graphs and homology rings of affine Grassmannians

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Abstract. Let $\mathfrak{g}$ be an untwisted affine Lie algebra with associated Weyl group W_a and let W be the Weyl group of the corresponding simple Lie algebra. To any level-0 weight γ we associate a rooted weighted graph Γ_γ that encodes the orbit of γ under the action W_a. From a combinatorial point of view, the graph Γ_γ is the weak version of the quantum Bruhat graph for the strong Bruhat order on either W or one of its parabolic quotients. We show that the graph Γ_γ encodes the periodic orientation of certain subsets of alcoves in W_a and therefore can be interpreted as an automaton determining the reduced expressions in these subsets. Then, by using some relevant quotients of the homology ring of affine Grassmannians, we show that the graph Γ_γ is positively multiplicative: there exists, in some sense, a maximal family of commuting adjacency matrices which also commute with the one of Γ_γ. This yields the key ingredients to study a large class of random walks which can be either seen as random paths on alcoves or as interacting particle systems.

Keywords. Root systems, affine Grassmannians, homology ring, oriented graphs, alcove walks

Mathematics Subject Classifications. 05E05, 05E10, 22E47, 57T15

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1. Introduction

The notion of positively multiplicative graphs naturally comes from the study of harmonic functions on infinite graphs and random walks on simple geometric configurations. As defined in [Ker03] for infinite graded graphs and in [GLT22a] in general, a graph Γ is positively multiplicative if there exist a commutative algebra $\mathsf{\Lambda}$ together with an element $x \in \mathsf{\Lambda}$ and a basis B such that the structure constants of $\mathsf{\Lambda}$ with respect to B are nonnegative and the matrix of the multiplication by x expressed in the basis B is the adjacency matrix of Γ . This is the case for example for the well-known Young lattice of partitions with Λ the algebra of symmetric functions, B its basis of Schur functions and $x = s_{(1)}$ the Schur function associated to the partition (1). As explained in [GLT22a], multiplicative graphs give a convenient tool to study various problems connected to representation theory of finite groups (character theory), representation theory of Lie algebras and their associated random walks, harmonic functions and Markov chains on infinite graded graphs, automatons recognizing reduced expressions in Coxeter groups, orbits in affine crystals etc. In [LT21] it is shown how the study of a natural class of random walks on the alcove tessellation of type A can be reduced to the determination of the Perron–Frobenius elements for a suitable positively multiplicative graph defined on k-irreducible partitions (i.e. partitions with at most k columns which do not contain any $i \times (k - i + 1)$-rectangle for $i = 1 \dots k$). One of the goal of this paper is to extend some results of [LT21] by showing that a much larger class of graphs defined from general affine Lie algebras are positively multiplicative.

More precisely, let $\mathfrak{g}$ be a non-twisted affine Lie algebra with associated affine Weyl group W_a and let $W \subset W_a$ be the finite Weyl group generated by the reflections of the root system Φ of the corresponding finite dimensional simple Lie algebra. To any level-0 weight γ of $\mathfrak{g}$, we associate an oriented weighted graph Γ_γ that encodes the orbit of γ under the action of W_a. From a combinatorial point of view, this graph is built as follows. Recall that the Weyl group W is a finite Coxeter group generated by simple reflections $\{s_1, \dots, s_n\}$ corresponding to a set S of simple roots in Φ. Moreover, there exists $J \subset \{1, \dots, n\}$ such that the orbit of γ under the action of W_a is in bijection with the parabolic quotient $W^J = W/W_J$, where $W_J = \langle s_j, j \in J\rangle$ is the parabolic subgroup induced by J. Then, Γ_γ is a weighted and colored extension of the cover graph of the weak Bruhat order on W^J. Namely, each equivalence class of W^J admits a minimal length representative w with respect to the length ℓ of the Coxeter group W, and the edge structure on Γ_γ is the following:

1. there is an arrow colored i with weight 1 from w to w' if $w' = s_i w$ for some $i \in \{1, \dots, n\} \setminus J$ and $\ell(w') > \ell(w)$,
2. there is an arrow colored 0 with some specific weight from w to w' if $w' = s_\theta w$ and $\ell(w') < \ell(w)$, where s_θ is the reflection associated to the longest root of the root system (Φ, S).

The weights of the arrows colored 0 belong to some polynomial algebra in variables indexed by $\{1, \dots, n\} \setminus J$, see Definition 4.7 for a precise statement. In type A and when $\gamma = \rho$ (the sum of all classical fundamental weights of $\mathfrak{g}$), the graph Γ_ρ is the positively multiplicative graph on k-irreducible partitions studied in [LT21]. The main result of the present paper is the following.

Theorem 1.1. *The graph Γ_γ is positively multiplicative.*

With this positivity property in hand, it becomes possible to generalize all the results of [LT21] by studying a large class of random walks on each graph Γ_γ. Before sketching the pattern of the proof of Theorem 1.1, let us review several interpretations of those graphs and their relevance from a combinatorial and probabilistic point of view.

Alcove paths models

The graph Γ_γ is defined using the level-0 action of W_a on the weight lattice. It does however contains a lot of information on the geometry of alcoves. It is well-known that W_a can be realised as a group of affine motions in an affine Euclidean space V generated by affine reflections. More precisely, W_a is generated by all reflections $\sigma_{\alpha,k}$ with respect to the affine hyperplane $H_{\alpha,k} = \{x \in V \mid (x, \alpha) = k\}$ where $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Moreover, there exists a finite root system $\Phi^\vee$ dual to Φ such that $W_a = W \ltimes Q^\vee$, where $Q^\vee$ is the coroot lattice generated by $\Phi^\vee$. The set of alcoves is the set of connected components of the set $V \setminus \cup_{\alpha\in\Phi,k\in\mathbb{Z}}\{H_{\alpha,k}\}$. The group W_a acts simply transitively on this set, hence, fixing the fundamental alcove A_0, there is a one to one correspondence between alcoves and elements of W_a. In the simplest case $\gamma = \rho$, where ρ the sum of all classical fundamental weights of $\mathfrak{g}$, the graph Γ_ρ encodes the periodic orientation of adjacent alcoves in the following way. For $u \in W$ and $\beta \in Q^\vee$, let us denote by ut_β the corresponding element in W_a. Let A and A' be two adjacent alcoves corresponding respectively to ut_β and $u't'_\beta$ in W_a and let $H_{\alpha,k}$ with $\alpha \in \Phi^+$ (the set of positive roots) and $k \in \mathbb{Z}$ be the unique hyperplane separating A and A'. Then the set of vertices of Γ_ρ is equal to W and there is an arrow from u to u' in the graph if and only if A lies on the positive side of $H_{\alpha,k}$ i.e the half-plane characterised by $(x, \alpha) > k$ and A' on the negative side. In the case where γ has a non-trivial stabiliser W_J in W, the graph Γ_γ encodes the orientation of adjacent alcoves in the so-called fundamental J-alcove $\mathcal{A}_J$ which is defined as the set of points that lie between hyperplanes $H_{\beta,0}$ and $H_{\beta,1}$ for all positive roots β in Φ_J, the parabolic root system of W_J.

It is interesting to note that since the graph Γ_ρ naturally encodes the orientation of adjacent alcoves, it can be interpreted as an automaton for reduced expressions of affine Grassmannians elements. Indeed, the set $W_a^{\Lambda_0}$ of affine Grassmannians elements is the set of elements of minimal length in the left cosets of W_a with respect to the finite Weyl group W. This set can also be caracterised as the set of alcoves that lies in the fundamental chamber $\mathcal{C}_0 = \{x \in V \mid \langle x, \alpha\rangle \geqslant 0,\ \forall \alpha \in \Phi^+\}$. It follows that if $w \in W_a^{\Lambda_0}$ and $s \in S$ are such that $sw, w \in W_a^{\Lambda_0}$ then we have $sw > w$ if and only if sw lies on the positive side of the unique hyperplane separating w and sw. As a consequence, reduced expression of elements in $W_a^{\Lambda_0}$ will be in bijection with certain paths in the graph Γ_ρ. In the case where γ has a non-trivial stabiliser W_J, we obtain an automaton for affine Grassmannians elements lying in the fundamental J-alcove $\mathcal{A}_J$. Observe there also exist automatons for the reduced expressions of any element of the affine Weyl groups (see [PY22, PY19] and the references therein).

Central random walks and interacting particle systems

The fact that Γ_γ is positively multiplicative has important consequences in the theory of random walks on alcoves that never cross twice the same hyperplane. We can expand the graph Γ_γ

to obtain an infinite graph Γ_γ^e which is the graph of the weak Bruhat order on the set of affine Grassmannians elements that lie in the J-alcove $\mathcal{A}_J$ and study Markov chains on this graph. The case where $\gamma = \rho$ and each new alcove (or each new vertex) is chosen uniformly among all the possible alcoves (vertices) was considered in [Lam15]. More in the spirit of the works by Kerov and Vershik [Ker03], one can also study central random walks on alcoves or central Markov chains on Γ_γ^e. This is another class of random walks that never cross twice the same hyperplane for which two trajectories with the same end points have the same probability. These are controlled by the positive harmonic functions on Γ_γ^e (see [LT21] for the case $\gamma = \rho$ in type A). Then, thanks to the positivity of Γ_γ, the determination of these positive harmonic functions is completely explicit. This is a particular case of a more general phenomenon for positively multiplicative graphs detailed in [GLT22a]. It then becomes possible to obtain a rational expression for the drift of the previous central random walks (i.e. for all types and for all level-0 weight γ) from the methods and results obtained in [LT21] for the affine Grassmannian in type A. We will not pursue in this direction in the present paper where we rather focus on the algebraic properties of the graphs Γ_γ.

In classical types, the case Γ_{ω_i} where ω_i is a fundamental classical weight is also of special interest since it describes the transitions of k particles moving with small steps on a circle (in type A) or on a segment (in type B, C, D) with n sites without intersecting each other. This problem in type A is studied in details in [GLT22b] using the fact the eigenvalues and the eigenvectors of the adjacency matrix of Γ_{ω_i} are known. A precise asymptotic behavior of the random walk on the graph is obtained and asymptotic formulas are deduced for the coefficients of the quantum cohomology of Grassmannian. It would be interesting to study the spectral properties of the adjacency matrices of the graphs Γ_γ in general.

Finite affine crystals

Finally, the graph Γ_γ also carries information on finite affine crystals. To any fundamental classical weight $\omega_i \in P$, there exists a finite-dimensional representation of the quantum group $U_q'(\mathfrak{g})$ called a Kirillov–Reshetikhin module. In many cases (in particular for the classical affine root systems), this module is known to admit a combinatorial skeleton $\widehat{B}(\omega_i)$, its so-called associated crystal (see [BS17] for a complete review on crystals). There is an action of the affine Weyl group W_a on $\widehat{B}(\omega_i)$ defined using its i-chains structure: for $v \in \widehat{B}(\omega_i)$, the image of v under the action of s_i is the symmetric of v in the i-chain containing v. The crystal $\widehat{B}(\omega_i)$ contains a unique vertex v_0 of weight ω_i and the orbit of v_0 under the action of W_a on this crystal is the same as the orbit of ω_i under the action of W_a in the following sense: the weight of the vertex $w \cdot v_0$ is $w(\omega_i)$. Thus the graph Γ_{ω_i} also encodes the action of W_a on v_0. In type A, since all the i-chains in $\widehat{B}(\omega_i)$ have length at most 1, we actually so recover the full crystal $\widehat{B}(\omega_i)$. More generally, one can associate an affine crystal $\widehat{B}(\gamma)$ to any dominant classical weight $\gamma = \sum \gamma_i \omega_i$ using tensor products of crystals by setting

$$\widehat{B}(\gamma) = \widehat{B}(\omega_1)^{\otimes \gamma_1} \otimes \cdots \otimes \widehat{B}(\omega_n)^{\otimes \gamma_n}.$$

As before, there is an action of W_a on $\widehat{B}(\gamma)$ and the orbit of the unique vertex of weight γ in $\widehat{B}(\gamma)$ under this action is the same as the orbit of γ under the action of W_a. Hence the graph Γ_γ also encodes the action of the affine Weyl group on $\widehat{B}(\gamma)$. Here again, it is a natural question to ask for the affine crystals $\widehat{B}(\gamma)$ that could be positively multiplicative when they do not reduce to one orbit. In connection with the alcove model for KR-crystals developed in [LNS+15, LNS+17] it would yield interesting generalisations of the previous alcove random walks.

Related works

Recall that the graph Γ_γ is a weighted version of the cover graph of the weak Bruhat order on the Weyl group W, with some extra edges coming from the action of s_θ, where θ is the highest root of Φ. They are the weak twins of the quantum Bruhat graph for the strong Bruhat order and its parabolic versions, see [BFP99]. They are also folded versions of similar graphs which have been introduced in [LS07] to study the weak Bruhat order on affine Lie algebras (see also [LLM+14] in type A). In [LS07], the authors have already studied the relation between the weak and the strong versions of these graphs in the affine case and for certain parabolic subcases.

The quantum Bruhat graph can also be shown to be positively multiplicative using the quantum cohomology ring $QH(G/B)$ of the flag variety associated to W and a particular basis $\{q^d\sigma_w\}_{w\in W,d\in\mathbb{N}^n}$ indexed by $W\times\mathbb{N}^n$. By [LS10], $QH(G/B)$ is isomorphic to the homology ring of affine Grassmannians up to a localization. As such, the graph Γ_ρ should also appear as the graph of the multiplication of a distinguished element σ of $QH(G/B)$ on the same basis as the one of the quantum Bruhat graph. We did not follow this path in the present paper, in particular because the geometric interpretation in terms of alcove paths of the strong Bruhat order is more complicated than for its weak version. Nevertheless, our results together with the isomorphism from [LS10] suggest that the corresponding distinguished element of $QH(G/B)$ should be σ_{s_θ}, where θ is the highest root previously introduced (see also Section 3.1). The same result should hold more generally for any graph Γ_γ by considering the quantum cohomology ring $QH(G/P)$ of the adequate partial flag variety G/P. In particular, the correspondence from [LS10] should also allow to compute the structure constants of the homology ring of the affine Grassmannians using known formulas for the ring of the quantum cohomology, see for example the algorithm of Mihalcea [Mih07]. We also expect it could be connected to the quantum alcove path model developed in [LNS+15, LNS+17].

Pattern of the proof

The main tool to prove Theorem 1.1 is the homology ring of affine Grassmannians. We refer to Section 5 for precise definitions and some references. The homology ring of affine Grassmannians Λ is a commutative $\mathbb{Q}$-algebra with a distinguished basis $\{\xi_w \mid w\in W_a^{\Lambda_0}\}$ such that the structure constants with respect to this basis are positive integers. In type A, this ring can be identified with a subalgebra of the symmetric functions and the previous distinguished basis then coincides with the basis of k-Schur functions (see [LLM+14]). This positive structure is the key to our result. The connection between Λ and our graph is made through the Pieri rules that describe in particular the multiplication by ξ_{s_0} in Λ:

$$\xi_{s_0}\xi_w=\sum a_i\xi_{s_iw}\quad\text{for all } w\in W_a^{\Lambda_0}$$

where the sum is over all $i \in \{0,\dots,n\}$ such that $s_iw > w$ and $s_iw \in W_a^{\Lambda_0}$. As it has been shown by Peterson (see also [LLM+14, Chapter 4, Cor. 4.14]), the algebra Λ has very nice factorisation properties: for all $w \in W_a^{\Lambda_0}$ and all ω_i, we have $\xi_{wt_{\omega_i}} = \xi_w\xi_{t_{\omega_i}}$. This property can be used to show that Λ is finite-dimensional over the ring of polynomials $\mathbb{Q}[\xi_{t_{\omega_1}},\dots,\xi_{t_{\omega_n}}]$. In turn, this allows us to extend the scalars in the definition of Λ to the ring of Laurent polynomials and to define a basis in bijection with the full set of alcoves, not only the affine Grassmannians elements. We then show that the adjacency matrix of Γ_ρ is essentially equal to the matrix of the multiplication by ξ_{s_0} in this extended algebra expressed in a well chosen basis, hence showing the positivity. In the more general case where γ has a non-trivial stabiliser in W, we introduce a quotient of some ideal of our extended algebra that has a basis in bijection with the set of alcoves in $\mathcal{A}_J$ and show that there are extra factorisation properties by "pseudo-translations" in this quotient. We then interpret the adjacency matrix of Γ_γ as the matrix of the multiplication by $\tilde{\xi}_{s_0}$, the image of ξ_{s_0} in the quotient, hence showing the positivity of Γ_γ. It is worth noticing here that one can apply to Γ_ρ the general procedure for computing positively multiplicative bases obtained in [GLT22a]. This gives a simple way to compute the structure constants of the homology ring of affine Grassmannians, at least in small ranks and this has been done in type G_2 [Gui23]. Such structure constants can also be computed using the correspondence between the homology ring of affine Grassmannians and the quantum cohomology ring of the flag variety associated to W, see [LS10] and the next paragraph for more details.

We now review the content of the paper. In Section 2, we give some results and definitions of [GLT22a] about positively multiplicative graphs. In Section 3, we introduce the notion of affine Weyl groups together with their action on the weight lattice P_a and their action on the set of alcoves. In Section 4, we define the graph Γ_γ associated to a level-0 weight γ and provide a lot of examples. In Section 5, we introduce the homology ring of affine Grassmannians and present some of its properties. In Section 6, we prove that Γ_γ is positively multiplicative for all level-0 weight γ. Finally, in Section 7, we present an interpretation of the graph Γ_γ for classical types in terms of particle moving on either a circle or a segment.

2. Multiplicative graphs

In this section we review some results of [GLT22a]. We start by defining the notion of positively multiplicative algebras and positively multiplicative graphs.

2.1. Multiplicative finite graphs

Let $\mathsf{Z} = \{\mathsf{z}_1,\dots,\mathsf{z}_N\}$ be a set (possibly empty) of formal indeterminates. Let $\mathbb{Q}[\mathsf{Z}^{\pm 1}]$ be the ring of Laurent polynomials in the variables Z and let $\mathbb{Q}(\mathsf{Z}^{\pm 1})$ be its fraction field. Write $\mathbb{Q}_+[\mathsf{Z}^{\pm 1}]$ for the subset of $\mathbb{Q}[\mathsf{Z}^{\pm 1}]$ containing the polynomials with nonnegative coefficients.

Let Λ be a unital algebra over $\mathbb{Q}[\mathsf{Z}^{\pm 1}]$ and let b be a basis of Λ containing 1. We say that Λ is positively multiplicative with respect to b, if the structure constants of the basis b are all in $\mathbb{Q}_+[\mathsf{Z}^{\pm 1}]$. We say that Λ is positively multiplicative if it is positively multiplicative with respect to some basis containing 1.

Let $\Gamma = (V, E)$ be a finite oriented weighted graph with set of vertices $V = \{v_1, \ldots, v_n\}$, set of arrows $E \subset V \times V$ and edge weight function with values in $\mathbb{Q}_+[\mathsf{Z}^{\pm 1}]$. We denote by $A_\Gamma = (a_{i,j})_{1 \leqslant i,j \leqslant n}$ the adjacency matrix of Γ, that is the coefficient $a_{i,j}$ is equal to 0 if there is no arrow from v_j to v_i and to the sum of the weights of the arrows from v_j to v_i otherwise. The adjacency algebra of Γ is the algebra $\mathbb{Q}[\mathsf{Z}^{\pm 1}][A_\Gamma]$ of polynomials in A_Γ with coefficients in $\mathbb{Q}[\mathsf{Z}^{\pm 1}]$. It will often be convenient to split any arrow in Γ weighted by a polynomial P in $\mathbb{Q}_+[\mathsf{Z}^{\pm 1}]$ into a multiple set of arrows, each of them being weighted by a monomial in $\mathbb{Q}_+[\mathsf{Z}^{\pm 1}]$.

In this section, we will assume that all graphs are strongly connected. This means that for each pair of vertices (u, v), there is an oriented path from u to v.

Definition 2.1. A graph $\Gamma = (V, E)$ is said to be *multiplicative* at $v_{i_0} \in V$ if its adjacency algebra is contained in a commutative algebra Λ of $n \times n$ matrices with coefficients in $\mathbb{Q}(\mathsf{Z}^{\pm 1})$ with a basis $\mathsf{b}_{i_0} = \{\mathsf{b}_i, i = 1, \ldots, n\}$ such that

$$\mathsf{b}_{i_0} = 1 \text{ and } A_\Gamma \mathsf{b}_j = \sum_{i=1}^{n} a_{i,j} \mathsf{b}_i \text{ for any } j = 1, \ldots, n.$$

In other words, the matrix $\mathrm{Mat}_{\mathsf{b}_{i_0}}(m_{A_\Gamma})$ of the multiplication by A_Γ in the basis b_{i_0} is A_Γ itself. In addition, if the algebra Λ is positively multiplicative with respect to b_{i_0} we say that Γ is positively multiplicative at v_{i_0}.

Let Λ be a commutative algebra and $x \in \Lambda$. As above, we write m_x for the linear map $\Lambda \to \Lambda$ defined by $m_x(a) = xa$. We start by stating a useful and easy result.

Proposition 2.2. *Let Γ be a finite oriented weighted graph with adjacency matrix $A_\Gamma = (a_{i,j})$. The graph Γ is multiplicative at v_{i_0} if and only if there exists a commutative n-dimensional algebra Λ over $\mathbb{Q}(\mathsf{Z}^{\pm 1})$, an element $x \in \Lambda$ and a basis $\mathsf{b} = (\mathsf{b}_1, \ldots, \mathsf{b}_n)$ of Λ such that $\mathsf{b}_{i_0} = 1$ and $\mathrm{Mat}_{\mathsf{b}}(m_x) = A_\Gamma$.*

Proof. Assume that there exists such an algebra Λ. The map

$$\begin{array}{rccc} \mathrm{Mat}_{\mathsf{b}} & : & \Lambda & \to & \mathrm{Mat}_n(\mathbb{Q}(\mathsf{Z}^{\pm 1})) \\ & & y & \mapsto & \mathrm{Mat}_{\mathsf{b}}(m_y) \end{array}$$

is an injective morphism of algebras. Therefore, $\mathrm{Mat}_{\mathsf{B}}(\Lambda)$ is a commutative subalgebra of $\mathrm{Mat}_n(\mathbb{Q}(\mathsf{Z}^{\pm 1}))$. We claim that Γ is multiplicative at v_{i_0} with associated algebra $\mathrm{Mat}_{\mathsf{b}}(\Lambda)$ and basis

$$\{\mathrm{Mat}_{\mathsf{b}}(m_{\mathsf{b}_i}) \mid i = 1, \ldots, n\}.$$

Indeed we have $x\mathsf{b}_j = m_x(\mathsf{b}_j) = \sum_{i=1}^n a_{i,j} \mathsf{b}_i$ so that $m_{x\mathsf{b}_j} = \sum a_{i,j} m_{\mathsf{b}_i}$. Then

$$\begin{aligned} A_\Gamma \mathrm{Mat}_{\mathsf{b}}(m_{\mathsf{b}_j}) = \mathrm{Mat}_{\mathsf{b}}(m_x) \mathrm{Mat}_{\mathsf{b}}(m_{\mathsf{b}_j}) = \mathrm{Mat}_{\mathsf{b}}(m_x m_{\mathsf{b}_j}) &= \mathrm{Mat}_{\mathsf{b}}(m_{x\mathsf{b}_j}) \\ &= \sum_{i=1}^{n} a_{i,j} \mathrm{Mat}_{\mathsf{b}}(m_{\mathsf{b}_i}). \end{aligned}$$

Since $\mathsf{b}_{i_0} = 1$, we have $\mathrm{Mat}_{\mathsf{b}}(m_{\mathsf{b}_{i_0}}) = 1_{\mathrm{Mat}_n(\mathbb{Q}(\mathsf{Z}^{\pm 1}))}$, hence the result. The converse is obvious by definition of a positively multiplicative graph. $\square$

2.2. Graph of maximal dimension

We say that a graph Γ is of maximal dimension when its adjacency algebra is of dimension n, the number of vertices in Γ and the size of A_Γ. This is the case if and only if the minimal polynomial μ_{A_Γ} of A_Γ is of degree n.

Proposition 2.3. *[GLT22a, Proposition 3.8] Let Γ be a graph of maximal dimension. Then there exists a basis $\mathsf{b} = (\mathsf{b}_1, \dots, \mathsf{b}_n)$ of the adjacency algebra of A_Γ such that $\mathrm{Mat}_{\mathsf{b}}(m_{A_\Gamma}) = A_\Gamma$. Further, if b_{i_0} is invertible, there exists a unique basis $\mathsf{b}' = (\mathsf{b}'_1, \dots, \mathsf{b}'_n)$ such that $b'_{i_0} = 1$ and $\mathrm{Mat}_{\mathsf{b}'}(m_{A_\Gamma}) = A_\Gamma$.*

Note that this is not always possible to find an index i_0 such that b_{i_0} is invertible, see [GLT22a, §3.2] for details.

A path of length $N \in \mathbb{N}$ in a graph Γ is a sequence $v_0, v_1, \dots, v_N$ of vertices together with a family of arrows $(e_1, \dots, e_N)$ where e_i is an arrow from v_{i-1} to v_i. The weight of a path is the product of the weights of the arrows, it is therefore an element of $\mathbb{Q}_+[\mathsf{Z}^{\pm 1}]$.

Let $k \in \{1, \dots, n\}$. We denote by M_k the $n \times n$ matrix whose coefficient $m_{i,j}$ is the sum of the weights of all paths of length $j-1$ starting at v_k and ending at v_i.

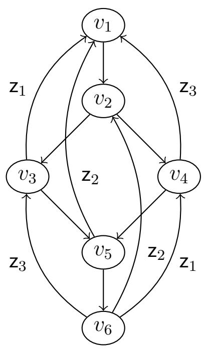
Theorem 2.4. *[GLT22a, Theorem 3.12]*

1. *The graph Γ is multiplicative at v_{i_0} and is of maximal dimension if and only if the matrix M_{i_0} is invertible. In this case, the columns of $M_{i_0}^{-1}$ define a basis $\mathsf{b}_{i_0} = \{\mathsf{b}_1, \dots, \mathsf{b}_{i_0} = 1, \dots, \mathsf{b}_n\}$ of the adjacency algebra (expressed in the basis $\{1, A_\Gamma, \dots, A_\Gamma^{n-1}\}$) that satisfies $\mathrm{Mat}_{\mathsf{b}_{i_0}}(m_{A_\Gamma}) = A_\Gamma$.*

2. *When M_{i_0} is invertible, the entries of the matrices in the basis b_{i_0} belong to $\frac{1}{\det M_{i_0}}\mathbb{Q}[\mathsf{Z}^{\pm 1}]$.*

3. *When μ_A is irreducible, the matrix M_{i_0} is invertible for any vertex v_{i_0}.*

Example 2.5. Consider the graph Γ with adjacency matrix A_Γ given as follows:



$$A_\Gamma = \begin{pmatrix} 0 & 0 & \mathsf{z}_1 & \mathsf{z}_3 & \mathsf{z}_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & \mathsf{z}_2 \\ 0 & 1 & 0 & 0 & 0 & \mathsf{z}_3 \\ 0 & 1 & 0 & 0 & 0 & \mathsf{z}_1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The minimal polynomial of A_Γ is

$$\mu_{A_\Gamma}(T) = T^6 - 2(\mathsf{z}_1 + \mathsf{z}_3)T^3 - 4\mathsf{z}_2 T^2 + (\mathsf{z}_1 - \mathsf{z}_3)^2$$

which is irreducible. The matrices M_1 and M_1^{-1} are respectively

$$\begin{pmatrix} 1 & 0 & 0 & \mathsf{z}_1+\mathsf{z}_3 & 2\mathsf{z}_2 & 0 \\ 0 & 1 & 0 & 0 & \mathsf{z}_1+\mathsf{z}_3 & 4\mathsf{z}_2 \\ 0 & 0 & 1 & 0 & 0 & \mathsf{z}_1+3\mathsf{z}_3 \\ 0 & 0 & 1 & 0 & 0 & 3\mathsf{z}_1+\mathsf{z}_3 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{\mathsf{z}_1+\mathsf{z}_3}{2} & -\mathsf{z}_2 \\ 0 & 1 & \frac{2\mathsf{z}_2}{\mathsf{z}_1-\mathsf{z}_3} & \frac{2\mathsf{z}_2}{\mathsf{z}_3-\mathsf{z}_1} & 0 & -\frac{\mathsf{z}_1+\mathsf{z}_3}{2} \\ 0 & 0 & \frac{3\mathsf{z}_1+\mathsf{z}_3}{2\mathsf{z}_1-2\mathsf{z}_3} & \frac{\mathsf{z}_1+3\mathsf{z}_3}{2\mathsf{z}_3-2\mathsf{z}_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{-1}{2\mathsf{z}_1-2\mathsf{z}_3} & \frac{-1}{2\mathsf{z}_3-2\mathsf{z}_1} & 0 & 0 \end{pmatrix}.$$

The graph Γ is multiplicative at v_1. Observe that in this particular example, that all the entries in $\{\mathsf{b}_1 = 1, \ldots, \mathsf{b}_6\}$ belong to $\mathbb{Q}_+[\mathsf{z}_1, \mathsf{z}_2, \mathsf{z}_3]$. For example, we have

$$b_3 = \frac{1}{\mathsf{z}_1 - \mathsf{z}_3}\left(2\mathsf{z}_2 A + \frac{1}{2}(3\mathsf{z}_1+\mathsf{z}_3)A^2 - \frac{1}{2}A^5\right) = \begin{pmatrix} 0 & \mathsf{z}_1 & \mathsf{z}_2 & 0 & 0 & \mathsf{z}_1\mathsf{z}_3 \\ 0 & 0 & \mathsf{z}_1 & 0 & \mathsf{z}_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathsf{z}_1 & \mathsf{z}_2 \\ 0 & 1 & 0 & 0 & 0 & \mathsf{z}_1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

2.3. Expansion of a graph

Let Γ be a strongly connected weighted graph with set of vertices $\{v_1, \ldots, v_n\}$. We assume that all the weights in Γ are positive monomials in $\mathbb{Q}_+[\mathsf{Z}^{\pm 1}]$.

Definition 2.6. The *expansion* of the graph Γ at v_1 is the graded graph with set of vertices $\Gamma_e = (\Gamma^\ell)_{\ell \in \mathbb{N}^*}$ constructed by induction as follows:

- $\Gamma_1 = \{(v_1, 1, 1)\}$
- if $(v_j, \mathsf{z}^\beta, \ell) \in \Gamma^\ell$ and there is an arrow $v_j \overset{m_{i,j}\mathsf{z}^{\beta_{i,j}}}{\rightarrow} v_i$ in Γ then $(v_i, \mathsf{z}^{\beta+\beta_{i,j}}, \ell+1) \in \Gamma^{\ell+1}$ and there is an arrow $(v_j, \mathsf{z}^\beta, \ell) \overset{m_{i,j}}{\rightarrow} (v_i, \mathsf{z}^{\beta+\beta_{i,j}}, \ell+1)$ in Γ_e.

Assume that Γ is a finite positively multiplicative graph. As in Definition 2.1, let Λ be a commutative n-dimensional algebra over $\mathbb{Q}(\mathsf{Z}^{\pm 1})$, $x \in \Lambda$ and $\mathsf{b} = (\mathsf{b}_1, \ldots, \mathsf{b}_n)$ a basis of Λ such that $\mathsf{b}_1 = 1$ and $\mathrm{Mat}_{\mathsf{b}}(m_x) = A_\Gamma$. Since Λ is an algebra over $\mathbb{Q}[\mathsf{Z}^{\pm 1}]$, it can also be seen as a $\mathbb{Q}$-algebra of infinite dimension with basis

$$\{\mathsf{z}^\beta \mathsf{b}_i \mid i \in \{1, \ldots, n\}, \beta \in \mathbb{Z}^m\}$$

where for all $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{Z}^m$ we have set $\mathsf{z}^\beta = \mathsf{z}_1^{\beta_1} \ldots \mathsf{z}_n^{\beta_n}$. We will denote by $\Lambda_{\mathbb{Q}}$ this algebra. We set $\Lambda'_e = \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}[\mathsf{q}]$ where q is a new indeterminate. It is easy to check that the set

$$\mathsf{b}'_e = \{\mathsf{q}^\ell \mathsf{z}^\beta \mathsf{b}_i \mid i \in \{1, \ldots, n\}, \beta \in \mathbb{Z}^m, \ell \in \mathbb{N}\}$$

is a basis of Λ'_e. We write Λ_e for the subspace of Λ'_e with basis $\mathsf{b}_e = \{\mathsf{q}^\ell \mathsf{z}^\beta \mathsf{b}_i \mid (v_i, \mathsf{z}^\beta, \ell) \in \Gamma_e\}$.

Proposition 2.7. *With the previous notation, the following assertions hold in $\mathsf{\Lambda}_e$.*

1. *For any vertex $(v_j, \mathsf{z}^\beta, \ell) \in \Gamma_e$*
$$\mathsf{q}x \times \mathsf{q}^\ell \mathsf{z}^\beta \mathsf{b}_j = \sum_{(v_j,\mathsf{z}^\beta,\ell) \overset{m_{i,j}}{\to} (v_i,\mathsf{z}^{\beta'},\ell+1)} m_{i,j} \mathsf{q}^{\ell+1} \mathsf{z}^{\beta'} \mathsf{b}_i.$$

2. *The element $1 = \mathsf{q}^0\mathsf{z}^0\mathsf{b}_1$ belongs to b_e and the product of two elements in the basis b_e expands on b_e with nonnegative coefficients. In particular $\mathsf{\Lambda}_e$ is a positively multiplicative subalgebra of $\mathsf{\Lambda}'_e$ with associated basis b_e.*

Example 2.8. The expansion of $\begin{tikzcd} v_1 \arrow[d] \\ v_2 \arrow[u, bend left=60, "\mathsf{z}_1"] \arrow[u, bend right=60, "\mathsf{z}_2"'] \end{tikzcd}$ is the graph

$$\begin{tikzcd}[column sep=tiny]
 & & & (v_1,1,1) \arrow[d] & & & \\
 & & & (v_2,1,2) \arrow[dl] \arrow[dr] & & & \\
 & & (v_1,\mathsf{z}_1,3) \arrow[d] & & (v_1,\mathsf{z}_2,3) \arrow[d] & & \\
 & & (v_2,\mathsf{z}_1,4) \arrow[dl] \arrow[dr] & & (v_2,\mathsf{z}_2,4) \arrow[dl] \arrow[dr] & & \\
 & (v_1,\mathsf{z}_1^2,5) \arrow[d] & & (v_1,\mathsf{z}_1\mathsf{z}_2,5) \arrow[d] & & (v_1,\mathsf{z}_2^2,5) \arrow[d] & \\
 & (v_2,\mathsf{z}_1^2,6) \arrow[dl] \arrow[dr] & & (v_2,\mathsf{z}_1\mathsf{z}_2,6) \arrow[dl] \arrow[dr] & & (v_2,\mathsf{z}_2^2,6) \arrow[dl] \arrow[dr] & \\
 (v_1,\mathsf{z}_1^3,7) & & (v_1,\mathsf{z}_1^2\mathsf{z}_2,7) & & (v_1,\mathsf{z}_1\mathsf{z}_2^2,7) & & (v_1,\mathsf{z}_2^3,7)
\end{tikzcd}$$

3. Affine Weyl group

3.1. Affine root systems and affine Lie algebras

Let $I = \{0, 1, \ldots, n\}$, $I^* = \{1, 2, \ldots, n\}$ and $A = (a_{i,j})_{(i,j)\in I^2}$ be a generalized Cartan matrix of a non-twisted affine root system. These affine root systems are classified in [Kac90] (see the Table 1 page 54). The matrix A has rank n and there exists a unique vector $v = (a_i)_{i\in I} \in \mathbb{Z}^{n+1}$ with $(a_i)_{i\in I}$ relatively primes and a unique vector $v^\vee = (a_i^\vee)_{i\in I} \in \mathbb{Z}^{n+1}$ with $(a_i^\vee)_{i\in I}$ relatively primes such that $v^\vee \cdot A = A \cdot {}^t v = 0$. Note that, since we are in the non-twisted case, we have $a_0 = a_0^\vee = 1$. We refer to [Car05, Kac90] for details and proofs of the results presented in this section.

Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A that is

1. $\mathfrak{h}$ is a complex vector space of dimension $n + 2$,
2. $\Pi = \{\alpha_0, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_0^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}$ are linearly independent subsets,
3. $a_{i,j} = \langle \alpha_i, \alpha_j^\vee \rangle$ for $0 \leqslant i, j \leqslant n$.

Here, $\langle\cdot,\cdot\rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ denotes the pairing $\langle\alpha, h\rangle = \alpha(h)$. Fix an element $d \in \mathfrak{h}$ such that $\langle\alpha_i, d\rangle = \delta_{0,i}$ for all $i \in I$ so that $\Pi^\vee \cup \{d\}$ is a basis of $\mathfrak{h}$. We denote by $\mathfrak{g}$ the affine Lie algebra associated to this datum and refer to [Kac90] for its definition.

Let $\Lambda_0, \ldots, \Lambda_n \subset \mathfrak{h}^*$ be the such that

$$\Lambda_i(\alpha_j^\vee) = \delta_{i,j} \quad \text{and} \quad \Lambda_i(d) = 0 \text{ for all } i, j \in I.$$

We set $\delta = \sum_{j=0}^n a_j\alpha_j$ so that

$$\delta(\alpha_i^\vee) = \sum_{j=0}^n a_j\alpha_j(\alpha_i^\vee) = \sum_{j=0}^n a_j a_{i,j} = [Av]_i = 0 \quad \text{and} \quad \delta(d) = a_0 = 1.$$

Then the family $(\Lambda_0, \ldots, \Lambda_n, \delta)$ is the dual basis of $(\alpha_0^\vee, \ldots, \alpha_n^\vee, d) \subset \mathfrak{h}$ and we have

$$\alpha_j = \sum_{i\in I}\langle\alpha_j, \alpha_i^\vee\rangle\Lambda_i + \langle\alpha_j, d\rangle\delta = \sum_{i\in I} a_{i,j}\Lambda_i \quad \text{for all } j \in I^*.$$

There exists an invariant nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$ such that $(\cdot,\cdot)$ is nondegenerate on $\mathfrak{h}$ and uniquely defined by

$$\begin{cases} (\alpha_i^\vee, \alpha_j^\vee) = \langle\alpha_i, \alpha_j^\vee\rangle a_i {a_i^\vee}^{-1} = a_j {a_j^\vee}^{-1} a_{j,i} & \text{for } i, j \in I \\ (\alpha_i^\vee, d) = 0 & \text{for } i \in I^* \\ (\alpha_0^\vee, d) = a_0 & \\ (d, d) = 0 & \end{cases}$$

It can be checked that $(\alpha_i^\vee, \alpha_j^\vee) = (\alpha_j^\vee, \alpha_i^\vee)$ for all $i, j \in I$. Let ν be the associated map from $\mathfrak{h}$ to its dual:

$$\begin{array}{cccc} \nu & : & \mathfrak{h} & \to & \mathfrak{h}^* \\ & & h & \mapsto & (h, \cdot) \end{array}$$

Then we have $\nu(d) = \Lambda_0$, $a_{i,j} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ and $\nu(\alpha_i^\vee) = \dfrac{2\alpha_i}{(\alpha_i, \alpha_i)}$. The form $(\cdot,\cdot)$ on $\mathfrak{h}$ then induces a form on $\mathfrak{h}^*$ via ν. We still denote this form $(\cdot,\cdot)$ and we have

$$\begin{cases} (\alpha_i, \alpha_j) = a_i^\vee {a_i}^{-1} a_{i,j} & \text{for } i, j \in I \\ (\alpha_i, \Lambda_0) = 0 & \text{for } i \in I^* \\ (\alpha_0, \Lambda_0) = a_0^{-1} & \\ (\Lambda_0, \Lambda_0) = 0 & \\ (\Lambda_0, \delta) = 1 & \end{cases}$$

There are different objects associated to the datum $(\mathfrak{h}, \Pi, \Pi^\vee)$, some of them live in $\mathfrak{h}$ other in $\mathfrak{h}^*$. We will as much as possible use the following convention: we will add the suffix “co” to the name of the object to indicate that it naturally lives in $\mathfrak{h}$ (eventhough we sometime think of it as an element of $\mathfrak{h}^*$) and we will add a superscript $^\vee$ to the notation. For instance, $\alpha_i^\vee$ is called a coroot as it is an element of $\mathfrak{h}$. We now introduce various lattices:

- the coweight lattice: $P_a^\vee := \bigoplus_{i\in I}\mathbb{Z}\alpha_i^\vee + \mathbb{Z}d \subset \mathfrak{h}$,
- the weight lattice $P_a = \{\gamma \in \mathfrak{h}^* \mid \gamma(P_a^\vee) \subset \mathbb{Z}\} = \bigoplus_{i\in I}\mathbb{Z}\Lambda_i + \mathbb{Z}\delta$,
- the dominant weights in $\mathfrak{h}^*$ are the elements of $P_a^+ = \bigoplus_{i\in I}\mathbb{Z}_{\geqslant 0}\Lambda_i + \mathbb{Z}\delta$,
- the root lattice $Q_a = \bigoplus_{i\in I}\mathbb{Z}\alpha_i$,
- the coroot lattice $Q_a^\vee = \bigoplus_{i\in I}\mathbb{Z}\alpha_i^\vee$.

For any $i \in I$, we define the simple reflection s_i on $\mathfrak{h}^*$ by

$$s_i(x) = x - \langle \alpha_i^\vee, x\rangle\alpha_i \text{ for any } x \in \mathfrak{h}^*.$$

The affine Weyl group W_a is the subgroup of $GL(\mathfrak{h}^*)$ generated by the reflections s_i. Since $(\Lambda_0, \dots, \Lambda_n, \delta)$ is the dual basis of $(\alpha_0^\vee, \dots, \alpha_n^\vee, d)$, we have for all $i \in I$

$$\begin{aligned} s_i(\delta) &= \delta - \langle \alpha_i^\vee, \delta\rangle\alpha_i = \delta \\ s_i(\Lambda_j) &= \Lambda_j \quad \text{if } i \neq j \\ s_i(\alpha_i) &= -\alpha_i \end{aligned}$$

The Weyl group W_a is acting on the weight lattice P_a.

Let $\mathring{A}$ be the matrix obtained from A by deleting the row and the column corresponding to 0. Then it is well-known that $\mathring{A}$ is a Cartan matrix of finite type. Let $\mathring{\mathfrak{h}}^*$ and $\mathring{\mathfrak{h}}$ be the vector spaces spanned by the subset $\mathring{\Pi} = \Pi \setminus \{\alpha_0\}$ and $\mathring{\Pi}^\vee = \Pi \setminus \{\alpha_0^\vee\}$. The root and weight lattices associated to $\mathring{A}$ are $Q = \bigoplus_{i\in I^*}\mathbb{Z}\alpha_i$ and $P = \bigoplus_{i\in I^*}\mathbb{Z}\omega_i$ where we have set $\omega_i = \Lambda_i - a_i^\vee\Lambda_0$ for all $i \in I^*$. We denote by W the finite Weyl group associated to $\mathring{A}$: it is generated by the orthogonal reflections s_i with respect to the hyperplane orthogonal to α_i in $\mathring{\mathfrak{h}}^*$. Finally we set $Q^\vee = \bigoplus_{i\in I^*}\mathbb{Z}\alpha_i^\vee \subset \mathring{\mathfrak{h}}^*$. Note that the reflection $s_i \in W_a$ for $i \in I^*$ stabilises $\mathring{\mathfrak{h}}^*$ so that the Weyl group W can be seen as the subgroup of W_a generated by $(s_i)_{i\in I^*}$.

Let θ be the highest root of the finite root system $\mathring{\Pi} = \{\alpha_1, \dots, \alpha_n\}$. Since we assumed that we are in the untwisted type, we have $\delta = \alpha_0 + \theta$. Let s_θ be the orthogonal reflection with respect to θ defined by $s_\theta(\gamma) = \gamma - \langle\theta^\vee, \gamma\rangle\theta$. For all $\gamma \in \mathfrak{h}^*$ we have

$$s_0 s_\theta(\gamma) = \gamma + (\gamma, \delta)\theta - ((\gamma, \theta) + \frac{1}{2}(\theta, \theta)(\gamma, \delta))\delta.$$

Consider $\beta \in \mathring{\mathfrak{h}}^*$. We define the map t_β on $\mathfrak{h}^*$ by

$$t_\beta(\gamma) = \gamma + (\gamma, \delta)\beta - ((\gamma, \beta) + \frac{1}{2}(\beta, \beta)(\gamma, \delta))\delta$$

for all γ in $\mathfrak{h}^*$. Note that if $(\gamma, \delta) = 0$ (as it is the case when $\gamma \in \mathring{\mathfrak{h}}^*$) we get a simpler formula

$$t_\beta(\gamma) = \gamma - (\gamma, \beta)\delta.$$

It is important to observe that t_β doesn't act as a translation on $\mathring{\mathfrak{h}}^*$. Nevertheless it can be shown that

- $t_\beta \circ t_{\beta'} = t_{\beta+\beta'}$ for all $\beta, \beta' \in \mathfrak{h}^*$,
- $w \circ t_\beta \circ w^{-1} = t_{w(\beta)}$ for all $\beta \in \mathfrak{h}^*$ and $w \in W$,
- $s_0 = t_\theta s_\theta = s_\theta t_{-\theta}$.

These relations tell us that W_a is the semi-direct product of the finite Weyl group W with a lattice that is generated by the W-orbit of θ. In the untwisted case, this orbit is equal to $\nu(Q^\vee)$ so that we have

$$W_a \simeq W \ltimes \nu(Q^\vee).$$

We will simply write $W_a \simeq W \ltimes Q^\vee$ where we identify $Q^\vee$ with the lattice $\nu(Q^\vee)$ given by:

$$\bigoplus_{i\in I} \mathbb{Z}\nu(\alpha_i^\vee) = \bigoplus_{i\in I} \mathbb{Z}\frac{2\alpha_i}{(\alpha_i, \alpha_i)} \subset \mathfrak{h}^*.$$

3.2. Affine action of W_a on alcoves

In the previous section, we have seen that the affine Weyl group W_a is a semi-direct product $W_a \simeq W \ltimes Q^\vee$. In this section, we construct an affine action of W_a on the set of alcoves in an Euclidean affine space V of dimension n. This action can be realised from the linear action in $\mathfrak{h}^*$ described in the previous section but the construction is technical and we will not need it here. We refer to [Kac90] and [Car05] for details of this construction. For the conventions in this section, we refer to [Bou07].

Remark 3.1. We will write the action of W_a on the affine space V and on the set of alcoves on the right. Let us briefly explain why. In this paper, we are studying the orbit of some weights $\gamma \in \mathfrak{h}^*$ under the action of W (and W_a). We have written this action on the left, so that if γ has a stabiliser of the form W_J where $J \subset I^*$, then the orbit of γ under the action of W will be in bijection with the set W^J of minimal length representatives of $W/W_J = \{xW_J \mid x \in W_a\}$. By considering a right action on the set of alcoves, the set W^J becomes connected (i.e. is made up of adjacent alcoves) and this makes it much easier to draw, see Examples 3.5 and 4.14.

We start with Φ, a reduced, irreducible, finite, crystallographic root system with set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$ in an n-dimensional $\mathbb{R}$-vector space V with inner product $(\cdot,\cdot)$. Let Φ^+ be the positive roots associated to Δ. For all $\alpha \in \Phi^+$, we define $H_{\alpha,0}$ to be the hyperplane orthogonal to α and $s_{\alpha,0}$ to be the orthogonal reflection with respect to $H_{\alpha,0}$. Setting $\alpha^\vee = 2\alpha/(\alpha,\alpha)$ for all $\alpha \in \Phi$ we have

$$H_{\alpha,0} = \{x \in V \mid (x,\alpha) = 0\} \quad \text{and} \quad (x)s_{\alpha,0} = x - (x,\alpha)\alpha^\vee.$$

The Weyl group W of Φ is the subgroup of $GL(V)$ generated by the orthogonal reflections s_α with $\alpha \in \Phi$. It is a Coxeter group with set of distinguished generators $S = \{s_1, \dots, s_n\}$ where we set $s_i := s_{\alpha_i,0}$.

The dual root system of Φ is $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ and the coroot lattice of Φ is $Q^\vee = \mathbb{Z}$-span of $\Phi^\vee$. The *fundamental weights* of Φ are the vectors $\omega_1, \dots, \omega_n$ where $(\omega_i, \alpha_j) = \delta_{ij}$. The weight lattice is $P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$, and the cone of *dominant weights* is $P^+ = \mathbb{N}\omega_1 + \cdots + \mathbb{N}\omega_n$. Note that $Q^\vee \subseteq P$.

The Weyl chambers are the closures of the open connected components of $V \backslash \mathcal{F}_0$ where

$$\mathcal{F}_0 = \bigcup_{\alpha \in \Phi^+} H_{\alpha,0}.$$

The fundamental Weyl chamber is defined by

$$\mathcal{C}_0 = \{x \in V \mid (x, \alpha) \geqslant 0 \text{ for all } \alpha \in \Phi^+\}.$$

The Weyl group W acts (on the right) simply transitively on the set of Weyl chambers.

The Weyl group W acts on $Q^\vee$ and the affine Weyl group W_a is $W_a = W \ltimes Q^\vee$ where we identify $\lambda \in Q^\vee$ with the translation $(x)t_\lambda = x + \lambda$. For $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, we define the hyperplane

$$H_{\alpha,k} = \{x \in V \mid (x, \alpha) = k\}$$

and write $s_{\alpha,k}$ for the (affine) orthogonal reflection with respect to $H_{\alpha,k}$. Explicitly, $(x)s_{\alpha,k} = x - ((x, \alpha) - k)\alpha^\vee$, so that $s_{\alpha,k} = s_\alpha t_{k\alpha^\vee}$. It is well-known that W_a is generated by the orthogonal reflections $s_{\alpha,k}$ and that it is a Coxeter group with distinguished set of generators $S_a = \{s_0, s_1, \dots, s_n\}$, where $s_0 = s_\theta t_{\theta^\vee}$, with θ the highest root of Φ. Any element of W_a can be uniquely written under the form ut_α where $u \in W$ and $\alpha \in Q^\vee$. We define two applications $\mathsf{f} : W_a \to W$ and $\mathsf{wt} : W_a \to Q^\vee$ via the equation $w = \mathsf{f}(w)t_{\mathsf{wt}(w)}$.

Each hyperplane $H_{\alpha,k}$ with $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$ divides V into two half spaces, denoted

$$H^+_{\alpha,k} = \{x \in V \mid (x, \alpha) > k\} \quad \text{and} \quad H^-_{\alpha,k} = \{x \in V \mid (x, \alpha) < k\}.$$

This is the "periodic orientation": it is invariant under translation by $\lambda \in Q^\vee$ since $H^+_{\alpha,k} t_\lambda = H^+_{\alpha,k+(\lambda,\alpha)}$.

The alcoves of W_a are the closure of the open connected components of $V \backslash \mathcal{F}$ where

$$\mathcal{F} = \bigcup_{\alpha \in \Phi^+, k \in \mathbb{Z}} H_{\alpha,k}.$$

The fundamental alcove is defined by

$$A_0 = \{x \in V \mid 0 < (x, \alpha) < 1 \text{ for all } \alpha \in \Phi^+\}.$$

The hyperplanes bounding A_0 are called the *walls* of A_0. Explicitly these walls are $H_{\alpha_i,0}$ with $i = 1, \dots, n$ and $H_{\theta,1}$. We say that a *face* of A_0 (that is, a codimension 1 facet) has type s_i for $i = 1, \dots, n$ if it lies on the wall $H_{\alpha_i,0}$ and of type s_0 if it lies on the wall $H_{\theta,1}$.

The affine Weyl group W acts simply transitively on the set of alcoves, and we use this action to identify the set of alcoves with W. We will denote this action on the right and identify the set of elements of W_a with the set of alcoves via $w \leftrightarrow A_0 w$. The fundamental alcove A_0

corresponds to the identity element e. Moreover, we use the action of W_a to transfer the notions of walls, faces, and types of faces to arbitrary alcoves. Alcoves A and A' are called *s-adjacent*, written $A \sim_s A'$, if $A \neq A'$ and A and A' share a common face of type s. Under the identification of alcoves with elements of W, the alcoves w and sw are s-adjacent.

Definition 3.2. Let $w \in W$, $s \in S$ and H be the unique hyperplane separating w and sw. We say that the crossing from w to sw is positive and write $w \overset{+}{\rightsquigarrow} sw$ if $w \in H^-$ and $sw \in H^+$. We say that the crossing is negative and write $w \overset{-}{\rightsquigarrow} sw$ otherwise.

Let $w \in W_a$ and let $(s_{i_\ell}, \dots, s_{i_1}) \in S^\ell$ be such that $w = s_{i_\ell} \dots s_{i_1}$. Then we have

$$e \sim_{s_{i_1}} s_{i_1} \sim_{s_{i_2}} s_{i_2} s_{i_1} \sim_{s_{i_3}} \cdots \sim_{s_{i_\ell}} s_{i_\ell} \cdots s_{i_1}.$$

In this way, the expression of $s_{i_\ell} \dots s_{i_1}$ representing w determine a sequence of adjacent alcoves starting at A_0 and finishing at $A_0 w$. It is a well known result that the length of $w \in W_a$ is equal to the number of hyperplanes that separate A_0 and $A_0 w$.

Example 3.3. Let Φ be a root system of type G_2 with simple roots α_1 and α_2 and fundamental weights ω_1 and ω_2. We have $P = Q^\vee$, and the dual root system is

$$\Phi^\vee := \pm\{\alpha_1^\vee, \alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee, \alpha_1^\vee + 3\alpha_2^\vee, 2\alpha_1^\vee + 3\alpha_2^\vee\}.$$

Recall that the affine Weyl group acts on the right on the set of alcoves. In the figure below

- the dark gray alcove is the fundamental alcove A_0;
- the thick arrows are the coroot system $\Phi^\vee$;
- the alcove $w = s_2 s_0 s_1 s_2 s_1$ is in dark green.
- the set of alcoves s_2, $s_2 s_0$, $s_2 s_0 s_1$, $s_2 s_0 s_1 s_2$ are in orange and numbered respectively from 1 to 4.
- the set of alcoves s_1, $s_2 s_1$, $s_1 s_2 s_1$, $s_0 s_1 s_2 s_1$, $s_2 s_0 s_1 s_2 s_1$ are in green.

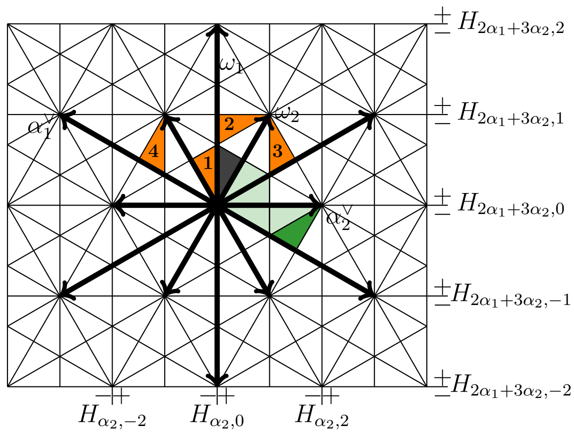


Figure 3.1: Alcoves and root system in type G_2.

3.3. Affine Grassmannians

The weak (left) Bruhat order on W_a is defined as follows: we have $u \leqslant v$ if and only if there exists a reduced expression of v ending by a reduced expression of u.

Definition 3.4. The affine Grassmannian elements are the minimal length representatives of the left cosets of W_a with respect to the finite Weyl group W. We will denote this set $W_a^{\Lambda_0}$. The weak Bruhat order induces a graph structure on $W_a^{\Lambda_0}$ and we will denote this graph by Γ_{Λ_0}.

When identifying the set of alcoves with W_a, the set $W_a^{\Lambda_0}$ is simply the set of alcoves that lies in the fundamental Weyl chambers $\mathcal{C}_0$. Two elements of $W_a^{\Lambda_0}$ are connected by a directed path in the graph Γ_{Λ_0} if and only if there is a path from one to the other such that all the crossing are positive.

The graph Γ_{Λ_0} also describes the orbit of the weight Λ_0 under the action of W_a. Indeed the stabiliser of Λ_0 in W_a is the finite Weyl group W, so that the orbit of Λ_0 is in bijection with W_a/W.

Example 3.5. Let Φ be a root system of type A_2 with simple roots α_1 and α_2. The dual root system is

$$\Phi^\vee := \pm\{\alpha_1^\vee, \alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee\}.$$

In type A, it can be useful to color the faces according to their types. In the picture below, we have colored the s_1, s_2 and s_0 faces in blue, green and red respectively. The gray alcove is the identity and the light gray alcoves represent $W_a^{\Lambda_0}$. We represent the graph Γ_{Λ_0} on the right hand-side. Note that it coincides with the expansion obtained in Example 2.8.

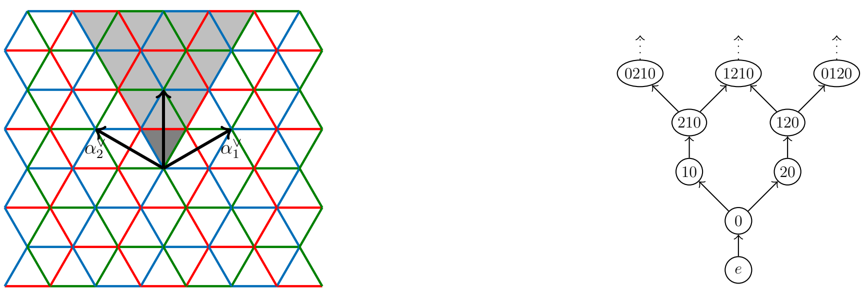


Figure 3.2: Affine grassmannians in type $\tilde{A}_2$ and its induced graph.

4. The graphs Γ_γ associated to level-0 weight

The level-0 weight lattice is the sublattice of P_a defined by

$$P_0 = \bigoplus_{i \in I^*} \mathbb{Z}\omega_i + \mathbb{Z}\delta.$$

The lattice P_0 is stabilized by W_a. In this section, we construct graphs encoding the orbit of the elements of P_0, the so-called level-0 weights, under the action of W_a. We denote by $\mathbb{Z}[Q^\vee]$ the group algebra of $Q^\vee$ over $\mathbb{Z}$ and write z^α for the element associated to $\alpha \in Q^\vee$ so that $\mathsf{z}^\alpha \mathsf{z}^{\alpha'} = \mathsf{z}^{\alpha+\alpha'}$. Recall that the affine Weyl group W_a is the semi-direct product $W \ltimes Q^\vee$ and that we defined $\mathsf{f} : W_a \to W$ and $\mathsf{wt} : W_a \to Q^\vee$ via the equation $w = \mathsf{f}(w) t_{\mathsf{wt}(w)}$.

Remark 4.1. We work over $\mathbb{Q}$ to fit with the setting of multiplicative graphs.

4.1. The graph Γ_ρ

We look at the level-0-weight $\rho = \omega_1 + \cdots + \omega_n$. The stabiliser of ρ under the action of W is reduced to the identity.

Definition 4.2. The graph Γ_ρ is the oriented weighted graph with weights in $\mathbb{Z}[Q^\vee]$ defined as follows

- the set of vertices is W
- there is an arrow $w \xrightarrow[a_i]{i} s_i w$ of type i and weight a_i for $i \in I^*$ when $\ell(s_i w) = \ell(w) + 1$
- there is an arrow $w \xrightarrow[\mathsf{z}^\alpha]{0} \mathsf{f}(s_0 w)$ of type 0 and weight z^α when $\ell(\mathsf{f}(s_0 w)) < \ell(w)$ and $\alpha = \mathsf{wt}(s_0 w)$.

There are different ways to interpret the graph Γ_ρ. First, if we forget about the weights and the orientations of the arrows, then the graph Γ_ρ is the Cayley graph of W with generators s_i and s_θ. Indeed we have $\mathsf{f}(s_0 w) = s_\theta w$ for all $w \in W$. As a consequence, since the stabiliser of ρ is reduced to the identity, this shows that Γ_ρ represents the orbit of ρ under the action of W. The edge weight function on Γ_ρ allows to recover the action of W_a on ρ. Let $w \in W$. If we set $u = \mathsf{f}(s_0 w)$ and $\alpha = \mathsf{wt}(s_0 w)$, we have

$$s_0 w(\rho) = u t_\alpha(\rho) = u(\rho - (\rho, \alpha)\delta) = u(\rho) - (\rho, \alpha)\delta$$

so that the weight α on the arrow from w to $\mathsf{f}(s_0 w)$ gives the coefficients of δ in the equality above. Next, if we forget about the 0-arrows, then the graph Γ_ρ is the graph of the weak Bruhat order on W. Finally, Γ_ρ also encodes encodes the orientation of adjacent alcoves as we will see later in this section.

In the following, when drawing examples of graphs Γ_ρ, we will omit the type $i \in I$ on the arrows as it can be deduced from the two extremities of the arrows. Further, we do not indicate the weight of an arrow when it is equal to 1.

Example 4.3. We draw the graph Γ_ρ in type $\tilde{A}_2$. We set $\mathsf{z}_1 = \mathsf{z}^{\alpha_1}$ and $\mathsf{z}_2 = \mathsf{z}^{\alpha_2}$ in $\mathbb{Z}[Q^\vee]$. The arrows going downward are all of type i with $i \in I^*$ and of weight 1 since we have $a_0 = a_1 = a_2 = 1$. The arrows going upward are all of type 0. On the left hand-side, we label the vertices with W and on the right hand-side with the corresponding weight $w(\rho)$. One can notice that when the arrows are going down (respectively up), so does the weights (mod δ).

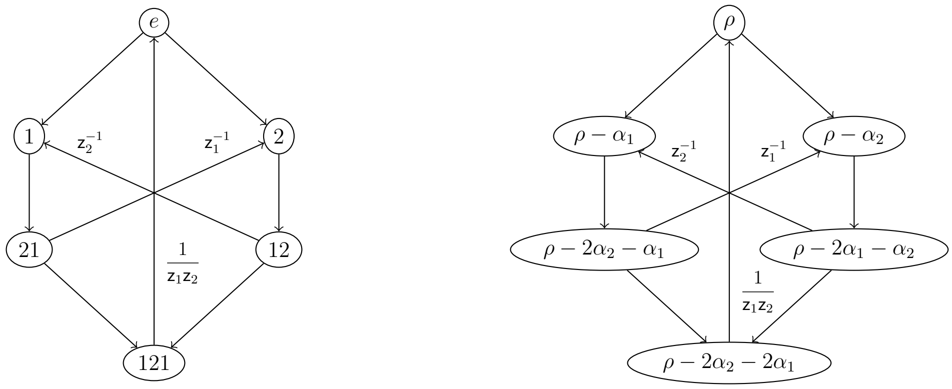


Figure 4.1: The graph Γ_ρ in type $\tilde{A}_2$.

Example 4.4. We draw the graphs Γ_ρ in type $\tilde{G}_2$. Again we set $\mathsf{z}_1 = \mathsf{z}^{\alpha_1}$ and $\mathsf{z}_2 = \mathsf{z}^{\alpha_2}$ in $\mathbb{Z}[Q^\vee]$ and we keep the same convention as in the previous example.

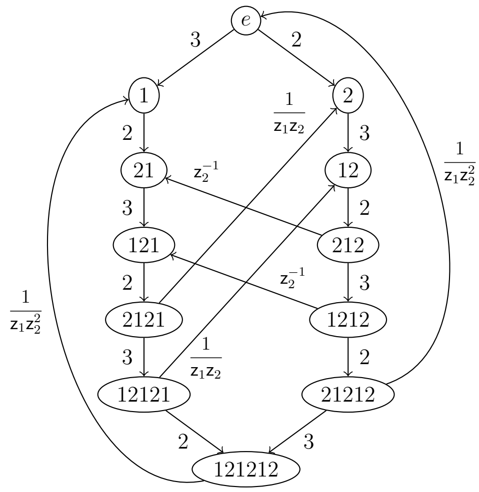


Figure 4.2: The graph Γ_ρ in type $\tilde{G}_2$.

All the arrows in the graph Γ_ρ are of the form $a_i\mathsf{z}^\alpha$ where α can be equal to 0. Recall that $\alpha \neq 0$ if and only if $i = 0$ and in this case, we have $a_0 = 1$. In the following proposition, we show how one can determine the sign of the crossing between two adjacent alcoves w and s_iw in W_a by looking at the direction of the arrow between $\mathsf{f}(w)$ and $\mathsf{f}(s_iw)$ in the graph Γ_ρ.

Proposition 4.5. *Let $w \in W_a$ and $i \in I$.*

1. *If there is an arrow from $\mathsf{f}(w)$ to $\mathsf{f}(s_iw)$ of weight $a_i\mathsf{z}^\alpha$ in Γ_ρ, then the crossing from w to s_iw is negative and $\mathsf{wt}(s_iw) = \mathsf{wt}(w) + \alpha$.*

2. *If there is an arrow from $\mathsf{f}(s_iw)$ to $\mathsf{f}(w)$ of weight $a_i\mathsf{z}^\alpha$ in Γ_ρ, then the crossing from w to s_iw is positive and $\mathsf{wt}(s_iw) = \mathsf{wt}(w) - \alpha$.*

Proof. We only need to prove (1). Let $\beta \in Q^\vee$ and $u \in W$ be such that $w = ut_\beta$. The crossing from w to $s_i w$ is of the same sign as the crossing from u to $s_i u$ since the orientation is left unchanged by translation. In the case where $i \in I^*$, we know that the crossing from u to $s_i u$ is positive if and only if $\ell(s_i u) < \ell(u)$. Further, we have $\mathsf{wt}(s_i w) = \mathsf{wt}(w)$ and $\alpha = 0$ so that $\mathsf{wt}(s_i w) = \mathsf{wt}(w) + \alpha$. Assume now that $i = 0$. The alcoves u and $s_0 u$ are separated by the hyperplane $(H_{\theta,1})u = H_{\theta u,1}$. We have

$$\begin{aligned} u \overset{-}{\rightsquigarrow} s_0 u &\iff (\theta)u \in \Phi^- \\ &\iff \ell(s_\theta u) < \ell(u) \\ &\iff \text{there is an arrow from } u \text{ to } s_\theta u \text{ in } \Gamma_\rho. \end{aligned}$$

The last equivalence comes from the fact that $\mathsf{f}(u) = u$ and $\mathsf{f}(s_0 u) = s_\theta u$. We prove the equality about the weight. By definition of Γ_ρ we have $\mathsf{wt}(s_0 u) = \alpha$ and we get $\mathsf{wt}(s_0 w) = \mathsf{wt}(s_0 u t_\beta) = \alpha + \beta$ as required. □

Let π be a (undirected) path on the graph Γ_ρ, that is, π is a sequence $u_0, u_1, \dots, u_N$ in W such that for all $k \in \{1, \dots, N\}$ there is an arrow from u_{k-1} to u_k or from u_k to u_{k-1} (π does not have to follow the directions of the arrows). We define the word of π to be $(i_1, \dots, i_N)$ where i_k is the type of the arrow between u_{k-1} and u_k and the group element of π to be $s_{i_1} \cdots s_{i_N} \in W_a$. Obviously, two different paths can have the same group element. Conversely, to any $w \in W$ and any expression (not necessarily reduced) $w = s_{i_1} \dots s_{i_n}$, we can associate a path $\pi_{\vec{w}}$ in Γ_ρ that starts at e and follows the arrow of type $i_1, \dots, i_N$.

With this in mind, the graph Γ_ρ can be interpreted as an automaton for the set of affine Grassmannians. This is a consequence of the following well known fact on affine Grassmannians. Let $w \in W_a$ such that $w = s_{i_1} \dots s_{i_N}$. Then $w \in W_a^{\Lambda_0}$ and the expression $s_{i_1} \dots s_{i_N}$ is reduced if and only if the crossing from $s_{i_k} \dots s_{i_N}$ to $s_{i_{k+1}} s_{i_k} \dots s_{i_N}$ is positive for all $1 \leqslant k \leqslant N-1$. Thus, $w \in W_a^{\Lambda_0}$ if and only if the path $\pi_{\vec{w}}$ (starting at e) corresponding to the expression $s_{i_1} \dots s_{i_N}$ of w only follows arrows in the opposite direction.

Example 4.6. We continue the previous example in type $\tilde{G}_2$. In order to picture the automaton more easily, we start at the identity (the black vertex) and we reverse the orientation of the arrows. We do not indicate the weight of the arrows (it doesn't play any role here) but we put the generator s_i when an arrow is of type i. One can obtain all reduced expressions of the affine Grassmannian elements simply by considering all the oriented paths starting at e and by concatenating the type of the arrows from right to left. For instance, one can check that the translations by the fundamental weights $t_{\omega_1}, t_{\omega_2}$ are in $W_a^{\Lambda_0}$ and that the expressions $t_{\omega_1} = s_1 s_2 s_1 s_2 s_0 s_1 s_2 s_1 s_2 s_0$ and $t_{\omega_2} = s_2 s_1 s_2 s_1 s_2 s_0$ are reduced.

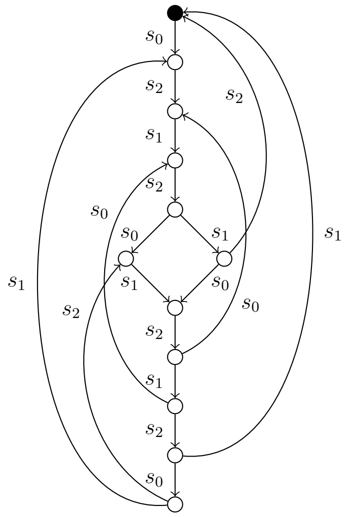


Figure 4.3: The graph Γ_ρ in type $\tilde{G}_2$.

4.2. The graphs Γ_γ

The construction of the previous section can be generalised to any level-0 weight. The main difference is that such weights can have a non-trivial stabiliser in W. We may assume that $\gamma = \omega_{i_1} + \cdots + \omega_{i_\ell} \in P$ and we set $J' = \{i_1, \dots, i_\ell\}$. Indeed, let γ be a level-0 weight so that $\gamma \in P + \mathbb{Z}\delta$. Since δ is stable under the action of W_a we may assume that $\gamma \in P$. Next, there exists a dominant weight which lies in the W-orbit of γ, hence we may assume that $\gamma = k_1\omega_1 + \cdots + k_\ell\omega_\ell$ where $(k_1, \dots, k_\ell) \in \mathbb{N}^\ell$. Now the orbit of $k_1\omega_1 + \cdots + k_\ell\omega_\ell$ under the action of W only depends on whether $k_i = 0$ or not. This is not true for the action of W_a but in the graphs Γ_γ that we will define, we may assume that $\gamma = \omega_{i_1} + \cdots + \omega_{i_\ell} \in P$ for some $J' = \{i_1, \dots, i_\ell\}$ just by changing the indeterminates labeling the arrows.

So, let $\gamma = \omega_{i_1} + \cdots + \omega_{i_\ell} \in P$, $J' = \{i_1, \dots, i_\ell\}$ and J be the complement of J' in $\{1, \dots, n\}$. Then the stabiliser of γ in W is W_J, the group generated by the set $\{s_j, j \in J\}$ and the orbit of γ under the action of W is in bijection with the set W^J of minimal length representatives of the left cosets W/W_J. It is a standard result that any $x \in W$ can be uniquely written as $x^J x_J$ where $x^J \in W^J$ and $x_J \in W_J$.

Recall that the action of t_β with $\beta \in Q^\vee$ on P_0 is given by $t_\beta(\gamma) = \gamma - (\gamma, \beta)\delta$. From there, we see that the group $\widetilde{W}_J := W_J \ltimes Q_J^\vee$ where

$$Q_J^\vee = \{\beta \in Q^\vee \mid (\gamma, \beta) = 0\} = \bigoplus_{j \in J} \mathbb{Z}\alpha_j^\vee$$

will stabilise γ (in fact it is strictly contained in the stabiliser of γ in W_a). The group $\widetilde{W}_J$ is an affine Weyl group. It is generated by the orthogonal reflections with respect to the hyperplanes $H_{\beta,k}$ with $\beta \in Q_J^\vee$ and $k \in \mathbb{Z}$. It acts simply transitively on the connected components of the set

$$\mathcal{F}_J := V \setminus \{H_{\alpha,k} \mid \alpha \in Q_J^\vee, k \in \mathbb{Z}\}.$$

Following Lusztig [Lus97], we call these connected components J-alcoves. The fundamental J-alcove is

$$\mathcal{A}_J = \bigcap_{\alpha\in\Phi_J^+} H_{\alpha,0}^+ \cap H_{\alpha,1}^- = \{x \in V \mid 0 < \langle x, \alpha\rangle < 1 \text{ for all } \alpha \in \Phi_J^+\}$$

where $\Phi_J^+ \subset \Phi^+$ is the positive root system of W_J. The closure of $\mathcal{A}_J$ is a fundamental domain for the action of $\widetilde{W}_J$ on V, that is, each $\lambda \in V$ is conjugate under the action of $\widetilde{W}_J$ to exactly one element in the closure of $\mathcal{A}_J$. Given $\beta \in Q^\vee$, we will slightly abuse the notation and write $\beta \in \mathcal{A}_J$ to mean that β lies in the closure of $\mathcal{A}_J$. Any $x \in \widetilde{W}_J$ can be uniquely written as ut_α where $u \in W_J$ and $\alpha \in Q_J^\vee$. Note that when $J = \{1, \ldots, n\}$ we recover the usual alcoves associated to W_a and $\mathcal{A}_J = A_0$, which explain the terminology of J-alcoves.

The affine Weyl group $\widetilde{W}_J$ and the set of J-alcoves have been studied in details in [GLP24] in the extended case where the authors studied positively folded alcove paths in $\mathcal{A}_J$ that bounce on the walls of $\mathcal{A}_J$. We also refer to [MST23] where the authors study retractions in buildings with respect to chimneys (which are our J-alcoves) in order to study orbits in affine flag varieties.

Let $\mathbb{Z}[Q^\vee/Q_J^\vee]$ be the group algebra of the quotient $Q^\vee/Q_J^\vee$. We write $\overline{\alpha}$ for the class of $\alpha \in Q^\vee$ modulo $Q_J^\vee$ and $\mathsf{z}^{\overline{\alpha}}$ for the corresponding element of $\mathbb{Z}[Q^\vee/Q_J^\vee]$. We now define our graphs Γ_γ and refer to §4.3 for some examples.

Definition 4.7. The graph Γ_γ is the oriented weighted graph with weights in $\mathbb{Z}[Q^\vee/Q_J^\vee]$ defined as follows

- the set of vertices is W^J
- there is an arrow $w \xrightarrow[a_i]{i} s_iw$ of type i for $i \in I^*$ if s_iw belong to W^J and $\ell(s_iw) = \ell(w)+1$
- there is an arrow $w \xrightarrow[z^{\overline{\alpha}}]{0} \mathsf{f}(s_0w)^J$ of type 0 if $\ell(\mathsf{f}(s_0w)^J) < \ell(w)$ and $\mathsf{wt}(s_0w) = \alpha$.

When $\gamma = \rho$ the graph Γ_ρ encodes the orbit of ρ and the structure of the semi-direct product $W \ltimes Q^\vee$ together with the periodic orientation of the alcoves in W_a. We will show that the graph Γ_γ encodes the orbit of γ under the action of W_a and the orientation of the alcoves in $\mathcal{A}_J$ in a very similar manner.

It is clear that if we forget about the 0 arrows then the graph Γ_γ encodes the orbit of γ under the action of W since the stabiliser of γ is W_J. Let $w \in W^J$ and consider the weight $w(\gamma)$. We want to show that the action of s_0 on $w(\gamma)$ is determined by the graph Γ_γ. First, assume that $\ell(\mathsf{f}(s_0w)^J) = \ell(w)$. In this case, since $s_\theta w$ and w are comparable in the Bruhat order, the elements $(s_\theta w)^J = \mathsf{f}(s_0w)^J$ and $w^J = w$ are also comparable in the Bruhat order, and therefore they must be equal. Further, the equality $(s_\theta w)^J = w$ implies that there exists $v \in W_J$ such that $s_\theta w = wv$ which in turn implies $v = w^{-1}s_\theta w$. This shows that $(\theta^\vee)w \in \Phi_J$. Finally we get

$$s_0w(\gamma) = s_\theta t_{\theta^\vee} w(\gamma) = s_\theta w t_{(\theta^\vee)w}(\gamma) = s_\theta w(\gamma) = (s_\theta w)^J(\gamma) = w(\gamma)$$

which explains why there is no arrow of type 0 leaving from w in this case. For the other cases, we see, doing exactly the same as in the case $\gamma = \rho$, that the weights of the 0 arrow determine the coefficient of δ in the action of s_0 on $w(\gamma)$.

We have seen that $W_J \ltimes Q_J^\vee$ stabilises the weight γ (but is not the full stabilizer of γ) so that to determine the orbit of γ under W_a it is sufficient to look at the action of $W_a/(W_J \ltimes Q_J^\vee)$. The later is in bijection with $W^J \times Q^\vee/Q_J^\vee$ and we will see that this set is closely related to the fundamental J-alcove $\mathcal{A}_J$. We also refer to [GLP24] for details.

Lemma 4.8. *Let $\alpha \in Q^\vee$ and $x \in W_a$.*

1. *There exists a unique element $\beta \in \overline{\alpha}$ such that $\beta \in \mathcal{A}_J$.*
2. *There exists a unique element $w \in \widetilde{W}_J$ such that $xw \in \mathcal{A}_J$. We write $p_J(x) = xw$.*
3. *If $\alpha \in \mathcal{A}_J$, there exists a unique element $v \in W_J$ such that $vt_\alpha \in \mathcal{A}_J$. We write $v = v_J(\alpha)$.*

Proof. (1) Let $\alpha \in Q^\vee$. There exists a unique element β in $\mathcal{A}_J$ in the orbit of α under the action of $\widetilde{W}_J$. Since $(\alpha)\sigma_{\kappa,k}$ is equal to α plus a multiple of κ, we see that β is equal to α plus a linear combination of roots in Φ_J, so that $\beta \equiv \alpha \mod Q_J^\vee$, that is $\beta \in \overline{\alpha}$.

(2) The alcove x lies in a connected component $\mathcal{U}$ of $V \setminus \{H_{\alpha,k} \mid \alpha \in Q_J^\vee, k \in \mathbb{Z}\}$ and there exists a unique element $w \in \widetilde{W}_J$ that sends this component to $\mathcal{A}_J$.

(3) Assume that $\alpha \in \mathcal{A}_J$. Let $w \in \widetilde{W}_J$ be such that $t_\alpha w \in \mathcal{A}_J$. Write $w = vt_\beta$ where $v \in W_J$ and $\beta \in Q_J^\vee$. Then $t_\alpha w = vt_{(\alpha)v+\beta} \in \mathcal{A}_J$. Note that since $\alpha \in \mathcal{A}_J$, the element w is a product of reflections with respect to hyperplanes going through α, hence $\mathsf{wt}(t_\alpha w) = \alpha$ and this forces the equality $(\alpha)v + \beta = \alpha$. $\square$

Let $x \in \mathcal{A}_J$ and $\alpha \in Q^\vee$. According to the second point of the lemma, there exists a unique $w \in \widetilde{W}_J$ such that $xt_\alpha w = p_J(xt_\alpha) \in \mathcal{A}_J$. It is easy to see that the element $p_J(xt_\alpha)$ only depends on x and the class of α modulo $Q_J^\vee$. Indeed, if $\beta = \alpha + \nu_J$ with $\nu_J \in Q_J^\vee$, then $t_{-\nu_J} w \in \tilde{W}_J$ and $xt_\beta t_{-\nu_J} w = xt_\alpha w \in \mathcal{A}_J$, hence showing that $p_J(xt_\beta) = p_J(xt_\alpha)$. We write $x \bullet t_{\overline{\alpha}} = p_J(xt_\alpha)$.

Proposition 4.9. *The map*

$$\begin{array}{ccc} \mathcal{A}_J \times Q^\vee/Q_J^\vee & \to & \mathcal{A}_J \\ (u, \overline{\alpha}) & \mapsto & u \bullet t_{\overline{\alpha}} \end{array}$$

defines a right group action of $Q^\vee/Q_J^\vee$ on the fundamental J-alcove $\mathcal{A}_J$.

Proof. We need to show that for all $u \in \mathcal{A}_J$ we have $\big(u \bullet t_{\overline{\beta}}\big) \bullet t_{\overline{\alpha}} = u \bullet t_{\overline{\alpha+\beta}}$. Fix $u \in \mathcal{A}_J$. There exists a unique $x \in \widetilde{W}_J$ such that $u \bullet t_{\overline{\beta}} = ut_\beta x \in \mathcal{A}_J$. Write $x = vt_\gamma$ with $v \in W_J$ and $\gamma \in Q_J^\vee$. The root $(\alpha)v$ is congruent to α modulo $Q_J^\vee$. There exists a unique $y \in \widetilde{W}_J$ such that $(u \bullet t_{\overline{\beta}})t_{(\alpha)v} y \in \mathcal{A}_J$. We have

$$\big(u \bullet t_{\overline{\beta}}\big) \bullet t_{\overline{\alpha}} = \big(u \bullet t_{\overline{\beta}}\big) \bullet t_{\overline{(\alpha)v}} = ut_\beta x t_{(\alpha)v} y = ut_\beta x t_{(\alpha)v} y = ut_{\beta+\alpha} xy \in \mathcal{A}_J.$$

By definition, there exists a unique $z \in \widetilde{W}_J$ such that $u \bullet t_{\overline{\alpha+\beta}} = ut_{\alpha+\beta} z \in \mathcal{A}_J$. This shows that $yx = z$ and $\big(u \bullet t_{\overline{\beta}}\big) \bullet t_{\overline{\alpha}} = u \bullet t_{\overline{\alpha+\beta}}$ as required. $\square$

Theorem 4.10. *The map $W^J \times Q^\vee/Q_J^\vee \to \mathcal{A}_J$ defined by $(u, \overline{\alpha}) \mapsto u \bullet t_{\overline{\alpha}}$ is a bijection.*

Proof. Let $x \in \mathcal{A}_J$. Let $x = uvt_\alpha$ where $u \in W^J$, $v \in W_J$ and $\alpha \in \mathcal{A}_J$ since $x \in \mathcal{A}_J$. The alcoves vt_α and uvt_α lie in the same connected component of $\mathcal{F}_J$, therefore $vt_\alpha \in \mathcal{A}_J$. This forces $v = v_J(\alpha)$ by the previous lemma. This shows that

$$\mathcal{A}_J \subset \{uv_J(\alpha)t_\alpha \mid \alpha \in \mathcal{A}_J, u \in W^J\}.$$

By definition, we have $v_J(\alpha)t_\alpha \in \mathcal{A}_J$ and this implies that $uv_J(\alpha)t_\alpha \in \mathcal{A}_J$ for all $u \in W^J$, showing the reverse inclusion. Let $\alpha \in \mathcal{A}_J$ and $u \in W^J$. We have $uv_J(\alpha)t_\alpha = ut_\beta v_J(\alpha) \in \mathcal{A}_J$ where $\beta \equiv \alpha$ modulo $Q_J^\vee$. It follows that $u \bullet t_{\overline{\beta}} = u \bullet t_{\overline{\alpha}} = uv_J(\alpha)t_\alpha$ hence showing that the map of the theorem is surjective.
Let $u, u' \in W^J$ and $\alpha, \beta \in Q^\vee$ such that $u \bullet t_{\overline{\alpha}} = u' \bullet t_{\overline{\beta}}$. Let $\alpha_0, \beta_0 \in \mathcal{A}_J$ be such that $\alpha_0 \equiv \alpha \mod Q_J^\vee$ and $\beta_0 \equiv \beta \mod Q_J^\vee$. We know that $uv_J(\alpha_0)t_{\alpha_0} = u \bullet t_{\overline{\alpha}} = u' \bullet t_{\overline{\beta}} = u'v_J(\beta_0)t_{\beta_0}$ which implies that $\alpha_0 = \beta_0$ since $uv_J(\alpha_0)$ and $u'v_J(\beta_0)$ lie in W. In particular we have $\alpha \equiv \beta \mod Q_J^\vee$. Now $uv_J(\alpha_0) = u'v_J(\beta_0)$ which implies that $u = u'$ and $v_J(\alpha_0) = v_J(\beta_0)$ since $u, u' \in W^J$ and $v_J(\alpha_0), v_J(\beta_0) \in W_J$. The map of the theorem is injective. $\square$

In the following theorem, we show how, given $w \in \mathcal{A}_J$ and $s_i \in S$, one can determine whether the alcove $s_i w$ lies in the J-alcove $\mathcal{A}_J$ by looking at the graph Γ_γ. Further we show that the sign of the crossing from w to $s_i w$ (when they are both in $\mathcal{A}_J$) is also determined by Γ_γ.

As in Proposition 4.5, we allow the weight $\overline{\alpha}$ to be equal to 0. This allows us to treat the cases $i \in I^*$ and $i = 0$ together.

Theorem 4.11. *Let $w \in \mathcal{A}_J$ and $i \in I$. We have $s_i w \in \mathcal{A}_J$ if and only if there is an arrow of type i adjacent to $\mathsf{f}(w)^J$ in Γ_γ. In this case*

1. *if $\mathsf{f}(w)^J \xrightarrow[a_i z^{\overline{\alpha}}]{i} \mathsf{f}(s_i w)^J$ then $\overline{\mathsf{wt}(s_i w)} = \overline{\mathsf{wt}(w) + \alpha}$ and the crossing is negative,*
2. *$\mathsf{f}(s_i w)^J \xrightarrow[a_i z^{\overline{\alpha}}]{i} \mathsf{f}(w)^J$ then $\overline{\mathsf{wt}(s_i w)} = \overline{\mathsf{wt}(w) - \alpha}$ and the crossing is positive.*

Proof. Write $w = uvt_\beta$ where $\beta \in \mathcal{A}_J$, $v = v_J(\beta)$ and $u = \mathsf{f}(w)^J \in W^J$. Let $H_{\alpha,k}$ be the hyperplane separating the alcoves $w = uvt_\beta$ and $s_i w = s_i uvt_\beta$. Then the hyperplane separating u and $s_i u$ is of direction $(\alpha)v^{-1}$. Since $v \in W_J$, we have $\alpha \in \Phi_J$ if and only if $(\alpha)v^{-1} \in \Phi_J$. Finally, we get that

$$s_i w \in \mathcal{A}_J \iff \alpha \notin \Phi_J \iff (\alpha)v^{-1} \notin \Phi_J \iff s_i u \in \mathcal{A}_J.$$

Assume first that $i \in I^*$. Then we have $s_i u \in \mathcal{A}_J$ if and only if $s_i u \in W^J$. By definition of Γ_γ, there is an arrow between u and $s_i u$ if and only if $s_i u \in W^J$. In this case, we obtain that $\mathsf{f}(s_i w)^J = s_i u$ and there is an arrow $u \xrightarrow[a_i]{i} s_i u$ when $s_i u > u$ in which case the crossing is negative and there is an arrow $s_i u \xrightarrow[a_i]{i} u$ when $s_i u < u$ in which case the crossing is positive. Since $\mathsf{wt}(s_i w) = \mathsf{wt}(w)$, this proves (1) when $i \in I^*$.

Assume that $i = 0$. The hyperplane separating u and $s_0 u$ is $H_{(\theta)u,1}$. It follows that

$$s_0 u \notin \mathcal{A}_J \iff (\theta)u \in \Phi_J \iff \exists v \in W_J, u^{-1} s_\theta u = v \iff \exists v \in W_J, s_\theta u = uv$$
$$\iff (s_\theta u)^J = u.$$

Assume that $s_0 w \in \mathcal{A}_J$. Then $s_0 u \in \mathcal{A}_J$ and $(s_\theta u)^J \neq u$. But since the elements $(s_\theta u)^J$ and $u^J = u$ are comparable in the Bruhat order this implies that $\ell((s_\theta u)^J) \neq \ell(u)$ and therefore there is an arrow between $u = \mathsf{f}(w)^J$ and $(s_\theta u)^J$. Conversely assume that there is an arrow of type 0 adjacent to u in Γ_γ. Then $\ell((s_\theta u)^J) \neq \ell(u)$ which implies that $(s_\theta u)^J \neq u$ and therefore $s_0 u \in \mathcal{A}_J$ as required.

Assume that there is an arrow $u \xrightarrow[z^{\overline{\alpha}}]{0} s_\theta u$, that is we have $\ell((s_\theta u)^J) < \ell(u)$. Then $s_\theta u < u$ and we have seen in the proof of Proposition 4.5 that this implies that the crossing from u to $s_0 u$ is negative. Finally since $\alpha = \mathsf{wt}(s_0 u)$ we have

$$\mathsf{wt}(s_0 w) = \mathsf{wt}(s_0 uvt_\beta) = \mathsf{wt}(s_\theta u t_\alpha v t_\beta) = \beta + v(\alpha) = \mathsf{wt}(w) + v(\alpha)$$

and since $v \in W_J$, we have $\overline{v(\alpha)} = \overline{\alpha}$ and $\overline{\mathsf{wt}(s_0 w)} = \overline{\mathsf{wt}(w) + \alpha}$ as required. This proves (1) in the case $i = 0$. The proof of (2) is similar. □

Corollary 4.12. *Let $w, u \in \mathcal{A}_J$ be two adjacent alcoves and $\alpha \in Q^\vee$. We have $w \overset{+}{\rightsquigarrow} u$ if and only if $w \bullet t_{\overline{\alpha}} \overset{+}{\rightsquigarrow} u \bullet t_{\overline{\alpha}}$*

Proof. According to the previous theorem, the sign of the crossing from w to u is determined by the element $\mathsf{f}(w)^J$ and $\mathsf{f}(u)^J$. Similarly, the sign of the crossing from $w \bullet t_{\overline{\alpha}}$ to $u \bullet t_{\overline{\alpha}}$ is determined by the element $\mathsf{f}(w \bullet t_{\overline{\alpha}})^J$ and $\mathsf{f}(u \bullet t_{\overline{\alpha}})^J$. There exists $x, y \in \widetilde{W}_J$ such that $w \bullet t_{\overline{\alpha}} = w t_\alpha x$ and $u \bullet t_{\overline{\alpha}} = u t_\alpha y$. Writing $x = v_1 t_{\beta_1}$ and $y = v_2 t_{\beta_2}$ with $v_1, v_2 \in W_J$ and $\beta_1, \beta_2 \in Q_J$, we see that $w \bullet t_{\overline{\alpha}} = w v_1 t_{(\alpha)v_1 + \beta_1}$ and $u \bullet t_{\overline{\alpha}} = u v_2 t_{(\alpha)v_2 + \beta_2}$. Thus we get $\mathsf{f}(w \bullet t_{\overline{\alpha}})^J = (wv_1)^J = w^J$ and $\mathsf{f}(u \bullet t_{\overline{\alpha}})^J = (uv_2)^J = u^J$ and the result. □

As in the case of $\gamma = \rho$, the graph Γ_γ can be interpreted as an automaton for reduced expression of the elements lying in $\mathcal{A}_J \cap W_a^{\Lambda_0}$. Indeed, we have $w \in \mathcal{A}_J \cap W_a^{\Lambda_0}$ and the expression $s_{i_1} \dots s_{i_N}$ is reduced if and only if $s_{i_k} \dots s_{i_N} \in \mathcal{A}_J$ for all $1 \leqslant k \leqslant N$ and if the crossing from $s_{i_k} \dots s_{i_N}$ to $s_{i_{k+1}} s_{i_k} \dots s_{i_N}$ is positive for all $1 \leqslant k \leqslant N-1$. Thus, in terms of the graph Γ_γ, we have $w \in \mathcal{A}_J \cap W_a^{\Lambda_0}$ if and only if there exists a path $\pi_{\vec{w}}$ in Γ_γ starting at e and following reversed arrows of type $i_1, \dots, i_N$. We have given two examples of such automaton in Example 4.14.

4.3. Some examples of Γ_γ

We now give a series of examples of graphs Γ_γ with a drawing of the corresponding strips in type $\tilde{G}_2$.

Example 4.13. Let W be of type $\tilde{A}_3$ and $\gamma = \omega_1$. The stabiliser of γ in W is the parabolic subgroup generated by $J = \{s_2, s_3\}$ and the set of minimal length representatives is $W^J = \{e, s_1, s_2 s_1, s_3 s_2 s_1\}$. We have $Q_J^\vee = \mathbb{Z}\alpha_2^\vee + \mathbb{Z}\alpha_3^\vee$ and the labels of the graph Γ_γ

are in the group algebra of $Q^\vee/Q_J^\vee = \mathbb{Z}\overline{\alpha_1}$. We represent the graph Γ_γ below where we have set $\mathsf{z}_1 = \mathsf{z}^{\overline{\alpha_1^\vee}}$. We omit the weights a_1, a_2, a_3 since they are all equal to 1.

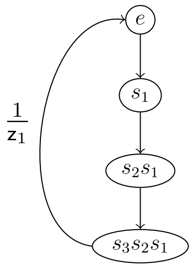


Figure 4.4: The graph Γ_{ω_1} in type $\tilde{A}_3$.

Next consider the weight $\gamma = \omega_2$. The stabiliser of γ in W is the parabolic subgroup generated by $J = \{s_1, s_3\}$ and the set of minimal length representatives is

$$W^J = \{e, s_2, s_1s_2, s_3s_2, s_1s_3s_2, s_2s_1s_3s_2\}.$$

We have $Q_J^\vee = \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_3^\vee$ and the labels of the graph Γ_γ are in the group algebra of $Q^\vee/Q_J^\vee = \mathbb{Z}\overline{\alpha_2}$. We represent the graph Γ_γ below where we have set $\mathsf{z}_2 = \mathsf{z}^{\overline{\alpha_2^\vee}}$. Again we omit the weights a_i's.

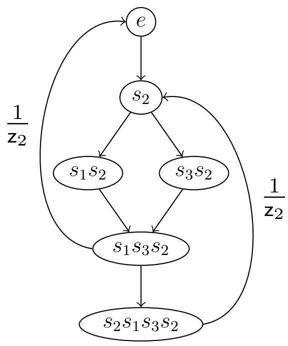


Figure 4.5: The graph Γ_{ω_2} in type $\tilde{A}_3$.

Example 4.14. Let W be of type $\tilde{G}_2$ as in Example 3.3. We first consider the level-0 weight $\gamma = \omega_2$. The stabiliser of γ in W is the parabolic subgroup generated by $J = \{s_1\}$ and the set of minimal length representatives is $W^J = \{e, s_2, s_1s_2, s_2s_1s_2, s_1s_2s_1s_2, s_2s_1s_2s_1s_2\}$. We have $Q_J^\vee = \mathbb{Z}\alpha_1^\vee$ and the labels of Γ_γ are in the group algebra of $Q^\vee/Q_J^\vee = \mathbb{Z}\overline{\alpha_2}$. We represent the graph Γ_γ below on the left hand-side where we have set $\mathsf{z}_2 = \mathsf{z}^{\overline{\alpha_2^\vee}}$. In the middle, we represent the strip $\mathcal{A}_J$. The subsets in green are from top to bottom $W^J \cdot t_{\overline{2\alpha_2}}$, W^J and $W^J \cdot t_{-\overline{2\alpha_2}}$ and the subsets in violet are from top to bottom $W^J \cdot t_{\overline{\alpha_2}}$, $W^J \cdot t_{-\overline{\alpha_2}}$ and $W^J \cdot t_{-\overline{3\alpha_2}}$. Finally, on the right handside, we represent the automaton for reduced expressions in $\mathcal{A}_J \cap W_a^{\Lambda_0}$. We keep the same convention as in Example 4.6 (we have inverted the direction of the arrows and the identity is the black vertex).

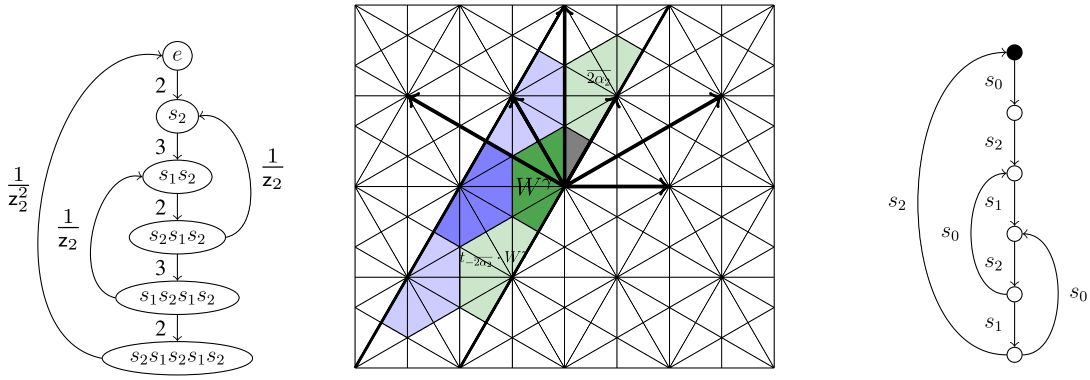


Figure 4.6: The graph Γ_{ω_2}, the set $\mathcal{A}_{\{s_1\}}$ and the associated automaton in type $\tilde{G}_2$.

Next consider the weight $\gamma = \omega_1$. The stabiliser of γ in W is the parabolic subgroup generated by $J = \{s_2\}$ and the set of minimal length representatives is:

$$W^\gamma = \{e, s_1, s_2s_1, s_1s_2s_1, s_2s_1s_2s_1, s_1s_2s_1s_2s_1\}.$$

We have $Q_J^\vee = \mathbb{Z}\alpha_2^\vee$ and the labels of Γ_γ are in the group algebra of $Q^\vee/Q_J^\vee = \mathbb{Z}\overline{\alpha_1}$. We represent the graph Γ_γ below on the left hand-side where we have set $\mathsf{z}_1 = \mathsf{z}^{\overline{\alpha_1^\vee}}$. In the middle, we represent the strip $\mathcal{A}_J$. The subsets in green are from top to bottom $W^J \cdot t_{\overline{\alpha_2}}, W^J, W^J \cdot t_{-\overline{\alpha_2}}$. Finally, on the right handside, we represent the automaton for reduced expressions in $\mathcal{A}_J \cap W_a^{\Lambda_0}$.

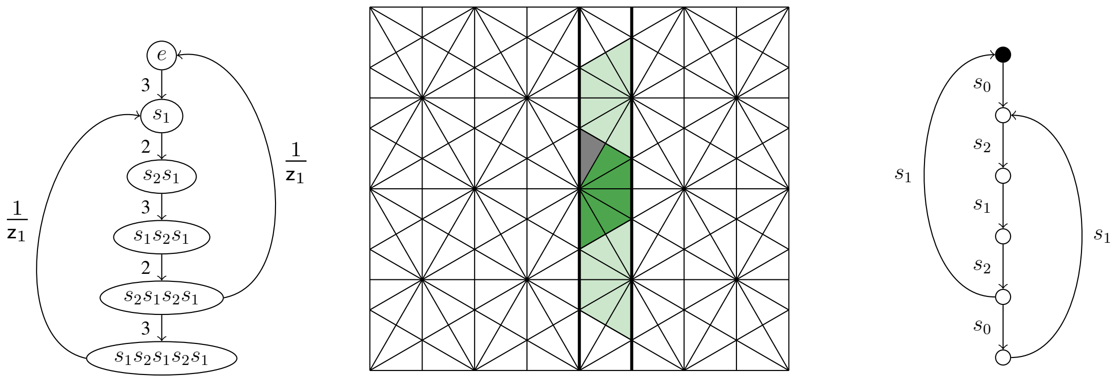


Figure 4.7: The graph Γ_{ω_1}, the set $\mathcal{A}_{\{s_2\}}$ and the associated automaton in type $\tilde{G}_2$.

4.4. KR graphs in type A and tableau combinatorics

As mentioned in the introduction, in type A_n, we can associate to any dominant weight $\lambda = \sum \lambda_i\omega_i \in \mathring{P}$ a finite affine crystal graph $\widehat{B}(\lambda)$. The affine Weyl group acts on $\widehat{B}(\lambda)$ and the image by the weight function of the orbit of the unique vertex of weight λ in $\widehat{B}(\lambda)$ under this action gives the orbit of λ under the action of W_a. Hence the graph Γ_λ also encodes the action of W_a on $\widehat{B}(\lambda)$.

In type A, there is a nice and simple combinatorial process that allows one to construct the graph Γ_γ (without the weights on its arrows). Let $\gamma = \omega_{i_1} + \cdots + \omega_{i_\ell} \in P_0$ and let $J' = \{i_1, \ldots, i_\ell\}$. We associate to γ the partition λ_γ whose Young diagram is made with ℓ columns of length $i_1, \ldots, i_\ell$. Recall that a semistandard tableau of shape λ_γ is a filling of λ_γ by integers between 1 and n whose columns strictly increase from top to bottom and weakly increase from left to right. We define T_γ to be the semistandard tableau of shape λ_γ that contains only i's on the i-th row. A key tableau of shape λ_γ is a semistandard tableau of shape λ_γ such that the entries in any column C also belong to the column located immediately to the left of C, except for the first column (see the figure below).

We can define an action of the affine Weyl group on the set of key tableaux of a given shape as follows. Let T be a key tableau. If $i \in \{1, \ldots, n\}$ then $s_i \bullet T$ is the key tableau obtained by changing the entries i into $i+1$ (respectively $i+1$ into i) in each column of T which contains i but not $i+1$ (respectively $i+1$ but not i). The tableau $s_0 \bullet T$ is the key tableau obtained by changing the entries $n+1$ into 1 (respectively 1 but not $n+1$) in the columns of T which contains $n+1$ but not 1 (respectively 1 but not $n+1$), and then by rearranging the columns in increasing order if needed.

Example 4.15. If $\gamma = \omega_1 + \omega_3$ in type $\tilde{A}_4$, then the partition λ_γ associated to γ is made of 2 columns of length $1, 3$ and we have

$$\lambda_\gamma = \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \\ \cline{1-1} \end{array} \quad \text{and} \quad T_\gamma = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}.$$

We have

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \xmapsto{s_1} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \xmapsto{s_3} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \xmapsto{s_2} \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \xmapsto{s_1} \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \xmapsto{s_0} \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}.$$

The graph $\widehat{B}(\gamma)$ can be obtain as follows:

- the vertices are the key tableaux which form the orbit of T_{λ_γ} under the action of W_a,
- there is an arrow between T and T' of type i if $T' = s_i \bullet T$ and $T \neq T'$.

Example 4.16. Let W be of type $\tilde{A}_3$ and $\gamma = \omega_1 + \omega_2$. The stabiliser of γ in W is the parabolic subgroup generated by $J = \{s_3\}$ and the set of minimal length representatives is:

$$W^J = \{e, s_1, s_2, s_1s_2, s_2s_1, s_3s_2, s_3s_2s_1, s_2s_1s_2, s_3s_1s_2, s_3s_2s_1s_2, s_2s_3s_1s_2, s_3s_2s_3s_1s_2\}$$

We obtain the following graphs

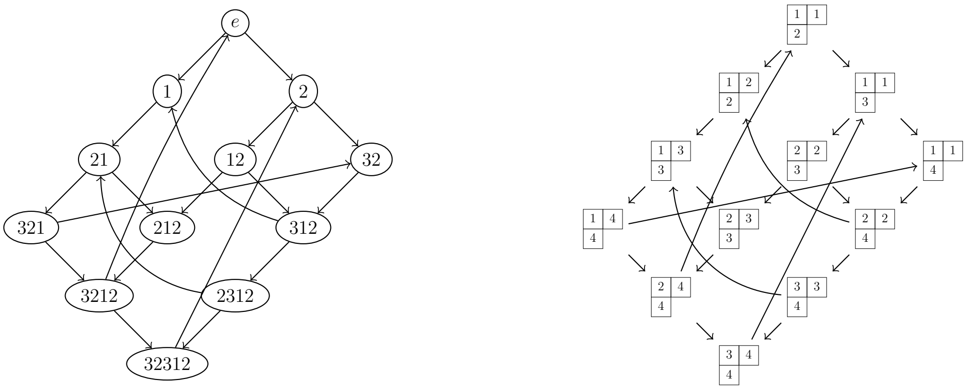


Figure 4.8: The unweighted graph $\Gamma_{\omega_1+\omega_2}$ in $\tilde{A}_3$ and the crystal $\widehat{B}(\omega_1+\omega_2)$.

Remark 4.17. There exist similar simple tableaux constructions for the classical types and for type G_2.

5. Homology ring of affine Grassmannians

In this section, we introduce the homology rings of affine Grassmannians which will be a key ingredient to show that the graphs that we have constructed in the previous sections are positively multiplicative.

5.1. Homology rings of Affine Grassmannians

Recall the definition of $W_a^{\Lambda_0}$ in Section 3.3. The homology algebra Λ associated to W_a is a commutative $\mathbb{Q}$-algebra with a distinguished basis $\{\xi_w, w \in W_a^{\Lambda_0}\}$ corresponding to affine Schubert classes whose associated structure coefficients are nonnegative integers:

$$\xi_w \xi_{w'} = \sum_{w'' \in W_a^{\Lambda_0}} c_{w,w'}^{w''} \xi_{w''} \quad \text{with } c_{w,w'}^{w''} \in \mathbb{Z}_{\geqslant 0} \text{ for all } w, w' \in W_a^{\Lambda_0}.$$

This ring is generated by a finite number of classes $\xi_{\rho_a}, a \in X = \{1, \ldots, N\}$ where $\{\rho_a, a \in X\}$ is a finite subset of $W_a^{\Lambda_0}$ whose description can be made explicit but depends on the affine root system considered ([Pon12] and also [LLM+14] Chapter 4 for a review on affine Schubert calculus in general affine types). We assume that $\rho_1 = s_0$. A general combinatorial description of the coefficients $c_{w,w'}^{w''}$ is not known in general but there exist simple formulas for the multiplication $\xi_{\rho_a}\xi_w$ where $a \in X$ (often referred as Pieri rules in the literature). In particular, as stated in [LLM+14], we have

$$\xi_{\rho_1}\xi_w = \xi_{s_0}\xi_w = \sum a_i \xi_{s_i w} \tag{5.1}$$

where the sums is over all $i \in I$ such that $s_i w > w$ and $s_i w \in W_a^{\Lambda_0}$. Recall here that the integers a_i have been defined in Section 3.1. More generally, we have for any $a \in X$

$$\xi_{\rho_a}\xi_w = \sum_{w'} c_{\rho_a,w}^{w'} \xi_{w'} \tag{5.2}$$

where the integers $c^{w'}_{\rho_a,w}$ can be positive only when $w' > w$ for the weak Bruhat order in $W_a^{\Lambda_0}$, that is if there exists $u \in W_a$ such that $w' = uw$ with $\ell(w') = \ell(u) + \ell(w)$. For the affine classical types, the rings of affine Grassmannians can be made more explicit by using subrings of symmetric functions and the previous Pieri rules then have a simple combinatorial description (see [LLM+14, Pon12]).

The set $W_a^{\Lambda_0}$ is stable by translation by classical dominant weights in P^+ in the following sense: given any alcove $A \in W_a^{\Lambda_0}$ and any dominant weight $\lambda \in P^+$, the alcove At_λ lies in $W_a^{\Lambda_0}$. In general, the element wt_λ with $w \in W_a$ and $\lambda \in P$ is not in the affine Weyl group but in the extended affine Weyl group. There exists however a unique element $u \in W_a$ such that $A_0 wt_\lambda = A_0 u$ with equality as subsets of V but not pointwise. We will denote this element of W_a by $w \star t_\lambda$. When $\lambda \in Q^\vee$, we have $wt_\lambda \in W_a$ and $w \star t_\lambda = wt_\lambda$. To lighten the notation we will write $\xi_{t_{\omega_i}}$ instead of $\xi_{e\star t_{\omega_i}}$. There is a nice factorization formula in Λ (see [LLM+14] Chapter 4 Corollary 4.14 and Equation (4.30))

$$\xi_{w\star t_{\omega_i}} = \xi_w \xi_{t_{\omega_i}} = \xi_{t_{\omega_i}} \xi_w \quad \text{for all } w \in W_a^{\Lambda_0} \text{ and } i \in I^*.$$

Definition 5.1. Let b_0 be the subset of $W_a^{\Lambda_0}$ of elements w such that $w \star t_{-\omega_i} \notin W_a^{\Lambda_0}$ for all $i \in I^*$.

Given $w \in W_a^{\Lambda_0}$, there exists a unique weight $\kappa = m_1\omega_1 + \cdots + m_n\omega_n \in P^+$ and a unique $u \in \mathsf{b}_0$ such that$w \star t_{-\kappa} = u \in \mathsf{b}_0$. As a consequence of the factorization above, we see that each Schubert class element ξ_w can be written under the form (recall that Λ is commutative)

$$\xi_w = \xi_{t_{\omega_1}}^{m_1} \cdots \xi_{t_{\omega_n}}^{m_n} \xi_u, u \in \mathsf{b}_0. \tag{5.3}$$

The set b_0 is finite with cardinality $\frac{1}{n_W}\mathrm{card}(W)$ where n_W is the cardinality of $P/Q^\vee$.

Example 5.2. In affine type A_n, the ring Λ can be identified with the subalgebra of the algebra of symmetric polynomials over $\mathbb{Q}$ in infinitely many variables generated by the complete symmetric functions $\xi_{\rho_a} = h_a$ with $a = 1, \dots, n$. The Schubert basis then corresponds to the n-Schur polynomials and the elements $\xi_{t_{\omega_i}}, i = 1, \dots, n$ to the ordinary Schur functions s_{R_i} indexed by the rectangular partitions $R_i = (n-i+1)^i$. We then have $n_W = n+1$ and the cardinality of b_0 is equal to $n!$. We refer to [LLM+14] for a complete exposition and for the related interesting and very rich combinatorics. Similar results exist for the other classical types and are stated in [Pon12].

Example 5.3. We represent the set of alcoves that lies in b_0 in type $\tilde{A}_2$ and $\tilde{G}_2$ in light gray (except for the fundamental alcove which is in dark gray). The fundamental weights ω_1, ω_2 are in blue. In type $\tilde{A}_2$ we have added 3 translates of b_0 by ω_1, ω_2 and $\omega_1 + \omega_2$ in green, blue and red respectively.

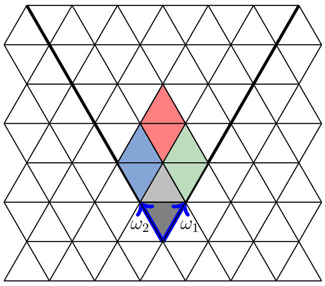


Figure 5.1: The set b_0 in type $\tilde{A}_2$.

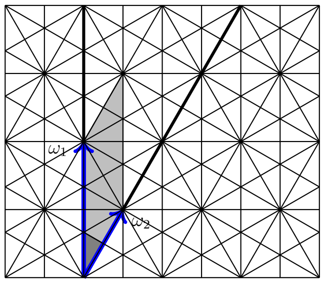


Figure 5.2: The set b_0 in type $\tilde{G}_2$.

Recall the definition of the graph Γ_{Λ_0} in Definition 3.4. The set of arrows in Γ_{Λ_0} is the set of pairs $(w, s_i w) \in W_a^{\Lambda_0} \times W_a^{\Lambda_0}$ such that $s_i w > w$. If we define the edge weight function p by setting $\mathsf{p}(w, s_i w) = a_i$ then the weighed graph Γ_{Λ_0} encodes the multiplication by ξ_{ρ_1} in the algebra Λ and in the basis $\{\xi_w \mid w \in W_a^{\Lambda_0}\}$. Indeed we have for $w \in W_a^{\Lambda_0}$

$$\xi_{\rho_1}\xi_w = \sum_{(w,w')\in E} \mathsf{p}(w, w')\xi_{w'}.$$

Example 5.4. We go back to the affine type $A_n^{(1)}$ case. The graph Γ_{Λ_0} can alternatively be regarded as the lattice of $(n+1)$-core partitions where the generators $s_i, i \in I$ act on the $(n+1)$-core partitions by either removing all the removable i-nodes or by adding all the addable i-nodes (this is well-defined since a core partition cannot contain addable i-nodes and removable i-nodes). There also exists a bijection between $(n + 1)$-core partitions and n-bounded partitions, that is partitions whose Young diagram has at most n columns. This yields another labeling of Γ_{Λ_0} (which does not coincide with the Young lattice on n-bounded partitions). The set b_0 contains $n!$ partitions which are exactly the n-bounded partitions that do not contain any rectangle R_i. Here again we refer to [LLM$^+$14] for a complete exposition.

5.2. Finite PM-graphs from affine Grassmannians

Lemma 5.5. *The elements $\xi_{t_{\omega_1}}, \dots, \xi_{t_{\omega_n}}$ are algebraically independent in the homology algebra Λ.*

Proof. Assume we have a polynomial P with coefficients in $\mathbb{Q}$ such that $P(\xi_{t_{\omega_1}}, \dots, \xi_{t_{\omega_n}}) = 0$. This gives a linear combination of Schubert classes equal to zero. Thus the polynomial P should be equal to zero since these Schubert classes gives a basis of Λ. $\square$

As we have seen in the previous section, any element ξ_w can be written under the form

$$\xi_w = \xi_{t_{\omega_1}}^{m_1} \cdots \xi_{t_{\omega_n}}^{m_n} \xi_u \quad \text{where} \quad u \in \mathsf{b}_0.$$

By the previous lemma, we see that the subring of Λ generated by $\xi_{t_{\omega_1}}, \dots, \xi_{t_{\omega_n}}$ is isomorphic to the ring of polynomials $\mathbb{Q}[\mathsf{z}_1, \dots, \mathsf{z}_n]$ in n variables via the map $\xi_{t_{\omega_i}} \leftrightarrow \mathsf{z}_i$ which in turn isomorphic to the monoid algebra of dominant weights $\mathbb{Q}[P^+] = \langle \mathsf{z}^\lambda \mid \lambda \in P^+\rangle_{\mathbb{Q}}$ via the maps $\mathsf{z}_i \leftrightarrow \mathsf{z}^{\omega_i}$. Identifying those rings, we see that Λ can be seen as a finite-dimensional ring over $\mathbb{Q}[P^+]$ with basis $\{\xi_w \mid w \in \mathsf{b}_0\}$. We extend the scalars to the group algebra $\mathbb{Q}[P]$ (or equivalently to the ring of Laurent polynomials $\mathbb{Q}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_n^{\pm 1}]$) and we denote by Λ_P this algebra. It is finite dimensional over $\mathbb{Q}[P]$ with basis $\{\xi_w \mid w \in \mathsf{b}_0\}$. Note that we are in the setting of Section 2 where the algebras are defined over $\mathbb{Q}$-rings of Laurent polynomials. Further, the ring Λ_P is positively multiplicative. For any $w \in \mathsf{b}_0$, we write m_{ξ_w} for the morphism of Λ_P defined by the multiplication by ξ_w and $\mathrm{Mat}_{\mathsf{b}_0}(m_{\xi_w})$ to for the matrix with respect to the basis $\{\xi_w \mid w \in \mathsf{b}_0\}$.

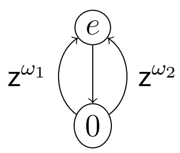
Definition 5.6. We define Γ_{b_0} to be the graph with set of vertices b_0 and adjacency matrix $\mathrm{Mat}_{\mathsf{b}_0}(m_{\xi_{\rho_1}})$.

Equivalently, the graph Γ_{b_0} can be defined as follows: we put a weighted arrow with weight $a_i\mathsf{z}^{\kappa}$ with $i \in I$ and $\kappa \in P_+$ from w to w' in b_0 whenever $\ell(s_iw) = \ell(w) + 1$ and $s_iw \star t_{-\kappa} = w'$. Note that κ is not equal to zero in the coefficient $a_i\mathsf{z}^{\kappa}$ only when $s_iw \notin \mathsf{b}_0$. We can see that Γ_{b_0} is strongly connected. Indeed, consider an element w in b_0. Then, there exists u in W_a and $\kappa \in Q_+^\vee$ such that $uw = t_\kappa$ with $\ell(t_\kappa) = \ell(u) + \ell(w)$. This yields a directed path in Γ_{b_0} from w to e. Since it is clear that there also exists a path from e to any element of b_0 in the graph Γ_{b_0}, this shows that Γ_{b_0} is strongly connected.

Proposition 5.7. *The graph Γ_{b_0} is positively multiplicative at e.*

Proof. By definition the adjacency matrix of Γ_{b_0} is the matrix of multiplication by ξ_{ρ_1} in the basis $\{\xi_w \mid w \in \mathsf{b}_0\}$ in Λ_P. Since the identity element ξ_e of Λ_P lies in b_0 and since Λ_P is positively multiplicative, we get the result by Proposition 2.2. $\square$

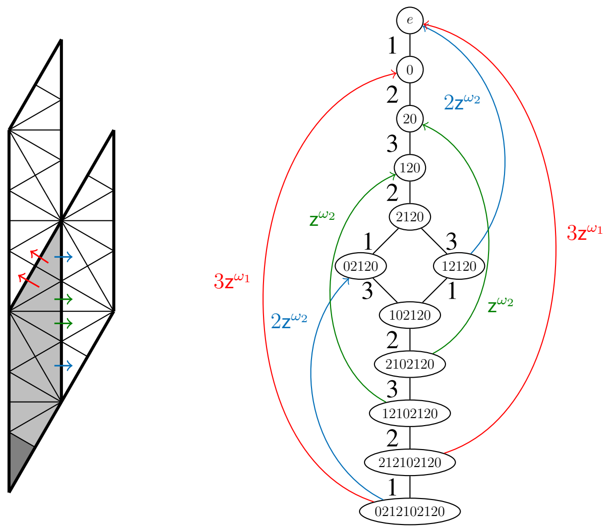
Example 5.8. We continue example 5.3. The graph Γ_{b_0} in type $\tilde{A}_2$ is particularly simple as there are only two alcoves in b_0 and only two pairs (w, s) in $\mathsf{b} \times I$ such that $s_iw \notin \mathsf{b}$. These pairs are $(s_0, 1)$ and $(s_0, 2)$ and we have $A_0s_1s_0 = A_0t_{\omega_1}$ (as sets) and $A_0s_2s_0 = A_0t_{\omega_2}$ (as sets) so that $s_1s_0 \star t_{-\omega_1} = e$ and $s_2s_0 \star t_{-\omega_2} = e$. We obtain the following graph and matrix



$$\begin{pmatrix} 0 & \mathsf{z}^{\omega_1} + \mathsf{z}^{\omega_2} \\ 1 & 0 \end{pmatrix}$$

Figure 5.3: The graph Γ_{b_0} in type $\tilde{A}_2$ and its transition matrix.

In type $\tilde{G}_2$, the situation is more complicated. There are 6 pairs $(w, i) \in \mathsf{b} \times I$ such that $s_iw \notin \mathsf{b}_0$. We represent these pairs below on the left hand-side where the gray area is the set b_0. For each pair (w, i) such that $s_iw \notin \mathsf{b}_0$ we put an arrow between (w, s_iw). We color the arrows in both figures to show which pairs corresponds to which arrow in the graph.



$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 2\mathsf{z}^{\omega_2} & 0 & 0 & 0 & 0 & 3\mathsf{z}^{\omega_1} & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\mathsf{z}^{\omega_1} \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \mathsf{z}^{\omega_2} & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & \mathsf{z}^{\omega_2} & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mathsf{z}^{\omega_2} \\
0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

Figure 5.4: The graph Γ_{b_0} in type $\tilde{G}_2$ and its adjacency matrix.

We also have the following interesting proposition whose proof is similar to Proposition 4.2 in [LT21]. We denote by $\mathrm{Frac}(\Lambda)$ and $\mathbb{Q}(P^+)$ the fraction fields of Λ and $\mathbb{Q}[P^+]$, respectively. Recall the definition of the elements $\xi_{\rho_a}, a \in X$ at the beginning of §5.1.

Proposition 5.9.

1. *Each $\xi_{\rho_a}, a \in X$ is algebraic over $\mathbb{Q}(P^+)$.*

2. *$\mathrm{Frac}(\Lambda)$ is an algebraic extension of $\mathbb{Q}(P^+)$ with degree $\mathrm{card}(\mathsf{b}_0)$. The set $\mathcal{I} = \{\xi_w \mid w \in \mathsf{b}_0\}$ is a basis of $\mathrm{Frac}(\Lambda)$ over $\mathbb{Q}(P^+)$.*

3. *The element ξ_{ρ_1} is primitive in $\mathrm{Frac}(\Lambda)$, that is we have $\mathrm{Frac}(\Lambda) = \mathbb{Q}(P^+)\big[\xi_{\rho_1}\big]$.*

4. *The expansion of Γ_{b_0} is the graph Γ_{Λ_0} with weights a_i.*

The fact that the expansion of Γ_{b_0} is the graph Γ_{Λ_0} allows one to use the multiplicative properties of the graph Γ_{b_0} to study random walks on Γ_{Λ_0}. This was done in type A in [LT21] in which case we have $a_i = 1$ for all i.

Corollary 5.10. *The graph Γ_{b_0} has maximal dimension, that is the degree of $\mu_{A_{\Gamma_{\mathsf{b}_0}}}$ is equal to $\mathrm{card}(\Gamma_{\mathsf{b}_0})$.*

Proof. The minimal polynomial of ξ_{ρ_1} has degree $\mathrm{card}(\Gamma_{\mathsf{b}_0})$ which is the size of the matrix $A_{\Gamma_{\mathsf{b}_0}}$. $\square$

Remark 5.11. The graph Γ_{b_0} has maximal dimension and is multiplicative at e. It follows by Theorem 2.4 (1) that the matrix M_e is invertible and the columns of M_e^{-1} give the vectors of a positively multiplicative basis $\mathsf{b}' = \{\mathsf{b}_0 = 1, \dots, \dots, \mathsf{b}_{n-1}\}$ expressed in the basis $\{1, A_{\Gamma_{\mathsf{b}_0}}, \dots, A^{n-1}_{\Gamma_{\mathsf{b}_0}}\}$. The coefficients appearing in these expressions are in $\frac{1}{\det M_e}\mathbb{Q}[P]$.

By uniqueness of the multiplicative basis satisfying $\mathsf{b}_0 = 1$, we see that the basis b' has to be equal to the basis $\mathsf{b} = \{\mathrm{Mat}_{\mathsf{b}_0}(\xi_w) \mid w \in \mathsf{b}_0\}$. As a consequence, since all the matrices in b have coefficients in $\mathbb{Q}[P^+]$, it follows that all the denominators in the expression of the b_i in $\{1, A_{\Gamma_\mathsf{b}}, \ldots, A_{\Gamma_\mathsf{b}}^{n-1}\}$ simplify (see Example 2.5). This also provides an explicit algorithm to compute the structure constants with respect to the basis b (these are given by the columns of the matrix in the basis b). We have computed explicitly the basis b in type $\tilde{G}_2$ [Gui23].

5.3. The $\mathbb{Q}$-algebra Λ_a

We have seen in the previous section that Λ_P is a finite-dimensional $\mathbb{Q}[P]$-algebra with basis $\{\xi_w \mid w \in \mathsf{b}_0\}$. The set b_0 is a fundamental domain for the action of P (by translations) on W_a, that is, any element of W_a can be uniquely written as $u \star t_\lambda$ where $u \in \mathsf{b}_0$ and $\lambda \in P$. We can therefore define two maps $\mathsf{wt}_{\mathsf{b}_0} : W_a \to P$ and $\mathsf{f}_{\mathsf{b}_0} : W_a \to \mathsf{b}_0$ by the equation $w = \mathsf{f}_{\mathsf{b}_0}(w) \star t_{\mathsf{wt}_{\mathsf{b}_0}(w)}$. Then by setting $\xi_w := \mathsf{z}^{\mathsf{wt}_{\mathsf{b}_0}(w)} \xi_{\mathsf{f}_{\mathsf{b}_0}(w)}$ for all $w \in W_a$ we can turn Λ_P into an infinite-dimensional algebra over $\mathbb{Q}$ with basis $\{\xi_w \mid w \in W_a\}$. We will denote by Λ_a this $\mathbb{Q}$-algebra. Recall the definition of $w \overset{+}{\rightsquigarrow} s_i w$ in 3.2.

Proposition 5.12. *For all $w \in W_a$, we have in Λ_a*

$$\xi_{\rho_1} \xi_w = \sum_{i \in I, w \overset{+}{\rightsquigarrow} s_i w} a_i \xi_{s_i w} \tag{5.4}$$

Proof. We know that for all $w \in W_a^{\Lambda_0}$, we have

$$\xi_{\rho_1} \xi_w = \sum_{i \in I, s_i w \in W_a^{\Lambda_0}, s_i w > w} a_i \xi_{s_i w}.$$

If $w \in W_a^{\Lambda_0}$, the assertion $s_i w \in W_a^{\Lambda_0}$ and $s_i w > w$ is equivalent to $w \overset{+}{\rightsquigarrow} s_i w$. Hence (5.4) holds for $w \in W_a^{\Lambda_0}$. Now, let $w \in W_a$. There exists $\gamma \in Q^\vee$ such that $x = w t_\gamma \in W_a^{\Lambda_0}$ so that $w = x t_{-\gamma}$ and $\xi_w = \mathsf{z}^{-\gamma} \xi_x$. We have

$$\xi_{\rho_1} \xi_w = \xi_{\rho_1} \mathsf{z}^{-\gamma} \xi_x = \mathsf{z}^{-\gamma} \sum_{s \in I, x \overset{+}{\rightsquigarrow} sx} a_s \xi_{sx} = \sum_{s \in I, x \overset{+}{\rightsquigarrow} sx} a_s \xi_{sxt_{-\gamma}} = \sum_{s \in I, w \overset{+}{\rightsquigarrow} sw} a_s \xi_{sw}.$$

The last equality comes from the fact that the orientation is unchanged by translation, that is we have $x \overset{+}{\rightsquigarrow} sx$ if and only if $w \overset{+}{\rightsquigarrow} sw$. ∎

The result above can be generalised to all ρ_a with $a \in X$. Let $w, w' \in W_a$. We write $w \overset{+}{\to} w'$ if and only if w lies on the positive side of all hyperplane separating w and w'. It is clear that if w and w' are adjacent then $w \overset{+}{\rightsquigarrow} w'$ is equivalent to $w \overset{+}{\to} w'$. Then, arguing as above we get for all $w \in W_a$ and $a \in X$:

$$\xi_{\rho_a} \xi_w = \sum_{w', w \overset{+}{\to} w'} a_{w'} \xi_{w'}.$$

Indeed, If $w \in W_a^{\Lambda_0}$, the assertion $w' \in W_a^{\Lambda_0}$ and $w' \geqslant w$ in the weak Bruhat order is equivalent to $w \overset{+}{\to} w'$.

5.4. Fundamental domain and basis of the homology algebra

In Section 5.2, we have used the lattice of fundamental weights and the fundamental domain b_0 to construct a (finite) positively multiplicative graph Γ_{b_0}. In this section we generalise this result to any $\mathbb{Z}$-sublattice L of P of rank n with basis $v_1, \ldots, v_n$ and to any fundamental domain.

Definition 5.13. A subset b of W_a is a fundamental domain for the action of L if and only if any element w of W_a can uniquely be written as a product $u \star t_\gamma$ where $\gamma \in L$ and $u \in \mathsf{b}$. To any fundamental domain b we associate two applications $\mathsf{wt}_\mathsf{b} : W_a \to L$ and $\mathsf{f}_\mathsf{b} : W_a \to \mathsf{b}$ defined by the equation $w = \mathsf{f}_\mathsf{b}(w) \star t_{\mathsf{wt}_\mathsf{b}(w)}$.

Let $\mathbb{Q}[L] = \langle \mathsf{z}^\alpha \mid \alpha \in L \rangle$ be the group algebra of L over $\mathbb{Q}$. It is a subalgebra of $\mathbb{Q}[P]$. By the factorisation property and since L is a sublattice of P, we have $\xi_w = \mathsf{z}^{\mathsf{wt}_\mathsf{b}(w)} \xi_{\mathsf{f}_\mathsf{b}(w)} \in \Lambda_a$ for all $w \in W_a$ and all fundamental domain b. This shows that Λ_a can be turned into a $\mathbb{Q}[L]$-algebra with basis $\{\xi_w \mid w \in \mathsf{b}\}$. We will write Λ_L when we consider Λ_a as an algebra over $Q[L]$. Note that Λ_L is finite-dimensional since L is assumed to be of rank n.

Remark 5.14. In the case where $L = Q^\vee$, W is a fundamental domain for the action of $Q^\vee$ so that $\{\xi_w \mid w \in W\}$ is a basis of $\Lambda_{Q^\vee}$.

Let $m_{\xi_{\rho_1}}$ be the linear endomorphism of Λ_L defined by multiplication by ξ_{ρ_1}. Given an L-fundamental domain b, we write $\mathrm{Mat}_\mathsf{b}(m_{\xi_{\rho_1}})$ for the matrix of $m_{\xi_{\rho_1}}$ in the basis $\{\xi_w \mid w \in \mathsf{b}\}$. We then define Γ_b to be the graph with set of vertices b and adjacency matrix $A_{\Gamma_\mathsf{b}} = \mathrm{Mat}_\mathsf{b}(m_{\xi_{\rho_1}})$. Equivalently, the graph Γ_b can be defined as follows: we put a weighted arrow with weight $a_i \mathsf{z}^\kappa$ with $i \in I$ from w to w' in b whenever $w \overset{+}{\rightsquigarrow} s_i w$, $s_i w \star t_{-\kappa} = w'$ and $\kappa \in L$. Note that κ is not equal to zero in the coefficient $a_i \mathsf{z}^\kappa$ only if $s_i w \notin \mathsf{b}$. We can see that Γ_b is strongly connected. Arguing as in Proposition 5.7 we get the following result.

Proposition 5.15. *Let* b *be a fundamental domain for the action of* L *containing the fundamental alcove* A_0*. The graph* Γ_b *with adjacency matrix* $\mathrm{Mat}_\mathsf{b}(m_{\xi_{\rho_1}})$ *is positively multiplicative at* e*.*

This proposition is the key to show that Γ_ρ is positively multiplicative. Indeed we will show that the adjacency matrix of Γ_ρ is essentially the matrix of $m_{\xi_{\rho_1}}$ expressed in the basis associated to the coroot lattice $Q^\vee$ and to the fundamental domain W.

Example 5.16. Let $L = Q^\vee$ in type $\tilde{A}_2$. We can choose as L-fundamental domain b the set of alcoves in green and the fundamental alcove A_0 which is in dark grey. We represent the graph with adjacency matrix $\mathrm{Mat}_\mathsf{b}(m_{\xi_{\rho_1}})$ on the right hand-side. We set $\mathsf{z}_1 = \mathsf{z}^{\alpha_1}$ and $\mathsf{z}_2 = \mathsf{z}^{\alpha_2}$.

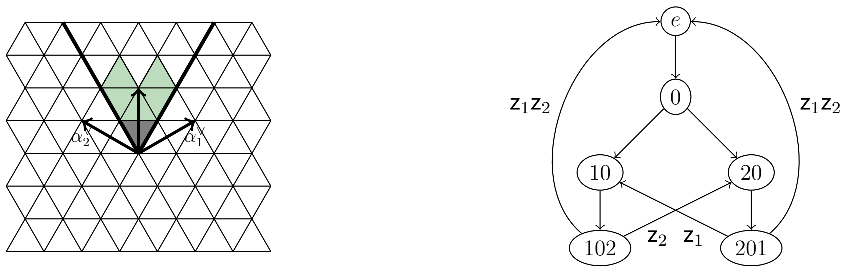


Figure 5.5: A fundamental domain for $Q^\vee$ and the associated graph in type $\tilde{A}_2$.

Example 5.17. Let $L = \langle \omega_1, 2\omega_2 \rangle_{\mathbb{Z}}$ in type $\tilde{A}_2$. Then L is a sublattice of P of rank 2. The set b defined by the alcoves in green and the fundamental alcove A_0 in dark grey is a L fundamental domain. We represent the graph with adjacency matrix $\mathrm{Mat}_{\mathsf{b}}(m_{\xi_{\rho_1}})$ on the right hand-side. We set $\mathsf{z}_1 = \mathsf{z}^{\omega_1}$ and $\mathsf{z}_2 = \mathsf{z}^{2\omega_2}$.

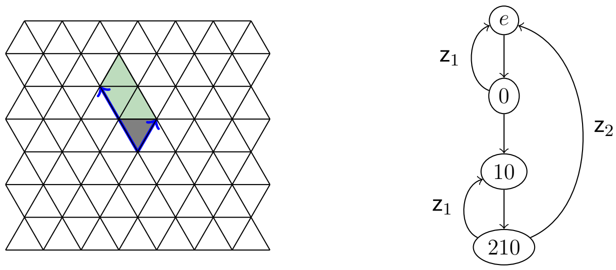


Figure 5.6: Fundamental domain for L and its associated the graph Γ_{b} in type $\tilde{A}_2$.

As in the case of P, we can prove the following result using similar methods as in [LT21]. Here $\mathbb{Q}(L)$ denotes the fraction field of $\mathbb{Q}[L]$.

Proposition 5.18. *Let L be a sublattice of P of rank n and b be a L-fundamental domain.*

1. *The field $\mathrm{Frac}(\Lambda)$ is a finite extension of $\mathbb{Q}(L)$ with degree $\mathrm{card}(\mathsf{b})$. The family $(\xi_w \mid w \in \mathsf{b})$ is a basis of $\mathrm{Frac}(\Lambda)$ over $\mathbb{Q}(L)$.*
2. *The element ξ_{ρ_1} is primitive in $\mathrm{Frac}(\Lambda)$ as an extension of $\mathbb{Q}(L)$, i.e we have $\mathrm{Frac}(\Lambda) = \mathbb{Q}(L)[\xi_{\rho_1}]$.*
3. *The graph Γ_{b} has maximal dimension, that is the degree of $\mu_{A_{\Gamma_{\mathsf{b}}}}$ is equal to $\mathrm{card}(\Gamma_{\mathsf{b}})$.*

5.5. Quotient of some subalgebra of the homology algebra

Let $J' \subset \{1, \dots, n\}$ and let J be the complement of J' in the set $\{1, \dots, n\}$. Recall the definition of $\mathcal{A}_J$ in Section 4.2. In this section, we define a quotient of an ideal of Λ_a which will have a basis in bijection with $\mathcal{A}_J$.

Lemma 5.19. *Let α_j be a simple root and let $\mathcal{I}^+_{\alpha_j,0}$ be the $\mathbb{Q}$-vector space $\langle \xi_w \mid w \subset H^+_{\alpha_j,0} \rangle_{\mathbb{Q}}$. Then $\mathcal{I}^+_{\alpha_j,0}$ is a subalgebra of Λ_a.*

Proof. We only need to prove that $\mathcal{I}^+_{\alpha_j,0}$ is stable by multiplication. Let $x, y \subset H^+_{\alpha_j,0}$. There exists $\lambda \in \oplus_{k \neq j} \mathbb{N}\omega_k$ such that $x \star t_\lambda \in W_a^{\Lambda_0}$. We have $\xi_x \xi_y = \xi_{t_{-\lambda}} \xi_{x \star t_\lambda} \xi_y$. For all $a \in X$, recall that

$$\xi_{\rho_a} \xi_w = \sum_{w', w \xrightarrow{+} w'} a_{w'} \xi_{w'}.$$

Since $x \star t_\lambda \in W_a^{\Lambda_0}$, we have $\xi_{x \star t_\lambda} \in \Lambda$ and since the ξ_{ρ_a}'s generate Λ, we can show that

$$\xi_{x \star t_\lambda} \xi_y = \sum_{w', y \xrightarrow{+} w'} a_{w'} \xi_{w'}.$$

Now $y \xrightarrow{+} w'$ and $y \subset H^+_{\alpha_j,0}$ implies that $w' \in H^+_{\alpha_j,0}$. Since $\lambda \in \oplus_{k\neq j}\mathbb{N}\omega_j$ we get that $w' \star t_{-\lambda}$ also lies in $H^+_{\alpha_j,0}$. Finally $\xi_x\xi_y$ lies in $I^+_{\alpha_j,0}$. $\square$

We set $\mathcal{C}_J = \cap_{j\in J} H^+_{\alpha_j,0}$ (this is the fundamental Weyl chamber for the group W_J). As a consequence of the previous lemma, we see that

$$\Lambda^+_{J,0} = \bigcap_{j\in J} \mathcal{I}^+_{\alpha_j,0} = \langle \xi_w \mid w \in \mathcal{C}_J\rangle_{\mathbb{Q}}$$

is a subalgebra of Λ_a. Note that if $J = \varnothing$ we have $\Lambda^+_{J,0} = \Lambda_a$.

Lemma 5.20. *Let $\alpha \in \Phi^+_J$ and let $\mathcal{I}^+_{\alpha,1} = \langle \xi_w \mid w \subset H^+_{\alpha,1} \cap \mathcal{C}_J\rangle_{\mathbb{Q}}$. Then $\mathcal{I}^+_{\alpha,1}$ is an ideal of $\Lambda^+_{J,0}$.*

Proof. Let $\xi_x \in \Lambda^+_{J,0}$ and $\xi_y \in \mathcal{I}^+_{\alpha,1}$. There exists $\lambda \in \oplus_{k\in J'}\mathbb{N}\omega_k$ such that $x \star t_\lambda \in W_a^{\Lambda_0}$. We have $\xi_x\xi_y = \xi_{t_{-\lambda}}\xi_{x\star t_\lambda}\xi_y$. Arguing as in the proof of the previous lemma we get

$$\xi_{x\star t_\lambda}\xi_y = \sum_{w', y\xrightarrow{+} w'} a_{w'}\xi_{w'}.$$

Now $y \xrightarrow{+} w'$ implies that $w' \in H^+_{\alpha,1} \cap \mathcal{C}_J$. Since $\lambda \in \oplus_{k\in J'}\mathbb{N}\omega_i$, we get that $w' \star t_{-\lambda} \in H^+_{\alpha,1} \cap \mathcal{C}_J$ since the root α is a positive sum of simple roots in $\{\alpha_j, j \in J\}$. It follows that $\xi_x\xi_y \in \mathcal{I}^+_{\alpha,1}$. $\square$

Definition 5.21. We define Λ_J to be the following quotient:

$$\Lambda_J := \Lambda^+_{J,0} / \sum_{\alpha\in\Phi^+_J} \mathcal{I}^+_{\alpha,1}.$$

It is clear that Λ_J admits $\{\tilde{\xi}_u \mid u \in \mathcal{A}_J\}$ as a basis where $\tilde{\xi}_u$ denotes the image of ξ_u in the quotient Λ_J and $\mathcal{A}_J$ is the fundamental J-alcove defined in Section 4.2. In the case where $J = \varnothing$ and $\gamma = \rho$, we get that $\Lambda^+_{J,0} = \Lambda_a$ and $\Lambda_J = \Lambda_a$. The following proposition is a consequence of Proposition 5.12.

Proposition 5.22. *For all $w \in \mathcal{A}_J \cap W_a^{\Lambda_0}$, we have*

$$\tilde{\xi}_{\rho_1}\tilde{\xi}_w = \sum_{i\in I, w \overset{+}{\rightsquigarrow} s_i w, s_i w\in\mathcal{A}_J} a_i\tilde{\xi}_{s_i w}.$$

We have defined an action of $Q^\vee/Q^\vee_J$ on $\mathcal{A}_J$ in Theorem 4.10 and showed that

$$\mathcal{A}_J = \{u \bullet t_{\overline{\alpha}} \mid u \in W^J, \alpha \in Q^\vee\}.$$

This action induces a map from Λ_J to itself defined by $\tilde{\xi}_w \mapsto \tilde{\xi}_{w\bullet t_{\overline{\alpha}}}$. We also denote this map $t_{\overline{\alpha}}$ and write $(\tilde{\xi}_w)t_{\overline{\alpha}} = \tilde{\xi}_{w\bullet t_{\overline{\alpha}}}$. Finally, we simply write $\tilde{\xi}_{t_{\overline{\alpha}}}$ instead of $\tilde{\xi}_{e\bullet t_{\overline{\alpha}}}$ where e is the identity element of W_a (corresponding to the fundamental alcove A_0).

Our strategy to establish that the graph Γ_ρ is positively multiplicative is to show that the adjacency matrix of Γ_ρ is essentially the matrix of $m_{\xi_{\rho_1}}$ in Λ_P in a well chosen basis. We want to do the same thing for Γ_γ using the quotient Λ_J, the multiplication by $\tilde{\xi}_{\rho_1}$ and the basis $\{\tilde{\xi}_w \mid w \in W^J\}$. To do so, we need to prove some factorisation properties in Λ_J using the element $\tilde{\xi}_{t_{\overline{\alpha}}}$ defined above.

Theorem 5.23. *We have* $(\tilde{\xi}_w)t_{\overline{\alpha}} = \tilde{\xi}_w\tilde{\xi}_{t_{\overline{\alpha}}} = \tilde{\xi}_{t_{\overline{\alpha}}}\tilde{\xi}_w$ *for all* $w \in W$ *and all* $\alpha \in Q^\vee$.

Proof. The map of $m_{\xi_{\rho_1}}$ commutes with $t_{\overline{\alpha}}$. Indeed, for all $w, u \in \mathcal{A}_J$ we have $w \overset{+}{\rightsquigarrow} u$ if and only if $w \bullet t_{\overline{\alpha}} \overset{+}{\rightsquigarrow} u \bullet t_{\overline{\alpha}}$ by Corollary 4.12. In other words, we have

$$\tilde{\xi}_{\rho_1} \cdot (\tilde{\xi}_w)t_{\overline{\alpha}} = (\tilde{\xi}_{\rho_1} \cdot \tilde{\xi}_w)t_{\overline{\alpha}}.$$

From there, using the generalised Pieri rules, we see that $t_{\overline{\alpha}}$ commutes with all the $\tilde{\xi}_{\rho_a}$ for $a \in X$ and therefore with $\tilde{\xi}_x$ for all $x \in \mathcal{A}_J$. We get for all $w \in \mathcal{A}_J$

$$\tilde{\xi}_{t_{\overline{\alpha}}} \cdot \tilde{\xi}_w = \tilde{\xi}_w \cdot \tilde{\xi}_{t_{\overline{\alpha}}} = \tilde{\xi}_w \cdot (\tilde{\xi}_e)t_{\overline{\alpha}} = (\tilde{\xi}_w\tilde{\xi}_e)t_{\overline{\alpha}} = (\tilde{\xi}_w)t_{\overline{\alpha}} = \xi_{w \bullet t_{\overline{\alpha}}}.$$

□

Let $w = u \bullet t_{\overline{\alpha}}$ where $u \in W^J$ and $\alpha \in Q^\vee$. There exists a unique family $(m_1, \ldots, m_k)$ of integers such that $\overline{\alpha} = m_1\overline{\alpha_{i_1}} + \ldots + m_k\overline{\alpha_{i_k}}$ where $i_1, \ldots, i_k \in J'$. It follows from the previous theorem that $\tilde{\xi}_w$ with $w \in \mathcal{A}_J$ can be factorised under the form

$$\tilde{\xi}_w = \tilde{\xi}^{m_1}_{t_{\overline{\alpha_{i_1}}}} \cdot \tilde{\xi}^{m_2}_{t_{\overline{\alpha_{i_2}}}} \cdots \tilde{\xi}^{m_k}_{t_{\overline{\alpha_{i_k}}}} \cdot \xi_u.$$

We see that the algebra Λ_J is finite-dimensional over $\mathbb{Q}[\tilde{\xi}^{\pm 1}_{t_{\overline{\alpha_j}}} \mid j \in J']$ with basis $\{\tilde{\xi}_w \mid w \in W^J\}$. Note that the algebra $\mathbb{Q}[\tilde{\xi}^{\pm 1}_{t_{\overline{\alpha_j}}} \mid j \in J']$ is isomorphic to the group algebra $\mathbb{Q}[Q^\vee/Q^\vee_J]$ via the map

$$\mathsf{z}^{\overline{\alpha}} \mapsto \prod_{j \in J'} \tilde{\xi}^{m_j}_{t_{\overline{\alpha_j}}} \quad \text{where } \overline{\alpha} = \sum_{j \in J'} m_j\overline{\alpha_j}.$$

We will identify these two algebras.

We define Γ_{W^J} to be the graph with set of vertices W^J and adjacency matrix $\mathrm{Mat}_{W^J}(m_{\tilde{\xi}_{\rho_1}})$ with coefficients in $\mathbb{Z}[Q^\vee/Q^\vee_J]$. We immediately get the following result.

Proposition 5.24. *The graph* Γ_{W^J} *is positively multiplicative at* e.

Remark 5.25. From there it is possible to study central random walk in the strip $\mathcal{A}_J \cap W_a^{\Lambda_0}$. Indeed, let b_0 be a subset of alcoves in $\mathcal{A}_J$ defined by $w \in \mathsf{b}_0$ if and only if $w \bullet t_{-\overline{\alpha}_{j'}} \notin \mathcal{A}_J \cap W_a^{\Lambda_0}$ for all $j' \in J'$. Then the set $\{\tilde{\xi}_w \mid w \in \mathsf{b}_0\}$ is a basis of Λ_J (viewed as a finite dimensional algebra over $\mathbb{Q}[Q^\vee/Q^\vee_J]$) and the graph Γ_{b_0} with adjacency matrix $\mathrm{Mat}_{\mathsf{b}_0}(m_{\tilde{\xi}_{\rho_1}})$ is positively multiplicative. It is not hard to see that the expansion of Γ_{b_0} will be the graph of the weak Bruhat order on $\mathcal{A}_J \cap W_a^{\Lambda_0}$ and therefore the methods of [LT21] should be generalisable to study central random walks on $\mathcal{A}_J$.

5.6. Automatons for affine Weyl group and random walks

We conclude this section with some considerations about finite automatons recognising the reduced expressions of any element in an affine Weyl group W. The existence of such automatons, their minimality and their explicit description is a difficult problem which has been considered in recent publications (see in particular [HNW16], [PY22, PY19] and the references therein).

As explained in section 4.2, the results in this paper yield similar automatons for reduced expressions of affine Grassmannians or affine Grassmannians contain in a strip $\mathcal{A}_J$. Moreover in both cases, we are able to prove that the associated graphs are positively multiplicative. By the results of [GLT22a], we thus have a complete description of the positive harmonic functions defined on their expanded version and a good understanding of the associated central Markov chains. This is in particular the case for $W_a^{\Lambda_0}$ where these Markov chains can be regarded as random walks on the alcoves located in the fundamental Weyl chamber which never cross two times the same hyperplane.

Now, this is a natural question to extend the previous results on random walks on the alcoves of the fundamental Weyl chamber to similar random walks on the full set of alcoves. Therefore, since we have finite automatons recognising the reduced expressions of all the elements in W_a, one could ask whether they are positively multiplicative. In fact this is easy to show this cannot be the case in general. Indeed, consider the infinite diedral group $W_a = \langle s_0, s_1 \rangle$, that is the affine Weyl group of type $\tilde{A}_1$. The reduced expressions in W have the form $w = (s_1 s_0)^l, w = s_0(s_1 s_0)^l$ with $l \geqslant 0$ or $w = (s_0 s_1)^k, w = s_1(s_0 s_1)^k$ with $k \geqslant 0$. It is not difficult to see that the finite automaton with 3 vertices v_1, v_2, v_3, start state v_1, accepting states v_1, v_2, v_3, and transition matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

recognizes exactly all the previous reduced expressions. But this automaton does not admit any weighted version

$$A' = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & d \\ b & c & 0 \end{pmatrix}$$

(where a, b, c, d positive reals) which could be positively multiplicative To see this, we can apply Theorem 3.1.2 in [GLT22a] showing that the graph with adjacency matrix A' is positively multiplicative if and only if the entries of the matrix

$$M = \frac{1}{a^2c - b^2d} \begin{pmatrix} 1 & 0 & 0 \\ 0 & ac & -bd \\ 0 & -b & a \end{pmatrix}$$

are all nonnegative which is clearly impossible. Observe also that the previous graph is not simply connected: once the vertex v_2 or v_3 is attained, it becomes impossible to come back at v_1.

This indicates that the results of the present paper cannot be extended in a direct way to the complete affine Weyl group. Therefore the determination of the positive harmonic functions on the alcove walks supported by all the alcoves needs a different approach. In the present case, they are particularly easy to determine due to the very simple structure of the Cayley graph for W_a

$$\begin{array}{ccc} & e & \\ \swarrow & & \searrow \\ s_0 & & s_1 \\ \downarrow & & \downarrow \\ s_1s_0 & & s_0s_1 \\ \downarrow & & \downarrow \\ s_0s_1s_0 & & s_1s_0s_1 \\ \downarrow & & \downarrow \\ s_1s_0s_1s_0 & & s_0s_1s_0s_1 \\ \vdots & & \vdots \end{array}$$

Indeed such an harmonic function f normalized so that $f(e) = 1$ satisfies $f(e) = f(s_0) + f(s_1) = 1$ and is constant on both subsets of reduced expressions starting by s_0 and s_1. Thus, it is completely determined by the choice of $f(s_0) \in]0,1[$ and we have an immediate parametrisation of the positive harmonic functions by the interval $]0,1[$. It would be very interesting to get a similar parametrisation in the general case of affine Weyl groups.

6. Proof of the main Theorem

In this section we prove that Γ_ρ and Γ_γ are positively multiplicative, therefore proving Theorem 1.1. To avoid confusion, let us recall how we have defined the different algebras that we have used so far. We started with Λ, the homology ring of affine Grassmannians, which is a commutative $\mathbb{Q}$-algebra with distinguished basis $\{\xi_w \mid w \in W_a^{\Lambda_0}\}$. From there, using the factorisation properties of Λ, we saw that Λ is a finite-dimensional algebra over $\mathbb{Q}[P^+]$. This result allowed us to extend the scalars to the ring $\mathbb{Q}[P]$ and to define a commutative $\mathbb{Q}$-algebra Λ_a with distinguished basis $\{\xi_w \mid w \in W_a\}$ where $\xi_w = \mathsf{z}^{\mathsf{wt}_{\mathsf{b}_0}(w)}\xi_{\mathsf{f}_{\mathsf{b}_0}(w)}$. Then we showed the following results:

1. The algebra Λ_a is a finite-dimensional algebra over $\mathbb{Q}[Q^\vee]$ with basis $\{\xi_w \mid w \in \mathsf{b}\}$ where b is a fundamental domain for the action of $Q^\vee$. We denote this algebra by $\Lambda_{Q^\vee}$
2. Given $J' = \{i_1, \dots, i_k\}$ and its complement J in $\{1, \dots, n\}$, we defined a quotient Λ_J of some ideal of Λ_a which is finite-dimensional over $\mathbb{Q}[Q^\vee/Q_J^\vee]$ with basis $\{\tilde{\xi}_w \mid w \in W^J\}$.

We use the first assertion to show that Γ_ρ is positively multiplicative and the second to show that Γ_γ where $\gamma = \sum_{j\in J'} \omega_j$ is positively multiplicative.

6.1. The graph Γ_ρ

Let $^-$ be the involution of $\mathbb{Q}[Q^\vee]$ defined by $\overline{\mathsf{z}^\alpha} = \mathsf{z}^{-\alpha}$ for all $\alpha \in Q^\vee$. Recall that

$$\xi_{\rho_1}\xi_w = \sum_{i\in I, w \overset{+}{\rightsquigarrow} s_i w} a_i \xi_{s_i w} \in \Lambda_a.$$

By definition the set W is a fundamental domain for the action of $Q^\vee$ so that $\{\xi_w \mid w \in W\}$ is a basis of the algebra $\Lambda_{Q^\vee}$. Recall that Γ_W is the graph with set of vertices W and adjacency matrix $\mathrm{Mat}_W(m_{\xi_{\rho_1}})$ where $m_{\xi_{\rho_1}}$ is the multiplication by ξ_{ρ_1} in $\Lambda_{Q^\vee}$ expressed in the basis $\{\xi_w \mid w \in W\}$.

Proposition 6.1. $\mathrm{Mat}_W(m_{\xi_{\rho_1}}) = {}^t\overline{A_{\Gamma_\rho}}$ *and* Γ_ρ *is positively multiplicative.*

Proof. Let $x, y \in W$. We have ${}^t\overline{A_{\Gamma_\rho}}[x,y] = p \neq 0$ if and only if there is an arrow in Γ_ρ from x to y of weight $\overline{p}$. There are two cases to consider:

- there exists $i \in I^*$ such that $y = s_i x > x$. In this case $\overline{p} = p = a_i \in \mathbb{N}$, $y \overset{+}{\rightsquigarrow} x$ and, according to the previous proposition, $a_i \xi_x$ will appear when developing $\xi_{\rho_1}\xi_y$, showing that $\mathrm{Mat}_W(m_{\xi_{\rho_1}})[x,y] = p$.
- we have $y = s_\theta x < x$. Then $s_0 x = \mathsf{f}(s_0 x) t_{\mathsf{wt}(s_0 x)} = y t_{\mathsf{wt}(s_0 x)}$ and $\overline{p} = \mathsf{z}^{\mathsf{wt}(s_0 w)}$. By Proposition 4.5, we know that $s_0 x \overset{+}{\rightsquigarrow} x$ hence showing that the term ξ_x will appear when developing $\xi_{\rho_1}\xi_{s_0 x}$. We have in Λ_a

$$\xi_{\rho_1}\xi_{s_0 x} = \mathsf{z}^{\mathsf{wt}(s_0 x)}\xi_{\rho_1}\xi_y = \xi_x + \cdots$$

  hence showing when factorising with respect to the fundamental domain W that the relation $\mathrm{Mat}_W(m_{\xi_{\rho_1}})[x,y] = \mathsf{z}^{-\mathsf{wt}(s_0 w)} = p$ holds as desired.

The fact that Γ_ρ is positively multiplicative follows easily from the fact that Γ_W is positively multiplicative (see Proposition 5.15). Indeed, it suffices to transpose and apply the bar involution to the positive basis associated to $\mathrm{Mat}_W(m_{\xi_{\rho_1}})$. $\square$

6.2. The graph Γ_γ

Let $\gamma = \omega_{i_1} + \cdots + \omega_{i_k}$ be a level 0 weight. Let $J' = \{i_1, \ldots, i_k\}$ and J be the complement of J' in $\{1, \ldots, n\}$. Let $^-$ be the involution of $\mathbb{Q}[Q^\vee/Q_J^\vee]$ defined by $\overline{\mathsf{z}^{\alpha}} = \mathsf{z}^{-\overline{\alpha}}$ for all $\alpha \in Q^\vee$. The set W^J is a fundamental domain for the action of $Q^\vee/Q_J^\vee$ on $\mathcal{A}_J$ so that $\{\tilde{\xi}_w \mid w \in W^J\}$ is a basis of the algebra Λ_J. Recall that Γ_{W^J} is the graph with set of vertices W^J and adjacency matrix $\mathrm{Mat}_{W^J}(m_{\tilde{\xi}_{\rho_1}})$ where $m_{\tilde{\xi}_{\rho_1}}$ is the multiplication by $\tilde{\xi}_{\rho_1}$ in Λ_J expressed in the basis $\{\tilde{\xi}_w \mid w \in W^J\}$.

Proposition 6.2. $\mathrm{Mat}_{W^J}(m_{\tilde{\xi}_{\rho_1}}) = {}^t\overline{A_{\Gamma_\gamma}}$ *and* Γ_γ *is positively multiplicative.*

Proof. Let $x, y \in W^J$. We have ${}^t\overline{A_{\Gamma_\gamma}}[x,y] = p \neq 0$ if and only if there is an arrow in Γ_γ from x to y of weight $\overline{p}$. There are two cases to consider:

- there exists $i \in I^*$ such that $y = s_i w > w$ and $y \in W^J$. In this case $\overline{p} = p = a_i \in \mathbb{N}$, $y \overset{+}{\rightsquigarrow} x$ and, according to Proposition 5.22, $a_i \tilde{\xi}_x$ will appear when developing $\tilde{\xi}_{\rho_1}\tilde{\xi}_y$, showing that $\mathrm{Mat}_W(m_{\tilde{\xi}_{\rho_1}})[x,y] = p$.
- we have $y = (s_\theta x)^J < x$ and $s_0 x \in \mathcal{A}_J$. We have $s_0 x = \mathsf{f}(s_0 x) t_{\mathsf{wt}(s_0 x)} = y v t_{\mathsf{wt}(s_0 x)} = y \bullet t_{\overline{\mathsf{w}(s_0 x)}}$ with $v \in W_J$ and $\overline{p} = \mathsf{z}^{\overline{\mathsf{wt}(s_0 w)}}$. By Theorem 4.11, we know that $s_0 x \overset{+}{\rightsquigarrow} x$ hence showing that the term $\tilde{\xi}_x$ will appear when developing $\tilde{\xi}_{\rho_1}\tilde{\xi}_{s_0 w}$. We have in Λ_J

$$\tilde{\xi}_{\rho_1}\tilde{\xi}_{s_0 x} = \mathsf{z}^{\overline{\mathsf{wt}(s_0 w)}}\tilde{\xi}_{\rho_1}\tilde{\xi}_y = \tilde{\xi}_x + \cdots .$$

  hence showing that $\mathrm{Mat}_W(m_{\xi_{\rho_1}})[x,y] = \mathsf{z}^{-\overline{\mathsf{wt}(s_0 w)}} = p$ as desired.

The fact that Γ_γ is positively multiplicative follows easily from the fact that Γ_{W^J} is positively multiplicative (see Proposition 5.24). $\square$

The statement of Theorem 1.1 is then the combination of both latter propositions.

7. Interpretation in terms of colored particle models

In this section, we give an interpretation of the parabolic Cayley graphs of classical types introduced in Section 4 in terms of particle moving on either a circle or a segment. The main tool to provide such interpretations is the combinatorial description of the affine Weyl groups of classical types as permutations of $\mathbb{Z}$, see [BB05, Section 8].

7.1. Type $\tilde{A}$

Let V be an euclidean space of dimension $n+1$ with basis $\varepsilon_1, \dots, \varepsilon_{n+1}$. The set of vectors

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant n+1\}$$

forms a positive root system of type A with associated simple system $\Pi = \{\alpha_1, \dots \alpha_n\}$ where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. We have $\theta = \theta^\vee = \varepsilon_1 - \varepsilon_{n+1}$. From there, we can construct the affine Weyl group of W_a of type $\tilde{A}_n$ as in Section 3.2.

The group W_a can be realised as the group of permutations σ of $\mathbb{Z}$ (acting on the right) that satisfy the following properties:

- $(i+n+1)\sigma = (i)\sigma + n + 1$ for all $i \in \mathbb{Z}$,
- $\sum_{i=1}^{n+1} (i)\sigma = \frac{(n+1)(n+2)}{2}$.

Any element $\sigma \in W_a$ is uniquely determined by the images of $1, 2, \dots, n+1$. We will sometime denote an element $\sigma \in W_a$ by its window notation $[(1)\sigma, \dots, (n+1)\sigma]$. The distinguished generators $s_0, s_1, \dots, s_n$ of W_a are given by $s_i = [1, \dots, i+1, i, \dots, n+1]$ and $s_0 = [0, 2, \dots, n, n+2]$.

The symmetric group W is the subgroup of W_a corresponding to the sequences $[j_1, \dots, j_{n+1}]$ with $1 \leqslant j_i \leqslant n+1$ for $1 \leqslant i \leqslant n+1$. We have the following formula for translations and affine symmetries

$$\begin{aligned} t_{\alpha_i} &= [1, \dots, (i+n+1), (i-n), \dots, n+1], \\ s_{\alpha_i,k} &= [1, \dots, (i+1) + k(n+1), i - k(n+1), \dots, n+1]. \end{aligned}$$

Given an element $w = [j_1, \dots, j_{n+1}]$, left multiplication by a distinguished generator is given by:

$$\begin{aligned} s_i[j_1, \dots, j_{n+1}] &= [j_1, \dots, j_{i+1}, j_i, \dots, j_{n+1}] \quad \text{and} \\ s_0[j_1, \dots, j_{n+1}] &= [j_{n+1} - (n+1), \dots, j_{i+1}, j_i, \dots, j_1 + (n+1)]. \end{aligned}$$

Right multiplication by σ is straightforward by definition

$$[j_1, \dots, j_{n+1}]\sigma = [(j_1)\sigma, \dots, (j_{n+1})\sigma].$$

Let $w \in W$. The equality $s_0 w = s_\theta t_\theta w = s_\theta w t_{w(\theta)}$ in W_a becomes when $j_{n+1} < j_1$

$$s_0[j_1, j_2, \ldots, j_{n+1}] = [j_{n+1}, j_2 \ldots, j_n, j_1] t_{-\sum_{r=j_{n+1}}^{j_1-1} \alpha_r}.$$

Further for $\sigma \in W$, we have $\ell(s_i\sigma) > \ell(\sigma)$ if and only if $j_i < j_{i+1}$ and $\ell(s_\theta\sigma) < \ell(\sigma)$ if and only if $j_1 < j_{n+1}$. Hence, the edge structure on Γ_ρ is given by arrows

- $[j_1, \ldots, j_i, j_{i+1}, \ldots, j_{n+1}] \xrightarrow[1]{i} [j_1, \ldots, j_{i+1}, j_i, \ldots, j_{n+1}]$ if $j_{i+1} > j_i$ and $1 \leqslant i \leqslant n$,
- $[j_1, \ldots, j_{n+1}] \xrightarrow[z^{-\alpha}]{0} [j_{n+1}, \ldots, j_1]$ with $\alpha = \sum_{s=j_{n+1}}^{j_1-1} \alpha_s$ if $j_{n+1} < j_1$.

The graphs Γ_γ

Let $J = \{i_1, \ldots, i_\ell\} \subset \{1, \ldots n\}$, J' be the complement of J in $\{1, \ldots n\}$ and $\gamma = \sum_{j \in J'} \omega_j$ be the corresponding dominant weight. The group W_J acts on the $\{1, \ldots, n+1\}$ and the set of orbits can be described as $\{B_m \mid 1 \leqslant m \leqslant r\}$ where all the elements in B_m are smaller than those in B_{m+1}. The set of minimal length representatives W^J is the subset of W corresponding to sequences $[j_1, \ldots, j_{n+1}]$ where i appears before j whenever $i < j$ and $i, j \in B_m$ for some $1 \leqslant m \leqslant r$. The definition of Γ_γ given in Definition 4.7 (see also Subsection 4.4 for another labeling by Key tableaux) translates then into the following rules:

- $[j_1, \ldots, j_i, j_{i+1}, \ldots, j_{n+1}] \xrightarrow[1]{i} [j_1, \ldots, j_{i+1}, j_i, \ldots, j_{n+1}]$ if $j_{i+1} > j_i$ and j_i, j_{i+1} belong to different orbits $B_m, B_{m'}$,
- and $[j_1, \ldots, j_{n+1}] \xrightarrow[z^{-\overline{\alpha}}]{0} ([j_{n+1}, \ldots, j_1])^J$ with $\alpha = \sum_{s=j_{n+1}}^{j_1-1} \alpha_s$ if $j_{n+1} < j_1$ and j_1, j_{n+1} belong to different orbits $B_m, B_{m'}$.

Recall here that the weights of Γ_γ can be regarded as elements of $Q^\vee/Q_J^\vee$ and that for $\alpha \in Q^\vee$, $\overline{\alpha}$ denotes the class of α modulo $Q_J^\vee$. In the last case, the notation $([k_1, \ldots, k_{n+1}])^J$ means that we take the minimal coset representative with respect to the set J.

This graph is isomorphic to a graph encoding the clockwise movement of $n+1$ colored particles on a discrete circle with non-intersecting conditions. Let $\mathsf{c} : [\![1, n]\!] \to [\![1, r]\!]$ be the color map defined by $i \in B_{\mathsf{c}(i)}$ for all $i \in \{1, \ldots, n\}$, and let $\mathcal{W}_J$ be the set of words in the alphabet $\{1, \ldots, r\}$ with $|B_c|$ occurrences of each letter c. Then, Γ_γ is isomorphic to the graph $\widetilde{\Gamma}_\gamma$ with vertices $\mathcal{W}_J$ and edge rules:

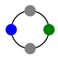
- $c_1 \ldots c_i c_{i+1} \cdots c_{n+1} \xrightarrow[1]{i} c_1 \cdots c_{i+1} c_i \cdots c_{n+1}$ if $c_i < c_{i+1}$,
- $c_1 \ldots c_{n+1} \xrightarrow[\prod_{c=c_{n+1}}^{c_1-1} \mathsf{z}^{-\overline{\alpha_{k_c}}}]{0} c_{n+1} \cdots c_1$ if $c_{n+1} < c_1$.

Example 7.1. Let W_a be the affine Weyl group $\tilde{A}_n$ and the weight $\gamma = \omega_i$ for some $1 \leqslant i \leqslant n-1$. We have $J = \{1, \ldots, n\} \setminus \{i\}$, $J' = \{i\}$, $B_1 = \{1, \ldots, i\}$ and $B_2 = \{i+1, \ldots, n+1\}$. The color map is defined by $\mathsf{c}(i) = 1$ if $i \in [\![1, i]\!]$ and $\mathsf{c}(i) = 2$ otherwise. The graph $\widetilde{\Gamma}_\gamma$ is the graph with vertices being words with i letters 1 and $n+1-i$ letters 2 and edge rules

- $\cdots 12 \cdots \xrightarrow[1]{i} \ldots 21 \cdots$,
- and $2c_2 \cdots c_{n-1}1 \xrightarrow[\mathsf{z}^{-\overline{\alpha_i}}]{0} 1c_2 \cdots c_{n-1}2$.

Considering letters 2 as holes and letters 1 as particles, $\widetilde{\Gamma}_\gamma$ encodes thus the clockwise movement of i non-intersecting particles on a discrete circle of size $n+1$, with an edge weighted by $\mathsf{z}^{-\overline{\alpha_i}}$ when a particle returns to the origin. Such graphs have been studied from a probabilistic point of view in [GLT22b].

Example 7.2. Let W_a be the affine Weyl group $\tilde{A}_3$ and let $\gamma = \omega_1 + \omega_2$. We have $J = \{3\}$, $J' = \{1, 2\}$, $B_1 = \{1\}$, $B_2 = \{2\}$ and $B_3 = \{3, 4\}$. The color map is given by $\mathsf{c}(1) = 1$, $\mathsf{c}(2) = 2$ and $\mathsf{c}(3) = \mathsf{c}(4) = 3$. The graph $\widetilde{\Gamma}_\gamma$ is the graph with vertices being words of length 4 with one letter 1, one letter 2 and two letters 3. In the graph below, the letter 1 is encoded by a blue particle, the letter 2 by a green particle and the color 3 by a hole (in gray) and we read the word starting from the north position and moving counterclockwise. For instance, we have the following correspondence

 $\leftrightarrow$ 3132.

We obtain the following graph where the arrows going up are all of type 0 and are the only arrows with a weight different from 1. Compare with Example 4.16.

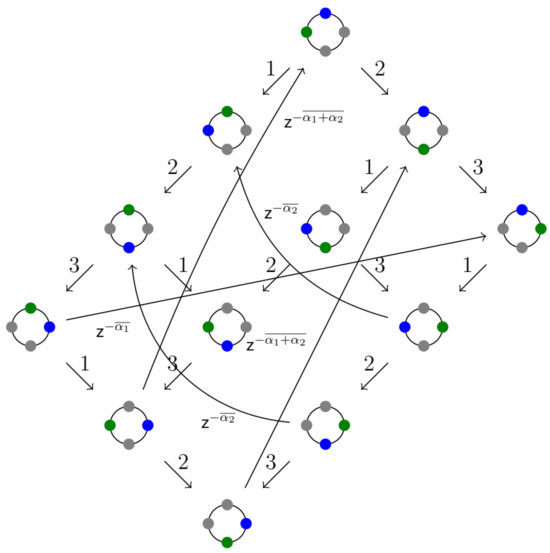


Figure 7.1: The graph $\widetilde{\Gamma}_{\omega_1+\omega_2}$ in type $\tilde{A}_3$.

7.2. Type $\tilde{C}$

Let V be an euclidean space of dimension n with basis $\varepsilon_1, \ldots, \varepsilon_n$. The set of vectors

$$\Phi^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant n-1\} \cup \{2\varepsilon_i \mid 1 \leqslant i \leqslant n\}$$

forms a positive root system of type C with associated simple system $\Pi = \{\alpha_1, \dots \alpha_{n-1}, \alpha_n\}$ where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $\alpha_n = 2\varepsilon_n$. We have $\theta = 2\varepsilon_1$ and $\theta^\vee = \varepsilon_1$. From there we can construct the affine Weyl group of W_a of type $\tilde{C}_n$ as in Section 3.2.

The group W_a can be realised as the group of permutations σ of $\mathbb{Z}$ (acting on the right) that satisfies the following properties

- $(-i)\sigma = -(i)\sigma$ for $i \in \mathbb{Z}$,
- $(i+N)\sigma = (i)\sigma + N$

where we have set $N = 2n+1$. We see that any element $\sigma \in W_a$ is completely determined by the image of $1, \dots, n$. Indeed we have $\sigma(kN) = kN$ for all $k \in \mathbb{Z}$ and if $n+1 \leqslant i \leqslant 2n$ we have

$$(i)\sigma = (N-(N-i))\sigma = N-(N-i)\sigma \quad \text{and} \quad 1 \leqslant N-i \leqslant n.$$

For later use, we define the following map

$$\mathsf{s_C} \; : \; \begin{array}{ccc} \{1, \dots, 2n\} & \to & \{1, \dots, 2n\} \\ i & \mapsto & N-i \end{array}.$$

We have $(i)\sigma = -(\mathsf{s_C}(i))\sigma + N$ for all $i \in \{1, \dots, 2n\}$. We will sometime denote an element $\sigma \in W_a$ by its window notation: $[(1)\sigma, \dots, (n)\sigma]$. The window notation of the distinguished generators $s_0, s_1, \dots, s_n$ of W_a is

$$\begin{aligned} s_0 &= [-1, 2, \dots, n], \\ s_i &= [1, \dots, i+1, i, \dots, n] \quad \text{for all } 1 \leqslant i \leqslant n-1, \\ s_n &= [1, 2, \dots, n-1, n+1]. \end{aligned}$$

The finite Weyl group W is the subgroup of W_a corresponding to the sequences $[j_1 \dots j_n]$ with $1 \leqslant j_i \leqslant 2n$ for $1 \leqslant i \leqslant n$. We have the following formula for translations

$$\begin{aligned} t_{\alpha_0} &= [1+N, 2, \dots, n], \\ t_{\alpha_i} &= [1, \dots, (i+1)+N, i-N, \dots, n] \quad \text{for } 1 \leqslant i \leqslant n-1, \\ t_{\alpha_n} &= [1, 2, \dots, n+N]. \end{aligned}$$

Given an element $w = [j_1, \dots, j_n]$, we have the following rules for right multiplication:

$$\begin{aligned} s_0[j_1, \dots, j_n] &= [-j_1, \dots, j_n], \\ s_i[j_1, \dots, j_n] &= [j_1, \dots, j_{i+1}, j_i, \dots, j_n] \quad \text{for } 1 \leqslant i \leqslant n+1, \\ s_n[j_1, \dots, j_n] &= [j_1, \dots, N-j_n]. \end{aligned}$$

Let $w \in W$. We have $s_\theta w < w$ if and only if $j_1 > n$ and the equality $s_0 w = s_\theta t_{\theta^\vee} w = s_\theta w t_{(\theta^\vee)w}$ in W_a becomes in this case:

$$s_0[j_1, j_2, \dots, j_n] = [N-j_1, \dots, j_n] t_{\alpha^\vee} \quad \text{where} \quad \alpha^\vee = -\alpha_n^\vee - \sum_{i=N-j_1}^{n-1} \alpha_i^\vee.$$

Hence, the edge structure on Γ_ρ is given by

- $[j_1, \dots, j_i, j_{i+1}, \dots, j_n] \xrightarrow[2]{i} [j_1, \dots, j_{i+1}, j_i, \dots, j_n]$ if $j_{i+1} > j_i$ and $1 \leqslant i \leqslant n-1$
- $[j_1, \dots, j_n] \xrightarrow[1]{n} [j_1, \dots, N-j_n]$ if $j_n \leqslant n$.
- $[j_1, \dots, j_n] \xrightarrow[z^{-\alpha^\vee}]{} [N-j_1, \dots, j_n]$ if $j_1 > n$, with $\alpha^\vee = \alpha_n + \sum_{i=N-j_1}^{n-1} \alpha_i^\vee$.

The graph Γ_γ

Let $J = \{i_1, \dots, i_\ell\} \subset \{1, \dots n\}$, J' be the complement of J in $\{1, \dots n\}$ and $\gamma = \sum_{j \in J'} \omega_j$ be the corresponding weight. The group W_J acts on $\{1, \dots, 2n\}$. In the case where $n \notin J_r$, the orbits of this action can be described as follows:

$$
\begin{aligned}
B_m &= \{k_m, \dots, k_{m+1}-1\} \quad \text{for } m = 1, \dots, r, \\
B_{2r-m} &= \{\mathsf{s_C}(k_{m+1}-1), \dots, \mathsf{s_C}(k_m)\} \quad \text{for } m = 1, \dots, r
\end{aligned}
$$

where $k_{n+1} = n+1$ and $k_m < k_{m+1}$. Note that we must have $n \in B_r$ and $n+1 \in B_{r+1}$ in this case. In the case where $n \in J_r$ we have

$$
\begin{aligned}
B_m &= \{k_m, \dots, k_{m+1}-1\} \quad \text{for } m = 1, \dots, r-1 \\
B_{2r-m} &= \{\mathsf{s_C}(k_{m+1}-1), \dots, \mathsf{s_C}(k_m)\} \quad \text{for } m = 1, \dots, r-1 \\
B_r &= \{k_r, \dots, n, \mathsf{s_C}(n), \dots, \mathsf{s_C}(k_r)\}.
\end{aligned}
$$

where $k_0 = 0$ and $k_m < k_{m+1}$. Note that we must have $\{n, n+1\} \subset B_r$ in this case.

The set of minimal length representatives W^J is the subset of W corresponding to sequences $[j_1, \dots, j_n]$ such that (1) i appears before j whenever $i < j$ and i and j belong to the same orbit B_k and (2) if $n \in J$ then $B_r \cap \{j_1, \dots, j_n\} \subset \{1, \dots, n\}$. The definition of Γ_γ given in 4.7 translates then into the following rules:

- $[j_1, \dots, j_i, j_{i+1}, \dots, j_n] \xrightarrow[2]{i} [j_1, \dots, j_{i+1}, j_i, \dots, j_n]$ if $j_{i+1} > j_i$ and j_i, j_{i+1} belong to two different orbits $B_c, B_{c'}$
- $[j_1, \dots, j_n] \xrightarrow[1]{n} [j_1, \dots, N-j_n]$ with $j_n \leqslant n$ and $j_n \notin B_r$ if $n \in J$
- $[j_1, \dots, j_n] \xrightarrow[z^{-\alpha^\vee}]{0} \Big([N-j_1, \dots, j_n]\Big)^J$ if $j_1 > n$, with $\alpha^\vee = \alpha_n^\vee + \sum_{i=N-j_1}^{n-1} \alpha_i^\vee$

In the last case, the notation $([k_1, \dots, k_n])^J$ means that we take the minimal coset representative with respect to the set J.

This graph is isomorphic to a graph encoding the movement on a segment of colored particles with a spin $\{+, -\}$ which exchange their position according to their spin and color. Let $\mathsf{c} : \{1, \dots, n\} \to \{1, \dots, r\}$ be the color map defined by $i \in B_{\mathsf{c}(i)} \cup B_{2r-\mathsf{c}(i)}$ and

$\mathsf{sp} : \{1,\dots,2n\} \to \{-,+\}$ be the spin map defined by $\mathsf{sp}(i) = -$ if $1 \leqslant i \leqslant n$ and $\mathsf{sp}(i) = +$ otherwise.

When $n \notin J$, let $\mathcal{W}_J$ be the set of words in $\{1^\pm,\dots,r^\pm\}$ with $|B_c|$ occurrences of the letter c (i.e. either c^+ or c^-). When $n \in J$, let $\mathcal{W}_J$ be the set of words in $\{1^\pm,\dots,r-1^\pm,r\}$ with $|B_c|$ occurrences of the letter c for $c \leqslant r-1$ and $|B_r|/2$ occurrences of the letter r. Set $q_c = \mathsf{z}^{-\overline{\alpha^\vee}}$ with $\alpha^\vee = \alpha_n^\vee + \sum_{1\leqslant r\leqslant c}\alpha_r^\vee$. Then, Γ_γ is isomorphic to the graph $\widetilde{\Gamma}_\gamma$ with vertices $\mathcal{W}_J$ and edge rules

- $\cdots c_i^{\mathsf{sp}_i} c_{i+1}^{\mathsf{sp}_{i+1}} \cdots x_n \xrightarrow[2]{i} x_1 \cdots c_{i+1}^{\mathsf{sp}_{i+1}} c_i^{\mathsf{sp}_i} \cdots x_n$ if $\mathsf{sp}_i(N/2 - c_i) < \mathsf{sp}_{i+1}(N/2 - c_{i+1})$,
- $x_1 \cdots x_{n-1} c_n^- \xrightarrow[1]{n} x_1 \cdots x_{n-1} c_n^+$ if $c_n \neq r$ when $n \in J$,
- $c_1^+ x_2 \cdots x_n \xrightarrow[q_{c_n}]{0} c_1^- x_2 \cdots x_n$ if $c_1 \neq r$ when $n \in J$.

The first condition in the above rules can be fulfilled in two different ways: either $\mathsf{sp}_i = -$ and $\mathsf{sp}_{i+1} = +$, or $\mathsf{sp}_i = \mathsf{sp}_{i+1}$ and $\mathsf{sp}_i c_i > \mathsf{sp}_i c_{i+1}$. In the simple case where $r = 2$ and $n \in J$, this graph encodes the movement of $|B_1|$ particles with a spin $\{+,-\}$ on a discrete segment of size n, where particles with spin $-$ go to the right and particles with spin $+$ go to the left.

Example 7.3. Let W_a be the affine Weyl group $\tilde{C}_3$ and let $\gamma = \omega_2$. We have $J = \{1,3\}$ and $J' = \{2\}$. The group W_J acts on $\{1,2,3,4,5,6\}$ and the orbits are $B_1 = \{1,2\}$, $B_2 = \{3,4\}$ and $B_3 = \{5,6\}$. The elements of W^J are

$$\begin{aligned}
&e = [1,2,3],\ s_2 = [1,3,2],\ s_1s_2 = [3,1,2],\ s_3s_2 = [1,3,5],\ s_2s_3s_2 = [1,5,3],\\
&s_1s_3s_2 = [3,1,5],\ s_1s_2s_3s_2 = [5,1,3],\ s_2s_1s_3s_2 = [3,5,1],\ s_1s_2s_1s_3s_2 = [5,3,1],\\
&s_3s_2s_1s_3s_2 = [3,5,6],\ s_1s_3s_2s_1s_3s_2 = [5,3,6],\ s_2s_1s_3s_2s_1s_3s_2 = [5,6,3].
\end{aligned}$$

We have $s_\theta = s_1s_2s_3s_2s_1$. In the list above, the elements that satisfy $s_\theta w < w$ are those with window notation starting by 5. For all such w, we have $(\theta^\vee)w = -\varepsilon_2 = -(\alpha_3^\vee + \alpha_2^\vee)$ and $s_0 w = s_\theta t_{\theta^\vee} w = s_\theta w t_{-\varepsilon_2}$ so that all the 0-arrows will have weight $\mathsf{z}^{-\overline{\varepsilon_2}}$. We thus omit this information when drawing the Γ_γ in the figure below. The color map is defined by $\mathsf{c}(1) = \mathsf{c}(2) = 1$ and $\mathsf{c}(3) = 2$. Since $3 \in J$, the particle 2 doesn't have spin. In the graph $\widetilde{\Gamma}_{\omega_2}$ below, the particle 1^- is represented in blue, 1^+ in green and 2 in gray. We do not indicate the weights $a_i \in \mathbb{N}$ but we do indicate the i when $1 \leqslant i \leqslant n$.

Example 7.4. Let W_a be the affine Weyl group $\tilde{C}_3$ and let $\gamma = \omega_3$. We have $J = \{1,2\}$ and $J' = \{3\}$. The group W_J acts on $\{1,2,3,4,5,6\}$ and the orbits are $B_1 = \{1,2,3\}$ and $B_2 = \{4,5,6\}$. The elements of W^J are

$$\begin{aligned}
&e = [1,2,3],\ s_3 = [1,2,4],\ s_2s_3 = [1,4,2],\ s_1s_2s_3 = [4,1,2],\ s_3s_2s_3 = [1,4,5],\\
&s_1s_3s_2s_3 = [4,1,5],\ s_2s_1s_3s_2s_3 = [4,5,1],\ s_3s_2s_1s_3s_2s_3 = [4,5,6].
\end{aligned}$$

We have $s_\theta = s_1s_2s_3s_2s_1$ and in the list above, the set of elements that satisfy $s_\theta w < w$ are exactly those whose window notation starts by 4. As before, the weight of each 0 arrow will

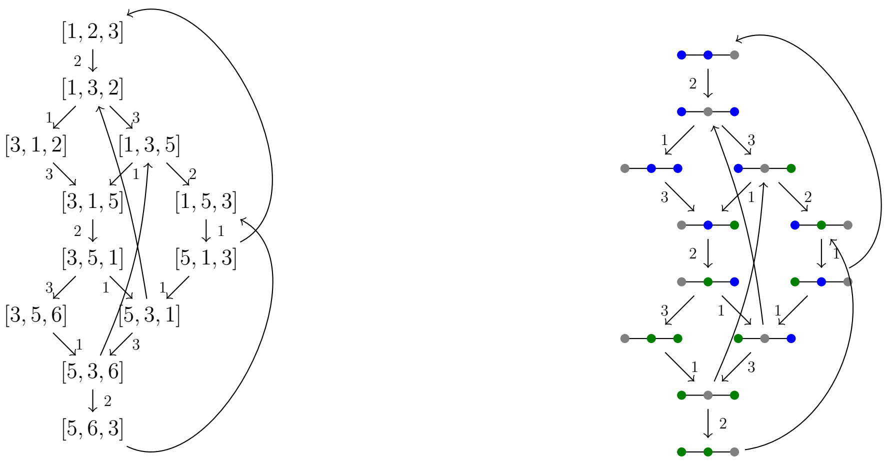


Figure 7.2: The graph Γ_{ω_2} and $\tilde{\Gamma}_{\omega_2}$ in type $\tilde{C}_3$.

Figure 7.3: The graph Γ_{ω_3} and $\tilde{\Gamma}_{\omega_3}$ in type $\tilde{C}_3$.

be $\mathsf{z}^{\overline{-\varepsilon_3}}$ and will not be indicated on the graphs below. The associated color map is constant defined by $\mathsf{c}(1) = \mathsf{c}(2) = \mathsf{c}(3) = 1$. Since $3 \notin J$, the particle 1 does have spin. In the graph $\tilde{\Gamma}_{\omega_2}$ below, 1^- is represented in blue and 1^+ in green. Again, we do not indicate the weights $a_i \in \mathbb{N}$ but we do indicate the i when $1 \leqslant i \leqslant n$.

There are similar interpretations of the graphs Γ_γ in type $\tilde{B}_n$ and $\tilde{D}_n$.

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