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Authors
Ascuitto, Robert J.
Glendenning, Norman K.

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Lawrence Radiation Laboratory
University of California
Berkeley, California 94720

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ABSTRACT

There are certainly some levels in all nuclei, and all levels in some nuclei for which the usual treatment of particle transfer reactions, which neglects inelastic effects, is invalid. Here a practical method for taking these effects into account is described. The method is discussed in terms of the (d,p) reaction, but it has a much broader application than this.

1. INTRODUCTION

The usual distorted wave Born approximation (DWBA) for transfer reactions makes three basic assumptions. First, it assumes that the transfer takes place directly from the elastic entrance channel to the residual channel. Only the transferred particles are treated explicitly, while all the others, which we shall refer to collectively as the core, are regarded as passive. The reaction is assumed to proceed only to the extent that the core state is unchanged. Second, it assumes that the elastic optical potential provides wave functions for the relative motion which are good <u>inside</u> the nucleus, or at least in the surface region, since this is where the transfer process is concentrated. Third, it assumes that the transfer process itself is weak so that it can be treated in first order. With these assumptions the transition amplitude can be computed for (d,p) reactions, from

$$T_{p,d} = \int \psi_p^{(-)*} \langle \Phi_p(A+1) | V_{np} | \Phi_d(A) \rangle \psi_d^{(+)} \, d\underline{r}_n \, d\underline{r}_p \qquad (1.1)$$

where ψ_d and ψ_p are distorted waves describing only the elastic scattering in the channels d and p, and Φ_p and Φ_d describe the nuclear states between which the reaction takes place.[1]

There are certainly situations where one or both of the first two assumptions are false, although the third assumption is probably always valid for particle transfer reactions. As concerns the first assumption, the theory will fail to the extent that the reaction of interest does take place between states, one of which is not the parent of the other. This situation is illustrated, in idealized form, for the (d,p) reaction in Fig. 1.

In this example the lower states of the residual nucleus, A+1, do have the target A as parent, so that the direct transition can occur. However the third state has as its parent an excited state of the target, so that it can be reached only through excitation of the core to the parent state either before or after the transfer takes place. The usual DWBA may provide good answers for the former transitions but it cannot treat the latter. As long as one knows which states have the ground state as their parent and restricts attention to them, the usual treatment may provide valid results.

However if there are strongly enhanced inelastic transitions, the usual DWBA will fail in any case because of the second assumption. This can be understood as follows. The (one-channel) optical potential is chosen so as to reproduce the elastic cross section. This assures that the wave function for the relative motion is correct in the external region. However, for the purpose of calculating the reaction, it is needed in the interior region, or at least in the neighborhood of the nuclear surface. It is here that the one-channel optical model will break down due to de-excitation of other channels back to the elastic channel, if their coupling is sufficiently strong.

One solution to the problem of including excitation of the core in transfer reactions was suggested by Penny and Satchler.[2] They propose evaluating the amplitude for d,p reactions from

$$T_{p,d} = \sum_{p',d'} \int \psi_{p,p'}^{(-)*} \langle \Phi_{p'}(A+1)|V_{np}|\Phi_{d'}(A)\rangle \psi_{d,d'}^{(+)} \, d\underline{r}_n \, d\underline{r}_p \qquad (1.2)$$

where $\psi_{d,d'}^{(+)}$ is a generalized distorted wave in the channel d' found by
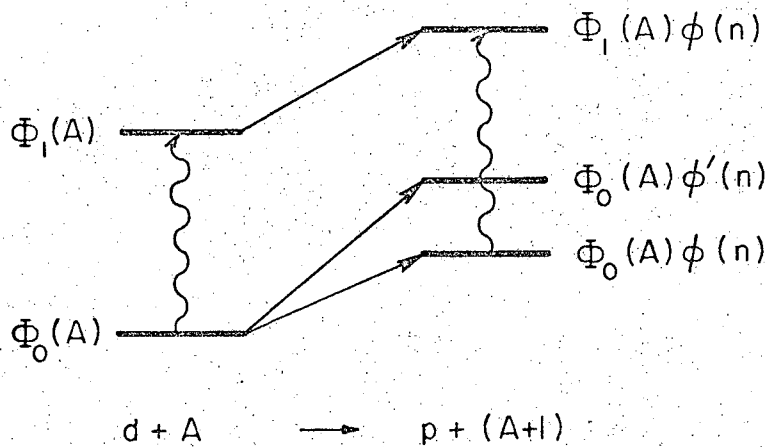
$$\Phi_1(A)\phi(n)$$

$$\Phi_1(A)$$

$$\Phi_0(A)\phi'(n)$$

$$\Phi_0(A)\phi(n)$$

$$\Phi_0(A)$$

$$d + A \longrightarrow p + (A+1)$$

Fig. 1. The (d,p) reaction between idealized nuclei is illustrated. The lowest two levels of (A+1) have the same core configuration $\Phi_0(A)$ as the target and so can be produced directly by the stripping process as indicated by the arrows. The third level, as indicated by its wave function, has an excited core, and so can be reached only through an inelastic collision with the deuteron before or with the proton after the transfer reaction. Only the lowest order routes are illustrated.

$$P''$$

$$d''$$

$$P'$$

$$d'$$

$$P$$

$$d$$

$$d + A \longrightarrow p + (A+1)$$

Fig. 2. For a typical deuteron channel d' and proton channel p', this figure illustrates by the arrows leading to these levels, what processes have to be described by the equations of motion in these channels. See section 2 for a full discussion.

solving the coupled equations for deuteron inelastic scattering by A with an incident wave in the channel d, and $\psi_{p,p'}^{(-)}$, is similarly defined for the proton-(A+1) system. However a numerical solution to the full 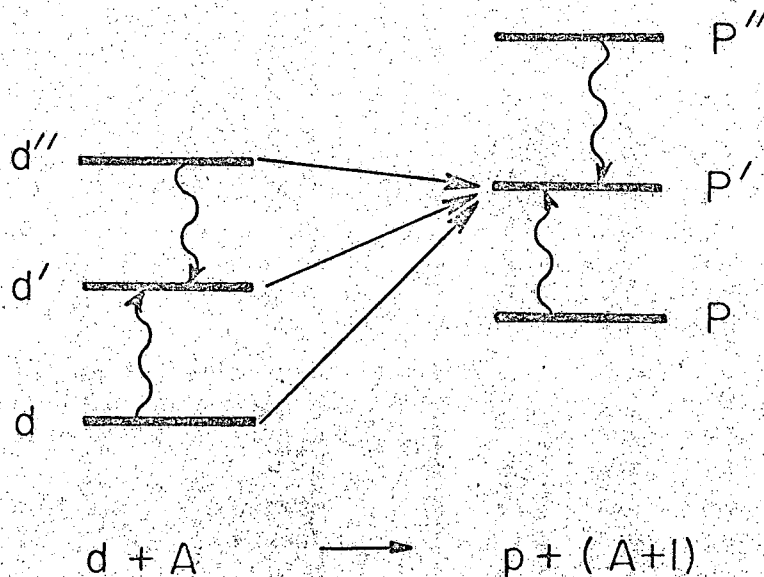problem has never been obtained. Iano and Austern considered the problem from a similar point of view, but solved it only to first order as concerns the inelastic transitions.[3] (That is to say, their generalized distorted waves correspond to the DWBA approximation for the inelastic processes.) Therefore they do not remedy the second cause of failure of the usual method. Kozlowsky and de-Shalit[4] and Levin[5] also have worked on this problem. They too treat inelastic effects only in first order, and only in the exit channel. The evaluation of (1.2) evidently is an extremely troublesome numerical problem.

In this paper we present a different solution which apparently is more amenable to numerical calculation. We are able to treat inelastic effects to all orders in both entrance and exit channels, and to treat spin-dependent interactions in the optical potentials and in the direct interactions.[6] The S-matrix is obtained directly from the application of the physical boundary conditions to the solution of an inhomogeneous system of equations describing the scattering in the residual system. The inhomogeneity is a source term which describes the production of the residual particle by the transfer process. The method is very general for scattering problems because the specification of the type of reaction enters only in the structure of the source term. The equations otherwise describe inelastic scattering.

In this paper we describe our method in terms of the (d,p) reaction, though it is applicable to all particle transfer reactions. In section 2 the rationale for the equations that we use to describe the reaction is

given, followed by a detailed formulation of their structure in section 3.

There also it is shown that the solution reduces to the usual DWBA in case

the core excitation effects are neglected. The source term due to the trans-

fer reaction is explicitly calculated in the zero-range approximation. In

section 4 the numerical problem posed by the systems of equations used to

describe the reaction is discussed.

2. SCHEMATIC PRESENTATION OF THE SOLUTION

In this section the reasoning is given by which we arrive at the equations we use to describe the transfer reaction where inelastic effects are included. For definiteness we consider the (d,p) reaction

$$d + A \rightarrow p + (A+1) \quad . \tag{2.1}$$

First focus attention on a typical deuteron channel d' in Fig. 2. We use d' to label all the quantum numbers needed to define the channel, such as the state of the nucleus A, and the angular momentum of the deuteron. The equation describing the motion in this channel is

$$(T_{d'} + V_{d'd'} - E_{d'}) u_{d'} = - \sum_{d'' \neq d'} V_{d'd''} u_{d''} \quad . \tag{2.2}$$

in which the various terms on the right represent the feeding of the channel d' by inelastic processes from other channels d", illustrated in the figure by the wiggly arrows. The equation (2.2) together with those describing all the other channels constitute the usual system of coupled equations for inelastic scattering.

Now focus attention on a typical proton channel p'. Again this channel is fed by inelastic processes leading from other proton channels (the wiggly arrows in the figure) but as well, it is fed by the transfer reaction from the various deuteron channels. Therefore, there will be two types of source terms in the equations for the proton motion. Accordingly we write,

$$(T_{p'} + V_{p'p'} - E_{p'}) \, w_{p'} = - \sum_{p''\neq p'} V_{p'p''} \, w_{p''} - \sum_{d'} \rho_{p'}^{d'} \, . \qquad (2.3)$$

Here $\rho_{p'}^{d'}$ represents the source of protons in channel p' as a result of the stripping reaction in channel d', illustrated by the straight arrow in the figure. Of course ρ will depend upon the solutions of the deuteron equations (2.2). We write down its detailed structure in section 3.

Since the reaction is initiated by a beam of deuterons incident on the ground state of nucleus A, the deuteron system (2.2) is to be solved subject to the boundary conditions that only a ground channel has an incoming wave while all channels may have outgoing waves. The proton system is subject to the condition that there are only outgoing waves.

The amplitudes of the outgoing proton waves are of course the S-matrix elements for the (d,p) reactions, in which the inelastic scattering is calculated to all orders among the retained channels in both the deuteron and proton systems, while the transfer reaction itself is treated as the weak process it is, only in first order. That is, we have not included the reaction back on the deuteron channels of pickup reactions. Therefore the solution to the problem can be obtained in two steps. First, the coupled system (2.2) for the deuterons is solved so that the source terms ρ can be constructed. Second, the inhomogeneous system (2.3) describing the protons is solved.

Of course one could think of extending the system of equations to include additional particle channels like triton and alpha channels in which each of these systems is described by equations like (2.3).

3.   DETAILED FORMULATION FOR THE (d,p) REACTION

In order to illustrate our method without unnecessary detail we con-
sider the (d,p) reaction in the absence of spin-dependent interactions.
This means we can treat the deuteron and proton as spinless particles. The
general case is treated in the appendix.

3.1.   Inelastic Deuteron Scattering

Let the target nucleus (A) be governed by the Hamiltonian H_A,
whose eigenfunctions are denoted by $\Phi_{\alpha J}(\underline{A})$, where $\underline{A}$ denotes all the nuclear
coordinates, and α denotes all nuclear quantum numbers additional to J, M
such as the parity π_α.

$$(H_A - E_{\alpha_d J_d}) \, \Phi_{\alpha_d J_d}(\underline{A}) = 0 \ . \tag{3.1}$$

[We use the subscript d on all quantum numbers referring either to the
deuteron itself or the nucleus (A) and p for quantum numbers describing
the proton and the nucleus (A+1).] For the total system we have

$$H = H_A + T + V(\underline{d}, \underline{A}) \tag{3.2}$$

where T is the energy of relative motion and V is the deuteron-nucleus
interaction. It should be understood that V is an effective interaction
in the sense discussed elsewhere, which carries the affect of all the chan-
nels we neglect on those we do treat explicitly.[7,8]

With the eigenfunction of total angular momentum $\underline{I} = \underline{\ell}_d + \underline{J}_d$ and
parity $\pi = (-)^{\ell_d} \pi_{\alpha_d}$

$$\phi_{d\pi I}^{M} (\hat{\underline{R}}, \underline{A}) = \left[Y_{\ell_d} (\hat{\underline{R}}) \, \Phi_{\alpha_d J_d} (\underline{A}) \right]_{I}^{M} \tag{3.3}$$

we expand a solution Ψ of

$$(H - E) \, \Psi = 0 \tag{3.4}$$

corresponding to incident particles in the channel d as

$$\Psi_{d\pi I}^{M} = \frac{1}{R} \sum_{d'} u_{d'}^{d\pi I}(R) \, \phi_{d'\pi I}^{M}(\hat{\underline{R}}, \underline{A}) \tag{3.5}$$

where we use d to denote the whole collection of quantum numbers in a
deuteron target channel

$$d \equiv \alpha_d \, J_d \, \ell_d \tag{3.6}$$

and by d' some other state of intrinsic motion $\alpha_d' \, J_d'$ and/or relative

motion ℓ_d'. In the usual way from (3.4) and (3.5) we get, for each total

angular momentum and parity I, π a system of coupled equations for the

radial functions $u(r)$. They have the form, for each channel d'

$$(T_{d'} + V_{d'd'}^{\pi I}(R) - E_{d'}) \, u_{d'}^{d\pi I}(R) = - \sum_{d'' \neq d'} V_{d'd''}^{\pi I}(R) \, u_{d''}^{d\pi I}(R) \, . \tag{3.7}$$

(A sum over a primed subscript will always include the ground (unprimed)

channel unless otherwise indicated.) In the above equation,

$$V_{dd'}^{\pi I}(R) = \langle \phi_{d\pi I}^{M}(\hat{\underline{R}}, \underline{A}) | V(\underline{d}, \underline{A}) | \phi_{d'\pi I}^{M}(\hat{\underline{R}}, \underline{A}) \rangle \tag{3.8}$$

and

$$T_d = \frac{\hbar^2}{2m_d} \left(-\frac{d^2}{dR^2} + \frac{\ell_d(\ell_d + 1)}{R^2} \right) \tag{3.9}$$

$$E_d = E - E_{\alpha_d J_d} . \tag{3.10}$$

These are to be solved subject to the boundary conditions that there are incoming waves (I) only in the channels containing the nucleus in its ground state, while all channels may have outgoing waves (0)

$$u_{d'}^{d\pi I}(R) \rightarrow \delta_{d'd} I_{\ell_d}(k_d R) - \left(\frac{v_d}{v_{d'}} \right)^{1/2} S_{d',d}^{\pi I} O_{\ell_{d'}}(k_{d'}, R) . \tag{3.11}$$

Here

$$I_\ell^*(kr) = O_\ell(kr) = G_\ell(kr) + iF_\ell(kr) \rightarrow e^{i(kr - \eta \ell n(2kr) - \ell\pi/2 + \sigma_\ell)} \tag{3.12}$$

where G and F are the irregular and regular Coulomb functions, $\eta = ZZ'e^2/\hbar v$, σ_ℓ is the Coulomb phase shift, and v the particle velocity.

The total wave function has all angular momentum and both parities, corresponding to the fact that a _beam_ of particles is incident on the target.

$$\Psi = \sum_{\ell_d IM} A_{d\pi I}{}^M \Psi_{d\pi I}{}^M .$$ (3.13)

If we choose

$$A_{d\pi I}{}^M = - \frac{1}{2ik_d} C_0{}^{\ell_d}{}_{M_d}{}^{J_d}{}_M{}^I \sqrt{4\pi(2\ell_d + 1)} \ i^{\ell_d} e^{i\sigma_d}$$ (3.14)

then, in the absence of the Coulomb force,

$$\Psi \rightarrow \Phi_{J_d}{}^{M_d}(A) \ e^{ik_d Z} + \text{scattered wave}$$

which is the desired solution.

The detailed structure of the coupled equations (3.7) based on micro-scopic nuclear descriptions has been discussed elsewhere.[7,8]

3.2. Proton Production and Inelastic Scattering

The nucleus A+1 we suppose is described by

$$\left(H_{A+1} - E_{\alpha_p J_p} \right) \Phi_{\alpha_p J_p} \ (\underline{A+1}) = 0$$ (3.15)

and with notation and definitions analogous to the above, the proton motion

in the channel p would be described, for each total angular momentum and

parity $I\pi$, by

$$(T_p + V_{pp}^{\pi I}(r_p) - E_p)\, w_p^{\pi I}(r_p) = -\sum_{p' \neq p} V_{pp'}^{\pi I}(r_p)\, w_{p'}^{\pi I}(r_p) \qquad (3.16)$$

were it not for the fact that there is a source of protons in the vicinity of the nucleus due to the stripping of the neutron from the deuteron. We must instead solve the inhomogeneous system

$$(T_p + V_{pp}^{\pi I}(r_p) - E_p)\, w_p^{d\pi I}(r_p) = -\sum_{p' \neq p} V_{pp'}^{\pi I}(r_p)\, w_{p'}^{d\pi I}(r_p) - \rho_p^{d\pi I}(r_p) \qquad (3.17)$$

where ρ denotes the proton source. We calculate this from

$$\rho_p^{d\pi I}(r_p) = r_p \sum_{d'} \langle \phi_{p\pi I}^{M}(\hat{r}_p, A+1) | V_{np}(r) | \phi_0(r)\, \phi_{d'\pi I}^{M}(\hat{R}, A)\, u_{d'}^{d\pi I}(R)/R \rangle \qquad (3.18)$$

and will justify this form later in section 3.4. Here $r = r_n - r_p$ and $2R = r_n + r_p$ and ϕ_0 is the deuteron wave function. In this matrix element, all coordinates (including $\hat{r}_p$) are integrated except r_p. The various terms in this sum represent the transitions indicated by arrows in Fig. 2 leading from the various channels d' in the deuteron system to the channel p in the proton system.

We have placed d as a superscript on w to correspond to the fact that the reaction is initiated by a beam of deuterons in the ground channel d, which information is carried in the source terms through their dependence on solutions of (3.7) with boundary conditions (3.11).

The equations (3.17) are to be solved with the physical boundary conditions that proton channels of this reaction have only outgoing waves

$$w_p^{d\pi I} \to - \left(\frac{v_d}{v_p} \right)^{1/2} S_{p,d}^{\pi I} \; O_{\ell_p}(k_p r_p) \quad . \qquad (3.19)$$

The amplitude of

$$\left(\frac{v_d}{v_p} \right)^{1/2} \Phi_{J_p}^{M_p} \; e^{i[k_p r_p - \eta \ell n(2k_p r)]}$$

in $\; w_p^{d\pi I} \; \phi_{p\pi I}^{M}$

is

$$- S_{p,d}^{\pi I} \; i^{-\ell_p} \; e^{i\sigma_p} \sum_{m_p} C_{m_p \; M_p \; M}^{\ell_p \; J_p \; I} \; Y_{\ell_p}^{m_p}(\hat{\underline{r}}_p) \quad .$$

On the other hand, the sources (3.18) were constructed with unit amplitude for $u_{d'}^{d\pi I}$, whereas its actual amplitude in the incident wave is given by (3.14). Hence the total amplitude for the process is

$$f(J_d M_d \to J_p M_p) = \frac{1}{2ik_d} \sum_{\ell_d \ell_p m_p IM} \sqrt{4\pi(2\ell_d + 1)} \; i^{\ell_d - \ell_p} \; e^{i(\sigma_d + \sigma_p)}$$

$$C_{0 \; M_d \; M}^{\ell_d \; J_d \; I} \; C_{m_p \; M_p \; M}^{\ell_p \; J_p \; I} \; S_{p,d}^{\pi I} \; Y_{\ell_p}^{m_p}(\hat{\underline{k}}_p) \quad . \qquad (3.20)$$

(We have replaced $\hat{\underset{\sim}{r}}_p$ by $\hat{\underset{\sim}{k}}_p$ since they become coincident asymptotically.)
The flux through the surface $r_p^2 d\Omega$ in the direction $\hat{\underset{\sim}{k}}_p$ is therefore
$v_d |f|^2 d\Omega$, while the incident deuteron flux is v_d. Whence

$$\frac{d\sigma}{d\Omega} = \frac{1}{2J_d+1} \sum_{M_p M_d} |f(J_d M_d \rightarrow J_p M_p)|^2 . \tag{3.21}$$

We should perhaps note that $S_{p,d}^{\pi I}$ is a short hand notation for

$$S_{p,d}^{\pi I} \equiv \langle (\ell_p J_p) IM | S | (\ell_d J_d) IM \rangle \tag{3.22}$$

(which is independent of M). It is then transparent that

$$\sum_{IM} C_{0 \ M_d \ M}^{\ell_p \ J_p \ I} \ C_{m_p \ M_p \ M}^{\ell_p \ J_p \ I} \ S_{p,d}^{\pi I} = \langle \ell_p m_p, J_p M_p | S | \ell_d \ 0, J_d M_d \rangle . \tag{3.23}$$

3.3. Check with DWBA

We can check the construction of the source term in all its details by
considering the limit of our equations when the inelastic effects in both
proton and deuteron channels are neglected. Since in any case, the transfer
process is calculated in first order, this limit should lead exactly to the
DWBA cross section. Our equations become

$$(T_d + V_{dd}^{\pi I} - E_d) \ u_d^{0\pi I}(R) = 0 \tag{3.24}$$

$$(T_p + V_{pp}^{\pi I} - E_p) \; w_p^{d\pi I}(r_p) = - \rho_p^{d\pi I}(r_p) \qquad (3.25)$$

where d refers to _elastic_ deuteron channels and p to _any_ proton channel. Introduce the solution of the homogeneous equation corresponding to the second of these

$$(T_p + V_{pp}^{\pi I} - E_p) \; w_p^{o\pi I}(r_p) = 0 \qquad (3.26)$$

which is regular at the origin and has the asymptotic behaviour,

$$w_p^{o\pi I} \to - 2i \; e^{i\delta_p} \; \sin(k_p r_p - \ell_p \pi/2 + \delta_p) \qquad (3.27)$$

where δ_p is the phase shift introduced by V_{pp}. Then the asymptotic solution of (3.25) can be written down immediately in terms of $u_d^{o\pi I}$ and $w_p^{o\pi I}$ (See for example Ref. 7). It is

$$u_p^{d\pi I} \to - \left(\frac{v_d}{v_p} \right)^{1/2} S_{p,d}^{\pi I} \; O_{\ell_p}(k_p r_p) \qquad (3.28)$$

where O_{ℓ_p} is an outgoing wave and S is given by the explicit formula

$$S_{p,d}^{\pi I} = \frac{i}{\hbar^2} \left(\frac{m_p m_d}{k_p k_d} \right)^{1/2} \int w_p^{o\pi I} \; \rho_p^{d\pi I} \; d \, r_p \qquad (3.29)$$

and $k_p = (2m_p E_p / \hbar^2)^{1/2}$ is the wave number. The superscript o on the solutions to (3.24, 3.26) indicate that they describe elastic scattering only. Introducing (3.18) for ρ into this expression for S gives,

$$S_{p,d}^{\pi I} = \frac{i}{\hbar^2} \left(\frac{m_p m_d}{k_p k_d} \right)^{1/2} \int \frac{w_p^{o\pi I}(r_p)}{r_p} \phi_{p\pi I}^{M*} (\hat{r}_p, \underline{A+1})$$

$$V_{np}(r) \phi_0(r) \phi_{d\pi I}^M(\hat{\underline{R}}, \underline{A}) \frac{u_d^{o\pi I}(R)}{R} d\underline{r}_n d\underline{r}_p d\underline{A} \quad . \tag{3.30}$$

Inserting this into (3.20) we have

$$f = \frac{1}{2\hbar^2 k_d} \left(\frac{m_p m_d}{k_p k_d} \right)^{1/2} \sum_{\ell_d \ell_p m_p} \sqrt{4\pi(2\ell_d + 1)} \; i^{\ell_d - \ell_p} \; e^{i(\sigma_p + \sigma_d)} \; Y_{\ell_p}^{m_p}(\hat{\underline{k}}_p)$$

$$\int \frac{w_p^{o\pi I}(r_p)}{r_p} \; Y_{\ell_p}^{m_p *}(\hat{\underline{r}}_p) \; \langle \Phi_{\alpha_p J_p}^{M_p}(\underline{A+1}) | V_{np}(r) | \phi_0(r) \; \Phi_{\alpha_d J_d}^{M_d}(\underline{A}) \rangle$$

$$\frac{u_d^{o\pi I}(R)}{R} \; Y_{\ell_d}^0(\hat{\underline{R}}) \; d\underline{r}_n \; d\underline{r}_p \quad . \tag{3.31}$$

Now let us write

$$\psi_d^{(+)} \equiv - \sum_{\ell_d} \sqrt{4\pi(2\ell_d + 1)} \; i^{\ell_d} \; e^{i\sigma_d} \; \frac{u_d^{o\pi I}(R)}{2ik_d R} \; Y_{\ell_d}^0(\underline{R}) \tag{3.32}$$

which is the partial wave expansion of the deuteron wave, distorted by V_{dd}. In the absence of distortion it goes to

$$\psi_d^{(+)} \to e^{ikZ} \quad .$$

(3.33)

Also

$$\psi_p^{(-)*} \cong -\ 4\pi \sum_{\ell_p m_p} i^{-\ell_p} e^{i\sigma_p} \frac{w_p^{o\pi I}(r_p)}{2i\ k_p\ r_p} Y_{\ell_p}^{m_p *}(\hat{r}_p)\ Y_{\ell_p}^{m_p}(\hat{k}_p)$$

(3.34)

which, in the absence of distortion goes to

$$\psi_p^{(-)*} \to e^{-i\ \underline{k}_p \cdot \underline{r}_p} \quad .$$

(3.35)

Equation (3.31) can be written now as,

$$f = \frac{1}{2\pi\hbar^2} \left(\frac{k_p}{k_d}\ m_p m_d \right)^{1/2} T_{p,d}$$

(3.36)

where

$$T_{p,d} \equiv \int \psi_p^{(-)*}(\underline{k}_p,\ \underline{r}_p)\ \langle \Phi_{\alpha_p J_p}^{M_p}(A+1)|V_{np}|\phi_0(r)\ \Phi_{\alpha_d J_d}^{M_d}(\underline{A})\rangle\ \psi_d^{(+)}(\underline{k}_d,\ \underline{R})\ d\ \underline{r}_n\ d\ \underline{r}_p$$

(3.37)

is precisely the usual DWBA expression for the T-matrix. The cross section according to (3.21) is

$$\frac{d\sigma}{d\Omega} = \frac{m_p m_d}{(2\pi\hbar^2)^2} \quad \frac{k_p}{k_d} \quad \frac{1}{2J_d+1} \sum_{M_d M_p} |T_{p,d}|^2 \qquad (3.38)$$

which is the usual DWBA result for the (d,p) reaction with spinless nucleons.

Thus we have confirmed, to the last factor, that our construction of the source term is the one which, in the _absence_ of inelastic effects leads to the DWBA cross section for the transfer reaction. Consequently, although our equations are approximate, they embody no approximations additional to those of the usual DWBA, whereas they do take account of inelastic effects, to all orders among the retained channels.

3.4. The Source Term

The source term can be explicitly evaluated under the usual zero-range approximation,

$$V_{np}(r) \; \phi_0(r) = D_0 \; \delta(r) \qquad (3.39)$$

where D_0 can be related to the binding energy of the deuteron.

To carry out the integration of the coordinates $\underset{\sim}{A}$ we make a parentage expansion

$$\Phi_{J_p}^{M_p}(A+1) = \sum_{\ell_n \alpha_d J_d} \beta_{\ell_n}(J_d, J_p) \left[\Phi_{\alpha_d J_d}(\underset{\sim}{A}) \; Y_{\ell_n}(\hat{\underset{\sim}{r}}_n) \right]_{J_p}^{M_p} \chi_{\ell_n}(r_n) \qquad (3.40)$$

where χ_{ℓ_n} is a single-particle radial function for the neutron in the nucleus $(A+1)$. Inserting these into the source (3.18) gives,

$$\rho_p^{d\pi I}(r_p) = D_O \sum_{d'} \left\langle \phi_{p\pi I}^M(\hat{r}_p, \underset{\sim}{A}, \underset{\sim}{r}_p) \middle| \phi_{d'\pi I}^M(\hat{r}_p, \underset{\sim}{A}) \right\rangle u_{d'}^{d\pi I}(r_p) \qquad (3.41)$$

where

$$\left\langle \phi_{p\pi I}^M(\hat{r}_p, \underset{\sim}{A}, \underset{\sim}{r}_p) \middle| \phi_{d\pi I}^M(\hat{r}_p, \underset{\sim}{A}) \right\rangle = \sum \beta_{\ell_n}(J_d, J_p) \chi_{\ell_n}(r_p)$$

$$\left\langle Y_{\ell_p}(\hat{r}_p) \left[\Phi_{J_d}(\underset{\sim}{A}) \; Y_{\ell_n}(\hat{r}_p) \right]_{J_p} \right]_I^M \middle| \left[Y_{\ell_d}(\hat{r}_p) \; \Phi_{J_d}(\underset{\sim}{A}) \right]_I^M \right\rangle . \qquad (3.42)$$

The integrations are over $\underset{\sim}{A}$ and $\hat{r}_p$. We write the bracket in this expression symbolically as

$$\langle \ell_p, (J_d, \ell_n) J_p; IM | \ell_d J_d; IM \rangle \qquad (3.43)$$

where the arguments of $\ell_p \, \ell_n$ and ℓ_d are $\hat{r}_p$ while those of J_d are $\underset{\sim}{A}$. To carry out the A integrations we recouple on the left

$$| \ell_p, (J_d, \ell_n) J_p; IM \rangle = \sum_K (\hat{J}_p \, \hat{K})^{1/2} (-)^{\ell_p + J_p + I} \begin{Bmatrix} \ell_p & \ell_n & K \\ J_d & I & J_p \end{Bmatrix} | (\ell_p \ell_n) K, J_d; IM \rangle.$$

$$(3.44)$$

where $\hat{K} = 2K+1$.

For the following bracket we have

$$\langle (\ell_p \ell_n)\, K,\, J_d;\, IM|\, \ell_d J_d;\, IM \rangle = \sum_{m_d M_d} C^{K\ \ J_d\ \ I}_{m_d\ M_d\ M}\ C^{\ell_d\ J_d\ \ I}_{m_d\ M_d\ M}\ \langle \ell_p \ell_n;\, K\, m_d|\, \ell_d m_d \rangle \ .$$

$$(3.45)$$

The $\hat{r}_p$ integration of the last bracket yields,

$$\langle \ell_p \ell_n;\, K\, m_d|\, \ell_d m_d \rangle = (-)^{\ell_d} \left(\frac{\hat{\ell}_p \hat{\ell}_n}{4\pi} \right)^{1/2} \begin{pmatrix} \ell_p & \ell_n & \ell_d \\ 0 & 0 & 0 \end{pmatrix} \delta_{K\ell_d}\ .$$

$$(3.46)$$

Thus we find for the source term in the channel p,

$$\rho_p^{d\pi I}(r) = D_0 \sum_{d'\ell_n} \beta_{\ell_n}(J_{d'},\, J_p)\, (-)^{\ell_n + J_p + I}$$

$$\times \left(\frac{\hat{J}_p\, \hat{\ell}_p\, \hat{\ell}_n\, \hat{\ell}_{d'}}{4\pi} \right)^{1/2} \begin{Bmatrix} \ell_p & \ell_n & \ell_{d'} \\ J_{d'} & I & J_p \end{Bmatrix} \begin{pmatrix} \ell_p & \ell_n & \ell_{d'} \\ 0 & 0 & 0 \end{pmatrix} \chi_{\ell_n}(r)\, u_{d'}^{d\pi I}(r)\ .$$

$$(3.47)$$

Here we see explicitly the radial form of the source which contains the product of the neutron bound state and the deuteron scattering state, as expected. That they have the same coordinate r, follows of course from the zero-range approximation. For simplicity we have omitted the usual center-of-mass correction.

Of course the strength with which the various levels d' contribute

to the source in the proton channel p depends on the extent to which each

of them is a parent of the level p. In the idealized example of Fig. 1, the

first two levels have only the target ground state as parent, while the third

state also has a pure parentaged, but based on the excited target state. In

general the parentage will be spread over several or even many states, though

it may still reside dominantly in several states. At any rate, this

information is inserted in the source term through the parentage coefficients

$\beta_{\ell_n}(J_{d'}, J_p)$.

3.5. The Inverse Reaction

The reaction, inverse to the one of primary interest can be calculated

at little extra trouble, as we shall see in the next section. The source

for deuterons corresponding to the (p,d) reaction in the channel labeled d

is

$$\rho_d^{p\pi I}(R) = R \sum_{p'} \langle \phi_{d\pi I}^M(\hat{R}, A) \, \phi_0(r) | V_{np}(r) | \phi_{p'\pi I}^M(\hat{r}_p, \underline{A+1}) \ w_{p'}^{p\pi I}(r_p)/r_p \rangle$$

$$(3.48)$$

in which $w_{p'}^{p\pi I}$ are solutions of the proton inelastic scattering problem

defined by (3.16) with boundary conditions

$$w_{p'}^{p\pi I}(r_p) \rightarrow \delta_{p'p} \, I_{\ell_p}(k_p \, r_p) - \left(\frac{v_p}{v_{p'}} \right)^{1/2} S_{p',p}^{\pi I} \, O_{\ell_{p'}}(k_{p'} \, r_p) \ . \qquad (3.49)$$

The equations describing the deuteron motion following the pickup reaction

are for each channel d,

$$(T_d + V_{dd}^{\pi I}(R) - E_d)\, u_d^{p\pi I}(R) = -\sum_{d'\neq d} V_{dd'}^{\pi I}(R)\, u_{d'}^{p\pi I}(R) - \rho_d^{p\pi I}(R) \qquad (3.50)$$

which are to be solved with the boundary condition

$$u_d^{p\pi I}(R) \to -\left(\frac{v_p}{v_d}\right)^{1/2} S_{d,p}^{\pi I}\, O_{\ell_d}(k_d R)\ . \qquad (3.51)$$

The source term can be evaluated explicitly as before, under the zero-range approximation, and is

$$\rho_d^{d\pi I}(R) = D_0 \sum_{p'\ell_n} \beta_{\ell_n}(J_d\, J_{p'})(-)^{\ell_n + J_{p'} + I}$$

$$\times \left(\frac{\hat{J}_{p'}\, \hat{\ell}_{p'}\, \hat{\ell}_n\, \hat{\ell}_d}{4\pi}\right)^{1/2} \begin{Bmatrix} \ell_{p'} & \ell_n & \ell_d \\ J_d & I & J_{p'} \end{Bmatrix} \begin{pmatrix} \ell_{p'} & \ell_n & \ell_d \\ 0 & 0 & 0 \end{pmatrix} \chi_{\ell_n}(R)\, w_{p'}^{p\pi I}(R)\ .$$

$$(3.52)$$

The cross section can be computed from (3.20) and (3.21) if everywhere the subscript d is replaced by p and vice versa.

4. THE NUMERICAL PROBLEM AND ITS SOLUTIONS

In brief the problem of calculating the (d,p) reaction including inelastic effects, amounts to solving the coupled equations (3.7) describing deuteron elastic and inelastic scattering, constructing the proton source terms (3.47) from this solution, and then solving the inhomogeneous coupled equations (3.17) for the proton scattering by the residual nucleus. Each of these two systems of coupled equations is defined by two boundary conditions, one at the origin, where all solutions must vanish, and one at some suitably large radius, say R, outside the range of the nuclear interactions. Let us denote the solution to the deuteron equations (3.7) by the vector

$$U(r) = \begin{pmatrix} u_1(r) \\ \cdot \\ \cdot \\ \cdot \\ u_N(r) \end{pmatrix} \qquad (4.1)$$

where u_1 denotes the entrance channel function, and there are altogether N channels. Clearly it is not known a priori what slope to assign to the components of this vector at the origin, so that in the asymptotic region it will have the required form

$$U(r) = U_\infty(r) \equiv \begin{pmatrix} I_1 -S_{11} O_1(r) \\ -S_{21} O_2(r) \\ -S_{N1} O_N(r) \end{pmatrix} \qquad , \quad r \geq R \quad . \qquad (4.2)$$

Consequently a linearly independent set of vector solutions $\underset{\sim}{U}_q(r)$ must

be generated from which the desired one can be constructed. This is

conveniently done by using the initial values

$$\underset{\sim}{U}_q(r) = \frac{(k_q r)^{\ell_q+1}}{(2\ell_q+1)!!} \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 1 \\ \cdot \\ 0 \end{pmatrix} \quad , \quad r \stackrel{\sim}{-} 0 \qquad (4.3)$$

where the 1 occurs in the qth position. Solving the system N times

corresponding to q = 1, .. N generates the desired set of independent

solutions. The particular form and normalization is chosen in accordance

with the nature of the solution near the origin, and it insures that the

solution does not grow out of bounds with increasing r. In practice, the

integration is started at some value of r, say R_{min} chosen to be as

small as possible but such that $U_q(R_{min})$ is not smaller than some assigned

small number, say of the order 10^{-30}.

With this set of solutions $\underset{\sim}{U}_q(r)$, q = 1, ... N, we seek the

linear combinations such that

$$\sum_{q=1}^{N} a_q \underset{\sim}{U}_q(R) = \underset{\sim}{U}_\infty(R) \quad . \qquad (4.4)$$

This, together with the derivative, constitute 2N linear algebraic

equations in the a_q and S_q, of which there are N each. From these

S-matrix elements the elastic and inelastic deuterons cross sections may be

calculated, while with the a_q, the desired solution can be constructed for use in the proton source term.

The solution of the inhomogeneous problem is obtained in a similar fashion, since a general solution can be obtained from a particular solution say $\underset{\sim}{W}(r)$ of the inhomogenous system (3.17) plus a general solution of the corresponding homogeneous system (3.16). One may start the particular solution with zero value and slope at the origin. The solutions $\underset{\sim}{W}_q$ to the homogeneous problem are obtained by the same means as above, and the solution with the desired asymptotic boundary condition is found by solving the linear algebraic equations

$$\sum_{q=1}^{M} b_q \underset{\sim}{W}_q(R) + \underset{\sim}{W}(R) = - \begin{pmatrix} S_1 \, O_1(R) \\ \cdot \\ \cdot \\ S_M \, O_M(R) \end{pmatrix} \tag{4.5}$$

for the unknowns b and S. From these last S-matrix elements the cross sections of the various proton channels in the (d,p) reaction can be calculated.

It is clear that one can calculate the proton elastic and inelastic scattering by the residual nucleus (A+1) by simply solving a set of linear algebraic equations.

$$\sum_{q=1}^{M} c_q \underset{\sim}{W}_q(R) = \underset{\sim}{W}_\infty(R) \tag{4.6}$$

where $W_\infty(R)$ is defined similarly to (4.2).

Finally the inverse (p,d) reaction can be calculated at only the cost
of finding a particular solution of (3.50) with the source term (3.48) con-
structed from the physical solution of (3.16) already found in connection
with the scattering boundary conditions (4.6) and then solving a set of
algebraic equations analogous to (4.5).

5. SUMMARY

We have shown how to go beyond the usual DWBA treatment of transfer

reactions to include the effects of inelastic scattering in both entrance and

exit channels. As discussed in more detail in the introduction, this may be

necessary in some cases for one or both of two reasons. The one of these is

that the coupling of inelastic channels to the ground is in some nuclei very

strong, especially in deformed regions. The other is that most certainly in

every nucleus some states produced in particle transfer reactions will have

a parentage based more on an excited rather than the ground state of the

target. This will be especially true of higher lying states.

Our basic equations cannot be rigorously derived from the Schroedinger

equation for the system. However we can say that they embody no approximations

beyond those made in the usual distorted wave method, whereas they do carry

the effects of inelastic processes on the transfer reaction. These effects

are carried to all orders among the retained channels so that both causes of

failure of the usual DWBA are covered. However like the DWBA, the transfer

reaction is treated as a weak process, only in first order.[9]

We have illustrated our method by a detailed example, the (d,p)

stripping and inverse pickup reaction, in which, for didactic purposes, we

have omitted reference to the nucleon and deuteron spins. The S-matrix

elements for the deuteron and proton elastic and inelastic scattering as well

as the stripping and pickup reactions are obtained by applying the appropriate

boundary conditions to the solutions of a homogeneous and inhomogeneous

system of coupled differential equations for both the deuteron and proton

channels. One particular solution only is required for each of the two

inhomogeneous systems while general solutions of the two homogeneous problems
are required.

The method can be applied to other problems besides the one considered
here. Our first application is to the (p,t) reaction using a microscopic
nuclear description and a spin-dependent direct interaction as well as
spin-orbit coupling in the optical potentials.[6]

Appendix: The General Case with Spins

Having considered the (d,p) reaction in detail in the absence of spin-dependent interactions, we now give the changes necessary to include such interactions as spin-orbit terms in the optical potentials and spin-dependent direct-interactions.

The channel functions ϕ_d and ϕ_p used in section 3 are now constructed with the use of spin-orbit functions as

$$\phi_{p\pi I}^{M} = \left[\mathcal{Y}_{(\ell_p 1/2)j_p} \Phi_{\alpha_p J_p} \right]_I^M \tag{A.1}$$

$$\phi_{d\pi I}^{M} = \left[\mathcal{Y}_{(\ell_d 1)j_d} \Phi_{\alpha_d J_d} \right]_I^M \tag{A.2}$$

where

$$\mathcal{Y}_{(\ell_p 1/2)j_p}^{m_p} = \left[Y_{\ell_p}(\hat{r}_p)\, \eta(\sigma_p) \right]_{j_p}^{m_p} \tag{A.3}$$

$$\mathcal{Y}_{(\ell_d 1)j_d}^{m_d} = \left[Y_{\ell_d}(\hat{R}) \left[\eta(\sigma_p)\, \eta(\sigma_n) \right]_1 \right]_{j_d}^{m_d} \tag{A.4}$$

where η is the spin function for a nucleon. The channel quantum numbers d and p now include the additional specification of j_d and j_p respectively. The source term for the (d,p) reaction becomes

$$\rho_p^{d\pi I}(r) = D_0 \sum_{d' \ell_n j_n} \beta_{\ell_n j_n}(J_{d'}, J_p) (-)^{\ell_{d'}+J_p+j_p+I}$$

$$\left(\frac{\hat{J}_p \hat{j}_{d'} \hat{\ell}_p \hat{\ell}_n}{4\pi}\right)^{1/2} \begin{pmatrix} \ell_p & \ell_n & \ell_{d'} \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} j_p & j_n & j_{d'} \\ J_{d'} & I & J_p \end{Bmatrix} \begin{bmatrix} \ell_p & 1/2 & j_p \\ \ell_n & 1/2 & j_n \\ \ell_{d'} & 1 & j_{d'} \end{bmatrix}$$

$$\chi_{\ell_n j_n}(r) \; u_{d'}^{d\pi I}(r) \tag{A.5}$$

where

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = (\hat{c}\hat{f}\hat{g}\hat{h})^{1/2} \begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} \tag{A.6}$$

is a recoupling coefficient with { } a 9-j symbol.

Corresponding to the incident wave

$$\Psi_0 = \Phi_{J_d}^{M_d}(A) \left[\eta(\sigma_p)\eta(\sigma_n) \right]_1^{\mu_d} e^{ik_d Z} \tag{A.7}$$

The amplitude of $u_{d'}^{d\pi I}$ in the source is

$$A_{d\mu_d M_d}^{\pi I M} = -\frac{1}{2ik_d} \; C_{0 \; \mu_d \; \mu_d}^{\ell_d \; 1 \; j_d} \; C_{\mu_d \; M_d \; M}^{j_d \; J_d \; I} \tag{A.8}$$

Hence the amplitude for the transition is

$$f(J_d M_d \mu_d \rightarrow J_p M_p \mu_p) = \frac{1}{2ik_d} \sum_{\ell_d j_d \ell_p j_p I} \sqrt{4\pi(2\ell_d+1)} \; i^{\ell_d - \ell_p} \; e^{i(\sigma_d + \sigma_p)}$$

$$C^{\ell_d \; 1 \; j_d}_{0 \; \mu_d \; \mu_d} \; C^{j_d \; J_d \; I}_{\mu_d \; M_d, M_d + \mu_d} \sum_{m_p m} C^{\ell_p \; 1/2 \; j_p}_{m_p \; \mu_p \; m} \; C^{j_p \; J_p \; I}_{m \; M_p, M_d + \mu_d} \; S^{\pi I}_{p,d} \; Y^{m_p}_{\ell_p}(\hat{k}_p)$$

(A.9)

so that the cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{3(2J_d+1)} \sum_{M_d \mu_d M_p \mu_p} \left| f \right|^2 \tag{A.10}$$

The source term for the inverse reaction can be written down immediately
from (A.5) by analogy to the relationship between (3.47) and (3.52). The
corresponding amplitude can be obtained from (A.9) by everywhere interchanging
the subscripts d and p, and interchanging 1 and 1/2 in the two
Clebsch-Gordon coefficients.

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FIGURE CAPTIONS

Fig. 1. The (d,p) reaction between idealized nuclei is illustrated. The

lowest two levels of (A+1) have the same core configuration $\Phi_o(A)$ as

the target and so can be produced directly by the stripping process as

indicated by the arrows. The third level, as indicated by its wave function,

has an excited core, and so can be reached only through an inelastic

collision with the deuteron before or with the proton after the transfer

reaction. Only the lowest order routes are illustrated.

Fig. 2. For a typical deuteron channel d' and proton channel p', this

figure illustrates by the arrows leading to these levels, what processes

have to be described by the equations of motion in these channels. See

section 2 for a full discussion.