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Authors
Friedman, Yaakov
Russo, Bernard

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FUNCTION REPRESENTATION OF COMMUTATIVE OPERATOR TRIPLE SYSTEMS

YAAKOV FRIEDMAN AND BERNARD RUSSO

1. *Introduction and notation*

By an operator triple system we mean a *complex* linear subspace of $\mathscr{L}(H, K)$, the bounded linear operators from a Hilbert space H to a Hilbert space K, which is closed in some topology and under some triple product of its elements. In this paper we shall be interested primarily in the class of J*-algebras, though we also deal with abstract Jordan triple systems. One of the principal consequences of our main result (Theorem 1) is a Gelfand Naimark representation theorem for associative Jordan triple systems (Theorem 2).

A J*-*algebra* is a norm closed complex subspace of $\mathscr{L}(H, K)$ which contains aa^*a if it contains the element a. Simple identities [6, p. 17] show that J*-algebras are closed under the triple product $(a, b, c) \to ab^*c + cb^*a$. The subclass of *ternary algebras*, that is those closed under $(a, b, c) \to ab^*c$, is more tractable but occurs less frequently than J*-algebras. Any collection of operators which is closed under a binary product gives rise to an operator triple system. Thus the class of J*-algebras is large enough to contain all C*-algebras, JC*-algebras, and many important Lie algebras of operators; and small enough to possess nice properties.

It is elementary that every Banach space can be embedded isometrically into a space of continuous functions satisfying some symmetry properties. We prove below (Theorem 1) that this embedding is onto when the Banach space is a *commutative* J*-algebra (defined below).

This result answers a question posed by Harris in [7] and has several immediate consequences. We show that the range of a contractive projection on a commutative J*-algebra is itself a commutative J*-algebra. This is a complete solution in a special case to a recent problem, cf. [1, 3, 4, 5]. We also prove a Stone–Weierstrass theorem in the setting of commutative J*-algebras, and use it to determine all closed ideals of a commutative J*-algebra. In the last section we extend our results to Jordan triple systems.

Some of the applications in analysis and geometry of J*-algebras and their generalizations are indicated in the introduction of [5].

If M is a J*-algebra and v is a partial isometry in M, then with $l = vv^*$ and $r = v^*v$, projections $E = E(v)$, $F = F(v)$ and $G = G(v)$ on M are defined by

$$Ex = lxr, \qquad Fx = (1-l)x(1-r) \qquad Gx = lx(1-r)+(1-l)xr\,.$$

If g belongs to the normed dual M' of M then by an abuse of notation, Eg denotes $g \circ E$, and similarly for F and G.

The second dual M'' has a natural structure of weakly closed J*-algebra [5, Lemma 2.1] and each $f \in M'$ has an enveloping polar decomposition

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as follows [5, Theorem 1]. There is a unique partial isometry $v \in M''$, denoted by $v(f)$, such that, with $r = v^*v$, we have

(1.1) $N_v \equiv v^*M''r$ is an ultra-weakly closed Jordan *-subalgebra of $\mathscr{L}(H)$, with unit r;

(1.2) ϕ_v, defined by $\phi_v(a) = f(va)$, for $a \in N_v$, is a faithful ultra-weakly continuous positive functional on N_v;

(1.3) $f(x) = \phi_v(v^*xr)$, for $x \in M''$.

In particular, $\|f\| = f(v)$. Note that (1.1) is valid for any partial isometry $v \in M''$ [5, Lemma 2.3]. For $f \in M'$ we shall use the abbreviations $E(f) = E(v(f))$, $F(f) = F(v(f))$ and $G(f) = G(v(f))$.

A J*-*homomorphism* is a linear map T satisfying $T(xx^*x) = Tx(Tx)^*Tx$. Such a map satisfies $\|T\| \leqslant 1$ and

$$T(xy^*z+zy^*x) = Tx(Ty)^*Tz+Tz(Ty)^*Tx$$

(cf. [6, pp. 17–19]).

We shall use the following additional notation: T is the unit circle in $\mathbb{C}$; V is a complex Banach space; $K = (V')_1$ is the unit ball in V' with the ω^*-topology; $S = S(V) = \operatorname{ext} K$ is the set of extreme points of K. A function $f \in C_{\mathbb{C}}(K)$ is said to be T-*homogeneous* if $f(\alpha k) = \alpha f(k)$ for all $\alpha \in T$, $k \in K$. The class of T-homogeneous functions in $C_{\mathbb{C}}(K)$ is denoted by $C_{\text{hom}}(K)$. Similar remarks hold for functions on S. If $f \in C_{\mathbb{C}}(K)$, then the function

$$[\hom_T f](k) = \int_T \alpha^{-1} f(\alpha k) d\alpha, \qquad k \in K,$$

where $d\alpha$ is the unit Haar measure on T, is continuous and T-homogeneous. It follows that $\hom_T$ is a norm-decreasing projection of $C_{\mathbb{C}}(K)$ onto $C_{\text{hom}}(K)$.

2. *Representation of commutative* J*-*algebras*

For any Banach space V the map $V \ni x \to \hat{x} \in C_{\text{hom}}(K)$, defined by $\hat{x}(f) = f(x)$, for $f \in K$, is a linear isometry of V onto a closed subspace $\hat{V}$ of $C_{\text{hom}}(K)$. Let $\Psi : V \to C_{\text{hom}}(S)$ be defined by $\Psi(x) = \hat{x}|S$. Then Ψ is a linear isometry of V onto a closed subspace of $C_{\text{hom}}(S)$. This follows from the Krein–Milman theorem.

Suppose now that V is the underlying Banach space of a commutative J*-algebra (to be defined). It follows from Proposition 2.3 below that Ψ is a J*-homomorphism of V into $C_{\text{hom}}(S)$, that is $\Psi(aa^*a) = \Psi(a)\overline{\Psi(a)}\Psi(a)$, and (Corollary 2.4) that $S \cup \{0\}$ is ω^*-closed. We show in Proposition 2.6 that V is a Lindenstrauss space, that is, the dual V^* is isometric to some L^1 space. Thus a commutative J*-algebra satisfies (1) of [10, Theorem 9]. Application of (2) of this theorem will show that Ψ maps V onto $C_{\text{hom}}(S)$.

Our first task is to define a commutative J*-algebra. We are guided by the following proposition.

PROPOSITION 2.1. *Let A be a C*-algebra. The following are equivalent:*

(1) *A is commutative;*

(2) *$G(v) = 0$ for all partial isometries $v \in A''$;*

(3) *$G(f) = 0$ for all $f \in A'$.*

Proof. (1) ⇒ (2) If A is commutative, then so is A''. Thus if $v \in A''$, then $l = vv^* = v^*v = r$, and for $x \in A''$ we have $G(v)x = lx(1-r)+(1-l)xr = 0$.

(2) ⇒ (3) This is obvious.

(3) ⇒ (1) Let e be an arbitrary projection in A''. Then $e = \sup\{e_\phi : \phi \in A'_+, e_\phi \leqslant e\}$ where e_ϕ denotes the support in A'' of the positive functional ϕ. By assumption, for each $x \in A''$,

$$0 = G(\phi)x = e_\phi x(1-e_\phi)+(1-e_\phi)xe_\phi .$$

Therefore $xe_\phi = e_\phi xe_\phi = e_\phi x$, and so $xe = ex$ for arbitrary $x \in A''$.

DEFINITION. A J*-algebra M is said to be *commutative* if $G(f) = 0$ for all $f \in M'$ (cf. Example 4.1).

Throughout the rest of this section, M will denote a commutative J*-algebra.

LEMMA 2.2. *For each $v \in M''$, N_v is a commutative von Neumann algebra.*

Proof. By (1.1), it suffices to prove that $ae = ea$ for an arbitrary projection e in N_v and an arbitrary element a in N_v. By the approximation argument in the proof of Proposition 2.1 we may assume that e is the support of some $\phi \in (N_v)_*^+$. Also, as in the proof of Proposition 2.1 it suffices to show that $G(\phi) = 0$. This follows from [5, Remark 3.2].

PROPOSITION 2.3. *For $f \in M'$, the following are equivalent:*

(1) *f is an extreme point of the unit ball of M';*

(2) *f is a non-zero J*-homomorphism of M into $\mathbb{C}$;*

(3) *$\hat{f}$ (the canonical image of f in M''') is a non-zero J*-homomorphism of M'' into $\mathbb{C}$.*

Proof. (1) ⇒ (3) By [5, Proposition 3.7],

$$E(f)x = \langle x,f\rangle v \qquad \text{for } x \in M'' . \tag{2.1}$$

Since M is commutative, $G(f) = 0$. Thus for each $x \in M''$, with $E = E(f)$ and $F = F(f)$, we can write $x = Ex+Fx$ and therefore $E(xx^*x) = Ex(Ex)^*Ex$. Using (2.1) we have

$$\langle xx^*x,f\rangle v = E(xx^*x) = \langle x,f\rangle v(\langle x,f\rangle v)^*(\langle x,f\rangle v) = \langle x,f\rangle^2\overline{\langle x,f\rangle} v$$

and therefore $\hat{f}(xx^*x) = \hat{f}(x)^2\overline{\hat{f}(x)}$.

(3) $\Rightarrow$ (2) This is trivial.
(2) $\Rightarrow$ (3) As noted in Section 1, f satisfies

$$f(xy^*z+zy^*x) = 2f(x)\overline{f(y)}f(z) \qquad \text{for } x, y, z \in M. \tag{2.2}$$

Now multiplication is separately ultraweakly continuous on M'' and the involution is ultraweakly continuous. Therefore (2.2) implies that $\hat{f}$ is a J*-homomorphism of M'' into $\mathbb{C}$.

(2) $\Rightarrow$ (1) We first show that $\|f\| = 1$. As noted above, $\|f\| \leqslant 1$. By (3), $\hat{f}$ is a J*-homomorphism of M'' so that, with $v = v(f)$, $|\hat{f}(v)| = |\hat{f}(vv^*v)| = |\hat{f}(v)|^3$, that is $\|f\| = \|f\|^3$; therefore $\|f\| = 1$ since $f \neq 0$.

Now write $f = \frac{1}{2}(g+h)$, with $g, h \in M'_1$. Then

$$1 = \|f\| \leqslant \tfrac{1}{2}(\|E(f)g\|+\|E(f)h\|) \leqslant \tfrac{1}{2}(\|g\|+\|h\|) = 1$$

and so by [5, Lemma 3.1], $g = E(f)g$ and $h = E(f)h$. Since $E(f)M' \simeq (N_v)_*$ (by [5, Remark 3.2]) we see that $f \in \operatorname{ext} M'_1$ if and only if $f \in \operatorname{ext} E(f)M'_1$, and this is the case if and only if $\phi_v \in \operatorname{ext}(N_v)_{*1}$. It remains to prove that $\phi_v \in \operatorname{ext}(N_v)_{*1}$.

For $x \in M''$ and $a = v^*xr \in N_v$, we have

$$\phi_v(a^2) = \phi_v(v^*xv^*xr) = \hat{f}(xv^*x) = \hat{f}(x)^2\overline{\hat{f}(v)} = \hat{f}(x)^2 = \phi_v(a)^2.$$

Thus ϕ_v is a multiplicative linear functional on the abelian von Neumann algebra N_v and therefore ϕ_v is a normal pure state of N_v. It follows that $\phi_v \in \operatorname{ext}(N_v)_{*1}$.

COROLLARY 2.4. $(\operatorname{ext} M'_1) \cup \{0\}$ *is weak*-compact.*

Proof. The weak*-limit of J*-homomorphisms of M is a J*-homomorphism of M.

LEMMA 2.5. *Let $f, g \in \operatorname{ext} M'_1$. Then f and g are either orthogonal (that is $f = F(g)f$) or linearly dependent.*

Proof. For any $f, g \in M'$, we have $f = E(g)f+F(g)f$ (since $G(g) = 0$) and $\|f\| = \|E(g)f\|+\|F(g)f\|$, by [9, Lemma 1.1]. Suppose that $E(g)f \neq 0$ and $F(g)f \neq 0$. Then $f = \alpha(\alpha^{-1}E(g)f)+(1-\alpha)((1-\alpha)^{-1}F(g)f)$ where $\alpha = \|E(g)f\|$ and since f is extreme, it follows that $f = (1-\alpha)^{-1}F(g)f$ and $f = \alpha^{-1}E(g)f$. Therefore $f = \alpha^{-1}E(g)f = \alpha^{-1}E(g)(1-\alpha)^{-1}F(g)f = 0$, a contradiction. Thus either $f = F(g)f$ or $f = E(g)f = \langle f, v(g)\rangle g$ since g is extreme (by [5, Proposition 3.7]).

PROPOSITION 2.6. *M is a Lindenstrauss space.*

Proof. Let $S = \operatorname{ext}(M')_1$. Choose, by the axiom of choice, a set $S_0 \subset S$ such that every $f \in S$ is a scalar multiple of exactly one element of S_0. By Lemma 2.5, the elements of S_0 are orthogonal, so that $v = \sum_{f \in S_0} v(f)$ exists in M'' and is a partial isometry with the property that $E(v)f = f$ for every $f \in S$. It follows, by the Krein–Milman theorem, that $E = E(v)$ is an isometry of M onto $E(M) \subset M''$, and therefore that $M' \simeq E(M)' \simeq E(M') \simeq (N_v)_* \simeq L^1$. Here $\simeq$ denotes isometric isomorphism and the last step follows from Lemma 2.2.

The following remark was pointed out by the referee.

REMARK 2.7. The proof of Proposition 2.6 shows that M'' is isometric to N_v, which is a commutative von Neumann algebra. By [6, Theorem 4] M'' and N_v are J*-isomorphic (cf. Proposition 3.1).

THEOREM 1. *Let M be a commutative* J*-*algebra, and let $S = \operatorname{ext} M'_1$. Then the map $\Psi : M \to C_{\text{hom}}(S)$ defined by $\Psi(x) = \hat{x}|S$, for $x \in M$, is a* J*-*isomorphism onto.*

Proof. By Proposition 2.3, Ψ is a J*-isomorphism of M into $C_{\text{hom}}(S)$. By Corollary 2.4 and Proposition 2.6, M satisfies (1) of [10, Theorem 9]. Now let $\phi \in C_{\text{hom}}(S)$. Since $S \cup \{0\}$ is w^*-closed, ϕ has a continuous extension $\tilde{\phi}$ to $C(K)$. By (2) of [10, Theorem 9] there is an $x \in M$ such that $\Psi(x) = \hat{x}|S = \text{hom}_T \tilde{\phi}|S = \phi$.

EXAMPLE 2.8. Let X be a T_σ-space, that is a compact Hausdorff space together with a continuous map $\sigma : T \times X \to X$ satisfying $\sigma(\alpha, \sigma(\beta, x)) = \sigma(\alpha\beta, x)$, and $\sigma(1, x) = x$, for $\alpha, \beta \in T$ and $x \in X$. Let

$$C_\sigma(X) = \{\phi \in C(X) : \phi(\sigma(\alpha, x)) = \alpha\phi(x), \text{ for all } (\alpha, x) \in T \times X\}.$$

A Banach space V isometric to a space of the form $C_\sigma(X)$ is called a *C_σ-space* (cf. Olsen [10]). Since $C_\sigma(X) \subset C(X)$, it follows that $C_\sigma(X)$ has a faithful representation as a commutative J*-algebra. Let $M = C_\sigma(X)$ and $S = \operatorname{ext} M'_1$. For $x \in X$, let $\rho(x) \in M'_1$ be defined by $\langle \rho(x), \phi \rangle = \phi(x)$. Then $\rho : X \to M'_1$ is continuous and obviously $\rho(x)$ is a J*-homomorphism of M into $\mathbb{C}$. Therefore, by Proposition 2.3, either $\rho(x) = 0$ or $\rho(x) \in S$. Thus $\rho : X \to S \cup \{0\}$ and ρ is one-to-one on $\rho^{-1}(S)$. Let $\tau : S \to K$ be the one-to-one map defined by $\rho(\tau(f)) = f$ for $f \in S$. Then the map $\phi \to \hat{\phi}$ of $C_\sigma(X)$ onto $C_{\text{hom}}(S)$ is also given by $\phi \to \phi \circ \tau$.

3. *Applications*

Our first application of Theorem 1 is a result of Banach–Stone type. For arbitrary J*-algebras, the implication (2) $\Rightarrow$ (1) is a deep result of Harris [6, Theorem 4].

PROPOSITION 3.1. *Let M and N be commutative* J*-*algebras and let $T : M \to N$ be a linear map. The following are equivalent:*

(1) *T is a* J*-*isomorphism of M onto N;*

(2) *T is an isometry of M onto N;*

(3) *there is a homeomorphism $\sigma : S(N) \to S(M)$ with $\sigma(\alpha f) = \alpha\sigma(f)$ for $(\alpha, f) \in T \times S(N)$ such that $(Ta)\hat{} = \hat{a} \circ \sigma$, for $a \in M$.*

Proof. (1) $\Rightarrow$ (2) Any J*-homomorphism is norm decreasing. Since T^{-1} is also a J*-homomorphism, (2) follows.

(2) $\Rightarrow$ (3) The map $\alpha = T'|S(N)$ has the stated properties.

(3) $\Rightarrow$ (1)
$$(T(aa^*a))\hat{} = (aa^*a)\hat{} \circ \sigma = (\hat{a}\bar{\hat{a}}\hat{a}) \circ \sigma = (Ta)\hat{}\,\overline{(Ta)\hat{}}\,(Ta)\hat{} = (Ta(Ta)^*Ta)\hat{}.$$

In [4, Theorem 5], the authors showed that the range of a contractive projection on a commutative C*-algebra is a C_σ-space. Using that result and Theorem 1 we can prove the following.

PROPOSITION 3.2. *If P is a contractive projection on a commutative* J*-*algebra, then $P(M)$ is isometric to a commutative* J*-*algebra.*

Proof. Identify M with $C_{\mathrm{hom}}(S)$. Then $P(M) = P(\mathrm{hom}_T(C_0(S)))$. Since $P \circ \mathrm{hom}_T$ is a contractive projection on $C_0(S)$, it follows that $P(M)$ is a complex C_σ-space and is therefore a commutative J*-algebra.

CONJECTURE. On an arbitrary J*-algebra M, each bicontractive projection P has the form $Px = \frac{1}{2}(x+\theta x)$, for $x \in M$, where θ is a J*-automorphism with $\theta^2 = \mathrm{id}$.

We prove this conjecture below in the case of a commutative J*-algebra. Other cases when this conjecture is true are given in the introduction of [5].

PROPOSITION 3.3. *Let M be a commutative* J*-*algebra and let P be a bicontractive projection on M. Then there is a homeomorphism σ of $S(M)$, with $\sigma^2 = \mathrm{id}$ and $\sigma(\alpha f) = \alpha\sigma(f)$, for $(\alpha, f) \in T \times S$, such that*

$$(Px)\hat{\ } = \tfrac{1}{2}(\hat{x} + \hat{x} \circ \sigma), \qquad x \in M.$$

Proof. Since M is a Lindenstrauss space by Proposition 2.6, the result follows from the main result of Bernau and Lacey [2], and Proposition 3.1.

The following is a Stone–Weierstrass theorem for commutative J*-algebras.

PROPOSITION 3.4. *Let B be a commutative* J*-*algebra and let A be a commutative* J*-*subalgebra of B. Let $X = \mathrm{ext}\, B_1' = S(B)$. Suppose that A separates the points of X, that is $f, g \in X$, $f \neq g$, implies that there is an element $a \in A$ such that $f(a) \neq g(a)$. Suppose also that for each $f \in X$ there is an element $a \in A$ with $f(a) \neq 0$. Then $A = B$.*

Proof. Let $Y = S(A)$. The two assumptions and Proposition 2.3 imply that the map $X \ni f \to f|A \in Y$ is a continuous bijection. Since $X \cup \{0\}$ is compact, this map is a homeomorphism and therefore $C_{\mathrm{hom}}(X) \simeq C_{\mathrm{hom}}(Y)$. By Theorem 1, $A \simeq C_{\mathrm{hom}}(Y)$ and $B \simeq C_{\mathrm{hom}}(X)$. Since the diagram

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \simeq\downarrow & & \downarrow\simeq \\ C_{\mathrm{hom}}(Y) & \xrightarrow[\simeq]{} & C_{\mathrm{hom}}(X) \end{array}$$

commutes, we must have $A = B$.

We shall now use the Stone–Weierstrass result to develop ideal theory for commutative J*-algebras. An *ideal* in any J*-algebra is a linear subspace I which contains ab^*c+cb^*a whenever it contains at least one of a, b, c.

For an ideal I in a commutative J*-algebra M, let $S_I = \{f \in S(M) : f(a) = 0$ for all $a \in I\}$. Then S_I is a closed homogeneous subset of $S \equiv S(M)$.

LEMMA 3.5. *Let I be a closed ideal in a commutative* J*-*algebra M, and let $r : M' \to I'$ be the restriction map. Then*

(1) *r is a homeomorphism of $S - S_I$ onto $S(I)$;*

(2) *the map $c \to \hat{c} \circ r$ is a* J*-*isomorphism of I onto $C_{\text{hom}}(S - S_I)$.*

Proof. If $f \in S - S_I$ then $f|I \neq 0$ so that $f \in S(I)$ by Proposition 2.3. Thus r maps $S - S_I$ into $S(I)$. Suppose that $f, g \in S - S_I$, and $f \neq g$. We shall prove that $f|I \neq g|I$. Since $f \in S - S_I$ there is $a_1 \in I$ with $f(a_1) = 1$. Suppose that $f(a) = g(a)$ for all $a \in I$. Then $1 = f(a_1) = g(a_1)$. Let $\phi \in C_0(S)$ satisfy $\phi(\alpha f) = 0$, $\phi(\alpha g) = \alpha$ for $\alpha \in T$ and choose $b \in M$ such that $\hat{b} = \text{hom}_T \phi$. Then $c \equiv a_1 b^* b + b b^* a_1 \in I$,

$$f(c) = 2f(a_1)|f(b)|^2 = 0 \qquad \text{and} \qquad g(c) = 2g(a_1)|g(b)|^2 = 2,$$

a contradiction. Thus $f|I \neq g|I$ and r is injective on $S - S_I$. Now let $h \in S(I)$ and let $\Gamma = \{k \in M_1' : k|I = h\}$. Then Γ is weak* compact and non-empty, and any extreme point $\tilde{h}$ of Γ lies in $S(M)$. Since necessarily $\tilde{h} \in S - S_I$, and since r is obviously continuous, we have proved that r is a continuous bijection of $S - S_I$ onto $S(I)$. Now observe that $\{\hat{c} \circ r : c \in I\}$ is a J*-subalgebra of $C_{\text{hom}}(S - S_I)$ which by the preceding part of this proof satisfies the two conditions of Proposition 3.4. Therefore (2) is proved and (1) follows from (2) and Proposition 3.1.

For any subset R of $S = S(M)$ let $I(R) = \{a \in M : \hat{a}(R) = 0\}$. Then $I(R)$ is a closed ideal in M and $R \subset S_{I(R)}$.

REMARK 3.6. If R is closed and homogeneous, we have $R = S_{I(R)}$. Indeed, if $g \in S_{I(R)}$ and $g \notin R$ then there is $\phi \in C_{\text{hom}}(S)$ with $\phi(R) = 0$, $\phi(g) = 1$. With $c \in M$ such that $\hat{c} = \phi$, we have $c \in I(R)$ but $g(c) = \hat{c}(g) = \phi(g) = 1$, contradicting the fact that $g \in S_{I(R)}$.

PROPOSITION 3.7. *Let M be a commutative* J*-*algebra, let $\mathscr{H}$ be the collection of all closed homogeneous subsets of S, and let $\mathscr{I}$ be the collection of all closed ideals in M. Then $R \to I(R)$ is a bijection of $\mathscr{H}$ onto $\mathscr{I}$, with inverse $I \to S_I$.*

Proof. We shall show that for each ideal $I \in \mathscr{I}$ we have $I = I(S_I)$. This will show that the map $R \to I(R)$ maps $\mathscr{H}$ onto $\mathscr{I}$. Since $S_{I(R)} = R$, it will follow that $R \to I(R)$ is a bijection.

We thus prove that $I = I(S_I)$. Since $\hat{a}(S_I) = 0$ for all $a \in I$, we have $I \subseteq I(S_I)$. For notational convenience let $J = I(S_I)$. Then $S_J = S_{I(S_I)} = S_I$ as remarked above. We have, by Lemma 3.5,

$$\begin{array}{ccc} I & \subseteq & J \\ \simeq \big\downarrow & & \big\downarrow \simeq \\ C_{\text{hom}}(S - S_I) & = & C_{\text{hom}}(S - S_J) \end{array}$$

and the diagram commutes. Therefore $I = J$.

REMARK 3.8. A *quadratic ideal* in a J*-algebra M is a linear subspace J which contains ab^*a whenever $a \in J$ and $b \in M$ (equivalently $ab^*c+cb^*a \in J$ whenever $a, c \in J$ and $b \in M$). Using the preceding techniques it is easy to show the following.

(a) In a commutative J*-algebra, the closed ideals coincide with the closed quadratic ideals.

(b) In a weakly closed commutative J*-algebra M, every weakly closed ideal I is of the form $I = E(v)(M)$ for some partial isometry $v \in M$ (take v to be an extreme point of the unit ball of I and use [6, Theorem 11]).

REMARK 3.9. We have the following simple consequences of Proposition 3.7 concerning maximal ideals.

(1) Since $I(R_1) \subseteq I(R_2)$ if and only if $R_1 \supseteq R_2$, there is a one-one correspondence between maximal ideals in M and $S/\sim$, where $f \sim g$ means that f and g are linearly dependent.

(2) Every closed ideal is the intersection of the maximal ideals which contain it.

(3) If I is a maximal ideal then $M/I \simeq \mathbb{C}$.

Finally we answer partially another question of Harris [7].

PROPOSITION 3.10. *Let M be a commutative J*-algebra and let I be a closed ideal. Then M/I is a J*-algebra, that is M/I is J*-isomorphic to $C_{\mathrm{hom}}(R)$ where $R = S_I$.*

Proof. The map $M/I \ni a+I \to \hat{a}|R \in C_{\mathrm{hom}}(R)$ is easily seen to be a J*-homomorphism onto. To show that $\|a+I\| = \|\hat{a}|R\|$, let $\hat{c} \in C_{\mathrm{hom}}(S)$ be a Tietze extension of $\hat{a}|R$. Then $c \in a+I$ and $\|a+I\| \leqslant \|c\| \leqslant \|\hat{a}|R\|$. The opposite inequality is obvious.

4. *Representation of associative Jordan triple systems*

Let U be a *complex* vector space endowed with a triple product $\{\cdot,\cdot,\cdot\}: U \times U \times U \to U$ which is linear in the two outer variables and conjugate linear in the middle variable. Then $(U, \{\ \})$ is called a *Jordan triple system* if the following two identities hold:

$$\{xy\{uvz\}\} - \{uv\{xyz\}\} = \{\{xyu\}vz\} - \{u\{yxv\}z\}; \tag{4.1}$$

$$\{xyz\} = \{zyx\}. \tag{4.2}$$

Kaup, in [8], defines an *hermitian Jordan triple system* to be a complex Banach space $(U, \|\ \|)$ endowed with a Jordan triple system structure $\{\cdot,\cdot,\cdot\}: U \times U \times U \to U$ in such a way that $\{x, x, \cdot\}$ is an hermitian operator on U for each x in U. He shows that this category is equivalent to the category of simply connected, symmetric, complex Banach manifolds with base point. More recently he has given a classification of 'atomic' hermitian Jordan triple systems of finite rank [9].

The pair $(U, \{\ \})$ is said to be an *associative triple system* if (instead of (4.1) and (4.2))

$$\{xy\{zuv\}\} = \{x\{uzy\}v\} = \{\{xyz\}uv\}. \tag{4.3}$$

A real subsystem V of a Jordan triple system U is called *flat* if

$$\{xyz\} = \{yxz\}. \tag{4.4}$$

If V is flat then $V^{\mathbb{C}} = V+iV$ is an associative subsystem of U.

Kaup [8] and Vigué [11] prove that if V is a flat subsystem of an hermitian Jordan triple system U whose corresponding complex Banach manifold is isomorphic to a *bounded* domain, then $V^{\mathbb{C}} = V+iV$ is J*-isomorphic to $C_0(S)$, with S locally compact. In this setting a J*-homomorphism is a linear map ψ satisfying $\psi(\{xyz\}) = \psi(x)\overline{\psi(y)}\psi(z)$.

In Theorem 2 below we prove that any associative Jordan triple system with a norm which satisfies some natural conditions is J*-isomorphic to the space of all homogeneous, continuous functions on a locally compact space.

Theorem 2 differs from the results of Kaup and Vigué in the following way. Although the norm conditions (4.5), (4.6) imply that U is an hermitian Jordan triple system, we do not make any assumption on the Banach manifold corresponding to U nor do we assume the existence of a flat generating subspace.

EXAMPLE 4.1. Let M be a J*-algebra. Then M becomes a Jordan triple system in the triple product $\{xyz\} = \frac{1}{2}(xy^*z+zy^*x)$. It follows from Theorem 1 that M is commutative as a J*-algebra if and only if $(M, \{\ \})$ is an associative Jordan triple system. Indeed if M is commutative then Theorem 1 implies that $(M, \{\ \})$ is associative. Conversely, if M is associative then so is M'' by the separate ultraweak continuity of multiplication and continuity of the involution on M''. Therefore, if v is any partial isometry in M'' and $x \in M''$, then

$$\begin{aligned} G(v)x &= vv^*x(1-v^*v)+(1-vv^*)xv^*v \\ &= vv^*x - vv^*xv^*v + xv^*v - vv^*xv^*v \\ &= 2\{vvx\} - 2\{v\{vxv\}v\} = 0 \end{aligned}$$

since

$$\{v\{vxv\}v\} = \{vv\{xvv\}\} = \{vv\{vvx\}\} = \{v\{vvv\}x\} = \{vvx\}.$$

EXAMPLE 4.2. Let U be a linear subspace of $C_0(S)$, with S locally compact, and let $c : S \to \mathbb{R}$ be bounded and continuous. Suppose that U is closed under the triple product $\{xyz\}_c = cx\bar{y}z$, for $x, y, z \in U$. Then $(U, \{\ \}_c)$ is an associative Jordan triple system.

Moreover, if $|c| = 1$, then $(U, \{\ \}_c)$ satisfies

$$\|\{xyz\}\| \leqslant \|x\|\,\|y\|\,\|z\|, \tag{4.5}$$

$$\|\{xxx\}\| = \|x\|^3, \tag{4.6}$$

where $\|\cdot\|$ denotes the sup norm on $C_0(S)$.

Note that if $S_+ = \{c = 1\}$ and $S_- = \{c = -1\}$ then $S = S_+ \cup S_-$ and

$$\{xyz\}_c = \begin{cases} x\bar{y}z & \text{on } S_+ , \\ -x\bar{y}z & \text{on } S_- . \end{cases} \tag{4.7}$$

The following theorem, due to H. Zettl [12], gives an abstract characterization of a ternary algebra, or in Zettl's terminology a norm closed ternary ring of operators (TRO).

A *ternary* C*-*ring* $(\mathscr{X}, \{\cdot, \cdot, \cdot\}, \|\cdot\|)$ consists of a complex Banach space and an associative triple system satisfying (4.5) and (4.6). A ternary C*-ring $(\mathscr{X}, \{\cdot, \cdot, \cdot\}, \|\cdot\|)$ is *isomorphic* (respectively *anti-isomorphic*) to a TRO R if there is a linear onto isometry $U : \mathscr{X} \to R$ satisfying $U(\{x, y, z\}) = U(x)U(y)^* U(z)$, for $x, y, z \in \mathscr{X}$ (respectively $U(\{x, y, z\}) = -U(x)U(y)^* U(z)$).

THEOREM A (Zettl). *Every ternary* C*-*ring* $\mathscr{X}$ *is the direct sum of two ternary* C*-*subrings* $\mathscr{X}_+$ *and* $\mathscr{X}_-$ *in such a way that* $\mathscr{X}_+$ *(respectively* $\mathscr{X}_-$*) is isomorphic (respectively anti-isomorphic) to a* TRO. *The subspaces* $\mathscr{X}_+$ *and* $\mathscr{X}_-$ *are orthogonal in the sense that for all* $z \in \mathscr{X}$,

$$\{x, y, z\} = 0 = \{y, x, z\}, \qquad \text{for } x \in \mathscr{X}_+, y \in \mathscr{X}_- . \tag{4.8}$$

Moreover, let P *be the projection of* $\mathscr{X}$ *onto* $\mathscr{X}_+$. *Then the map* $T = 2P - I$ *satisfies* $T^2 = \mathrm{id}$ *and*

$$T\{x, y, z\} = \{Tx, Ty, Tz\}, \qquad x, y, z \in \mathscr{X} . \tag{4.9}$$

LEMMA 4.3. *Let* $\mathscr{X}$ *be a ternary* C*-*ring with decomposition* $\mathscr{X} = \mathscr{X}_+ \oplus \mathscr{X}_-$ *as in Theorem* A, *and let* P *be the projection of* $\mathscr{X}$ *onto* $\mathscr{X}_+$. *Then*

(1) $\|x\| = \max\{\|Px\|, \|(1-P)x\|\}$, *for* $x \in \mathscr{X}$;

(2) $\|f\| = \|P'f\| + \|(1-P')f\|$, *for* $f \in \mathscr{X}'$;

(3) $\operatorname{ext} \mathscr{X}'_1 = \operatorname{ext}(\mathscr{X}'_+)_1 \cup \operatorname{ext}(\mathscr{X}'_-)_1$

in the sense that, with $f_+ = P'f$, $f_- = (1-P')f$, *we have* $f \in \operatorname{ext} \mathscr{X}'_1$ *if and only if either* $f_+ = 0$ *and* $f_- \in \operatorname{ext}(\mathscr{X}'_-)_1$, *or* $f_+ \in \operatorname{ext}(\mathscr{X}'_+)_1$ *and* $f_- = 0$.

Proof. It is elementary that (2) follows from (1) and (3) follows from (2). To prove (1) let $x \in \mathscr{X}$ and write $x = x_+ + x_-$ with $x_+ \in \mathscr{X}_+$, $x_- \in \mathscr{X}_-$. For notational convenience write $y^3 = \{y, y, y\}$. Then by (4.8), the linearity properties of $\{\ \}$, and (4.6) we have $x^3 = x_+^3 + x_-^3$, and $\|x\|^3 = \|x^3\| = \|x_+^3 + x_-^3\| \leqslant \|x_+^3\| + \|x_-^3\|$, that is $\|x\| \leqslant (\|x_+\|^3 + \|x_-\|^3)^{1/3}$. Iteration of this argument yields that $\|x\| \leqslant (\|x_+\|^{3^n} + \|x_-\|^{3^n})^{3^{-n}}$ for $n = 1, 2, \dots$, and this implies that $\|x\| \leqslant \max\{\|x_+\|, \|x_-\|\}$. But (4.9) implies that $\|T\| \leqslant 1$ where $T = 2P - I$. Therefore $\|x_+\| = \|Px\| = \|\frac{1}{2}(x + Tx)\| \leqslant \|x\|$, and $\|x_-\| = \|(I-P)x\| = \|\frac{1}{2}(x - Tx)\| \leqslant \|x\|$.

THEOREM 2. *Let* $(U, \{\ \}, \|\ \|)$ *be an associative Jordan triple system with a complete norm satisfying* (4.5) *and* (4.6). *Let* $S = \text{ext}\, U'_1$. *Then*

(1) *the map* $\Psi : U \to C_{\text{hom}}(S)$ *defined by* $\Psi(x) = \hat{x}|S$ *is an isometry onto;*

(2) $S = S_+ \cup S_-$, *where* S_+ *(respectively* S_-*) is the set of all non-zero* J**-homomorphisms (respectively* J**-anti-homomorphisms (this means* $f(xx^*x) = -f(x)\overline{f(x)}f(x)$*)) of* U *into* $\mathbb{C}$, *so that* $S \cup \{0\}$ *is* w**-compact;*

(3) *when we equip* $C_{\text{hom}}(S)$ *with the triple product* (4.7), Ψ *is a* J**-isomorphism.*

Proof. Since $(U, \{\ \}, \|\ \|)$ is a ternary C*-ring we have, by Zettl's theorem, $U = U_+ \oplus U_-$, where U_+, U_- are ternary C*-subrings of U and there exist an isomorphism $\pi_+ : U_+ \to R_+$ and an anti-isomorphism $\pi_- : U_- \to R_-$ onto ternary rings of operators R_+, R_-. By Example 4.1, R_+ and R_- are commutative J*-algebras. By Theorem 1 there exist onto J*-isomorphisms $\Psi_+ : R_+ \to C_{\text{hom}}(\hat{S}_+)$ and $\Psi_- : R_- \to C_{\text{hom}}(\hat{S}_-)$, where $\hat{S}_\pm = \text{ext}\,(R'_\pm)_1$. By (3) of Lemma 4.3 we can identify S_+ with $\hat{S}_+$ and S_- with $\hat{S}_-$ and therefore $C_{\text{hom}}(S_+ \cup S_-)$ with $C_{\text{hom}}(\hat{S}_+ \cup \hat{S}_-) = C_{\text{hom}}(\hat{S}_+) \oplus C_{\text{hom}}(\hat{S}_-)$. It follows that the map $\Psi : U \to C_{\text{hom}}(S_+ \cup S_-)$ given by $\Psi(x) = (\Psi_+ \circ \pi_+(x)) \oplus (\Psi_- \circ \pi_-(x))$ is a J*-isomorphism of $(U, \{\ \}, \|\ \|)$ onto $(C_{\text{hom}}(S_+ \cup S_-), \{\ \}_c, \|\ \|_\infty)$.

REMARK 4.4. All of the results in Section 3 of this paper can be extended to an associative Jordan triple system with a norm satisfying (4.5) and (4.6). The proofs are immediate..

Note added in proof, March 21, 1983. The authors have recently given an affirmative answer to the conjecture in Section 3. Also they have extended Proposition 3.2 to arbitrary J*-algebras.

References

1. J. ARAZY and Y. FRIEDMAN, *Contractive projections in* C_1 *and* C_∞, Memoir 200 (American Mathematical Society, Providence, 1978).
2. S. J. BERNAU and H. E. LACEY, 'Bicontractive projections and re-ordering of L_p spaces', *Pacific J. Math.*, 69 (1977), 291–302.
3. E. EFFROS and E. STØRMER, 'Positive projections and Jordan structure in operator algebras', *Math. Scand.*, 45 (1979), 127–138.
4. Y. FRIEDMAN and B. RUSSO, 'Contractive projections on $C_0(K)$', *Trans. Amer. Math. Soc.*, 273 (1982), 57–73.
5. Y. FRIEDMAN and B. RUSSO, 'Contractive projections on operator triple systems', *Math. Scand.*, to appear.
6. L. HARRIS, 'Bounded symmetric homogeneous domains in infinite dimensional spaces', *Proceedings on Infinite Dimensional Holomorphy*, Lecture Notes in Mathematics 364 (Springer, Berlin, 1973), pp. 13–40.
7. L. HARRIS, 'A generalization of C*-algebras', *Proc. London Math. Soc.*, 42 (1981), 331–361.
8. W. KAUP, 'Algebraic characterization of symmetric complex Banach Manifolds', *Math. Ann.*, 228 (1977), 39–64.
9. W. KAUP, 'Über die Klassifikation der symmetrischen hermiteschen Mannigfaltigkeiten unendlicher Dimension, I', *Math. Ann.*, 257 (1981), 463–486.
10. G. OLSEN, 'On the classification of complex Lindenstrauss spaces', *Math. Scand.*, 35 (1974), 237–258.
11. J.-P. VIGUÉ, 'Les domaines bornés symetriques d'un espace de Banach complexe et les systèmes triples de Jordan', *Math. Ann.*, 229 (1977), 223–231.

12. H. ZETTL, 'A characterization of ternary rings of operators', preprint, Universität Erlangen-Nürnberg, 1982.

Department of Mathematics,
University of California,
Irvine,
California 92717,
U.S.A.