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Author
Piltner, Reinhard

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BY

R. PILTNER

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DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA
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The derivation of a thick and thin plate formulation without ad hoc assumptions

REINHARD PILTNER

Department of Civil Engineering, University of California at Berkeley, Berkeley, CA 94720, U.S.A.

Abstract. For the plate formulation considered in this paper appropriate three-dimensional elasticity solution representations for isotropic materials are constructed. No a priori assumptions for stress or displacement distributions over the thickness of the plate are made. The strategy used in the derivation is to separate functions of the thickness variable z from functions of the coordinates x and y lying in the midplane of the plate. Real and complex 3-dimensional elasticity solution representations are used to obtain three types of functions of the coordinates x,y and the according differential equations. The separation of the functions of the thickness coordinate can be done by considering separately homogeneous and nonhomogeneous boundary conditions on the upper and lower faces of the plate.

1. Introduction

The aim of plate theories is to avoid the difficult task of solving the 3-dimensional elasticity equations under given surface boundary conditions. The goal of plate theories is first to obtain one or more differential equations for functions depending only on the two space coordinates x,y lying in the midplane of the plate and second to solve these differential equations with subject to boundary conditions which are altered (for example to conditions for stress resultants) according to the considered theory. In order to reach this goal assumptions on stress, strain and displacement distributions over the thickness of the plate are made. Of the numerous contributions to the subject of plate theories and solutions especially the work of Kirchhoff [1], Reissner [2,3], Mindlin [4], Hencky[5] and Lo, Christiansen, Wu [6] ought to be mentioned.

A different concept of deriving a plate formulation is to separate in a 3-dimensional analysis functions of the thickness coordinate z from functions of the remaining coordinates x and y. The evaluation of the functions of the thickness coordinate is governed by the differential equations of the space problem and by the boundary conditions on the upper and lower faces of the plate.

In order to derive a plate formulation without ad hoc assumptions one can start with 3-dimensional elasticity solution representations. Here, the real solution representation of Neuber/Papkovich [7,8] as well as results obtained via the complex representation of the author [9-12] will be used.

The basic step for the considered plate formulation is the decomposition of the solution into different parts. The displacement field **u**, which has to satisfy the Navier-equations

$$\mathbf{D}^T \mathbf{E} \mathbf{D} \mathbf{u} = -\bar{\mathbf{f}}, \tag{1.1}$$

is decomposed into the form

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_h + \mathbf{u}_p \\ &= \mathbf{u}_h^0 + \mathbf{u}_h^p + \mathbf{u}_p , \end{aligned} \tag{1.2}$$

where $\mathbf{u}_h$ is a solution of the homogeneous system of differential equations and $\mathbf{u}_p$ is a solution of the nonhomogeneous differential equations. In the used notation $\bar{\mathbf{f}}$ is the body force vector, **D** is a differential operator matrix and **E** is the matrix of material coefficients. The solution parts are constructed such that

$$\begin{aligned} \mathbf{D}^T \mathbf{E} \mathbf{D} \mathbf{u}_h^0 &= \mathbf{0}, \\ \mathbf{D}^T \mathbf{E} \mathbf{D} \mathbf{u}_h^p &= \mathbf{0}, \\ \mathbf{D}^T \mathbf{E} \mathbf{D} \mathbf{u}_p &= -\bar{\mathbf{f}} \end{aligned} \tag{1.3}$$

are satisfied. Moreover $\mathbf{u}_h^0$ and $\mathbf{u}_p$ ensure the satisfaction of the homogeneous stress boundary conditions on the upper and lower faces of the plate whereas $\mathbf{u}_h^p$ is a particular homogeneous solution ensuring the satisfaction of the load conditions on the lower and upper plate faces. With the sum of the solution parts $\mathbf{u}_h^0$, $\mathbf{u}_h^p$ and $\mathbf{u}_p$ we have to fulfill the remaining boundary conditions on the lateral faces of a plate under consideration.

Three types of solution functions for the solution part $\mathbf{u}_h^0$ are constructed in paragraphs 4,5 and 6. Examples for the construction of the solution parts $\mathbf{u}_h^p$ and $\mathbf{u}_p$ (involving the treatment of surface loads and body forces) are given in paragraphs 7,8 and 9.

2. Representation of the 3-dimensional stress and displacement fields

With the aid of the real stress function

$$F = H_0 + xH_1 + yH_2 + zH_3 , \tag{2.1}$$

which contains the harmonic functions $H_0(x,y,z)$, $H_1(x,y,z)$, $H_2(x,y,z)$, $H_3(x,y,z)$, the displacement components u, v, w for three-dimensional elasticity problems can be expressed with the Neuber/Papkovich representation [7,8]:

$$\begin{aligned} 2\mu u &= -F_x + 4(1-\nu)H_1, \\ 2\mu v &= -F_y + 4(1-\nu)H_2, \\ 2\mu w &= -F_z + 4(1-\nu)H_3, \end{aligned} \tag{2.2}$$

where $\mu = E/(1+\nu)$. In terms of the functions H_j $(j=0,1,2,3)$ satisfying $\Delta H_j = 0$ we can write the displacements as follows:

$$\begin{aligned} 2\mu u &= -H_{0x} + (3-4\nu)H_1 - xH_{1x} - yH_{2x} - zH_{3x}, \\ 2\mu v &= -H_{0y} + (3-4\nu)H_2 - xH_{1y} - yH_{2y} - zH_{3y}, \\ 2\mu w &= -H_{0z} + (3-4\nu)H_3 - xH_{1z} - yH_{2z} - zH_{3z}. \end{aligned} \tag{2.3}$$

The according stresses are

$$\begin{aligned} \sigma_{xx} &= H_{0yy} + H_{0zz} + 2(1-\nu)H_{1x} + x(H_{1yy} + H_{1zz}) \\ &\quad + 2\nu H_{2y} + y(H_{2yy} + H_{2zz}) \\ &\quad + 2\nu H_{3z} + z(H_{3yy} + H_{3zz}), \\ \sigma_{yy} &= H_{0xx} + H_{0zz} + 2\nu H_{1x} + x(H_{1xx} + H_{1zz}) \\ &\quad + 2(1-\nu)H_{2y} + y(H_{2xx} + H_{2zz}) \\ &\quad + 2\nu H_{3z} + z(H_{3xx} + H_{3zz}), \\ \sigma_{zz} &= H_{0xx} + H_{0yy} + 2\nu H_{1x} + x(H_{1xx} + H_{1yy}) \\ &\quad + 2\nu H_{2y} + y(H_{2xx} + H_{2yy}) \\ &\quad + 2(1-\nu)H_{3z} + z(H_{3xx} + H_{3xx}), \\ \tau_{xy} &= -H_{0xy} + (1-2\nu)H_{1y} - xH_{1xy} + (1-2\nu)H_{2x} - yH_{2xy} - zH_{3xy}, \\ \tau_{xz} &= -H_{0xz} + (1-2\nu)H_{1z} - xH_{1xz} - yH_{2xz} + (1-2\nu)H_{3x} - yH_{3xz}, \\ \tau_{yz} &= -H_{0yz} - xH_{1yz} + (1-2\nu)H_{2z} - yH_{2yz} + (1-2\nu)H_{3y} - yH_{3yz}. \end{aligned} \tag{2.4}$$

The displacement functions (2.3) satisfy the homogeneous Navier-equations

$$\begin{aligned} (1-2\nu)\Delta u + \frac{\partial e}{\partial x} &= 0, \\ (1-2\nu)\Delta v + \frac{\partial e}{\partial y} &= 0, \end{aligned} \tag{2.5}$$

$$(1-2\nu)\Delta w + \frac{\partial e}{\partial z} = 0,$$

where

$$e = u_x + v_y + w_z \,,$$

and the stresses (2.4) satisfy the homogeneous equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0,$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = 0, \qquad (2.6)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0.$$

Moreover the stresses (2.4) and the displacements (2.3) are compatible and satisfy the constitutive equations $\sigma = \mathbf{EDu}$.

3. Separation of variables for space harmonic functions

Now we want to separate functions of the thickness variable z from the harmonic functions $H_j(x,y,z)$ appearing in the solution representation (2.3) and (2.4). Omitting the index j we can write a harmonic function H in the form

$$H(x,y,z) = f(z)\, h(x,y). \qquad (3.1)$$

Substitution of (3.1) into the equation $\Delta H = 0$ gives us

$$H_{xx} + H_{yy} + H_{zz} = f\, h_{xx} + f\, h_{yy} + f''\, h = 0, \qquad (3.2)$$

from which we obtain the two possible solutions

$$\text{i)} \quad h_{xx} + h_{yy} = 0 \quad \text{and} \quad f(z) = c_0 + c_1 z \qquad (3.3)$$

and

$$\text{ii)} \quad h_{xx} + h_{yy} + q^2 h = 0 \quad \text{and} \quad f - q^2 f = 0. \qquad (3.4)$$

So f(z) is linear or contains the hyperbolic functions sinh qz and cosh qz or the trigonometric functions $\sin \omega z$ and $\cos \omega z$ if $q = i\omega$. Using the separation form (3.1) we see that f(z) can not contain a term of the form z^j with $j>1$. In order to construct a space harmonic function which contains polynomial terms in z we have to represent H(x,y,z) as a sum in the form

$$H(x,y,z) = h_0(x,y) + zh_1(x,y) + z^2h_2(x,y) + \cdots + z^nh_n(x,y) = \sum_{i=0}^{n} z^k h_k(x,y) \qquad (3.5)$$

and then to look for the relations between the h_k ($k=0,1,2,3,\ldots,n$). Substitution of (3.5) into the Laplace-equation gives

$$\begin{aligned}\Delta H = \; & H_{xx} + H_{yy} + H_{zz} \\ = \; & \Delta h_0 + 2h_2 \\ & + z[\Delta h_1 + 6h_3] \\ & + z^2[\Delta h_2 + 12h_4] \\ & + z^3[\Delta h_3 + 20h_5] \\ & + \cdots \\ & + z^{n-2}[\Delta h_{n-2} + n(n-1)h_n] \\ & + z^{n-1}\Delta h_{n-1} \\ & + z^n \Delta h_n = 0. \end{aligned} \qquad (3.6)$$

From (3.6) we obtain the relations

$$\begin{aligned} h_2 &= -\frac{1}{2}\Delta h_0 \\ h_3 &= -\frac{1}{6}\Delta h_1 \\ h_4 &= -\frac{1}{12}\Delta h_2 = \frac{1}{2\cdot 12}\Delta\Delta h_0 \\ h_5 &= -\frac{1}{20}\Delta h_3 = \frac{1}{6\cdot 20}\Delta\Delta h_1 \\ h_6 &= -\frac{1}{30}\Delta h_4 = -\frac{1}{2\cdot 12\cdot 30}\Delta\Delta\Delta h_0 \\ & \cdots \\ h_n &= -\frac{1}{n(n-1)}\Delta h_{n-2} = \frac{1}{n(n-1)(n-2)(n-3)}\Delta\Delta h_{n-4} = \cdots \end{aligned} \qquad (3.7)$$

and

$$\begin{aligned} \Delta h_{n-1} &= 0, \\ \Delta h_n &= 0. \end{aligned} \qquad (3.8)$$

Depending on whether k is odd or even we can write h_k as

$$h_k = \begin{cases} (-1)^j \dfrac{1}{(2j)!}\Delta^j h_0 & \text{for } k = 2j \\ (-1)^j \dfrac{1}{(2j+1)!}\Delta^j h_1 & \text{for } k = 2j+1 \end{cases} \qquad (3.9)$$

We see that for H(x,y,z) given with expression (3.5) there remain only two functions, one relating the powers of z with even exponents and the other connecting the powers of z with odd exponents. Therefore we decompose the space harmonic function H(x,y,z) in the form

$$H(x,y,z) = H_{even}(x,y,z) + H_{odd}(x,y,z) \tag{3.10}$$

where

$$H_{even} = \sum_{j=0}^{N} z^{2j} h_{2j} \tag{3.11}$$

and

$$H_{odd} = \sum_{j=0}^{N} z^{2j+1} h_{2j+1}. \tag{3.12}$$

The requirement (3.8) reads now

$$\Delta h_{2N} = 0,$$

$$\Delta h_{2N+1} = 0, \tag{3.13}$$

which gives us from (3.9) the characteristic differential equations

$$\Delta^{N+1} h_0 = 0 \tag{3.14}$$

for a series with even powers of z and the largest exponent 2N and

$$\Delta^{N+1} h_1 = 0 \tag{3.15}$$

for a series with odd powers of z and the largest exponent 2N+1.

In order to determine the functions h_0 and h_1 for a chosen maximum N we use the following representation:

$$h_0 = a_0 G + a_1 \Delta G + a_2 \Delta\Delta G + \cdots + a_N \Delta^N G = \sum_{k=0}^{N} a_k \Delta^K G(x,y), \tag{3.16}$$

$$h_1 = b_0 g + b_1 \Delta g + b_2 \Delta\Delta g + \cdots + b_N \Delta^N g = \sum_{k=0}^{N} b_k \Delta^K g(x,y),$$

where $\Delta^{N+1}G(x,y) = 0$ and $\Delta^{N+1}g(x,y) = 0$. Using (3.16) and (3.9) we can write the space harmonic function parts H_{even} and H_{odd} in the form

$$\begin{aligned} H_{even} = \; & [a_0 G + a_1 \Delta G + a_2 \Delta\Delta G + \cdots + a_{N-2}\Delta^{N-2} G + a_{N-1}\Delta^{N-1} G + a_N \Delta^N G] \\ & -\frac{z^2}{2}[a_0 \Delta G + a_1 \Delta\Delta G + a_2 \Delta\Delta\Delta G + \cdots + a_{N-2}\Delta^{N-1} G + a_{N-1}\Delta^N G + 0 \;] \\ & +\frac{z^4}{4!}[a_0 \Delta\Delta G + a_1 \Delta\Delta\Delta G + a_2 \Delta\Delta\Delta\Delta G + \cdots + a_{N-2}\Delta^N G + 0 \; + \; 0 \;] \end{aligned}$$

$$\begin{aligned}
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.17)\\
&+ (-1)^N \frac{z^{2N}}{(2N)!} a_0 \Delta^N G \\
&= \sum_{j=0}^{N} (-1)^j \frac{z^{2j}}{(2j)!} \Delta^j \Big[\sum_{k=0}^{N} a_k \Delta^K G \Big]
\end{aligned}$$

and

$$\begin{aligned}
H_{odd} = \;& z[b_0 g + b_1 \Delta g + b_2 \Delta\Delta g + \cdots + b_{N-2}\Delta^{N-2} g + b_{N-1}\Delta^{N-1} g + b_N \Delta^N g] \\
&- \frac{z^3}{3!}[b_0 \Delta g + b_1 \Delta\Delta b + b_2 \Delta\Delta\Delta g + \cdots + b_{N-2}\Delta^{N-1} g + b_{N-1}\Delta^N g + 0 \;] \\
&+ \frac{z^5}{5!}[b_0 \Delta\Delta g + b_1 \Delta\Delta\Delta g + b_2 \Delta\Delta\Delta\Delta g + \cdots + b_{N-2}\Delta^N g + 0 + 0 \;] \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.18)\\
&+ (-1)^N \frac{z^{2N+1}}{(2N+1)!} b_0 \Delta^N g \\
&= \sum_{j=0}^{N} (-1)^j \frac{z^{2j+1}}{(2j+1)!} \Delta^j \Big[\sum_{k=0}^{N} b_k \Delta^K g \Big].
\end{aligned}$$

The coefficients a_j and b_j $(j=0,1,2,\ldots,N)$ will be determined later from the stress boundary conditions for the upper and lower faces of the plate.

It should be mentioned that it will also be helpful to use for the representation of the four functions H_j in (2.1) the integrated forms

$$\int H_{even} dx, \quad \int H_{even} dy, \quad \int H_{odd} dx, \quad \int H_{odd} dy,$$

which are also harmonic functions.

Since the separation of variables for the space harmonic functions gives us the possibility to construct solution representations involving polynomials in z, trigonometric and hyperbolic functions of the thickness coordinate z, we will look separately at the three types of plate solutions belonging to the $\mathbf{u}_h^0$-part of the displacement field.

4. Homogeneous solution parts involving powers of z

4.1 Bending solution

In the Kirchhoff-plate theory we have to deal with biharmonic solution functions. So let us first see what we get in the 3-dimensional analysis with the aid of a function $G(x,y)=g(x,y)$ which satisfies $\Delta\Delta G = 0$ and is used to represent a space harmonic function in the forms (3.17) and (3.18). We see that only 2 terms in each of the representations (3.17) and (3.18) remain. Accordingly let us choose the space harmonic functions of the stress and displacement representation (2.3), (2.4) as

$$H_0(x,y,z) = z\,[b_0 G(x,y) + b_1 \Delta G(x,y)] - \frac{z^3}{6} b_0 \Delta G(x,y),$$

$$H_3(x,y,z) = [a_0 G(x,y) + a_1 \Delta G(x,y)] - \frac{z^2}{2} a_0 \Delta G(x,y), \tag{4.1}$$

$$H_1(x,y,z) = H_2(x,y,z) = 0.$$

The possibility of higher order solutions will be considered later. Since in the representation for F according equation (2.1) only H_3 is multiplied with z, the order of the polynomials of H_3 and H_1 differ in one.

The coefficients a_0, a_1, b_0, b_1 are to be calculated from the following homogeneous boundary conditions on the upper and lower plate faces:

$$\sigma_{zz}(x,y,z=\pm h/2) = 0,$$

$$\tau_{xz}(x,y,z=\pm h/2) = 0, \tag{4.2}$$

$$\tau_{yz}(x,y,z=\pm h/2) = 0.$$

The substitution of (4.1) into (2.4) gives us

$$\sigma_{zz} = z\,[b_0 - (1-2\nu)a_0]\Delta G \tag{4.3}$$

$$\tau_{xz} = -[b_0 - (1-2\nu)a_0]\frac{\partial}{\partial x}G + \left[\left\{b_0 - (1-2\nu)a_0 + 2a_0\right\}\frac{z^2}{2} - b_1 + (1-2\nu)a_1\right]\frac{\partial}{\partial x}\Delta G,$$

$$\tau_{yz} = -[b_0 - (1-2\nu)a_0]\frac{\partial}{\partial y}G + \left[\left\{b_0 - (1-2\nu)a_0 + 2a_0\right\}\frac{z^2}{2} - b_1 + (1-2\nu)a_1\right]\frac{\partial}{\partial y}\Delta G,$$

so that we obtain from the stress boundary condition (4.2) the following relationships for the coefficients a_0, a_1, b_0, b_1 :

$$b_0 - (1-2\nu)a_0 = 0, \tag{4.4}$$

$$\frac{h^2}{4}a_0 - b_1 + (1-2\nu)a_1 = 0. \tag{4.5}$$

Two coefficients can be chosen. In order to find a proper normalization involving nonzero values we write the displacement w to obtain

$$2\mu w = [-b_0 + (3-4\nu)a_0\,]\,G + [\,b_0 - (3-4\nu)a_0 + 2a_0\,]\,\frac{z^2}{2}\,\Delta G$$

$$+ [-b_1 + (3-4\nu)a_1\,]\,\Delta G \qquad (4.6)$$

For our normalization we require the coefficient of G in relation (4.6) to be

$$-b_0 + (3-4\nu)a_0 = 1, \qquad (4.7)$$

which gives in connection with (4.4)

$$a_0 = \frac{1}{2(1-\nu)}, \qquad (4.8)$$

$$b_0 = \frac{1-2\nu}{2(1-\nu)}. \qquad (4.9)$$

Requiring the coefficient expression of $z^0\,\Delta G$ in (4.6) to be

$$-b_1 + (3-4\nu)a_1 = 0 \qquad (4.10)$$

gives us in connection with (4.8) and (4.5)

$$a_1 = \frac{h^2}{4}\,\frac{1}{4(1-\nu)^2}, \qquad (4.11)$$

$$b_1 = \frac{h^2}{4}\,\frac{3-4\nu}{4(1-\nu)^2}. \qquad (4.12)$$

Since we can decompose our function G, which satisfies $\Delta\Delta G=0$, through

$$G = G_H + G_{BH} \qquad (4.13)$$

into a harmonic function part G_H (i.e. $\Delta G_H=0$) and a true biharmonic function part G_{BH} (i.e. $\Delta G_{BH}\neq 0$ and $\Delta\Delta G_{BH}=0$) we can write H_0 and H_3 as $H_0 = H_0^H + H_0^{BH}$ and $H_3 = H_3^H + H_3^{BH}$ to obtain

$$H_0^H = z\,b_0\,G_H,$$

$$H_3^H = a_0\,G_H, \qquad (4.14)$$

and

$$H_0^{BH} = z\,[b_0 G_{BH} + b_1\Delta G_{BH}\,] - \frac{z^3}{6}\,b_0\Delta G_{BH},$$

$$H_3^{BH} = [a_0 G_{BH} + a_1\Delta G_{BH}\,] - \frac{z^2}{2}\,a_0\Delta G_{BH}. \qquad (4.15)$$

So there exists a harmonic function part which satisfies the required homogeneous boundary conditions on the lower and upper faces of the plate and is associated with the terms z^0 and z^1, respectively. The other solution part (including biharmonic functions) is associated with the terms z^0, z^2 and z^1, z^3, respectively.

The harmonic solution part involving G_H gives us the displacement representation

$$\begin{aligned} 2\mu u &= -z \frac{\partial G_H}{\partial x} \\ 2\mu v &= -z \frac{\partial G_H}{\partial y} \\ 2\mu w &= G_H \end{aligned} \tag{4.16}$$

which basically represents the relationships of the Kirchhoff-plate theory.

Since the harmonic solution part is included in the biharmonic function G we omit further distinctions and give now the displacements and the associated stresses with the aid of G and its partial derivatives. Using the coefficients of (4.8), (4.9), (4.11) and (4.12) we obtain

$$\begin{aligned} 2\mu u &= -z \frac{\partial}{\partial x} G - \frac{1}{4(1-\nu)} [\, h^2 z - 2(2-\nu)\frac{z^3}{3} \,] \frac{\partial}{\partial x} \Delta G, \\ 2\mu v &= -z \frac{\partial}{\partial y} G - \frac{1}{4(1-\nu)} [\, h^2 z - 2(2-\nu)\frac{z^3}{3} \,] \frac{\partial}{\partial y} \Delta G, \\ 2\mu w &= G + \frac{\nu}{2(1-\nu)} z^2 \Delta G, \end{aligned} \tag{4.17}$$

$$\begin{aligned} \sigma_{xx} &= -\frac{1}{1-\nu} z\, [\, G_{xx} + \nu G_{yy} \,] - \frac{1}{4(1-\nu)} [\, h^2 z - 2(2-\nu)\frac{z^3}{3} \,] \frac{\partial^2}{\partial x^2} \Delta G, \\ \sigma_{yy} &= -\frac{1}{1-\nu} z\, [\, G_{yy} + \nu G_{xx} \,] - \frac{1}{4(1-\nu)} [\, h^2 z - 2(2-\nu)\frac{z^3}{3} \,] \frac{\partial^2}{\partial y^2} \Delta G, \\ \sigma_{zz} &= 0, \\ \tau_{xy} &= -z G_{xy} - \frac{1}{4(1-\nu)} [\, h^2 z - 2(2-\nu)\frac{z^3}{3} \,] \frac{\partial^2}{\partial x \partial y} \Delta G, \\ \tau_{xz} &= \frac{1}{2(1-\nu)} [\, z^2 - \frac{h^2}{4} \,] \frac{\partial}{\partial x} \Delta G, \\ \tau_{yz} &= \frac{1}{2(1-\nu)} [\, z^2 - \frac{h^2}{4} \,] \frac{\partial}{\partial y} \Delta G. \end{aligned} \tag{4.18}$$

For thin plates with the thickness h being small compared to the other plate dimensions we arrive with the biharmonic function G at the relationships for the Kirchhoff-theory if we neglect all terms involving h^2, z^2 and z^3.

The biharmonic function G(x,y) appearing in (4.17) and (4.18) can be expressed with the aid of two complex functions $\Phi(\zeta)$ and $\chi(\zeta)$ in the form

$$G = \frac{1}{2} [\bar{\zeta} \Phi(\zeta) + \zeta \overline{\Phi(\zeta)} + \chi(\zeta) + \overline{\chi(\zeta)}] = \mathrm{Re}[\bar{\zeta} \Phi + \chi], \tag{4.19}$$

where $\zeta = x + iy$. For the substitution of the partial derivatives of the function G in the displacements and stresses in terms of derivatives of the complex functions we can use the following relationships:

$$\begin{aligned}
G_x &= \mathrm{Re}[\Phi + \zeta \overline{\Phi'} + \overline{\chi'}], \\
G_y &= \mathrm{Im}[\Phi + \zeta \overline{\Phi'} + \overline{\chi'}], \\
G_{xx} &= \mathrm{Re}[\Phi' + \overline{\Phi'} + \zeta \overline{\Phi''} + \overline{\chi''}], \\
G_{yy} &= \mathrm{Re}[\Phi' + \overline{\Phi'} - \zeta \overline{\Phi''} - \overline{\chi''}], \\
G_{xy} &= \mathrm{Im}[\zeta \overline{\Phi''} + \overline{\chi''}], \\
\Delta G &= 4\,\mathrm{Re}[\Phi'], \\
\frac{\partial}{\partial x}\Delta G &= 4\,\mathrm{Re}[\Phi''], \\
\frac{\partial}{\partial y}\Delta G &= -4\,\mathrm{Im}[\Phi''], \\
\frac{\partial^2}{\partial x^2}\Delta G &= 4\,\mathrm{Re}[\Phi'''], \\
\frac{\partial^2}{\partial y^2}\Delta G &= -4\,\mathrm{Re}[\Phi'''], \\
\frac{\partial^2}{\partial x \partial y}\Delta G &= -4\,\mathrm{Im}[\Phi'''].
\end{aligned} \tag{4.20}$$

Using (4.20) we can write (4.17) and (4.18) as

$$\begin{aligned}
2\mu u &= -z\,\mathrm{Re}[\Phi + \zeta \overline{\Phi'} + \overline{\chi'}] - \frac{1}{1-\nu} [h^2 z - 2(2-\nu)\frac{z^3}{3}]\,\mathrm{Re}[\Phi''], \\
2\mu v &= -z\,\mathrm{Im}[\Phi + \zeta \overline{\Phi'} + \overline{\chi'}] + \frac{1}{1-\nu} [h^2 z - 2(2-\nu)\frac{z^3}{3}]\,\mathrm{Im}[\Phi''], \\
2\mu w &= \mathrm{Re}[\bar{\zeta} \Phi + \chi] + \frac{2\nu}{1-\nu} z^2\,\mathrm{Re}[\Phi'],
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
\sigma_{xx} = &- \frac{1}{1-\nu} z\,\mathrm{Re}[2(1+\nu)\Phi' + (1-\nu)(\zeta\overline{\Phi''} + \overline{\chi''})] \\
&- \frac{1}{1-\nu} [h^2 z - 2(2-\nu)\frac{z^3}{3}]\,\mathrm{Re}[\Phi'''],
\end{aligned}$$

$$\sigma_{yy} = -\frac{1}{1-\nu} z \,\mathrm{Re}[\, 2(1+\nu)\Phi' - (1-\nu)(\, \zeta\overline{\Phi''} + \overline{\chi''}\,)\,]$$

$$+ \frac{1}{1-\nu} [\, h^2 z - 2(2-\nu)\frac{z^3}{3}\,] \,\mathrm{Re}[\, \Phi'''\,],$$

$$\sigma_{zz} = 0, \tag{4.22}$$

$$\tau_{xy} = -z \,\mathrm{Im}[\, \zeta\, \overline{\Phi''} + \overline{\chi''}\,] + \frac{1}{1-\nu} [\, h^2 z - 2(2-\nu)\frac{z^3}{3}\,] \,\mathrm{Im}[\, \Phi'''\,],$$

$$\tau_{xz} = \frac{2}{1-\nu} [\, z^2 - \frac{h^2}{4}\,] \,\mathrm{Re}[\, \Phi''\,],$$

$$\tau_{yz} = -\frac{2}{1-\nu} [\, z^2 - \frac{h^2}{4}\,] \,\mathrm{Im}[\, \Phi''\,].$$

Now we want to consider a displacement representation involving higher order terms. The next higher order stress functions following (4.1) would be

$$H_0(x,y,z) = z\,[\, b_0 g + b_1\Delta g + b_2\Delta\Delta g\,] - \frac{z^3}{6} [\, b_0\Delta g + b_1\Delta\Delta g\,] + \frac{z^5}{120} b_0\Delta\Delta g,$$

and (4.23)

$$H_3(x,y,z) = [\, a_0 g + a_1\Delta g + a_2\Delta\Delta g\,] - \frac{z^2}{2} [\, a_0\Delta g + a_1\Delta\Delta g\,] + \frac{z^4}{24} a_0\Delta\Delta g,$$

where $g(x,y)$ is a solution of $\Delta\Delta\Delta g = 0$. For the stresses σ_{zz}, τ_{xz}, τ_{yz} we obtain

$$\sigma_{zz} = z\,[\,[\, b_0 - (1-2\nu)a_0\,]\Delta g + [\, b_1 - (1-2\nu)a_1\,]\Delta\Delta g\,] - \frac{z^3}{6} [\, b_0 + (1+2\nu)a_0\,]\Delta\Delta g$$

$$\tau_{xz} = -[\, b_0 - (1-2\nu)a_0\,]\frac{\partial}{\partial x} g - [\, b_1 - (1-2\nu)a_1\,]\frac{\partial}{\partial x}\Delta g$$

$$-[\, b_2 - (1-2\nu)a_2\,]\frac{\partial}{\partial x}\Delta\Delta g$$

$$+ \frac{z^2}{2} [\,[\, b_0 + (1+2\nu)a_0\,]\frac{\partial}{\partial x}\Delta g + [\, b_1 + (1+2\nu)a_1\,]\frac{\partial}{\partial x}\Delta\Delta g\,]$$

$$- \frac{z^4}{24} [\, b_0 + (3+2\nu)a_0\,]\frac{\partial}{\partial x}\Delta\Delta g\,] \tag{4.24}$$

$$\tau_{yz} = -[\, b_0 - (1-2\nu)a_0\,]\frac{\partial}{\partial y} g - [\, b_1 - (1-2\nu)a_1\,]\frac{\partial}{\partial y}\Delta g -$$

$$-[\, b_2 - (1-2\nu)a_2\,]\frac{\partial}{\partial y}\Delta\Delta g$$

$$+ \frac{z^2}{2} [\,[\, b_0 + (1+2\nu)a_0\,]\frac{\partial}{\partial y}\Delta g + [\, b_1 + (1+2\nu)a_1\,]\frac{\partial}{\partial y}\Delta\Delta g\,]$$

$$-\frac{z^4}{24}\,[\,b_0 + (3+2\nu)a_0\,]\frac{\partial}{\partial y}\Delta\Delta g\,].$$

The stresses (4.24) can only vanish on $z=\pm h/2$ if $a_0=b_0=0$ or if $\Delta\Delta g=0$. But both cases mean that in (4.23) we have no terms z^4 and z^5 involved. A look at the use of other higher order polynomials in z for H_0 and H_3 shows that there are no further possibilities to satisfy the homogeneous conditions on the upper and lower faces of the plate than through the stress functions of (4.1) involving z^3 as the highest power of z.

4.2 Membrane solution

The bending solution part constructed in the previous paragraph contains the antisymmetric terms z^1, z^3 for the displacement components u and v whereas the displacement component w has the symmetric terms z^0, z^2. For the membrane solution part involving polynomials in z we will construct a solution which contains the symmetric terms z^0, z^2 in u and v, and the antisymmetric term z in w. The stress function F for the membrane case contains only even powers of z. Using two terms in the series representations (3.17) and (3.18) we can choose the four harmonic functions H_j in the biharmonic stress function (2.1) as

$$H_0 = [\,a_0G + a_1\Delta G\,] - \frac{z^2}{2}\,a_0\Delta G,$$

$$H_1 = \int\left\{[\,c_0G + c_1\Delta G\,] - \frac{z^2}{2}\,c_0\Delta G\right\}dx,$$

$$H_2 = \int\left\{[\,d_0G + d_1\Delta G\,] - \frac{z^2}{2}\,d_0\Delta G\right\}dy, \qquad (4.25)$$

and

$$H_3 = z\,[\,b_0G + b_1\Delta G\,] - \frac{z^3}{6}\,b_0\Delta G,$$

where G(x,y) has to satisfy $\Delta\Delta G=0$ according to $\Delta^{N+1}G=0$ and the choice of N, which has the value $N=1$.

From the boundary conditions $\sigma_{zz}=\tau_{xz}=\tau_{yz}=0$ on $z=\pm h/2$ we have to calculate the coefficients a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1. For σ_{zz} at the lower and upper faces of the plate we obtain

$$\sigma_{zz}(x,y,z=\pm h/2) = [\,a_0 + 2\nu(c_1 + d_1) - \nu\frac{h^2}{4}(c_0 + d_0) + 2(1-\nu)b_1 + \nu\frac{h^2}{4}b_0\,]\Delta G$$
$$+ [\,2\nu(c_0 + d_0) + 2(1-\nu)b_0\,]G$$

$$+ x\, c_0 \int \Delta G\, dx + y\, d_0 \int \Delta G\, dy = 0. \tag{4.26}$$

Since (4.26) has to hold for all values of x and y we get

$$c_0 = d_0 = 0, \tag{4.27}$$

and from this we obtain

$$b_0 = 0. \tag{4.28}$$

Using (4.27) and (4.28) we can write the boundary conditions for $z = \pm h/2$ as

$$\sigma_{zz}(x,y,z=\pm h/2) = [\, a_0 + 2\nu(c_1 + d_1) + 2(1-\nu) b_1 \,] \Delta G = 0, \tag{4.29}$$

$$\tau_{xz}(x,y,z=\pm h/2) = (\pm \frac{h}{2}) [\, a_0 - 2\nu b_1 \,] \frac{\partial}{\partial x} \Delta G = 0, \tag{4.30}$$

$$\tau_{yz}(x,y,z=\pm h/2) = (\pm \frac{h}{2}) [\, a_0 - 2\nu b_1 \,] \frac{\partial}{\partial y} \Delta G = 0.$$

For the satisfaction of these equations we can distinguish between the two cases $\Delta G \neq 0$ and $\Delta G = 0$. Now let us treat the two cases separately and use the indices 1 and 2 for the function G (G_1 for the case $\Delta G_1 \neq 0$ and G_2 for the case $\Delta G_2 = 0$).

For the case $\Delta G_1 \neq 0$ we obtain from (4.30)

$$a_0 = 2\nu b_1. \tag{4.31}$$

Substitution of (4.31) into (4.29) gives

$$b_1 = -\nu(c_1 + d_1) \tag{4.32}$$

so that a_0 becomes

$$a_0 = -2\nu^2(c_1 + d_1). \tag{4.33}$$

The coefficient $(c_1 + d_1)$ can be chosen for a proper normalization, which will be done later. For a simplification of the solution representation we choose $c_1 = d_1$.

For the case $\Delta G_2 = 0$ the boundary conditions (4.29), (4.30) are satisfied for arbitrary choices of a_0, a_1, c_1, d_1. Here we choose $a_0 = 1$ and $a_1 = b_1 = c_1 = d_1 = 0$.

Adding the functions of the two cases according to $G = G_1 + G_2$ we obtain the following set of stress functions, which guarantee the satisfaction of the boundary conditions (4.29)-(4.30):

$$H_0 = -4\nu^2 c_1 G_1 + a_1 \Delta G_1 + 2c_1 \nu^2 z^2 \Delta G_1 + G_2,$$

$$H_1 = c_1 \int \Delta G_1\, dx,$$

$$H_2 = c_1 \int \Delta G_1 \, dy,$$

$$H_3 = -2\nu c_1 z \Delta G_1, \tag{4.34}$$

$$F = c_1 \left[x \int \Delta G_1 \, dx + y \int \Delta G_1 \, dy - 4\nu^2 G_1 - 2z^2\nu(1-\nu)\Delta G_1 \right] + a_1 \Delta G_1 + G_2.$$

The integral terms in (4.34) can be expressed in a simple form by using a complex function representation. With the aid of the complex functions $\Phi_1(\zeta)$, $\chi_1(\zeta)$, $\chi_2(\zeta)$ we can write G_1 and G_2 as

$$G_1 = \frac{1}{2} \left[\bar{\zeta} \, \Phi_1(\zeta) + \zeta \, \overline{\Phi_1(\zeta)} + \chi_1(\zeta) + \overline{\chi_1(\zeta)} \right] \tag{4.35}$$

and

$$G_2 = \frac{1}{2} \left[\chi_2(\zeta) + \overline{\chi_2(\zeta)} \right]. \tag{4.36}$$

Using the relationships

$$\Delta G_1 = 2 \left[\Phi_1' + \overline{\Phi_1'} \right],$$

$$\int \Delta G_1 \, dx = 2 \left[\Phi_1 + \overline{\Phi_1} \right], \tag{4.37}$$

$$\int \Delta G_1 \, dy = -2i \left[\Phi_1 - \overline{\Phi_1} \right],$$

and

$$x \int \Delta G_1 \, dx + y \int \Delta G_1 \, dy = 2(x - iy)\Phi_1 + 2(x + iy)\overline{\Phi_1} \tag{4.36}$$

we can write F as

$$F = 2c_1 \left\{ (1-\nu^2)\bar{\zeta}\Phi_1 + (1-\nu^2)\zeta\overline{\Phi_1} - \nu^2\chi_1 - \nu^2\overline{\chi_1} - 2z^2\nu(1-\nu) \left[\Phi_1' + \overline{\Phi_1'} \right] \right\}$$

$$+ 2a_1 \left[\Phi_1' + \overline{\Phi_1'} \right] + \frac{1}{2} \left[\chi_2(\zeta) + \overline{\chi_2(\zeta)} \right]. \tag{4.39}$$

where $\zeta = x + iy$. For a proper normalization c_1 is chosen such that the factor of the term $\bar{\zeta}\Phi_1$ in equation (4.39) becomes 1/2. Choosing

$$c_1 = \frac{1}{4(1-\nu^2)} \tag{4.40}$$

and using the substitutions

$$\chi = \chi_2 - \frac{\nu^2}{1-\nu^2}\chi_1,$$

$$\Phi = \Phi_1, \tag{4.41}$$

(4.39) becomes

$$F = \frac{1}{2} [\bar{\zeta}\Phi + \zeta\bar{\Phi} + \chi + \bar{\chi}] + [2a_1 - \frac{\nu}{1+\nu} z^2][\Phi' + \bar{\Phi}'] \tag{4.42}$$

The remaining coefficient a_1 can be chosen arbitrarily. For convenience we choose a_1 such that the mean value of F across the thickness of the plate is equal to the stress function F^* used in the Muskhelishvili formulation for plane stress problems [13]. So we require that

$$F^* = \frac{1}{h} \int_{-h/2}^{h/2} F \, dz = \frac{1}{2} [\bar{\zeta}\Phi + \zeta\bar{\Phi} + \chi + \bar{\chi}] \tag{4.43}$$

and get

$$\int_{-h/2}^{h/2} [2a_1 - \frac{\nu}{1+\nu} z^2] \, dz = 0, \tag{4.44}$$

from which we obtain

$$a_1 = \frac{h^2}{24} \frac{\nu}{1+\nu}. \tag{4.45}$$

The treatment of the homogeneous boundary conditions on the upper and lower faces of the plate led us to the following stress functions:

$$\begin{aligned} H_0 &= \frac{1}{2} [\chi + \bar{\chi}] - \frac{\nu^2}{2(1-\nu^2)} [\bar{\zeta}\Phi + \zeta\bar{\Phi}] + [\frac{h^2}{12} \frac{\nu}{1+\nu} + \frac{\nu^2}{1-\nu^2} z^2][\Phi' + \bar{\Phi}'] \\ &= \mathrm{Re}[\chi] - \frac{\nu^2}{1-\nu^2} \mathrm{Re}[\bar{\zeta}\Phi] + 2[\frac{h^2}{12} \frac{\nu}{1+\nu} + \frac{\nu^2}{1-\nu^2} z^2]\mathrm{Re}[\Phi'] \\ H_1 &= \frac{1}{2(1-\nu^2)} [\Phi + \bar{\Phi}] = \frac{1}{1-\nu^2} \mathrm{Re}[\Phi], \\ H_2 &= \frac{-i}{2(1-\nu^2)} [\Phi - \bar{\Phi}] = \frac{1}{1-\nu^2} \mathrm{Im}[\Phi], \\ H_3 &= -z \frac{\nu}{1-\nu^2} [\Phi' + \bar{\Phi}'] = -2z \frac{\nu}{1-\nu^2} \mathrm{Re}[\Phi'], \end{aligned} \tag{4.46}$$

and

$$F = \mathrm{Re}[\bar{\zeta}\Phi + \chi] + 2 \frac{\nu}{1+\nu} [\frac{h^2}{12} - z^2] \mathrm{Re}[\Phi']. \tag{4.47}$$

With the aid of (4.46), (2.3) and (2.4) we get the displacements

$$\begin{aligned} 2\mu u &= \mathrm{Re}[\frac{3-\nu}{1+\nu} \Phi - \zeta\bar{\Phi}' - \bar{\chi}'] - 2 \frac{\nu}{1+\nu} [\frac{h^2}{12} - z^2] \mathrm{Re}[\Phi''], \\ 2\mu v &= \mathrm{Im}[\frac{3-\nu}{1+\nu} \Phi - \zeta\bar{\Phi}' - \bar{\chi}'] + 2 \frac{\nu}{1+\nu} [\frac{h^2}{12} - z^2] \mathrm{Im}[\Phi''], \\ 2\mu w &= -4z \frac{\nu}{1+\nu} \mathrm{Re}[\Phi'], \end{aligned} \tag{4.48}$$

and the stresses

$$\sigma_{xx} = \mathrm{Re}[\ \Phi' + \overline{\Phi'} - \zeta\overline{\Phi''} - \overline{\chi''}\ ] - 2\frac{\nu}{1+\nu}[\ \frac{h^2}{12} - z^2\ ]\ \mathrm{Re}[\Phi'''],$$

$$\sigma_{yy} = \mathrm{Re}[\ \Phi' + \overline{\Phi'} + \zeta\overline{\Phi''} + \overline{\chi''}\ ] - 2\frac{\nu}{1+\nu}[\ \frac{h^2}{12} - z^2\ ]\ \mathrm{Re}[\Phi'''], \quad (4.49)$$

$$\tau_{xy} = -\mathrm{Im}[\ \zeta\overline{\Phi''} + \overline{\chi''}\ ] + 2\frac{\nu}{1+\nu}[\ \frac{h^2}{12} - z^2\ ]\ \mathrm{Im}[\Phi'''],$$

$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0.$$

We observe that due to the properties of the constructed space stress function F(x,y,z) the general three-dimensional relations (2.4) reduce here to the expressions $\sigma_{xx} = F_{yy}$, $\sigma_{yy} = F_{xx}$ and $\tau_{xy} = -F_{xy}$, which have the form of the Airy-stress relationships for plane problems.

Taking the average of the displacements (4.48) and the stresses (4.49) by integrating with respect to the thickness coordinate z and dividing by the thickness h we obtain the Muskhelishvili formulas for plane stress [13].

5. Homogeneous solution parts involving trigonometric functions of z

In order to use a multiplicative separation of a space harmonic function in the form (3.1) with $\sin \omega_n z$ or $\cos \lambda_m z$ as separated functions we need to construct a displacement representation which involves no sums in the displacement components and which does not have the factors x,y or z. This can be obtained with $F_x = F_y = F_z = H_3 = 0$. As we see from (2.1) F_x, F_y and F_z vanish if we choose H_0, H_1 and H_2 such that the stress function F vanishes. This can be reached by the choice of the harmonic functions H_j in the form

$$H_0 = -x\frac{\partial\Psi}{\partial y} + y\frac{\partial\Psi}{\partial x},$$

$$H_1 = \frac{\partial\Psi}{\partial y},$$

$$H_2 = -\frac{\partial\Psi}{\partial x}, \quad (5.1)$$

$$H_3 = 0,$$

where $\Psi = \Psi(x,y,z)$ satisfying $\Delta\Psi = 0$. Substitution of (5.1) into (2.2) gives us the simple displacement representation

$$2\mu u = 4(1-\nu)\ \Psi_y,$$

$$2\mu v = -4(1-\nu)\,\Psi_x, \tag{5.2}$$

$$2\mu w = 0,$$

or

$$2\mu u = \Phi_y,$$

$$2\mu v = -\Phi_x, \tag{5.3}$$

$$2\mu w = 0,$$

if we use the substitution $\Phi(x,y,z) = 4(1-\nu)\Psi(x,y,z)$. For the function $\Phi(x,y,z)$ satisfying $\Delta\Phi = 0$ we can use the following multiplicative separation

$$\Phi(x,y,z) = \begin{cases} g_n(x,y) \sin \omega_n z \\ \hat{g}_m(x,y) \cos \lambda_m z \end{cases} \tag{5.4}$$

which leads to

$$\frac{\partial^2 g_n}{\partial x^2} + \frac{\partial^2 g_n}{\partial y^2} - \omega_n^2\, g_n = 0,$$

$$\frac{\partial^2 \hat{g}_m}{\partial x^2} + \frac{\partial^2 \hat{g}_m}{\partial y^2} - \lambda_m^2\, \hat{g}_m = 0 \tag{5.5}$$

as the characterizing differential equations for $g_n(x,y)$ and $\hat{g}_m(x,y)$. It is obvious that one solution part in equation (5.4) is symmetric with respect to the origin of the z-axis and the other part is antisymmetric.

5.1 Bending solution

For thin plate bending applications we are interested in the solution part leading to antisymmetric stress distributions for σ_{xx}, σ_{yy}, τ_{xy} and symmetric stress distributions for τ_{xz}, τ_{yz} over the thickness of the plate. The according displacements and stresses are

$$2\mu u = \frac{\partial g_n}{\partial y} \sin \omega_n z,$$

$$2\mu v = -\frac{\partial g_n}{\partial x} \sin \omega_n z, \tag{5.6}$$

$$2\mu w = 0,$$

$$\sigma_{xx} = \frac{\partial^2 g_n}{\partial x \partial y} \sin \omega_n z,$$

$$\sigma_{yy} = -\frac{\partial^2 g_n}{\partial x \partial y} \sin \omega_n z,$$

$$\sigma_{zz} = 0, \qquad (5.7)$$

$$\tau_{xy} = \frac{1}{2} [\frac{\partial^2 g_n}{\partial y^2} - \frac{\partial^2 g_n}{\partial x^2}] \sin \omega_n z,$$

$$\tau_{xz} = \frac{1}{2} \omega_n \frac{\partial g_n}{\partial y} \cos \omega_n z,$$

$$\tau_{yz} = -\frac{1}{2} \omega_n \frac{\partial g_n}{\partial x} \cos \omega_n z,$$

where $\omega_n = n\pi/h$ $(n=1,3,5,....)$. In order to satisfy the boundary conditions for τ_{xz} and τ_{yz} at $z = \pm h/2$ we have to choose odd numbers for n.

5.2 Membrane solution

The membrane solution satisfying the stress boundary conditions (4.2) is obtained when we write in relationships (5.6) and (5.7) $\hat{g}_m$ instead of g_n, λ_m instead of ω_n, $\cos \lambda_m z$ instead of $\sin \omega_n z$, and $\sin \lambda_m z$ instead of $\cos \omega_n z$, where $\lambda_m = m\pi/h$ $(m=0,2,4,....)$.

6. Homogeneous solution parts involving hyperbolic functions

In the paper [11] discussing the application of the author's complex 3-dimensional elasticity solution representation [9-12] for the analysis of a rectangular plate it was found that a homogeneous solution involving hyperbolic functions of z can be written in the form

$$2\mu u = -\text{Re}[G_x(x,y) f(z)],$$

$$2\mu v = -\text{Re}[G_y(x,y) f(z)], \qquad (6.1)$$

$$2\mu w = \text{Re}[G(x,y) h(z)],$$

where G(x,y) satisfies $\Delta G + q^2 G = 0$ and q is a complex eigenvalue which can be computed from the homogeneous boundary conditions $\sigma_{zz}(x,y,z=\pm h/2) = \tau_{xz}(x,y,z=\pm h/2) = \tau_{yz}(x,y,z=\pm h/2) = 0$.

In order to get simple expressions during the derivation of the solution (6.1), we make use of the coordinate transformation

$$\hat{z} = z + \frac{h}{2} \tag{6.2}$$

so that the boundary conditions in the new coordinates are

$$\begin{aligned} \sigma_{zz}(x,y,\hat{z}=0) &= \sigma_{zz}(x,y,\hat{z}=h) = 0, \\ \tau_{xz}(x,y,\hat{z}=0) &= \tau_{xz}(x,y,\hat{z}=h) = 0, \\ \tau_{yz}(x,y,\hat{z}=0) &= \tau_{yz}(x,z,\hat{z}=h) = 0, \end{aligned} \tag{6.3}$$

($\hat{z}$ corresponds to the thickness coordinate z used in [11] .) From the results of the example given in article [11] we can see how we have to choose the harmonic functions $H_j(x,y,z)$ if we want to start with the displacement representation (2.2). Let us include now also the case of certain nonhomogeneous stress boundary conditions on the upper and lower faces of the plate so that we are also able to construct particular solutions in the same time.

Choosing the harmonic functions H_j in the form

$$\begin{aligned} H_0(x,y,\hat{z}) &= G(x,y)\ [\ c \cosh q\hat{z} + d \sinh q\hat{z}\ ], \\ H_1(x,y,\hat{z}) &= 0, \\ H_2(x,y,\hat{z}) &= 0, \\ H_3(x,y,\hat{z}) &= G(x,y)\ [\ a \cosh q\hat{z} + b \sinh q\hat{z}\ ], \end{aligned} \tag{6.4}$$

we obtain the following displacement and stress representation:

$$\begin{aligned} 2\mu u &= -G_x\ [\ a\,\hat{z} \cosh q\hat{z} + b\,\hat{z} \sinh q\hat{z} + c \cosh q\hat{z} + d \sinh q\hat{z}\ ], \\ 2\mu v &= -G_y\ [\ a\,\hat{z} \cosh q\hat{z} + b\,\hat{z} \sinh q\hat{z} + c \cosh q\hat{z} + d \sinh q\hat{z}\ ], \end{aligned} \tag{6.5}$$

$$\begin{aligned} 2\mu w = \ & G\ [\ a\left\{(3-4\nu) \cosh q\hat{z} - q\hat{z} \sinh q\hat{z}\right\} + b\left\{(3-4\nu) \sinh q\hat{z} - q\hat{z} \cosh q\hat{z}\right\} \\ & - c\,q \sinh q\hat{z} - d\,q \cosh q\hat{z}\ ], \end{aligned}$$

$$\begin{aligned} \sigma_{xx} = \ & -G_{xx}\ [\ a\,\hat{z} \cosh q\hat{z} + b\,\hat{z} \sinh q\hat{z} + c \cosh q\hat{z} + d \sinh q\hat{z}\ ] \\ & + G\ 2\nu q\ [\ a \sinh q\hat{z} + b \cosh q\hat{z}\ ], \end{aligned}$$

$$\begin{aligned} \sigma_{yy} = \ & -G_{yy}\ [\ a\,\hat{z} \cosh q\hat{z} + b\,\hat{z} \sinh q\hat{z} + c \cosh q\hat{z} + d \sinh q\hat{z}\ ] \\ & + G\ 2\nu q\ [\ a \sinh q\hat{z} + b \cosh q\hat{z}\ ], \end{aligned}$$

$$\begin{aligned} \sigma_{zz} = \ & G\,q\ [\ a\left\{ -q\hat{z} \cosh q\hat{z} + 2(1-\nu)\sinh q\hat{z}\right\} + b\left\{ -q\hat{z} \sinh q\hat{z} + 2(1-\nu)\cosh q\hat{z}\right\} \\ & - c\,q \cosh q\hat{z} - d\,q \sinh q\hat{z}\ ], \end{aligned}$$

$$\tau_{xy} = -G_{xy} [\, a\,\hat{z} \cosh q\hat{z} + b\,\hat{z} \sinh q\hat{z} + c \cosh q\hat{z} + d \sinh q\hat{z} \,], \tag{6.6}$$

$$\tau_{xz} = G_x [\, a \left\{(1-2\nu) \cosh q\hat{z} - q\hat{z} \sinh q\hat{z} \right\} + b \left\{(1-2\nu) \sinh q\hat{z} - q\hat{z} \cosh q\hat{z} \right\}$$

$$- c\, q \sinh q\hat{z} - d\, q \cosh q\hat{z} \,],$$

$$\tau_{yz} = G_y [\, a \left\{(1-2\nu) \cosh q\hat{z} - q\hat{z} \sinh q\hat{z} \right\} + b \left\{(1-2\nu) \sinh q\hat{z} - q\hat{z} \cosh q\hat{z} \right\}$$

$$- c\, q \sinh q\hat{z} - d\, q \cosh q\hat{z} \,].$$

The substitution of the stresses (6.6) into the equilibrium equations (2.6) gives us

$$-\frac{\partial}{\partial x} [\, G_{xx} + G_{yy} + q^2 G \,][\, a\,\hat{z} \cosh q\hat{z} + b\,\hat{z} \sinh q\hat{z} + c \cosh q\hat{z} + d \sinh q\hat{z} \,] = 0,$$

$$-\frac{\partial}{\partial y} [\, G_{xx} + G_{yy} + q^2 G \,][\, a\,\hat{z} \cosh q\hat{z} + b\,\hat{z} \sinh q\hat{z} + c \cosh q\hat{z} + d \sinh q\hat{z} \,] = 0, \tag{6.7}$$

$$[\, G_{xx} + G_{yy} + q^2 G \,]\, 2(1-\nu)\, [\, a \cosh q\hat{z} + b \sinh q\hat{z} \,] = 0,$$

which are satisfied if

$$\Delta G + q^2 G = 0. \tag{6.8}$$

If we have nonhomogeneous stress boundary conditions we choose values for q and calculate the unknown coefficients a, b, c, d. An example for such a solution will be given in chapter 7.

Here we want to get solutions for the homogeneous boundary conditions (6.3). Since the boundary condition for τ_{xz} gives the same equations as the condition for τ_{yz} we obtain four homogeneous equations involving the four unknown coefficients a, b, c, d. The system of equations can be written as

$$\begin{bmatrix} K_{11} & -K_{12} & -2q \cosh qh & -2q \sinh qh \\ 0 & 4(1-\nu) & -2q & 0 \\ 2(1-2\nu) & 0 & 0 & -2q \\ K_{41} & -K_{42} & -2q \sinh qh & -2q \cosh qh \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{6.9}$$

where

$$K_{11} = -2qh \cosh qh + 4(1-\nu) \sinh qh,$$

$$K_{12} = 2qh \sinh qh - 4(1-\nu) \cosh qh,$$

$$K_{41} = -2qh \sinh qh + 2(1-2\nu) \cosh qh, \tag{6.10}$$

$$K_{42} = 2qh \cosh qh - 2(1-2\nu) \sinh qh.$$

The equation (6.9) has only nontrivial solutions if the q-dependent determinant of the coefficient matrix vanishes. This requirement gives us the equation

$$q^2 h^2 - \sinh^2 qh = 0, \tag{6.11}$$

from which we can compute two infinite series of complex solution values for q associated to $qh = -\sinh qh$ and $qh = \sinh qh$. The lowest zeros of $qh + \sinh qh = 0$ and $qh - \sinh qh = 0$ are given in tables 1 and 2, respectively. One series of zeros is associated to symmetric solutions and the other to antisymmetric solutions. The conjugate complex values of q are also solutions of the characteristic equations.

In order to satisfy homogeneous stress boundary conditions on the upper and lower plate faces we obtained only complex values for q to be used for the solution. Since q is complex and G(x,y) has to satisfy $\Delta G + q^2 G = 0$, G is also complex. Using the complex q and its complex conjugate value $\bar{q}$ we can formulate a real solution for the displacements as indicated in (6.1).

6.1 Bending solution

The bending solution is obtained from the zeros of $qh - \sinh qh = 0$. The lowest five values for qh are given in Table 1.

Table 1: Lowest zeros of $qh - \sinh qh = 0$, where $q = q_r + iq_i$

$q_r h$	$q_i h$
2.76867828298732	7.49767627777639
3.35220988485351	13.8999597139765
3.71676767975250	20.2385177078300
3.98314164033996	26.5545472654916
4.19325147043121	32.8597410050699

For every complex eigenvalue we can evaluate the corresponding eigenvector which has the form

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = A \begin{bmatrix} 1 \\ \dfrac{1 - \cosh qh}{qh} \\ \dfrac{1-2\nu}{q} \\ \dfrac{2(1-\nu)}{q} \dfrac{1 - \cosh qh}{qh} \end{bmatrix}, \tag{6.12}$$

where A is an arbitrary factor, which for our purposes can be chosen as A=1.

6.2 Membrane solution

Using the zeros of qh + sinh qh = 0 (Table 2) and the corresponding eigenvector

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = A \begin{bmatrix} 1 \\ \frac{1 + \cosh qh}{qh} \\ \frac{1-2\nu}{q} \\ \frac{2(1-\nu)}{q} \; \frac{1 + \cosh qh}{qh} \end{bmatrix} \tag{6.13}$$

for the displacement representation (6.5) and the stresses (6.6) we get a membrane solution.

Table 2: Lowest zeros of qh + sinh qh = 0, where $q = q_r + iq_i$

$q_r h$	$q_i h$
2.25072861160186	4.21239223049066
3.10314874582525	10.7125373972793
3.55108734702208	17.0733648531518
3.85880899310557	23.3983552256513
4.09370492476533	29.7081198252760

7. Particular solution for a normal load p(x,y) acting on the upper face of the plate

The use of real values q and a real function G in the displacement and stress representation (6.5)-(6.6) is possible for the construction of particular solutions satisfying nonhomogeneous stress boundary conditions on the upper and lower faces of the plate. A particular solution for the boundary conditions

$$\begin{aligned} &\sigma_{zz}(x,y,z=-h/2) = -p(x,y), \\ &\sigma_{zz}(x,y,z=h/2) = 0, \\ &\tau_{xz}(x,y,z=\pm h/2) = \tau_{yz}(x,y,z=\pm h/2) = 0, \end{aligned} \tag{7.1}$$

can be constructed by choosing

$$G(x,y) = \sin \alpha_m x \sin \beta_n y,$$

$$\alpha_m = \frac{m\pi}{a}, \quad \beta_n = \frac{n\pi}{b}, \tag{7.2}$$

$$q_{mn} = \sqrt{\alpha_m^2 + \beta_n^2},$$

and by using relationships (6.5)-(6.6). The domain for the particular solution is chosen to be $0 \le x \le a$, $0 \le y \le b$, $0 \le \hat{z} \le h$ and the boundary of the particular solution in the x-y-plane can be considered as an auxiliary boundary. The actual boundary (of the x-y-plane) of a plate under consideration has to lie inside the auxiliary boundary. The rectangular auxiliary boundary is chosen as we have a solution [11] available for the boundary conditions $w(0,y,\hat{z}) = w(a,y,\hat{z}) = w(x,0,\hat{z}) = w(x,b,\hat{z}) = v(0,y,\hat{z}) = v(a,y,\hat{z}) = u(x,o,\hat{z}) = u(x,b,\hat{z}) = \sigma_{xx}(0,y,\hat{z}) = \sigma_{xx}(a,y,\hat{z}) = \sigma_{yy}(x,0,\hat{z}) = \sigma_{yy}(x,b,\hat{z}) = 0$. The solution can be expressed with the aid of the coefficients of the double Fourier-series of the load given by

$$p(x,y) = \sum_m \sum_n a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \tag{7.3}$$

where for example $a_{mn} = 16p_0/(\pi^2 mn)$ for a constant load p_0 and $m = 1,3,5,\ldots$, $n = 1,3,5,\ldots$. The particular solution for the displacements taken from article [11] is

$$2\mu u = \sum_m \sum_n \alpha_m \cos \alpha_m x \sin \beta_n y \Big[-4A_{mn}\, \hat{z} \cosh q_{mn}\hat{z} + 4B_{mn}\, \hat{z} \sinh q_{mn}\hat{z} + \frac{2}{q_{mn}} E_{mn} \sinh q_{mn}\hat{z} - \frac{2}{q_{mn}} F_{mn} \cosh q_{mn}\hat{z} \Big],$$

$$2\mu v = \sum_m \sum_n \beta_n \sin \alpha_m x \cos \beta_n y \Big[-4A_{mn}\, \hat{z} \cosh q_{mn}\hat{z} + 4B_{mn}\, \hat{z} \sinh q_{mn}\hat{z} + \frac{2}{q_{mn}} E_{mn} \sinh q_{mn}\hat{z} - \frac{2}{q_{mn}} F_{mn} \cosh q_{mn}\hat{z} \Big], \tag{7.4}$$

$$2\mu w = \sum_m \sum_n \sin \alpha_m x \sin \beta_n y \Big[4A_{mn} \Big\{ (3-4\nu) \cosh q_{mn}\hat{z} - q_{mn}\hat{z} \sinh q_{mn}\hat{z} \Big\} + 4B_{mn} \Big\{ -(3-4\nu) \sinh q_{mn}\hat{z} + q_{mn}\hat{z} \cosh q_{mn}\hat{z} \Big\} + 2E_{mn} \cosh q_{mn}\hat{z} - 2F_{mn} \sinh q_{mn}\hat{z} \Big],$$

where $\hat{z} = z + h/2$ and

$$A_{mn} = \frac{a_{mn}[\, q_{mn}h + \cosh q_{mn}h \sinh q_{mn}h \,]}{4q_{mn}[\, \sinh^2 q_{mn}h - q_{mn}^2 h^2 \,]},$$

$$B_{mn} = \frac{a_{mn} \sinh^2 q_{mn}h}{4q_{mn}[\, \sinh^2 q_{mn}h - q_{mn}^2 h^2 \,]}, \tag{7.5}$$

$$E_{mn} = -2(1-2\nu)A_{mn},$$

$$F_{mn} = -4(1-\nu)B_{mn} + \frac{a_{mn}}{2q_{mn}}.$$

The according stresses can be found in [11].

The particular solution (7.4) satisfies the load conditions (7.1) on the upper and lower faces of the plate. In general it satisfies of course not the boundary conditions on the actual lateral faces of the plate under consideration. But through our solution strategy of superposing a particular solution satisfying nonhomogeneous boundary conditions on the upper and lower plate faces and homogeneous solutions satisfying homogeneous boundary conditions on the same faces we reduce our plate problem to the satisfaction of given boundary conditions on the side faces of the plate.

8. Particular solution for a constant normal load p on the upper face of the plate

Assuming the same boundary conditions as in (7.1) but now with $p(x,y) = p = \text{const.}$ we can construct a simple particular solution involving polynomial terms in x, y, z. Using odd and even functions of the forms (3.17) and (3.18), respectively, we obtain a particular solution with the following choice of stress functions H_j:

$$\begin{aligned} H_0(x,y,\hat{z}) = {} & [\, a_0 + a_1\Delta g + a_2\Delta\Delta g \,] - \frac{\hat{z}^2}{2!}[\, a_0\Delta g + a_1\Delta\Delta g \,] + \frac{\hat{z}^4}{4!} a_0\Delta\Delta g \\ & + \hat{z}[\, c_0 g + c_1\Delta g + c_2\Delta\Delta g \,] - \frac{\hat{z}^3}{3!}[\, c_0\Delta g + c_1\Delta\Delta g \,] + \frac{\hat{z}^5}{5!} c_0\Delta\Delta g, \\ H_1(x,y,\hat{z}) = {} & H_2(x,y,\hat{z}) = 0, \\ H_3(x,y,\hat{z}) = {} & \hat{z}[\, b_0 + b_1\Delta g + b_2\Delta\Delta g \,] - \frac{\hat{z}^3}{3!}[\, b_0\Delta g + b_1\Delta\Delta g \,] + \frac{\hat{z}^5}{5!} b_0\Delta\Delta g \\ & + [\, d_0 g + d_1\Delta g + d_2\Delta\Delta g \,] - \frac{\hat{z}^2}{2!}[\, d_0\Delta g + d_1\Delta\Delta g \,] + \frac{\hat{z}^4}{4!} d_0\Delta\Delta g. \end{aligned} \tag{8.1}$$

In order to get harmonic functions H_0 and H_3 the function $g(x,y)$ has to satisfy

$$\Delta\Delta\Delta g = 0. \tag{8.2}$$

For a convenient treatment of the boundary conditions we use again the coordinate $\hat{z} = z + h/2$ lying in the interval $0 \le \hat{z} \le h$. If we choose $g(x,y)$ such that

$$\Delta\Delta g = p = \text{const.}, \tag{8.3}$$

we automatically satisfy (8.2) and have the properties

$$\frac{\partial}{\partial x}\,\Delta\Delta g = 0,$$

$$\frac{\partial}{\partial y}\,\Delta\Delta g = 0. \qquad (8.4)$$

The substitution of (8.1) into (2.4), using (8.4), and setting $b_0 = 0$ (to make σ_{zz} a third order polynomial in z) gives us

$$\begin{aligned}\sigma_{zz} = \; & [\, a_0 + 2(1-\nu)b_1 \,]\Delta g + [\, a_1 + 2(1-\nu)b_2 \,]\Delta\Delta g \\ & + \hat{z}\,[\, c_0 - (1-2\nu)d_0 \,]\Delta g + \hat{z}\,[\, c_1 - (1-2\nu)d_1 \,]\Delta\Delta g \\ & - \frac{\hat{z}^3}{6}\,[\, c_0 - (1-2\nu)d_0 \,]\Delta\Delta g - \frac{1}{3}\, d_0\, \hat{z}^3\, \Delta\Delta g \\ & - \frac{\hat{z}^2}{2}\,[\, a_0 + 2(1-\nu)b_1 \,]\Delta\Delta g + b_1\, \hat{z}^2\, \Delta\Delta g, \end{aligned} \qquad (8.5)$$

$$\begin{aligned}\tau_{xz} = \frac{\partial}{\partial x}\Big\{ & -[\, c_0 - (1-2\nu)d_0 \,]g - [\, c_1 - (1-2\nu)d_1 \,]\Delta g \\ & + \hat{z}\,[\, a_0 + 2(1-\nu)b_1 \,]\Delta g - 2\hat{z}\, b_1\, \Delta g \\ & + \frac{\hat{z}^2}{2}\,[\, c_0 - (1-2\nu)d_0 \,]\Delta g + \hat{z}^2\, d_0\, \Delta g \Big\}. \end{aligned} \qquad (8.6)$$

The expression for τ_{yz} is obtained if we take the right-hand side of (8.6) and write the partial derivative with respect to y instead of x.

From (8.3), $\sigma_{zz}(x,y,\hat{z}=0) = -p$ and $\tau_{xz}(x,y,\hat{z}=0) = \tau_{yz}(x,y,\hat{z}=0) = 0$ we get

$$\begin{aligned} & a_0 + 2(1-\nu)b_1 = 0, \\ & a_1 + 2(1-\nu)b_2 = -1, \\ & c_0 - (1-2\nu)d_0 = 0, \\ & c_1 - (1-2\nu)d_1 = 0. \end{aligned} \qquad (8.7)$$

Using these relationships we obtain from $\sigma_{zz}(x,y,\hat{z}=h) = \tau_{xz}(x,y,\hat{z}=h) = \tau_{yz}(x,y,\hat{z}=h) = 0$ the equations

$$\begin{aligned} & -\frac{1}{3}\, h^3\, d_0 + h^2\, b_1 = 1, \\ & h^2\, d_0 - 2\, h\, b_1 = 0, \end{aligned} \qquad (8.8)$$

which have the solution

$$b_1 = \frac{3}{h^2},$$

$$d_0 = \frac{6}{h^3}. \tag{8.9}$$

Choosing $c_1 = 0$ and $a_1 = -1$ we get $d_1 = 0$, $b_2 = 0$, and with (8.9) and (8.7) we obtain

$$c_0 = (1-2\nu)\,\frac{6}{h^3},$$

$$a_0 = -\frac{6(1-\nu)}{h^2}. \tag{8.10}$$

a_2, c_2 and d_2 did not appear in the considered boundary conditions and can be set to zero.

For $g(x,y)$ satisfying (8.3) we choose

$$g(x,y) = \frac{p}{64}\,(x^2 + y^2)^2. \tag{8.11}$$

With the aid of the chosen and evaluated quantities we obtain a particular solution for the displacements involving the coordinate $\hat{z}$. Using the substitution $\hat{z} = z + h/2$ we get

$$2\mu u = \frac{p\,x}{4h^3}\,[\,(2-\nu)(4z^3 - 3h^2 z) - 3(1-\nu)(x^2 + y^2)z + \nu h^3\,],$$

$$2\mu v = \frac{p\,y}{4h^3}\,[\,(2-\nu)(4z^3 - 3h^2 z) - 3(1-\nu)(x^2 + y^2)z + \nu h^3\,], \tag{8.12}$$

$$2\mu w = \frac{p}{32h^3}\,[\,-16(1+\nu)z^4 + 48\nu(x^2 + y^2)z^2 + 24h^2(1+\nu)z^2$$
$$+ 6(1-\nu)(x^2 + y^2)^2 - 12h^2\,\nu(x^2 + y^2) - 16h^3(1-\nu)z - h^4(13-3\nu)\,].$$

A look at the according normal stresses

$$\sigma_{xx} = \frac{p}{4h^3}\,[\,4(2+\nu)z^3 - (9+3\nu)x^2 z - (3+9\nu)y^2 z - 3h^2(2+\nu)z - \nu h^3\,] \tag{8.13}$$

and

$$\sigma_{yy} = \frac{p}{4h^3}\,[\,4(2+\nu)z^3 - (3+9\nu)x^2 z - (9+3\nu)y^2 z - 3h^2(2+\nu)z - \nu h^3\,] \tag{8.14}$$

shows that there are constant stress terms in σ_{xx} and σ_{yy}. In order to get a particular solution for which $\sigma_{xx}(x,y,z=0)$ and $\sigma_{yy}(x,y,z=0)$ vanish we have to superpose a membrane solution of the form (4.48) and (4.49). This can be done by choosing $\chi(\zeta) = 0$ and $\Phi(\zeta) = c\zeta$ in equations (4.48) and (4.49). Calculating the coefficient c to be $c = \nu p/8$ we obtain the following membrane displacements and stresses:

$$2\mu u = p\,\frac{\nu(1-\nu)}{4(1+\nu)}\,x,$$

$$2\mu v = p\,\frac{\nu(1-\nu)}{4(1+\nu)}\,y, \tag{8.15}$$

$$2\mu w = -p \frac{\nu^2}{2(1+\nu)} z,$$

$$\sigma_{xx} = \sigma_{yy} = p \frac{\nu}{4},$$

$$\sigma_{zz} = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0. \qquad (8.16)$$

Superposition of the functions (8.12) and (8.15) and omitting the rigid body term $[ph(13-3\nu)/32]$ in the deflection w of relationships (8.12) gives us the following particular solution for a constant normal load p on the upper face of the plate:

$$2\mu u = \frac{p\,x}{4h^3} \left[(2-\nu)(4z^3 - 3h^2 z) - 3(1-\nu)(x^2 + y^2)z + \frac{2\nu h^3}{1+\nu} \right],$$

$$2\mu v = \frac{p\,y}{4h^3} \left[(2-\nu)(4z^3 - 3h^2 z) - 3(1-\nu)(x^2 + y^2)z + \frac{2\nu h^3}{1+\nu} \right], \qquad (8.17)$$

$$2\mu w = \frac{p}{16h^3} \Big[-8(1+\nu)z^4 + 24\nu(x^2 + y^2)z^2 + 12h^2 (1+\nu)z^2$$
$$+ 3(1-\nu)(x^2 + y^2)^2 - 6h^2 \nu(x^2 + y^2) - \frac{8h^3}{1+\nu} z \Big],$$

$$\sigma_{xx} = \frac{p}{4h^3} \left[4(2+\nu)z^3 - (9+3\nu)x^2 z - (3+9\nu)y^2 z - 3h^2(2+\nu)z \right],$$

$$\sigma_{yy} = \frac{p}{4h^3} \left[4(2+\nu)z^3 - (3+9\nu)x^2 z - (9+3\nu)y^2 z - 3h^2(2+\nu)z \right],$$

$$\sigma_{zz} = -\frac{p}{2h^3} (z + h)(2z - h)^2, \qquad (8.18)$$

$$\tau_{xy} = -\frac{3p}{2h^3} (1-\nu)xyz,$$

$$\tau_{xz} = \frac{3p}{4h^3} x(2z - h)(2z + h),$$

$$\tau_{yz} = \frac{3p}{4h^3} y(2z - h)(2z + h).$$

9. Particular solution for a constant body force acting in the thickness direction

A solution for the case of a constant body force $\bar{f}_z$ (i.e. the right-hand side vector in (2.5) and (2.6) becomes $[0, 0, -\bar{f}_z]$) can be written as

$$2\mu u = -\bar{f}_z\, m\, \gamma_2\, \nu\, x\, z,$$

$$2\mu v = -\bar{f}_z (1 - m)\gamma_2\, \nu\, y\, z, \qquad (9.1)$$

$$2\mu w = \bar{f}_z \left\{ m \left[\frac{1}{2}\gamma_2 \nu - (1+\gamma_2) \right] x^2 + (1-m) \left[\frac{1}{2}\gamma_3 \nu - (1+\gamma_3) \right] y^2 + \frac{1}{2}(1-\nu) \left[m\gamma_2 + (1-m)\gamma_3 \right] z^2 \right\}$$

$$\begin{aligned}
\sigma_{xx} &= \bar{f}_z (1-m)\gamma_3 \nu z, \\
\sigma_{yy} &= \bar{f}_z m \gamma_2 \nu z, \\
\sigma_{zz} &= \bar{f}_z [m\gamma_2 + (1-m)\gamma_3] z \\
\tau_{xy} &= 0, \\
\tau_{xz} &= -\bar{f}_z m(1+\gamma_2)x, \\
\tau_{yz} &= -\bar{f}_z (1-m)(1+\gamma_3)y,
\end{aligned} \qquad (9.2)$$

where m, γ_2 and γ_3 are arbitrary constants. In order to have a symmetric structure in the displacements and stresses with respect to the x- and y-coordinates we choose $m = 1/2$, $\gamma_2 = -1/2$ and $\gamma_3 = -1/2$. Since the resulting stresses $\sigma_{zz} = -\bar{f}_z z/2$, $\tau_{xz} = -\bar{f}_z x/4$ and $\tau_{yz} = -\bar{f}_z y/4$ do not satisfy the stress free boundary conditions (4.2) on the upper and lower faces of the plate we have to superpose a solution of the homogeneous differential equations (2.5).

The homogeneous solution part can be constructed with the aid of the stress functions H_0 and H_3 given with equations (4.23) which lead to the stresses σ_{zz}, τ_{xz} and τ_{yz} given in relationships (4.24). After the superposition of the stresses (4.24) and (9.2) we get the boundary conditions

$$\sigma_{zz}(x,y,z=\pm h/2) = (\pm\frac{h}{2}) \left\{ -\frac{1}{2}\bar{f}_z + [\, b_0 - (1-2\nu)a_0 \,]\Delta g + \left[[\, b_1 - (1-2\nu)a_1 \,] - \frac{h^2}{24} [\, b_0 + (1+2\nu)a_0 \,] \right] \Delta\Delta g \right\} = 0,$$

$$\begin{aligned}
\tau_{xz}(x,y,z=\pm h/2) = {} & -\frac{1}{4}\bar{f}_z x - [\, b_0 - (1-2\nu)a_0 \,]\frac{\partial g}{\partial x} \\
& - \left[[\, b_1 - (1-2\nu)a_1 \,] - \frac{h^2}{8} [\, b_0 + (1+2\nu)a_0 \,] \right] \frac{\partial}{\partial x}\Delta g \\
& - \left[[\, b_2 - (1-2\nu)a_2 \,] - \frac{h^2}{8} [\, b_1 + (1+2\nu)a_1 \,] \right.
\end{aligned}$$

$$+ \frac{h^4}{384}\,[\, b_0 + (3+2\nu)a_0 \,]\Bigg]\frac{\partial}{\partial x}\Delta\Delta g = 0, \tag{9.3}$$

$$\tau_{yz}(x,y,z=\pm h/2) = -\frac{1}{4}\bar{f}_z\, y - [\, b_0 - (1-2\nu)a_0 \,]\frac{\partial g}{\partial y}$$

$$- \Bigg[[\, b_1 - (1-2\nu)a_1 \,] - \frac{h^2}{8}\,[\, b_0 + (1+2\nu)a_0 \,]\Bigg]\frac{\partial}{\partial y}\Delta g$$

$$- \Bigg[[\, b_2 - (1-2\nu)a_2 \,] - \frac{h^2}{8}\,[\, b_1 + (1+2\nu)a_1 \,]$$

$$+ \frac{h^4}{384}\,[\, b_0 + (3+2\nu)a_0 \,]\Bigg]\frac{\partial}{\partial y}\Delta\Delta g = 0,$$

Here we should recall that the function g has to fulfill $\Delta\Delta\Delta g = 0$, which ensures that the functions H_0 and H_3 of the form (4.23) are harmonic. Choosing

$$\Delta\Delta g = 1 \tag{9.4}$$

we satisfy $\Delta\Delta\Delta g = 0$ and obtain the properties

$$\frac{\partial}{\partial x}\Delta\Delta g = 0 \tag{9.5}$$

and

$$\frac{\partial}{\partial y}\Delta\Delta g = 0, \tag{9.6}$$

which simplify the boundary conditions (9.3). If we choose now g as

$$g = \frac{1}{64}(x^2 + y^2)^2 \tag{9.7}$$

we get

$$\Delta g = \frac{1}{4}(x^2 + y^2), \tag{9.8}$$

and from this we obtain the following relations, which are useful for the treatment of the boundary conditions for τ_{xz} and τ_{yz}:

$$\frac{\partial}{\partial x}\Delta g = \frac{1}{2}x, \tag{9.9}$$

$$\frac{\partial}{\partial y}\Delta g = \frac{1}{2}y. \tag{9.10}$$

Using relationships (9.4) to (9.10) we obtain from the stress boundary conditions on the plate faces $z = \pm h/2$ the equations

$$b_0 - (1-2\nu)a_0 = 0,$$

$$-\frac{1}{2}\bar{f}_z + [\, b_1 - (1-2\nu)a_1 \,] - \frac{h^2}{12}a_0 = 0, \qquad (9.11)$$

$$-\frac{1}{4}\bar{f}_z - \frac{1}{2}[\, b_1 - (1-2\nu)a_1 \,] + \frac{h^2}{8}a_0 = 0.$$

We can set

$$a_1 = 0 \qquad (9.12)$$

and obtain

$$b_1 = \bar{f}_z,$$

$$a_0 = \frac{6}{h^2}\bar{f}_z, \qquad (9.13)$$

and

$$b_0 = (1-2\nu)\frac{6}{h^2}\bar{f}_z.$$

The coefficients a_2 and b_2 did not appear in the boundary conditions (9.3) and can be set to zero.

With the function g given in equation (9.7) and the coefficients calculated above our stress functions H_0 and H_3 of the form given with relationships (4.23) are well defined and we can derive the following particular solution for a constant body force $\bar{f}_z$:

$$2\mu u = \frac{\bar{f}_z}{4h^2}[\, 4(2-\nu)z^3 - 3(1-\nu)(x^2 + y^2)z - 2h^2(1-\nu)z \,]x,$$

$$2\mu v = \frac{\bar{f}_z}{4h^2}[\, 4(2-\nu)z^3 - 3(1-\nu)(x^2 + y^2)z - 2h^2(1-\nu)z \,]y, \qquad (9.14)$$

$$2\mu w = \frac{\bar{f}_z}{16h^2}[\, -8(1+\nu)z^4 + 24\nu(x^2 + y^2)z^2 + 4h^2(1+\nu)z^2$$

$$- 2h^2(1+\nu)(x^2 + y^2) + 3(1-\nu)(x^2 + y^2)^2 \,],$$

$$\sigma_{xx} = \frac{\bar{f}_z}{4h^2}[\, 4(2+\nu)z^3 - (9+3\nu)x^2z - (3+9\nu)y^2z - h^2(2+\nu)z \,],$$

$$\sigma_{yy} = \frac{\bar{f}_z}{4h^2}[\, 4(2+\nu)z^3 - (3+9\nu)x^2z - (9+3\nu)y^2z - h^2(2+\nu)z \,],$$

$$\sigma_{zz} = -\frac{\bar{f}_z}{2h^2}z(2z-h)(2z+h), \qquad (9.15)$$

$$\tau_{xy} = -\frac{3\bar{f}_z}{2h^2}(1-\nu)xyz,$$

$$\tau_{xz} = \frac{3\bar{f}_z}{4h^2}x(2z-h)(2z+h)$$

$$\tau_{yz} = \frac{3\bar{f}_z}{4h^2}y(2z-h)(2z+h)$$

10. Conclusions

Decomposing the 3-dimensional displacement field into different parts a systematic way has been shown in order to get a plate formulation without ad hoc assumptions. From the procedure of separating functions of the thickness variable z from functions of the remaining two other space coordinates x, y we obtained for the bending and membrane cases the characteristic differential equations which are the biharmonic equation and Helmholtz-equations with real and complex wave parameters, respectively.

The use of solution series of the characteristic differential equations for the representation of the displacements and stresses ensures the satisfaction of the Navier-equations and the equilibrium equations, respectively. Moreover, the constructed representation ensures the satisfaction of the stress boundary conditions on the upper and lower faces of the plate. For the solution of a concrete plate problem with the aid of the presented plate formulation involving series of solution functions of the mentioned characteristic differential equations the task remains to satisfy the actual lateral boundary conditions for the given plate geometry.

If it is not possible to find an analytical solution for a given plate geometry one can look for a numerical scheme in which the remaining lateral plate boundary conditions are satisfied in a (defined) optimal sense to obtain an approximate solution. Examples for solution methods which are based on the use of a system of linear independent functions satisfying a priori the Navier-equations and the equilibrium equations, respectively, are given in the references [14,15]

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