Title
APPLICATION OF GENERAL THEORETICAL PRINCIPLES TO EXPERIMENTS - Lecture 5. APPLICATION OF CAUSALITY TO SCATTERING

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APPLICATION OF GENERAL THEORETICAL PRINCIPLES TO EXPERIMENTS - Lecture 5

APPLICATION OF CAUSALITY TO SCATTERING

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I. Introduction

By making use of certain basic physical principles, it is possible to say a great deal about the outcome of a particular scattering experiment. For example, the principle that the interaction between particles must be invariant to a translation of the coordinate system may be used to prove that the total momentum of the system must be conserved. Another basic principle which has received much attention recently is the causality principle, which states that no signal may be transmitted with a speed exceeding the speed of light. Gell-Mann, Goldberger and collaborators have made use of the causality principle to give a quantum mechanical derivation of some dispersion relations, which had previously been derived, using classical methods, by Kramers, Kroning, and other authors.[1] The essential features of this quantum mechanical derivation will be discussed later in these lectures.

II. The Scattering Matrix.

To simplify the discussion we shall discuss only reactions in which there are two particles in the initial state, and two particles in the final

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1 See, for example, M. L. Goldberger, Phys. Rev. 99, 979 (1955), and Gell-Mann, Goldberger, and Thirring, Phys. Rev. 95, 1612 (1954). Their works contain references to previous discussions of dispersion relations.

state. (Examples: $\pi^- + p \rightarrow \pi^- + p$; $\pi^- + p \rightarrow \Lambda^0 + \theta^0$.) We shall treat the masses of the heavier initial particle and the heavier final particle as if they were infinite, though this is not necessary for most of the conclusions derived. The possible initial and final states of the system will be denoted by one of the following two sets of quantum numbers,

(a) $(\alpha , \vec{p})$ and

(b) $(\alpha , E, \ell, \ell_z)$,

where α denotes the nature of the particles and their "internal quantum numbers" such as spin and isotopic spin. The vector $\vec{p}$ denotes the momentum of the light particle, and E, ℓ, and ℓ_z represent the energy, orbital angular momentum, and z component of the orbital angular momentum of the light particle. Primes will be used to denote the final states.

Let the heavy target particle be located at the origin of the coordinate system. Then, in the region of large $\vec{r}$, we may express the wave function of the system ψ as a superposition of an incoming wave, $\psi^{(-)}$, and an outgoing wave $\psi^{(+)}$. Each of these waves may be expanded in the eigenstates of the $(\alpha , E, \ell, \ell_z)$ representation. The complex matrix elements of the scattering matrix S are defined by the equation,

$$\psi = \sum_{q,E} A_{q,E} \left(\psi_{q,E}^{(-)} - \sum_{q'} S_{qq'} \psi_{q'E}^{(+)} \right) . \qquad (1)$$

The quantum numbers q and q' denote all the quantum numbers other than the energy, i.e., $q = (\alpha , \ell, \ell_z)$; $q' = (\alpha ', \ell', \ell'_z)$. We are defining S only when energy is conserved.

The elements $S_{qq'}$ are independent of the coefficients $A_{q,E}$, which give the form of the arbitrary incoming wave. In actual practice, $\psi^{(-)}$ is the incoming part of a plane wave.

III. Relation of the S Matrix to Scattering Cross Sections.

If there is no interaction between the projectile and target particle, the requirement that the wave function be finite at the origin may be used to show that

$$\psi_{q,E}^{(-)} = -\psi_{q'E}^{(+)} \quad . \tag{2}$$

The wave function will be a plane wave, which, expanded in spherical waves, will have the form

$$\psi = e^{i\vec{k}_\alpha \cdot \vec{r}} = \sum_{qE} A_{q\,E} \left(\psi_{q\,E}^{(-)} - \psi_{q\,E}^{(+)} \right) . \tag{3}$$

Thus in this case the S matrix is the unit matrix,

$$S = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix} .$$

We see from the above argument that the matrix element

$$(S - 1)_{qq'} = S_{qq'} - \delta_{qq'}$$

is proportional to the amplitude for the reaction $q \rightarrow q'$. We define the scattering amplitude $T_{qq'}$ for the transition $q \rightarrow q'$ by

$$T_{qq'} = \frac{(S - 1)_{qq'}}{2ik} \tag{4}$$

where $k = p/\hbar$, p being the initial momentum.

The cross section for the process $q \rightarrow q'$ is then given by

$$\sigma_{qq'} = 4\pi(2\ell + 1) \left| T_{qq'} \right|^2 \quad . \tag{5}$$

IV. Unitarity of the S-Matrix

A well-known physical law states that if a beam of particles is used to bombard a target particle, each particle in the beam must do something, either miss the target, be scattered, or induce some other reaction. The number of final states must equal the number of initial states, which may be expressed by the equation

$$\sum_i \left| A_i \right|^2 = \sum_j \left| B_j \right|^2$$

where we are letting one index represent a complete set of states, and A and B refer to the incoming and outgoing states respectively. In this notation the S matrix is given by

$$B_j = \sum_k S_{kj} A_k \quad .$$

Taking the complex conjugate we get

$$B_j^* = \sum_\ell S_{\ell j}^* A_\ell^* \quad .$$

Substituting these expressions for B_j and B_j^* our equation becomes

$$\sum_i A_i A_i^* = \sum_{jk\ell} A_k A_\ell^* S_{kj} S_{\ell j}^* \qquad (6)$$

Since the coefficients representing the incoming wave A_i are arbitrary, the above relation necessitates

$$\sum_j S_{kj} S_{\ell j}^* = \delta_{k\ell} \quad . \qquad (7)$$

This equation expresses the unitary condition on the S-matrix.

If only ~~one reaction~~ elastic scattering is possible and the spins of the particles may be neglected, the S matrix becomes very simple. There is no mixing between states [illegible] angular momentum and [illegible]

[illegible]

magnitude one. We write the matrix

$$S = \begin{pmatrix} e^{2i\delta_0} & 0 & \cdots \\ 0 & e^{2i\delta_1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

The real number δ_ℓ is the scattering phase shift corresponding to the orbital angular momentum ℓ. From Eq. (4) the scattering amplitude for a given ℓ is

$$T_\ell = \frac{e^{i\delta} \sin\delta}{k} \tag{8}$$

and the cross section for a given ℓ is

$$\sigma_\ell = \frac{4\pi(2\ell + 1)}{k^2} \sin^2 \delta_\ell \tag{9}$$

Now let us investigate the consequences of unitarity when there are two possible final states which may be reached from a particular initial state. For example we consider the initial state to be $\pi^- + P$ in a S state, and consider only the two final states $\pi^- + P$ and $\pi^0 + N$, both also in S states. We denote these states by the numbers 1 and 2 respectively. Unitarity requires that

$$\left| S_{11} \right|^2 + \left| S_{12} \right|^2 = 1 . \tag{10}$$

The total S wave cross section for the two possible final states is

$$\sigma_t = \frac{\pi}{k^2} \left[\left| S_{11} \right|^2 + \left| S_{12} \right|^2 + 1 - 2 \operatorname{Re} S_{11} \right] . \tag{11}$$

As can be seen from Eq. (4) and (5), and the fact that

$\left| S_{11} - 1 \right|^2$ [illegible] $2 \operatorname{Re} S_{11}$

Combining these two equations we see that

$$\sigma_T = \frac{\pi}{k^2}\left[-2\ \mathrm{Re}\ S_{11} + 2\right] \tag{12}$$

and, using Eq. (4)

$$\sigma_T = \frac{4\pi}{k}\ \mathrm{Im}\ T_{11} \quad . \tag{13}$$

Thus, the imaginary part of the amplitude for the elastic process is related to the total cross section for all processes. The above result is independent of the number of final states involved.

A similar relation holds if we use the plane wave $(\alpha, \vec{p})$ representation. In this representation (again neglecting spins) the initial state is completely in the state

$$\vec{p} = \vec{i}_z\, p \quad ,$$

where $\vec{i}_z$ is a unit vector in the z direction. The total cross section then is proportional to the sum $\sum_{\alpha' p'} \left| (S - 1)_{\alpha p, \alpha' p'} \right|^2$, where $\alpha' p'$ range over all possible final states. The unitarity condition is again

$$\sum_{\alpha' p'} \left| S_{\alpha \vec{p}, \alpha' \vec{p}'} \right|^2 = 1 .$$

The matrix element of (S - 1) differs from that of S only when the initial and final states are the same, which in this case refers to scattering in the forward direction. So the unitarity condition in this case gives a relation similar to Eq. (13), except that here the imaginary part of the forward elastic scattering amplitude is related to the total cross section for all processes. The relation is

$$\sigma_T = \frac{4\pi}{k}\ \mathrm{Im}\ T_f \quad , \tag{14}$$

where T_f, the forward scattering amplitude is related to the differential cross section in the forward direction by

$$\frac{d\sigma}{d\Omega} = \left| T_f \right|^2 . \tag{15}$$

This relationship shows that whenever any kind of reaction takes place, there must be some scattering in the forward direction. This result is well-known in the theory of optical diffraction.

It should be emphasized that the unitary property of the S-matrix, and the relation Eq. (14), are correct independently of the assumption that the only possible final states are two particle states. If final states involving more particles are possible, the total cross section in Eq. (14) must include transitions to these states.