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DUALITY AND HIGH ENERGY BEHAVIOR OF A REGGEON-
REGGEON AMPLITUDE IN THE DUAL RESONANCE MODEL*

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May 20, 1976

ABSTRACT: The analytic structure and high energy behavior of the 2-reggeon $\rightarrow$ 2-particle amplitude is studied using a double helicity pole limit of the B_6 function. In the high energy limit the amplitude gets a contribution from the exchange of a new Regge trajectory β . The properties of β exchange and its implications for Regge theory and phenomenology are discussed.

Because of the presence of cuts in the Toller variables, Finite Energy Sum Rules for reggeon-reggeon amplitudes are in general not saturated by s-channel dynamical singularities (resonances, thresholds). The additional contributions are, however, shown to be absent for certain discontinuities of the six-point amplitude which therefore have ordinary duality properties. The theoretical discussion is supplemented by numerical examples of the B_6 function.

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1. INTRODUCTION

From recent calculations using various types of multi-Regge cluster models it has become apparent that an understanding of the analytic properties of reggeon-reggeon amplitudes is important in evaluating unitarity integrals in these models^{1,2,3,4}). Specifically, one needs to know whether the production amplitude of a cluster (fig. 1a) may be approximated by the crossed-channel Regge exchange (fig. 1b), in the semilocal duality sense familiar from $2 \rightarrow 2$ reactions. Since semilocal duality usually implies the existence of Finite Energy Sum Rules (FESR's), this question may be answered by studying the analytic structure of the reggeon-reggeon amplitude. Conversely, for amplitudes that do satisfy FESR's, experience has shown that the duality relation holds semilocally. A first investigation of the validity of FESR's has already been made⁵), based on the properties of a certain class of Feynman diagrams.

In this paper we make a complete study, within the Dual Resonance Model (DRM), of the analytic properties and high-energy behavior of the 2-reggeon $\rightarrow$ 2-particle amplitude in the helicity-pole limit shown in fig. 2a. We then discuss to what extent the s-channel resonances (fig. 2b) are dual to t-channel Regge exchange (fig. 2c). The DRM is a convenient model amplitude as it has a self-consistent singularity structure and Regge behavior in all channels, the functional form of the amplitude being nevertheless relatively simple. The DRM is, of course, particularly suitable for investigations of duality in many-body amplitudes. In cases where also other models, such as Feynman diagrams, have been studied, the general conclusions concerning the analytic structure and validity of FESR's have turned out to be model-independent.

The 2-reggeon $\rightarrow$ 2-particle amplitude in fig. 2a is a special case of the 4-reggeon amplitude discussed above (fig. 1). We shall here address ourselves to this simpler amplitude, as many novel complications in analytic structure occur already at this stage. This investigation may also be regarded as a generalization of earlier work^{6,7,8)} on duality in reggeon-particle amplitudes, which has led to many phenomenological verifications⁹⁾ of duality in reggeon amplitudes. Similarly, one may hope that the sum rules derived below will have observable consequences for the strength and spin structure of the 2-reggeon-resonance coupling in fig. 2b.

The essential new feature in the analytic structure of a reggeon-reggeon amplitude, as compared to the previously studied reggeon-particle amplitudes, is the existence of a nontrivial constraint on the invariants, arising from the four-dimensionality of space-time. The exact form of this constraint is derived in the next section. It relates the s-channel c.m. energy s to the variables that are asymptotic in the limit of fig. 2a. Thus the singularities of the amplitude in these variables are reflected onto the s-plane, and in general contribute to the FESR. However, in any practical application of the FESR's one is mainly concerned about the s-channel dynamical singularities, which in the DRM are represented by resonance poles. One must therefore try to write the FESR for functions that are free of singularities in the asymptotic variables, but whose s-channel resonance poles are the same as those of the full amplitude. As we shall see below, this can in general be done for the s-channel, but not for the u-channel resonances.

The sum rules we derive turn out to have a r.h.s. contribution corresponding to the exchange of a new family of Regge trajectories. Due to their rather unusual properties, these new trajectories have been overlooked in previous derivations of Regge limits in the DRM¹⁰⁾. However, they become manifest in a detailed study of the analytic properties of the amplitude, such as the present one. The new parent trajectory β is related to the usual trajectory α according to $\beta = \frac{1}{2} \alpha - \frac{1}{2}$. Thus β gives the leading asymptotic behavior for $\alpha < -1$. It has positive signature and factorizes. The coupling of β to any two-particle state vanishes. It can therefore contribute only in a three-particle channel of B_6 as indicated in fig. 2c. A brief account of the new trajectories has already been given¹¹⁾. The detailed derivation of the asymptotic limits involving β -exchange are given below and in appendices A and B. In sect. 7 we also comment on the existence of β -exchange in amplitudes which correspond more closely to the real world than does the DRM. In particular, we argue that since more realistic amplitudes should contain negative-signature (ρ and ω) trajectories with intercepts greater than zero, we may expect them to contain higher-lying β -trajectories than those in the DRM, of the form $\beta = \frac{1}{2} \alpha$, which would have correspondingly greater phenomenological implications.

The paper is organized as follows. In sect. 2 we derive the explicit form, in the double helicity pole limit of fig. 2a, of the four-dimensionality constraint that relates the nine linearly independent invariants of any six-point function. In sect. 3 we study the high energy behavior and analyticity properties of the $B_6(s,t)$ amplitude, i.e., the B_6 function that corresponds to the ordering of

external lines in fig. 2. We discuss the complications that in general arise in sum rules and show that a resonance-saturated FESR may nevertheless be written down. This FESR is generalized to an arbitrary planar amplitude in sect. 4. The $B_6(u,t)$ and $B_6(s,u)$ terms are studied in sects. 5 and 6, respectively. In particular, we show why there can be no simple FESR that involves resonances in the u -channel of the reggeon-reggeon amplitude. In sect. 7 we discuss several properties of the new β Regge exchange, based on the calculations of high energy limits in sects. 3, 5 and the appendices. Readers who are mainly interested in this aspect of the paper can proceed directly to sect. 7. Section 8 contains some further remarks concerning the consequences of the β reggeons, as well as our conclusions with regard to the FESR's. In appendix A some asymptotic limits involving the confluent hypergeometric function ${}_2F_2$ are derived. The calculation of a single Regge limit of the B_6 function that has a contribution from β exchange is presented in appendix B.

2. THE KINEMATIC CONSTRAINTS

In the limit of fig. 2a, the amplitude will in general retain a nontrivial dependence on the finite ratios s_{345}/s_{12} , s_{234}/s_{56} and $\kappa = s_{345} s_{234}/s_{34}$. The first two ratios may be thought of as specifying the azimuthal angles of the momenta of 2,3 and 4,5 with respect to the $\alpha_a + \alpha_b \rightarrow 1 + 6$ reaction plane. The value of κ is then determined by the four-dimensionality (Gram determinant) constraint that connects the nine linearly independent invariants of a six-point function. Now, in the $s = s_{61} \rightarrow \infty$ Regge limit of fig. 2c, all three ratios behave like s . In deriving the FESR through a dispersion relation in s we therefore need to know the analytic

structure of the amplitude when the ratios s_{345}/s_{12} , s_{234}/s_{56} and κ/s are held fixed. Thus the singularities in s_{345}/s_{12} , etc., are reflected onto the s plane, and give rise to new contributions to the FESR, which are not due to resonances (or thresholds) in s . It was shown in ref. 7) that the contributions due to the singularities in s_{345}/s_{12} and s_{234}/s_{56} vanish in the special case of a helicity-pole (HP) limit, defined by

$$s_{12}/s_{345} = s_{56}/s_{234} = 0 \quad (1)$$

in fig. 2a. Thus from here on we shall restrict ourselves to the helicity pole configuration of eq. (1).

The amplitude in fig. 2a is, however, more general than the one considered in ref. 7), and so the conditions (1) do not yet eliminate all extra singularities in s . The four-dimensionality constraint determines κ as a function of s , so that the singularities of the amplitude in κ must be regarded as singularities in s . The simplest method*) of finding the functional dependence of κ on

*) This is a straightforward extension of the corresponding calculation in ref. 8).

s is to evaluate the four-momenta in terms of the invariants in the c.m. of the t -channel process $4 + 5 + 6 \rightarrow 1 + 2 + 3$. Let

$$\begin{aligned}
 p_b &\equiv p_4 + p_5 = \left(\frac{t + t_b - m_6^2}{2\sqrt{t}}, 0, 0, p_t \right) \\
 p_6 &= \left(\frac{t - t_b + m_6^2}{2\sqrt{t}}, 0, 0, -p_t \right) \\
 p_a &\equiv p_2 + p_3 = \left(\frac{t + t_a - m_1^2}{2\sqrt{t}}, q_t \sin \theta_t, 0, q_t \cos \theta_t \right) \\
 p_1 &= \left(\frac{t - t_a + m_1^2}{2\sqrt{t}}, -q_t \sin \theta_t, 0, -q_t \cos \theta_t \right).
 \end{aligned} \tag{2}$$

Then to leading order in s_{345} the momentum of particle 3 is, in the limit (1),

$$p_3 = \frac{s_{345}}{2p_t \sin \theta_t} (0, -\cos \theta_t, \pm i, \sin \theta_t). \tag{3}$$

Next we rotate the coordinate system by an angle θ_t around the y-axis. In the new (primed) coordinate system the particle 3 momentum is

$$p'_3 = \frac{s_{345}}{2p_t \sin \theta_t} (0, -1, \pm i, 0). \tag{4}$$

Since $\vec{p}'_a$ lies along the $+z'$ axis, and $\vec{p}'_b$ makes an angle $-\theta_t$ with it, we can calculate p'_4 analogously to p_3 above. Thus

$$p'_4 = \frac{s_{234}}{2q_t \sin \theta_t} (0, \cos \theta_t, \pm i, \sin \theta_t). \tag{5}$$

To leading order we have $s_{34} = -2p'_3 \cdot p'_4$, hence

$$\begin{aligned}
 s_{345} s_{234} / s_{34} &= 2 p_t q_t (\cos \theta_t)^{(+)} 1 \\
 &= s - m_1^2 - m_6^2 + \frac{1}{2t} (t + m_1^2 - t_a)(t + m_6^2 - t_b) \\
 &\quad \left(\frac{+}{-} \right) \frac{1}{2t} [\lambda(t, t_a, m_1^2) \lambda(t, t_b, m_6^2)]^{\frac{1}{2}}
 \end{aligned} \tag{6}$$

where $\lambda(a, b, c) = (a + b - c)^2 - 4ab$. The minus sign in eq. (6) is the physical choice when $(m_1^2 - t_a)(m_6^2 - t_b) > 0$, since κ should be finite at $t = 0$.

Asymptotically as $s \rightarrow \infty$, eq. (6) assumes the familiar form

$$\eta \equiv \frac{s_{345} s_{234}}{s_{34} s} = 1. \tag{7}$$

The fact that κ , according to eq. (6), is a linear function of s even for finite s is crucial for the validity of the FESR's we shall derive. It means that no new singularities are introduced in the amplitude by the transformation from κ to s .

From the point of view of the resonance production process $3 + 4 \rightarrow 2 + s + 5$ in fig. 2b, eq. (6) fixes the transverse momentum $p_\perp$ of the resonance s . In the Regge limit we are studying,

$$s_{345} s_{234} / s_{34} = s + p_1^2$$

hence

$$\begin{aligned}
 p_1^2 &= -m_1^2 - m_6^2 + \frac{1}{2t} (t + m_1^2 - t_a)(t + m_6^2 - t_b) \\
 &\quad \left(\frac{+}{-} \right) \frac{1}{2t} [\lambda(t, t_a, m_1^2) \lambda(t, t_b, m_6^2)]^{\frac{1}{2}}.
 \end{aligned} \tag{8}$$

In general, this value of $p_{\perp}^2$ will not be in the physical region of the $3 + 4 \rightarrow 2 + s + 5$ process, and $p_{\perp}^2$ can in fact be negative. Fortunately, for the physically interesting range of the variables in eq. (8), $p_{\perp}^2$ is close to the physical region. Thus, with the minus sign in eq. (8) as the physical choice, $p_{\perp}^2$ is positively semi-definite if $t \leq 0$ and $m_1 = m_6$. Furthermore, $p_{\perp}^2 = 0$ if any one of the following conditions hold:

- (i) $m_1 = m_6$ and $t_a = t_b$
- (ii) $m_1 = m_6 = 0$, and $(t - t_a)(t - t_b) \geq 0$
- (iii) $t_a = t_b = 0$, and $(m_1^2 - t)(m_6^2 - t) \geq 0$.

Hence, in practical applications $p_{\perp}^2$ will assume small positive values.

3. THE $B_6(s, t)$ TERM

Just as in the case of the B_4 function, there are three distinct B_6 functions, corresponding to different orderings of the external lines, that contribute to the reggeon amplitude we are studying. We label them by the channels containing resonances as $B_6(s, t)$, $B_6(u, t)$ and $B_6(s, u)$. Here the channels are defined by their particle content in fig. 2 as $s = 16$, $t = 123$, $u = 236$. As we shall see, these three B_6 functions have quite different singularity structures in the asymptotic variables. Since, according to our discussion in sect. 2, such singularities are important for the FESR's we want to derive, we shall discuss each B_6 term separately, beginning with the $B_6(s, t)$ term shown in fig. 2a.

The integral representation of B_6 in the Regge limit of fig. 2a has been given in ref. ¹²⁾. Imposing the helicity pole constraint (1) we get ^{*}

^{*}) The amplitude is well-behaved in the helicity pole limit only provided $\alpha_a - \alpha_t > 0$ and $\alpha_b - \alpha_t > 0$. This restriction is not important for our present considerations, however. For a discussion of this point see sect. 3.1 of ref. ⁷⁾.

$$B_6(s, t) = (-\alpha_{345})^{\alpha_a} (-\alpha_{234})^{\alpha_b} \int_0^1 dx x^{-\alpha_s-1} (1-x)^{-\alpha_t+\alpha_a+\alpha_b-1} \\ \times \int_0^{\infty} dz_1 dz_2 z_1^{-\alpha_a-1} z_2^{-\alpha_b-1} \exp \left[-z_1 - z_2 + \frac{z_1 z_2}{\kappa(1-x)} \right] \quad (9)$$

where ^{*})

$$\kappa = \alpha_{345} \alpha_{234} / \alpha_{34} \quad (10)$$

^{*}) The κ defined in eq. (10), which we shall use in the rest of this paper, differs by a factor $\alpha' =$ trajectory slope from the κ referred to in sect. 2.

From eq. (9) we can see that $B_6(s, t)$ has a singularity at $\kappa = 0$. Through the four-dimensionality constraint (6) this singularity will appear also in the α_s plane. Thus it is important to display explicitly the analytic structure of B_6 in κ . This can be achieved by applying the identity

$$e^z = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\lambda \Gamma(-\lambda) (-z)^{\lambda} \quad (11)$$

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to the last factor in eq. (9), to get

$$E_6(s, t) = (-\alpha_{345})^{\alpha_a} (-\alpha_{234})^{\alpha_b} \frac{1}{2\pi i} \int_{-100}^{100} d\lambda \Gamma(-\lambda) \Gamma(\lambda - \alpha_a) \Gamma(\lambda - \alpha_b) \times \frac{\Gamma(-\alpha_s) \Gamma(-\alpha_t + \alpha_a + \alpha_b - \lambda)}{\Gamma(-\alpha_s - \alpha_t + \alpha_a + \alpha_b - \lambda)} (-\kappa)^{-\lambda}. \quad (12)$$

The integration contour can be closed to the left, giving

$$B_6(s, t) = (-\alpha_{345})^{\alpha_a} (-\alpha_{234})^{\alpha_b} \Gamma(-\alpha_a) \Gamma(-\alpha_b) \left[(-\kappa)^{-\alpha_a} \gamma_{ab} + (-\kappa)^{-\alpha_b} \gamma_{ba} \right] \quad (13a)$$

where

$$\gamma_{ab} = \frac{\Gamma(\alpha_a - \alpha_b)}{\Gamma(-\alpha_b)} \frac{\Gamma(-\alpha_s) \Gamma(-\alpha_t + \alpha_b)}{\Gamma(-\alpha_s - \alpha_t + \alpha_b)} \times {}_2F_2(-\alpha_a, -\alpha_t + \alpha_b; 1 - \alpha_a + \alpha_b, -\alpha_s - \alpha_t + \alpha_b; -\kappa) \quad (13b)$$

$$\gamma_{ba} = \gamma_{ab}(a \leftrightarrow b).$$

Here ${}_2F_2$ denotes the confluent hypergeometric function¹³⁾

$${}_2F_2(a, b; c, d; x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+a) \Gamma(n+b) \Gamma(c)}{\Gamma(a) \Gamma(b) \Gamma(n+c)} \frac{\Gamma(d)}{\Gamma(n+d)} \frac{x^n}{n!} \quad (14)$$

which is entire in its argument x .

Equation (13) represents a limit of $B_6(s, t)$ in which all the large variables tend to $-\infty$. To reach the limit appropriate for the

physical reaction $3 + 4 \rightarrow 2 + 1 + 6 + 5$, some of the variables have to be analytically continued to $+\infty + i\epsilon$. This can be done in the standard way^{*)}. Adding also the diagrams with twists on the reggeons

*) As we have noted previously¹¹⁾, this procedure of finding the asymptotic behavior by analytic continuation of a limit taken in another direction is not always correct. We do not expect any difficulties in the present situation, however.

α_a, α_b we find the physical amplitude with signaturized reggeons to be

$$B_6^{\tau_a \tau_b}(s, t) = \alpha_{345}^{\alpha_a} \alpha_{234}^{\alpha_b} \Gamma(-\alpha_a) \Gamma(-\alpha_b) \xi_a \xi_b \times \left[\xi_b^{-1} \xi_{ab} \kappa^{-\alpha_a} \gamma_{ab} + \xi_a^{-1} \xi_{ba} \kappa^{-\alpha_b} \gamma_{ba} \right] \quad (15)$$

where the signature factors are

$$\xi_a = \tau_a + e^{-i\pi\alpha_a} \quad (16)$$

$$\xi_{ab} = \tau_a \tau_b + e^{-i\pi(\alpha_b - \alpha_a)}$$

From eq. (13) or, equivalently, eq. (15) we can see that there is no FESR that relates the resonances in $B_6(s, t)$ to the asymptotic $(\alpha_s \rightarrow \infty)$ discontinuity in α_s . If in fact we write an FESR for $B_6(s, t)$, then the cut $0 \leq \kappa \leq \infty$, which by eq. (6) must be regarded as a cut in α_s , will contribute to the α_s -discontinuity at both finite and asymptotic values of α_s ^{*)}. Consequently no exact

*) This would be analogous to the Continuous Moment Sum Rules of four-point amplitudes. Since in the present case we believe that any practical application would involve only the discontinuity due to the resonances in s we shall not consider this possibility further.

statement can be made about the part of the discontinuity that comes from the resonances. As we shall next discuss, however, one may still expect a semilocal duality relation between the resonances and the Regge term to be approximately valid under some circumstances.

In order to get a duality constraint on the resonance contributions alone, we have to write the FESR for a function that has the same resonance poles as $B_6(s, t)$, but no singularities in κ . This can be done by noting that the discontinuity of $B_6(s, t)$ in the asymptotic variable α_{234} is, from eq. (13),

$$\begin{aligned} D_{234} B_6(s, t) &= B_6(s, t; \alpha_{234} \rightarrow \infty + i\epsilon) - B_6(s, t; \alpha_{234} \rightarrow \infty - i\epsilon) \\ &= (-\alpha_{345})^{\alpha_a} \alpha_{234}^{\alpha_b} \Gamma(-\alpha_a) \Gamma(-\alpha_b) \kappa^{-\alpha_a} 2i \sin \pi(\alpha_a - \alpha_b) \nu_{ab}. \end{aligned} \quad (17)$$

Here all other asymptotic variables except α_{234} approach $-\infty$. Just as was done in eq. (15) for the amplitude, we could of course analytically continue the discontinuity to positive values of these variables. Since this would not change any of our conclusions, we shall for simplicity discuss the uncontinued form (17) of the discontinuity.

Comparing eqs. (13) and (17) we see that the poles in α_s coming from ν_{ab} have the same residues in $B_6(s, t)$ and in $D_{234} B_6(s, t)$, apart from the constant factor $2i \sin \pi(\alpha_a - \alpha_b)$. Furthermore, $\kappa^{\alpha_a} D_{234} B_6(s, t)$ has no singularities in κ , and thus satisfies an FESR with only resonance contributions to the low-energy integral. This leads us to expect that the resonances in $\kappa^{\alpha_a} D_{234} B_6(s, t)$ will be semilocally dual to the extrapolated Regge term. To the extent that all resonances at a given mass average the

Regge behavior in $\kappa^{\alpha_a} D_{234} B_6(s, t)$, the same will be true for the resonances in $D_{234} B_6(s, t)$, since the resonances and Regge term are scaled by the same factor $\kappa^{-\alpha_a}$. In this sense the resonances in the first term of $B_6(s, t)$ (eq. (13a)) are semilocally dual to the leading t-channel Regge exchange. By repeating the above argument for the discontinuity $D_{345} B_6(s, t)$ we may conclude that the full $B_6(s, t)$ has resonance-Regge duality, both before (eq. (13a)) and after (eq. (15)) the analytic continuation.

It should be borne in mind, however, that strict semilocal duality, as we used it above, can at best hold only over a limited range of momentum transfers. Consider, for example, a value of α_t such that the FESR for $\kappa^{\alpha_a} D_{234} B_6(s, t)$ is superconvergent. Then the sum over resonances will equal zero not because each resonance vanishes separately, but because of cancellations between neighboring resonance contributions. A scaling factor $\kappa^{-\alpha_a}$ will destroy this cancellation, so that the sum over resonances in $D_{234} B_6(s, t)$, or in $B_6(s, t)$, will not vanish. Thus the fact that there is no exact resonance-saturated sum rule for $B_6(s, t)$ means that duality, as we know it for four-point amplitudes must be applied with caution to the reggeon-reggeon amplitudes.

Let us now examine in more detail the exact sum rule for $\kappa^{\alpha_a} D_{234} B_6(s, t)$ or, equivalently, for ν_{ab} . The resonance contributions can be obtained directly from eq. (13b). The Regge behavior of ν_{ab} as $\alpha_s \rightarrow \infty$ may be found by replacing the ${}_2F_2$ function by its asymptotic form, given in eq. (A12). Deriving the FESR in the usual way through a contour integral in α_s we get the (zeroth moment) sum rule for ν_{ab} :

$$\sum_{n=0}^N \frac{\Gamma(\alpha_a - \alpha_b)}{\Gamma(-\alpha_b)} \frac{\Gamma(n+1+\alpha_t - \alpha_b)}{n! \Gamma(1+\alpha_t - \alpha_b)} 2^{F_2(-\alpha_a, -\alpha_t+\alpha_b; 1-\alpha_a+\alpha_b, -n-\alpha_t+\alpha_b; -\kappa)}$$

$$= \int_0^N dn \left[\frac{\sin \pi \alpha_b}{\sin \pi(\alpha_b - \alpha_a)} \frac{\Gamma(1+\alpha_t)}{\Gamma(1+\alpha_t - \alpha_a) \Gamma(1+\alpha_t - \alpha_b)} n^{\alpha_t - \alpha_b} \right.$$

$$\left. + \frac{\sin \pi(\alpha_b - \beta_t)}{\sin \pi(\alpha_a - \alpha_b)} \frac{\Gamma(-\beta_t - 1)}{\Gamma(-\alpha_a) \Gamma(-\alpha_b)} 2^{-\beta_t - 2} n^{\beta_t - \alpha_b} \right] \quad (18)$$

Here $\beta_t = \frac{1}{2} \alpha_t - \frac{1}{2}$ is the new Regge trajectory, which we discuss more extensively in sect. 7. Equation (18) is valid to leading order in N as $N \rightarrow \infty$.

The FESR (18) relates the s -channel resonances in fig. 2b to the Regge term in fig. 2c. Thus we have, for the first time, a duality constraint involving the reggeon-reggeon-resonance coupling. It is well-known¹⁰⁾ that this coupling has two parts, as shown by the general structure of a $2 \rightarrow 3$ amplitude in the double Regge limit (fig. 3):

$$T(2 \rightarrow 3) = (-s_1)^{\alpha_1} (-s_2)^{\alpha_2} \left[(-\kappa)^{-\alpha_1} V_{12}(t_1, t_2; \kappa) \right.$$

$$\left. + (-\kappa)^{-\alpha_2} V_{21}(t_1, t_2; \kappa) \right] \quad (19)$$

Here V_{12} and V_{21} are entire functions of $\kappa = s_1 s_2 / s$. This structure corresponds precisely to the decomposition of B_6 in eq. (13a). Thus the two terms (19) of the reggeon-reggeon-resonance

coupling are constrained separately by the FESR's for V_{ab} and V_{ba} . We should also note that the value of κ in the FESR is fixed in the HP limit by the four-dimensionality constraint (6). As we already discussed in sect. 2, this value generally lies within or close to the physical region of the resonance production process.

There are two Regge terms on the r.h.s. of eq. (18), corresponding to the exchange of the usual α_t trajectory and the new β_t trajectory. Both exchanges are built up simultaneously by the resonances; there is no way to associate either exchange with a specific part of the resonance contribution^{*)}. The relative magnitude

*) As was observed in ref.¹¹⁾ (see also sect. 7), the α_t - and β_t -exchanges are intricately connected to each other. Both of them have unphysical singularities in t , which cancel in the combined $(\alpha_t + \beta_t)$ contribution. Thus it should in general not be possible to separate the α - and β -exchanges from each other.

of the two Regge terms is determined, on the one hand, by their intercepts. The α_t (β_t) trajectory is leading for $\alpha_t > -1$ ($\alpha_t < -1$). On the other hand, their Regge residues have a different dependence on the momentum transfers. Thus, for example, the α_t -exchange term vanishes when $\alpha_a - \alpha_t = 1, 2, \dots$ or $\alpha_b - \alpha_t = 1, 2, \dots$, in which case the resonances are dual only to β_t -exchange. Similarly, β_t -exchange is absent when $\alpha_b - \beta_t = 0, \pm 1, \pm 2, \dots$ or when either of the reggeons is on-shell, i.e. α_a or $\alpha_b = 0, 1, 2, \dots$.

It is instructive to illustrate some of the general arguments given above by numerical examples of the duality relation (18). A "typical" situation with small negative momentum transfers is shown in

fig. 4a. The resonances are semilocally dual to the sum of the α_t and β_t Regge exchanges. As discussed above, this implies that semilocal duality holds to a good approximation also for the function $\kappa^{-\alpha_a} \sqrt{ab}$ and for the full $B_6(s,t)$ term. In fig. 4b, on the other hand, the momentum transfers are such that eq. (18) becomes a superconvergence relation. As shown by the solid histogram, the sum over resonances in $\sqrt{ab}$ does indeed tend to zero as the cutoff is increased. The resonances do not cancel semilocally, however, and consequently $\kappa^{-\alpha_a} \sqrt{ab}$ is not even approximately superconvergent (dashed histogram).

4. A PLANAR FESR

In this section we would like to discuss how a sum rule analogous to (18) may be derived for a fairly general class of six-point amplitudes. From the derivation of the $B_6(s,t)$ FESR (18) we can see that for this purpose we require an amplitude which satisfies the following conditions in the HP limit (1) of fig. 2a:

- (i) The amplitude should have a decomposition analogous to eq. (13a), $\sqrt{ab}$ and $\sqrt{ba}$ being entire functions of κ . This ensures that the contributions from singularities in κ can be eliminated from the FESR.
- (ii) In the $s \rightarrow \infty$ Regge limit (fig. 2c), the amplitude (and its discontinuities) should factorize into a product of reggeon propagators and vertices. This means that the Regge parameters in the FESR can be related to other processes.

Condition (i) is automatically satisfied by all planar amplitudes $A(s,t)$, with the external particles ordered as in fig. 2.

By their definition¹⁾, these amplitudes have singularities only in the planar variables corresponding to that ordering, such as $s, t, s_{234}, s_{345}, s_{12},$ and s_{56} . In the HP limit (1), the cuts in s_{12} and s_{56} are eliminated. Since s_{234} and s_{345} are overlapping, $A(s,t)$ can according to the generalized Steinmann relations¹⁰⁾ have no (asymptotic) double discontinuity in these two variables. This implies a decomposition of the form (13a) for $A(s,t)$.

Because the amplitude $A(s,t)$ is planar and the ordering of the external lines is fixed, the reggeons in fig. 2c are unsigned. Thus, the factorization condition (ii) is nontrivial. However, within the DRM it has been shown¹⁴⁾ that such unsigned amplitudes and all their discontinuities do factorize in multi-Regge limits, provided only that all the planar variables (in which the amplitude has singularities) tend to $+\infty + i\epsilon$. This factorization is probably related to the existence of planar unitarity equations¹⁾, and may thus be expected to be quite general. Hence condition (ii) is satisfied if $A(s,t)$ is evaluated for values of the invariants in the physical region of the process $3 + 4 \rightarrow 5 + 6 + 1 + 2$. In the FESR considerations that follow we shall assume this to be the case.

The existence of a decomposition (13a) for $A(s,t)$, together with the assumption of a factorizing Regge behavior as $s \rightarrow \infty$, allows us to derive an FESR for the s_{234} -discontinuity of $A(s,t)$ that is analogous to eq. (18):

$$\int_0^N ds \kappa^{\alpha_a} D_s D_{234} A(s,t) = \sum_1 \int_0^N ds s^{\alpha_a} V(\alpha_a) P(\alpha_a) V(\alpha_a, \bar{\alpha}_1) P(\bar{\alpha}_1) V(\bar{\alpha}_1, \bar{\alpha}_b) P(\bar{\alpha}_b) V(\bar{\alpha}_b) . \quad (20)$$

Just as for $B_6(s, t)$, only physical singularities in s (i.e., resonances and thresholds) contribute to the s -discontinuity on the left-hand side. The subenergies s_{234} and s_{345} are held at a fixed (asymptotic) value, with κ given by eq. (6). The helicity pole constraint (1) is imposed.

On the r.h.s. of eq. (20) the sum is over all Regge poles α_1 in the t -channel. Based on our experience with the DRM (eq. (18)) and on the general arguments given below in sect. 7, we expect these to be of two kinds: the ordinary α -type reggeons that appear in four-point amplitudes, and the new β reggeons that do not couple to two particles. We have used the factorization property discussed above to express the Regge term as a product of the external one-reggeon vertices $V(\alpha)$, the internal two-reggeon vertices $V(\alpha_1, \alpha_2)$ and the reggeon propagators $P(\alpha)$. The definition of the cut and uncut vertices and propagators is given in fig. 5. (We recall that they are all defined with respect to unsigned reggeons.)

The reggeon propagators and vertices appearing in eq. (20) can, in the case of α -type exchanges, be determined using simpler amplitudes. Thus the vertices $V(\alpha)$ and the propagators $P(\alpha)$ are fixed by the high energy behavior of the planar four-point amplitude. Since it is of the form $f(t)(-s)^\alpha$, the cut and uncut propagators for any α -type reggeon are simply related,

$$P(\bar{\alpha}) = 2i e^{i\pi\alpha} \sin \pi(-\alpha) P(\alpha) \quad (21)$$

where we defined the vertices to be unchanged by cutting (cf. fig. 5c).

The vertex $V(\alpha_1, \bar{\alpha}_2)$ for a cut reggeon α_2 (fig. 5e) is simply related to the uncut vertex $V(\alpha_1, \alpha_2)$ (fig. 5d), whenever

α_1 is a helicity pole. From eq. (19) we can see that in the limit when $s_1/s \rightarrow 0$, where α_1 becomes a helicity pole, the planar $2 \rightarrow 3$ amplitude is (assuming $\alpha_1 > \alpha_2$)

$$T(2 \rightarrow 3) = (-s)^{\alpha_1} (-s_2)^{\alpha_2 - \alpha_1} v_{12}(t_1, t_2; \kappa = 0). \quad (22)$$

Thus any discontinuity of $T(2 \rightarrow 3)$ in s or s_2 is simply related to the full amplitude. Using the relation (21) between the propagators, one can easily derive a relation between the two-reggeon vertices,

$$V(\alpha_1, \bar{\alpha}_2) = e^{-i\pi\alpha_1} \frac{\sin \pi(\alpha_1 - \alpha_2)}{\sin \pi(-\alpha_2)} V(\alpha_1, \alpha_2) \quad (23)$$

where, we emphasize, the helicity pole α_1 and the Regge pole α_2 are both α -type exchanges. If we define the singly and doubly cut propagators to be equal (fig. 5b), we similarly obtain

$$V(\bar{\alpha}_1, \bar{\alpha}_2) = V(\alpha_1, \bar{\alpha}_2). \quad (24)$$

We see, then, that for α -type exchanges α_1 the Regge term in eq. (20) can be determined by studies of $2 \rightarrow 2$ and $2 \rightarrow 3$ processes. The most practical way of finding the two-reggeon coupling $V(\alpha_a, \bar{\alpha}_1)$ is probably through an FESR⁷⁾ for a reggeon (α_a)-particle process (obtained, e.g., by putting α_b on-shell in fig. 2a). It is also interesting to observe that the Regge term in the FESR for $D_{234}A(s, t)$ (eq. (20)) is simply related to that in the analogous FESR for $D_{345}A(s, t)$. Because of eq. (24), the only difference lies in which of the propagators $P(\alpha_a)$, $P(\alpha_b)$ is cut. Since the two kinds of propagator are related by eq. (21), the relation between the full Regge terms can be easily worked out. Through semilocal duality, this

relation implies a connection between the two terms of the reggeon-reggeon-resonance couplings (cf. eq. (19)), which determine the resonance contributions to the corresponding FESR's.

For β -type exchanges α_i in eq. (20), no information on the Regge couplings can be obtained from 4- or 5-point amplitudes.

However, we still expect the Regge terms in the FESR's for $D_{234}A(s,t)$ and $D_{345}A(s,t)$ to be related. In the DRM (eq. (18)) we see that the Regge residue of β_t is antisymmetric under the interchange $\alpha_a \leftrightarrow \alpha_b$, apart from the factor $\sin \pi(\alpha_b - \beta_t)$. This symmetry is a consequence of the relation¹¹⁾

$$(-\alpha_s)^{-\alpha_a} \gamma_{ab}^{(\beta)} = -(-\alpha_s)^{-\alpha_b} \gamma_{ba}^{(\beta)} \quad (\alpha_s \rightarrow -\infty) \quad (25)$$

which ensures that β_t does not contribute to untwisted^{*)} B_6 .

*) A twist on a reggeon line reverses the ordering of all particles on one side of the twist. Here and in the following we take the untwisted diagram to have the ordering of fig. 2, and the high energy limit to be such that all planar variables corresponding to that ordering tend to $+\infty$ (or remain finite).

amplitudes. Now the unphysical singularities in t of the α -type exchanges, which require the presence of β exchanges, are in general expected to occur only in amplitudes with twisted reggeons α_a, α_b .^{*)}

*) We discuss this point in more detail in sect. 7. Note that this statement only holds for the amplitude with all asymptotic variables evaluated above their cuts. Discontinuities of the untwisted amplitude in general do have contributions from β -exchange.

It is therefore reasonable to assume that there will be no β -type exchanges in any untwisted amplitude. This requires eq. (25) to hold in general, γ_{ab} and γ_{ba} being defined by the decomposition (13a) of the planar amplitude $A(s,t)$. It then follows that the relation between the Regge terms in the FESR's for $D_{234}A(s,t)$ and $D_{345}A(s,t)$ will be the same as in the DRM.

Finally, we note that the α -type Regge poles in eq. (20) occur in degenerate pairs of opposite-signed trajectories. The leading β -trajectories, on the other hand, have well-defined signature (positive in the DRM). Thus duality does not always imply signature degeneracy. In the four-point function this degeneracy is required because the amplitude is real in the channel without a discontinuity, so that the Regge poles must cancel in the imaginary part. Similarly for the six-point amplitude, γ_{ab} in eq. (13a) is real when $\alpha_s \rightarrow -\infty$. The signed amplitude cannot in this case be constructed by combining the limits $\alpha_s \rightarrow +\infty$ and $\alpha_s \rightarrow -\infty$ of a given amplitude, however. Rather, we have to add the new diagram shown in fig. 6a, in which the reggeons are not adjacent (and take the limit $\alpha_s \rightarrow +\infty$). Thus the reality of γ_{ab} for $\alpha_s \rightarrow -\infty$ has no bearing on the signature of the exchanged trajectories.

5. THE $B_6(u,t)$ TERM

The ordering of external lines in the $B_6(u,t)$ reggeon amplitude is as shown in fig. 6a. In the HP limit (1) we have the kinematic relations $s_{36} = -s_{345}$, $s_{14} = -s_{234}$ and $s_{356} = -s_{34}$. Thus the finite ratio κ_u of these asymptotic variables is simply related to κ (eq. (10)),

$$\kappa_u = \frac{\alpha_{36} \alpha_{14}}{\alpha_{356}} = -\kappa. \quad (26)$$

According to eq. (6), κ_u is a linear function of α_u .

We can right away see that the analytic structure, and hence the duality properties, of the $B_6(u, t)$ amplitude will be quite different from those of the $B_6(s, t)$ term considered in sect. 3. First, the two asymptotic variables α_{36} and α_{14} in which $B_6(u, t)$ has cuts are not overlapping. Thus, since we have to allow for a double discontinuity in α_{36} and α_{14} , the structure of $B_6(u, t)$ in κ_u will be more complicated than the structure of $B_6(s, t)$ in κ (cf. eq. (13a)). Second, the variable α_{356} overlaps with the u-channel resonances (fig. 6b). The residues will therefore be polynomials in $1/\kappa_u$. Thus each u-channel residue is singular at $\kappa_u = 0$, and it is clearly impossible to find a function that has the same resonances as $B_6(u, t)$ but is free of singularities in κ_u . This means that there can be no FESR with only u-channel resonance contributions to the low-energy integral.

Let us now study explicitly the properties of $B_6(u, t)$. Imposing the HP constraints (1) on the integral representation¹²⁾ of the amplitude in the limit of fig. 6a, we obtain

$$B_6(u, t) = (-\alpha_{36})^{\alpha_a} (-\alpha_{14})^{\alpha_b} \int_0^1 du u^{-\alpha_t + \alpha_a + \alpha_b - 1} (1-u)^{-\alpha_u - 1} \times \int_0^\infty dz_1 dz_2 z_1^{-\alpha_a - 1} z_2^{-\alpha_b - 1} \exp \left[-z_1 - z_2 + \frac{1-u}{u \kappa_u} z_1 z_2 \right]. \quad (27)$$

Using the identity (11) we get the Mellin-Barnes integral

$$B_6(u, t) = (-\alpha_{36})^{\alpha_a} (-\alpha_{14})^{\alpha_b} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\lambda \Gamma(-\lambda) \Gamma(\lambda - \alpha_a) \Gamma(\lambda - \alpha_b)$$

$$\times \frac{\Gamma(\lambda - \alpha_u) \Gamma(-\lambda - \alpha_t + \alpha_a + \alpha_b)}{\Gamma(-\alpha_u - \alpha_t + \alpha_a + \alpha_b)} (-\kappa_u)^{-\lambda}.$$

The integration contour can be closed to the left, giving a representation in terms of the confluent hypergeometric function ${}_2F_2$:

$$B_6(u, t) = (-\alpha_{36})^{\alpha_a} (-\alpha_{14})^{\alpha_b} \Gamma(-\alpha_a) \Gamma(-\alpha_b) \times \left\{ (-\kappa_u)^{-\alpha_a} \frac{\Gamma(\alpha_a - \alpha_b)}{\Gamma(-\alpha_b)} \frac{\Gamma(-\alpha_u + \alpha_a) \Gamma(-\alpha_t + \alpha_b)}{\Gamma(-\alpha_u - \alpha_t + \alpha_a + \alpha_b)} \times {}_2F_2(-\alpha_a, -\alpha_t + \alpha_b; 1 - \alpha_a + \alpha_b, 1 + \alpha_u - \alpha_a; \kappa_u) + (-\kappa_u)^{-\alpha_b} \frac{\Gamma(\alpha_b - \alpha_a)}{\Gamma(-\alpha_a)} \frac{\Gamma(-\alpha_u + \alpha_b) \Gamma(-\alpha_t + \alpha_a)}{\Gamma(-\alpha_u - \alpha_t + \alpha_a + \alpha_b)} \times {}_2F_2(-\alpha_b, -\alpha_t + \alpha_a; 1 + \alpha_a - \alpha_b, 1 + \alpha_u - \alpha_b; \kappa_u) + (-\kappa_u)^{-\alpha_u} \frac{\Gamma(\alpha_u - \alpha_a) \Gamma(\alpha_u - \alpha_b)}{\Gamma(-\alpha_a) \Gamma(-\alpha_b)} \Gamma(-\alpha_u) \times {}_2F_2(-\alpha_u, -\alpha_u - \alpha_t + \alpha_a + \alpha_b; 1 - \alpha_u + \alpha_a, 1 - \alpha_u + \alpha_b; \kappa_u) \right\}. \quad (28)$$

Comparing this expression with eq. (13), we see that the more complicated structure of the $B_6(u,t)$ amplitude, compared to $B_6(s,t)$, is due essentially to the last term in eq. (28). This term has a nonvanishing double discontinuity in α_{36} and α_{14} , and contains all the u-channel resonance poles. It vanishes whenever either of the reggeons α_a, α_b is on-shell. All three terms in eq. (28) also have "spurious" singularities in α_u at $\alpha_u - \alpha_a = 0, \pm 1, \pm 2, \dots$ and $\alpha_u - \alpha_b = 0, \pm 1, \pm 2, \dots$. The residues of these poles cancel in the sum of the terms, so that the poles do not appear in $B_6(u,t)$. When one of the external reggeons is on-shell, say at $\alpha_a = 0$, the "spurious" poles of the first term at $\alpha_u - \alpha_a = 0, 1, 2, \dots$ turn into the "real" u-channel poles of the on-shell amplitude. Thus the various terms in eq. (28) are intimately connected to each other.

If we wished to derive an FESR involving only the u-channel resonances, analogous to that for $B_6(s,t)$ in sect. 3, the first step would be to isolate the (third) term in $B_6(u,t)$, which contains the physical resonances in α_u . This can be done by taking the double discontinuity

$$D_{14} D_{36} B_6(u,t) = \frac{\alpha_a}{\alpha_{36}} \frac{\alpha_b}{\alpha_{14}} (2\pi i)^2 \frac{(-\kappa_u)^{-\alpha_u} \Gamma(-\alpha_u)}{\Gamma(1 - \alpha_u + \alpha_a) \Gamma(1 - \alpha_u + \alpha_b)} \times {}_2F_2(-\alpha_u, -\alpha_u - \alpha_t + \alpha_a + \alpha_b; 1 - \alpha_u + \alpha_a, 1 - \alpha_u + \alpha_b; \kappa_u). \quad (29)$$

However, we are now unable to remove the factor $(-\kappa_u)^{-\alpha_u}$, which has a cut in κ_u , because it is not power behaved in the Regge limit

$|\alpha_u| \rightarrow \infty$. Indeed, it is precisely this factor which ensures that the resonance residues are polynomials in $1/\kappa_u$. We conclude, therefore, that the u-channel resonances alone are not required by analyticity to build up the t-channel Regge exchanges, in the usual FESR sense. For the same reason, the resonances need not saturate a superconvergence relation, even though the amplitude vanishes arbitrarily fast for $|\alpha_u| \rightarrow \infty$.

To further illustrate the properties of $B_6(u,t)$, let us calculate the Regge limit when $\alpha_u \rightarrow +\infty + i\epsilon$, and see how the imaginary part of the amplitude is built up by the resonances in α_u and the cut in κ_u . The high energy behavior of the first two terms in eq. (28) can be obtained directly using eq. (A9) for the ${}_2F_2$ functions. Since $\kappa_u = \alpha_u + \text{const.}$, the factor in front of the ${}_2F_2$ function in the third term behaves like

$$\lim_{\substack{\alpha_u \rightarrow \infty + i\epsilon \\ \kappa_u \rightarrow \infty + i\epsilon}} (-\kappa_u)^{-\alpha_u} \Gamma(-\alpha_u) \Gamma(\alpha_u - \alpha_a) \Gamma(\alpha_u - \alpha_b) = \frac{1}{i(2\pi)} e^{3/2} e^{i\pi\alpha_u(1\pm 1)} \alpha_u^{-\alpha_a - \alpha_b - 3/2} e^{-\kappa_u}. \quad (30)$$

The exponential $e^{-\kappa_u}$ multiplied by the ${}_2F_2$ function gives a power behavior in α_u according to eq. (A15). Thus for $\kappa_u \rightarrow \infty + i\epsilon$ the third term is exponentially damped as $\text{Im } \alpha_u \rightarrow +\infty$. The contribution from the first two terms is

$$\lim_{\substack{\alpha_u \rightarrow \infty + i\epsilon \\ \kappa_u \rightarrow \infty + i\epsilon}} B_6(u, t) = (-\alpha_{36})^{\alpha_a} (-\alpha_{14})^{\alpha_b} \Gamma(-\alpha_a) \Gamma(-\alpha_b) \\ \times e^{-i\pi(\alpha_t - \alpha_a - \alpha_b)} \alpha_u^{\alpha_t - \alpha_a - \alpha_b} \frac{\Gamma(-\alpha_t + \alpha_a) \Gamma(-\alpha_t + \alpha_b)}{\Gamma(-\alpha_t)}. \quad (31)$$

We note that β_t -exchange does not contribute in this limit.

The imaginary part of $B_6(u, t)$ in eq. (31) is due both to the resonance poles and to the $0 \leq \kappa_u \leq \infty$ cut in κ_u . To see their separate contributions we next evaluate the discontinuities of $B_6(u, t)$ in α_u and κ_u , by combining limits for $\alpha_u \rightarrow \infty \pm i\epsilon$ and $\kappa_u \rightarrow \infty \pm i\epsilon$. The third term in eq. (28) contributes to both of these discontinuities, and can be calculated using eqs. (30) and (A15). Since there is no simultaneous discontinuity in α_u and κ_u , the discontinuity in α_u may be evaluated with κ_u on either side of its cut, and vice versa. We obtain, in the Regge limit of fig. 6c,

$$D_u B_6(u, t) = (-\alpha_{36})^{\alpha_a} (-\alpha_{14})^{\alpha_b} \Gamma(-\alpha_a) \Gamma(-\alpha_b) \\ \times 2\pi i \left\{ \alpha_u^{\alpha_t - \alpha_a - \alpha_b} \frac{\Gamma(1 + \alpha_t)}{\Gamma(1 + \alpha_t - \alpha_a) \Gamma(1 + \alpha_t - \alpha_b)} \right. \\ \left. + \alpha_u^{\beta_t - \alpha_a - \alpha_b} \frac{\Gamma(-\beta_t - 1)}{\Gamma(-\alpha_a) \Gamma(-\alpha_b)} 2^{-\beta_t - 2} \right\}, \quad (32)$$

$$D_\kappa B_6(u, t) = (-\alpha_{36})^{\alpha_a} (-\alpha_{14})^{\alpha_b} \Gamma(-\alpha_a) \Gamma(-\alpha_b) \\ \times 2\pi i \left\{ \alpha_u^{\alpha_t - \alpha_a - \alpha_b} \frac{\Gamma(-\alpha_t + \alpha_a) \Gamma(-\alpha_t + \alpha_b)}{\pi \Gamma(-\alpha_t)} \frac{\sin \pi \alpha_a \sin \pi \alpha_b}{\sin \pi \alpha_t} \right. \\ \left. - \alpha_u^{\beta_t - \alpha_a - \alpha_b} \frac{\Gamma(-\beta_t - 1)}{\Gamma(-\alpha_a) \Gamma(-\alpha_b)} 2^{-\beta_t - 2} \right\}. \quad (33)$$

It is straightforward to verify that the sum $(D_u + D_\kappa) B_6(u, t)$ gives rise to the full imaginary part of $B_6(u, t)$ in eq. (31).

The α_u -discontinuity in eq. (32) is indeed equal to $2\pi i$ times the leading behavior, when $\alpha_u \rightarrow \infty$, of the u -channel pole residues in $B_6(u, t)$. But the absence of a resonance-saturated FESR means the the sum of all pole residues need not be equal to the integral of the Regge term (32), even when the sum is cut off at large α_u . Nevertheless, it is of course possible that such a relation may hold approximately in a limited range of momentum transfers. To investigate this we have made a numerical study of the residues in $B_6(u, t)$. Figure 7 shows a comparison between the residues and the Regge term, for the same values of momentum transfers that we used to illustrate duality in the $B_6(s, t)$ term (fig. 4). We see in fig. 7a that semilocal duality works surprisingly well at small momentum transfers. In fig. 7b on the other hand, the superconvergence relation is very badly violated. Most of the violation is due to the large value of the residue at $\alpha_u = 3$. For the momentum transfers we have chosen, κ_u is given in terms of α_u by $\kappa_u = \alpha_u - 3.1$. For $\alpha_u = 3$,

κ_u is quite small and the residue, being of order $\mathcal{O}(1/\kappa_u^3)$, grows large. Thus fig. 7b illustrates in a striking way the effect that the singularities in κ_u can have on the FESR.

6. THE $B_6(s,u)$ TERM

The $B_6(s,u)$ amplitude has resonance poles both in the s- and u-channels (fig. 8). It has no t-channel Regge poles, and is therefore expected to vanish exponentially in the high energy limit $|\alpha_s| \rightarrow \infty$. When both reggeons are put on-shell ($\alpha_a = \alpha_b = 0$ in fig. 8), $B_6(s,u)$ reduces to the four-point amplitude $B_4(s,u)$, which is known to satisfy a superconvergence relation. $B_4(s,u)$ also has a wrong-signature fixed pole, which means that the sum over the s-channel pole residues alone (when convergent) is equal to a nonzero constant. The u-channel residues sum up to the same constant. This ensures that the difference between the s- and u-channel contributions vanishes, as required by the superconvergence relation.

In sect. 3 we saw that $\kappa^{\alpha_a} D_{234} B_6(s,t)$ satisfies a resonance-saturated FESR. It is obviously important to understand whether this FESR holds also for the $B_6(s,u)$ amplitude. Since $B_6(s,u)$ has no t-channel Regge poles, the FESR for $\kappa^{\alpha_a} D_{234} B_6(s,u)$ would be reduced to a superconvergence relation. Moreover, $D_{234} B_6(s,u)$ can have no u-channel resonance poles, because the variables α_{234} and α_u are overlapping. By continuity arguments (in the reggeon masses) the s-channel resonances alone cannot saturate a superconvergence relation, since they do not do so for $B_4(s,u)$. We are thus led to

expect fixed pole type contributions ^{*}) in the FESR for

^{*}) Here and in the following we shall by a "fixed pole contribution" understand a cut-off independent term on the r.h.s. of the FESR. We have not investigated whether such a term can be related to the J-plane structure of the amplitude.

$\kappa^{\alpha_a} D_{234} B_6(s,u)$. Unlike in the case of the four-point amplitude, however, the magnitude of the fixed pole contributions cannot be precisely related to the u-channel resonances. As we discussed in sect. 5, any FESR involving u-channel poles must also include terms due to the singularities of the amplitude in κ_u , because the pole residues are singular in κ_u .

The expression for $B_6(s,u)$ in the Regge limit of fig. 8a is ¹²⁾

$$B_6(s,u) = \int_0^1 dx x^{-\alpha_s-1} (1-x)^{-\alpha_u-1} \left[-\alpha_{26}^x - \alpha_{345}(1-x) \right]^{\alpha_a} \times \left[-\alpha_{15}^x - \alpha_{234}(1-x) \right]^{\alpha_b} \times \int_0^\infty dz_1 dz_2 z_1^{-\alpha_a-1} z_2^{-\alpha_b-1} \times \exp \left\{ -z_1 - z_2 + \frac{\alpha_{34}(1-x)z_1 z_2}{[\alpha_{26}^x + \alpha_{345}(1-x)][\alpha_{15}^x + \alpha_{234}(1-x)]} \right\}. \quad (34)$$

Using eq. (11) we get the Mellin-Barnes representation

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$$\begin{aligned}
 B_6(s, u) &= (-\alpha_{345})^{\alpha_a} (-\alpha_{234})^{\alpha_b} \frac{1}{2\pi i} \\
 &\times \int_{-i\infty}^{i\infty} d\lambda \Gamma(-\lambda) \Gamma(\lambda - \alpha_a) \Gamma(\lambda - \alpha_b) (-\kappa)^{-\lambda} \\
 &\times \int_0^1 dx x^{-\alpha_s-1} (1-x)^{-\alpha_u+\lambda-1} \left[1 + \left(\frac{\alpha_{26}}{\alpha_{345}} - 1 \right) x \right]^{\alpha_a-\lambda} \\
 &\times \left[1 + \left(\frac{\alpha_{15}}{\alpha_{234}} - 1 \right) x \right]^{\alpha_b-\lambda}.
 \end{aligned} \tag{35}$$

In the HP limit (1) we have the kinematic relations

$$\frac{\alpha_{26}}{\alpha_{345}} = \frac{\alpha_{15}}{\alpha_{234}} = 1 \tag{36}$$

Imposing these constraints on the amplitude in eq. (35), and closing the λ -integration contour to the left, we obtain

$$\begin{aligned}
 B_6(s, u) &= (-\alpha_{345})^{\alpha_a} (-\alpha_{234})^{\alpha_b} \Gamma(-\alpha_a) \Gamma(-\alpha_b) \\
 &\times \left\{ (-\kappa)^{-\alpha_a} \frac{\Gamma(\alpha_a - \alpha_b)}{\Gamma(-\alpha_b)} \frac{\Gamma(-\alpha_s) \Gamma(-\alpha_u + \alpha_a)}{\Gamma(-\alpha_s - \alpha_u + \alpha_a)} \right. \\
 &\times {}_2F_2(-\alpha_a, 1 + \alpha_s + \alpha_u - \alpha_a; 1 - \alpha_a + \alpha_b, 1 + \alpha_u - \alpha_a; -\kappa)
 \end{aligned}$$

(Eq. 37 continued)

$$\begin{aligned}
 &+ (-\kappa)^{-\alpha_b} \frac{\Gamma(\alpha_b - \alpha_a)}{\Gamma(-\alpha_a)} \frac{\Gamma(-\alpha_s) \Gamma(-\alpha_u + \alpha_b)}{\Gamma(-\alpha_s - \alpha_u + \alpha_b)} \\
 &\times {}_2F_2(-\alpha_b, 1 + \alpha_s + \alpha_u - \alpha_b; 1 + \alpha_a - \alpha_b, 1 + \alpha_u - \alpha_b; -\kappa) \\
 &+ (-\kappa)^{-\alpha_u} \frac{\Gamma(\alpha_u - \alpha_a)}{\Gamma(-\alpha_a)} \frac{\Gamma(\alpha_u - \alpha_b)}{\Gamma(-\alpha_b)} \Gamma(-\alpha_u) \\
 &\times {}_2F_2(-\alpha_u, 1 + \alpha_s; 1 - \alpha_u + \alpha_a, 1 - \alpha_u + \alpha_b; -\kappa) \Big\}.
 \end{aligned} \tag{37}$$

The structure of $B_6(s, u)$ in κ can be seen to be quite similar to that of $B_6(u, t)$ (eq. (28)). In particular, the third term in eq. (37) allows for a nonvanishing double discontinuity in the nonoverlapping variables α_{26} and α_{15} . Using the high energy behaviors of the ${}_2F_2$ functions derived in Appendix A, one can readily show that $B_6(s, u)$ is exponentially damped in $|\text{Im } \alpha_s|$ as $|\alpha_s| \rightarrow \infty$. The resonances along the real axis alternate in sign and are power behaved as $\alpha_s \rightarrow \pm\infty$. Thus the asymptotic properties of $B_6(s, u)$ are completely analogous to those of $B_4(s, u)$.

As in the case of the $B_6(s, t)$ and $B_6(u, t)$ amplitudes, the explicit singularities in κ of $B_6(s, u)$ preclude the derivation of a resonance-saturated superconvergence relation for the full amplitude. The discontinuity of $B_6(s, u)$ in α_{234} , on the other hand, should have a simpler structure in κ . We define this discontinuity to be

$$D_{234} B_6 = B_6(\alpha_{234} \rightarrow \infty + i\epsilon, \alpha_{15} \rightarrow \infty + i\epsilon) - B_6(\alpha_{234} \rightarrow \infty - i\epsilon, \alpha_{15} \rightarrow \infty + i\epsilon) \tag{38}$$

with all other large variables approaching $-\infty$. It is clear that α_{234} and α_{15} have to be treated as independent variables in calculating $D_{234}B_6$. The kinematic relation (36) can be imposed only after the variables have been analytically continued above or below their respective cuts. Thus we cannot use eq. (37) in deriving $D_{234}B_6(s,u)$, but must go back to the expression (35) for $B_6(s,u)$. When we put $\alpha_{26} = \alpha_{345}$ and use the identity

$$\Gamma(-a)(1+x)^a = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\mu \Gamma(-\mu) \Gamma(\mu - a) x^\mu \quad (39)$$

on the factor $\left[1 + \frac{x}{1-x} \frac{\alpha_{15}}{\alpha_{234}}\right]^{\alpha_b - \lambda}$ in eq. (35), we get

$$B_6(s,u) = (-\alpha_{345})^{\alpha_a} (-\alpha_{234})^{\alpha_b} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} d\lambda \Gamma(-\lambda) \Gamma(\lambda - \alpha_a) (-\kappa)^{-\lambda} \times \int_{-i\infty}^{i\infty} d\mu \Gamma(-\mu) \Gamma(\mu + \lambda - \alpha_b) \frac{\Gamma(\mu - \alpha_s) \Gamma(-\mu - \alpha_u + \alpha_b)}{\Gamma(-\alpha_s - \alpha_u + \alpha_b)} \left(\frac{\alpha_{15}}{\alpha_{234}}\right)^\mu. \quad (40)$$

In this expression we may continue α_{234} and α_{15} to positive values and calculate $D_{234}B_6(s,u)$ as defined by eq. (38). This gives rise to the phase factor

$$\frac{e^{-i\pi(\alpha_b - \lambda)}}{e^{-i\pi(\alpha_b - \lambda - 2\mu)}} = 2i e^{-i\pi\mu} \sin \pi(\mu + \lambda - \alpha_b).$$

If we then let $\alpha_{15} = \alpha_{234}$ and close the μ -contour to the left, the poles at $\mu = \alpha_s - k$ ($k = 0, 1, 2, \dots$) give rise to a sum which is

proportional to ${}_2F_1(-\alpha_s, -\alpha_s - \alpha_u + \alpha_b; 1 - \lambda - \alpha_s + \alpha_b; 1)$.

Expressing the ${}_2F_1$ function in terms of Γ -functions and closing the λ -contour to the left finally gives

$$D_{234}B_6(s,u) = 2i(-\alpha_{345})^{\alpha_a} \alpha_{234}^{\alpha_b} \Gamma(-\alpha_a) \Gamma(-\alpha_b) \kappa^{-\alpha_a} \times \left\{ \frac{\pi e^{-i\pi\alpha_s}}{\Gamma(-\alpha_b) \Gamma(1 - \alpha_a + \alpha_b)} \frac{\Gamma(-\alpha_s) \Gamma(1 + \alpha_s + \alpha_u - \alpha_a)}{\Gamma(1 + \alpha_u - \alpha_a)} \times {}_2F_2(-\alpha_a, 1 + \alpha_s + \alpha_u - \alpha_a; 1 - \alpha_a + \alpha_b, 1 + \alpha_u - \alpha_a; -\kappa) \right\}. \quad (41)$$

From eq. (41) we can explicitly see that $\kappa^{\alpha_a} D_{234}B_6(s,u)$ is free of singularities in κ , and has no u -channel resonance poles. Thus if we derive an FESR by means of the usual contour integral in α_s , only the s -channel resonances will contribute to the low-energy integral. However, we cannot neglect the contribution from the asymptotic (circular) part of the contour. Due to the factor $e^{-i\pi\alpha_s}$ in eq. (41), $D_{234}B_6(s,u)$ grows exponentially as $\text{Im } \alpha_s \rightarrow \infty$. The rest of the factors in eq. (41) are power behaved as $|\alpha_s| \rightarrow \infty$. Using Cauchy's theorem we may therefore estimate the integral over the circle, as its radius N tends to infinity, by an integral of the form

$$\lim_{N \rightarrow \infty} \int_0^N dv \sin \pi v v^\alpha = \Gamma(\alpha + 1) \pi^{-\alpha-1} \cos \pi\alpha/2 - \frac{1}{\pi} N^\alpha \cos \pi N. \quad (42)$$

Thus when the power $\alpha < 0$, so that the sum over s -channel residues

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converges, the asymptotic part of the FESR integration contour contributes a constant. The ensuing FESR will then have a fixed pole-type term on the r.h.s., in close analogy to the "wrong-signature" sum rules for four-point amplitudes.¹⁵⁾

The size of the fixed pole contribution to the FESR for $\kappa^{\alpha_a} D_{234} B_6(s, u)$ is difficult to calculate analytically. Since the constant term on the r.h.s. of eq. (42) is of the same order of magnitude for all powers α , the nonleading (daughter) terms in the asymptotic expansion of $\kappa^{\alpha_a} D_{234} B_6(s, u)$ cannot be neglected. In the special case of an on-shell reggeon $\alpha_a = 0$, the sum over s-channel residues in the curly brackets of eq. (41) can be done explicitly:

$$\sum_{n=0}^{\infty} \sin \pi(-\alpha_b) \frac{\Gamma(1 + \alpha_s + \alpha_u)}{n! \Gamma(1 + \alpha_s + \alpha_u - n)} = \sin \pi(-\alpha_b) 2^{\alpha_s + \alpha_u},$$

(for $\alpha_s + \alpha_u > -1$).

(43)

In fig. 9a we show a numerical example of how the fixed pole contribution varies as a function of α_a . Within a moderate range of α_a the variation with respect to the on-shell value (43) is rather small.

In our study of the $B_6(s, t)$ and $B_6(u, t)$ amplitudes above, we found that although resonance-saturated FESR's for the full amplitudes were not strictly valid, semilocal duality still worked quite well for a typical set of (small) momentum transfers. It is thus of some interest to see whether an analogous statement is true for the $B_6(s, u)$ amplitude. We consider the s- and u-channel resonance contributions to the expression in curly brackets in eq. (37),

and ask whether their respective residues sum up to constants which cancel each other. If we would have a resonance-saturated superconvergence relation for $B_6(s, u)$ this cancellation would be exact. Since $\kappa \rightarrow +\infty$ for $\alpha_s \rightarrow +\infty$, the s-channel residues in the first two terms of eq. (37) are complex. They would therefore have to cancel in the imaginary part, and build up a real part which is equal and opposite to the contribution from the u-channel resonances in the third term. From fig. 9b we see that this is approximately the case, for the small values of momentum transfers that we used previously. Thus in a limited kinematic region, the singularities in κ of $B_6(s, u)$ give a fairly small contribution to the exact superconvergence relation.

7. THE β REGGE POLE

A very surprising and perhaps important result of the calculations of high energy limits presented in this paper is the existence of a new Regge pole $\beta = \frac{1}{2} \alpha - \frac{1}{2}$ in the B_6 function. This new reggeon has many unusual features, some of which have in the past been implicitly or explicitly assumed not to exist in physical amplitudes (this may partly explain why β was overlooked in B_6 until now). We have already briefly mentioned the properties of β in a separate publication¹¹⁾. Here we would like to give a more comprehensive discussion. In particular, we shall discuss the possible presence of β -type Regge poles in amplitudes that are more general than the DRM.

The most unconventional property of β -exchange is that it may or may not contribute to B_6 in a given high energy limit, depending on the sign of the asymptotic variables and on whether they are above or below their respective cuts. Thus it can be readily seen from

eqs. (13) and (A12) that the β_t exchange term is nonvanishing in only one of the four limits corresponding to twisted or untwisted^{*)}

^{*)} See footnote below eq. (25).

reggeons α_a, α_b in fig. 2c. This is the limit where both reggeons are twisted, so that the incoming and outgoing lines are ordered as in fig. 10. The reason for this anomalous behavior can be traced back to the fact that the β contribution is singular if viewed as a function of the variable η in eq. (7). For any value of $\eta \neq 1$ the β Regge pole term is absent¹¹⁾. Now it has been realized for a long time that the continuation of the asymptotic variables along semicircles from $-\infty$ to $+\infty$ sometimes implies that η rotates around the origin, acquiring a phase 2π . Thus the continuation from one limit to another requires that the analytic dependence of the amplitude on η be retained, the constraint (7)^{*)} being imposed only after the

^{*)} Although eq. (7) is commonly referred to as a "four-dimensionality" constraint, we should note that the same constraint would have to be satisfied in a space of arbitrary dimensions provided only the limit is one where the momentum components in a given longitudinal direction tend to ∞ , while all the transverse components remain fixed. Thus β -type exchanges are present in Regge limits of the DRM regardless of the dimensionality of the underlying space-time.

continuation. The new and unexpected property of β exchange, which shows that the asymptotic behavior of an amplitude can be singular in η , means that any two limits which are connected to each other by an analytic continuation in η need not be simply related. In our

example above it is easy to verify that the limit with both reggeons α_a, α_b twisted in fig. 2c is the only one which cannot be reached from the limit with all large variables approaching $-\infty$ without an analytic continuation in η . Conversely, in all examples we have so far considered, it has turned out that two limits will be correctly related to each other if $\eta = 1$ can be held fixed during the analytic continuation of the large variables along semicircles. Furthermore, the standard method always gives the right result for the α -exchange contribution, even when η is continued analytically, because that contribution is analytic in η .

The above conclusions are based on the expression (13) for B_6 in the HP limit (1) of fig. 2a, which was derived by first letting the relevant asymptotic variables approach $-\infty$. In order to be completely sure that our conclusions are correct, we have calculated the single Regge limit of B_6 shown in fig. 10, starting from the basic integral representation of the amplitude. The limit is taken in the proper direction from the beginning, so that no continuation of asymptotic forms is needed. Since the same limit has been calculated previously¹²⁾ for the case that all asymptotic variables approach $-\infty$, we are able to investigate directly whether the asymptotic behavior can depend on the direction in which the limit is taken. Moreover, the limit of fig. 10 is quite general, and should include the HP limit considered above as a special case. Based on the above arguments, we expect the diagram with the external lines ordered as in fig. 10 to give the entire β exchange contribution to the amplitude (apart from the diagram with a twist on the β reggeon which, as we shall see, is simply related to the diagram in fig. 10).

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The derivation of the single Regge limit is given in appendix B. We find that the asymptotic behavior is given by two terms, corresponding to α_t and β_t exchange in fig. 10. The α_t exchange term is related by analytic continuation to the behavior of B_6 in the limit where all large variables approach $-\infty$. This is expected, as that term is analytic in η . The term $B_6^{(\beta)}$ corresponding to β_t exchange is new and given by eq. (B21). The derivation makes it clear that the condition $\eta = 1$ is crucial for the existence of $B_6^{(\beta)}$.

From the factorized form (B21) of $B_6^{(\beta)}$ we can see that the coupling of β to three particles has no poles in any two-particle subchannel. Thus β does not couple to two particles. This is required by consistency, since there is no β Regge pole in the B_4 amplitude. The β propagator has poles at $\beta = -1, 0, 1, 2, \dots$. The residues of these poles are not, however, polynomials in the overlapping variables α_{34} , α_{61} , α_{345} and α_{234} . Since the residues of the full B_6 are polynomials, we conclude that the asymptotic contribution due to α exchange must cancel the nonpolynomial residues of $B_6^{(\beta)}$. This can happen because the β trajectory crosses a trajectory of the α family at each pole position (fig. 11), so that the energy dependence of both contributions is the same. In particular, the first pole at $\beta = \alpha = -1$ must cancel completely, as it is not present in B_6 . Analogous cancellations will occur for the daughter trajectories of β , which are parallel and spaced by $1/2$ unit. We have explicitly verified the cancellations (in the helicity pole limit) at the first few intersections in fig. 11.

In the further Regge limit $\alpha_{12}, \alpha_{56} \rightarrow -\infty$, fig. 10 reduces to fig. 2c. The expression for $B_6^{(\beta)}$ in this limit is given by

eq. (B23), and allows us to determine the reggeon (α)- reggeon (β)- particle coupling. It has the general structure given by eq. (19), as required by the Steinmann relations. The part of the vertex that has the factor $(-\kappa)^{-\beta}$ is zero, however. This is a consequence of the absence of poles in the variables α_{12} and α_{56} , which implies that there can be no asymptotic discontinuity in these variables.

We show in appendix B that the leading β trajectory has purely positive signature. Thus the β reggeons do not, like the α reggeons, occur in degenerate pairs of opposite signature. As we discussed at the end of sect. 4, this can happen in a planar model like the DRM essentially because the β Regge pole appears only in six-point and higher amplitudes.

To determine the degeneracy level of the positively signed β trajectory one has to consider the couplings of β to an arbitrary number of particles. From B_6 we can obtain only the three-particle coupling, so this question remains open. However, we can already infer that β must be at least doubly degenerate. Whereas the signed amplitude $B_6^{(\beta)\tau_a\tau_b}$ can be expressed as a single factorizing term (eq. (B23)), two terms are required for the signed discontinuity $D_s B_6^{(\beta)\tau_a\tau_b}$. This can be most easily seen using the HP limit formula (13). Substituting the $\alpha_s \rightarrow \infty \pm i\epsilon$ behavior (A12) of the 2^F_2 functions, forming the discontinuity and adding the four terms corresponding to twisted and untwisted reggeons α_a, α_b we get

$$D_s B_6^{(\beta)\tau_a\tau_b}(s, t) = -\alpha_{345}^{\alpha_a} \alpha_{234}^{\alpha_b} \alpha_s^{\beta_t - \alpha_a - \alpha_b} \Gamma(-\beta_t - 1) 2^{-\beta_t - 1} \times \left[i\pi e^{i\pi\beta_t} (\tau_a + e^{-i\pi\alpha_a})(\tau_b + e^{-i\pi\alpha_b}) + \tau_a \tau_b 2\pi \sin \pi \beta_t \right]. \quad (44)$$

We interpret this as the contribution of two cut β propagators with degenerate trajectories but distinct couplings to a signatured α reggeon and a particle.

What can be said about the presence of β -type Regge poles in amplitudes more general than the DRM? First of all we note that the α and β exchange contributions are very closely connected in B_6 . This is best illustrated by the cancellations that occur at all intersections of the two trajectory families in fig. 11, as we discussed above. We also saw in sect. 3 that the α and β reggeons cannot be separated in FESR's. Thus any combination of B_6 amplitudes and/or satellite terms will contain the β Regge pole.

A more model-independent argument for the existence of β exchange is provided by the presence of unphysical singularities in the α exchange contribution. This argument was given in detail in ref.¹¹⁾ for the DRM. We shall review it here with an emphasis on the generality of the properties that the vertex functions are assumed to have. First, we used the fact that the four-point amplitude vanishes at nonsense wrong-signature points of the exchange trajectory. In general, wrong-signature zeros (NWSZ's) can be removed by fixed pole contributions. For certain exchanges, however, such as ρ - A_2 and ω - f in πN elastic scattering, one can use duality and exchange degeneracy constraints to show that the fixed poles must be absent. This is also supported by the data, as exemplified by the dip at $\alpha_p = 0$ in $\pi^- p \rightarrow \pi^0 n$. We conclude that at least in some cases the NWSZ's are expected to occur in the physical amplitude.

The second property we needed was that the full $2 \rightarrow 3$ amplitude does not vanish at the nonsense wrong-signature points of

one of the exchanged trajectories. This implies that the 2-reggeon-particle coupling has a pole, which removes the zero coming from the Regge propagator. This is precisely what happens in the DRM (B_5). A general study¹⁶⁾ of the $2 \rightarrow 3$ amplitude indicates that the structure of the 2-reggeon-particle vertex should be essentially the same as in the DRM, modified only by smooth, unknown functions of the momentum transfers. In particular, if NWSZ's were present in the $2 \rightarrow 3$ amplitude they would not have any natural interpretation, but would have to be regarded as "accidental". Since there is an infinite number of nonsense wrong-signature points, we find such a possibility rather unpalatable.

The question concerning the existence of the NWSZ's in $2 \rightarrow 3$ amplitudes can of course also be answered experimentally. For example, in the reaction $\pi^+ p \rightarrow \pi^0(\pi^+ p)$ we expect no zero at $t_{\pi^+ \pi^0} \approx -0.6 \text{ GeV}^2$, when the $\pi^+ p$ effective mass is above the resonance region^{*)}. The

*) More precisely, the $2 \rightarrow 3$ amplitude has two terms (cf. eq. (19)), one of which is expected to have the NWSZ's, while the other is finite. Since both terms have the same energy dependence, some structure at $\alpha_p = 0$ should persist at large $\pi^+ p$ masses.

existing data¹⁷⁾ do indeed suggest that the dip, which is quite strong in the reaction $\pi^+ p \rightarrow \pi^0 \Delta^{++}$, is quickly filled in as the $\pi^+ p$ mass is increased.

If we assume that the $2 \rightarrow 3$ amplitude is finite at the nonsense wrong-signature points of the exchanged trajectories, factorization leads to poles at these values of momentum transfer in $2 \rightarrow 4$ amplitudes. This can be seen from fig. 2c, where at a nonsense

0.0 0.4 5 0 2 3 0 7

wrong-signature point for the α_t trajectory there are two singular couplings and only one vanishing propagator. Since this unphysical pole cannot appear in the full $2 \rightarrow 4$ amplitude, it must be canceled by the contribution from another exchange β_t .

The above argument can be repeated for all trajectories that are exchanged in the $2 \rightarrow 2$ amplitude. Associated with each such α type reggeon we thus expect a family of β -type trajectories that cross the α trajectory at the positions of the unphysical poles. This does not yet determine the slope of the β trajectory. However, if a cancellation between the α and β pole residues is to occur that is analogous to the one in B_6 , then all intersections must be at integer values of the angular momentum (cf. fig. 11). This limits the ratio β'/α' of the slopes to be $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$. Such cancellations may well be required, because at least the unphysical poles on the parent α trajectory have nonpolynomial residues, just as in B_6 . The slope of β could be fixed by determining which (if any) of the poles on the first α daughter have nonpolynomial residues. We shall not pursue this subject further here, however.

As a physical example, consider the leading β trajectory associated with the ρ reggeon. It should have the same signature as the ρ ($\tau = -$), and intersect the ρ trajectory at $\alpha_\rho = 0$. Assuming $\alpha_\rho = 0.5 + t$ and $\beta'_\rho/\alpha'_\rho = \frac{1}{2}$ we get $\beta_\rho = 0.25 + 0.5 t$. Thus the α_ρ and β_ρ trajectories lie quite close to each other in the small $|t|$ region.

Experimentally, β exchange could in principle be observed e.g., in reactions of the form $a + b \rightarrow (c + d) + (e + f)$, where the subsystems $c + d$ and $e + f$ are nonresonant. The sparse data on

such reactions, together with the high accuracy needed to separate the α_ρ and β_ρ trajectories, makes it rather unlikely that such evidence will be available soon. On the other hand, the absence of NWSZ's in $2 \rightarrow 3$ reactions, which as we saw above implies ^{*)} the existence of a

^{*)} This is of course only true as long as Regge cuts can be neglected.

new exchange contribution in the $2 \rightarrow 4$ amplitude, is much simpler to test experimentally. In fact, it is sufficient to establish whether the nonsense wrong-signature dips are absent in the triple-Regge region of one-particle inclusive reactions. To see this, let us consider again the process $\pi^+ p \rightarrow \pi^0(\pi^+ p)$, which is a particular contribution to the inclusive reaction $\pi^+ p \rightarrow \pi^0 + X$. If the $\pi^+ p \rightarrow \pi^0(\pi^+ p)$ amplitude is finite at $\alpha_\rho = 0$, the same must be true of the inclusive cross section. In the triple-Regge language (fig. 12), this means that the triple-Regge coupling has a (double) pole which cancels the zero in the ρ propagators. By two-component duality we expect the ρ and f exchanges in fig. 12 to be built up from resonance contributions to the missing mass, which do have the NWSZ's. Hence it is the $\rho\rho f$ coupling which is singular at $\alpha_\rho = 0$. In view of the fact that the pomeron and f couplings have generally been found to be proportional^{2,18)}, this is a somewhat surprising conclusion. It also indicates how the $\alpha_\rho = 0$ dip should disappear as the missing mass is increased and pomeron exchange becomes more important. Such a trend is indeed seen in the data¹⁹⁾ on $\pi^- p \rightarrow \pi^0 X$ at 5 GeV/c, although the energy is too low for a conclusive test.

3. DISCUSSION

The original purpose of this work was to investigate, within the DRM, the new, nonresonant contributions that appear in FESR's for reggeon-reggeon amplitudes, owing to their more complicated analytic structure. Our most surprising find, however, was the presence of a new contribution also on the r.h.s. of the FESR's, corresponding to the exchange of a previously unknown Regge trajectory β in the DRM. This trajectory was not seen in earlier derivations of Regge limits in the DRM because of an erroneous technical assumption, namely that high energy limits can be calculated by first letting the relevant variables approach $-\infty$ and then analytically continuing the limit to $+\infty$, if necessary. In view of this, new and correct methods have to be developed for calculating high energy limits of multiparticle amplitudes. Taking the limit in the proper direction from the beginning is certainly possible (see appendix B), but is unfortunately rather cumbersome. Based on our results so far we believe that all the expressions previously obtained for the standard α exchange contributions are correct, the only mistake being that the β exchange term has been overlooked. The existence of even more Regge trajectories in the DRM, which decouple from 2- and 3-particle states, remains at this stage an intriguing open possibility.

The importance of the β trajectories naturally depends on whether they are present in more general amplitudes than the DRM, and on the value of their intercepts. In sect. 7 we used the factorization property of Regge pole residues to show that β -type exchanges are indeed required in general, provided only that the reggeon-reggeon-particle vertices have the same overall structure as in the DRM. This

also led to some interesting phenomenological predictions, such as the absence of nonsense wrong-signature dips in the triple-Regge region of inclusive reactions.

The factorization argument shows that the α and β trajectories must intersect at nonsense wrong-signature points. Thus the highest-lying β trajectories will be those corresponding to ρ and ω exchange:

$$\beta_{\rho,\omega}(t) = \frac{1}{2} \alpha_{\rho,\omega}(t). \quad (45)$$

It is interesting to note that the Neveu-Schwarz dual model for pion scattering²⁰⁾, which is in many respects more realistic than the conventional DRM, does in fact contain ρ and ω trajectories which have intercepts greater than zero and wrong-signature zeros at $\alpha = 0$. Hence one may expect that in this dual model there should be β trajectories associated with the ρ and ω , of the form (45), although we have not explicitly verified the existence of these new trajectories in the model.

If, as seems likely, high-lying β trajectories of the form (45) do exist in realistic amplitudes, one would have to consider their effects in models that are based on a multi-Regge representation of the amplitude. The β reggeons can be exchanged in any leg or loop of a multi-Regge diagram, subject only to the constraint that they decouple from two on-shell particles. They would, in particular, affect the factorization properties of the diagram, especially as the β trajectories may themselves be degenerate (as is the case in the DRM).

0.0 0.0 0.4 5.0 2.3 0.8

The analytic structure of the reggeon-reggeon amplitude turned out, as expected, to be more complicated than that of reggeon-particle amplitudes. We restricted our study to the leading helicity pole limit for the reggeons, because even the reggeon-particle amplitude is FESR analytic only in that limit. The important new feature of the 2-reggeon $\rightarrow$ 2-particle amplitude, which is obtained by factorization from a six-point amplitude, is that there is a four-dimensionality (Gram determinant) constraint on the invariants. This constraint expresses the ratio κ of asymptotic variables (eq.(10)) as a linear function of the reggeon-reggeon energy s . To prevent the singularities in κ from contributing to the FESR we must therefore write the FESR for functions that are entire in κ , but have the same singularities in s (resonances, thresholds) as the full amplitude.

We made a comprehensive study of the analytic structure in κ of the DRM B_6 function. The $B_6(s,t)$ amplitude, which corresponds to the ordering of the external lines in fig. 2 and thus has no u-channel resonances, turned out to have a two-term structure (13) in κ . The functions $\sqrt{\nu_{ab}}$ and $\sqrt{\nu_{ba}}$ are entire in κ and thus satisfy resonance-saturated FESR's. As we discussed in sect. 4, any planar amplitude $A(s,t)$ that has no u-channel discontinuity will have the same structure in κ as $B_6(s,t)$. Consequently the discontinuities of $A(s,t)$ in s_{234} and s_{345} (which isolate the first and second terms in eq. (13a), respectively) satisfy ordinary FESR's of the form (20).

The $B_6(u,t)$ and $B_6(s,u)$ amplitudes, which do contain u-channel resonances (figs. 6b and 8c), have a three-term structure in κ (cf. eqs. (28) and (37)). We found that there was no FESR linking

the u-channel poles directly to the Regge behavior. This can be understood in general, because the pole residues must be polynomials in $1/\kappa$. Thus any function that has the same u-channel poles as the full amplitude must also have a singularity at $\kappa = 0$, which contributes to the FESR.

Taking a discontinuity in s_{234} or s_{345} eliminates u-channel resonances and so avoids the above problem. From our experience with four-point functions we know, however, that fixed pole contributions to the FESR's in general exist¹⁵⁾ and can only be canceled by considering the correct combination of s- and u-channel discontinuities. Thus an FESR of the type (20), which involves only the s-channel discontinuity, will have an additional r.h.s. contribution if the fixed pole residue is nonzero. We verified that such a contribution indeed is present for the DRM $B_6(s,u)$ amplitude.

In conclusion, we have found that discontinuities of the reggeon-reggeon amplitude in s_{234} and in s_{345} do satisfy ordinary FESR's, which are saturated by dynamical singularities in s . In general, there is a fixed pole contribution on the r.h.s. of the sum rules. As in the case of four-point amplitudes, the fixed pole residue may be expected to vanish when the u-channel is exotic²¹⁾. The two terms in the coupling of an s-channel resonance to the external reggeons contribute to separate FESR's (written for the s_{234} - and s_{345} -discontinuities of the amplitude, respectively). This fact has a natural interpretation. Taking, say, the s_{234} -discontinuity of the amplitude means, for nonpomeron exchanges α_b in fig. 2, that we get the contribution of a heavy resonance in the 234 -channel, which cascade decays through the resonance s (fig. 13). The FESR (20) for

$D_{234}A(s,t)$ thus imposes a duality constraint on this subprocess of the $3+4 \rightarrow 5+6+1+2$ reaction. Similarly, the FESR for $D_{345}A(s,t)$ constrains the production and cascade decay of a heavy resonance in the 345 -channel.

In the case of FESR's that do have contributions from singularities in κ , we estimated the magnitude of these nonresonant terms by making a numerical study of the B_6 function. Our results suggest that at small values of the momentum transfers ordinary semi-local duality between the resonances and the (α and β) Regge terms works fairly well. Thus the new contributions are not very big in this kinematic region. At larger momentum transfers, however, the nonresonant contributions are quite important. Similarly, as long as the reggeon masses are not very far from their on-shell values, the fixed pole residue in the $B_6(s,u)$ amplitude is close to its on-shell value, calculated from the $B_4(s,u)$ amplitude.

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APPENDIX A

Here we shall derive some asymptotic limits involving the confluent hypergeometric function¹⁵⁾ ${}_2F_2$ that are needed in the text. We use the integral representation

$${}_2F_2(a,b; c,d; x) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(d-b)} \int_0^1 dt du t^{a-1} (1-t)^{c-a-1} u^{b-1} (1-u)^{d-b-1} e^{tux} \quad (A1)$$

which may be easily verified by expanding the exponential in x and comparing with the standard power series representation of ${}_2F_2$.

$$1. \quad \lim_{d \rightarrow +\infty} {}_2F_2(a,b; c,d+\lambda; d+\mu) \quad (a, b, c, \lambda, \mu \text{ constant})$$

In the integral of eq. (A1), $(1-u)e^{tu} < 1$ for $0 < t, u \leq 1$. Hence the integrand vanishes exponentially with $d \rightarrow \infty$ unless $u \rightarrow 0$. If we expand in u ,

$$\begin{aligned} I &= \int_0^1 dt du t^{a-1} (1-t)^{c-a-1} u^{b-1} (1-u)^{d+\lambda-b-1} e^{tu(d+\mu)} \\ &\approx \int_0^1 dt du t^{a-1} (1-t)^{c-a-1} u^{b-1} \\ &\quad \times \exp \left[tu(d+\mu) - (d+\lambda-b-1)(u + \frac{1}{2}u^2) \right] \end{aligned} \quad (A2)$$

the terms of order $O(u^3)$ in the exponent do not contribute to the leading asymptotic behavior, and have been neglected. Similarly, for

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$d \rightarrow \infty$ we may omit the terms involving μ , λ and b in the exponent of eq. (A2).

To find the $d \rightarrow \infty$ behavior of I , we divide the t integration region into two parts α and β , given by $0 \leq t \leq t_0$ and $t_0 \leq t \leq 1$, respectively, where

$$1 - t_0 = d^{-\gamma} \quad (0 < \gamma < 1/2). \quad (A3)$$

In the α region, the effective range of the u -integration is determined by the exponent $-d(1-t)u \leq -d^{1-\gamma}u$ to be $u \leq \mathcal{O}(d^{\gamma-1})$. Since $\gamma - 1 < -1/2$, the u^2 term in the exponent can be neglected:

$$\begin{aligned} I^{(\alpha)} &\simeq \int_0^{t_0} dt \int_0^1 du t^{a-1} (1-t)^{c-a-1} u^{b-1} \exp[-d(1-t)u] \\ &= d^{-b} \int_0^{t_0} dt t^{a-1} (1-t)^{c-a-b-1} \int_0^{d(1-t)} dv v^{b-1} e^{-v} \end{aligned} \quad (A4)$$

where we defined $v = d(1-t)u$. When $d \rightarrow \infty$, $t_0 \rightarrow 1$ and $d(1-t) \geq d(1-t_0) \rightarrow \infty$. If $a+b-c < 0$, the integral converges at these limiting values, and we have

$$I^{(\alpha)} \simeq d^{-b} \frac{\Gamma(a)\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-b)} \quad (a+b-c < 0). \quad (A5)$$

On the other hand, when $a+b-c > 0$, the leading behavior of $I^{(\alpha)}$ in eq. (A4) comes from $t \simeq t_0$, so that

$$I^{(\alpha)} \simeq d^{-b+\gamma(a+b-c)} \frac{\Gamma(b)}{a+b-c} \quad (a+b-c > 0). \quad (A6)$$

The contribution from the β integration region of eq. (A2) is, after a substitution $x = d(1-t)u$; $y = \frac{1}{2} du^2$,

$$I^{(\beta)} \simeq d^{\frac{1}{2}(a-b-c)} 2^{\frac{1}{2}(a+b-c-2)} \int_0^{d^{1-\gamma}} dx \int_{y_0}^{d/2} dy x^{c-a-1} y^{\frac{1}{2}(a+b-c)-1} e^{-x-y}$$

where $y_0 = \frac{1}{2} x^2 d^{2\gamma-1}$. Since only finite values of x contribute as $d \rightarrow \infty$, the lower limit $y_0 \rightarrow 0$. For $a+b-c > 0$ the integral is regular at $y_0 = 0$ so that

$$I^{(\beta)} \simeq d^{-b} d^{\frac{1}{2}(a+b-c)} 2^{\frac{1}{2}(a+b-c-2)} \Gamma(c-a) \Gamma\left[\frac{1}{2}(a+b-c)\right], \quad (a+b-c > 0). \quad (A7)$$

When $a+b-c < 0$ the leading term comes from $y \approx y_0$:

$$I^{(\beta)} \simeq d^{-b+\gamma(a+b-c)} \frac{\Gamma(b)}{c-a-b} \quad (a+b-c < 0). \quad (A8)$$

From eqs. (A5)–(A8) it is clear that the leading contribution to $I = I^{(\alpha)} + I^{(\beta)}$ is given by eq. (A5) ($a+b-c < 0$) or eq. (A7) ($a+b-c > 0$). Thus for the ${}_2F_2$ function in eq. (A1) we have found

$$\begin{aligned} \lim_{d \rightarrow \infty} {}_2F_2(a, b; c, d + \lambda; d + \mu) &= \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \\ &+ \frac{\Gamma(c) \Gamma\left[\frac{1}{2}(a+b-c)\right]}{2\Gamma(a) \Gamma(b)} (2d)^{\frac{1}{2}(a+b-c)}. \end{aligned} \quad (A9)$$

The same result was obtained in ref. ¹¹⁾ using the power series representation of ${}_2F_2$.

$$2. \quad \lim_{d \rightarrow -\infty - i\epsilon} {}_2F_2(a, b; c, d + \lambda; d + \mu) \quad (a, b, c, \lambda, \mu \text{ constant})$$

As usual in the DRM, this limit should be calculated by letting $d \rightarrow \infty e^{-i(\pi-\delta)}$ with $\delta \rightarrow 0^+$. It could be obtained most easily by analytic continuation of eq. (A9). However, since we have seen that such a procedure is not always correct^{*)}, we prefer to take

*) Compare ref. 11). We do not expect these complications in the present case, as the ratio d/x in the ${}_2F_2$ function (A1) can be kept equal to unity during the continuation.

the limit directly in the indicated direction. This calculation is also useful since a generalization of it will be required in appendix B, where we evaluate a single Regge limit of the B_6 function.

The integral representation (A1) of ${}_2F_2$ is convergent only when $d > b > 0$. Thus before taking the $d \rightarrow -\infty$ limit we must find a new representation that remains valid for large negative d . A convenient way of doing this is to regard the integrand in eq. (A1) as a discontinuity:

$$(1-u)^{d-b-1} = \frac{1}{2i \sin \pi(d-b-1)} \text{Disc}_u (u-1)^{d-b-1} \quad (0 \leq u \leq 1). \quad (A10)$$

The u -integration is then equivalent to the contour integral shown in fig. 14a

$${}_2F_2(a, b; c, d + \lambda; d + \mu) = \frac{\Gamma(c) \Gamma(d + \lambda)}{\Gamma(a) \Gamma(b) \Gamma(c-a) \Gamma(d + \lambda - b)}$$

$$\times \frac{1}{\sin \pi(d + \lambda - b - 1)}$$

$$\times \frac{1}{2i} \int_0^1 dt \int_C du t^{a-1} (1-t)^{c-a-1} u^{b-1} (u-1)^{d+\lambda-b-1} e^{tu(d+\mu)}. \quad (A11)$$

This representation is convergent for all d . As $d \rightarrow -\infty$, however, the integrand will grow exponentially in parts of the integration domain. In order to find the asymptotic limit we therefore distort the u -contour further, as shown in fig. 14b. For $d < \min\{-\lambda, -\mu\}$ the integral along the semicircle in fig. 14b vanishes as the radius tends to ∞ . We are thus left with a contour along the imaginary u -axis. On this contour the integrand vanishes exponentially with $d \rightarrow -\infty$ unless $u \rightarrow 0$, so that we need keep only the lowest powers in u of the integrand. Furthermore, since $\arg(u-1) \rightarrow -\pi$ when $u \rightarrow i0^-$, the integral along the negative imaginary u -axis vanishes exponentially as $\text{Im } d \rightarrow -\infty$.

Putting $u = iv$ in eq. (A11), and keeping only the lowest powers of v , we have for $d \rightarrow -\infty - i\epsilon$,

$${}_2F_2(a, b; c, d + \lambda; d + \mu) \simeq \frac{\Gamma(c)}{\Gamma(a) \Gamma(b) \Gamma(c-a)} e^{-i\pi b/2} (-d)^b \times \int_0^1 dt t^{a-1} (1-t)^{c-a-1} \int_0^\infty dv v^{b-1} \exp\left[-idv(1-t) + \frac{1}{2} dv^2\right].$$

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The limiting behavior as $d \rightarrow -\infty$ of this integral can be evaluated analogously to the $d \rightarrow +\infty$ limit of eq. (A2), by dividing the t -integration region into two parts. We shall not repeat those arguments here. The result is

$$\lim_{d \rightarrow -\infty - i\epsilon} {}_2F_2(a, b; c, d + \lambda; d + \mu) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} + \frac{\Gamma(c) \Gamma\left[\frac{1}{2}(a + b - c)\right]}{2\Gamma(a) \Gamma(b)} e^{-i\pi(a+b-c)/2} (-2d)^{\frac{1}{2}(a+b-c)} \quad (A12)$$

As expected, this agrees with what one obtains by letting $d \rightarrow e^{-i\pi}d$ in eq. (A9). It is also easy to verify that the limit of ${}_2F_2(a, b; c, d + \lambda; d + \mu)$ where $d \rightarrow -\infty + i\epsilon$ is the complex conjugate of eq. (A12).

$$3. \quad \lim_{a \rightarrow +\infty} e^a {}_2F_2(a + \lambda, a + \mu; a + \nu, a + \sigma; -a) \quad (\lambda, \mu, \nu, \sigma \text{ constant})$$

Using the integral representation (A1), we have for $a \rightarrow +\infty$,

$$e^a {}_2F_2(a + \lambda, a + \mu; a + \nu, a + \sigma; -a) \simeq \frac{a^{v+\sigma-\lambda-\mu}}{\Gamma(v - \lambda) \Gamma(\sigma - \mu)} \times \int_0^1 dt du t^{a+\lambda-1} (1-t)^{v-\lambda-1} u^{a+\mu-1} (1-u)^{\sigma-\mu-1} \times \exp[a(1-tu)] .$$

Since $tu e^{1-tu} < 1$ for $tu < 1$, the integrand vanishes exponentially as $a \rightarrow \infty$ unless $tu \rightarrow 1$. To lowest nonvanishing order in $1 - tu$ we have

$$e^a {}_2F_2(a + \lambda, a + \mu; a + \nu, a + \sigma; -a) \simeq \frac{a^{v+\sigma-\lambda-\mu}}{\Gamma(v - \lambda) \Gamma(\sigma - \mu)}$$

$$\times \int_0^1 dt du (1-t)^{v-\lambda-1} (1-u)^{\sigma-\mu-1} \exp\left[-\frac{1}{2} a(1-tu)^2\right] .$$

In terms of the new variables $1 - t = x_1/\sqrt{a}$, $1 - u = x_2/\sqrt{a}$ this becomes, for large a ,

$$e^a {}_2F_2(a + \lambda, a + \mu; a + \nu, a + \sigma; -a) \simeq \frac{a^{\frac{1}{2}(v+\sigma-\lambda-\mu)}}{\Gamma(v - \lambda) \Gamma(\sigma - \mu)}$$

$$\times \int_0^\infty dx_1 dx_2 x_1^{v-\lambda-1} x_2^{\sigma-\mu-1} \exp\left[-\frac{1}{2} (x_1 + x_2)^2\right] . \quad (A13)$$

The integral in eq. (A13) can be evaluated using the further substitution $y = \frac{1}{2} (x_1 + x_2)^2$, $z = x_1/x_2$. The result is

$$\lim_{a \rightarrow +\infty} e^a {}_2F_2(a + \lambda, a + \mu; a + \nu, a + \sigma; -a) = \left(\frac{1}{2} a\right)^{\frac{1}{2}(v+\sigma-\lambda-\mu)} \frac{\sqrt{\pi}}{\Gamma\left[\frac{1}{2}(v + \sigma - \lambda - \mu) + \frac{1}{2}\right]} . \quad (A14)$$

$$4. \quad \lim_{a \rightarrow -\infty - i\epsilon} e^a {}_2F_2(a + \lambda, a + \mu; a + \nu, a + \sigma; -a) \quad (\lambda, \mu, \nu, \sigma \text{ constant})$$

The exact definition of the limit is again $a \rightarrow \infty e^{-i(\pi-\delta)}$ with $\delta \rightarrow 0^+$. In order to get a convergent integral representation we use the trick (A10), now applied to both the t^{a-1} and u^{b-1}

factors in eq. (A1):

$$e^a {}_2F_2(a + \lambda, a + \mu; a + \nu, a + \sigma; -a) \\ \approx \frac{(-a)^{\nu+\sigma-\lambda-\mu} e^{-i\pi(\nu+\sigma-\lambda-\mu)}}{\Gamma(\nu - \lambda)\Gamma(\sigma - \mu)\sin \pi(a + \lambda)\sin \pi(a + \mu)}$$

$$\times \frac{1}{(2i)^2} \int_{\mathcal{C}} dt du (-t)^{a+\lambda-1} (1-t)^{\nu-\lambda-1} (-u)^{a+\mu-1} (1-u)^{\sigma-\mu-1} \\ \times \exp[a(1-tu)] .$$

The t - and u -integration contours $\mathcal{C}$ can be chosen to be the unit circle around the origin (fig. 15). Then the integrand vanishes exponentially with $\text{Re}(a) \rightarrow -\infty$ unless $ut \rightarrow 1$. Furthermore, due to the sine factors in front of the integral, the expression is exponentially damped for $\text{Im}(a) \rightarrow -\infty$ except if $\arg(-t) \rightarrow \pi$ and $\arg(-u) \rightarrow \pi$. Hence we may substitute $-t = \exp[i(\pi - \phi_1)]$; $-u = \exp[i(\pi - \phi_2)]$ and keep only the lowest powers of ϕ_1, ϕ_2 :

$$e^a {}_2F_2(a + \lambda, a + \mu; a + \nu, a + \sigma; -a) \approx \frac{(-a)^{\nu+\sigma-\lambda-\mu} e^{-i\pi \frac{1}{2}(\nu+\sigma-\lambda-\mu)}}{\Gamma(\nu - \lambda)\Gamma(\sigma - \mu)} \\ \times \int_0^{2\pi} d\phi_1 d\phi_2 \phi_1^{\nu-\lambda-1} \phi_2^{\sigma-\mu-1} \exp\left[\frac{1}{2}a(\phi_1 + \phi_2)^2\right] .$$

The substitution $\phi_1 = x_1/\sqrt{-a}$; $\phi_2 = x_2/\sqrt{-a}$ makes this integral identical to the one in eq. (A13). The final result is

$$\lim_{a \rightarrow -\infty - i\epsilon} e^a {}_2F_2(a + \lambda, a + \mu; a + \nu, a + \sigma; -a) \\ = \left(-\frac{1}{2}a\right)^{\frac{1}{2}(\nu+\sigma-\lambda-\mu)} \frac{e^{-i\pi \frac{1}{2}(\nu+\sigma-\lambda-\mu)} \sqrt{\pi}}{\Gamma\left[\frac{1}{2}(\nu + \sigma - \lambda - \mu) + \frac{1}{2}\right]} . \quad (A15)$$

This would have been obtained by the analytic continuation $a \rightarrow e^{-i\pi}a$ of eq. (A14). In the limit $a \rightarrow -\infty + i\epsilon$ the result is the complex conjugate of eq. (A15).

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APPENDIX B

Here we want to calculate the single Regge limit of B_6 shown in fig. 10, and discuss some properties of the contribution from β exchange. From the results of the double helicity-pole limit in sect. 3, and the discussion in sect. 7 concerning the analytic continuation of high energy limits, we believe that the limit specified by the ordering of the lines in fig. 10 is the only one to which β exchange contributes. A twist on the reggeon line in fig. 10 reverses the ordering of the lines in the lower cluster and so preserves the order of incoming vs. outgoing lines in the full diagram. Thus the twisted diagram can be obtained from the untwisted one by a simple substitution of variables. This will enable us to determine the signature of the β trajectory.

In the limit of fig. 10,

$$\alpha_{34}, \alpha_{61} \rightarrow \infty e^{i\delta} \quad (B1)$$

$$\alpha_{234}, \alpha_{345} \rightarrow -\infty$$

where δ is a small positive number. Furthermore, the constraint (7) must be satisfied. In order to derive an integral representation of B_6 that is valid in the region (B1), we shall interpret the integral over the variables conjugate to α_{34} and α_{61} as a contour integral, analogously to what was done in appendix A for the ${}_2F_2$ function^{*}).

^{*}) The same method has been used by Mandelstam²²⁾ for the B_4 function.

We therefore start from the twisted representation²³⁾ of B_6 shown in fig. 16:

$$B_6 = \int_0^1 du \, du_1 \, du_2 \, u_1^{-\alpha_{34}-1} (1-u_1)^{-\alpha_{23}-1} u^{-\alpha_{234}-1} (1-u)^{-\alpha_{345}-1} \\ \times u_2^{-\alpha_{61}-1} (1-u_2)^{-\alpha_{56}-1} \\ \times (1-uu_1)^{-\alpha_t+\alpha_{12}+\alpha_{23}} (1-uu_2)^{-\alpha_t+\alpha_{45}+\alpha_{56}} \\ \times (1-uu_1-uu_2+uu_1u_2)^{-\alpha_{45}-\alpha_{12}+\alpha_t+\alpha_{345}} \quad (B2)$$

where $\alpha_t \equiv \alpha_{123}$. Using the identity (A10) on the factors $u_1^{-\alpha_{34}-1}$ and $u_2^{-\alpha_{61}-1}$ we get

$$B_6 = \frac{(1/2i)^2}{\sin \pi \alpha_{34} \sin \pi \alpha_{61}} \int_C du_1 \, du_2 \int_0^1 du (-u_1)^{-\alpha_{34}-1} (-u_2)^{-\alpha_{61}-1} \\ \times (1-u_1)^{-\alpha_{23}-1} (1-u_2)^{-\alpha_{56}-1} \\ \times u^{-\alpha_{234}-1} (1-u)^{-\alpha_t+\alpha_{23}+\alpha_{56}-1} \left[1 + \frac{u}{1-u} (1-u_1) \right]^{-\alpha_t+\alpha_{12}+\alpha_{23}} \\ \times \left[1 + \frac{u}{1-u} (1-u_2) \right]^{-\alpha_t+\alpha_{45}+\alpha_{56}} \\ \times \left[1 + \frac{u}{1-u} (1-u_1)(1-u_2) \right]^{-\alpha_{45}-\alpha_{12}+\alpha_t+\alpha_{345}} \quad (B3)$$

Here the u_1 and u_2 integration contours C are loops that begin

and end at +1, and circle origo once in the negative direction (without crossing the positive real axis).

The integral (B3) is convergent for all values of $\alpha_{34}, \alpha_{61} > 0$ and $\alpha_{234}, \alpha_{345} < 0$. For the integrand to remain power bounded in the limit (B1) we must have

$$|u_1| \geq 1; \quad |u_2| \geq 1 \quad (B4a)$$

$$|u| \leq 1 \quad (B4b)$$

$$\left| 1 + \frac{u}{1-u} (1-u_1)(1-u_2) \right| \geq 1 \quad (B4c)$$

everywhere in the (suitably modified) integration region. We therefore distort each of the contours C in (B3) into two rays, emanating from +1 at a small positive and negative angle, respectively, w.r.t. the negative real axis (for sufficiently large values of the variables in (B1) the arcs connecting the rays at infinity can be neglected.).

Let us for the moment concentrate on the rays that lie in the lower half u_1 and u_2 planes. The conditions (B4a) require that each ray be rotated so that it makes an angle $\geq \pi/2$ with the negative real axis. Actually, when both rays are parallel to the imaginary axis (fig. 17a) a singularity of the integrand, coming from the last factor in (B3), starts entering the integration region. Hence the rays cannot be simply rotated further. At the same time, condition (B4c) is violated.

To get out of this dilemma, we shall modify the path of the u -integration in (B3). In fact, if we integrate along the semicircle shown in fig. 17b:

$$u = \frac{1}{2} (1 - e^{i\phi}) \quad \phi: 0 \rightarrow \pi \quad (B5)$$

the condition (B4b) is still satisfied and

$$\frac{u}{1-u} = -i \frac{\sin \phi}{1 + \cos \phi}$$

has a constant phase $-\pi/2$. Thus,

$$\arg \left[\frac{u}{1-u} (1-u_1)(1-u_2) \right] = \frac{\pi}{2}$$

and the inequality (B4c) is satisfied.

To recapitulate, we have found an integral representation of the form (B3) for B_6 that remains valid in the limit (B1). When the paths of integration are chosen as in fig. 17, the integrand is exponentially damped in the variables of (B1) everywhere except near $u_1, u_2, u \simeq 1$, which is therefore the region that determines the leading asymptotic behavior of B_6 . We recall that in the above argument the rays in the upper half u_1, u_2 planes were left out. These contributions can be treated in an analogous manner, leading again to an integral that is dominated by the $u_1, u_2, u \simeq 1$ region in the limit (B1). However, the phases of $-u_1$ and $-u_2$ in (B3) are then $\simeq -\pi$, leading to factors $\exp(i\pi\alpha_{34})$ and $\exp(i\pi\alpha_{61})$ that are exponentially damped as $\text{Im } \alpha_{34}, \text{Im } \alpha_{61} \rightarrow +\infty$. The rays in the lower half u_1, u_2 planes, on the other hand, produce phase factors $\exp(-i\pi\alpha_{34})$ and $\exp(-i\pi\alpha_{61})$ which precisely balance the behavior of the $\sin \pi \alpha_{34} \sin \pi \alpha_{61}$ factor in the denominator of (B3). Thus the paths shown in fig. 17 are sufficient for calculating the leading behavior of B_6 in the high energy limit (B1).

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At this stage we can already see how the β exchange contribution is going to emerge. As usual, the α contribution comes from the region where $(1 - u_1)\alpha_{34}$, $(1 - u_2)\alpha_{61}$ and $(1 - u)(-\alpha_{234})$ are finite. Because the paths of integration in fig. 17 are perpendicular to the real axes, the various factors in the integrand of (B3) acquire rapidly rotating phases as soon as u , u_1 or u_2 get outside this region. The phases rotate in different directions, however, so that there can be a small part of the integration domain where the full phase is stationary. The effective range of integration in u_1 , u_2 and u is then enlarged, and is finally cut off by terms that describe the curvature of the path away from the unit circle. These terms require $(1 - u_1)^2\alpha_{34}$, etc. to be finite, and thus give rise to the $\frac{1}{2}$ unit slope of the β trajectory.

According to this discussion we should expand the factors in (B3) as powers of $(1 - u_1)$, $(1 - u_2)$ and $(1 - u)$, keeping the linear and quadratic terms. Thus, e.g.,

$$u_1^{-\alpha_{34}-1} = \exp \left\{ (-\alpha_{34} - 1) \log[1 - (1 - u_1)] \right\} \\ \simeq \exp \left\{ \alpha_{34} \left[(1 - u_1) + \frac{1}{2} (1 - u_1)^2 \right] \right\}.$$

Similarly, using eq. (B5) and defining $x = \frac{1}{2}(\pi - \phi)$,

$$u^{-\alpha_{234}-1} = \left[\frac{1}{2} (1 - e^{i\phi}) \right]^{-\alpha_{234}-1} \simeq \exp \left[\alpha_{234} \left(ix + \frac{1}{2} x^2 \right) \right].$$

In terms of the new variables

$$t_1 = -i \frac{1 - u_1}{x}$$

$$t_2 = -i \frac{1 - u_2}{x}$$

we get in this way an approximate form of eq. (B3), which is correct to leading order in the limit (B1):

$$B_6 \simeq e^{-i\pi\alpha_t/2} \int_0^\infty dt_1 dt_2 dx t_1^{-\alpha_{23}-1} t_2^{-\alpha_{56}-1} (1 + t_1)^{-\alpha_t+\alpha_{12}+\alpha_{23}} \\ \times (1 + t_2)^{-\alpha_t+\alpha_{45}+\alpha_{56}} x^{-\alpha_t-1} \exp \left[ix\xi_1 - \frac{1}{2} x^2 \xi_2 \right] \quad (B6)$$

where

$$\xi_1 = \alpha_{34} t_1 + \alpha_{61} t_2 + \alpha_{234} + \alpha_{345} t_1 t_2 \quad (B7) \\ \xi_2 = \alpha_{34} t_1^2 + \alpha_{61} t_2^2 - \alpha_{234} - \alpha_{345} t_1^2 t_2^2.$$

When the constraint (7) is imposed, ξ_1 and ξ_2 factorize:

$$\xi_1 = \alpha_{345} (t_1 - t_1^0)(t_2 - t_2^0) \quad (B8)$$

$$\xi_2 = -\alpha_{345} (t_1^2 + t_1^0)(t_2^2 + t_2^0)$$

where

$$t_1^0 = -\frac{\alpha_{234}}{\alpha_{34}} \quad (B9) \\ t_2^0 = -\frac{\alpha_{234}}{\alpha_{61}}.$$

In our limit (B1), t_1^0 and t_2^0 are positive, so that ξ_1 vanishes inside the integration region when $t_1 = t_1^0$ or $t_2 = t_2^0$. These are the lines where the full phase of the integrand is stationary and, as we argued above, have to be treated with extra care. Consequently, we divide the integration region into several parts by the lines (dashed in fig. 18):

$$|t_1 - t_1^0| = \alpha_{34}^{-\epsilon} \quad (0 < \epsilon < \frac{1}{4}) \quad (B10)$$

$$|t_2 - t_2^0| = \alpha_{34}^{-\epsilon}$$

which isolate the zeros of ξ_1 . This is closely analogous to the treatment of the ${}_2F_2$ function (cf. eq. (A3)). In particular, we find that whenever the leading asymptotic behavior in any one region comes from one of the boundaries (B10), then that behavior is non-leading compared to the contribution from the region on the other side of the boundary in question. This is reasonable, as the precise location of the boundary is anyhow immaterial. For conciseness, we shall not verify this explicitly below.

We shall now discuss in detail the contributions to the integral (B6) from the various regions in fig. 18. It should be noted that the limit (B1) can at this stage be taken with α_{34} and α_{61} on the real axis (i.e., $\delta = 0$ in (B1)).

Region I: $|t_1 - t_1^0| \geq \alpha_{34}^{-\epsilon}$ and $|t_2 - t_2^0| \geq \alpha_{34}^{-\epsilon}$

From the qualitative discussion above, we expect the entire α -exchange term to come from this region. We can slightly tilt the path of integration in x of eq. (B6):

$$\int_0^\infty dx \rightarrow \int_0^\infty e^{\pm i\gamma} dx \quad (\gamma > 0) \quad (B11)$$

where the $+$ ($-$) sign is chosen when $\xi_1 > 0$ (< 0). Then $x\xi_1$ in the exponent of (B6) acquires a positive imaginary part, and so this term cuts off the x -integration at

$$x \lesssim \mathcal{O}(\alpha_{34}^{-1+2\epsilon}) \quad (B12)$$

(all large variables α_{34} , α_{345} , etc. are taken to be proportional to each other.). Since $-1+2\epsilon < -\frac{1}{2}$, the term $-\frac{1}{2}x^2\xi_2$ can be neglected in the whole range (B12). The x -integration can then be done explicitly, after rotating the path through an additional angle $\frac{\pi}{2} - \gamma$ ($\xi_1 > 0$) or $-\frac{\pi}{2} + \gamma$ ($\xi_1 < 0$). The result is

$$B_6^{(I)} \simeq \Gamma(-\alpha_t) \int_I dt_1 dt_2 t_1^{-\alpha_{23}-1} t_2^{-\alpha_{56}-1} (1+t_1)^{-\alpha_t+\alpha_{12}+\alpha_{23}} \\ \times (1+t_2)^{-\alpha_t+\alpha_{45}+\alpha_{56}} |\xi_1|^{\alpha_t} \left[e^{-i\pi\alpha_t} \theta(\xi_1) + \theta(-\xi_1) \right] \quad (B13)$$

For $\alpha_t > -1$, the leading behavior of $B_6^{(I)}$ is obtained by letting

$$\int_I dt_1 dt_2 \rightarrow \int_0^\infty dt_1 dt_2$$

in eq. (B13). After a change of integration variables,

$$x_1 = \frac{1}{1+t_1} ; \quad x_2 = \frac{t_2}{1+t_2}$$

one can see that $B_6^{(I)}$ agrees with the α -exchange contribution as given by eq. (44) of ref. 12). Thus the method, used in ref. 12), of analytically continuing asymptotic limits gives the right answer for α -exchange.

When $\alpha_t < -1$, the leading behavior of $B_6^{(I)}$ in eq. (B13) comes from the boundaries (B10) of the integration region. As we already remarked, this behavior is nonleading compared to the β -exchange term that we shall find below.

Region II: $|t - t_1^0| < \alpha_{34}^{-\epsilon}$ and $|t_2 - t_2^0| \geq \alpha_{34}^{-\epsilon}$

We can put $t_1 = t_1^0$ in all factors of (B6) except in ξ_1 , which is proportional to $t_1 - t_1^0$. The t_1 integration can then be trivially done. After the substitution

$$y = -x |t_2 - t_2^0| \alpha_{345}^{-\epsilon} \quad (B14)$$

we get the expression

$$B_6^{(II)} \simeq e^{-i\pi\alpha_t/2} (t_1^0)^{-\alpha_{23}-1} (1+t_1^0)^{-\alpha_t+\alpha_{12}+\alpha_{23}} 1(-\alpha_{345})^{\alpha_t-\epsilon(\alpha_t+1)} \\ \times \int_{II} dt_2 t_2^{-\alpha_{56}-1} (1+t_2)^{-\alpha_t+\alpha_{45}+\alpha_{56}} |t_2 - t_2^0|^{\alpha_t} \\ \times \int_0^\infty dy y^{-\alpha_t-2} \left[e^{-iy} - e^{iy} \right] e^{-\frac{1}{2}x^2 \xi_2} \quad (B15)$$

From eqs. (B8) and (B14) one can see that the cutoff in y provided by the factor $\exp(-\frac{1}{2}x^2 \xi_2)$ tends to infinity in the limit (B1).

Hence the y -integral in (B15) tends to a constant. For $-1 < \alpha_t < 0$ the t_2 -integral also approaches a constant, so that the behavior of $B_6^{(II)}$ is given by the effective trajectory

$$\alpha_t - \epsilon(\alpha_t + 1) < \alpha_t \quad (\alpha_t > -1)$$

and is thus nonleading compared to $B_6^{(I)}$. When $\alpha_t < -1$, the t_2 -integral is dominated by the $|t_2 - t_2^0| \simeq \alpha_{34}^{-\epsilon}$ region, and the effective trajectory is

$$\alpha_t - 2\epsilon(\alpha_t + 1) = \frac{1}{2}\alpha_t - \frac{1}{2} + (\frac{1}{2} - 2\epsilon)(\alpha_t + 1) < \frac{1}{2}\alpha_t - \frac{1}{2},$$

$$(\alpha_t < -1).$$

This behavior is nonleading compared to β_t -exchange (to be found below). Hence the contribution to the integral (B6) from region II in fig. 18 gives a nonleading behavior in the limit (B1) for all values of t , and can be ignored.

Region III: $|t_1 - t_1^0| \geq \alpha_{34}^{-\epsilon}$ and $|t_2 - t_2^0| < \alpha_{34}^{-\epsilon}$

By symmetry arguments, the treatment and conclusions of region II are also applicable here. Thus there is no leading contribution from this region of integration.

Region IV: $|t_1 - t_1^0| < \alpha_{34}^{-\epsilon}$ and $|t_2 - t_2^0| < \alpha_{34}^{-\epsilon}$

We can put $t_1 = t_1^0$ and $t_2 = t_2^0$ in all factors of (B6) except in ξ_1 . Performing the t_2 -integration we get

$$B_6^{(IV)} \approx e^{-i\pi\alpha_t/2} (t_1^0)^{-\alpha_{23}-1} (t_2^0)^{-\alpha_{56}-1} (1+t_1^0)^{-\alpha_t+\alpha_{12}+\alpha_{23}} \\ \times (1+t_2^0)^{-\alpha_t+\alpha_{45}+\alpha_{56}} I \quad (B16)$$

where

$$I = \int_{IV} d(t_1 - t_1^0) \int_0^\infty dx \frac{x^{-\alpha_t-1}}{1 - \alpha_{345} x(t_1 - t_1^0)} \\ \times \left[\exp(i\alpha_{345} \alpha_{34}^{-\epsilon} (t_1 - t_1^0)) - \exp(-i\alpha_{345} \alpha_{34}^{-\epsilon} (t_1 - t_1^0)) \right] \\ \times \exp\left(\frac{1}{2} x^2 \alpha_{345} c\right). \quad (B17)$$

Here $c = t_1^0 t_2^0 (1+t_1^0)(1+t_2^0)$. Since the integrand is symmetric under $(t_1 - t_1^0) \rightarrow -(t_1 - t_1^0)$ we have

$$\int_{IV} d(t_1 - t_1^0) = 2 \int_0^{\alpha_{34}^{-\epsilon}} d(t_1 - t_1^0).$$

Next we introduce the new variables

$$y = -\alpha_{345} \alpha_{34}^{-\epsilon} x(t_1 - t_1^0)$$

$$z = -\frac{1}{2} x^2 \alpha_{345} c$$

and get

$$I = 12^{-\frac{1}{2} \alpha_t - \frac{1}{2}} c^{\frac{1}{2} \alpha_t} (-\alpha_{345})^{\frac{1}{2} \alpha_t - \frac{1}{2}} \int_0^\infty \frac{dy}{y} \left[e^{-iy} - e^{iy} \right] \\ \times \int_{z_0}^\infty dz z^{-\frac{1}{2} \alpha_t - \frac{3}{2}} e^{-z} \quad (B18)$$

where

$$z_0 = \frac{1}{2} y^2 (-\alpha_{345})^{-1} \alpha_{34}^{4\epsilon} c$$

satisfies $z_0 \rightarrow 0$ for fixed y in the limit (B1).

When $-1 < \alpha_t < 0$, the z -integral in eq. (B18) is dominated by the $z \approx z_0$ contribution. This corresponds to the boundary $t_1 - t_1^0 \approx \alpha_{34}^{-\epsilon}$ in fig. 18, and the behavior of $B_6^{(IV)}$ can readily be seen to be nonleading compared to α -exchange.

For $\alpha_t < -1$, we can let $z_0 \rightarrow 0$ in (B18). The z -integral then gives $\Gamma(-\frac{1}{2} \alpha_t - \frac{1}{2})$, and the y -integral is $-i\pi$. With this result eq. (B16) becomes, using (B9),

$$B_6^{(IV)} \approx -i\pi e^{-i\pi\beta_t} 2^{-\beta_t-1} \Gamma(-\beta_t-1) (-\alpha_{345})^{\beta_t} \left(-\frac{\alpha_{234}}{\alpha_{34}}\right)^{\beta_t-\alpha_{23}} \\ \times \left(-\frac{\alpha_{234}}{\alpha_{61}}\right)^{\beta_t-\alpha_{56}} \left(1 - \frac{\alpha_{234}}{\alpha_{34}}\right)^{-\beta_t+\alpha_{12}+\alpha_{23}} \left(1 - \frac{\alpha_{234}}{\alpha_{61}}\right)^{-\beta_t+\alpha_{45}+\alpha_{56}} \\ = B_6^{(\beta)} \quad (B19)$$

where $\beta_t = \frac{1}{2} \alpha_t - \frac{1}{2}$. This expression represents the contribution

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$B_6^{(\beta)}$ of β -exchange in the high energy limit (B1).

It is important to notice that the entire β -exchange term originates from region IV in fig. 18, which isolates the intersection of the straight lines $t_1 = t_1^0$ and $t_2 = t_2^0$. According to eq. (B8), these lines are the solution to the equation $\xi_1(t_1, t_2) = 0$. This is, however, only the case when the constraint (7) is satisfied. From (B7) we see that in general the solution to $\xi_1(t_1, t_2) = 0$ is a hyperbola in the (t_1, t_2) -plane, the branches of which do not intersect. Just as there is no β -exchange term coming from regions II and III in fig. 18, we do not expect any β -exchange to arise from regions surrounding the branches of a hyperbola. Thus the $B_6^{(\beta)}$ term would be absent for any value $\eta \neq 1$ in eq. (7). This is a generalization of the corresponding result in the double HP limit¹¹⁾.

The signature of the β trajectory can be determined by calculating also the diagram with a twist on the reggeon line in fig. 10. The twist reverses the ordering of, say, the lines 4, 5 and 6 in fig. 10. It is clear that the expression for the β -exchange contribution to the twisted diagram can be obtained from eq. (B19) by the interchange of variables

$$\alpha_{34} \leftrightarrow -\alpha_{345}; \quad \alpha_{61} \leftrightarrow -\alpha_{234}; \quad \alpha_{45} \leftrightarrow \alpha_{56}. \quad (B20)$$

Using the constraint (7) one can readily show that eq. (B19) is invariant under the interchange (B20). Thus when we add the untwisted and twisted diagrams with weights 1 and τ_t , respectively, the signature factor for β -exchange becomes simply $1 + \tau_t$, i.e., the β -trajectory has pure positive signature.

There are several equivalent expressions for $B_6^{(\beta)}$, which can be obtained from eq. (B19) using the constraint (7). Thus we have, for example,

$$B_6^{(\beta)} = -i\pi e^{-1\pi\beta_t} (1 + \tau_t)^2 t^{-\beta_t-1} \Gamma(-\beta_t - 1) \alpha_{61}^{\beta_t} \\ \times \left[\left(-\frac{\alpha_{345}}{\alpha_{61}} \right)^{\beta_t - \alpha_{12}} \left(1 - \frac{\alpha_{245}}{\alpha_{61}} \right)^{-\beta_t + \alpha_{12} + \alpha_{23}} \right] \\ \times \left[\left(-\frac{\alpha_{234}}{\alpha_{61}} \right)^{\beta_t - \alpha_{56}} \left(1 - \frac{\alpha_{234}}{\alpha_{61}} \right)^{-\beta_t + \alpha_{45} + \alpha_{56}} \right]. \quad (B21)$$

Here we included the signature factor for β_t and wrote the residue in an explicitly factorized form. The existence of these equivalent forms for $B_6^{(\beta)}$ ensures that the various symmetries of the diagram in fig. 10 are satisfied, although the symmetries are not explicit in either (B19) or (B21).

Finally we shall study how $B_6^{(\beta)}$ behaves under the further Regge limit corresponding to fig. 2c. Denoting the Toller variables at the particle 1 and 6 reggeon vertices by

$$\kappa_1 = \frac{\alpha_{12} \alpha_{61}}{\alpha_{345}} \\ \kappa_6 = \frac{\alpha_{56} \alpha_{61}}{\alpha_{234}} \quad (B22)$$

the limit is most easily obtained from eq. (B21) by letting

$\alpha_{12}, \alpha_{56} \rightarrow -\infty$ with κ_1 and κ_6 fixed (in the helicity pole limit $\kappa_1 = \kappa_6 = 0$). We get for the limit of fig. 2c,

$$\begin{aligned}
 {}_{B_6}^{(\beta)} \tau_a \tau_b &\simeq \tau_a (-\alpha_{12})^{\alpha_{23}} \begin{bmatrix} -\alpha_{23} & -\kappa_1 \\ \kappa_1 & e \end{bmatrix} \begin{bmatrix} -i\pi e^{-i\pi\beta_t} (1 + \tau_t) 2^{-\beta_t-1} \\ -i\pi e^{-i\pi\beta_t} (1 + \tau_t) 2^{-\beta_t-1} \end{bmatrix} \\
 &\times \Gamma(-\beta_t - 1) \alpha_{61}^{\beta_t} \begin{bmatrix} -\alpha_{45} & -\kappa_6 \\ \kappa_6 & e \end{bmatrix} \tau_b (-\alpha_{56})^{\alpha_{45}}. \quad (B23)
 \end{aligned}$$

The factor $\tau_a \tau_b$ has been included to indicate that when the reggeons α_a, α_b in fig. 2c are signaturized, only the term with twists on both reggeons contributes to β_t -exchange. The expression (B23) is factorizable into a product of Regge propagators and vertices. The reggeon-reggeon-particle vertices have a correct analytic structure in κ_1 and κ_6 .

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FIGURE CAPTIONS

- Fig. 1 (a) Diagram visualizing the unitarity sum over a cluster of produced particles in reggeon-reggeon scattering.
- (b) The process in (a) approximated through semilocal duality by crossed channel Regge exchange.
- Fig. 2 (a) The reggeon-reggeon $\rightarrow$ particle-particle amplitude as obtained from a six-point amplitude.
- (b) An s-channel resonance contribution to the amplitude in (a).
- (c) A t-channel Regge exchange contribution to the amplitude in (a).
- Fig. 3 The double Regge limit of a $2 \rightarrow 3$ amplitude.
- Fig. 4 (a) The semilocal duality property of $\mathcal{V}_{ab}$ as described by eq. (18). The example is for $t_a = -0.2$, $t_b = 0.0$, $t = -0.3$ and $\alpha_i(t_i) = -0.2 + t_i$. The horizontal lines are the contributions from the resonance residues at $\alpha_s = n$. The curves are the corresponding Regge terms (powers of $n + 0.5$) for α exchange only (dashed line) and for both α and β exchange (solid line).
- (b) Sum of the N first s-channel resonance residues of $\mathcal{V}_{ab}$ for values of t_i ($t_a = +0.8$, $t_b = +0.7$ and $t = -1.4$) for which eq. (18) becomes a superconvergence relation. The sum over the residues (solid histogram) tends to zero as $N \rightarrow \infty$. When each residue is multiplied by $(\kappa)^{-\alpha_a}$ the sum (dashed histogram) no longer tends to zero as $N \rightarrow \infty$, showing that the superconvergence relation cannot be saturated by resonances alone.

Fig. 5 Definition of the cut and uncut propagators and vertices.
 (a) The reggeon propagator $P(\alpha)$. (b) The cut and twice cut propagators $P(\bar{\alpha})$. (c) The particle-particle-reggeon vertex $V(\alpha)$, which is by definition unaffected by cuts.
 (d) The reggeon-reggeon-particle vertex $V(\alpha_1, \alpha_2)$.
 (e,f) The reggeon-reggeon-particle vertex cut in different ways.

Fig. 6 (a) The reggeon-reggeon amplitude corresponding to the $B_6(u, t)$ term.
 (b) A u-channel resonance contribution to the amplitude in (a).
 (c) A t-channel Regge exchange contribution to the amplitude in (a).

Fig. 7 (a) The u-channel resonance residues (horizontal lines) of the expression in the curly bracket of eq. (28) ($\tilde{B}_6(u, t)$), compared with the corresponding Regge terms for only α -exchange (dashed curve), and the sum of α and β exchange (solid curve). The trajectories and the momentum transfers are the same as in fig. 4(a) and the Regge terms written as powers of $n + 0.5$.
 (b) Sum of the u-channel residues for values of t_1 as in fig. 4(b). It can be seen that the superconvergence relation is not saturated by resonance contributions.

Fig. 8 (a) The reggeon-reggeon amplitude corresponding to the $B_6(s, u)$ term.
 (b,c) s- and u-channel resonance contributions to the amplitude in (a).

Fig. 9 (a) The fixed pole contribution as computed from the sum of the s-channel resonance residues of the curly bracket in eq. (41).

(b) The near-cancellation of the sum of all s- and u-channel resonance residues of eq. (37). The three vectors refer to the complex numerical values of the sum of residues coming from the three terms in eq. (37). The example is for the same values of t_a , t_b and t as in fig. 4(a).

Fig. 10 A single Regge limit of the DRM six-point function to which the new trajectory $\beta_t = \frac{1}{2} \alpha_t - \frac{1}{2}$ contributes.

Fig. 11 A Chew-Frautschi plot of the α and β trajectories.

Fig. 12 Triple-Regge diagram for the reaction $\pi^+ p \rightarrow \pi^0 X$, showing the exchanged trajectories.

Fig. 13 Production and cascade decay of a heavy resonance in the 234 -channel.

Fig. 14 (a) The integration interval in (A1) between $u = 0$ and 1 replaced by a contour integral using eq. (A10).

(b) A distortion of the contour in (a) to a piece along the imaginary axis and a semicircle.

Fig. 15 The integration contour in the u and t planes used in the text.

Fig. 16 Diagram defining the standard integral representation (B2) of B_6 .

Fig. 17 The modified integration contours for B_6 in the u_1, u_2 and u planes.

Fig. 18 Definition of the t_1, t_2 integration regions in eq. (B6). The dashed lines correspond to eqs. (B10).

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Fig. 1

Fig. 2

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XBL 764-2633

Fig. 3

XBL 764-2634

Fig. 4

$$w^\alpha = P(\alpha) \quad (a)$$

(b) $\mathcal{P}^a = \mathcal{P}(\bar{a})$

$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} = V(a) \quad (c)$$

$$\alpha_1 \mid \alpha_2 \quad \text{---} = V(\alpha_1, \alpha_2) \quad \text{(d)}$$

$$\text{wavy line with } a_1' \text{ and } a_2 \text{ labels} = \text{wavy line with } a_1 \text{ and } \bar{a}_2 \text{ labels} = V(a_1, \bar{a}_2) \quad (e)$$

$$a_1' \mid a_2 \quad \cancel{f(a_1, a_2)} = V(\bar{a}_1, \bar{a}_2) \quad (f)$$

XBL 764-2636

Fig. 6

Fig. 8

XBL 764-2638

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Fig. 9

XBL 764-2639

Fig. 10

XBL 761-2027

Fig. 11

XBL 764-2640

XBL 764-2641

Fig. 12

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XBL 764-2642

Fig. 13

XBL 764-2643

Fig. 14

0 0 0 0 4 5 0 2 3 2 4

XBL 764-2644

Fig. 15

XBL 764-2645

Fig. 16

XBL 764-2647

Fig. 18

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