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UNIVERSITY OF CALIFORNIA
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SIMPLICIAL DISTANCE IN BRUHAT-TITS BUILDINGS OF SPLIT CLASSICAL TYPE

A dissertation submitted in partial satisfaction
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

Xu Gao

June 2023

The Dissertation of Xu Gao
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2023

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Abstract

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This dissertation studies the notion of *simplicial distance* on Bruhat-Tits buildings. That is a measure of proximity between vertices in the simplicial structure. The purpose of this research is three-fold: (i). to provide a concrete characterization of the simplicial distance; (ii). to better understand simplicial balls, and (iii). to derive a formula for the simplicial volume and explore its asymptotic growth.

To accomplish these goals, a comprehensive examination of vertices becomes necessary. They are analyzed using three frameworks: root systems, norms, and lattices. By leveraging concave functions, we interpret simplicial balls as fixed-point sets of Moy-Prasad subgroups and deduce a formula for the simplicial volume. Additionally, the theory of q-exponential polynomials is developed to facilitate the asymptotic study.

Through this research, we focus on split classical types (namely, types of A_n, B_n, C_n, D_n, and any combination thereof) over a local field. The presented findings contribute to the advancement of our understanding of Bruhat-Tits buildings.

To my parents

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Chapter 1.

Introduction

From the datum of a reductive algebraic group G over a non-Archimedean valued field K, there is an outstanding space $\mathscr{B}(\mathsf{G})$, called its *Bruhat-Tits building*, capturing the intricate structures of G and playing an essential role in the study of it. The main part of this theory was primarily developed in the 1960s-1980s by Goldman-Iwahori [GI63], Iwahori-Matsumoto [IM65], Hijikata [Hij75], and finally Bruhat-Tits [BT-1, BT-2, BT-3, BT-4]. Since then, numerous advancements have emerged, such as functorial properties [Lan00, PY02, Pra20a, Pra20b], compactifications [BS73, Lan96, GR06, Wer07], and Moy-Prasad filtrations [MP94, MP96, Yu15, FR17, Fin21a]. Different interpretations of Bruhat-Tits buildings has been found through various approaches, including incidence geometry of lattices [Gar97], tropical linear algebra [JSY07, Wer11], and analytic geometry of reductive groups [RTW15, RTW22]. In the decades since, Bruhat-Tits buildings has become an essential tool in many areas of mathematics, such as representation theory [SS97, Adl98, Yu01, DeB02a, DeB02c, Fin21b] and harmonic analysis [BM00, DeB02b, AD02] of reductive groups, arithmetic geometry [BS76, GS92, Rag94, Tei95, Ji08], and mathematical physics [GKP+17]. We refer to [Tit79, Yu09, Ji12] for surveys of Bruhat-Tits theory.

A Bruhat-Tits building $\mathscr{B}$ over a local field K is, in particular, a polysimplicial complex. From the perspective of incidence geometry, we have a *simplicial distance* on it. Specifically, given any two vertices x and y in $\mathscr{B}$, a *path* from x to y is a sequence of adjacent vertices $x_0, x_1, \cdots, x_l$ with $x_0 = x$ and $x_l = y$. The number l is called the *length* of the path. Then the *simplicial distance* between x and y is the length of a shortest path from x to y, and we denote it by $d(x, y)$. The notion of simplicial distance plays a central role in this research, whose purpose is three-fold.

The first purpose is to provide a characterization of the simplicial distance in a Bruhat-Tits building $\mathscr{B}$ of *split classical type*. When $\mathscr{B}$ is of split type A_n and thus can be realized as the building of the *general linear group* $\mathsf{GL}(V)$ of a finite-dimensional vector space V over K, this is clear. Following [Gar97, §19.1; RTW15, 2.22], we can interpret a vertex in $\mathscr{B}$ as a *homothety* class of lattices in V. Then two vertices x and y are *adjacent* if there exist representatives L and L', respectively, such that $L \supseteq L' \supseteq \varpi L$, where ϖ is any uniformizer of the local field K. From this interpretation of the incidence relation, one can prove that $d(x, y) \leqslant d$ if and only if $L \supseteq L' \supseteq \varpi^d L$ (see [Suh22, 2.1.1] for a proof). In Bruhat-Tits buildings of other split classical types, similar interpretations of vertices and their incidence relations are provided in [Gar97, §20.1 – 20.3]. However, given the complexity of these descriptions, it is difficult to obtain a portrayal of the simplicial distance directly from it.

In [BT-1, BT-2] and their extensive follow-up literature, a Bruhat-Tits building $\mathscr{B}$ is constructed as a metric space equipped with a natural isometry group action. This metric space is obtained by gluing together copies of a Euclidean affine space in a way that respects the associated root system. This geometric description of $\mathscr{B}$ allows us to analyze its simplicial structure explicitly, by utilizing the geometry of the corresponding root system. As a result, we obtain a characterization of the simplicial distance when $\mathscr{B}$ is of split classical type.

Theorem 1.1. *In an irreducible Bruhat-Tits building of split classical type, two vertices x and y have simplicial distance at most d if and only if they are separated by at most $d-1$ parallel walls. In particular, fixing a fixed special vertex o as the reference point, for any vertex x, we have*

$$d(x, o) \leqslant d \iff a_0(x - o) \leqslant d, \tag{1.1}$$

where a_0 is the highest root relative to a Weyl chamber covering x.

The geometric description of a Bruhat-Tits building in [BT-1, BT-2] and its lattice interpretation in [Gar97] are related through the methods of *maximinorante norms* developed in [BT-3, BT-4]. By leveraging these connections, we are able to obtain a portrayal (see Theorems 7.1, 7.3, and 7.4) of the simplicial distance in the context of [Gar97], namely expressed in the language of lattices.

The second purpose is to sort out several notions related to simplicial balls. Let $\mathscr{B}$ be a Bruhat-Tits building and x a vertex in it. The *simplicial ball* with center x and radius r is the set of all vertices with simplicial distance at most r from x:

$$B(x, r) := \{y \text{ is a vertex in } \mathscr{B} \mid d(x, y) \leqslant r\}.$$

The *simplicial sphere* with center x and radius r is the set of all vertices with simplicial distance exactly r from x:

$$\partial(x, r) := \{y \text{ is a vertex in } \mathscr{B} \mid d(x, y) = r\}.$$

In a Bruhat-Tits building, a vertex is either *special* or adjacent to a special one. If two vertices x and y are adjacent, then we have

$$B(x, r-1) \subseteq B(y, r) \subseteq B(x, r+1).$$

Thus, we may focus on special vertices.

In the rest of the dissertation, o will be a fixed special vertex. The set $B(o, r)$ will be denoted by $B(r)$ for short, and its cardinality will be denoted by $\mathrm{SV}(r)$. Likewise, the set $\partial(o, r)$ and its cardinality will be denoted by $\partial(r)$ and $\mathrm{SSA}(r)$ respectively. The functions $\mathrm{SV}(\,\cdot\,)$ and $\mathrm{SSA}(\,\cdot\,)$ are called the *simplicial volume* and the *simplicial surface area* in $\mathscr{B}$ respectively. The lattice description (Theorems 7.1, 7.3, and 7.4) gives us a sense about what them count. For instance, in the Bruhat-Tits building of $\mathsf{GL}(V)$, if L_0 is in the homothety class of lattices corresponding to the reference point o, then the quantity $\mathrm{SV}(r)$ counts, up to homotheties, the lattices L between L_0 and $\varpi^r L_0$. On the other hand, Theorem 1.1 gives us the following geometric description of $B(r)$.

Theorem 1.2. *In an irreducible Bruhat-Tits building of split classical type, the simplicial subset of the building $\mathscr{B}$ generated by the simplicial ball $B(r)$ is precisely the fixed-point set of the Moy-Prasad subgroups of level r at point o.*

We will go into the details of *Moy-Prasad subgroups* in § 4.4. Here, please allow the author to only mention that in the case of $\mathsf{GL}(V)$, they are precisely the *principal congruent subgroups*.

The last purpose of this dissertation is to analyze the asymptotic growths of the functions $\mathrm{SV}(\,\cdot\,)$ and $\mathrm{SSA}(\,\cdot\,)$. Note that

$$\partial(r) = B(r) \setminus B(r-1).$$

Therefore, for sufficiently large r, we have

$$C_1 \cdot \mathrm{SSA}(r) \leqslant \mathrm{SV}(r) \leqslant C_2 \cdot \mathrm{SSA}(r),$$

where C_1, C_2 are positive constants. We use the asymptotic notation $\mathrm{SV}(r) \asymp \mathrm{SSA}(r)$ to denote this fact.

One of the main theorems in this dissertation is the following.

Theorem 1.3. *Let $\mathscr{B}$ be an irreducible Bruhat-Tits building of split classical type over a local field K with residue cardinality q. Then the simplicial volume* $\mathrm{SV}(\,\cdot\,)$ *and the simplicial surface area* $\mathrm{SSA}(\,\cdot\,)$ *in it have the following asymptotic dominant relation:*

$$\mathrm{SV}(r) \asymp \mathrm{SSA}(r) \asymp r^{\varepsilon(n)} q^{\pi(n) r},$$

where $\varepsilon(n)$ and $\pi(n)$ are shown in the following table.

Split type of $\mathscr{B}$	$\varepsilon(n)$	$\pi(n)$
A_n *(n is odd)*	0	$(\frac{n+1}{2})^2$
A_n *(n is even)*	1	$\frac{n}{2}(\frac{n}{2}+1)$
B_n $(n = 3)$	0	5
B_n $(n \geqslant 4)$	0	$\frac{n^2}{2}$
C_n $(n \geqslant 2)$	0	$\frac{n(n+1)}{2}$
D_n $(n = 4)$	2	6
D_n $(n \geqslant 5)$	1	$\frac{n(n-1)}{2}$

Table 1.1. Asymptotic dominant of $\mathrm{SV}(\,\cdot\,)$ and $\mathrm{SSA}(\,\cdot\,)$

Remark. This theorem only talks about *irreducible* Bruhat-Tits buildings. However, we will see in § 6.1 that general asymptotic results can be deduced from the irreducible ones.

Remark. When the Bruhat-Tits building $\mathscr{B}$ is of split type A_n, this asymptotic dominant relation is given in [Suh22, 2.1.2]. We refer to [Suh21, Suh22] for an application of it.

In order to analyze the asymptotic behaviors of the functions $\mathrm{SV}(\,\cdot\,)$ and $\mathrm{SSA}(\,\cdot\,)$, we need formulas for them in terms of the root system Φ and the ground local field K. This is achieved by the following theorem.

Theorem 1.4. *Let $\mathscr{B}$ be a Bruhat-Tits building of split type Φ over a local field K with residue cardinality q. Then the simplicial volume $\mathrm{SV}(\,\cdot\,)$ and the simplicial surface area $\mathrm{SSA}(\,\cdot\,)$ in it can be computed by the following formulas:*

$$\mathrm{SV}(r) = \sum_{I\subseteq\Delta} \frac{\mathscr{P}_{\Phi;I}(q)}{q^{\deg(\mathscr{P}_{\Phi;I})}} \sum_{x\in B(r,{}^{v}C,I)} \prod_{a(x)>0} q^{\lceil a(x)\rceil},$$
$$\mathrm{SSA}(r) = \sum_{I\subseteq\Delta} \frac{\mathscr{P}_{\Phi;I}(q)}{q^{\deg(\mathscr{P}_{\Phi;I})}} \sum_{x\in \partial(r,{}^{v}C,I)} \prod_{a(x)>0} q^{\lceil a(x)\rceil},$$

where

- *$\lceil\,\cdot\,\rceil$ is the ceiling function,*
- *Δ is a basis of the root system Φ,*
- *$\mathscr{P}_{\Phi;I}$ is the Poincaré polynomial associated to the pair (Φ, I),*
- *${}^{v}C$ is a Weyl chamber of Φ,*
- *and the index sets $B(r,{}^{v}C,I)$ (resp. $\partial(r,{}^{v}C,I)$) consists of the vertices in $\overline{o+{}^{v}C}$ having type I with simplicial distance at most r (resp. exactly r) from o.*

In order to apply the formulas, we need to find explicit descriptions of the index sets $B(r,{}^{v}C,I)$ and $\partial(r,{}^{v}C,I)$. When Φ is classical, such descriptions follow from Theorem 1.1. Then we immediately see that each formula in Theorem 1.4 can be expanded into a finite linear combination of multi-summations of the form

$$\sum_{c_1,\cdots,c_t} q^{L(c_1,\cdots,c_t)+e(c_1,\cdots,c_t)},$$

where $L(c_1, \cdots, c_t)$ is a linear form of the variables $c_1, \cdots, c_t$ and $e(c_1, \cdots, c_t)$ is a parity function of $c_1, \cdots, c_t$. In order to handle such multi-summations, the notion of *(super) q-exponential polynomials* is introduced and studied. Applying general result on (super) q-exponential polynomials, we are able to prove Theorem 1.3 and the following improvement.

Theorem 1.5. *Notations are as in Theorem 1.3.*

(i). *Suppose $\mathscr{B}$ is of split type A_n, C_n, B_3, or D_4. Then the simplicial volume* $\mathrm{SV}(\,\cdot\,)$ *in it has the following asymptotic growth as $r \longrightarrow \infty$:*

$$\mathrm{SV}(r) \sim \tilde{C}(n) \cdot r^{\varepsilon(n)} q^{\pi(n)r},$$

where $\tilde{C}(n)$ is a positive number that is a rational function of q. Similarly, the simplicial surface area $\mathrm{SSA}(\,\cdot\,)$ *has the following asymptotic growth as $r \longrightarrow \infty$:*

$$\mathrm{SSA}(r) \sim C(n) \cdot r^{\varepsilon(n)} q^{\pi(n)r},$$

where $C(n)$ is a positive number that is a rational function of q.

(ii). *Suppose $\mathscr{B}$ is of split type B_n ($n \geqslant 4$) or D_n ($n \geqslant 5$). Then the simplicial volume* $\mathrm{SV}(\,\cdot\,)$ *in it has the following asymptotic growth as $r \longrightarrow \infty$:*

$$\mathrm{SV}(2r) \sim \tilde{C}_0(n) \cdot r^{\varepsilon(n)} q^{2\pi(n)r},$$

$$\mathrm{SV}(2r+1) \sim \tilde{C}_1(n) \cdot r^{\varepsilon(n)} q^{2\pi(n)r},$$

where $\tilde{C}_0(n)$ and $\tilde{C}_1(n)$ are positive numbers that are rational functions of q. Similarly, the simplicial surface area $\mathrm{SSA}(\,\cdot\,)$ in $\mathscr{B}$ has the following asymptotic growth as $r \to \infty$:

$$\mathrm{SSA}(2r) \sim C_0(n) \cdot r^{\varepsilon(n)} q^{2\pi(n)r},$$
$$\mathrm{SSA}(2r+1) \sim C_1(n) \cdot r^{\varepsilon(n)} q^{2\pi(n)r},$$

where $C_0(n)$ and $C_1(n)$ are positive numbers that are rational functions of q.

Indeed, we will prove stronger results (see Theorems 11.1, 12.1, 12.2, 13.1, and 13.2) and give explicit formulas for the involved leading coefficients.

Plan This dissertation is organized as follows.

Chapters 2 to 5 form Part I. In this part, we will review the theory of Bruhat-Tits buildings and fix conventions.

Chapters 6 to 8 form Part II. In this part, we will analyze vertices in apartments of split classical types, give a concrete characterization of the simplicial distance, and prove the formulas of the simplicial volume and the simplicity surface area shown in Theorem 1.4.

The rest chapters form Part III, which focuses on the asymptotic behaviors of the functions $\mathrm{SV}(\,\cdot\,)$ and $\mathrm{SSA}(\,\cdot\,)$. In Chapter 10, we will introduce the concept of *(super) q-exponential polynomials*, providing a foundation for the subsequent chapters. Chapters 11 to 14 then utilize this concept to analyze and understand the asymptotic behaviors of the simplicial volume and simplicial surface area in Bruhat-Tits buildings of split types A_n, B_n, C_n, and D_n.

Part I.

Preliminaries

Chapter 2.

General Theory of Buildings

We begin with some generality on buildings.

§ 2.1. Projective geometry over $\mathbb{F}_q$ and $\mathbb{F}_1$

Back in the 1950s, Jacques Tits noticed the following interesting phenomenons [Tit57].

2.1.1. Let $\mathbb{P}\mathbb{F}_q^n$ be the projective space associated to the vector space $\mathbb{F}_q^n$. Then its cardinality (or equivalently, the number of one-dimensional subspaces of $\mathbb{F}_q^n$) can be presented by the *quantum number* $[n]_q := \sum_{i=0}^{n-1} q^i$. If we pass to the limit $q \to 1$, then we get n, the number of coordinate labels $\{1, 2, \cdots, n\}$. Recalling how we count the cardinality of $\mathbb{P}\mathbb{F}_q^n$ using the coordinates, we can view the set $P_n = \{1, 2, \cdots, n\}$ as the analogue of $\mathbb{P}\mathbb{F}_q^n$ over the imaginary "prime field of characteristic one*" $\mathbb{F}_1$.

More generally, we can count points, lines, planes, ... in $\mathbb{P}\mathbb{F}_q^n$. They correspond to points of the Grassmannians $\mathrm{Gr}\left(1, \mathbb{F}_q^n\right), \mathrm{Gr}\left(2, \mathbb{F}_q^n\right), \mathrm{Gr}\left(3, \mathbb{F}_q^n\right), \ldots$. In general, the *Grassmannian* $\mathrm{Gr}\left(k, \mathbb{F}_q^n\right)$ consists of subspaces of $\mathbb{F}_q^n$ having dimension k and its cardi-

*Namely, the addition collapses. For an introduction, see [Lor18] especially §1.1.

nality can be presented by the *quantum binomial* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ (see 2.1.4). If we pass to the limit $q \to 1$, then we get $\binom{n}{k}$, which is the number of k-subsets of P_n.

2.1.2. The above can be organized into incidence geometry: namely the combinatorial gadget describing which proper subspace belongs to which. On the $\mathbb{F}_q$-side, a nontrivial proper subspace of $\mathbb{F}_q^n$ is of *color* k if it is k-dimensional and two such subspaces are said to be *incident* if one of them belongs to another properly. In this way, we organize nontrivial proper subspaces of $\mathbb{F}_q^n$ into a colored simplicial complex $\mathscr{B}(n, q)$, in which a k-simplex is a *flag*

$$\mathbb{F}_q^n = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_{k+1} \supsetneq 0$$

of subspaces of $\mathbb{F}_q^n$. On the $\mathbb{F}_1$-side, a nonempty proper subset of P_n is of *color* k if it has cardinality k and two such subsets are said to be *incident* if one of them belongs to another properly. In this way, we organize nonempty proper subsets of P_n into a colored simplicial complex $\mathscr{B}(n, 1)$, in which a k-simplex is a *flag*

$$P_n = I_0 \supsetneq I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_{k+1} \supsetneq \emptyset$$

of subsets of P_n.

The two sides are related as follows. Fix a basis $\boldsymbol{e}$ of $\mathbb{F}_q^n$ (for example, the standard basis). Then to take a nontrivial proper subspace V of $\mathbb{F}_q^n$ having a basis which is part of $\boldsymbol{e}$ amounts to taking a nonempty proper subset I of $\boldsymbol{e}$ (which is in bijection to P_n) and V is k-dimensional if and only if I has cardinality k. Moreover, to take a flag respecting the basis $\boldsymbol{e}$ in the sense that each V_i has a basis being part of $\boldsymbol{e}$ amounts to taking a flag of nonempty proper subsets of $\boldsymbol{e}$.

However, different choices of bases may give the same subcomplex: for instance, when the two bases are different by a diagonal matrix. To avoid this, it is better to keep in the region of projective geometry. So instead of fixing a basis, we fix a *frame* $\mathbf{\Lambda}$, that is an n-set of points $\{x_1, x_2, \cdots, x_n\}$ in $\mathbb{P}\mathbb{F}_q^n$ in general position (namely, they do not belong to a common hyperplane), or equivalently, an n-set of one-dimensional subspaces $\{\lambda_1, \cdots, \lambda_n\}$ of $\mathbb{F}_q^n$ spanning $\mathbb{F}_q^n$. Then different choices of frames do give different subcomplexes of $\mathscr{B}(n, q)$.

In this way, we associate to each frame $\mathbf{\Lambda}$ a subcomplex $\mathscr{A}(\mathbf{\Lambda})$ of $\mathscr{B}(n, q)$ isomorphic to $\mathscr{B}(n, 1)$ and the complex $\mathscr{B}(n, q)$ is the union of them. They are the prototypes of *buildings* and *apartments*.

2.1.3. There is a natural action of $G = \mathrm{GL}\left(\mathbb{F}_q^n\right)$, the *general linear group* (but essentially, it is the action of $\mathrm{PGL}\left(\mathbb{F}_q^n\right)$, the *projective linear group*) on $\mathscr{B}(n, q)$. This action comes from the action of $\mathrm{PGL}\left(\mathbb{F}_q^n\right)$ on $\mathbb{P}\mathbb{F}_q^n$ and hence on each Grassmannian $\mathrm{Gr}\left(k, \mathbb{F}_q^n\right)$.

Fix a frame $\mathbf{\Lambda}$ (for example, the one given by the standard basis), then the stabilizer of the subcomplex $\mathscr{A}(\mathbf{\Lambda})$ is precisely the stabilizer of the frame itself. Let's denote it by $N(\mathbf{\Lambda})$ (in our example of standard basis, it is the group of *monomial matrices*, i.e. matrices that have precisely one nonzero entry in each row and each column). The fixator of $\mathbf{\Lambda}$ acts trivially on $\mathscr{A}(\mathbf{\Lambda})$. Let's denote it by $Z(\mathbf{\Lambda})$ (in our example of standard basis, it is the group of diagonal matrices). The quotient group $W(\mathbf{\Lambda}) := N(\mathbf{\Lambda})/Z(\mathbf{\Lambda})$ is called the *Weyl group* associated to $\mathbf{\Lambda}$. Then one finds that $W(\mathbf{\Lambda}) \cong \mathfrak{S}_n$, the *symmetric group*, which acts naturally on P_n and hence on $\mathscr{B}(n, 1)$ exactly as $W(\mathbf{\Lambda})$ acts on $\mathscr{A}(\mathbf{\Lambda})$.

2.1.4. Let's consider the maximal simplices in $\mathscr{B}(n, q)$. From the description in 2.1.2, we see that a maximal simplex is nothing but a *complete flag*

$$\mathbb{F}_q^n = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_{n-1} \supsetneq 0$$

of subspaces of $\mathbb{F}_q^n$. Using an induction argument, it is not difficult to see that the number of complete flags is presented by the *quantum factorial* $[n]_q! := \prod_{i=1}^n [i]_q$. The quantum factorials are related to quantum binomials by the formula

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

This can be seen by picking the k-dimensional subspace V_{n-k} from a complete flag, breaking it into a complete flag of V_{n-k} and a complete flag of $\mathbb{F}_q^n/V_{n-k}$.

The maximal simplices in $\mathscr{B}(n,1)$ are *complete flags*

$$P_n = I_0 \supsetneq I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_{n-1} \supsetneq \emptyset$$

of subsets of P_n. There are $n!$ such complete flags. The number $n!$ is precisely the $q \to 1$ limit of $[n]_q!$.

The stabilizer of a complete flag is called a *Borel subgroup* of G. Note that the action of G on complete flags is transitive. Hence, the number of complete flags is the index of a Borel subgroup in G.

Let's take the standard basis $\boldsymbol{e} = (e_1, \cdots, e_n)$ of $\mathbb{F}_q^n$ and let $V_k = \bigoplus_{i=1}^{n-k} \mathbb{F}_q e_i$. Then we get a complete flag whose stabilizer B is precisely the group of invertible upper triangular matrices. Straightforward computation shows that B has order $q^{\binom{n}{2}}(q-1)^n$ and thus G has order $q^{\binom{n}{2}}(q-1)^n[n]_q!$.

2.1.5. We summarize above as follows.

(i). On the $\mathbb{F}_q$-side, we have the "building" $\mathscr{B}(n,q)$, which is the union of "apartments" $\mathscr{A}(\boldsymbol{\Lambda})$, one for each frame $\boldsymbol{\Lambda}$, and the number of them is

$$\frac{\#G}{\#N(\boldsymbol{\Lambda})} = \frac{\#B \cdot \#\{\text{complete flags}\}}{\#Z(\boldsymbol{\Lambda}) \cdot \#\mathfrak{S}_n} = \frac{q^{\binom{n}{2}}(q-1)^n[n]_q!}{(q-1)^n n!} = \frac{q^{\binom{n}{2}}[n]_q!}{n!}.$$

Each "apartment" $\mathscr{A}(\mathbf{\Lambda})$ is isomorphic to $\mathscr{B}(n,1)$, the one on the $\mathbb{F}_1$-side. Hence, the "building" $\mathscr{B}(n,q)$ can be seen as so many copies of $\mathscr{B}(n,1)$ glued together. By passing to the limit $q \to 1$, this quantity gives 1, coinciding with the number of "apartments" in $\mathscr{B}(n,1)$.

(ii). The quantum factorial $[n]_q!$ counts the maximal simplices in $\mathscr{B}(n,q)$, which becomes $n!$, the number of maximal simplices in $\mathscr{B}(n,1)$ by taking the limit $q \to 1$.

(iii). The quantum binomial $\begin{bmatrix} n \\ k \end{bmatrix}$ counts the vertices of color k in $\mathscr{B}(n,q)$, which becomes $\binom{n}{k}$, the number of vertices of color k in $\mathscr{B}(n,1)$ by taking the limit $q \to 1$.

(iv). There are more combinatorial quantities in $\mathscr{B}(n,q)$ becoming ones for $\mathscr{B}(n,1)$ by taking the limit $q \to 1$.

Tits's observations are not limited to $\mathrm{PGL}\left(\mathbb{F}_q^n\right)$. In fact, he did for all semisimple groups over $\mathbb{F}_q$. Of course, there was no $\mathbb{F}_1$-geometry back in Tits' time, but it seems the above observations inspired him to develop the theory of buildings with the following principle:

> Buildings are multifold apartments and apartments are $q \to 1$ limit case of buildings, which can be thought as forgetting the additive arithmetic of the base field.

§ 2.2. Abstract Buildings

Before moving on, we now give a formal definition of polysimplicial complexes.

Definition 2.2.1. An *(abstract) simplicial complex* is a nonempty poset $\mathcal{S}$ (whose members are called *simplices*) satisfying

S1. any two simplices σ, τ have an infimum $\sigma \cap \tau$;

So there is a unique smallest element in $\mathcal{S}$, called the *empty simplex*, denoted by $\emptyset$.

S2. for each simplex σ the poset $\mathcal{S}_{\leqslant\sigma}$ of simplices smaller than σ (they are called *faces* of σ) form a *Boolean lattice* of rank k, namely isomorphic to the power set of a k-set, for some finite k. In this case, we say σ is of *dimension* $k-1$ and is a *$(k-1)$-simplex*.

The *dimension* of $\mathcal{S}$ is the supremum of dimensions of its simplices. The minimal nonempty simplices are of dimension 0 and are thus called *vertices*. Let $\mathcal{V}$ denote the set of vertices. Then $\mathcal{S}$ can be identified with a poset of nonempty subsets of $\mathcal{V}$.

A *morphism* between simplicial complexes is a map preserving infima, suprema and the empty simplex $\emptyset$. Note that this implies that such a morphism is determined by its restriction to vertices. So equivalently, such a morphism is a map between vertices extending to a monotonic map preserving simplices. A morphism $\varphi\colon \mathcal{S} \to \mathcal{S}'$ is said to *fix a simplex $\sigma \in \mathcal{S} \cap \mathcal{S}'$ pointwise* if it induces an identity from $\mathcal{S}_{\leqslant\sigma}$ to $\mathcal{S}'_{\leqslant\sigma}$.

A *polysimplicial complex* is a Cartesian product of simplicial complexes (in the category of posets) and morphisms between polysimplicial complexes are therefore defined.

Example 2.2.2. One can verify that $\mathscr{B}(n, q)$ and $\mathscr{B}(n, 1)$ are simplicial complexes.

Let's analyze how the "apartments" $\mathscr{A}(\mathbf{\Lambda})$ are glued into the "building" $\mathscr{B}(n, q)$.

Proposition 2.2.3. *For any two simplices F, F' in $\mathscr{B}(n, q)$, there is an "apartment" $\mathscr{A}(\mathbf{\Lambda})$ containing both of them.*

Proof. We may assume F, F' are maximal, i.e. being complete flags:

$$F\colon \mathbb{F}_q^n = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_{n-1} \supsetneq 0,$$
$$F'\colon \mathbb{F}_q^n = V_0' \supsetneq V_1' \supsetneq V_2' \supsetneq \cdots \supsetneq V_{n-1}' \supsetneq 0.$$

Then we may view them as composition series for $\mathbb{F}_q^n$. Therefore, by the *Jordan-Hölder Theorem*, there is a permutation π of $P_n = \{1, 2, \cdots, n\}$ such that whenever $j = \pi(i)$, we have isomorphisms

$$\frac{V_{n-i}}{V_{n-i+1}} \xleftarrow{\sim} \frac{V_{n-i} \cap V'_{n-j}}{(V_{n-i+1} \cap V'_{n-j}) + (V_{n-i} \cap V'_{n-j+1})} \xrightarrow{\sim} \frac{V'_{n-j}}{V'_{n-j+1}}$$

induced from inclusions. Let λ_i be the one-dimensional subspace of $V_{n-i} \cap V'_{n-j}$ whose image in above quotients are non-trivial. Then $\boldsymbol{\Lambda} = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is a frame with $\mathscr{A}(\boldsymbol{\Lambda})$ containing both F and F'. □

Proposition 2.2.4. *If $\mathscr{A}(\boldsymbol{\Lambda})$ and $\mathscr{A}(\boldsymbol{\Lambda}')$ are two "apartments" containing both F and F', then there is an isomorphism between them fixing F and F' pointwise.*

Proof. Again, we may assume F, F' are maximal and let V_i, V_i', λ_i be as above. Then $i \mapsto \lambda_i$ induces an isomorphism $\phi_{\boldsymbol{\Lambda}}\colon \mathscr{B}(n, 1) \to \mathscr{A}(\boldsymbol{\Lambda})$. The inverse of it can be described by vertices as

$$\psi_{\boldsymbol{\Lambda}}\colon U \mapsto \{i \in P_n \mid U \cap V_{n-i+1} \neq U \cap V_{n-i}\}.$$

Similarly, we have an isomorphism $\phi_{\boldsymbol{\Lambda}'}\colon \mathscr{B}(n, 1) \to \mathscr{A}(\boldsymbol{\Lambda}')$ and its inverse $\psi_{\boldsymbol{\Lambda}'}$. Note that the morphism $\psi_{\boldsymbol{\Lambda}}$ (and similarly $\psi_{\boldsymbol{\Lambda}'}$) is determined by the complete flag F, we conclude that $\psi_{\boldsymbol{\Lambda}}$ and $\psi_{\boldsymbol{\Lambda}'}$ coincide on the intersection of $\mathscr{A}(\boldsymbol{\Lambda})$ and $\mathscr{A}(\boldsymbol{\Lambda}')$. Then $\phi_{\boldsymbol{\Lambda}'} \circ \psi_{\boldsymbol{\Lambda}}$ is an isomorphism between $\mathscr{A}(\boldsymbol{\Lambda})$ and $\mathscr{A}(\boldsymbol{\Lambda}')$ fixing F and F' pointwise. □

Then the buildings can be defined as follows.

Definition 2.2.5. A *(abstract) building* is a polysimplicial complex $\mathscr{B}$ equipped with a family $\mathcal{A}$ of subcomplexes of $\mathscr{B}$, whose members are called *apartments*, such that the following axioms are satisfied.

B0. Each apartment $A \in \mathcal{A}$ is isomorphic to an abstract apartment $\mathscr{A}$.

B1. For any two simplices F, F', there is an apartment A containing them.

B2. If A, A' are two apartments containing both F and F', then there is an isomorphism between A and A' fixing F and F' pointwise.

A *morphism* between buildings is a morphism of the underlying polysimplicial complexes which maps apartments in apartments.

Of course, one has to define what is an apartment to make this definition sense.

Example 2.2.6. $\mathscr{B}(n, q)$ is a building with apartments isomorphic to $\mathscr{B}(n, 1)$.

Let's analyze what the "apartment" $\mathscr{B}(n, 1)$ looks like.

Proposition 2.2.7. *All maximal simplices have the same dimension.*

Proof. This is clear, they are precisely the $(n-1)$-subsets of P_n. $\square$

Proposition 2.2.8. *Any two maximal simplices C, C' are connected by a sequence $(C_0, C_1, \cdots, C_s)$ with $C_0 = C$ and $C_s = C'$ such that for each i, $C_{i-1} \cap C_i$ has codimension 1 in both C_{i-1} and C_i.*

Proof. Note that a maximal simplex in $\mathscr{B}(n, 1)$ is a complete flag, hence a sequence $(i_1, i_2, \cdots, i_{n-1})$, which can be identified with an ordering of P_n. Hence, any two such simplices are different by a permutation $\pi \in \mathfrak{S}_n$. But any permutation can be written as the composition of transpositions while two sequences different by a transposition meet in a sequence with one term being removed. $\square$

In general, a polysimplicial complex which has above properties is called a *chamber complex* and its maximal simplices are called *chambers*. A one-codimensional face of a chamber is called a *panel*. A sequence $(C_0, C_1, \cdots, C_s)$ connecting two chambers by panels is called a *gallery*. Note that any Boolean lattice is a chamber complex with a unique chamber: its maximal element. A *chamber map* between chamber complexes is a morphism mapping chambers to chambers.

Proposition 2.2.9. *There is a coloring, namely a chamber map from the complex to a Boolean lattice.*

Proof. It suffices to define colors for vertices. Then the color of a simplex would be the set of the colors of its vertices. For instance, one can define the *color* of a vertex as its cardinality as in 2.1.2. $\square$

In general, a chamber complex which has this property is said to be *colorable*. It is worth noticing that any two colorings are different by an isomorphism of Boolean lattices (in other words, up to a permutation of the colors of vertices).

Proposition 2.2.10. *The Weyl group acts transitively on the simplices of the same color.*

Proof. Two simplices $F = (I_i)$ and $F' = (I'_i)$ are of the same color means two things: first, they have the same number of entries; second, each pair of entries (I_i, I'_i) have the same cardinality. This is precisely the condition for the existence of a permutation $\pi \in \mathfrak{S}_n$ interchanging them. $\square$

Proposition 2.2.11. *Fix a chamber C, then all the stabilizers of its panels in the Weyl group are of order 2 and their generators s_j form a generating system S of the Weyl group with generating relations of the form $(s_i s_j)^{m_{ij}} = 1$.*

Proof. As in Proposition 2.2.8, a chamber C is a sequence $(i_1, i_2, \cdots, i_{n-1})$. Let i_n be the complement of this sequence in P_n. Then for each panel obtained from C by deleting i_j, let s_j be the transposition (i_j, i_n). Then this panel's stabilizer is precisely $\{1, s_j\}$ and one can verify the system $S = \{s_1, \cdots, s_{n-1}\}$ satisfies the requirement. □

Note that it follows from Proposition 2.2.11 that the stabilizer of a face of C is generated by those s_j with j being not a color of its vertex. Furthermore, the complex $\mathscr{B}(n, 1)$ can be built from the pair (W, S) of the Weyl group $W = \mathfrak{S}_n$ and the system $S = (s_j)$ of generators in Proposition 2.2.11. Indeed, any face of the chamber C corresponds to the subset I of S generating its stabilizer and any simplex is translated to such a face by an element of W, unique up to the stabilizer $\langle I \rangle$. Therefore, the simplices in $\mathscr{B}(n, 1)$ can be identified with the cosets $w\langle I \rangle$ with $w \in W$ and $I \subseteq S$.

Definition 2.2.12. A *Coxeter system* is a pair (W, S) of a group W and a system of its generators $S = \{s_1, s_2, \cdots, s_n\}$ such that all s_i are of order 2 and the generating relations for S are of the form $(s_i s_j)^{m_{ij}} = 1$. Its *Coxeter complex* $\Sigma(W, S)$ is the polysimplicial complex defined as the complex of cosets of the form $w\langle I \rangle$ with $w \in W$ and $I \subseteq S$, where the order is given by reverse inclusion.

Then Propositions 2.2.7 to 2.2.11 shows that $\mathscr{B}(n, 1)$ is isomorphic to the Coxeter complex $\Sigma(\mathfrak{S}_n, S)$, where S can be chosen to be any generating system of transpositions, for instance $S = \{(1, n), (2, n), \cdots, (n-1, n)\}$.

A *morphism* between Coxeter systems (W, S) and (W', S') is a group homomorphism

$f\colon W \to W'$ such that $f(S) \subseteq S'$. In this category, a Coxeter system (W,S) is a *product* of subsystems $(W_i,S_i)_{1\leqslant i\leqslant m}$ if we have a group decomposition $W = W_1 \times \cdots \times W_m$ and a set decomposition $S = S_1 \sqcup \cdots \sqcup S_m$. A Coxeter system is *irreducible* if it can not be decomposed into proper subsystems.

One can see that morphisms between Coxeter systems induce morphisms between their Coxeter complexes and such a functor is compatible with the decomposition. In particular, a Coxeter complex of an irreducible Coxeter system is simplicial.

Now, we can complete Definition 2.2.5 by defining an *apartment* to be a polysimplicial complex isomorphic to the Coxeter complex of some Coxeter system.

A Coxeter complex $\Sigma(W,S)$ is *finite* if and only if W is finite if and only if all m_{ij} are finite. If this is the case, this Coxeter complex is said to be *spherical*, otherwise it is *affine*. A building is said to be *spherical* (resp. *affine*) if its apartments are isomorphic to a spherical (resp. affine) Coxeter complex.

We have seen that $\mathscr{B}(n,q)$ is such a building: its apartments are isomorphic to the Coxeter complex $\Sigma(\mathfrak{S}_n,S)$. This is not an accident. In fact, any reductive group over an arbitrary field gives rise to such a building. They are called the *Tits buildings*. We refer to [Bourbaki, chap.IV] for the theory of Coxeter systems and [Tit74] for a treatment of Tits buildings in the language of Coxeter complexes.

§ 2.3. Euclidean apartments

Although buildings can be defined and studied in a pure combinatorial way, it would be more intuitive and convenient if we can also define them geometrically.

2.3.1. One way to visualize the Coxeter complex $\Sigma(\mathfrak{S}_n,S)$ is the follows. The group $\mathfrak{S}_n$

acts faithfully on $\mathbb{R}^n$ as permutations of the coordinates. For any transposition (i, j), its set of fixed points is the hyperplane $\left\{(x_0, \cdots, x_{n-1}) \in \mathbb{R}^n \,\middle|\, x_i = x_j\right\}$, and it thus acts as the reflection respect to this hyperplane. Therefore, the group $\mathfrak{S}_n$ can be determined by the reflections/hyperplanes defined by the transpositions. Moreover, the hyperplanes partition $\mathbb{R}^n$ into pieces of various dimensions with an obvious order relation: one such a piece belongs to the closure of another. This gives rise to a complex isomorphic to $\Sigma(\mathfrak{S}_n, S)$. The system S can be obtained as the reflections respect to a chamber.

With this example in mind, we make the following definition.

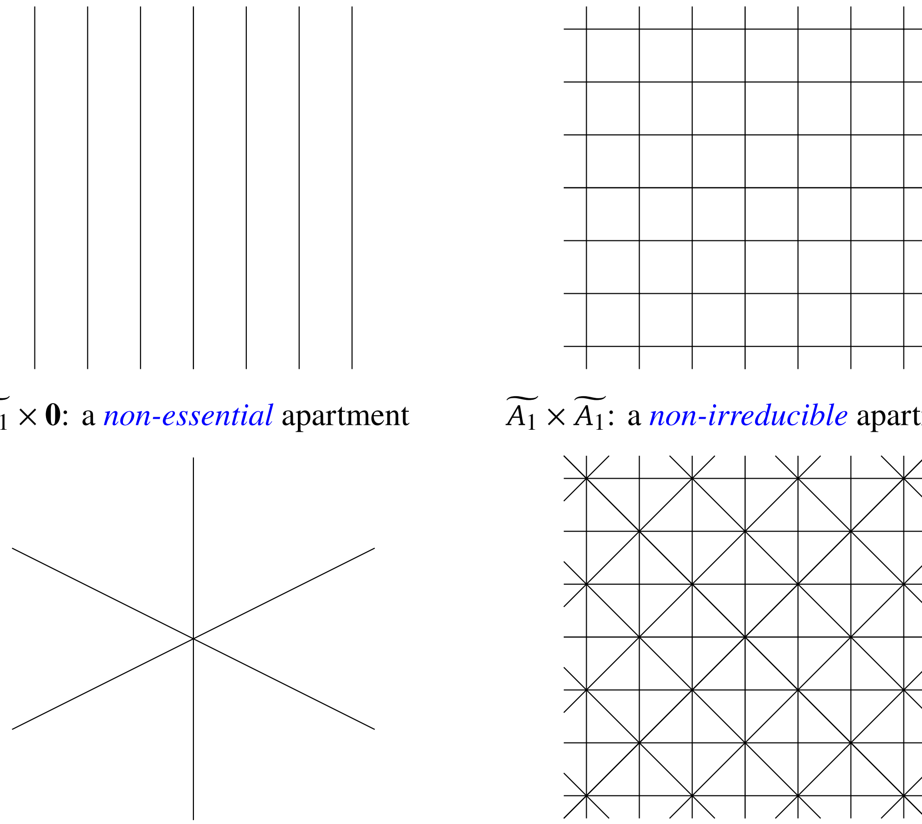
Definition 2.3.2. A *(Euclidean) apartment* $\mathscr{A}$ is a Euclidean affine space $\mathbb{A}$ equipped with a reflection group W (called its *Weyl group*) on it.

Let $\mathbb{A}$ be a Euclidean affine space. We use ${}^v\mathbb{A}$ to denote its associated vector space. For an affine transformation f on $\mathbb{A}$, we use ${}^vf$ to denote its *vectorial part*. For an affine subspace X of $\mathbb{A}$, we use ${}^vX$ to denote its *direction*.

A *reflection* on $\mathbb{A}$ is an affine isometry whose fixed points form a hyperplane. Any hyperplane H is associated with a reflection r_H with respect to it.

A *reflection group* W is a group of affine isometries generated by reflections and such that its vectorial part ${}^vW$ is finite. W is said to be *irreducible* if ${}^vW$ acts irreducibly on ${}^v\mathbb{A}$ and is said to be *essential* if ${}^vW$ acts essentially on ${}^v\mathbb{A}$ (that is, there is no nonzero fixed point). An apartment is said to be *irreducible* (resp. *essential*, *trivial*, etc.) if its reflection group is so.

The kernel $T = \ker(W \to {}^vW)$ consists of translations and hence is called the *translation group*. It is then a subgroup of ${}^v\mathbb{A}$. The apartment $\mathscr{A}$ is said to be *spherical* (resp. *affine*, *discrete*) if T is finite (resp. infinite, discrete).



$\widetilde{A_1} \times \mathbf{0}$: a *non-essential* apartment

$\widetilde{A_1} \times \widetilde{A_1}$: a *non-irreducible* apartment

A_2: a *spherical* apartment

$\widetilde{C_2}$: contains *non-special* vertices

Figure 2.1. Some examples of apartments

2.3.3. A *morphism* between apartments $(\mathbb{A}, W)$ and $(\mathbb{A}', W')$ is a continuous affine map $f\colon \mathbb{A} \to \mathbb{A}'$ with a group homomorphism $\phi\colon W \to W'$ such that $\phi(w).f(x) = f(w.x)$ for all $w \in W$ and $x \in \mathbb{A}$. In this category, an apartment $(\mathbb{A}, W)$ is said to be a *product* of apartments $(\mathbb{A}_i, W_i)_{1 \leqslant i \leqslant m}$ if we have an orthogonal decomposition $\mathbb{A} = \mathbb{A}_1 \times \cdots \times \mathbb{A}_m$ and a group decomposition $W = W_1 \times \cdots \times W_m$ such that each W_i acts trivially on the orthogonal complement of $\mathbb{A}_i$.

Any apartment $\mathscr{A}$ admits a decomposition [Bourbaki, chap.V, §3, no.8]

$$\mathscr{A} = \mathscr{A}_0 \times \mathscr{A}_1 \times \cdots \times \mathscr{A}_m,$$

where $\mathscr{A}_0$ is trivial and each $\mathscr{A}_i$ (for $1 \leqslant i \leqslant m$) is irreducible.

Throughout this dissertation, all apartments are assumed to be discrete. This is equivalent to saying that in each irreducible component of it, the translation group T is either finite or a full-rank lattice in ${}^v\mathbb{A}$.

2.3.4. Let $\mathbb{A} = (\mathbb{A}, W)$ be an apartment.

The hyperplanes of fixed points of reflections in W are called the *walls* in $\mathbb{A}$. The set $\mathcal{H}$ of walls is stable under W and completely determines it.

A *half-apartment* (also called an *affine root* in [BT-1, 1.3.3]) is a closed half-space α of $\mathbb{A}$ bounded by a wall $\partial\alpha$, called its *wall*.

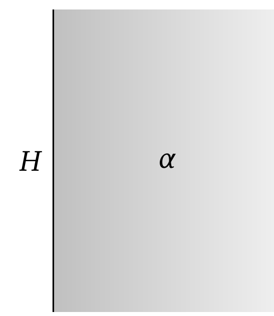


Figure 2.2. An affine root α and its wall (boundary) H

A *facet* in $\mathbb{A}$ is an equivalence class in $\mathbb{A}$ for the relation "x and y are contained in the same half-apartments". A facet F is an open convex subset of the affine subspace (called the *support* of F) that it spans.

The set $\mathcal{F}$ of facets admits an order: a facet F is said to be a *face* of another F', denoted by $F \leqslant F'$, if F is *covered by* F', namely contained in the closure of F'. Such an order gives rise to a polysimplicial complex. To see this, first notice that facets in an apartment are compatible with its decomposition into irreducible components. Hence, we may assume our apartment $\mathbb{A}$ is irreducible and essential. Then this can be seen from the fact that any triangulation of a topological space gives rise to a simplicial complex (indeed, this is where the notion comes from). When $\mathbb{A}$ is discrete affine, its facets already triangulate the ambient space. When $\mathbb{A}$ is spherical, its facets triangulate the unit sphere. This is why it is called spherical.

2.3.5. The maximal facets are called *chambers* (or *alcoves*). They are the connected components of the complement of the union of all walls in $\mathbb{A}$. The Weyl group W acts simply transitively on the set C of chambers [Bourbaki, chap.V, §3, no.2, th.1].

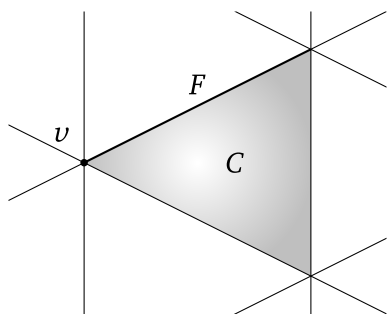
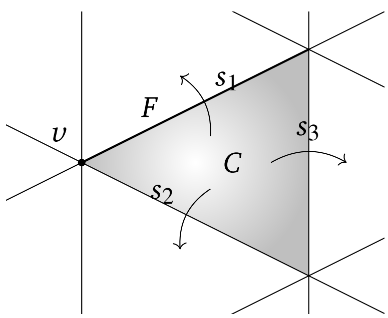


Figure 2.3. A vertex v, a facet F, and an alcove C

Let C be a chamber. Then its closure $\overline{C}$ is a fundamental domain of W in $\mathbb{A}$ [Bourbaki, chap.V, §3, no.3, th.2] and is the intersection of some half-apartments, whose walls are called the *walls* of C. Equivalently, the walls of C are the supports of panels of it, where a *panel* means a maximal proper face of C. Moreover, W is generated by the set S of reflections with respect to the walls of C and the pair (W, S) is a Coxeter system [Bourbaki, chap.V, §3, no.2, th.1]. The projection of C onto an irreducible component $\mathbb{A}_i$ is again a chamber in it and induces an irreducible Coxeter system (W_i, S_i). Then (W, S) is the product of them. In other words, decomposition of the pair of $(\mathscr{A}, C)$ of an apartment and a chamber is compatible with the decomposition of the Coxeter system (W, S) it defines.

A *type function* on $\mathscr{A}$ is a morphism τ from the complex $\mathcal{F}$ of facets to a Boolean lattice, which maps chambers to the maximal element and is *W-stable* in the sense that for any facet F and any $w \in W$, $\tau(F) = \tau(w.F)$. The image of this function is denoted by $\mathcal{T}$ and its members are called *types*. This notion is essentially the same as a *coloring* as in Proposition 2.2.9 plus Proposition 2.2.10. They differ in one respect: for a coloring, the target Boolean lattice is viewed as a power set $\mathscr{P}(\mathfrak{I})$ with its usual order $\subseteq$, while for a type function, we use the reverse order $\supseteq$. In other words, a face of type I is of color $\neg I := \mathfrak{I} \setminus I$.



The alcove C defines a generating set

$$S = \{s_1, s_2, s_3\},$$

where the reflections s_1, s_2, s_3 are shown in the picture. The facet F in the figure is then a *panel* of C. It has *type* 1 and *color* $\{2, 3\}$. Hence, $C_1 = F$. The vertex v in the figure has *type* $\{1, 2\}$ and *color* 3. Hence, $C_{\{1,2\}} = v$.

Figure 2.4. The generating set and the type function associated to C

Since any facet is transformed by W to a unique face of C, the type function τ is completely determined by the types of its panels, which we may view as an indexing of S. Indeed, let I be a type, then the set C_I of points $x \in \overline{C}$ such that the reflections $s \in S$ fixing x are indexed by I is a face of C of type I and its stabilizer is the subgroup W_I of W generated by the reflections indexed by I [Bourbaki, chap.V, §3, no.3, prop.1]. Then $\tau(F) = I$ if and only if F is transformed to C_I.

2.3.6. A reflection group W is said to be *linear* if it fixes a point. This is the case if and only if W is finite [Bourbaki, chap.V, §3, no.9]. If this is the case, we can identify W

with its vectorial part ${}^vW$ by choosing the fixed point to be the origin of $\mathbb{A}$.

Conversely, the vectorial part ${}^vW$ of the Weyl group W can be viewed as a linear reflection group on ${}^v\mathbb{A}$. The spherical apartment ${}^v\mathscr{A} = ({}^v\mathbb{A}, {}^vW)$ obtained in this way is called the *vectorial apartment* of $\mathscr{A}$. The walls (resp. facets, chambers) in ${}^v\mathscr{A}$ are called the *vectorial walls* (resp. *vectorial facets*, *vectorial chambers*) and the set of them is denoted by ${}^v\mathcal{H}$ (resp. ${}^v\mathcal{F}$, ${}^v\mathcal{C}$). Note that the vectorial walls are precisely the directions of walls in $\mathscr{A}$.

2.3.7. Let x be a point in $\mathscr{A}$. The stabilizer W_x of x is a linear reflection group whose vectorial part ${}^vW_x$ is a subgroup of ${}^vW$. The apartment $\mathscr{A}_x = (\mathbb{A}, W_x)$ is called the *spherical apartment* at x. The walls in $\mathscr{A}_x$ are precisely the walls in $\mathscr{A}$ passing through x and the set of them is denoted by $\mathcal{H}_x$. The facets (resp. chambers) in $\mathscr{A}_x$ are called the *vectorial facets* with *base point* x (resp. *vectorial chambers* with *base point* x) and the set of them is denoted by $\mathcal{F}_x$ (resp. $\mathcal{C}_x$).

A point $x \in \mathbb{A}$ is said to be *special* if the spherical apartment $\mathscr{A}_x$ is isomorphic to ${}^v\mathscr{A}$, or equivalently, the set $\mathcal{H}_x$ is a complete set of representatives of ${}^v\mathcal{H}$. This can happen only if x belongs to a minimal facet.

2.3.8. The minimal facets are called *vertices*. The set of vertices is denoted by $\mathcal{V}$. When the apartment is essential, they are points. From now on, all apartments are assumed to be essential unless otherwise specified*.

Under this assumption, every special point is a vertex. Furthermore, any special vertex is an extreme point of the closure of some chamber. Conversely, any chamber admits a special point as an extreme point of its closure [Bourbaki, chap.V, §3, no.10,

*This means we will only focus on *reduced* buildings, rather than *extended* buildings.

prop.11's cor]. However, not all extreme points, hence not all vertices are special (see $\widetilde{C}_2$ in Fig. 2.1 for an example).

§ 2.4. Root systems

Before moving on to the definition of buildings, let's look at some examples of Euclidean apartments arising from root systems (as well as root data). They are the key examples used in the study of reductive groups.

2.4.1. Let $\mathbb{V}$ be a Euclidean vector space and $\mathbb{V}^*$ its dual space. For any $a \in \mathbb{V}^* \setminus \{0\}$, let r_a be the reflection with respect to the hyperplane $H_a := \mathrm{Ker}(a)$ and $a^\vee$ the vector orthogonal to H_a satisfying $a(a^\vee) = 2$. So for any $\mathbf{v} \in \mathbb{V}$, we have

$$r_a(\mathbf{v}) = \mathbf{v} - a(\mathbf{v})a^\vee.$$

Note that r_a also induces a reflection on $\mathbb{V}^*$, namely $f \mapsto f - f(a^\vee)a$. A finite spanning subset $\Phi \subseteq \mathbb{V}^* \setminus \{0\}$ is called a *root system* on $\mathbb{V}$ if

RS1. for any $a \in \Phi$, $r_a(\Phi) = \Phi$;

RS2. for any $a, b \in \Phi$, $a(b^\vee) \in \mathbb{Z}$;

and is *reduced* if

RS3. for any $a \in \Phi$, $\mathbb{R}a \cap \Phi = \{\pm a\}$.

From now on, all root systems are assumed to be reduced*.

Elements of Φ are called *roots* in Φ. For a root $a \in \Phi$, the vector $a^\vee$ is called its *coroot*; they form a root system $\Phi^\vee$ on $\mathbb{V}^*$, called the *coroot system*. A subset $\Psi \subseteq \Phi$ is

*This means we will only focus on split reductive groups.

called a *subroot system* if for any $a \in \Psi$, $r_a(\Psi) = \Psi$, and is said to be *closed* if for any $a, b \in \Psi$ such that $a + b$ is a root, $a + b \in \Psi$.

Any root system Φ admits the *Weyl group* ${}^{v}W(\Phi)$, that is the reflection group of $\mathbb{V}$ generated by r_a for $a \in \Phi$. It is a linear reflection group with walls H_a for $a \in \Phi$. In this way, we get a spherical apartment ${}^{v}\mathscr{A}(\Phi) := (\mathbb{V}, {}^{v}W(\Phi))$. Note that not all spherical apartments arise in this way (see [Bourbaki, chap.VI, §2, no.5, prop.9]) and non-isomorphic root systems may have isomorphic Weyl groups (for instance root systems of types B_n and C_n).

2.4.2. A root system Φ is said to be *irreducible* if it cannot be written as the union of two proper subsets such that they are orthogonal to each other. A root system Φ is irreducible if and only if its Weyl group ${}^{v}W(\Phi)$ is [Bourbaki, chap.VI, §1, no.2, prop.5's cor]. Any root system decomposes into disjoint union of irreducible ones and such a decomposition is compatible with the decomposition of Weyl groups and hence of apartments [Bourbaki, chap.VI, §1, no.2, prop.6 and 7].

2.4.3. Let Φ be a root system. Then there is a closed subset Φ^+ of Φ such that for any $a \in \Phi$, either $a \in \Phi^+$ or $-a \in \Phi^+$. This set is called a system of *positive roots*. Once such a set is chosen, elements in the set $\Phi^- := -\Phi^+$ are called *negative roots*. A positive root is called a *simple root* if it cannot be written as the sum of two positive roots. The set Δ of simple roots form a *basis* of Φ in the sense that any root is a $\mathbb{Z}$-linear combination of simple roots and its coefficients are either all non-negative or all non-positive [Bourbaki, chap.VI, §1, no.6, th.3]. The cardinality of the set Δ is called the *rank* of Φ and is independent of the choice of Δ. Indeed, it equals $\dim \mathbb{V}$.

Let Δ be a basis of Φ. Then the set

$$^{v}C = \{\mathbf{v} \in \mathbb{V} \mid \forall a \in \Delta : a(\mathbf{v}) > 0\}$$

is a vectorial chamber, called the *Weyl chamber* associated to Δ [Bourbaki, chap.VI, §1, no.5, th.2]. Conversely, let ^{v}C be a vectorial chamber. Then for any $\mathbf{v} \in {}^{v}C$, the sets

$$\Phi^{+} = \{a \in \Phi \mid a(\mathbf{v}) > 0\} \qquad \text{and} \qquad \Phi^{-} = \{a \in \Phi \mid a(\mathbf{v}) < 0\}$$

form a partition of Φ into positive and negative roots and are independent of the choice of $\mathbf{v}$. Then one can obtain a basis Δ by taking the simple roots. But there is a more geometric description: they are the roots defining the walls of ^{v}C pointing inside. As vectorial chambers are Weyl chambers associated to some choice of basis, we call them *Weyl chambers* to specify that they are chambers in the spherical apartment $^{v}\mathscr{A}(\Phi)$.

2.4.4. The relation between simple roots and types is the following. First, the Weyl group ^{v}W is generated by r_a for $a \in \Delta$ as they are the roots defining walls of ^{v}C and point inside. Therefore, a type $I \in \mathcal{T}$ corresponds to a subset of Δ. From now on, we do not distinguish them. Then the face of ^{v}C corresponding to I is the set

$$^{v}C_I = \{\mathbf{v} \in \mathbb{V} \mid \forall a \in I : a(\mathbf{v}) = 0; \forall a \in \Delta \setminus I : a(\mathbf{v}) > 0\}.$$

Let Φ_I be the subroot system of Φ generated by I, then the stabilizer $^{v}W_I$ is the Weyl group of it. The set $\Psi = \Phi_I \cup \Phi^{+}$ has the property that $\Psi \cup (-\Psi) = \Phi$ and is closed. Such kind of subsets of Φ are said to be *parabolic*. Given a parabolic subset Ψ of Φ containing Φ^{+}, then the simple roots in $\Psi \cap (-\Psi) \cap \Phi^{+}$ gives the type I. See [Bourbaki, chap.VI, §1, no.7].

Convention 2.4.5. Given a basis $\Delta = \{a_1, \cdots, a_n\}$, a *type* is a subset of Δ, and is identified with a subset of $\{1, \cdots, n\}$. For a type I of Δ, we use t_I to denote the cardinality of $\Delta \setminus I$ and $\ell_i(I)$ $(1 \leqslant i \leqslant t_I)$ the i-th index in $\Delta \setminus I$. We use the convention that $\ell_0 = 0$. We will omit I if there is no ambiguity.

2.4.6. Given a basis Δ of a root system Φ, its *Dynkin diagram* is defined as follows. The vertices are the simple roots of Φ and the number of edges between two vertices is $4\cos^2(\theta)$ if the angle between them is θ. Furthermore, these edges are decorated with arrows pointing from the longer root to the shorter root. It turns out that, up to graph isomorphisms, the Dynkin diagram is independent of the choice of the basis Δ.

From above description, we see that Φ is irreducible if and only if its Dynkin diagram is connected. The Dynkin diagrams of irreducible root systems are classified as follows [Bourbaki, chap.VI, §4, no.2, th.3], where the subscription n in the notation X_n denotes the rank of it.

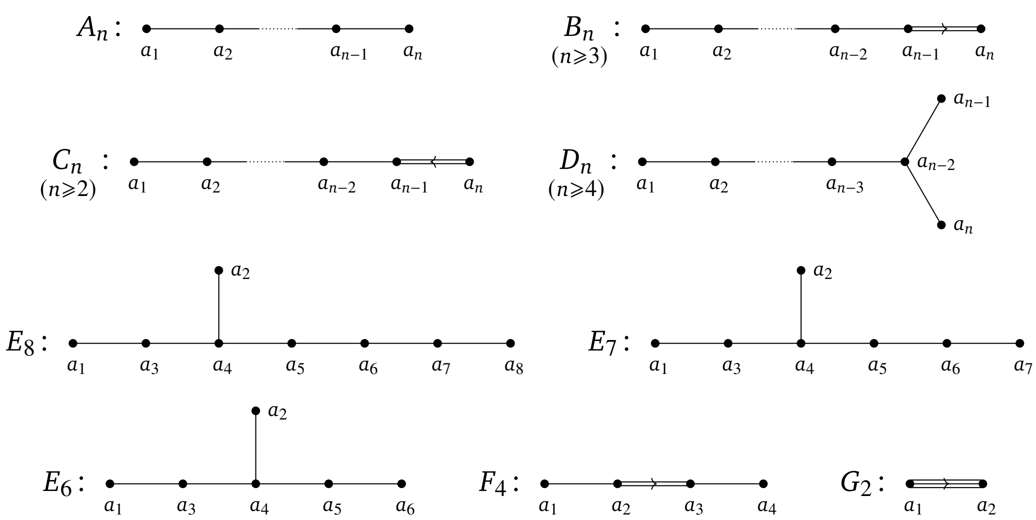


Figure 2.5. Dynkin diagrams of irreducible root systems

A spherical apartment is said to be of *type* X_n if it is isomorphic to ${}^v\mathscr{A}(\Phi)$ for an irreducible root system Φ of type X_n. A spherical apartment is of *classical type* if its every irreducible component is of type A_n, B_n, C_n, or D_n.

2.4.7. Let $\mathbb{A}$ be an affine space such that ${}^v\mathbb{A} = \mathbb{V}$ with a specified point o. For any $a \in \mathbb{V}^*$ and $k \in \mathbb{R}$, denote the affine function $x \mapsto a(x - o) + k$ on $\mathbb{A}$ by $a + k$ and denote the closed half-space $\{x \in \mathbb{A} \mid (a + k)(x) \geqslant 0\}$ by α_{a+k}.

For each $a \in \Phi$, let Γ_a be a fixed nonempty subset of $\mathbb{R}$. The affine function $a + k$ is called an *affine root* if $a \in \Phi$ and $k \in \Gamma_a$. Let Σ denote the set of closed half-spaces α_{a+k} with $a + k$ an affine root. Then $a + k \mapsto \alpha_{a+k}$ gives rise to a bijection between the set of affine roots and Σ. For this reason, we will not distinguish the affine root $a + k$ and the closed half-space α_{a+k} and will call Σ the *affine root system**. The roots are vectorial part of affine roots. Hence, we denote Φ by ${}^v\Sigma$ and call it the *vectorial root system* of Σ.

For $\alpha = \alpha_{a+k}$ an affine root, let ${}^v\alpha$ denote its vectorial part a, let $\partial\alpha$ denote its boundary $\{x \in \mathbb{A} \mid (a + k)(x) = 0\}$, let r_α denote the reflection with respect to $\partial\alpha$, let α^* denote the other affine root sharing the same boundary with α, that is $\overline{\mathbb{A} \setminus \alpha}$, and let α_+ denote the intersection of all the affine roots containing a neighborhood of α.

2.4.8. Let Σ be an affine root system on a Euclidean affine space $\mathbb{A}$, its *affine Weyl group* $W(\Sigma)$ is the reflection group on $\mathbb{A}$ generated by r_α for all $\alpha \in \Sigma$. In this way, we obtain an apartment $\mathscr{A}(\Sigma) := (\mathbb{A}, W(\Sigma))$ with vectorial apartment ${}^v\mathscr{A}({}^v\Sigma)$. Suppose all the subsets Γ_a are taken to be the same discrete subgroup $\Gamma \neq 0$ of $\mathbb{R}$, then the walls in the apartment $\mathscr{A}(\Sigma)$ are precisely the boundaries $\partial\alpha$ with $\alpha \in \Sigma$ [Bourbaki, chap.VI, §2, no.1, prop.2]. For x a point in the apartment $\mathscr{A}(\Sigma)$, let Σ_x be the set of affine roots α such that $x \in \partial\alpha$ and let ${}^v\Sigma_x$ be the set of vectorial parts of affine roots in Σ_x. Then Σ_x

*Note that, there is a notion called *affine root system*, defined similarly as root system, but for affine spaces. In this dissertation, this terminology is restricted to those arise from (reduced) root systems.

can be identified with ${}^v\Sigma_x$ by $\alpha \mapsto {}^v\alpha$. In particular, the roots in ${}^v\Sigma$ can be identified with the affine roots in Σ_o. Note that ${}^v\Sigma_x$ is a closed subroot system of ${}^v\Sigma$. Then the spherical apartment $\mathscr{A}_x$ at x can be identified with ${}^v\mathscr{A}\left({}^v\Sigma_x\right)$.

2.4.9. Notations as before. Suppose $\Phi = {}^v\Sigma$ is irreducible and all Γ_a are the same discrete subgroup of $\mathbb{R}$. Let ${}^vC$ be a Weyl chamber of Φ and Δ be the set of simple roots it defines. Then there is a unique root a_0 such that $\|a_0\| \geqslant \|a\|$ for all root a [Bourbaki, chap.VI, §1, no.8, prop.25]. This a_0 is called the *highest root* with respect to Δ or ${}^vC$. The set

$$C = (o + {}^vC) \setminus \alpha^*_{-a_0+} = \text{interior of } \left(\bigcap_{a\in\Delta} \alpha_a\right) \cap \alpha_{-a_0+}$$

is a chamber in $\mathscr{A}(\Sigma)$ [Bourbaki, chap.VI, §2, no.2, prop.5] and is called the *fundamental alcove* for Δ.

Let $\widetilde{\Delta}$ denote the set of affine roots α defining the walls of C, which means $C \subseteq \alpha$ and $\partial\alpha$ is a wall of C. Then $\widetilde{\Delta}$ consists of the simple roots and the affine root $\alpha_0 = \alpha_{-a_0+}$. Such a set $\widetilde{\Delta}$ is a *basis* of Σ in the sense that any affine root is a $\mathbb{Z}$-linear combination of its elements and the coefficients are either all non-negative or all non-positive.

Conversely, let C be a chamber in $\mathscr{A}(\Sigma)$ and x a special vertex which is also an extreme point of $\overline{C}$. The affine roots defining walls of C form a basis $\widetilde{\Delta}$ of the affine root system Σ. Among these affine roots, those vanishing at x give rise to a basis Δ of the root system Φ by taking their vectorial parts and the rest one gives rise to the highest root with respect to Δ by taking the negation of its vectorial part. Since chambers in $\mathscr{A}(\Sigma)$ are fundamental alcoves for some basis, we call them *alcoves* to avoid confusion with Weyl chambers.

2.4.10. The type function is introduced as follows. The affine Weyl group $W(\Sigma)$ is generated by r_α for $\alpha \in \widetilde{\Delta}$ as they are the affine roots defining walls of C. Therefore, a type $I \in \mathcal{T}$ corresponds to a proper subset of $\widetilde{\Delta}$. From now on, we do not distinguish them. Then the face of C corresponding to I is the set

$$C_I = \overline{C} \cap \Big(\bigcap_{\alpha \in I} \partial\alpha\Big) \setminus \Big(\bigcup_{\alpha \in \widetilde{\Delta}\setminus I} \partial\alpha\Big).$$

Convention 2.4.11. Given a basis $\Delta = \{a_1, \cdots, a_n\}$ of an irreducible Φ, the highest root relative to it is denoted by a_0 and the coefficients are denoted by h_i, namely

$$a_0 = h_1 a_1 + \cdots + h_n a_n. \tag{2.4.1}$$

Given a basis $\widetilde{\Delta} = \{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ of Σ, a *color* of a vertex v is the index i ($0 \leqslant i \leqslant n$) such that v is mapped to $\widetilde{\Delta} \setminus \{\alpha_i\}$ through the type function in 2.4.10.

2.4.12. Let Σ be an irreducible affine root system with $\widetilde{\Delta}$ a basis. Then the *extended Dynkin diagram* of it is defined similarly to Dynkin diagram except in the case of $\widetilde{A}_1$, where there is a left-right double arrow between the two vertices.

The following are the extended Dynkin diagrams of all irreducible affine root systems [Bourbaki, chap.VI, §4, no.3, prop.4], where the notation $\widetilde{X}_n$ indicates the affine root system that arises from the root system of type X_n. Note that the Dynkin diagrams are decorated in the following way: the part consisting of bold vertices is the ordinary Dynkin diagram and its vertices represent the simple roots a_i ($1 \leqslant i \leqslant n$), then the extra hollow vertex presents (the affine root α_0 defined by) the highest root a_0 and each simple root a_i is labelled by its coefficient h_i.

An affine apartment is said to be of *type* $\widetilde{X}_n$ (or *split type* X_n) if it is isomorphic to $\mathscr{A}(\Sigma)$ for an irreducible affine root system Σ of type $\widetilde{X}_n$. An affine apartment is of *split*

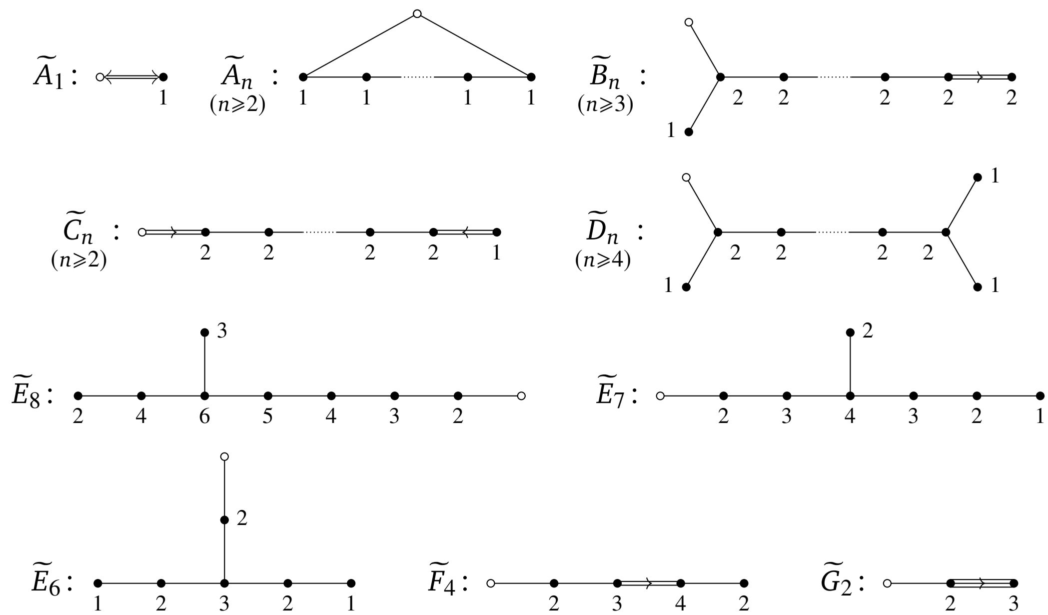


Figure 2.6. Extended Dynkin diagrams of irreducible affine root systems

classical type if its every irreducible component is of split type A_n, B_n, C_n, or D_n.

§ 2.5. Root data

Root systems can arise from root data. This section focus on root data.

Definition 2.5.1. A *(reduced) root datum** $\mathcal{R}$ is a quadruple $(\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ in which

- X and $\mathsf{X}^\vee$ are free $\mathbb{Z}$-modules of finite rank in duality by a pairing

$$\langle \cdot\,, \cdot \rangle \colon \mathsf{X} \times \mathsf{X}^\vee \to \mathbb{Z},$$

- Φ and $\Phi^\vee$ are finite subsets of $\mathsf{X} \setminus \{0\}$ and $\mathsf{X}^\vee \setminus \{0\}$ respectively, in bijection by a correspondence $a \leftrightarrow a^\vee$,

*in the sense of [SGA3, XXI, 1.1.1].

satisfying

RD1. for any $a \in \Phi$, $\langle a, a^\vee \rangle = 2$;

RD2. for any $a \in \Phi$, the "reflection" $r_a \colon x \mapsto x - \langle x, a^\vee \rangle a$ preserves Φ and the "reflection" $r_a \colon y \mapsto y - \langle a, y \rangle a^\vee$ preserves $\Phi^\vee$;

RD3. for any $a \in \Phi$, $\mathbb{Z}a \cap \Phi = \{\pm a\}$.

Note that we do not distinguish the two kinds of "reflections" in symbols since they form isomorphic finite groups of automorphisms on X and $\mathsf{X}^\vee$ respectively, and therefore it is better to view them as two representations of a same finite group ${}^v W(\mathcal{R})$. This group is called the *Weyl group* of the root datum.

2.5.2. If $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ is a root datum, then its Weyl group acts on the real vector space $\mathsf{X}^\vee_{\mathbb{R}} := \mathsf{X}^\vee \otimes \mathbb{R}$ and there is a unique inner product on it invariant under the action. Let $\mathbb{V}$ be the subspace of $\mathsf{X}^\vee_{\mathbb{R}}$ spanned by $\Phi^\vee$, called the *coroot space* of $\mathcal{R}$. Then Φ is a (reduced) root system on the Euclidean vector space $\mathbb{V}$.

In general, $\mathbb{V}$ is not the entire $\mathsf{X}^\vee_{\mathbb{R}}$. When it is, we say $\mathcal{R}$ is *semisimple*. So the apartment associated to root systems can also be viewed as the apartment associated to semisimple root data. As for the non-semisimple ones, they give rise to non-essential apartments and hence are ignored in this dissertation.

The quadruple $\mathcal{R}^\vee = (\mathsf{X}^\vee, \Phi^\vee, \mathsf{X}, \Phi)$ is also a root datum, called the *dual root datum* of $\mathcal{R}$. It is clear that dual root data give rise to coroot systems on the dual spaces.

2.5.3. Let $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ and $\mathcal{R}' = (\mathsf{X}', \Phi', \mathsf{X}'^\vee, \Phi'^\vee)$ be two root data. Then a *morphism* $f \colon \mathcal{R}' \to \mathcal{R}$ between them is a linear map $f \colon \mathsf{X}' \to \mathsf{X}$ inducing a bijection $\Phi \to \Phi'$ and its transpose ${}^\intercal f$ induces a bijection $\Phi'^\vee \to \Phi^\vee$. If f is a morphism of root data, then it also induces bijections between bases, systems of positive roots and

Weyl chambers [SGA3, XXI, 6.1.3]. As a consequence, it induces an isomorphism of spherical apartments ${}^{v}\mathscr{A}(\Phi') \cong {}^{v}\mathscr{A}(\Phi)$ (and also an isomorphism of affine apartments $\mathscr{A}(\Sigma') \cong \mathscr{A}(\Sigma)$ if $\Phi = {}^{v}\Sigma$ and $\Phi' = {}^{v}\Sigma'$ with covariant choice of Γ_a's).

A morphism of root data $f\colon \mathcal{R}' \to \mathcal{R}$ is an *isogeny** if the linear map f is injective and has finite cokernel $\mathrm{K}(f)$. This $\mathrm{K}(f)$ is also called the *cokernel* of f.

2.5.4. Let $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ be a root datum. Let $\mathsf{X}_0 = \{x \in \mathsf{X} \mid \langle x, \Phi^\vee \rangle = 0\}$ and $\mathsf{X}_0^\vee = \mathsf{X}^\vee/(\mathbb{V} \cap \mathsf{X}^\vee)$. Then X_0 and $\mathsf{X}_0^\vee$ are in duality by the pairing of $\mathcal{R}$ and thus give a trivial root datum $(\mathsf{X}_0, \emptyset, \mathsf{X}_0^\vee, \emptyset)$. It is called the *coradical* of $\mathcal{R}$ and is denoted by $\mathrm{corad}(\mathcal{R})$.

The dual root datum of the coradical of the dual $\mathcal{R}^\vee = (\mathsf{X}^\vee, \Phi^\vee, \mathsf{X}, \Phi)$ is called the *radical* of $\mathcal{R}$ and is denoted by $\mathrm{rad}(\mathcal{R})$. More precisely, let $\mathsf{Y}_0 = \left\{y \in \mathsf{X}^\vee \;\middle|\; \langle \Phi, y \rangle = 0\right\}$ and $\mathsf{Y}_0^\vee = \mathsf{X}/(\mathbb{V}^* \cap \mathsf{X})$, then $\mathrm{rad}(\mathcal{R})$ is the root datum $(\mathsf{Y}_0^\vee, \emptyset, \mathsf{Y}_0, \emptyset)$. It follows that $\mathcal{R}$ is semisimple if and only if $\mathrm{corad}(\mathcal{R}) = 0$ if and only if $\mathrm{rad}(\mathcal{R}) = 0$.

Let $\mathcal{R}^0$ denote the trivial root datum $(\mathsf{X}, \emptyset, \mathsf{X}^\vee, \emptyset)$. Then the inclusion and projection to X induce morphisms of root data

$$\mathrm{corad}(\mathcal{R}) \longrightarrow \mathcal{R}^0 \longrightarrow \mathrm{rad}(\mathcal{R}).$$

And the composition $\mathrm{corad}(\mathcal{R}) \to \mathrm{rad}(\mathcal{R})$ is an isogeny [SGA3, XXI, 6.3.4]. Its cokernel is denoted by $\mathrm{N}(\mathcal{R})$. Note that there is a pairing:

$$\mathrm{N}(\mathcal{R}) \times \mathrm{N}(\mathcal{R}^\vee) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

2.5.5. A *lattice* $\mathcal{L}$ in a $\mathbb{R}$-vector space $\mathbb{V}$ is a discrete finitely generated $\mathbb{Z}$-submodule of $\mathbb{V}$ spanning $\mathbb{V}$. Its *dual lattice* $\mathcal{L}^*$ is the lattice in the dual space $\mathbb{V}^*$ consisting of those

*in the sense of [SGA3, XXI, 6.2.1] and is called a *central isogeny* in [Mil17, 23.2] functionals $f \in \mathbb{V}^*$ such that $f(\mathcal{L}) \subseteq \mathbb{Z}$.

Given a root system Φ on a Euclidean vector space $\mathbb{V}$, there are four lattices:

- $\mathcal{Q}$ the *root lattice*, which is the lattice in $\mathbb{V}^*$ generated by the roots;
- $\mathcal{Q}^\vee$ the *coroot lattice*, which is the lattice in $\mathbb{V}$ generated by the coroots;
- $\mathcal{P}$ the *weight lattice*, which is the dual lattice of $\mathcal{Q}^\vee$ in $\mathbb{V}^*$;
- $\mathcal{P}^\vee$ the *coweight lattice*, which is the dual lattice of $\mathcal{Q}$ in $\mathbb{V}$.

Suppose the root system Φ is given by a root data $\mathcal{R}$. Then X contains $\mathcal{Q}$. If $\mathcal{R}$ is semisimple, then X is a lattice in $\mathbb{V}^*$ between $\mathcal{Q}$ and $\mathcal{P}$. In this case, the quotient $\mathcal{P}/\mathsf{X}$ is a finite group $\pi_1(\mathcal{R})$, called the *fundamental group* of $\mathcal{R}$; the quotient $\mathsf{X}/\mathcal{Q}$ is a finite group $Z(\mathcal{R})$, called the *centre* of $\mathcal{R}$.

2.5.6. Let $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ be a root datum.

Let Y be a submodule of X containing Φ and $i\colon \mathsf{Y} \to \mathsf{X}$ the inclusion. Its transpose is denoted by ${}^t i\colon \mathsf{X}^\vee \to \mathsf{Y}^\vee$ and denote $\Phi_{\mathsf{Y}} = \Phi$, $\Phi_{\mathsf{Y}}^\vee = {}^t i(\Phi^\vee)$. Then $(\mathsf{Y}, \Phi_{\mathsf{Y}}, \mathsf{Y}^\vee, \Phi_{\mathsf{Y}}^\vee)$ is a root datum and i is a morphism of root data. It is called the root datum *induced* by $\mathcal{R}$ on Y and is denoted by $\mathcal{R}_{\mathsf{Y}}$.

Let $\mathsf{Y}^\vee$ be a submodule of $\mathsf{X}^\vee$ containing $\Phi^\vee$. Then the dual root datum of the root datum induced by $\mathcal{R}^\vee$ on $\mathsf{Y}^\vee$ is called the root datum *coinduced* by $\mathcal{R}$ on $\mathsf{Y}^\vee$ and is denoted by $\mathcal{R}^{\mathsf{Y}^\vee}$.

The following are some special cases of above.

- $\mathrm{ad}(\mathcal{R})$ the root datum induced by $\mathcal{R}$ on the root lattice $\mathcal{Q}$;
- $\mathrm{ss}(\mathcal{R})$ the root datum induced by $\mathcal{R}$ on $\mathbb{V}^* \cap \mathsf{X}$;
- $\mathrm{der}(\mathcal{R})$ the root datum coinduced by $\mathcal{R}$ on $\mathbb{V} \cap \mathsf{X}^\vee$;
- $\mathrm{sc}(\mathcal{R})$ the root datum coinduced by $\mathcal{R}$ on the coroot lattice $\mathcal{Q}^\vee$.

2.5.7. We have seen various root data constructed from a given one $\mathcal{R}$. They form a diagram of morphisms of root data:

$$\begin{tikzcd}
\mathrm{ad}(\mathcal{R}) \arrow[r] & \mathrm{ss}(\mathcal{R}) \arrow[rr] \arrow[d] \arrow[dr] & & \mathrm{der}(\mathcal{R}) \arrow[r] \arrow[d] & \mathrm{sc}(\mathcal{R}) \\
& \mathrm{ss}(\mathcal{R}) \times \mathrm{corad}(\mathcal{R}) \arrow[r] & \mathcal{R} \arrow[r] \arrow[ur] & \mathrm{der}(\mathcal{R}) \times \mathrm{rad}(\mathcal{R}) &
\end{tikzcd}$$

Moreover, we have the following propositions [SGA3, XXI, 6.5.5 – 6.5.9].

(i). The horizontal ones are isogenies between root data.

(ii). The diagram is commutative.

(iii). $\mathrm{ad}(\mathcal{R})$ is *adjoint*, namely every isogeny to it is an isomorphism.

(iv). $\mathrm{sc}(\mathcal{R})$ is *simply-connected*, namely every isogeny from it is an isomorphism.

(v). $\mathcal{R}$ is semisimple if and only if the middle triangle consists of isomorphisms.

(vi). If $\mathcal{R}$ is semisimple, its centre $Z(\mathcal{R})$ and fundamental group $\pi_1(\mathcal{R})$ are the cokernels of the isogenies $\mathrm{ad}(\mathcal{R}) \to \mathcal{R}$ and $\mathcal{R} \to \mathrm{sc}(\mathcal{R})$ respectively.

(vii). The cokernels of the isogenies $\mathrm{ss}(\mathcal{R}) \times \mathrm{corad}(\mathcal{R}) \to \mathcal{R}$, $\mathcal{R} \to \mathrm{der}(\mathcal{R}) \times \mathrm{rad}(\mathcal{R})$ and $\mathrm{ss}(\mathcal{R}) \to \mathrm{der}(\mathcal{R})$ are all isomorphic to $\mathrm{N}(\mathcal{R})$.

(viii). $\mathcal{R}$ is the product of a semisimple root datum with a trivial root datum if and only if $\mathrm{N}(\mathcal{R}) = 0$.

(ix). All root data in this diagram have isomorphic root systems and hence isomorphic apartments.

§ 2.6. Euclidean buildings

In this section, a geometric definition of buildings (the Euclidean buildings) is given, and its properties are further discussed.

Definition 2.6.1. A *(Euclidean) building* is a set $\mathscr{B}$ equipped with a polysimplicial complex $\mathcal{F}$, whose members are subsets of $\mathscr{B}$ and are called *facets*, and a family $\mathcal{A}$ of subsets of $\mathscr{B}$, whose members are called *apartments*, such that the following axioms are satisfied.

EB0. For each apartment $A \in \mathcal{A}$, there is a Euclidean apartment $\mathscr{A}$ together with a bijection between them, exchanging the complex $\mathcal{F}_A$ of facets contained in A and the complex of facets in $\mathscr{A}$.

Note that, this allows us to view apartments in $\mathscr{B}$ as Euclidean affine spaces and hence it makes sense to talk about isometries between them.

EB1. For any two facets F, F', there is an apartment A containing them.

EB2. If A, A' are two apartments containing both F and F', then there is an isomorphism between A and A' fixing F and F' pointwise.

Here an isomorphism between A and A' is an isometry between them exchanging the posets $\mathcal{F}_A$ and $\mathcal{F}_{A'}$.

Note that, from the definition, all apartments $A \in \mathcal{A}$ are isomorphic to an abstract one $\mathscr{A}$. Then $\mathscr{B}$ is said to be of *type* $\mathscr{A}$ and is said to be *spherical* (resp. *discrete*, *affine*, etc.) if so is $\mathscr{A}$. The Weyl group W of $\mathscr{A}$ is also called the *Weyl group* of $\mathscr{B}$.

Remark. The notions of *walls*, *chambers*, *vertices* and *types* in a building is defined similarly as in an apartment, and we will use the same notations as there. Furthermore, there is a *type function* $\tau\colon \mathcal{F} \to \mathcal{T}$ extending the type function on an apartment to the entire building uniquely.

Remark. We have assumed that apartments are essential. In particular, the buildings in Bruhat-Tits theory used in this dissertation are the *reduced buildings*, rather than

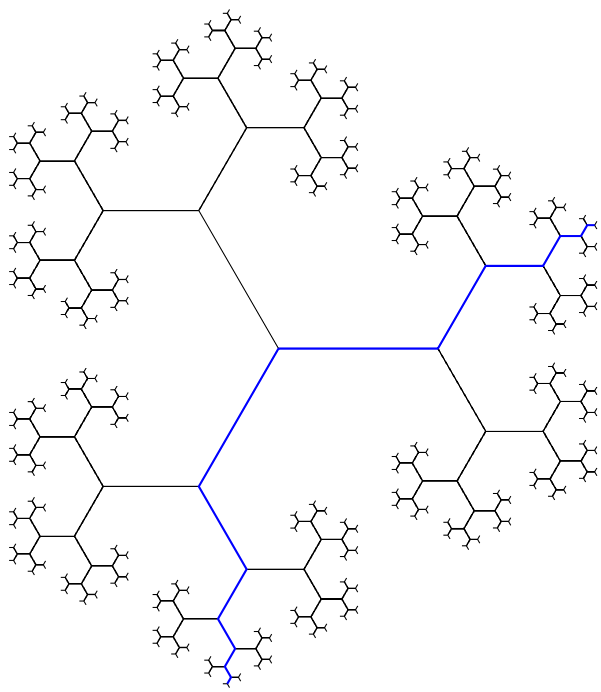


Figure 2.7. An example of Euclidean building (the Bruhat-Tits tree of $\mathsf{GL}_2(\mathbb{Q}_3)$) with an apartment specified by blue color

extended buildings. However, this is harmless as we focus more on the polysimplicial structure, and we do want the vertices being points.

Remark. One can see that a discrete building $\mathscr{B}$ is completely determined by its combinatorial information, which is encoded in the polysimplicial complex $\mathcal{F}$ up to a choice of the family $\mathscr{A}$. To see this, one can compare the axioms **EB0.** to **EB2.** with **B0.** to **B2.**. Therefore, to give a discrete Euclidean building $(\mathscr{B}, \mathscr{A})$ is equivalent to give an abstract building $(\mathcal{F}, \{\mathcal{F}_A\}_{A\in\mathscr{A}})$.

2.6.2. The apartments are Euclidean affine spaces, hence have metrics. Those metrics are compatible in the sense that they agree on any overlap, hence are glued into a metric $d(-,-)$ on the entire building $\mathscr{B}$ consistently. Then $\mathscr{B}$ equipped with this metric is a complete metric space having the *CAT(0)-property* [Rou09, 6.5], which means that geodesic triangles in $\mathscr{B}$ are at least as thin as in a Euclidean plane: saying x, y, z are three points in $\mathscr{B}$ forming a geodesic triangle and $\bar{x}, \bar{y}, \bar{z}$ are three points in a Euclidean plane having the same pointwise distance as x, y, z, then for any point m in the *geodesic segment* $[x, y]$ in the triangle and $\bar{m}$ the corresponding point in the segment $[\bar{x}, \bar{y}]$ (namely, $d(\bar{x}, \bar{m}) = d(x, m)$), then $d(z, m) \leqslant d(\bar{z}, \bar{m})$.

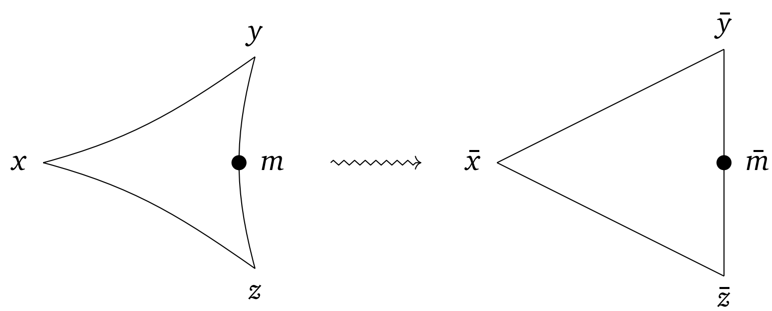


Figure 2.8. The CAT(0)-property

Consequences of the CAT(0)-property include:

(i). [Rou09, 6.6] the geodesic segments between points are unique;

(ii). [Rou09, 7.1] any group of isometries stabilizing a nonempty bounded subset has a fixed point;

(iii). [Rou09, 7.3] the distance from a point to a nonempty closed convex subset is achieved by a unique point.

For more details, see [Rou09, §6 and 7].

2.6.3. A *morphism* between buildings $\mathscr{B}$ and $\mathscr{B}'$ is a continuous map inducing a *chamber map* between $\mathcal{F}$ and $\mathcal{F}'$ and maps apartments in apartments. Then an *automorphism* of a building is an isometry transforming a facet (resp. apartment) in a facet (resp. apartment). Any building can be decomposed into a product of a trivial building with irreducible ones, similarly as in 2.3.3. However, there is no guarantee that such a decomposition gives a good corresponding decomposition on the family $\mathcal{A}$.

Remark. With above definition of morphisms, we obtain an equivalence of categories between discrete Euclidean buildings and abstract buildings.

2.6.4. An automorphism is said to be *type-preserving* if it leaves the type function τ invariant. For instance, any $w \in W$ is such an automorphism. A group G of automorphisms is said to be *strongly transitive* if it acts transitively on the pairs (C, A) where C is a chamber in the apartment A. This is the case if and only if G acts transitively on apartments and in any apartment A, the following conditions for a pair of chambers C, C' in A are equivalent:

(i). C and C' are conjugated by the Weyl group W;

(ii). C and C' are conjugated by the stabilizer $N_G(A)$ of A in G;

(iii). C and C' are conjugated by G.

When a group G of automorphisms is strongly transitive and type-preserving, we have

$$W \cong N_G(A)/C_G(A),$$

where $C_G(A)$ is the fixator of an apartment A in G.

2.6.5. Let G be a strongly transitive and type-preserving group of automorphisms and F be a facet in an apartment A. The stabilizer (which is also the fixator)

$$G_F := N_G(F) = C_G(F)$$

of F is called a *parabolic* subgroup of G. The parabolic group G_F acts transitively on the apartments containing F. Indeed, one can deduce this from the fact that G_F acts transitively on chambers containing F since G is strongly transitive. Here the former is due to that G acts transitively on chambers and is type-preserving.

Moreover, we have the *Bruhat decomposition* [Rou09, 6.9]

$$G = G_F.N_G(A).G_F.$$

In particular, if $F = C$ is a chamber, then

$$G = \bigsqcup_{w \in W} G_C w G_C.$$

2.6.6. Let x be a point in an affine building $\mathscr{B}$. The *link* of x is the subcomplex $\mathcal{F}_x$ of the facets covering x. For any apartment A, let $\mathcal{F}_{x,A}$ be the subcomplex $\mathcal{F}_x \cap \mathcal{F}_A$, where $\mathcal{F}_A$ is as in Definition 2.6.1. Then $\mathcal{F}_x$ is an abstract spherical building with the system of apartments $\{\mathcal{F}_{x,A}\}_{A \in \mathcal{A}}$. To see this, recall that for any vertex x in an affine Euclidean apartment, the facets in the spherical apartment at x can be identified with the facets covering x through a radially shrinking with center x. In this way, we obtain a spherical building $\mathscr{B}_x$, called the *spherical building* at x. Note that the embedding $\mathscr{B}_x \longrightarrow \mathscr{B}$ is not isometric, only conformal.

2.6.7. A *bornology* on a set X is a collection $\mathcal{B}$ of subsets of X such that it covers X and is stable under inclusion and finite unions. Once such a bornology is chosen, its members are called *bounded subsets* of X. For instance, any metric space has a canonical bornology induced by its metric. Another example is any locally compact topological space, where the bornology consists of all relatively compact subsets. A *morphism* between bornological sets is a map preserving the bornologies.

A *bornological group* is a group G equipped with a bornology on it stable under multiplication. For instance, let G be an isometry group on a metric space X, then there is a canonical bornology whose members are subsets M such that the set $M.x$ is bounded in X for some $x \in X$.

Let $\varphi\colon G' \to G$ be a group homomorphism and G a bornological group. Then we can canonically pull back the bornology on G to G': a subset of G' is bounded when its image is bounded in G.

So we can talk about *bounded subgroups* of a group G acting on the building $\mathscr{B}$ regardless its own topology or bornology. But if G is topological or bornological, it makes sense to ask if its bornology is the same as the pullback one. It is worth to point out that this is the case when G acts continuously on $\mathscr{B}$.

Let G be a strongly transitive and type-preserving group of automorphisms of $\mathscr{B}$, then for any subgroup H of G, the following conditions are equivalent [Rou09, 7.2].

(i). H is bounded;

(ii). H fixes a point in $\mathscr{B}$;

(iii). H is contained in a parabolic subgroup of G.

In particular, the maximal bounded subgroups of G are the maximal parabolic subgroups and hence the stabilizers of vertices.

In general, even if G is not type-preserving, then maximal bounded subgroups of G are still stabilizers of points, but: (i). not all such stabilizers are maximal; and (ii). not all such stabilizers are stabilizers of vertices.

Chapter 3.

Reductive Groups and Tits Buildings

Tits' building theory [Tit74, Bourbaki] was applied to study the structure of reductive groups over an arbitrary field, a family of linear algebraic groups which play important roles in mathematics. We refer to [Mil17] for algebraic groups and reductive groups over an arbitrary field and [SGA3] for group schemes and reductive group schemes over general base.

Throughout this chapter, we fix a ground field K and an algebraic closure K^{alg} (resp. separable closure K^{sep}) of it.

§ 3.1. Algebraic groups

We first recall some basic notions on algebraic groups.

Definition 3.1.1. By an *algebraic group*[*] (defined over K), we mean a group object in the category of schemes of finite type over K.

[*]Generally, a *group scheme* over a base S is a group object in the category of schemes over S. The materials covered in this section works over general base, not only over K. Just in case, we write $(\,\cdot\,)/S$ to emphasize the base S.

3.1.2. An algebraic group is said to be *affine* (resp. *smooth*, *connected*, etc.) if so is its underlying scheme.* But it is often useful to have another viewpoint: an algebraic group is in particular a group-valued functor from the category $\mathbf{Alg}_K$ of finitely generated K-algebras. In particular, affine algebraic groups are precisely the representable group-valued functors.

We will use bold letters like G to denote algebraic groups defined over K. For any K-algebra R, the group scheme obtained by base change $\mathsf{G} \otimes_K R$ is denoted by G_R and the group of R-points is denoted by $\mathsf{G}(R)$ (but if we use notations with parenthesis, e.g. $\mathsf{GL}(V)$, to denote an algebraic group, then its group of R-points is denoted by padding R into the parenthesis as the last parameter, e.g. $\mathsf{GL}(V, R)$). Moreover, $\mathsf{G}(K)$ is simply denoted by G and $\mathsf{G}_R(R) \cong \mathsf{G}(R)$ is simply denoted by G_R. We also write $g \in \mathsf{G}$ to mean that g is an R-point of G for some K-algebra R.

Many group-theoretical constructions apply to algebraic groups. For G an algebraic group and H a subgroup, we use $\mathsf{N}_{\mathsf{G}}(\mathsf{H})$ (resp. $\mathsf{Z}_{\mathsf{G}}(\mathsf{H})$) to denote *normalizer* (resp. *centralizer*) of H in G. In particular, $\mathsf{Z}(\mathsf{G})$ denote the centre of G.

3.1.3. Let G be an algebraic group. Its *neutral component* G° is the largest connected subgroup of G. Its *component group* $\pi_0(\mathsf{G})$ is the universal étale scheme under G. Then there is an exact sequence [Mil17, 2.37]:

$$1 \longrightarrow \mathsf{G}^\circ \longrightarrow \mathsf{G} \longrightarrow \pi_0(\mathsf{G}) \longrightarrow 1.$$

The above formations are compatible with field extensions and products.

*It is worth to emphasize that the topological terminology such as *connected* talks about the underlying scheme of G, not the underlying set of G (although it may carry a topological structure). For instance, when the ground field K is a local field, the multiplicative group $\mathbb{G}_{\mathrm{m}}$ (see Example 3.1.4) is connected while the topological group $K^\times$ is totally-disconnected.

The following conditions on an algebraic group G are equivalent [Mil17, 1.36]:

(i). G is irreducible;

(ii). G is connected;

(iii). G is geometrically connected;

(iv). $\pi_0(\mathsf{G})$ equals the trivial group 1.

Example 3.1.4. Here we give some algebraic groups presented as functors.

(i). The functor $R \rightsquigarrow (R, +)$ mapping a K-algebra to its underlying abelian group defines an algebraic group $\mathbb{G}_{\mathrm{a}}$, called the *additive group*.

(ii). The functor $R \rightsquigarrow (R^{\times}, \times)$ mapping a K-algebra to its unit group defines an algebraic group $\mathbb{G}_{\mathrm{m}}$, called the *multiplicative group*.

(iii). The functor $R \rightsquigarrow \{r \in R \mid r^n = 1\}$ mapping a K-algebra to its set of n-th roots of unity defines an algebraic group μ_n, called the *group of n-th roots of unity*.

(iv). Let G be a finite group. The constant functor $R \rightsquigarrow G$ is not a scheme, but its sheafification $R \rightsquigarrow \mathrm{Map}(\pi_0(R), G)$, where $\pi_0(R)$ is the set of connected components of $\mathrm{Spec}(R)$, defines an algebraic group $\underline{G}$. Such an algebraic group is called a *constant algebraic group*.

(v). Let V be a finite-dimensional vector space over K, then the functor $R \rightsquigarrow V_R := V \otimes_K R$ defines an algebraic group $\mathbb{W}(V)$, called the *additive group* of V. Any choice of basis of V gives rise to an isomorphism from this group to a product of copies of $\mathbb{G}_{\mathrm{a}}$.

(vi). The functor mapping a K-algebra R to the additive group of $m \times n$ matrices with entries in R defines an algebraic group $\mathsf{M}_{m \times n}$.

(vii). Let V be a finite-dimensional vector space over K, then the functor $R \rightsquigarrow \mathrm{End}(V_R)$ defines an algebraic group $\mathsf{End}(V)$. When V is of dimension n, any choice of basis of V gives an isomorphism from this group to $\mathsf{M}_{n\times n}$.

(viii). The functor mapping a K-algebra R to the group of invertible $n \times n$ matrices with entries in R defines an algebraic group GL_n, called the *general linear group*.

(ix). Let V be a finite-dimensional vector space over K, then the functor $R \rightsquigarrow \mathrm{Aut}(V_R)$ defines an algebraic group $\mathsf{GL}(V)$, called the *general linear group* of V. When V is of dimension n, any choice of basis of V gives rise to an isomorphism from this group to GL_n.

All above functors are representable. Hence, above algebraic groups are affine.

3.1.5. A *representation* of an algebraic group G is a homomorphism of group-valued functors $\rho\colon \mathsf{G} \longrightarrow \mathsf{GL}(V)$, where V is a vector space over K and $\mathsf{GL}(V)$ is the functor $R \rightsquigarrow \mathrm{Aut}(V_R)$. When V is finite-dimensional, this is a homomorphism of algebraic groups. Such a representation is *faithful* if ρ is injective.

An algebraic group is *linear* if it admits a finite-dimensional faithful representation. Equivalently, an algebraic group is linear if it is isomorphic to an algebraic subgroup of some GL_n. It turns out that [Mil17, 1.43 and 4.10]:

$$\text{affine algebraic group} = \text{linear algebraic group}.$$

Example 3.1.6. Here we give some linear algebraic groups.

(i). The functor $R \rightsquigarrow \{g \in \mathsf{GL}_n(R) \mid \det(g) = 1\}$ mapping a K-algebra R to the group of invertible $n \times n$ matrices with entries in R and determinant 1 defines an algebraic subgroup SL_n of GL_n, called the *special linear group*.

(ii). The functor $R \rightsquigarrow \left\{(g_{ij}) \in \mathsf{GL}_n(R) \,\middle|\, g_{ij} = 0 \text{ if } i > j\right\}$ mapping a K-algebra R to the group of upper triangular invertible $n \times n$ matrices with entries in R defines an algebraic subgroup T_n of GL_n.

(iii). The functor $R \rightsquigarrow \left\{(g_{ij}) \in \mathsf{GL}_n(R) \,\middle|\, g_{ij} = 0 \text{ if } i > j \text{ and } g_{ij} = 1 \text{ if } i = j\right\}$ mapping a K-algebra R to the group of upper triangular invertible $n \times n$ matrices with entries in R and diagonal entries 1 defines an algebraic subgroup U_n of T_n.

(iv). The functor $R \rightsquigarrow \{\operatorname{diag}(t_1, \cdots, t_n) \in \mathsf{GL}_n(R)\}$ mapping a K-algebra R to the group of invertible diagonal $n \times n$ matrices with entries in R defines an algebraic subgroup D_n of T_n. Note that $\mathsf{D}_n \cong \mathbb{G}_{\mathrm{m}}^n$.

(v). The functor $R \rightsquigarrow \{g \in \mathsf{GL}(V, R) \mid \det(g) = 1\}$ mapping a K-algebra R to the group of R-automorphisms of V_R having determinant 1 defines an algebraic subgroup $\mathsf{SL}(V)$ of $\mathsf{GL}(V)$, called the *special linear group* of V.

(vi). The quotient of GL_n (resp. $\mathsf{GL}(V)$) by the normal subgroup of scalars is a linear algebraic group. It is denoted by PGL_n (resp. $\mathsf{PGL}(V)$) and is called the *projective linear group*.

Example 3.1.7. Here are more examples of linear algebraic groups. They form so-called *classical groups*.

(i). Let D be a division algebra over K and V be a finite-dimensional D-vector space, namely free right D-module of finite rank. Then the functor $R \rightsquigarrow \operatorname{Aut}_{D_R}(V_R)$ defines an algebraic group $\mathsf{GL}_D(V)$, called the *general linear group* of V. One also have the *special linear group* $\mathsf{SL}_D(V)$, specified by the condition $\det(g) = 1$, and the *projective linear group* $\mathsf{PGL}_D(V)$.

(ii). Let D be a division algebra over K with an involution σ such that K is precisely the σ-fixed-point subfield of the center of D. Let V be a finite-dimensional D-vector space and fix $\epsilon = \pm 1$, a *hermitian form* on V is a K-bilinear form $\langle \cdot \mid \cdot \rangle \colon V \times V \to D$ such that

$$\langle v.\xi \mid w.\eta \rangle = \xi^\sigma \langle v \mid w \rangle \eta \qquad \text{for all} \qquad v, w \in V, \xi, \eta \in D$$

and that

$$\langle w \,|\, v\rangle = \epsilon\langle v \,|\, w\rangle^{\sigma} \qquad \text{for all} \qquad v, w \in V.$$

When V is equipped with a hermitian form, we say it is a *hermitian space*. Then the functor $R \rightsquigarrow \{g \in \mathsf{GL}_D(V,R) \mid \langle g.v \,|\, g.w\rangle = \langle v \,|\, w\rangle \text{ for all } v, w \in V_R\}$ defines an algebraic group $\mathsf{O}(V)$, called the *orthogonal group* of V. One also have the *special orthogonal group* $\mathsf{SO}(V)$, specified by the condition $\det(g) = 1$.

(iii). More general, if we replace the condition $\langle g.v \,|\, g.w\rangle = \langle v \,|\, w\rangle$ for all $v, w \in V_R$ by $\exists \nu(g) \in \mathbb{G}_{\mathrm{m}}(R)$ such that $\langle g.v \,|\, g.w\rangle = \nu(g)\langle v \,|\, w\rangle$ for all $v, w \in V_R$, we get another algebraic group $\mathsf{GO}(V)$, called the *similitude group* of V.

Example 3.1.8. We are more interested in the following special cases of 3.1.7.(ii):

(i). $\sigma = \mathrm{id}$ and $\epsilon = -1$: namely, $\langle \cdot \mid \cdot \rangle$ is *alternative*. We say V is a *symplectic space* and the groups $\mathsf{O}(V)$ (equals to $\mathsf{SO}(V)$) and $\mathsf{GO}(V)$ are denoted by $\mathsf{Sp}(V)$ and $\mathsf{GSp}(V)$ respectively. In particular, we use the notations Sp_{2n} and GSp_{2n} for the case $V = K^{2n}$ (we use the index set $\{\pm 1, \cdots, \pm n\}$) and $\langle \cdot \mid \cdot \rangle$ is given by

$$\langle \mathbf{x} \,|\, \mathbf{y}\rangle = x_1 y_{-1} - x_{-1} y_1 + \cdots + x_n y_{-n} - x_{-n} y_n.$$

(ii). $\sigma = \mathrm{id}$ and $\epsilon = -1$: namely, $\langle \cdot \mid \cdot \rangle$ is *symmetric*. We should assume that V is equipped with a non-degenerate quadratic form $\mathfrak{q}$ so that

$$\langle u \,|\, v\rangle = \mathfrak{q}(u+v) - \mathfrak{q}(u) - \mathfrak{q}(v) \qquad \text{for all} \qquad u, v \in V.$$

If this is the case, we say V is a *quadratic space*. In particular, we use the notations O_{2n}, SO_{2n}, and GO_{2n} for the case $V = K^{2n}$ (we use the index set $\{\pm 1, \cdots, \pm n\}$) and $\mathfrak{q}$

is given by

$$\mathfrak{q}(\mathbf{x}) = x_1 x_{-1} + \cdots + x_n x_{-n}.$$

We also use the notations O_{2n+1}, SO_{2n+1}, and GO_{2n+1} for the case $V = K^{2n+1}$ (we use the index set $\{0, \pm 1, \cdots, \pm n\}$) and $\mathfrak{q}$ is given by

$$\mathfrak{q}(\mathbf{x}) = x_0^2 + x_1 x_{-1} + \cdots + x_n x_{-n}.$$

3.1.9. An algebraic group G is *unipotent* if every finite-dimensional representation $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$ is *unipotent*, namely there exists a G-stable flag

$$V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_{m-1} \supsetneq V_m = 0,$$

such that G acts trivially on each factor V_i/V_{i+1}. Equivalently, an algebraic group is unipotent if it is isomorphic to an algebraic subgroup of some U_n.

For any $g \in \mathsf{G}(K^{\mathrm{alg}})$, we have *Jordan-Chevalley decomposition* [Mil17, 9.18]: there exist unique elements $g_s, g_u \in \mathsf{G}(K^{\mathrm{alg}})$ such that

$$g = g_s g_u = g_u g_s,$$

and for any representation $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$, the linear operator $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent. An element $g \in \mathsf{G}(K^{\mathrm{alg}})$ is said to be *semisimple* (reps. *unipotent*) if $g = g_s$ (resp. $g = g_u$). A smooth algebraic group G is unipotent if and only if all elements of $\mathsf{G}(K^{\mathrm{alg}})$ are unipotent [Mil17, 14.12].

3.1.10. An algebraic group is a *torus* if it becomes isomorphic to a product of copies of $\mathbb{G}_{\mathrm{m}}$ over some field containing K. A torus over K is *split* if it is already isomorphic to a product of copies of $\mathbb{G}_{\mathrm{m}}$ over K.

An algebraic group G is *diagonalizable* if its every representation is *diagonalizable*, namely it is a sum of one-dimensional representations. Equivalently, an algebraic group is diagonalizable if it is isomorphic to an algebraic subgroup of some D_n.

An algebraic group G is *of multiplicative type* if it becomes diagonalizable over some field containing K. All tori are of multiplicative type. A smooth commutative algebraic group G is of multiplicative type if and only if all elements of $\mathsf{G}(K^{\mathrm{alg}})$ are semisimple [Mil17, 12.21].

A *character* of an algebraic group G is a homomorphism $\chi\colon \mathsf{G} \to \mathbb{G}_{\mathrm{m}}$. Let χ and χ' be two characters of G, then the sum $\chi + \chi'$ is defined as

$$(\chi + \chi')(g) = \chi(g) \cdot \chi'(g), \qquad \forall g \in \mathsf{G}\,.$$

This is again a character and the set of characters is an abelian group, denoted by $\mathsf{X}(\mathsf{G})$. The *character group* of G is the abelian group

$$\mathsf{X}^*(\mathsf{G}) := \mathrm{Hom}\big(\mathsf{G}_{K^{\mathrm{sep}}}, \mathbb{G}_{\mathrm{m},K^{\mathrm{sep}}}\big).$$

A *cocharacter* of an algebraic group G is a homomorphism $\lambda\colon \mathbb{G}_{\mathrm{m}} \to \mathsf{G}$. Suppose G is commutative. Then the sum $\lambda + \lambda'$ of two cocharacters of G is defined as:

$$(\lambda + \lambda')(z) = \lambda(z) \cdot \lambda'(z), \qquad \forall z \in \mathbb{G}_{\mathrm{m}}\,.$$

This is again a cocharacter. The *cocharacter group* of G is then the abelian group

$$\mathsf{X}_*(\mathsf{G}) := \mathrm{Hom}\big(\mathbb{G}_{\mathrm{m},K^{\mathrm{sep}}}, \mathsf{G}_{K^{\mathrm{sep}}}\big).$$

Example 3.1.11. Let $\mathsf{G} = \mathsf{D}_n$. For each $1 \leqslant i \leqslant n$, define $\chi_i \colon \mathsf{D}_n \to \mathbb{G}_{\mathrm{m}}$ as the character

$$\operatorname{diag}(t_1, \cdots, t_n) \mapsto t_i$$

and $\lambda_i \colon \mathbb{G}_{\mathrm{m}} \to \mathsf{D}_n$ as the cocharacter

$$t \mapsto \operatorname{diag}(1, \cdots, t, \cdots, 1)$$

with t at the i-th position. Then

(i). characters of D_n are $\mathbb{Z}$-linear combinations of $\chi_1, \cdots, \chi_n$;

(ii). cocharacters of D_n are $\mathbb{Z}$-linear combinations of $\lambda_1, \cdots, \lambda_n$.

Therefore, if T is a torus of dimension n, then its character group $\mathsf{X}^*(\mathsf{T})$ (resp. cocharacter group $\mathsf{X}_*(\mathsf{T})$) is isomorphic to $\mathbb{Z}^n$ and furthermore consists of all characters (resp. cocharacters) providing T is split.

Let χ be a character and λ be a cocharacter of T. Then the composition $\chi \circ \lambda$ is an endomorphism $t \mapsto t^{\langle \chi, \lambda \rangle}$ of $\mathbb{G}_{\mathrm{m}}$, which can be identified with the integer $\langle \chi, \lambda \rangle \in \mathbb{Z}$. In this way, we get a perfect pairing of $\mathbb{Z}$-modules

$$\langle \cdot\, , \cdot \rangle \colon \mathsf{X}^*(\mathsf{T}) \times \mathsf{X}_*(\mathsf{T}) \to \mathbb{Z}$$

making $\mathsf{X}^*(\mathsf{T})$ and $\mathsf{X}_*(\mathsf{T})$ in duality.

Example 3.1.12. Let G be a diagonalizable algebraic group. Then $\mathsf{X}(\mathsf{G})$ is a finitely generated abelian group and (here p is the characteristic of K) [Mil17, 12.5]

(i). G is smooth if and only if $\mathsf{X}(\mathsf{G})$ has no p-torsion;

(ii). G is connected if and only if $\mathsf{X}(\mathsf{G})$ has no torsion other than p-torsions;

(iii). G is smooth and connected if and only if $\mathsf{X}(\mathsf{G})$ is free.

Moreover, the functor $\mathsf{G} \rightsquigarrow \mathsf{X}(\mathsf{G})$ gives a contravariant equivalence from the category of diagonalizable algebraic groups to the category of finitely generated abelian groups [Mil17, 12.9].

More general, the functor $\mathsf{G} \rightsquigarrow \mathsf{X}^*(\mathsf{G})$ gives a contravariant equivalence from the category of algebraic groups of multiplicative type over K to the category of finitely generated $\mathbb{Z}$-modules equipped with a continuous action of the absolute Galois group of K [Mil17, 12.23].

In particular, an algebraic group of multiplicative type is a torus if and only if it is smooth and connected.

3.1.13. An algebraic group G is *trigonalizable* if its every finite-dimensional representation $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$ is *trigonalizable*, namely there exists a G-stable maximal flag

$$V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_{\dim V-1} \supsetneq V_{\dim V} = 0.$$

Equivalently, an algebraic group is trigonalizable if it is isomorphic to an algebraic subgroup of some T_n [Mil17, 16.2]. All unipotent algebraic groups are trigonalizable.

An algebraic group G is *solvable* if it has a subnormal series

$$\mathsf{G} \supsetneq \mathsf{G}_0 \supsetneq \mathsf{G}_1 \supsetneq \cdots \supsetneq \mathsf{G}_m = 1$$

such that each factor $\mathsf{G}_i/\mathsf{G}_{i+1}$ is commutative. A solvable algebraic group G is *split* if it has a subnormal series (G_i) in which each factor $\mathsf{G}_i/\mathsf{G}_{i+1}$ is isomorphic to either $\mathbb{G}_\mathrm{a}$ or $\mathbb{G}_\mathrm{m}$. Hence, split solvable algebraic groups are trigonalizable [Mil17, 16.52].

Any trigonalizable algebraic group G has a subnormal series (G_i) in which G_0 is unipotent, G/G_0 is diagonalizable and each factor $\mathsf{G}_i/\mathsf{G}_{i+1}$ is $(\mathsf{G}/\mathsf{G}_0)$-equivariantly embedded into $\mathbb{G}_a$ [Mil17, 16.21]. Therefore, trigonalizable algebraic groups are solvable. Conversely, every smooth connected solvable algebraic group becomes trigonalizable after some finite filed extension [Mil17, 16.30].

Example 3.1.14. T_n is trigonalizable and hence solvable. It has a normal series

$$\mathsf{T}_n \supsetneq \mathsf{U}_n = \mathsf{U}_n^{(0)} \supsetneq \mathsf{U}_n^{(1)} \supsetneq \cdots \supsetneq \mathsf{U}_n^{(m)} = 1,$$

where $m = \binom{n}{2}$ and for each $0 \leqslant r \leqslant m$,

$$\mathsf{U}_n^{(r)} : R \rightsquigarrow \left\{ (u_{ij}) \in \mathsf{U}_n(R) \,\middle|\, u_{ij} = 0 \text{ for } \tfrac{1}{2}(j-i-1)(2n-j+i)+i \leqslant r \right\}.$$

In which, U_n is the largest solvable normal subgroup of T_n (and is in fact smooth and connected), the quotient $\mathsf{U}_n/\mathsf{T}_n$ is isomorphic to D_n and each factor $\mathsf{U}_n^{(r)}/\mathsf{U}_n^{(r+1)}$ is isomorphic to $\mathbb{G}_a$.

3.1.15 (Cohomology of algebraic groups)**.** Definition 3.1.1 is equivalent to say that an algebraic group is a locally presentable group-valued sheaf on the site K_{fppf} whose underlying category is $\mathbf{Alg}_K^{\mathrm{op}}$ and is equipped with the *fppf topology**. Let R be an object in this site, its *fppf covering* is a family of K-algebra homomorphisms $R \to R_i$ of finite presentation such that $R \to \prod_i R_i$ is faithfully flat. Therefore, a functor F from $\mathbf{Alg}_K$ is a sheaf on K_{fppf} if and only if it satisfies the following [Mil17, 5.65].

(i). (*Local*) For any K-algebras $R_1, \cdots, R_m$,

$$\mathsf{F}(R_1 \times \cdots \times R_m) \cong \mathsf{F}(R_1) \times \cdots \times \mathsf{F}(R_m).$$

*The name *fppf* is short of “fidèlement plate de présentation finie”, that is, "faithfully flat and of finite presentation" in English. Note that any finitely generated K-algebra R is noetherian, hence all morphisms of finite type in the category $\mathbf{Alg}_K$ are actually of finite presentation. This is not true for general base S and the two sheaf conditions (*Local* and *Decent*) need to be presented in more general form as in [SGA3, IV, 6.3.1].

(ii). (*Descent*) For any faithfully flat K-algebra homomorphism $R' \to R$, the sequence

$$\mathsf{F}(R) \longrightarrow \mathsf{F}(R') \rightrightarrows \mathsf{F}(R' \otimes_R R')$$

is exact, where the homomorphisms $R' \to R' \otimes_R R'$ are $r \mapsto r \otimes 1$ and $r \mapsto 1 \otimes r$ respectively.

Saying a sequence of algebraic groups

$$1 \longrightarrow \mathsf{N} \longrightarrow \mathsf{G} \longrightarrow \mathsf{Q} \longrightarrow 1$$

is *exact* means N is isomorphic to the kernel of $\mathsf{G} \to \mathsf{Q}$ and $\mathsf{G} \to \mathsf{Q}$ is surjective as a sheaf homomorphism. The latter turns out to say that $\mathsf{G} \to \mathsf{Q}$ is faithfully flat [Mil17, 5.43]. When we consider homomorphisms between smooth algebraic groups, this is equivalent to say that $\mathsf{G} \to \mathsf{Q}$ is surjective on closed points [Mil17, 1.71]. Hence, to verify a homomorphism between smooth algebraic groups is surjective, it is sufficient to verify on K^{alg}-points.

Hence, in general we do not have a short exact sequence of the groups of K-points. Instead, there is a long exact sequence:

$$1 \longrightarrow \mathsf{N}(K) \longrightarrow \mathsf{G}(K) \longrightarrow \mathsf{Q}(K) \longrightarrow \mathrm{H}^1(K, \mathsf{N}) \longrightarrow \mathrm{H}^1(K, \mathsf{G}) \longrightarrow \mathrm{H}^1(K, \mathsf{Q}).$$

It turns out that [Mil17, 3.50], if G is a smooth algebraic group, then the group cohomology $\mathrm{H}^1(K, \mathsf{G})$ is canonically isomorphic to the *Galois cohomology* $\mathrm{H}^1(\Gamma, \mathsf{G}(K^{\mathrm{sep}}))$ with $\Gamma = \mathsf{Gal}(K^{\mathrm{sep}}/K)$. The following are some useful results in Galois cohomology.

(i). (*Hilbert's theorem 90*) If L/K is a Galois extension, then $\mathrm{H}^1(\mathsf{Gal}(L/K), L^\times) = 0$.

(ii). (*Lang's theorem* [Mil17, 17.98]) If G is a smooth connected algebraic group over a finite field K, then $\mathrm{H}^1(K, \mathsf{G}) = 0$.

(iii). ([Mil17, 25.61; BT-3, 4.3]) If G is a simply-connected semisimple group over a local field K, then $\mathrm{H}^1(K, \mathsf{G})$ vanishes.

3.1.16. An algebraic group is *vectorial* if it is isomorphic to a product of copies of $\mathbb{G}_{\mathrm{a}}$. Let V be a finite-dimensional vector space over K, then the algebraic group $\mathbb{W}(V)$ is a vectorial group. A vectorial group is in particular a vector bundle on K_{fppf}.

For V a vector bundle on K_{fppf}, let $\mathsf{V}^{\times}$ denote the open subscheme of V obtained by deleting the zero section. Then the action of $\mathbb{G}_{\mathrm{a}}$ on V induces an action of $\mathbb{G}_{\mathrm{m}}$ on $\mathsf{V}^{\times}$. In particular, if L is a one-dimensional vector space over K, then $\mathbb{W}(L)$ is a line bundle and $\mathbb{W}(L)^{\times}$ is a homogeneous principal $\mathbb{G}_{\mathrm{m}}$-bundle [SGA3, XIX, 4.3-4.4].

3.1.17. For R a K-algebra, its *algebra of dual numbers* is the algebra $R[\epsilon]/(\epsilon^2)$. Let $\mathscr{D}$ denote the functor sending each R to its algebra of dual numbers. For X a K-scheme, the composition $\mathsf{X} \circ \mathscr{D}$ is also a K-scheme, called the *tangent bundle* of X and is denoted by $\mathsf{T}(\mathsf{X})$. For any point x of X, the pullback of $\mathsf{T}(\mathsf{X})$ along $x \hookrightarrow \mathsf{X}$ is called the *tangent space* of X at x and is dented by $\mathsf{T}_x(\mathsf{X})$.

Let G be an algebraic group and e be its identity. Then both $\mathsf{T}(\mathsf{G})$ and $\mathsf{T}_e(\mathsf{G})$ are algebraic groups, and we have a split short exact sequence [SGA3, II, 3.9.0.2]

$$\begin{tikzcd} 1 \arrow[r] & \mathsf{T}_e(\mathsf{G}) \arrow[r, "i"] & \mathsf{T}(\mathsf{G}) \arrow[r, "\mathrm{pr}"] & \mathsf{G} \arrow[r] \arrow[l, bend left, dotted, "s"] & 1. \end{tikzcd}$$

Let $\varphi\colon \mathsf{G} \to \mathsf{G}'$ be a homomorphism of algebraic groups, then there is a unique morphism of K_{fppf}-vector bundles $\mathrm{d}\,\varphi\colon \mathsf{T}_e(\mathsf{G}) \to \mathsf{T}_e(\mathsf{G}')$ making the following diagram commute

$$\begin{tikzcd} 1 \arrow[r] & \mathsf{T}_e(\mathsf{G}) \arrow[r] \arrow[d, dotted, "\mathrm{d}\,\varphi"] & \mathsf{T}(\mathsf{G}) \arrow[r] \arrow[d, "\mathsf{T}(\varphi)"] & \mathsf{G} \arrow[r] \arrow[d, "\varphi"] & 1. \\ 1 \arrow[r] & \mathsf{T}_e(\mathsf{G}') \arrow[r] & \mathsf{T}(\mathsf{G}') \arrow[r] & \mathsf{G}' \arrow[r] & 1. \end{tikzcd}$$

The morphism $\mathrm{d}\,\varphi$ is called the *differential* of φ. We will not distinguish it from the K-linear map on K-points $\mathrm{d}\,\varphi(K)\colon\ \mathsf{T}_e(\mathsf{G},K) \to \mathsf{T}_e(\mathsf{G}',K)$.

Let $\mathfrak{g}$ denote the vector space $\mathsf{T}_e(\mathsf{G},K)$, hence $\mathsf{T}_e(\mathsf{G}) = \mathbb{W}(\mathfrak{g})$. Then the action of G on itself by conjugations induces a representation $\mathrm{Ad}\colon\ \mathsf{G} \to \mathsf{GL}(\mathfrak{g})$ of G on $\mathfrak{g}$: for any $g \in \mathsf{G}$, the endomorphism $\mathrm{Ad}(g)$ is the differential of $\mathrm{inn}(g)$ (conjugated by g). This representation is called the *adjoint representation* of G [SGA3, II, 4.1]. Let $\mathrm{ad}\colon\ \mathsf{T}_e(\mathsf{G}) \to \mathsf{End}(\mathfrak{g})$ denote the differential of the adjoint representation and for any $X, Y \in \mathfrak{g}$, define $[X,Y]$ as $\mathrm{ad}(X).Y$. Then this gives rise to a Lie bracket

$$\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}\colon \qquad X, Y \longmapsto [X,Y].$$

This Lie algebra is called the *Lie algebra* of G and is denoted by $\mathrm{Lie}(\mathsf{G})$.

Example 3.1.18. The Lie algebras of $\mathbb{G}_{\mathrm{m}}$ and $\mathbb{G}_{\mathrm{a}}$ are the trivial Lie algebra K. The Lie algebras of GL_n, SL_n, T_n, U_n and D_n are the Lie algebras $\mathfrak{gl}_n$ of all matrices, $\mathfrak{sl}_n$ of trace zero matrices, $\mathfrak{t}_n$ of all upper triangular matrices, $\mathfrak{u}_n$ of strict upper triangular matrices and $\mathfrak{d}_n$ of all diagonal matrices respectively.

3.1.19. The above constructions give rise to an equivalence of categories between vectorial groups and finite-dimensional vector spaces over K [Mil17, 10.9]. Moreover, when K is of characteristic zero and G is an unipotent group over it, there is an isomorphism of schemes (and of algebraic groups if G is further commutative)

$$\exp\colon\ \mathsf{T}_e(\mathsf{G}) = \mathbb{W}(\mathrm{Lie}(\mathsf{G})) \longrightarrow \mathsf{G},$$

called the *exponential map* [Mil17, 14.32].

§ 3.2. Reductive groups

Let's introduce the notion of reductive groups.

3.2.1. Let G be a smooth connected linear algebraic group.

(i). [Mil17, 6.44] There is a largest smooth connected solvable norm subgroup $\mathscr{R}(\mathsf{G})$ of G. It is called the *radical* of G.

(ii). [Mil17, 6.46] There is a largest smooth connected unipotent norm subgroup $\mathscr{R}_u(\mathsf{G})$ of G. It is called the *unipotent radical* of G.

Since unipotent groups are solvable, $\mathscr{R}_u(\mathsf{G})$ is a subgroup of $\mathscr{R}(\mathsf{G})$.

Definition 3.2.2. An algebraic group G is *reductive* (resp. *semisimple*) if its *geometric unipotent radical* $\mathscr{R}_u(\mathsf{G}_{K^{\mathrm{alg}}})$ (resp. *geometric radical* $\mathscr{R}(\mathsf{G}_{K^{\mathrm{alg}}})$) is trivial.

It turns out that the formations of $\mathscr{R}_u(\mathsf{G})$ and $\mathscr{R}(\mathsf{G})$ commute with separable field extensions [Mil17, 19.1 and 19.9]. Hence, when K is perfect, G is reductive (resp. semisimple) if and only if $\mathscr{R}_u(\mathsf{G})$ (resp. $\mathscr{R}(\mathsf{G})$) is trivial.

Example 3.2.3. For any finite-dimensional vector space V, $\mathsf{SL}(V)$ is semisimple, while $\mathsf{GL}(V)$ is reductive but not semisimple.

Since any torus becomes a product of copies of $\mathbb{G}_{\mathrm{m}} = \mathsf{GL}_1$ over a finite field extension, it is reductive. Conversely, if G is a solvable reductive group, then since $\mathscr{R}_u(\mathsf{G}_{K^{\mathrm{alg}}})$ is trivial, it is a torus by [Mil17, 16.33].

Example 3.2.4. In Example 3.1.7,

(i). $\mathsf{SL}_D(V)$, $\mathsf{Sp}(V)$, and $\mathsf{SO}(V) = \mathsf{O}(V)^\circ$ are semisimple;

(ii). $\mathsf{GL}_D(V)$, $\mathsf{GSp}(V)$, and $\mathsf{GO}(V)^\circ$ are reductive but not semisimple.

3.2.5. Let G be a reductive group. There are various semisimple groups related to it.

(i). The radical $\mathscr{R}(\mathsf{G})$ is a *central* torus, namely it is contained in the centre $\mathsf{Z}(\mathsf{G})$. Therefore, the quotient $\mathsf{G}/\mathsf{Z}(\mathsf{G})$ is semisimple. It is furthermore *adjoint*, namely it is semisimple with trivial centre, and is called the *adjoint group* of G with notation G^{ad}.

(ii). The radical $\mathscr{R}(\mathsf{G})$ turns out to be the largest subtorus of $\mathsf{Z}(\mathsf{G})$ and hence the formation of $\mathscr{R}(\mathsf{G})$ commute with field extensions [Mil17, 19.21]. Therefore, the quotient $\mathsf{G}^{\mathrm{ss}} := \mathsf{G}/\mathscr{R}(\mathsf{G})$ is semisimple.

(iii). The derived group $\mathsf{G}^{\mathrm{der}}$ is semisimple [Mil17, 19.21]. Indeed, its geometric radical $\mathscr{R}\left(\mathsf{G}^{\mathrm{der}}_{K^{\mathrm{alg}}}\right)$ is normal in $\mathsf{G}_{K^{\mathrm{alg}}}$ hence $\mathscr{R}\left(\mathsf{G}^{\mathrm{der}}_{K^{\mathrm{alg}}}\right) \subseteq \mathscr{R}(\mathsf{G}_{K^{\mathrm{alg}}})$ and is central. But $\mathsf{Z}(\mathsf{G}) \cap \mathsf{G}^{\mathrm{der}}$ is finite hence $\mathscr{R}\left(\mathsf{G}^{\mathrm{der}}_{K^{\mathrm{alg}}}\right)$ is trivial.

Example 3.2.6. The above semisimple groups associated to $\mathsf{G} = \mathsf{GL}_n$ are the following:

(i). $\mathsf{Z}(\mathsf{GL}_n) \cong \mathbb{G}_{\mathrm{m}}$, hence $\mathsf{G}^{\mathrm{ad}} = \mathsf{PGL}_n$;

(ii). $\mathscr{R}(\mathsf{GL}_n) = \mathsf{Z}(\mathsf{GL}_n)$, hence we obtain PGL_n again;

(iii). the derived group of GL_n is SL_n.

3.2.7. Let G be a reductive group with $\mathsf{Z}(\mathsf{G})$ its centre, G^{ad} its adjoint group, $\mathsf{G}^{\mathrm{der}}$ its derived group, G^{Ab} its abelianization and let $\mathsf{Z}\left(\mathsf{G}^{\mathrm{der}}\right)$ be the centre of $\mathsf{G}^{\mathrm{der}}$. We have the following *deconstruction* of G [Mil17, 19.25]:

$$\begin{tikzcd}
\mathsf{Z}\left(\mathsf{G}^{\mathrm{der}}\right) \arrow[r, hook] \arrow[d, hook] & \mathsf{G}^{\mathrm{der}} \arrow[d, hook] \arrow[dr, two heads, bend left] & \\
\mathsf{Z}(\mathsf{G}) \arrow[r, hook] \arrow[dr, two heads, bend right] & \mathsf{G} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{G}^{\mathrm{ad}} \\
& \mathsf{G}^{\mathrm{Ab}} &
\end{tikzcd}$$

where the square is bicartesian, namely $\mathsf{Z}\left(\mathsf{G}^{\mathrm{der}}\right) = \mathsf{Z}(\mathsf{G}) \cap \mathsf{G}^{\mathrm{der}}$ and $\mathsf{G} = \mathsf{Z}(\mathsf{G}) \cdot \mathsf{G}^{\mathrm{der}}$, and all rows and columns are exact sequences.

Conversely, suppose we have a triple $(\mathsf{H}, \mathsf{D}, \varphi)$ with H a semisimple algebraic group, D an algebraic group of multiplicative type, and $\varphi\colon \mathsf{Z}(\mathsf{H}) \to \mathsf{D}$ a monomorphism whose cokernel is a torus T. Then the homomorphism

$$\mathsf{Z}(\mathsf{H}) \longrightarrow \mathsf{H} \times \mathsf{D}\colon z \longmapsto (z, \varphi(z)^{-1})$$

is normal and its cokernel, denoted by G, is reductive and with the following deconstruction [Mil17, 19.27]

$$\begin{tikzcd}
\mathsf{Z}(\mathsf{H}) \arrow[r, hook] \arrow[d, hook] & \mathsf{H} \arrow[d, hook] \arrow[dr, two heads, bend left] & \\
\mathsf{D} \arrow[r, hook] \arrow[dr, two heads, bend right] & \mathsf{G} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{H}^{\mathrm{ad}} \\
& \mathsf{T} &
\end{tikzcd}$$

Namely, $\mathsf{Z}(\mathsf{G}) \cong \mathsf{D}, \mathsf{G}^{\mathrm{ad}} \cong \mathsf{H}^{\mathrm{ad}}, \mathsf{G}^{\mathrm{der}} \cong \mathsf{H}$ and $\mathsf{G}^{\mathrm{Ab}} \cong \mathsf{T}$.

More generally, one can start from a triple $(\mathsf{H}, \mathsf{D}, \varphi)$ with φ not necessarily injective. Then we can replace H by the $\mathsf{H}/\mathrm{Ker}(\varphi)$ and everything follows.

3.2.8. Let G be a reductive group with radical $\mathscr{R}(\mathsf{G})$, semisimple quotient G^{ss}, derived group $\mathsf{G}^{\mathrm{der}}$ and abelianization G^{Ab}. Then by [Mil17, 12.46], $\mathsf{G} = \mathscr{R}(\mathsf{G}) \cdot \mathsf{G}^{\mathrm{der}}$ and hence we have another deconstruction of G:

$$\begin{tikzcd}
\mathscr{R}(\mathsf{G}) \cap \mathsf{G}^{\mathrm{der}} \arrow[r, hook] \arrow[d, hook] & \mathsf{G}^{\mathrm{der}} \arrow[d, hook] \arrow[dr, bend left, two heads] & \\
\mathscr{R}(\mathsf{G}) \arrow[r, hook] \arrow[dr, bend right, two heads] & \mathsf{G} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{G}^{\mathrm{ss}} \\
& \mathsf{G}^{\mathrm{Ab}} &
\end{tikzcd}$$

In particular, a reductive group G is a product of a semisimple group and a torus if and only if $\mathscr{R}(\mathsf{G}) \cap \mathsf{G}^{\mathrm{der}} = 1$.

Example 3.2.9. Let $\mathsf{G} = \mathsf{GL}_n$. Then we have the following deconstruction

$$\begin{tikzcd}
\mu_n \arrow[r, hook] \arrow[d, hook] & \mathsf{SL}_n \arrow[d, hook] \arrow[dr, bend left, two heads] & \\
\mathbb{G}_{\mathrm{m}} \arrow[r, hook, "\lambda \mapsto \lambda I_n"] \arrow[dr, bend right, two heads, "\lambda \mapsto \lambda^n"'] & \mathsf{GL}_n \arrow[r, two heads] \arrow[d, two heads, "\det"] & \mathsf{PGL}_n \\
& \mathbb{G}_{\mathrm{m}} &
\end{tikzcd}$$

Conversely, GL_n can be recovered from the triple $(\mathsf{SL}_n, \mathbb{G}_{\mathrm{m}}, \mu_n \hookrightarrow \mathbb{G}_{\mathrm{m}})$.

Similar conclusion applies to $\mathsf{GL}(V)$.

3.2.10. Let G be a reductive group. It is *splittable* if it has a split maximal torus. A *split reductive group* is a pair (G, T) of a reductive group and a split maximal torus in it. A *homomorphism* between split reductive groups is a homomorphism of algebraic group preserving the split maximal torus. It turns out that, any two maximal split tori (hence split maximal tori if G is splittable) in G are conjugate by an element of G [Mil17, 25.10], while two (not necessarily split) maximal tori are only conjugate over a finite separable extension [Mil17, 17.87].

Let G be a splittable reductive group. Then its *rank* is the dimension of one (hence any) split maximal torus in it and its *semisimple rank* is the rank of $\mathsf{G}/\mathscr{R}(\mathsf{G})$. Since the centre $\mathsf{Z}(\mathsf{G})$ is contained in every maximal torus [Mil17, 17.61], the semisimple rank of G equals $\operatorname{rank}(\mathsf{G}) - \dim \mathsf{Z}(\mathsf{G})$.

Example 3.2.11. D_n is a split maximal torus in GL_n, and it induces a split maximal torus in PGL_n by quotienting out the scalars $\mathbb{G}_\mathrm{m}$ and a split maximal torus in SL_n by intersecting with it. Hence, GL_n is splittable with rank n and semisimple rank $n-1$.

Example 3.2.12. Notations are as in Example 3.1.8. A split maximal torus in Sp_{2n} is $\mathsf{D}_{2n} \cap \mathsf{Sp}_{2n}$, which consists of diagonal matrices $\mathbf{t} \in \mathsf{D}_{2n}$ such that $\chi_i(\mathbf{t})\chi_{-i}(\mathbf{t}) = 1$. Similarly, a split maximal torus in SO_{2n} is $\mathsf{D}_{2n} \cap \mathsf{SO}_{2n}$, which consists of diagonal matrices $\mathbf{t} \in \mathsf{D}_{2n}$ such that $\chi_i(\mathbf{t})\chi_{-i}(\mathbf{t}) = 1$. Finally, a split maximal torus in SO_{2n+1} is $\mathsf{D}_{2n+1} \cap \mathsf{SO}_{2n+1}$, which consists of diagonal matrices $\mathbf{t} \in \mathsf{D}_{2n+1}$ such that $\chi_i(\mathbf{t})\chi_{-i}(\mathbf{t}) = 1$ and that $\chi_0(\mathbf{t}) = 1$.

Example 3.2.13 ([Mil17, 17.89])**.** Let V be a vector space over K of dimension n. Then the conjugacy classes of maximal tori in $\mathsf{GL}(V)$ are one-one corresponding to the isomorphism classes of étale K-algebras of degree n: a maximal torus T gives a decomposition $V = \bigoplus_i V_i$ into simple T-modules and thus finite separable extensions $K_i = \operatorname{End}_{\mathsf{T}}(V_i)$ and an étale K-algebra $A = \prod_i K_i$ of degree n; conversely, as V is a free A-module of rank 1, it decomposes into vector spaces V_i, one-dimensional over K_i, and the A-equivariant automorphisms preserving this decomposition form a maximal torus T such that $\mathsf{T}(K) = A^\times$.

In particular, the only conjugacy class of split maximal tori in $\mathsf{GL}(V)$ corresponds to the étale algebra K^n.

3.2.14. A homomorphism between smooth connected algebraic groups is said to be an *isogeny* if it is surjective and has finite kernel. An *isogeny* of split reductive groups $(\mathsf{G}', \mathsf{T}') \to (\mathsf{G}, \mathsf{T})$ is a homomorphism of split reductive groups such that $\varphi\colon \mathsf{G}' \to \mathsf{G}$ is an isogeny.

An isogeny is *central* if its kernel is central, namely contained in the centre, and is *multiplicative* if its kernel is of multiplicative type. A multiplicative isogeny is central (since every normal multiplicative subgroup of a connected algebraic group is central [Mil17, 12.38]) and the converse is true if its domain is reductive (since the centre of a reductive group is of multiplicative type [Mil17, 17.62]).

Let G be a smooth connected algebraic group. A *universal covering* on it is a multiplicative isogeny $\widetilde{\mathsf{G}} \to \mathsf{G}$ universal in the sense that no other multiplicative isogeny can factor through it. When the universal covering exists, its kernel is called the *fundamental group* $\pi_1(\mathsf{G})$ of G. If this group is trivial, namely, every multiplicative isogeny to G is an isomorphism, then we say G *simply connected*.

Example 3.2.15. In 3.2.7 and 3.2.8, the homomorphisms $\mathsf{G}^{\mathrm{der}} \to \mathsf{G}^{\mathrm{ad}}$, $\mathsf{G}^{\mathrm{der}} \to \mathsf{G}^{\mathrm{ss}}$, $\mathsf{Z}(\mathsf{G}) \to \mathsf{G}^{\mathrm{Ab}}$ and $\mathscr{R}(\mathsf{G}) \to \mathsf{G}^{\mathrm{Ab}}$ are isogenies. In particular, the homomorphism $\mathsf{SL}_n \to \mathsf{PGL}_n$ in Example 3.2.9 is a universal covering (hence $\pi_1(\mathsf{PGL}_n) \cong \mu_n$) and it induces a central isogeny of split reductive groups $(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \to (\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_{\mathrm{m}})$.

§ 3.3. Root systems and root groups

Given a split reductive group, there is a root system associated to it.

3.3.1 ([BT-2, 1.1.2 and 1.1.3])**.** Let G be an affine algebraic group and T be a split torus in it. Since T is diagonalizable, it acts (via the adjoint representation) on $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$

diagonalizably, and we have a decomposition:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{a \in \mathsf{X}^*(\mathsf{T})} \mathfrak{g}_a, \tag{3.3.1}$$

where $\mathfrak{g}_0 = \mathfrak{g}^{\mathsf{T}}$ and $\mathfrak{g}_a$ is the subspace on which T acts through a nontrivial character a. A character a is a *root* if $\mathfrak{g}_a$ is nontrivial. The class of all positive real multiples of a root is called a *radical ray*. The set of all radical rays is denoted by $\Phi(\mathsf{G}, \mathsf{T})$, called the *root system* of the pair (G, T).

If (G, T) is a split reductive group, then $\mathfrak{g}_0 = \mathfrak{t} := \mathrm{Lie}(\mathsf{T})$ [Mil17, 10.34]. Moreover, any radical ray contains exactly one root and $\Phi(\mathsf{G}, \mathsf{T})$ can be further identified with a (reduced) root system, justifying its name.

Example 3.3.2. The pair $(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ is a split reductive group with Lie algebra the one dimensional vector space K. The adjoint action of $\mathbb{G}_{\mathrm{m}}$ on K is trivial, hence the root system of $(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ is empty.

Example 3.3.3. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. The action of D_n on $\mathfrak{gl}_n := \mathrm{Lie}(\mathsf{GL}_n)$ is

$$(\mathrm{diag}(t_1, \cdots, t_n), (g_{ij})_{i,j}) \longmapsto (t_i g_{ij} t_j^{-1})_{i,j}.$$

By Example 3.1.11, the characters of D_n are of the form $c_1\chi_1 + \cdots + c_n\chi_n$. If $(g_{ij})_{i,j}$ is an eigenvector of $c_1\chi_1 + \cdots + c_n\chi_n$, then for any $t_1, \cdots, t_n \in R$, we have

$$\forall i, j : t_i g_{ij} t_j^{-1} = (t_1^{c_1} \cdots t_n^{c_n}) g_{ij}.$$

Therefore: (i). the Lie algebra $\mathfrak{d}_n$ of D_n consists of all diagonal matrices; (ii). the root system $\Phi(\mathsf{GL}_n, \mathsf{D}_n) = \left\{\chi_i - \chi_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$; (iii). for each $a = \chi_i - \chi_j$, the Lie algebra $\mathfrak{g}_a$ is generated by E_{ij}, the matrix with 1 in the ij position and 0 elsewhere.

Example 3.3.4. Let's consider the split reductive group $(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m})$. A character $c_1\chi_1 + \cdots + c_n\chi_n$ of D_n factors through $\mathsf{D}_n/\mathbb{G}_\mathrm{m}$ if and only if $c_1 + \cdots + c_n = 0$. Hence,

$$\mathsf{X}^*(\mathsf{D}_n/\mathbb{G}_\mathrm{m}) = \{c_1\chi_1 + \cdots + c_n\chi_n \mid c_1, \cdots, c_n \in \mathbb{Z}, c_1 + \cdots + c_n = 0\}$$

and we see that $\Phi(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m}) = \left\{\chi_i - \chi_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$.

The Lie algebra of PGL_n and $\mathsf{D}_n/\mathbb{G}_\mathrm{m}$ are $\mathfrak{pgl}_n := \mathfrak{gl}_n/KI_n$ and $\mathfrak{d}_n/KI_n$. For each $a = \chi_i - \chi_j$, the Lie algebra $\mathfrak{g}_a$ is generated by E_{ij}.

Example 3.3.5. Let's consider the split reductive group $(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$. Then two characters $c_1\chi_1 + \cdots + c_n\chi_n$ and $c'_1\chi_1 + \cdots + c'_n\chi_n$ of D_n may give rise to the same character of $\mathsf{D}_n \cap \mathsf{SL}_n$. This is the case precisely when $c_i - c'_i$ is a constant. Hence,

$$\mathsf{X}^*(\mathsf{D}_n \cap \mathsf{SL}_n) = (\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n)/\mathbb{Z}(\chi_1 + \cdots + \chi_n)$$

and we see that $\Phi(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) = \left\{\overline{\chi_i} - \overline{\chi_j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$.

The Lie algebra of SL_n and $\mathsf{D}_n \cap \mathsf{SL}_n$ are $\mathfrak{sl}_n$, consisting of matrices with trace 0, and $\mathfrak{d}_n \cap \mathfrak{sl}_n$. For each $a = \overline{\chi_i} - \overline{\chi_j}$, the Lie algebra $\mathfrak{g}_a$ is generated by E_{ij}.

3.3.6. Let G be an affine algebraic group and T be a split torus in it. Then the normalizer $\mathsf{N} = \mathsf{N}_\mathsf{G}(\mathsf{T})$ acts on T, hence on $\mathsf{X}^*(\mathsf{T})$ by conjugations. The centralizer $\mathsf{Z}_\mathsf{G}(\mathsf{T})$ is the neutral component of N [Mil17, 12.40]. Therefore, $\pi_0(\mathsf{N})$ acts on $\mathsf{X}^*(\mathsf{T})$.

If (G, T) is a split reductive group. Then $\mathsf{Z}_\mathsf{G}(\mathsf{T}) = \mathsf{T}$ [Mil17, 17.84] and hence $\pi_0(\mathsf{N}) = \mathsf{N}/\mathsf{T}$, which is constant [Mil17, 21.1]. The finite group N/T is denoted by ${}^vW(\mathsf{G}, \mathsf{T})$ and is called the *Weyl group* of the pair (G, T).

Example 3.3.7. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. Then N consists of invertible monomial matrices and the regular representation of $\mathfrak{S}_n$ gives a semi-direct product $\mathsf{N} = \mathsf{D}_n \rtimes \underline{\mathfrak{S}_n}$. Hence, the Weyl group ${}^{v}W(\mathsf{GL}_n, \mathsf{D}_n)$ is isomorphic to $\mathfrak{S}_n$.

Similar arguments apply to $(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_{\mathrm{m}})$ and $(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$ and their Weyl groups are ${}^{v}W(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_{\mathrm{m}}) \cong \mathfrak{S}_n$ and ${}^{v}W(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \cong \mathfrak{S}_n$.

3.3.8. Let (G, T) be a split reductive group and $a \in \Phi(\mathsf{G}, \mathsf{T})$ a root of it. Then there is a unique homomorphism $u_a \colon \mathbb{W}(\mathfrak{g}_a) \to \mathsf{G}$ such that its differential $\mathrm{d}\, u_a$ is the inclusion $\mathfrak{g}_a \hookrightarrow \mathfrak{g}$. Let U_a denote the image of u_a. It is called the *root group* of G and satisfies the following properties [Mil17, 21.11 and 21.19; SGA3, XX, 1.5, XXII, 1.1].

(i). U_a has Lie algebra $\mathfrak{g}_a$ and a smooth subgroup of G contains U_a if and only if its Lie algebra contains $\mathfrak{g}_a$.

(ii). U_a is normalized by T and T acts on U_a through the character a:

$$\mathrm{inn}(\mathbf{t}).u_a(X) = u_a(a(\mathbf{t})X),$$

for all $\mathbf{t} \in \mathsf{T}$ and $X \in \mathbb{W}(\mathfrak{g}_a)$.

(iii). Let L_a be the algebraic subgroup of G generated by U_a, U_{-a} and T, called the *Levi subgroup* associated to a. Then the morphism

$$\mathbb{W}(\mathfrak{g}_{-a}) \times \mathsf{T} \times \mathbb{W}(\mathfrak{g}_a) \longrightarrow \mathsf{L}_a$$

defined by $(Y, \mathbf{t}, X) \mapsto u_{-a}(Y) \cdot \mathbf{t} \cdot u_a(X)$ is an open immersion.

Moreover, if $m \in \mathsf{N}_{\mathsf{G}}(\mathsf{T})$. Then $b = a \circ \mathrm{inn}(m)$ is a root, and we have the following commutative diagram [SGA3, XXII, 1.4].

$$\begin{tikzcd} \mathbb{W}(\mathfrak{g}_a) \arrow[r, "u_a"] \arrow[d, "\mathrm{Ad}(m)"'] & \mathsf{G} \arrow[d, "\mathrm{inn}(m)"] \\ \mathbb{W}(\mathfrak{g}_b) \arrow[r, "u_b"] & \mathsf{G} \end{tikzcd}$$

Indeed, both u_a and $\mathrm{inn}(m)^{-1} \circ u_b \circ \mathrm{Ad}(m)$ have the differential $\mathfrak{g}_a \hookrightarrow \mathfrak{g}$.

Remark ([BT-2, 1.1.3 and 1.1.9]). In general, let G be an affine algebraic group and T be a split torus in it. Then a *root subgroup* associated to a *radical ray* $a \in \Phi(\mathsf{G}, \mathsf{T})$ is the largest connected closed subgroup U_a normalized by T and all characters appearing in the adjoint representation on $\mathrm{Lie}(\mathsf{U}_a)$ belong to a. The notion of Levi subgroups still make sense, and we have open immersion

$$\mathsf{U}_a \times \mathsf{T} \times \mathsf{U}_{-a} \longrightarrow \mathsf{L}_a\,.$$

However, U_a is merely split unipotent in general, not necessary vectorial.

Example 3.3.9. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ and its root $a = \chi_i - \chi_j$. Then U_a is the algebraic group

$$R \rightsquigarrow I_n + RE_{ij}.$$

The homomorphism $u_a \colon \mathbb{W}(\mathfrak{g}_a) \to \mathsf{GL}_n$ is

$$xE_{ij} \mapsto I_n + xE_{ij}.$$

For any $\mathbf{t} = \mathrm{diag}(t_1, \cdots, t_n) \in \mathsf{T}$, we have

$$\mathrm{inn}(\mathbf{t}).u_a(xE_{ij}) = I_n + t_i x t_j^{-1} E_{ij} = u_a(t_i t_j^{-1} x E_{ij}) = u_a(a(\mathbf{t}) x E_{ij}).$$

3.3.10. Notations as in 3.3.8. Then there is a natural duality on the one-dimensional vectorial groups

$$\mathbb{W}(\mathfrak{g}_a) \times \mathbb{W}(\mathfrak{g}_{-a}) \longrightarrow \mathbb{G}_{\mathrm{a}} : \qquad (X, Y) \longmapsto \langle X, Y \rangle$$

and a unique cocharacter $a^\vee : \mathbb{G}_{\mathrm{m}} \to \mathsf{T}$ such that [SGA3, XX, 2.1]:

(i). for any $X \in \mathbb{W}(\mathfrak{g}_a)$ and $Y \in \mathbb{W}(\mathfrak{g}_{-a})$, the product $u_a(X) \cdot u_{-a}(Y)$ lies in the image of the open immersion in 3.3.8.(iii) if and only if $1 + \langle X, Y \rangle \in \mathbb{G}_{\mathrm{m}}$;

(ii). under these conditions we have the formula

$$u_a(X) \cdot u_{-a}(Y) = u_{-a}((1 + \langle X, Y \rangle)^{-1} Y) \cdot a^\vee(1 + \langle X, Y \rangle) \cdot u_a((1 + \langle X, Y \rangle)^{-1} X);$$

(iii). $\langle a, a^\vee \rangle = 2$.

The above duality induces a pairing of $\mathbb{G}_{\mathrm{m}}$-bundles [SGA3, XX, 2.6]:

$$\mathbb{W}(\mathfrak{g}_a)^\times \times \mathbb{W}(\mathfrak{g}_{-a})^\times \longrightarrow \mathbb{G}_{\mathrm{m}} : \qquad (X, Y) \longmapsto XY.$$

Then for any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, there is a unique $X^{-1} \in \mathbb{W}(\mathfrak{g}_{-a})^\times$ such that $XX^{-1} = 1$. This gives rise to an isomorphism $(-)^{-1}$ compatible with the action of $\mathbb{G}_{\mathrm{m}}$. Then for any $x \in \mathbb{G}_{\mathrm{m}}$ and $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, we have [SGA3, XX, 2.7]:

$$a^\vee(x) = u_{-a}((x^{-1} - 1) X^{-1}) u_a(X) u_{-a}((x - 1) X^{-1}) u_a(-x^{-1} X).$$

This cocharacter is called the *coroot* associated to the *root* a.

The root a and its coroot $a^\vee$ induces the following Lie algebra homomorphisms

$$K \xrightarrow{\mathrm{d}\, a^\vee} \mathfrak{t} \xrightarrow{\mathrm{d}\, a} K.$$

The vector $H_a := \mathrm{d}\, a^\vee(1)$ is called the *infinitesimal coroot vector*. Then $H_{-a} = -H_a$ and for any $X \in \mathbb{W}(\mathfrak{g}_a)$, $Y \in \mathbb{W}(\mathfrak{g}_{-a})$ and $H \in \mathbb{W}(\mathfrak{t})$, we have [SGA3, XX, 2.10]:

$$[H,X] = \mathrm{d}\, a(H)X, \qquad [H,Y] = -\mathrm{d}\, a(H)Y, \qquad [X,Y] = \langle X, Y\rangle H_a.$$

Hence, if $H_a \neq 0$, then for any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, the following define an embedding from the Lie algebra $\mathfrak{sl}_2$:

$$E_{12} \mapsto X, \qquad E_{21} \mapsto X^{-1}, \qquad E_{11} - E_{22} \mapsto H_a.$$

Remark. For each $a \in \Phi$, one can fix such an embedding and hence fix a choice of basis of $\mathfrak{g}_a$ (as well as $\mathfrak{g}_{-a}$). However, such choices for various roots are not independent.

Definition 3.3.11. A *pinning* on the pair (G, T) is a basis Δ of Φ together with a family of isomorphisms $(u_a \colon \mathbb{G}_\mathrm{a} \to \mathsf{U}_a)_{a\in\Delta}$. Given a pinning, the family $(u_a)_{a\in\Delta}$ extends uniquely to a coherent system of isomorphisms $(u_a)_{a\in\Phi}$, called a *Chevalley system*. We refer to [BT-2, 3.2.2] or [SGA3, XXIII, 6.2] for more details. Note that when such a system is given, we ambiguously use the notation u_a to denote either the inclusion $\mathbb{W}(\mathfrak{g}_a) \to \mathsf{G}$ or the composition $\mathbb{G}_\mathrm{a} \cong \mathbb{W}(\mathfrak{g}_a) \to \mathsf{G}$.

Example 3.3.12. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ and its root $a = \chi_i - \chi_j$. To take the advantage of calculations on GL_2, we can define a homomorphism ξ_{ij} mapping a 2×2-matrix $M = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathsf{GL}_2(R)$ to the $n \times n$-matrix $\xi_{ij}(M)$ satisfying

$$\xi_{ij}(M).e_k = \begin{cases} xe_i + ze_j & \text{if } k = i, \\ ye_i + we_j & \text{if } k = j, \\ e_k & \text{otherwise,} \end{cases}$$

where $e_1, e_2, \cdots, e_n$ is the standard basis of R^n. Then we have

$$u_a(x) = \xi_{ij}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \qquad \text{and} \qquad xE_{ij} = \mathrm{d}\,\xi_{ij}\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right).$$

Also note that $\xi_{ji} = \xi_{ij} \circ$ transpose.

Then the duality is

$$\langle xE_{ij}, yE_{ji}\rangle = xy.$$

The coroot $a^\vee$ associated to a is $\lambda_i - \lambda_j$ and one can verify that

$$\begin{aligned}
&u_a(x) \cdot u_{-a}(y) \\
=&\xi_{ij}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\right) \\
=&\xi_{ij}\left(\begin{pmatrix} 1+xy & x \\ y & 1 \end{pmatrix}\right) \\
=&\xi_{ij}\left(\begin{pmatrix} 1 & 0 \\ (1+xy)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1+xy & 0 \\ 0 & (1+xy)^{-1} \end{pmatrix}\begin{pmatrix} 1 & (1+xy)^{-1} \\ 0 & 1 \end{pmatrix}\right) \\
=&u_{-a}((1+xy)^{-1}) \cdot a^\vee(1+xy) \cdot u_a((1+xy)^{-1}).
\end{aligned}$$

The differentials of a and $a^\vee$ are

$$\begin{aligned}
\mathrm{d}\,a = \mathrm{d}\,\chi_i - \mathrm{d}\,\chi_j &\colon\ \mathrm{diag}(t_1, \cdots, t_n) \mapsto t_i - t_j, \\
\mathrm{d}\,a^\vee = \mathrm{d}\,\lambda_i - \mathrm{d}\,\lambda_j &\colon z \mapsto \xi_{ij}\left(\begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}\right).
\end{aligned}$$

In particular, the infinitesimal coroot vector associated to a is

$$H_a = \xi_{ij}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right).$$

For any $H = \mathrm{diag}(t_1, \cdots, t_n) \in \mathfrak{t}$, we have

$$[H, xE_{ij}] = \xi_{ij}\left(\begin{pmatrix} 0 & (t_i x - t_j x) \\ 0 & 0 \end{pmatrix}\right) = \mathrm{d}\, a(H) x E_{ij},$$

$$[H, yE_{ji}] = \xi_{ij}\left(\begin{pmatrix} 0 & 0 \\ (t_j y - t_i y) & 0 \end{pmatrix}\right) = -\mathrm{d}\, a(H) y E_{ji},$$

$$[xE_{ij}, yE_{ji}] = \xi_{ij}\left(\begin{pmatrix} xy & 0 \\ 0 & -xy \end{pmatrix}\right) = xyH_a.$$

3.3.13. Notations as in 3.3.8 and 3.3.10. Then L_a is the centralizer of the largest subtorus of $\mathrm{Ker}(a)$ and the pair $(\mathsf{L}_a, \mathsf{T})$ is a split reductive group [SGA3, XIX, 1.12 and XXII, 1.1; Mil17, 21.11 and 21.23]. The Lie algebra of it admits a decomposition

$$\mathrm{Lie}(\mathsf{L}_a) = \mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$$

and the Weyl group ${}^{v}W(\mathsf{L}_a, \mathsf{T})$ contains exactly one nontrivial element r_a given by the formula

$$r_a \colon \chi \longmapsto \chi - \langle \chi, a^\vee \rangle a.$$

Moreover, for any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, let

$$m_a(X) = u_a(X) \cdot u_{-a}(-X^{-1}) \cdot u_a(X).$$

Then we have [SGA3, XX, 3.1; Mil17, 20.39]:

(i). $m_a(X) \in \mathsf{N}_{\mathsf{L}_a}(\mathsf{T})$;

(ii). let M_a° be the image of $m_a\colon \mathbb{W}(\mathfrak{g}_a)^\times \to \mathsf{N}_{\mathsf{L}_a}(\mathsf{T})$, then $\mathsf{M}_a = \mathsf{T}\cdot\mathsf{M}_a^\circ$ is a right congruence class modulo T: indeed, for any $z \in \mathbb{G}_\mathrm{m}$ and $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, we have

$$m_a(zX) = a^\vee(z)m_a(X);$$

(iii). this right congruence class is precisely r_a: indeed, for any $t \in \mathsf{T}$ and $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, we have

$$\mathrm{inn}(m_a(X)).t = t \cdot a^\vee(a(t))^{-1};$$

(iv). for any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$ and $Y \in \mathbb{W}(\mathfrak{g}_{-a})^\times$, we have

$$m_a(X)m_{-a}(Y) = a^\vee(XY).$$

Example 3.3.14. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ and its root $a = \chi_i - \chi_j$. Then the algebraic subgroup L_a is

$$R \rightsquigarrow \mathsf{D}_n(R) + RE_{ij} + RE_{ji}.$$

Its Lie algebra is $\mathfrak{d}_n + KE_{ij} + KE_{ji}$, which is precisely $\mathfrak{d}_n \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$.

The normalizer of D_n in L_a is precisely the monomial matrices belonging to L_a. Hence, the Weyl group ${}^{\upsilon}W(\mathsf{L}_a, \mathsf{T})$ contains exactly one nontrivial element $r_a = (i, j)$, the permutation of i-th and j-th coordinates.

The map m_a is

$$xE_{ij} \longmapsto \xi_{ij}\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -x^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = \xi_{ij}\left(\begin{pmatrix}0 & x\\ -x^{-1} & 0\end{pmatrix}\right).$$

We have

$$\begin{aligned}
m_a(x) &= \xi_{ij}\left(\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}\right) \\
&= \xi_{ij}\left(\begin{pmatrix} x & 0 \\ 0 & -x^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = a^\vee(x)m_a(1).
\end{aligned}$$

The action of $m_a(x)$ on $t = \mathrm{diag}(t_1, \cdots, t_n) \in \mathsf{T}$ is

$$\begin{aligned}
\mathrm{inn}(m_a(x)).t &= \mathrm{inn}\left(\xi_{ij}\left(\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}\right)\right). \mathrm{diag}(t_1, \cdots, t_n) \\
&= \mathrm{inn}\big(a^\vee(x)\big). \mathrm{inn}\left(\xi_{ij}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\right). \mathrm{diag}(t_1, \cdots, t_n) \\
&= \mathrm{inn}\big(a^\vee(x)\big). \mathrm{diag}\big(t_{\sigma(1)}, \cdots, t_{\sigma(1)}\big) \qquad \text{with } \sigma = (i, j) \\
&= \mathrm{diag}\big(t_{\sigma(1)}, \cdots, t_{\sigma(1)}\big) \qquad \text{with } \sigma = (i, j) \\
&= \mathrm{diag}(t_1, \cdots, t_n)\xi_{ij}\left(\begin{pmatrix} t_i^{-1}t_j & 0 \\ 0 & t_it_j^{-1} \end{pmatrix}\right) \\
&= \mathrm{diag}(t_1, \cdots, t_n)a^\vee(a(t))^{-1}.
\end{aligned}$$

We also have

$$\begin{aligned}
m_a(x) \cdot m_{-a}(y) &= \xi_{ij}\left(\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & -y^{-1} \\ y & 0 \end{pmatrix}\right) \\
&= \xi_{ij}\left(\begin{pmatrix} xy & 0 \\ 0 & (xy)^{-1} \end{pmatrix}\right) = a^\vee(xy).
\end{aligned}$$

§ 3.4. Root data

Definition 3.4.1. Let (G, T) be a split reductive group. Then there is a *root datum* $\mathcal{R}(\mathsf{G}, \mathsf{T}) = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ associated to it [SGA3, XXII, 1.14; Mil17, 21.c], where

- the $\mathbb{Z}$-module X is the character group $\mathsf{X}^*(\mathsf{T})$;
- the root system Φ is the root system $\Phi(\mathsf{G}, \mathsf{T})$;
- the dual $\mathbb{Z}$-module $\mathsf{X}^\vee$ is the cocharacter group $\mathsf{X}_*(\mathsf{T})$;
- the coroot system $\Phi^\vee$ is the set $\Phi^\vee(\mathsf{G}, \mathsf{T})$ of coroots $a^\vee$ associated to the roots $a \in \Phi(\mathsf{G}, \mathsf{T})$.

Let $\mathbb{V}$ denote the subspace of $\mathsf{X}_*(\mathsf{T}) \otimes \mathbb{R}$ spanned by $\Phi^\vee(\mathsf{G}, \mathsf{T})$ equipped with a ${}^vW(\mathsf{G}, \mathsf{T})$-invariant inner product, called the *coroot space*. Then we get a spherical apartment ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$ with underlying Euclidean vector space $\mathbb{V}$ on which the Weyl group ${}^vW(\mathsf{G}, \mathsf{T})$ acts as its reflection group.

3.4.2. The *rank* of a root datum $\mathcal{R} = (\mathsf{X}, \Phi, \mathsf{X}^\vee, \Phi^\vee)$ is the rank of the $\mathbb{Z}$-module X and its *semisimple rank* is the dimension of the coroot space $\mathbb{V}$. Let (G, T) be a split reductive group. Then the rank (resp. semisimple rank) of the root datum $\mathcal{R}(\mathsf{G}, \mathsf{T})$ is the rank (resp. semisimple rank) of G.

Example 3.4.3. Let's consider the split reductive group $(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$. Then by Example 3.3.2, the root datum $\mathcal{R}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ is $(\mathbb{Z}, \emptyset, \mathbb{Z}, \emptyset)$.

Example 3.4.4. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. Then by Examples 3.3.3, 3.3.9, and 3.3.12, the root datum $\mathcal{R}(\mathsf{GL}_n, \mathsf{D}_n)$ is

- $\mathsf{X} = \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n$;

- $\Phi = \left\{\chi_i - \chi_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$;
- $\mathsf{X}^\vee = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n$;
- $\Phi^\vee = \left\{\lambda_i - \lambda_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$.

The coroot space is

$$\mathbb{V} = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{R}, c_1 + \cdots + c_n = 0\}.$$

Example 3.4.5. Let's consider the split reductive group $(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m})$. Two cocharacters $c_1\lambda_1 + \cdots + c_n\lambda_n$ and $c_1'\lambda_1 + \cdots + c_n'\lambda_n$ of D_n give rise to the same cocharacter of $\mathsf{D}_n/\mathbb{G}_\mathrm{m}$ precisely when $c_i - c_i'$ is a constant. Hence,

$$\mathsf{X}_*(\mathsf{D}_n/\mathbb{G}_\mathrm{m}) = (\mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n)/\mathbb{Z}(\lambda_1 + \cdots + \lambda_n)$$

and we see that $\Phi^\vee(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m}) = \left\{\overline{\lambda_i} - \overline{\lambda_j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$ with coroot space

$$\mathbb{V} = \left\{c_1\overline{\lambda_1} + \cdots + c_n\overline{\lambda_n} \,\middle|\, c_1, \cdots, c_n \in \mathbb{R}, c_1 + \cdots + c_n = 0\right\}.$$

Then, by Example 3.3.4, the root datum $\mathcal{R}(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m})$ is

- $\mathsf{X} = \{c_1\chi_1 + \cdots + c_n\chi_n \mid c_1, \cdots, c_n \in \mathbb{Z}, c_1 + \cdots + c_n = 0\}$;
- $\Phi = \left\{\chi_i - \chi_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$;
- $\mathsf{X}^\vee = (\mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n)/\mathbb{Z}(\lambda_1 + \cdots + \lambda_n)$;
- $\Phi^\vee = \left\{\overline{\lambda_i} - \overline{\lambda_j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$.

Example 3.4.6. Let's consider the split reductive group $(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$. A cocharacter $c_1\lambda_1 + \cdots + c_n\lambda_n$ of D_n factors through $\mathsf{D}_n \cap \mathsf{SL}_n$ if and only if $c_1 + \cdots + c_n = 0$. Hence,

$$\mathsf{X}_*(\mathsf{D}_n \cap \mathsf{SL}_n) = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{Z}, c_1 + \cdots + c_n = 0\}$$

and we see that $\Phi^\vee(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) = \left\{\lambda_i - \lambda_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$ with coroot space

$$\mathbb{V} = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{R}, c_1 + \cdots + c_n = 0\}.$$

Then, by Example 3.3.5, the root datum $\mathcal{R}(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$ is

- $\mathsf{X} = (\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n)/\mathbb{Z}(\chi_1 + \cdots + \chi_n)$;
- $\Phi = \left\{\overline{\chi_i} - \overline{\chi_j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$;
- $\mathsf{X}^\vee = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{Z}, c_1 + \cdots + c_n = 0\}$;
- $\Phi^\vee = \left\{\lambda_i - \lambda_j \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$.

Example 3.4.7. Notations are as in Example 3.1.8.(i). Let's consider the split reductive group $(\mathsf{Sp}_{2n}, \mathsf{D}_{2n} \cap \mathsf{Sp}_{2n})$. Note that characters χ_i and $-\chi_{-i}$ of D_{2n} restrict to the same character on $\mathsf{D}_{2n} \cap \mathsf{Sp}_{2n}$. We use $\chi_{\pm i}$ to denote this character. On the other hand, the cocharacter $\lambda_i - \lambda_{-i}$ of D_{2n} lands in $\mathsf{D}_{2n} \cap \mathsf{Sp}_{2n}$. We use $\lambda_{\pm i}$ to denote this cocharacter.

Then the root datum $\mathcal{R}(\mathsf{Sp}_{2n}, \mathsf{D}_{2n} \cap \mathsf{Sp}_{2n})$ is

- $\mathsf{X} = \mathbb{Z}\chi_{\pm 1} \oplus \cdots \oplus \mathbb{Z}\chi_{\pm n}$;
- $\Phi = \left\{\pm\chi_{\pm i} \pm \chi_{\pm j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\} \cup \{\pm 2\chi_{\pm i} \mid 1 \leqslant i \leqslant n\}$;
- $\mathsf{X}^\vee = \mathbb{Z}\lambda_{\pm 1} \oplus \cdots \oplus \mathbb{Z}\lambda_{\pm n}$;
- $\Phi^\vee = \left\{\pm\lambda_{\pm i} \pm \lambda_{\pm j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\} \cup \{\pm\lambda_{\pm i} \mid 1 \leqslant i \leqslant n\}$;

Example 3.4.8. Notations are as in Example 3.1.8.(ii). Let's consider the split reductive group $(\mathsf{SO}_m, \mathsf{D}_m \cap \mathsf{SO}_m)$ (where $m = 2n$ or $2n + 1$). Note that characters χ_i and $-\chi_{-i}$ of D_m restrict to the same character on $\mathsf{D}_m \cap \mathsf{SO}_m$. We use $\chi_{\pm i}$ to denote this character. On the other hand, the cocharacter $\lambda_i - \lambda_{-i}$ of D_m lands in $\mathsf{D}_m \cap \mathsf{SO}_m$. We use $\lambda_{\pm i}$ to denote this cocharacter.

Then the root datum $\mathcal{R}(\mathsf{SO}_{2n}, \mathsf{D}_{2n} \cap \mathsf{SO}_{2n})$ is

- $\mathsf{X} = \mathbb{Z}\chi_{\pm 1} \oplus \cdots \oplus \mathbb{Z}\chi_{\pm n}$;
- $\Phi = \left\{\pm\chi_{\pm i} \pm \chi_{\pm j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$;
- $\mathsf{X}^\vee = \mathbb{Z}\lambda_{\pm 1} \oplus \cdots \oplus \mathbb{Z}\lambda_{\pm n}$;
- $\Phi^\vee = \left\{\pm\lambda_{\pm i} \pm \lambda_{\pm j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$;

While the root datum $\mathcal{R}(\mathsf{SO}_{2n+1}, \mathsf{D}_{2n+1} \cap \mathsf{SO}_{2n+1})$ is

- $\mathsf{X} = \mathbb{Z}\chi_{\pm 1} \oplus \cdots \oplus \mathbb{Z}\chi_{\pm n}$;
- $\Phi = \left\{\pm\chi_{\pm i} \pm \chi_{\pm j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\} \cup \{\pm\chi_{\pm i} \mid 1 \leqslant i \leqslant n\}$;
- $\mathsf{X}^\vee = \mathbb{Z}\lambda_{\pm 1} \oplus \cdots \oplus \mathbb{Z}\lambda_{\pm n}$;
- $\Phi^\vee = \left\{\pm\lambda_{\pm i} \pm \lambda_{\pm j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\} \cup \{\pm 2\lambda_{\pm i} \mid 1 \leqslant i \leqslant n\}$;

3.4.9. Let $\varphi\colon (\mathsf{G}, \mathsf{T}) \to (\mathsf{G}', \mathsf{T}')$ be a homomorphism between split reductive groups. Then it induces a linear map $f = \varphi^*\colon\ \mathsf{X}^*(\mathsf{T}') \to \mathsf{X}^*(\mathsf{T})$. Then f is a morphism of root data if and only if there is a bijection $u\colon\ \Phi(\mathsf{G}, \mathsf{T}) \to \Phi(\mathsf{G}', \mathsf{T}')$ such that

$$f(u(a)) = a, \qquad {}^{\mathsf{T}}\!f(a^\vee) = u(a)^\vee.$$

A homomorphism of split reductive groups $\varphi\colon (\mathsf{G}, \mathsf{T}) \to (\mathsf{G}', \mathsf{T}')$ induces an isogeny of root data $\varphi^*\colon\ \mathcal{R}(\mathsf{G}', \mathsf{T}') \to \mathcal{R}(\mathsf{G}, \mathsf{T})$ if and only if it is a central isogeny. Moreover, all isogenies of root data arise in this way [SGA3, XXII, 4.2.11; Mil17, 23.25].

Example 3.4.10. Let's consider the inclusion

$$\iota\colon (\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \longhookrightarrow (\mathsf{GL}_n, \mathsf{D}_n).$$

Then the linear map $f = \iota^*\colon\ \mathsf{X}^*(\mathsf{GL}_n, \mathsf{D}_n) \to \mathsf{X}^*(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ is the linear map

$$\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n \longrightarrow \mathbb{Z}$$

mapping χ_i to 1. Then this is not a morphism of root data since it does not induce a bijection on roots.

Example 3.4.11. Let's consider the determinant homomorphism

$$\det\colon (\mathsf{GL}_n, \mathsf{D}_n) \longrightarrow (\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m}).$$

Then the linear map $f = \det^*\colon\ \mathsf{X}^*(\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m}) \to \mathsf{X}^*(\mathsf{GL}_n, \mathsf{D}_n)$ is the linear map

$$\mathbb{Z} \longrightarrow \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n$$

mapping 1 to $\chi_1 + \cdots + \chi_n$. Then this is not a morphism of root data since it does not induce a bijection on roots.

Example 3.4.12. Let's consider the isogeny

$$\varphi\colon (\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m}) \longrightarrow (\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m})\colon t \mapsto t^n.$$

Then the linear map $f = \varphi^*\colon\ \mathsf{X}^*(\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m}) \to \mathsf{X}^*(\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m})$ is the linear map

$$\mathbb{Z} \longrightarrow \mathbb{Z}\colon 1 \longmapsto n.$$

It is an isogeny of root data with finite cokernel μ_n.

Example 3.4.13. Let's consider the inclusion

$$\iota\colon (\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \hookrightarrow (\mathsf{GL}_n, \mathsf{D}_n).$$

Then the linear map $f = \iota^*\colon\ \mathsf{X}^*(\mathsf{GL}_n, \mathsf{D}_n) \to \mathsf{X}^*(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$ is the projection

$$\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n \twoheadrightarrow (\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n)/\mathbb{Z}(\chi_1 + \cdots + \chi_n).$$

This is a morphism of root data but not an isogeny since f is not injective.

Example 3.4.14. Let's consider the quotient map

$$\pi\colon (\mathsf{GL}_n, \mathsf{D}_n) \twoheadrightarrow (\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m}).$$

Then the linear map $f = \pi^*\colon \mathsf{X}^*(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m}) \to \mathsf{X}^*(\mathsf{GL}_n, \mathsf{D}_n)$ is the inclusion

$$\{c_1 + \cdots + c_n = 0\} \cap \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n \hookrightarrow \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n.$$

This is a morphism of root data but not an isogeny since f has infinite cokernel.

Example 3.4.15. Let's consider the composition

$$\varphi\colon\ (\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \xhookrightarrow{\iota} (\mathsf{GL}_n, \mathsf{D}_n) \xrightarrow{\pi} (\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m})$$

of previous two homomorphisms. Then the linear map

$$f = \varphi^*\colon\ \mathsf{X}^*(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_\mathrm{m}) \longrightarrow \mathsf{X}^*(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$$

is the following restriction of the projection π^*

$$\{c_1 + \cdots + c_n = 0\} \cap \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n \longrightarrow (\mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_n)/\mathbb{Z}(\chi_1 + \cdots + \chi_n)$$

It turns out that this is an isogeny of root data with finite cokernel $\mu_n\overline{\chi_1}$.

3.4.16. Let (G, T) be a split reductive group. Applying [Mil17, 17.86] to 3.2.7 and 3.2.8, we have the following commutative diagram of split reductive groups, where the horizontal arrows are isogenies, the diagonals are short exact sequences and

$$\begin{tikzcd}
\mathrm{sc}(\mathsf{G},\mathsf{T}) \arrow[r] & \mathrm{der}(\mathsf{G},\mathsf{T}) \arrow[r] \arrow[dr] & & \mathrm{ss}(\mathsf{G},\mathsf{T}) \arrow[r] & \mathrm{ad}(\mathsf{G},\mathsf{T}) \\
& & (\mathsf{G},\mathsf{T}) \arrow[ur] \arrow[dr] & & \\
& \mathrm{rad}(\mathsf{G},\mathsf{T}) \arrow[ur] \arrow[rr] & & \mathrm{corad}(\mathsf{G},\mathsf{T}) &
\end{tikzcd}$$

Figure 3.1. Split reductive groups with isomorphic root systems.

$\mathrm{ad}(\mathsf{G},\mathsf{T})$	the adjoint group G^{ad} and the image of T in it $(\mathsf{T}/\mathsf{Z}(\mathsf{G}))$;
$\mathrm{ss}(\mathsf{G},\mathsf{T})$	the semisimple quotient G^{ss} and the image of T in it $(\mathsf{T}/\mathscr{R}(\mathsf{G}))$;
$\mathrm{der}(\mathsf{G},\mathsf{T})$	the derived group $\mathsf{G}^{\mathrm{der}}$ and the preimage of T in it $(\mathsf{T}\cap\mathsf{G}^{\mathrm{der}})$;
$\mathrm{sc}(\mathsf{G},\mathsf{T})$	the universal covering of all above;
$\mathrm{rad}(\mathsf{G},\mathsf{T})$	the radical $\mathscr{R}(\mathsf{G})$ and the trivial torus 1;
$\mathrm{corad}(\mathsf{G},\mathsf{T})$	the abelianization G^{Ab} and the trivial torus 1.

Moreover, we have [SGA3, XXII, 4.3.7, 6.2.1 and 6.2.3; Mil17, 23.a]

(i). $\mathcal{R}(\mathrm{ad}(\mathsf{G},\mathsf{T})) = \mathrm{ad}(\mathcal{R}(\mathsf{G},\mathsf{T}))$;

(ii). $\mathcal{R}(\mathrm{ss}(\mathsf{G},\mathsf{T})) = \mathrm{ss}(\mathcal{R}(\mathsf{G},\mathsf{T}))$;

(iii). $\mathcal{R}(\mathrm{der}(\mathsf{G},\mathsf{T})) = \mathrm{der}(\mathcal{R}(\mathsf{G},\mathsf{T}))$;

(iv). $\mathcal{R}(\mathrm{sc}(\mathsf{G},\mathsf{T})) = \mathrm{sc}(\mathcal{R}(\mathsf{G},\mathsf{T}))$;

(v). $\mathcal{R}(\mathrm{rad}(\mathsf{G},\mathsf{T})) = \mathrm{rad}(\mathcal{R}(\mathsf{G},\mathsf{T}))$;

(vi). $\mathcal{R}(\mathrm{corad}(\mathsf{G},\mathsf{T})) = \mathrm{corad}(\mathcal{R}(\mathsf{G},\mathsf{T}))$;

(vii). the morphisms between above root data come from the homomorphisms between corresponding split reductive groups.

Example 3.4.17. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. Then the deconstruction in Example 3.2.9 gives the following isogenies of root data.

$$\mathcal{R}(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n) \xleftarrow{\textit{Example } 3.4.13} \mathcal{R}(\mathsf{GL}_n, \mathsf{D}_n) \xleftarrow{\textit{Example } 3.4.14} \mathcal{R}(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_{\mathrm{m}})$$
$$\mathcal{R}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \xleftarrow{\textit{Example } 3.4.12} \mathcal{R}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$$

where $\mathcal{R}(\mathsf{SL}_n, \mathsf{D}_n \cap \mathsf{SL}_n)$ is described in Example 3.4.4, $\mathcal{R}(\mathsf{GL}_n, \mathsf{D}_n)$ in Example 3.4.5, $\mathcal{R}(\mathsf{PGL}_n, \mathsf{D}_n/\mathbb{G}_{\mathrm{m}})$ in Example 3.4.6, and $\mathcal{R}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$ in Example 3.4.3 respectively.

§ 3.5. Tits buildings

Let G be a reductive group. Associated to it, there is a spherical building ${}^{v}\mathscr{B}(\mathsf{G})$ equipped with a natural G-action, called its ***Tits building***. In this section, Tits buildings will be introduced for splittable reductive groups, and we will see that the underlying building only depends on the root system and the ground field.

3.5.1. Let G be a reductive group. A ***parabolic subgroup*** of it is a smooth subgroup P such that G/P is a complete variety. A subgroup T of G is ***Borel*** if it is smooth, connected, solvable, and parabolic. It turns out that a smooth subgroup P is parabolic if and only if $\mathsf{P}_{K^{\mathrm{alg}}}$ contains a Borel subgroup in $\mathsf{G}_{K^{\mathrm{alg}}}$ [Mil17, 17.16] and every parabolic subgroup is connected and equal to its own normalizer since this is so over K^{alg} [Mil17, 17.49]. When G has a Borel subgroup, it is said to be ***quasi-split***. In this case, Borel subgroups are exactly the minimal parabolic subgroups and maximal connected solvable subgroups [Mil17, 17.19] and any two of them are conjugated by an element of G [Mil17, 25.8]. If the Borel subgroup is furthermore split (as a solvable algebraic group, namely it admits a normal series whose factors are isomorphic to either $\mathbb{G}_{\mathrm{a}}$ or $\mathbb{G}_{\mathrm{m}}$), then G is said to be *split*. It turns out that, G is split if and only if it is splittable [Mil17, 21.64].

Let $\pi\colon \mathsf{G} \to \mathsf{Q}$ be a quotient map and H a smooth subgroup of G. Then if H is parabolic (resp. Borel), so is $\pi(\mathsf{H})$. Moreover, every such subgroup of Q arises in this way [Mil17, 17.20]. This allows us to reduce the study of (the poset of) parabolic subgroups from reductive groups to simply-connected semisimple groups. The *Tits building* of a reductive group is essentially this poset [Tit74, 5.2].

Proposition 3.5.2. *Let (G, T) be a split reductive group. Then there are natural one-to-one correspondences between the following sets:*

(i). *The set of Borel subgroups B of G containing T.*

(ii). *The set of Weyl chambers ${}^{v}C$ in the vectorial apartment ${}^{v}\mathscr{A}$.*

(iii). *The set of systems of positive roots Φ^+ in the root system Φ.*

(iv). *The set of bases Δ of Φ.*

The Weyl group ${}^{v}W$ acts simply transitively on each set. Moreover, after choosing a quadruple $(\mathsf{B}, {}^{v}C, \Phi^+, \Delta)$, we have the following isomorphic posets.

(i). *The poset of parabolic subgroups P of G containing B.*

(ii). *The poset of faces ${}^{v}F$ of the Weyl chamber ${}^{v}C$.*

(iii). *The poset of parabolic subsets Ψ of Φ containing Φ^+.*

(iv). *The poset of subsets I of Δ.*

Proof. If a system of positive roots Φ^+ is given, then B is generated by T and U_a for all $a \in \Phi^+$ and if a Borel subgroup B containing T is given, then the set of roots a whose Lie algebra $\mathfrak{g}_a$ is contained in the Lie algebra of T forms a system of positive roots Φ^+ [Mil17, 21.d].

If a parabolic subset Ψ is given, then P is generated by T and U_a for all $a \in \Psi$ and if a parabolic subgroup P containing B is given, then the set of roots a whose Lie algebra $\mathfrak{g}_a$ is contained in the Lie algebra of P forms a parabolic subset Ψ [Mil17, 21.i]. $\square$

Convention 3.5.3. If $(\mathsf{P}, {}^vF, \Psi, I)$ is a quadruple as above, then we say that each of them has *type* I, where I is identified as a subset of $\{1, \cdots, n\}$ in Convention 2.4.5.

3.5.4. Fix a Borel subgroup B of G containing T. Let I be a type and P_I the parabolic subgroup corresponding to it. Then the unipotent radical of P_I is generated by U_a for all $a \in \Phi^+ \setminus \Psi$ and the reductive quotient of P_I is isomorphic to the centralizer L_I of the largest subtorus contained in $\mathrm{Ker}(a)$ for all $a \in I$ [Mil17, 21.91]. This reductive group is called the *Levi subgroup* associated to I and $(\mathsf{L}_I, \mathsf{T})$ is a split reductive group with root datum $(\mathsf{X}^*, \Phi_I, \mathsf{X}_*, \Phi_I^\vee)$ and Weyl group ${}^vW_I$ [Mil17, 21.90].

Example 3.5.5. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$. Then the subgroup T_n of upper triangular invertible matrices is a Borel subgroup containing D_n. It corresponds to the system of positive roots $\Phi^+ = \left\{\chi_i - \chi_j \,\middle|\, 1 \leqslant i < j \leqslant n\right\}$ with basis $\Delta = \{a_1 = \chi_1 - \chi_2, \cdots, a_{n-1} = \chi_{n-1} - \chi_n\}$. The Weyl chamber corresponding to it is

$$ {}^vC = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 > x_2 > \cdots > x_n\}. $$

Let $I = \Delta \setminus \{l_1 = k_1, l_2 = k_1 + k_2, \cdots, l_{t-1} = k_1 + k_2 + \cdots + k_{t-1}\}$ be a type on the apartment, identified with a subset of Δ. Then the parabolic subgroup P_I, its unipotent radical $\mathscr{R}_u(\mathsf{P}_I)$ and the Levi subgroup L_I consist of the matrices of the following forms respectively

$$\begin{pmatrix} A_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & A_t \end{pmatrix}, \qquad \begin{pmatrix} I_{k_1} & * & * \\ 0 & \ddots & * \\ 0 & 0 & I_{k_t} \end{pmatrix}, \qquad \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_t \end{pmatrix},$$

where A_i is a $k_i \times k_i$ matrix. The facet corresponding to them is

$${}^vF = \left\{(x_1, \cdots, x_n) \in \mathbb{R}^n \,\middle|\, x_1 = \cdots = x_{l_1} > x_{l_1+1} \cdots x_{l_{t-1}} > x_{l_{t-1}+1} = \cdots = x_n\right\}.$$

Theorem 3.1 ([Rou09, §10; Tit74, §5])**.** *Let* (G, T) *be a split reductive group with Weyl group* ${}^vW$*, coroot space* $\mathbb{V}$*, normalizer* N *of* T *and the root groups* U_a*. Then there is a unique (up to unique isomorphism) G-set* ${}^v\mathscr{B}(\mathsf{G})$ *containing* $\mathbb{V}$ *and satisfying the following.*

(i). ${}^v\mathscr{B}(\mathsf{G}) = \bigcup_{g \in G} g.\mathbb{V}$*;*

(ii). *N stabilizes* $\mathbb{V}$ *and acts on it through* ${}^vW$*;*

(iii). *For every* $a \in \Phi$*, the fixator of* $\alpha_{a+0} := \{\mathbf{v} \in \mathbb{V} \mid a(\mathbf{v}) \geqslant 0\}$ *is* $T \cdot U_a$*.*

Then ${}^v\mathscr{B}(\mathsf{G})$ is a building of type ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T})$. Indeed, since N stabilizes $\mathbb{V}$ and preserves its apartment structure, each $g.\mathbb{V}$ is endowed with such a structure and moreover they agree on intersections.

Definition 3.5.6. The building ${}^v\mathscr{B}(\mathsf{G})$ is called the *Tits building* of G.

Remark. Apartments in ${}^v\mathscr{B}(\mathsf{G})$ are one-one corresponding to split maximal tori. In fact, each $g.\mathbb{V}$ endowed with its apartment structure is precisely the spherical apartment ${}^v\mathscr{A}(\mathsf{G}, \mathsf{T}^g)$.

The action of G on ${}^v\mathscr{B}(\mathsf{G})$ is strongly transitive and type-preserving. It is also worth to mention that ${}^v\mathscr{B}(\mathsf{G})$ is further an $\mathrm{Aut}(\mathsf{G})$-set. Indeed, if φ is an automorphism of

G, then $\varphi(\mathsf{T})$ is also a split maximal torus and the pushforward along φ defines an isomorphism from ${}^v\mathscr{A}(\mathsf{G},\mathsf{T})$ to ${}^v\mathscr{A}(\mathsf{G},\varphi(\mathsf{T}))$.

Example 3.5.7. The simplicial complex structure on the Tits building of GL_n can be described as in 2.1.2.

3.5.8. Let G be a splittable reductive group. Let φ be a homomorphism in the following sequence.

$$\mathsf{G}^{\mathrm{sc}} \longrightarrow \mathsf{G}^{\mathrm{der}} \longrightarrow \mathsf{G} \longrightarrow \mathsf{G}^{\mathrm{ss}} \longrightarrow \mathsf{G}^{\mathrm{ad}}$$

Then for any split maximal torus T in G, its image or preimage under φ is again a split maximal torus T' and such a corresponding $\mathsf{T} \mapsto \mathsf{T}'$ gives rise to a bijection between the set of maximal tori. Therefore, by 2.5.6 and 3.4.16, we see that all above reductive groups have isomorphic Tits buildings.

Conversely, any root datum arises from a splittable reductive group [Mil17, 23.55; SGA3, XXV, 1.2]. Hence, we see that the Tits building ${}^v\mathscr{B}(\mathsf{G})$ depends only on the root system Φ and the ground field K and any root system gives rise to such a building. So we can denote this building by ${}^v\mathscr{B}(\Phi, K)$.

Chapter 4.

Bruhat-Tits Theory

The datum of a split reductive group over a local field gives rise to a root group datum and a valuation on it [BT-2]. Bruhat and Tits [BT-1] introduced an affine building based on such purely group-theoretical data. They also show in [BT-2] that these data come with some extra schematic structures, which turn out to be an important ingredient in the theory of reductive groups over local fields.

In the rest of this dissertation, the ground field K is assumed to be equipped with a discrete valuation $\mathrm{val}(\,\cdot\,)\colon K \to \mathbb{R} \cup \{\infty\}$. Its valuation group $\mathrm{val}(K^\times)$ is denoted by Γ, and we fix the following associated notations.

$$\mathcal{O}_K := \{x \in K \mid \mathrm{val}(x) \geqslant 0\},$$

$$\mathfrak{m}_K := \{x \in K \mid \mathrm{val}(x) > 0\},$$

$$\kappa := \mathcal{O}_K/\mathfrak{m}_K.$$

We also fix a *uniformizer* ϖ, namely a generator of $\mathfrak{m}_K$. Let $\gamma = \mathrm{val}(\varpi)$.

§ 4.1. Valuations on root group data

Definition 4.1.1. Let Φ be a root system and G be a group. A *root group datum** (of type Φ in G) is a system $(T, (U_a, M_a)_{a\in\Phi})$, where

- T is a subgroup of G and for each $a \in \Phi$,
- U_a is a non-trivial subgroup of G and
- M_a is a right congruence class modulo T,

satisfying the following axioms.

RGD1. For any $a, b \in \Phi$, the commutator group $[U_a, U_b]$ is contained in the group generated by the U_c for $c = ia + jb \in \Phi$ with $i, j > 0$.

RGD2. For each $a \in \Phi$, the class M_a satisfies $U^*_{-a} := U_{-a} \setminus \{1\} \subseteq U_a M_a U_a$.

RGD3. For any $a, b \in \Phi$ and each $m \in M_a$, we have $\mathrm{inn}(m).U_b \subseteq U_{r_a(b)}$, where r_a is the reflection associated to a.

RGD4. Let Φ^+ be a system of positive roots in Φ and if U^+ (resp. U^-) is the subgroup of G generated by the U_a for $a \in \Phi^+$ (resp. $a \in \Phi^-$), then $TU^+ \cap U^- = \{1\}$.

This root group datum is said to be *generating* when G is generated by the subgroups T and U_a for $a \in \Phi$.

4.1.2. Let $(T, (U_a, M_a)_{a\in\Phi})$ be a root group datum. We have the following consequences of above axioms [BT-1, 6.1.2].

(i). $U_a \neq U_{-a}$ and $U_a M_a U_a \cap N_G(U_a) = \emptyset$.

(ii). For any $u \in U^*_{-a}$, there is a unique triple $(u', m, u'') \in U_a \times G \times U_a$ such that $u = u'mu''$, $\mathrm{inn}(m).U_a = U_{-a}$ and $\mathrm{inn}(m).U_{-a} = U_a$. Moreover, $m \in M_a$ and $u' \neq 1$.

*It is called a *reduced root datum* in [BT-1, 6.1.1]. We only focus on reduced root datum as we focus on split reductive groups. As for general case, see the original papers [BT-1].

Let $m(-)\colon U_{-a}^{*} \to M_a$ denote the map $u \mapsto m$ in above and put M_a° being its image.

(iii). T normalizes U_a and M_a.

(iv). $M_a = M_a^{-1} = M_{-a}$ and $T \cup M_a$ is a subgroup of G.

(v). Let L_a be the subgroup of G generated by U_a, U_{-a} and T. Then

$$L_a = U_a M_a U_a \cup TU_a.$$

(vi). $N_G(U_a) \cap L_a = TU_a$ and

$$M_a = \{g \in L_a \mid \mathrm{inn}(g).U_a = U_{-a} \text{ and } \mathrm{inn}(g).U_{-a} = U_a\}.$$

So M_a is completely determined by U_a, U_{-a} and T. Hence, we can say $(T, (U_a)_{a\in\Phi})$ is a root group datum without mentioning M_a.

(vii). Let N be the subgroup of G generated by T and M_a for all $a \in \Phi$. Then, if Φ is nonempty, N is already generated by M_a's and normalizes T. Moreover, there is an epimorphism ${}^{v}\nu\colon N \to {}^{v}W(\Phi)$ such that for each $a \in \Phi$ and $m \in N$, we have $\mathrm{inn}(m).U_a = U_b$ with $b = {}^{v}\nu(m).a$. In particular, we have ${}^{v}\nu(M_a) = \{r_a\}$. Also note that $\mathrm{Ker}({}^{v}\nu) = T$ [BT-1, 6.1.11].

(viii). Suppose Φ is nonempty. Let N° be the subgroup of G generated by M_a° for all $a \in \Phi$ and let $T^{\circ} = N^{\circ} \cap T$. Then $(T^{\circ}, (U_a, M_a^{\circ})_{a\in\Phi})$ is a generating root group datum on the subgroup G° of G generated by U_a for all $a \in \Phi$.

Example 4.1.3 ([BT-1, 6.1.3.b; BT65])**.** Let (G, T) be a split reductive group over K, $(\mathsf{U}_a)_{a\in\Phi}$ be the root groups associated to the root system Φ of (G, T) and $(\mathsf{M}_a)_{a\in\Phi}$ be the right congruence classes in 3.3.13. Then $(T, (U_a, M_a)_{a\in\Phi})$ forms a generating root group datum in G:

RGD1. Let a and b be two roots in Φ. Then for any $i, j > 0$ such that $ia + jb \in \Phi$, there is a linear function [SGA3, XXII, 5.5.4]

$$f_{a,b;i,j} \colon \mathfrak{g}_a^{\otimes i} \otimes_K \mathfrak{g}_b^{\otimes j} \longrightarrow \mathfrak{g}_{ia+jb}$$

such that for any $X \in \mathbb{W}(\mathfrak{g}_a)$ and $Y \in \mathbb{W}(\mathfrak{g}_b)$, we have

$$[u_a(X), u_b(Y)] = \prod_{ia+jb\in\Phi} u_{ia+jb}(f_{a,b;i,j}(X^i \otimes Y^j))$$

where the product is taken in any order.

RGD2. Let $a \in \Phi$. Let $\mathsf{U}_{-a}^* = u_{-a}(\mathbb{W}(\mathfrak{g}_{-a})^\times)$, then $\mathsf{U}_{-a}^*(K) = U_{-a}^*$. Taking any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, then we have $X^{-1} \in \mathbb{W}(\mathfrak{g}_a)^\times$ and

$$\begin{aligned} u_{-a}(X^{-1}) &= u_a(X)u_a(-X)u_{-a}(-(-X)^{-1})u_a(-X)u_a(X) \\ &= u_a(X)m_a(-X)u_a(X). \end{aligned}$$

RGD3. This follows from 3.3.8 and 3.3.13.(iii).

RGD4. There are closed immersions [SGA3, XXII, 5.5.1 and 5.6.5; Mil17, 21.68]

$$\mathsf{T} \times \prod_{a\in\Phi^+} \mathsf{U}_a \longrightarrow \mathsf{G} \qquad \text{and} \qquad \prod_{a\in\Phi^+} \mathsf{U}_a \longrightarrow \mathsf{G},$$

with images $\mathsf{T}\cdot\mathsf{U}_+$ and U_+. Where $\mathsf{T}\cdot\mathsf{U}_+$ is a Borel subgroup of G corresponding to the system of positive roots Φ^+, while U_+ is its unipotent radical and is generated by the root groups U_a for all $a \in \Phi^+$. Similarly, we have Borel subgroup $\mathsf{T}\cdot\mathsf{U}_-$ and its unipotent radical U_-. Then the Borel subgroups $\mathsf{T}\cdot\mathsf{U}_+$ and $\mathsf{T}\cdot\mathsf{U}_-$ are *opposite*, namely their intersection is T [SGA3, 5.9.2; Mil17, 21.84]. Therefore, $\mathsf{T}\cdot\mathsf{U}_+ \cap \mathsf{U}_-$ is trivial.

Moreover, G is generated by T and the root groups U_a for all $a \in \Phi$ [Mil17, 21.11].

We also verify the corollaries in 4.1.2:

(i). This is clear.

(ii). We have already seen in above discussion that if $u = u_{-a}(X^{-1})$, then the triple $(u_a(X), m_a(-X), u_a(X))$ satisfies the requirements. Suppose (u', m, u'') is another triple, then $m_a(-X) = u_a(-X)u'mu''u_a(-X) \in M_a$ and hence it maps U_a to U_{-a} by conjugate. Then $u_a(-X)u'$ normalizes U_{-a} and $u''u_a(-X)$ normalizes U_a, hence $u' = u_a(X)$ and $u'' = u_a(X)$ by 4.1.2.(i).

(iii). The first follows from 3.3.8.(ii) and as for the second: let $\mathbf{t} \in \mathsf{T}$ and $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, then we have

$$\begin{aligned}
\mathrm{inn}(\mathbf{t}).m_a(X) &= \mathrm{inn}(\mathbf{t}).(u_a(X)u_{-a}(-X^{-1})u_a(X)) \\
&= (\mathrm{inn}(\mathbf{t}).u_a(X))(\mathrm{inn}(\mathbf{t}).u_{-a}(-X^{-1}))(\mathrm{inn}(\mathbf{t}).u_a(X)) \\
&= u_a(a(\mathbf{t})X) \cdot u_{-a}((-a)(\mathbf{t})(-X^{-1})) \cdot u_a(a(\mathbf{t})X) \\
&= u_a(a(\mathbf{t})X) \cdot u_{-a}(-(a(\mathbf{t})X)^{-1}) \cdot u_a(a(\mathbf{t})X) \\
&= m_a(a(\mathbf{t})X).
\end{aligned}$$

(iv). For any $X \in \mathbb{W}(\mathfrak{g}_a)^\times$, we have

$$\begin{aligned}
m_a(X)^{-1} &= (u_a(X)u_{-a}(-X^{-1})u_a(X))^{-1} \\
&= u_a(-X)u_{-a}(X^{-1})u_a(-X) \\
&= m_a(-X).
\end{aligned}$$

By 3.3.13.(iv), we also have

$$m_a(X)^{-1} = m_{-a}(X^{-1}).$$

These prove the first part. As for the second: let $X, Y \in \mathbb{W}(\mathfrak{g}_a)^\times$, then we have

$$m_a(X)m_a(Y) = m_a(X)m_{-a}(-Y^{-1}) = a^\vee(\langle X, -Y^{-1}\rangle) \in \mathsf{T}\,.$$

(v). This follows from the Bruhat decomposition [SGA3, XXII, 5.7.4; Mil17, 21.73] for L_a:

$$\mathsf{L}_a = \mathsf{B} \sqcup \mathscr{R}_u(\mathsf{B})w\,\mathsf{B},$$

where B is the Borel subgroup $\mathsf{T} \cdot \mathsf{U}_a$ of L_a with unipotent radical $\mathscr{R}_u(\mathsf{B}) = \mathsf{U}_a$ and w is the only nontrivial element of ${}^vW(\mathsf{L}_a, \mathsf{T})$, hence M_a.

(vi). The first follows from the normalizer theorem [Mil17, 17.50]. As for the second: suppose $g \in \mathsf{L}_a$ has the property that $\mathrm{inn}(g).\mathsf{U}_a = \mathsf{U}_{-a}$ and $\mathrm{inn}(g).\mathsf{U}_{-a} = \mathsf{U}_a$. Then $g \notin \mathsf{T} \cdot \mathsf{U}_a$ and hence $g \in \mathsf{U}_a \cdot \mathsf{M}_a \cdot \mathsf{U}_a$. But $\mathsf{U}_a \cap \mathsf{N}_\mathsf{G}(\mathsf{U}_{-a})$ is trivial. Hence, $g \in \mathsf{M}_a$.

(vii). $\mathsf{N} = \mathsf{N}_\mathsf{G}(\mathsf{T})$ is generated by M_a for all $a \in \Phi$, the epimorphism ${}^v\nu \colon N \to {}^vW(\Phi)$ is the quotient map $N \to {}^vW$ and the statement follows from 3.3.13.

(viii). By [Mil17, 21.49], $\mathsf{G}^{\mathrm{der}}$ is generated by U_a for all $a \in \Phi$. Then it is clear that $G^\circ = \mathsf{G}^{\mathrm{der}}(K)$, $T^\circ = T \cap G^\circ$ and $N^\circ = N_{G^\circ}(T^\circ)$.

Note that the above facts already will imply Theorem 3.1 using either *Tits system* or similar construction in Definition 4.2.9.

Definition 4.1.4. A *valuation* on the root group datum $(T, (U_a, M_a)_{a\in\Phi})$ is a family $\varphi = (\varphi_a)_{a\in\Phi}$ of functions $\varphi_a \colon U_a \to \mathbb{R} \cup \{\infty\}$ satisfying the following axioms.

V0. For each $a \in \Phi$, the image of φ_a contains at least three elements.

V1. For each $a \in \Phi$ and any $\lambda \in \mathbb{R} \cup \{\infty\}$, the set $U_{a,\lambda} := \varphi_a^{-1}([\lambda, \infty])$ is a subgroup of U_a and $U_{a,\infty} = \{1\}$.

V2. For each $a \in \Phi$ and any $m \in M_a$, the function $u \mapsto \varphi_{-a}(u) - \varphi_a(mum^{-1})$ is constant on U_{-a}^*.

V3. For any pair $a, b \in \Phi$ not proportional and any $\lambda, \mu \in \mathbb{R} \cup \{\infty\}$, the commutator group $[U_{a,\lambda}, U_{b,\mu}]$ is contained in the subgroup generated by $U_{ia+jb, i\lambda+j\mu}$ for all $i, j > 0$ such that $ia + jb \in \Phi$.

V4. For each $a \in \Phi$ and any $u \in U_a$, $u', u'' \in U_{-a}$ such that $u'uu'' \in M_a$, we have $\varphi_{-a}(u') = \varphi_{-a}(u'') = -\varphi_a(u)$.

For each $a \in \Phi$, let Γ_a denote the set $\varphi_a(U_a^*)$ and for any $k \in \Gamma_a$, let $M_{a,k}$ be the intersection of M_a and $U_{-a}\varphi_a^{-1}(k)U_{-a}$.

Example 4.1.5. Let $\mathsf{T} = \mathbb{G}_{\mathrm{m}}^n$. Then (T, T) is a split reductive group with empty root system. Then there is only one way to define a valuation on the root group datum $(T, \emptyset)$, namely the *trivial valuation* **0**.

Example 4.1.6. Let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ over K. Denote $a_{ij} = \chi_i - \chi_j \in \Phi$ and define $\varphi = (\varphi_{a_{ij}})_{a_{ij}\in\Phi}$ as

$$\varphi_{a_{ij}} = \mathrm{val} \circ u_{a_{ij}}^{-1}.$$

Note that we have

$$\varphi_{a_{ij}}\left(\xi_{ij}\left(\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right)\right)\right) = \mathrm{val}(t), \qquad \varphi_{-a_{ij}}\left(\xi_{ij}\left(\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right)\right)\right) = \mathrm{val}(t).$$

Then φ is a valuation on the root group datum $(D_n, (U_{a_{ij}}, M_{a_{ij}})_{a_{ij}\in\Phi})$ with $\Gamma_{a_{ij}} = \Gamma$:

V0. This clear as val is nontrivial.

V1. For any $\lambda \in \mathbb{R} \cup \{\infty\}$, we have

$$U_{a_{ij},\lambda} = \left\{ \xi_{ij}\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) \in U_{a_{ij}} \;\middle|\; \operatorname{val}(t) \geqslant \lambda \right\}.$$

Then $U_{a_{ij},\infty} = \{I_n\}$ and for any $x, y \in K$ with $\operatorname{val}(x), \operatorname{val}(y) \geqslant \lambda$, we have

$$u_{a_{ij}}(x) \cdot u_{a_{ij}}(y)^{-1} = u_{a_{ij}}(x - y),$$

and its valuation is $\operatorname{val}(x - y) \geqslant \min\{\operatorname{val}(x), \operatorname{val}(y)\} \geqslant \lambda$.

V2. For any $x, y, z \in K^\times$ and

$$m = \xi_{ij}\left(\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\right) \in M_{a_{ij}}, \qquad u = \xi_{ij}\left(\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right) \in U^*_{-a_{ij}},$$

we have

$$mum^{-1} = \xi_{ij}\left(\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} 0 & y^{-1} \\ x^{-1} & 0 \end{pmatrix}\right) = \xi_{ij}\left(\begin{pmatrix} 1 & xzy^{-1} \\ 0 & 1 \end{pmatrix}\right).$$

Therefore,

$$\begin{aligned} \varphi_{-a_{ij}}(u) - \varphi_{a_{ij}}(mum^{-1}) &= \varphi_{-a_{ij}}\left(u_{-a_{ij}}(z)\right) - \varphi_{a_{ij}}\left(u_{a_{ij}}(xzy^{-1})\right) \\ &= \operatorname{val}(z) - \operatorname{val}\left(xzy^{-1}\right) \\ &= -\operatorname{val}(x) + \operatorname{val}(y), \end{aligned}$$

independently of u.

V3. We need the following *commutator formula*:

$$[u_{a_{ij}}(x), u_{a_{kl}}(y)] = \begin{cases} u_{a_{il}}(xy) & \text{if } i \neq l, k = j, \\ u_{a_{kj}}(-xy) & \text{if } i = l, k \neq j, \\ I_n & \text{if } i \neq l, k \neq j. \end{cases}$$

From which, we see that if $\varphi_{a_{ij}}(u) \geqslant \lambda$ and $\varphi_{a_{kl}}(v) \geqslant \mu$, then either $[u, v] = I_n$ or $a_{ij} + a_{kl} \in \Phi$ and $\varphi_{a_{ij}+a_{kl}}([u, v]) \geqslant \lambda + \mu$.

V4. For any $x, y, z \in K$ and $u = u_{a_{ij}}(x)$, $u' = u_{-a_{ij}}(y)$, $u'' = u_{-a_{ij}}(z)$, we have

$$u'uu'' = \xi_{ij}\left(\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right) = \xi_{ij}\left(\begin{pmatrix} 1 + xz & x \\ y + z + xyz & 1 + xy \end{pmatrix}\right).$$

If $u'uu'' \in M_{a_{ij}}$, then we must have

$$1 + xz = 1 + xy = 0, \qquad \text{and} \qquad -x^{-1} = y + z + xyz.$$

Hence, we have $y = z = -x^{-1}$ and thus $\varphi_{-a_{ij}}(u') = \varphi_{-a_{ij}}(u'') = -\varphi_{a_{ij}}(u)$.

Now, let $\mathrm{val}(x) = \lambda$. Then we see that

$$M_{a_{ij},\lambda} = \left\{ \xi_{ij}\left(\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}\right) \middle| \, t \in K, \mathrm{val}(t) = \lambda \right\}.$$

Example 4.1.7 ([BT-1, 6.2.3])**.** Notations as in Example 4.1.3. Let $(u_a)_{a\in\Phi}$ be a Chevalley system. Then $\varphi = (\varphi_a := \mathrm{val} \circ u_a^{-1})_{a\in\Phi}$ is a valuation on the root group datum $(T, (U_a, M_a)_{a\in\Phi})$ with $\Gamma_a = \Gamma$. The proof is basically the same as in Example 4.1.6 plus the following facts:

(i). By 3.3.13.(ii), any $m \in M_a$ can be written as $m = \mathbf{t}m_a(1)$ for some $\mathbf{t} \in T$. Hence, for any $x \in K^\times$ and $u = u_{-a}(x)$, we have

$$\begin{aligned}\varphi_{-a}(u) - \varphi_a(mum^{-1}) &= \varphi_{-a}(u_{-a}(x)) - \varphi_a\left(\operatorname{inn}(\mathbf{t}m_a(1)).u_{-a}(x)\right)\\ &= \varphi_{-a}(u_{-a}(x)) - \varphi_a\left(\operatorname{inn}(\mathbf{t}).u_a(-x)\right)\\ &= \varphi_{-a}(u_{-a}(x)) - \varphi_a\left(u_a(-a(\mathbf{t})x)\right)\\ &= -\operatorname{val}(a(\mathbf{t})).\end{aligned}$$

(ii). For any $x, y \in K$, we have [SGA3, XXII, 5.5.2]

$$[u_a(x), u_b(y)] = \prod_{ia+jb\in\Phi} u_{ia+jb}(C_{a,b;i,j}x^i y^j),$$

where $C_{a,b;i,j} \in K$ is a constant.

4.1.8. Given a root group datum $(T, (U_a)_{a\in\Phi})$ in G and let φ be a valuation on it. Then for any vector $\mathbf{v}$ in the ambient space $\mathbb{V}$ of Φ, the family $\psi = (\psi_a)_{a\in\Phi}$ given by $\psi_a\colon u \mapsto \varphi_a(u) + a(\mathbf{v})$ is a valuation [BT-1, 6.2.5] and is denoted by $\varphi + \mathbf{v}$. The valuations φ and $\psi = \varphi + \mathbf{v}$ are said to be *equipollent*. The mapping $(\varphi, \mathbf{v}) \mapsto \varphi + \mathbf{v}$ defines an action of $\mathbb{V}$ on the set of valuations and each equipollent class is an orbit.

Let $\mathbb{A}$ denote the set of valuations equipollent to φ. Then $\mathbb{A}$ is an affine space with ${}^v\mathbb{A} = \mathbb{V}$ and 2.4.7 applies. For $\alpha = \alpha_{a+k}$ with $a \in \Phi$, $k \in \Gamma_a$, let $U_\alpha = U_{a,k}$ and $U_{\alpha+} = \bigcup_{h>k} U_{a,h}$ (note that $U_{\alpha+} = U_{\alpha_+}$ if Γ_a is discrete). It is clear that the affine root system Σ and the mapping $\alpha \mapsto U_\alpha$ depends only on the equipollent class of φ.

Example 4.1.9. Continue Example 4.1.6. The coroot space is (by Example 3.4.4)

$$\mathbb{V} = \{c_1\lambda_1 + \cdots + c_n\lambda_n \mid c_1, \cdots, c_n \in \mathbb{R}, c_1 + \cdots + c_n = 0\}.$$

Then for any $\mathbf{v} = c_1\lambda_1 + \cdots + c_n\lambda_n$, the valuation $\varphi + \mathbf{v}$ is given by

$$u_{a_{ij}}(x) \longmapsto \mathrm{val}(x) + a_{ij}(\mathbf{v}) = \mathrm{val}(x) + c_i - c_j.$$

Let $k \in \Gamma_a$, then $\varphi + \mathbf{v} \in \alpha_{a+k}$ if and only if $c_i - c_j + k \geqslant 0$.

Example 4.1.10. Notations as in Example 4.1.7. A valuation $\psi = (\psi_a)_{a\in\Phi}$ is said to be *compatible* with $\mathrm{val}(\,\cdot\,)$ if for all $u \in U_a$ and $\mathbf{t} \in T$,

$$\psi_a\left(\mathbf{t}u\mathbf{t}^{-1}\right) = \psi_a(u) + \mathrm{val}(a(\mathbf{t})).$$

It turns out that [BT-2, 4.2.9]: a valuation ψ is compatible with $\mathrm{val}(\,\cdot\,)$ if and only if it is equipollent to φ given in Example 4.1.7. Hence, $\mathbb{A}$ is precisely the set of all valuations compatible with $\mathrm{val}(\,\cdot\,)$.

4.1.11. Let $m \in N$ and $w = {}^v\nu(m) \in {}^vW$. Then the family $\psi = (\psi_a)_{a\in\Phi}$ given by $\psi_a\colon u \mapsto \varphi_{w^{-1}.a}(m^{-1}um)$ is a valuation [BT-1, 6.2.5] and is denoted by $m.\varphi$. We thus obtain an action of N on the set of valuations such that for any $m \in N$ and $\mathbf{v} \in \mathbb{V}$, we have $m.(\varphi + \mathbf{v}) = m.\varphi + {}^v\nu(m).\mathbf{v}$. Moreover, we have [BT-1, 6.2.10]:

(i). The action of N stabilizes $\mathbb{A}$ and for any $m \in N$, the map $\nu(m)\colon \varphi \mapsto m.\varphi$ is an automorphism of the Euclidean affine space $\mathbb{A}$ whose vectorial part is ${}^v\nu(m)$.

(ii). For each $a \in \Phi$ and $k \in \Gamma_a$, the image of $M_{a,k}$ under ν is the reflection r_{a+k}.

(iii). The automorphism $\nu(m)$ maps affine roots to affine roots. For any $\alpha \in \Sigma$, we have $mU_\alpha m^{-1} = U_{\nu(m).\alpha}$

In particular, for $u \in U_a^*$, $\nu(m(u)) = r_{a+\varphi_a(u)}$ [BT-1, 6.2.12]. Therefore, the valuation φ is completely determined by the homomorphism $\nu\colon N \longrightarrow \mathrm{Aut}(\mathbb{A})$.

Example 4.1.12. Continue Examples 4.1.6 and 4.1.9.

(i). The normalizer N is the group of monomial matrices in $\mathsf{GL}_n(K)$. Any $m \in N$ can be written as

$$m = \sum_{k=1}^{n} x_k E_{\sigma(k)k},$$

where $\sigma \in \mathfrak{S}_n$ is a permutation such that $w = {}^v\nu(m)$ is identified with σ through ${}^vW \cong \mathfrak{S}_n$. Then, for any $u = u_{a_{ij}}(t) \in U_{a_{ij}}$, we have

$$\begin{aligned} m^{-1}um &= \left(\sum_{k=1}^{n} x^{-1}_{\sigma^{-1}(k)} E_{\sigma^{-1}(k)k}\right)\left(I_n + tE_{ij}\right)\left(\sum_{k=1}^{n} x_k E_{\sigma(k)k}\right) \\ &= I_n + x^{-1}_{\sigma^{-1}(i)} t x_{\sigma^{-1}(j)} E_{\sigma^{-1}(i)\sigma^{-1}(j)} \\ &= u_{a_{\sigma^{-1}(i)\sigma^{-1}(j)}}\left(x^{-1}_{\sigma^{-1}(i)} t x_{\sigma^{-1}(j)}\right) \in U_{a_{\sigma^{-1}(i)\sigma^{-1}(j)}} = U_{w^{-1}.a_{ij}}. \end{aligned}$$

Hence, the valuation $m.\varphi$ is given by

$$(m.\varphi)_{a_{ij}} \colon u = u_{a_{ij}}(t) \longmapsto \varphi_{w^{-1}.a_{ij}}(m^{-1}um) = \mathrm{val}\Big(x^{-1}_{\sigma^{-1}(i)} t x_{\sigma^{-1}(j)}\Big).$$

From above computations, it is also clear that $mU_\alpha m^{-1} = U_{\nu(m).\alpha}$ holds for any affine root α.

(ii). For any $\mathbf{v} \in \mathbb{V}$, one can verify that

$$\begin{aligned} (m.(\varphi + \mathbf{v}))_{a_{ij}}(u) &= (\varphi + \mathbf{v})_{w^{-1}.a_{ij}}(m^{-1}um) \\ &= \varphi_{w^{-1}.a_{ij}}(m^{-1}um) + (w^{-1}.a_{ij})(\mathbf{v}) \\ &= (m.\varphi)_{a_{ij}}(u) + a_{ij}(w.\mathbf{v}) \\ &= (m.\varphi + w.\mathbf{v})_{a_{ij}}(u). \end{aligned}$$

Also note that

$$\begin{aligned}(m.\varphi)_{a_{ij}}(u) - \varphi_{a_{ij}}(u) &= \mathrm{val}\Big(x_{\sigma^{-1}(i)}^{-1} t x_{\sigma^{-1}(j)}\Big) - \mathrm{val}(t)\\ &= \mathrm{val}\Big(x_{\sigma^{-1}(i)}^{-1}\Big) - \mathrm{val}\Big(x_{\sigma^{-1}(j)}^{-1}\Big)\\ &= \Big\langle a_{ij}, \sum_{k=1}^{n} \mathrm{val}\Big(x_{\sigma^{-1}(k)}^{-1}\Big)\lambda_k\Big\rangle.\end{aligned}$$

Therefore, the affine transformation $\nu(m)\colon \varphi \mapsto m.\varphi$ has vectorial part $w = {}^{v}\nu(m)$ and translation part

$$\mathbf{v}_m = \sum_{k=1}^{n} \left(\mathrm{val}\Big(x_{\sigma^{-1}(k)}^{-1}\Big) + \tfrac{1}{n}\,\mathrm{val}(\det(m))\right) \lambda_k \in \mathbb{V}.$$

(iii). If $u = u_{a_{ij}}(x) \in U_{a_{ij}}^{*}$, then

$$m(u) = m_{a_{ij}}(x) = \xi_{ij}\left(\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}\right).$$

Hence, ${}^{v}\nu(m(u)) = (i, j)$ and the translation part is

$$\mathbf{v}_{m(u)} = \mathrm{val}\Big(x^{-1}\Big)\lambda_i + \mathrm{val}(-x)\lambda_j = -\,\mathrm{val}(x)a_{ij}^{\vee} = -\varphi_{a_{ij}}(u)a_{ij}^{\vee}.$$

Then we have

$$\begin{aligned}\nu(m(u)).(\varphi + \mathbf{v}) &= \varphi + (i, j).\mathbf{v} - \varphi_{a_{ij}}(u)a_{ij}^{\vee}\\ &= \varphi + \mathbf{v} - a_{ij}(\mathbf{v})a_{ij}^{\vee} - \varphi_{a_{ij}}(u)a_{ij}^{\vee}\\ &= r_{a_{ij}+\varphi_{a_{ij}}(u)}(\varphi + \mathbf{v}).\end{aligned}$$

Example 4.1.13. The condition of being compatible with $\mathrm{val}(\,\cdot\,)$ in Example 4.1.10 can be interpreted as follows. Let $\psi = (\psi_a)_{a\in\Phi}$ be a valuation and $\mathbb{A}$ the set of valuations equipollent to it. For any $\mathbf{t} \in T$, the automorphism $\nu(\mathbf{t}) : \mathbb{A} \to \mathbb{A}$ is a translation, denoted by $\mathbf{v_t}$. Then, for all $a \in \Phi$ and $u \in U_a$, we have

$$\psi_a\left(\mathbf{t}^{-1}u\mathbf{t}\right) - \psi_a(u) = \langle a, \mathbf{v_t}\rangle.$$

Therefore, ψ is compatible with $\mathrm{val}(\,\cdot\,)$ if and only if

$$\langle a, \mathbf{v_t}\rangle = -\,\mathrm{val}(a(\mathbf{t}))$$

for all $\mathbf{t} \in T$ and $a \in \Phi$.

Now, suppose ψ is compatible with $\mathrm{val}(\,\cdot\,)$. Then the above shows that for all $\chi \in \mathsf{X}_{\mathrm{ss}} := \mathbb{V}^* \cap \mathsf{X}$, we have

$$\langle \chi, \mathbf{v_t}\rangle = -\,\mathrm{val}(\chi(\mathbf{t})).$$

This implies that $\mathbf{v_t} \in \mathsf{X}_{\mathrm{ss}}^\vee \otimes \Gamma$, where $\mathsf{X}_{\mathrm{ss}}^\vee$ is the dual lattice of X_{ss} and is the cocharacter group of the semisimple quotient $(\mathsf{G}^{\mathrm{ss}}, \mathsf{T}/\mathscr{R}(\mathsf{G}))$.

4.1.14. Let $H = \mathrm{Ker}(\nu)$ and $\widehat{W} = \nu(N)$. Let W denote the subgroup of $\widehat{W}$ generated by r_{a+k} with $a \in \Phi$ and $k \in \Gamma_a$. It is a normal subgroup because N permutes $M_{a,k}$. Let $N' = \nu^{-1}(W)$. It is usually not the entire N. We say the root group datum (together with the valuation φ) is *simply-connected* when $N' = N$. Let $T' = T \cap N'$ and let G' be the subgroup of G generated by N' and the U_a for $a \in \Phi$. Since $M_a \cap N' \neq \emptyset$ for all $a \in \Phi$, we see that $(T', (U_a)_{a\in\Phi})$ is a simply-connected generating root group datum in G' [BT-1, 6.2.11]. Recall that (4.1.2.(viii)) N° is generated by M_a° for all $a \in \Phi$, hence $N^\circ \subseteq N'$ and therefore the generating root group datum $(T^\circ, (U_a)_{a\in\Phi})$ on G° is also simply-connected.

The valuation φ is said to be *special* if $0 \in \Gamma_a$ for all $a \in \Phi$. If this is the case, then the group W (resp. $\widehat{W}$) can be decomposed as $W = W_\varphi \ltimes \operatorname{Ker}\big(W \to {}^vW\big)$ (resp. $\widehat{W} = W_\varphi \ltimes \nu(T)$) [BT-1, 6.2.19], where W_φ is the stabilizer of φ.

The valuation φ is said to be *discrete* if Γ_a is a discrete subset of $\mathbb{R}$ for all $a \in \Phi$. If this is the case, then W is the affine Weyl group $W(\Sigma)$ for the affine root system Σ [BT-1, 6.2.22].

Suppose Φ is irreducible and φ is discrete and special. This is the case we most focus on. Then all Γ_a are the same discrete subgroup Γ of $\mathbb{R}$ [BT-1, 6.2.23]. So 2.4.8 applies and we get an apartment $\mathscr{A}(\Sigma)$. Then $\operatorname{Ker}\big(W \to {}^vW\big) = Q^\vee \otimes \Gamma$ and $\nu(T)$ is between $Q^\vee \otimes \Gamma$ and $\mathcal{P}^\vee \otimes \Gamma$ [BT-1, 6.2.20].

Example 4.1.15. Continue Example 4.1.5. Since $\mathbb{A} = \{\mathbf{0}\}$, we must have $N' = N = T$. On the other side, $N^\circ = \{1\}$ gives a smaller simply-connected root group datum. The trivial valuation $\mathbf{0}$ on them is both special and discrete.

Example 4.1.16. Continue Examples 4.1.6, 4.1.9, and 4.1.12. Let $m \in N$ with related notations as before. Then for $m \in \operatorname{Ker}(\nu)$, one must have both ${}^v\nu(m) = \mathrm{id}$ and $\mathbf{v}_m = 0$. Hence, m is diagonal and for all $1 \leqslant k \leqslant n$, $\operatorname{val}(x_k) = \frac{1}{n}\operatorname{val}(\det(m))$. Therefore,

$$H = \{\operatorname{diag}(x_1, \cdots, x_n) \in \mathsf{D}_n(K) \mid \operatorname{val}(x_1) = \cdots = \operatorname{val}(x_n)\}.$$

It is clear from the computations above that the translation group $\nu(T)$ is $\mathsf{X}_{\mathrm{ss}}^\vee \otimes \Gamma$. On the other hand, the translation group of W is clearly $Q^\vee \otimes \Gamma$, which has index n in the previous one.

Let $m \in N$ with related notations as before. Then for $m \in N'$, one must have $\mathbf{v}_m \in Q^\vee \otimes \Gamma$, which is equivalent to say that for all $1 \leqslant k \leqslant n$, $\operatorname{val}\big(x_k^{-1}\big) + \frac{1}{n}\operatorname{val}(\det(m)) \in \Gamma$, which is the case if and only if $\frac{1}{n}\operatorname{val}(\det(m)) \in \Gamma$. Therefore,

$$N' = \{m \in N \mid \operatorname{val}(\det(m)) \in n\Gamma\}.$$

Therefore, we have

$$G' = \{g \in \mathsf{GL}_n(K) \mid \operatorname{val}(\det(g)) \in n\Gamma\}.$$

It is worth to mention that there is a group

$$\mathsf{GL}_n(K)^1 := \{g \in \mathsf{GL}_n(K) \mid \operatorname{val}(\det(g)) = 0\},$$

between G' and $G^\circ = \mathsf{SL}_n(K)$. Hence, for this group, the generating root group datum $(D_n \cap \mathsf{GL}_n(K)^1, (U_{a_{ij}})_{a_{ij}\in\Phi})$ is simply-connected.

Example 4.1.17. Continue Examples 4.1.7, 4.1.10, and 4.1.13. Let $\mathbb{A}$ be the affine space of all valuations compatible with $\operatorname{val}(\cdot)$. For any $m \in N$, the automorphism $\nu(m)$ is trivial if and only if $m \in T$ (hence ${}^v\nu(m) = \mathrm{id}$) and the translation vector $\mathbf{v}_m = 0$. Therefore,

$$H = \{\mathbf{t} \in T \mid \operatorname{val}(\chi(\mathbf{t})) = 0 \text{ for all } \chi \in \mathsf{X}_{\mathrm{ss}}\}.$$

It follows from Example 4.1.13 that $\nu(T) \subseteq \mathsf{X}_{\mathrm{ss}}^\vee \otimes \Gamma$. Conversely, for any $\lambda \in \mathsf{X}_{\mathrm{ss}}^\vee$ and $t \in K^\times$, we have

$$\langle \chi, \mathbf{v}_{\lambda(t)}\rangle = -\operatorname{val}(\chi(\lambda(t))) = -\operatorname{val}(t)\langle \chi, \lambda\rangle.$$

Hence, $\nu(T) \supseteq \mathsf{X}_{\mathrm{ss}}^\vee \otimes \Gamma$. On the other hand, the translation group of W is clearly $Q^\vee \otimes \Gamma$.

One can replace $T = \mathsf{T}(K)$ by a suitable subgroup and obtain a different root group datum. The discussions on valuations still hold. It is also worth to mention that

[BT-2, 4.2.16]: there is a group

$$G^1 := \{g \in \mathsf{G}(K) \mid \mathrm{val}(\chi(g)) = 0 \text{ for all } \chi \in \mathsf{X}(\mathsf{G})\},$$

between G' and $G^\circ = \mathsf{G}^{\mathrm{der}}$. Hence, for this group, the generating root group datum $(T \cap G^1, (U_a)_{a\in\Phi})$ is simply-connected.

§ 4.2. Bruhat-Tits building

Given a root group datum $(T, (U_a, M_a)_{a\in\Phi})$ in G with a valuation $\varphi = (\varphi_a)_{a\in\Phi}$ on it, Bruhat and Tits [BT-1] associate an affine building equipped with natural G-action to these data.

4.2.1. Let Ω be a nonempty subset of $\mathbb{A}$ and let U_Ω denote the subgroup generated by U_α for all affine roots $\alpha \supseteq \Omega$. Then the image of $N \cap U_\Omega$ under $\nu\colon N \to \widehat{W}$ is generated by the reflections r_α for affine roots α such that $\Omega \subseteq \partial\alpha$ and is identified with the Weyl group of $\Phi_\Omega := \left\{a \in \Phi \,\middle|\, \exists\alpha, {}^v\alpha = a, \partial\alpha \supseteq \Omega\right\}$ [BT-1, 7.1.3]. Let N_Ω denote its preimage and let $P_\Omega = H \cdot U_\Omega$. Then

$$N_\Omega = N \cap P_\Omega.$$

Let $\widehat{N}_\Omega$ denote the fixator of Ω in N:

$$\widehat{N}_\Omega := \{n \in N \mid \nu(n).x = x \text{ for all } x \in \Omega\}.$$

Then $\widehat{N}_\Omega$ contains N_Ω and normalizes P_Ω. Hence,

$$\widehat{P}_\Omega := \widehat{N}_\Omega \cdot P_\Omega = \widehat{N}_\Omega \cdot U_\Omega$$

is a group having P_Ω and U_Ω as its normal subgroups. By 4.1.11, for any $n \in N$, we have

$$\mathrm{inn}(n).P_\Omega = P_{\nu(n).\Omega} \qquad \text{and} \qquad \mathrm{inn}(n).\widehat{P}_\Omega = \widehat{P}_{\nu(n).\Omega}.$$

Note that the map $\Omega \mapsto U_\Omega$ (resp. Φ_Ω, N_Ω, P_Ω, $\widehat{N}_\Omega$, $\widehat{P}_\Omega$) reverses the order of inclusions.

Remark. For $x \in \mathbb{A}$ a point, $\widehat{N}_x = \nu^{-1}(\widehat{W}_x)$. Hence, if the generating root group datum is simply-connected, we have $\widehat{N}_x = N_x$ and hence $\widehat{P}_x = P_x$.

Example 4.2.2. Continue Examples 4.1.6, 4.1.9, 4.1.12, and 4.1.16.

First consider $\Omega = \alpha_{a_{ij}+k} \in \Sigma$. Then $U_\Omega = U_{a_{ij},k}$, $\Phi_\Omega = \emptyset$ and

$$P_\Omega = H \cdot U_{a_{ij},k} = \left\{ \mathrm{diag}(x_1, \cdots, x_n) + tE_{ij} \in \mathsf{GL}_n(K) \;\middle|\; \begin{array}{l} \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n), \\ \mathrm{val}(t) - \mathrm{val}(x_i) \geqslant k \end{array} \right\}.$$

In particular, $N_\Omega = H$. Note that we also have $\widehat{N}_\Omega = H$ since Ω contains an open in $\mathbb{A}$. Therefore, $\widehat{P}_\Omega = P_\Omega$.

Next, consider $x = \varphi + \mathbf{v} \in \mathbb{A}$. Then U_x is generated by $U_{a_{ij},-a_{ij}(\mathbf{v})}$ for all $a_{ij} \in \Phi$ and $\Phi_x = \left\{ a_{ij} \in \Phi \;\middle|\; a_{ij}(\mathbf{v}) \in \Gamma \right\}$. Then W_x is generated by $r_{a,-a(\mathbf{v})}$ for all $a \in \Phi_x$ but $\widehat{W}_x$ may be larger in general: it contains $r_{a,-a(\mathbf{v})}$ even when $a(\mathbf{v}) \notin \Gamma$. Now, suppose x is special. Then $\widehat{W}_x = W_x \cong {}^{v}W$ and $\widehat{P}_x = P_x$ is generated by $H \cdot U_{a_{ij},-a_{ij}(\mathbf{v})}$ for all $a_{ij} \in \Phi$.

Example 4.2.3. In the above example, if we instead use the root group datum $(D_n \cap \mathsf{GL}_n(K)^1, (U_{a_{ij}})_{a_{ij}\in\Phi})$. Then for $\Omega = \alpha_{a_{ij}+k}$, we have

$$\widehat{P}_\Omega = P_\Omega = \left\{ \mathrm{diag}(x_1, \cdots, x_n) + tE_{ij} \in \mathsf{GL}_n(K) \;\middle|\; \begin{array}{l} \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n) = 0, \\ \mathrm{val}(t) - \mathrm{val}(x_i) \geqslant k \end{array} \right\}.$$

To see what are $\widehat{P}_\Omega$ and P_Ω in general, we need more knowledge on such subgroups.

Proposition 4.2.4 ([BT-1, 7.1.11])**.** *Let Ω be a nonempty subset of $\mathbb{A}$. Then*

$$\widehat{P}_\Omega = \bigcap_{x\in\Omega} \widehat{P}_x.$$

So in particular, $\widehat{P}_\Omega \cap \widehat{P}_{\Omega'} = \widehat{P}_{\Omega\cup\Omega'}$.

4.2.5. Let Ω be a nonempty subset of $\mathbb{A}$. The *enclosure* $\mathrm{cl}(\Omega)$ of Ω is the intersection of all affine roots α containing Ω. It turns out that [BT-1, 7.1.2 and 7.1.9]

$$U_{\mathrm{cl}(\Omega)} = U_\Omega, \qquad \widehat{N}_{\mathrm{cl}(\Omega)} = \widehat{N}_\Omega \qquad \text{and} \qquad \widehat{P}_{\mathrm{cl}(\Omega)} = \widehat{P}_\Omega.$$

From its definition, we see that $\mathrm{cl}(\Omega)$ must be a disjoint union of some facets in $\mathbb{A}$. With Proposition 4.2.4, we conclude that all the groups $\widehat{P}_\Omega$ are of the form

$$\bigcap_F \widehat{P}_F,$$

where F ranges over all facets in $\mathbb{A}$ such that $F \subseteq \overline{\mathrm{cl}(\Omega)}$.

Hence, to understand the subgroups $\widehat{P}_\Omega$, it suffices to understand those $\widehat{P}_F$.

4.2.6. Let ${}^vC$ be a Weyl chamber in $\mathbb{A}$ and $\Phi^+_{{}^vC}$ (resp. $\Phi^-_{{}^vC}$) the system of positive (resp. negative) roots in Φ defined by ${}^vC$. Let $U^+_{{}^vC}$ (resp. $U^-_{{}^vC}$) the subgroup of G generated by the U_a for $a \in \Phi^+_{{}^vC}$ (resp. $a \in \Phi^-_{{}^vC}$). Then for any $x \in \mathbb{A}$, we have $U_{x\pm{}^vC} \subseteq U^\pm_{{}^vC}$ and $\widehat{N}_{x\pm{}^vC} = N_{x\pm{}^vC} = H$. As a consequence, $\widehat{P}_{x+{}^vC} = P_{x+{}^vC}$. Denote it by $B_{x,{}^vC}$.

In general, let Ω be a nonempty subset of $\mathbb{A}$. Then we have [BT-1, 7.1.4]

$$P_\Omega \cap U^\pm_{{}^vC} = U_{\Omega\pm{}^vC} \qquad \text{and} \qquad P_\Omega = N_\Omega \cdot U_{\Omega+{}^vC} \cdot U_{\Omega-{}^vC}.$$

As a consequence, we have [BT-1, 7.1.8]

$$\widehat{P}_\Omega \cap U^\pm_{{}^vC} = U_{\Omega\pm{}^vC}, \qquad \widehat{P}_\Omega \cap N = \widehat{N}_\Omega \qquad \text{and} \qquad \widehat{P}_\Omega = \widehat{N}_\Omega \cdot U_{\Omega+{}^vC} \cdot U_{\Omega-{}^vC}.$$

Theorem 4.1 (Bruhat decomposition [BT-1, 7.3.4])**.** *Let ${}^{v}C$ and ${}^{v}C'$ be two Weyl chambers and x, x' be two points in $\mathbb{A}$.*

(i). *We have*

$$G = B_{x,{}^{v}C} \cdot N \cdot B_{x',{}^{v}C'}.$$

(ii). *More precisely, the canonical map from N to the set of double cosets induces a bijection from $\widehat{W} \cong N/H$ to $B_{x,{}^{v}C} \backslash G / B_{x',{}^{v}C'}$.*

Example 4.2.7. Continue Examples 4.1.6, 4.1.9, 4.1.12, 4.1.16, and 4.2.2. Let Ω be a nonempty subset of $\mathbb{A}$. We claim that * (with convention that $a_{ii} = 0$)

$$\widehat{P}_{\Omega} = \left\{ g = (g_{ij})_{i,j} \in \mathsf{GL}_n(K) \;\middle|\; \forall i, j : \operatorname{val}\big(g_{ij}\big) - \tfrac{1}{n} \operatorname{val}(\det(g)) \geqslant - \inf_{x \in \Omega} a_{ij}(x) \right\}.$$

Proof. Denote the right-hand side by L_{Ω}. Then it is clear that

$$L_{\Omega} = \bigcap_{x \in \Omega} L_x.$$

Therefore, it suffices to show $\widehat{P}_x = L_x$.

First, we have $H \subseteq L_x$ and for any $a_{ij} \in \Phi$,

$$L_x \cap U_{a_{ij}} = U_{a_{ij}, -a_{ij}(x)} = P_x \cap U_{a_{ij}}.$$

Therefore, $P_x \subseteq L_x$. Let ${}^{v}C$ be any Weyl chamber, then we have $B_{x,{}^{v}C} \subseteq P_x \subseteq L_x$. Hence, by Theorem 4.1, we have

$$L_x = B_{x,{}^{v}C} \cdot (L_x \cap N) \cdot B_{x,{}^{v}C}.$$

Therefore, it suffices to show $L_x \cap N = \widehat{N}_x$.

*Slightly different from [BT-1, 10.2.9] due to different conventions on the root group datum

If $m = \sum_{k=1}^{n} x_k E_{\sigma(k)k} \in L_x \cap N$ with $\sigma = {}^v\nu(m)$, then we have

$$\mathrm{val}\Big(x_{\sigma^{-1}(k)}\Big) - \frac{1}{n}\mathrm{val}(\det(m)) \geqslant -a_{k\sigma^{-1}(k)}(x) \qquad (1 \leqslant k \leqslant n).$$

This implies that

$$\sum_{k=1}^{n} \mathrm{val}(x_k) - \mathrm{val}(\det(m)) \geqslant 0,$$

which should be an equality. Therefore, for all $1 \leqslant k \leqslant n$, we have

$$\mathrm{val}\Big(x_{\sigma^{-1}(k)}^{-1}\Big) + \frac{1}{n}\mathrm{val}(\det(m)) = a_{k\sigma^{-1}(k)}(x).$$

Then, by Example 4.1.12, we have (writing x as $\varphi + \mathbf{v}$)

$$m.x - x = \sigma.\mathbf{v} - \mathbf{v} + \sum_{k=1}^{n} a_{k\sigma^{-1}(k)}(x)\lambda_k.$$

Note that

$$\sigma.\mathbf{v} - \mathbf{v} = \sum_{k=1}^{n} \langle a_{\sigma^{-1}(k)k}, \mathbf{v}\rangle \lambda_k.$$

Therefore, we have

$$m.x - x = \sum_{k=1}^{n} \Big(a_{\sigma^{-1}(k)k}(x) + a_{k\sigma^{-1}(k)}(x)\Big)\lambda_k = 0.$$

This shows $L_x \cap N \subseteq \widehat{N}_x$.

Conversely, if $m = \sum_{k=1}^{n} x_k E_{\sigma(k)k} \in \widehat{N}_x$ with $\sigma = {}^v\nu(m)$, then $m.x = x$. Which, by similar argument as above, implies

$$\mathrm{val}\Big(x_{\sigma^{-1}(k)}^{-1}\Big) + \frac{1}{n}\mathrm{val}(\det(m)) = a_{k\sigma^{-1}(k)}(x).$$

Since other entries of m are 0, the inequality holds trivially. Therefore, $m \in L_x$. □

Example 4.2.8. Consider the root group datum $(D_n \cap \mathsf{GL}_n(K)^1, (U_{a_{ij}})_{a_{ij}\in\Phi})$. Similar argument as in Example 4.2.7 shows that (note that it is simply connected)

$$P_\Omega = \left\{ (g_{ij})_{i,j} \in \mathsf{GL}_n(K)^1 \,\middle|\, \forall i,j : \mathrm{val}\big(g_{ij}\big) \geqslant -\inf_{x\in\Omega} \big(\chi_i(x) - \chi_j(x)\big) \right\}.$$

In particular, if we take Ω to be the origin $o = \varphi$, then we have

$$P_o = \left\{ (g_{ij})_{i,j} \in \mathsf{GL}_n(K)^1 \,\middle|\, \forall i,j : \mathrm{val}\big(g_{ij}\big) \geqslant 0 \right\} = \mathrm{GL}_n(\mathscr{O}_K).$$

More generally, a special point $x = \varphi + \mathbf{v}$ defines a $\mathscr{O}_K$-submodule

$$L = \{(x_1, \cdots, x_n) \in K^n \mid \mathrm{val}(x_i) + \chi_i(\mathbf{v}) \geqslant 0\}$$

of K^n, and we have $P_x = \left\{ g \in \mathsf{GL}_n(K)^1 \,\middle|\, g.L = L \right\}$.

Definition 4.2.9. The *Bruhat-Tits building* of a valuation φ on a root group datum $(T, (U_a)_{a\in\Phi})$ in G is the quotient set $\mathscr{B}(\varphi)$ of $G \times \mathbb{A}$ under the following equivalent relation [BT-1, 7.4.1]:

$$(g,x) \sim (h,y) \iff \exists n \in N : y = \nu(n).x,\ g^{-1}hn \in \widehat{P}_x.$$

We will simply denote this set by $\mathscr{B}$ if there is no ambiguity.

Remark. Let (G, T) be a split reductive group over K. By Example 4.1.10, there is essentially only one (up to equipollence) reasonable way to define a valuation on the *standard root group datum* * given in Example 4.1.3. Therefore, there is a unique affine building $\mathscr{B}(\mathsf{G})$ associated to it. It is called the *Bruhat-Tits building* of G.

Then following the same argument in 3.5.8, we see that the Bruhat-Tits building depends only on the root system Φ and the ground field K.

*as long as on root group data deduced from the standard one such as $(T \cap G^1, (U_a)_{a\in\Phi})$.

4.2.10. The left multiplication of G on the product $G \times \mathbb{A}$ is compatible with above equivalent relation, hence $\mathscr{B}$ inherits a G-action. Identifying $\mathbb{A}$ with the subset $\{1\} \times \mathbb{A}$ of $\mathscr{B}$, we have:

(i). $\mathscr{B} = \bigcup_{g \in G} g.\mathbb{A}$;

(ii). each U_α fixes $\alpha \in \Sigma$ pointwise [BT-1, 7.4.5];

(iii). for each nonempty $\Omega \subseteq \mathbb{A}$, its fixator is $\widehat{P}_\Omega$, and it acts transitively on apartments containing Ω [BT-1, 7.4.4, 7.4.9];

(iv). the stabilizer (resp. fixator) of $\mathbb{A}$ is N (resp. H) [BT-1, 7.4.10].

Then one can carry the apartment structure on $\mathbb{A}$ to each $g.\mathbb{A}$ and see that they agree on the intersections [BT-1, 7.4.18]. Hence, $\mathscr{B}$ is a building of type $\mathscr{A}(\Sigma)$. The action of G on it is strongly transitively by the construction but is not necessarily type-preserving since the affine Weyl group W of $\mathscr{A}(\Sigma)$ is usually not the entire $\widehat{W}$. The subgroup of type-preserving automorphisms is then the group $G' = \nu^{-1}(W)$ introduced in 4.1.14.

4.2.11. Let $\lambda\colon \Phi \to \mathbb{R}_{>0}$ be a function, constant on each irreducible component, and let $\mathbf{v} \in \mathbb{V}$. Then the family $u \mapsto \lambda(a)\varphi_a(u) + a(\mathbf{v})$ defines a valuation [BT-1, 6.2.5] which is denoted by $\lambda\varphi + \mathbf{v}$. A valuation ψ is said to be *equivalent* to φ if $\psi = \lambda\varphi + \mathbf{v}$ for some λ and $\mathbf{v}$. If this is the case, then there is a unique G-equivalent map $i\colon\ \mathscr{B}(\varphi) \to \mathscr{B}(\psi)$ such that its restriction to $\mathbb{A}$ is an *affinié* from $\mathbb{A} = \varphi + \mathbb{V}$ to $\psi + \mathbb{V}$ with homothetic ratio λ [BT-1, 7.4.3].

4.2.12. Let Φ_1 be a closed subroot system of Φ, N_1° be the subgroup generated by M_a° for all $a \in \Phi_1$ and let $T_1^\circ = N_1^\circ \cap T$. In addition, let T_1 be a subgroup of T containing T_1° and let G_1 be the subgroup of G generated by U_a for all $a \in \Phi_1$ and T_1. Then $(T_1, (U_a, M_a^\circ.T_1)_{a\in\Phi_1})$ is a generating root group datum on G_1 and φ induces a valuation on it. Let $\mathscr{B}_1$ be the Bruhat-Tits building associated to these data. Then the underlying set of $\mathscr{B}_1$ is canonically identified with the quotient of the subset $G_1.\mathbb{A}$ (as a G_1-set) of $\mathscr{B}$ by the intersection of the kernels of all $a \in \Phi_1$ [BT-1, 7.6.4]. The image of $\mathbb{A}$ in $\mathscr{B}_1$ is denoted by $\mathbb{A}_1$.

4.2.13. The *bornology* defined by φ is the bornology $\mathcal{B}(\varphi)$ on G induced from the action of G on the building $\mathscr{B}(\varphi)$ as in 2.6.7. It is the smallest bornology on G containing the bornology on N induced from the action of N on $\mathbb{A}$, the subgroups $U_{a,k}$ for all a and k and is compatible with the group law [BT-1, 8.1.4, 8.1.8]. This bornology makes G a bornological group and in which each $U_{a,k}$ is bounded while each U_a is not. Note that the subgroups $\widehat{P}_x$ are bounded but not necessarily maximal bounded. We refer last two paragraphs of 2.6.7 for a discussion of its bounded subgroups.

Let ψ be another valuation. Then the following are equivalent [BT-1, 8.1.10]

(i). ψ is equivalent to φ;

(ii). $\mathcal{B}(\psi) = \mathcal{B}(\varphi)$;

(iii). $\mathcal{B}(\psi)$ and $\mathcal{B}(\varphi)$ agrees on each U_a;

(iv). $\mathcal{B}(\psi)$ and $\mathcal{B}(\varphi)$ agrees on N.

Example 4.2.14. Continue Example 4.2.7. Before moving on, note that all discussions before apply if we replace $G = \mathsf{GL}_n(K)$ by a subgroup G_1 obtained as in 4.2.12. So we simply let G denote either $\mathsf{GL}_n(K)$ or such a subgroup.

The bornology $\mathcal{B}$ on G defined by φ can be described as follows: a subset M is bounded when the set

$$\left\{\mathrm{val}\big(g_{ij}\big) - \tfrac{1}{n}\,\mathrm{val}(\det(g)) \;\middle|\; g = (g_{ij}) \in M, 1 \leqslant i, j \leqslant n\right\}$$

is bounded from below. Indeed, it suffices to verify on N: for any $M \subseteq N$, the above set is $\{-\chi_i(v_m) \mid m \in M, 1 \leqslant i \leqslant n\}$, which is bounded from below if and only if it is bounded (since $\chi_1 + \cdots + \chi_n$ vanishes on $\mathbb{V}$) if and only if $M.\varphi$ is bounded.

§ 4.3. Concave functions

One important ingredient in Bruhat-Tits theory is the theory of various subgroups associated to concave functions. They are refinements of parabolic subgroups and generalizations of $\widehat{P}_*$ and P_* in previous subsection.

Definition 4.3.1 ([BT-1, 6.4.1])**.** Let's first introduce the *ordered monoid of extended real numbers* $\widetilde{\mathbb{R}}$. Formally, $\widetilde{\mathbb{R}}$ is the union of

$$\mathbb{R}, \qquad \mathbb{R}+ := \{k+ \mid k \in \mathbb{R}\} \qquad \text{and} \qquad \{\infty\}$$

The commutative addition on $\mathbb{R}$ is extended to $\widetilde{\mathbb{R}}$ as follows:

- for all $k, l \in \mathbb{R}$, $k + (l+) = (k+) + (l+) = (k + l)+$;
- for all $\lambda \in \widetilde{\mathbb{R}}$, $\lambda + \infty = \infty$.

The total order on $\mathbb{R}$ is extended to $\widetilde{\mathbb{R}}$ as follows:

- for all $k, l \in \mathbb{R}$ such that $k < l$, $k < k+ < l$;
- for all $\lambda \in \widetilde{\mathbb{R}}$ such that $\lambda \neq \infty$, $\lambda < \infty$.

Whenever we have a filtration $\{F_k\}_{k\in\mathbb{R}}$ (for instance, the filtration $\{U_{a,k}\}_{k\in\mathbb{R}}$ of a root subgroup U_a in 4.1.4), we can extend it to $\{F_\lambda\}_{\lambda\in\widetilde{\mathbb{R}}}$ by defining

$$F_\lambda = \bigcup_{k\in\mathbb{R}, k\geqslant\lambda} F_k, \qquad F_\infty = \bigcap_{k\in\mathbb{R}} F_k.$$

We say $k \in \mathbb{R}$ is a *jump* of the filtration if $F_{k+} \neq F_k$. In our most usage of filtrations, the jumps are elements of Γ. For any $\lambda \in \widetilde{\mathbb{R}}$, we use the notation $\lceil \lambda \rceil$ to denote the smallest $k \in \Gamma$ such that $\lambda \leqslant k$.

Definition 4.3.2 ([BT-1, 6.4.3; BT-2, 4.5.3])**.** Let Φ be a root system and denote $\widetilde{\Phi} = \Phi \cup \{0\}$. A *concave function on* $\widetilde{\Phi}$ is a function $f\colon \widetilde{\Phi} \to \widetilde{\mathbb{R}}$ such that

C. for any finite family (a_i) in $\widetilde{\Phi}$ such that $\sum_i a_i \in \widetilde{\Phi}$, we have

$$\sum_i f(a_i) \geqslant f(\sum_i a_i).$$

Note that the axiom is equivalent to the following:

C1. for any roots $a, b \in \Phi$ such that $a + b \in \Phi$, we have $f(a) + f(b) \geqslant f(a+b)$;

C2. for any root $a \in \Phi$, we have $f(a) + f(-a) \geqslant f(0)$;

C3. $f(0) \geqslant 0$.

A concave function f on $\widetilde{\Phi}$ is said to be a *concave function on* Φ if $f(0) = 0$ and $f(\Phi) \subseteq \mathbb{R}$. Equivalently, a concave function f on Φ is a function $f\colon \Phi \to \mathbb{R}$ satisfying **C1.** and **C2.**.

4.3.3. Let f be a concave function on $\widetilde{\Phi}$. We use U_f to denote the subgroup generated by $U_{a,f(a)}$ for all $a \in \Phi$. Given a choice of positive roots Φ^+ of Φ, we denote the intersection $U_f \cap U^+$ (resp. $U_f \cap U^-$) by U_f^+ (resp. U_f^-). Then we have the following facts [BT-1, 6.4.9].

(i). $U_f \cap U_a = U_{a,f(a)}$ for any $a \in \Phi$;

(ii). The homomorphisms

$$\prod_{a\in\Phi^+} U_{a,f(a)} \to U_f^+ \qquad \text{and} \qquad \prod_{a\in\Phi^-} U_{a,f(a)} \to U_f^-$$

are bijective regardless of the order of factors.

We refer to [BT-1, 6.4.38] for the condition of a *good filtration* $\{H_k\}_{k\geqslant 0}$ on H, under the name *prolongement de la valuation*. Note that one of the requirement is

$$H_{[0]} \subseteq H_0 \subseteq H,$$

where $H_{[0]}$ is the subgroup of H generated by $U_{a,k} \cup U_{-a,-k}$ [BT-1, 6.4.14].

We fix a *good filtration* H_k on H. Let P_f denote the subgroup $H_{f(0)} \cdot U_f$, then we have the following multiplication map:

$$\prod_{a\in\Phi^+} U_{a,f(a)} \times H_{f(0)} \times \prod_{a\in\Phi^-} U_{a,f(a)} \longrightarrow P_f. \tag{4.3.1}$$

It is injective in general and moreover bijective if $f(0) > 0$ [BT-1, 6.4.48].

Example 4.3.4 ([BT-1, 6.4.2; BT-2, 4.6.26])**.** Let Ω be a nonempty subset of $\mathbb{A}$. Define $f_\Omega \colon \Phi \to \mathbb{R} \cup \{\infty\}$ by

$$f_\Omega(a) = \inf\{k \in \mathbb{R} \mid \Omega \subseteq \alpha_{a+k}\}.$$

Then f_Ω is a concave function on Φ. We then have $U_{f_\Omega} = U_\Omega$ and the group P_{f_Ω} is a subgroup of P_Ω in general. The group P_{f_F} with F a facet in $\mathbb{A}$ is called a *parahoric subgroup*, but this terminology usually restricts to a specific choice of H_0.

Note that $f_\Omega \neq f_{\mathrm{cl}(\Omega)}$ in general, while $U_\Omega = U_{\mathrm{cl}(\Omega)}$ and $P_\Omega = P_{\mathrm{cl}(\Omega)}$.

Example 4.3.5. To make above more clear, let's consider the split reductive group $(\mathsf{GL}_n, \mathsf{D}_n)$ and refer Examples 4.1.6, 4.1.9, 4.1.12, 4.1.16, 4.2.2, 4.2.7, and 4.2.14. Then

we have

$$U_{a_{ij},k} = \left\{ \xi_{ij}\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) \in U_{a_{ij}} \;\middle|\; \mathrm{val}(t) \geqslant k \right\}.$$

$$H = \{\mathrm{diag}(x_1, \cdots, x_n) \in \mathsf{GL}_n \mid \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n)\},$$

$$H_{[0]} = \{\mathrm{diag}(x_1, \cdots, x_n) \in \mathsf{SL}_n \mid \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n) = 0\}.$$

We will also consider

$$H^\circ := \{\mathrm{diag}(x_1, \cdots, x_n) \in \mathsf{GL}_n \mid \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n) = 0\}.$$

Now, let $\Omega = \alpha_{a_{ij}+k}$. Then we have

$$U_\Omega = U_{\alpha_{a_{ij},k}} = U_{\alpha_{a_{ij},\lceil k\rceil}} = U_{\alpha_{a_{ij}+\lceil k\rceil}} = U_{\mathrm{cl}(\Omega)},$$

$$P_\Omega = \left\{ \mathrm{diag}(x_1, \cdots, x_n) + tE_{ij} \in \mathsf{GL}_n(K) \;\middle|\; \begin{array}{l} \mathrm{val}(x_1) = \cdots = \mathrm{val}(x_n), \\ \mathrm{val}(t) - \mathrm{val}(x_i) \geqslant k \end{array} \right\}.$$

On the other hand, the concave functions $f_\Omega, f_{\mathrm{cl}(\Omega)}$ are

$$f_\Omega(a) = \begin{cases} k & \text{if } a = a_{ij}, \\ \infty & \text{if } a \neq a_{ij}. \end{cases} \qquad f_{\mathrm{cl}(\Omega)}(a) = \begin{cases} \lceil k\rceil & \text{if } a = a_{ij}, \\ \infty & \text{if } a \neq a_{ij}. \end{cases}$$

Therefore, $f_\Omega \neq f_{\mathrm{cl}(\Omega)}$ in general, while $U_{f_\Omega} = U_{a_{ij},k} = U_\Omega$. It is then clear that $P_{f_\Omega} = P_\Omega$ if we take $H_0 = H$. But for other choices, namely $H = H^\circ$ or $H_{[0]}$, we have

$$P_{f_\Omega} = \left\{ \mathbf{h} + tE_{ij} \in \mathsf{GL}_n(K) \;\middle|\; \mathbf{h} \in H, \mathrm{val}(t) \geqslant k \right\}.$$

Example 4.3.6. Continue Example 4.3.5 with a general nonempty subset Ω. First note that $H \cap \mathsf{GL}_n(K)^1 = H^\circ$. Hence, using Example 4.2.8, we have

$$\begin{aligned} P_{f_\Omega} &= \left\{ (g_{ij})_{i,j} \in G \,\middle|\, \forall i,j : \operatorname{val}\big(g_{ij}\big) \geqslant -\inf_{x \in \Omega} \big(\chi_i(x) - \chi_j(x)\big) \right\} \\ &= \left\{ (g_{ij})_{i,j} \in G \,\middle|\, \forall i,j : \operatorname{val}\big(g_{ij}\big) \geqslant f_\Omega \big(\chi_i(x) - \chi_j(x)\big) \right\}, \end{aligned}$$

where $G = \mathsf{GL}_n(K)^1$ if we take $H_0 = H^\circ$ and $G = \mathsf{SL}_n(K)$ if we take $H_0 = H_{[0]}$.

4.3.7 ([BT-1, 6.4.10; BT-2, 4.5.2, 4.6.12])**.** Let f be a concave function on Φ. Define f' as follows:

$$f'(a) := \inf\{k \in \Gamma_a \mid k \geqslant f(a)\}.$$

Then f' is also a concave function on Φ, called the *optimization* of f. If $f' = f$, we say f is *optimal*. Note that under the assumption that $\Gamma_a = \Gamma$ for all $a \in \Phi$, we have $f'(a) = \lceil f(a) \rceil$.

The set of roots $a \in \Phi$ such that $f'(a) + f'(-a) = 0$ is denoted by Φ_f, called the *root system* associated to f.

Remark. Note that for any $a \in \Phi$, we have

$$f_\Omega(a) + f_\Omega(-a) = -\inf_{x \in \Omega} a(x) - \inf_{x \in \Omega}(-a(x)) = \sup_{x \in \Omega} a(x) - \inf_{x \in \Omega} a(x) \geqslant 0.$$

The equality holds if and only if $a(x)$ is a constant for $x \in \Omega$. Note that this constant may not be contained within Γ, hence the condition merely says that Ω is contained in a hyperplane parallel to the wall $\partial\alpha_{a+0}$. On the other side, from the definition of optimizes, $a \in \Phi_{f_\Omega}$ can be interpreted as $\Omega \subseteq \partial\alpha$ for some affine root α with vectorial part a. Therefore, $\Phi_{f_\Omega} = \Phi_\Omega$. Another way to see this is use the observation that $f'_\Omega = f_{\mathrm{cl}(\Omega)}$.

4.3.8 ([BT-1, 6.4.23; BT-2, 4.6.9])**.** Let f be a concave function on Φ. Define $f^* : \widetilde{\Phi} \to \widetilde{\mathbb{R}}$

as follows:

$$f^*(a) := \begin{cases} f(a) & \text{if } f(a) + f(-a) > 0, \\ f(a)+ & \text{if } f(a) + f(-a) = 0. \end{cases}$$

Then f^* is a concave function.

Let $\overline{G}_f$ denote the quotient P_f/P_{f^*} and let $\overline{U}_{f;a}$ (resp. $\overline{T}_f$) be the image of $U_{a,f(a)}$ (resp. $H_{f(0)}$) in $\overline{G}_f$. Then $(\overline{T}_f, (\overline{U}_{f;a})_{a\in\Phi_f})$ is a generating root group datum of type Φ_f on $\overline{G}_f$.

Example 4.3.9. In Example 4.3.5, we have $f^*_{\alpha_{a_{ij}+k}} = f_{\alpha_{a_{ij}+k}}$. Hence, $\overline{G}_{f_{\alpha_{a_{ij}+k}}}$ is the trivial group.

Example 4.3.10. Assume $\Omega = \mathrm{cl}(\Omega)$ in Example 4.3.6. Hence, f_Ω is optimal. Then

$$f^*_\Omega(a) = \begin{cases} f(a) & \text{if } a \notin \Phi_\Omega, \\ f(a)+ & \text{if } a \in \Phi_\Omega. \end{cases}$$

Take H_0 to be H°. Then the group $P_{f^*_\Omega}$ can be computed using Eq. (4.3.1):

$$P_{f^*_\Omega} = I_n + \left\{(g_{ij})_{i,j} \in \mathsf{GL}_n(\mathscr{O}_K) \;\middle|\; \forall i,j : \mathrm{val}(g_{ij}) \geqslant f^*_\Omega\left(\chi_i(x) - \chi_j(x)\right)\right\}.$$

In particular, if we take Ω to be the origin o, then we have

$$P_{f^*_o} = I_n + \varpi\, \mathsf{M}_{n\times n}(\mathscr{O}_K).$$

Therefore, $P_{f_o}/P_{f^*_o}$ is nothing other than $\mathsf{GL}_n(\kappa)$.

Now suppose Ω contains o, then for any $a \in \Phi$, either $f_\Omega(a) = 0$ or $f_\Omega(-a) = 0$. Then $\Psi_\Omega := \{a \in \Phi \mid f_\Omega(a) = 0\}$ is a parabolic subset. We can thus choose a system of positive roots Φ^+ such that $\Phi^+ \subseteq \Psi_\Omega$. Hence, we may assume $a_{ij}(x) \geqslant 0$ for all

$1 \leqslant i < j \leqslant n$. Then we have (identified as subgroups of $\mathsf{GL}_n(\kappa)$):

$$P_{f_\Omega}/(P_{f_o^*} \cap P_{f_\Omega}) = \left\{(g_{ij})_{i,j} \in \mathsf{GL}_n(\kappa) \,\middle|\, \forall i,j : a_{ij} \notin \Psi_\Omega \implies g_{ij} = 0\right\},$$
$$P_{f_\Omega}/P_{f_\Omega^*} = \left\{(g_{ij})_{i,j} \in \mathsf{GL}_n(\kappa) \,\middle|\, \forall i,j : a_{ij} \notin \Phi_\Omega \implies g_{ij} = 0\right\}.$$

Note that: through the above identification, $P_{f_\Omega}/(P_{f_o^*} \cap P_{f_\Omega})$ is (the group of κ-points of) a parabolic subgroup P_{I_Ω} of $\mathsf{GL}_{n,\kappa}$ and $P_{f_\Omega}/P_{f_\Omega^*}$ is (the group of κ-points of) its Levi subgroup L_{I_Ω}, where the type I_Ω is defined by the parabolic subset Ψ_Ω.

§ 4.4. Smooth models associated to concave functions

In this section, we will take (G, T) to be a split reductive group and $G = \mathsf{G}(K)$. Furthermore, we assume K is a *Henselian field* in the sense that $\mathscr{O}_K$ is a *Henselian ring*. We also assume that the residue field κ is perfect.

The second part [BT-2] of Bruhat-Tits theory says that there are more algebraic-geometric structures on its Bruhat-Tits building. We follow [Yu15] to state such result and deduce some properties which will be used later. We emphasize that since we focus on split reductive group only, a lot of difficulties vanish. However, we still keep the general statement unless we turn to specific examples.

4.4.1. Let $\mathfrak{T}^{\mathrm{NR}}$ denote the *Néron-Raynaud model* [CY01, 3.1] of the torus T, namely the neutral component of the standard *lft Néron model* [BLR90, 10.1.1]. In our case, since T is split, its Néron-Raynaud model $\mathfrak{T}^{\mathrm{NR}}$ can be characterized as the connected smooth model such that $\mathfrak{T}^{\mathrm{NR}}(\mathscr{O}_K)$ equals the subgroup [BLR90, 10.1.5]

$$H^\circ := \{\mathbf{t} \in \mathsf{T}(K) \mid \mathrm{val}(\chi(\mathbf{t})) = 0, \text{ for all } \chi \in \mathsf{X}(\mathsf{T})\}.$$

The *Moy-Prasad filtration* $\{\mathsf{T}(K)_k\}_{k\geqslant 0}$ [Yu15, 4.2; MP96, 3.2] is defined as:

$$\mathsf{T}(K)_k := \{\mathbf{t} \in H^\circ \mid \mathrm{val}(\chi(\mathbf{t}) - 1) \geqslant k, \text{ for all } \chi \in \mathsf{X}(\mathsf{T})\}.$$

It defines a good filtration $H_k := \mathsf{T}(K)_k$ on H. There is also a *Moy-Prasad filtration* $\{\mathfrak{t}_k\}_{k\geqslant 0}$ of the Lie algebra $\mathfrak{t} = \mathrm{Lie}(\mathsf{T})$:

$$\mathfrak{t}_k := \{T \in \mathfrak{t} \mid \mathrm{val}(\mathrm{d}\,\chi(T)) \geqslant k, \text{ for all } \chi \in \mathsf{X}(\mathsf{T})\}.$$

There is a family $\{\mathfrak{T}_k\}_{k\geqslant 0}$ of connected smooth models of T such that [Yu15, §4]:

(i). $\mathfrak{T}_k(\mathcal{O}_K) = \mathsf{T}(K)_k$;

(ii). the special fiber $(\mathfrak{T}_k)_\kappa$ is unipotent for all $k > 0$;

(iii). the congruence subgroup

$$\Gamma(\varpi^m, \mathfrak{T}_k) := \mathrm{Ker}(\mathfrak{T}_k(\mathcal{O}_K) \longrightarrow \mathfrak{T}_k(\mathcal{O}_K/\varpi^m))$$

equals $\mathsf{T}(K)_{k+m\gamma}$ for all $m \geqslant 0$;

(iv). the Lie algebra of $\mathfrak{T}_k$ equals $\mathfrak{t}_k$.

Remark. The scheme $\mathfrak{T}_k$ is constructed as follows [Yu15, 4.5]. First, consider the *higher unit group*

$$\mathbb{G}_\mathrm{m}(\mathcal{O}_K)_k := \{1 + t \in \mathbb{G}_\mathrm{m}(\mathcal{O}_K) \mid \mathrm{val}(t) \geqslant k\}.$$

It admits a smooth model of $\mathbb{G}_\mathrm{m}/K$ via a *dilatation* [BLR90, §3.2] in the Néron-Raynaud model $\mathbb{G}_\mathrm{m}/\mathcal{O}_K$.

$$(\mathbb{G}_\mathrm{m}/\mathcal{O}_K)^{(k)} := \mathrm{Spec}\left(\mathcal{O}_K\left[\frac{X-1}{\varpi^{\frac{\lceil k \rceil}{\gamma}}}, \frac{X^{-1}-1}{\varpi^{\frac{\lceil k \rceil}{\gamma}}}\right]\right).$$

Then $\mathfrak{T}_k$ can be obtained by extension the isomorphism

$$(\mathbb{G}_\mathrm{m}(\mathcal{O}_K)_k)^n \xrightarrow{\sim} \mathsf{T}(K)_k$$

to above smooth model.

Remark. The good filtration H_k can be taken fairly general, depending on which model of T to use. See [BT-2, §4.4] for discussion on models $\mathfrak{T}_0$ of T and [Yu15, §4 and §5] for discussion on the filtrations H_k and $\mathfrak{T}_k$.

4.4.2 ([Yu15, 6.2; BT-2, §4.3])**.** For each root subgroup U_a, the filtration $\{U_{a,k}\}_{k\in\mathbb{R}}$ extends to a family $\{\mathfrak{U}_{a,k}\}_{k\in\mathbb{R}}$ of connected smooth models of U_a such that:

(i). $\mathfrak{U}_{a,k}(\mathscr{O}_K) = U_{a,k}$;

(ii). the special fiber $(\mathfrak{U}_{a,k})_\kappa$ is unipotent for all k;

(iii). the congruence subgroup

$$\Gamma(\varpi^m, \mathfrak{U}_{a,k}) := \operatorname{Ker}\big(\mathfrak{U}_{a,k}(\mathscr{O}_K) \to \mathfrak{U}_{a,k}(\mathscr{O}_K/\varpi^m)\big)$$

equals $U_{a,k+m\gamma}$ for all $m \geqslant 0$;

(iv). the Lie algebras $\mathfrak{u}_{a,k}$ of $\mathfrak{U}_{a,k}$ form a filtration on the Lie algebra $\mathfrak{u}_a$ of U_a.

Remark. With our assumption, the scheme $\mathfrak{U}_{a,k}$ can be obtained by extending the isomorphism of one-dimensional free $\mathscr{O}_K$-modules*

$$K_k := \{x \in K \mid \operatorname{val}(x) \geqslant k\} \xrightarrow{u_a} U_{a,k}$$

to an isomorphism of vectorial $\mathscr{O}_K$-group schemes

$$u_a \colon \mathbb{W}_{\mathscr{O}_K}(K_k) \xrightarrow{\sim} \mathfrak{U}_{a,k}\,.$$

*This is not the case in general where U_a is merely split unipotent, not necessary vectorial. However, the scheme $\mathfrak{U}_{a,k}$ is still constructed explicitly. See [BT-2, §4.3] for details.

In particular, we have

$$\mathfrak{U}_{a,k}(\mathcal{O}_K/\varpi^m) \cong K_{\geqslant k} \otimes \mathcal{O}_K/\varpi^m \tag{4.4.1}$$

compatible with the filtrations. In particular, $\mathfrak{U}_{a,k}(\mathcal{O}_K/\varpi^m)$ is an m-dimensional vector space over κ.

At this stage, we have group schemes $\mathfrak{T}_k$ and $\mathfrak{U}_{a,k}$ ($a \in \Phi$). Such a datum is basically Bruhat-Tits' *schematic root group datum* [BT-2, 3.1.1].

The main theorem of the *schematic Bruhat-Tits theory* is

Theorem 4.2 ([Yu15, 8.3; BT-2, §4.6])**.** *Fix a choice of the schematic root group datum $(\mathfrak{T}_k, (\mathfrak{U}_{a,k})_{a\in\Phi})$. For a concave function f on $\widetilde{\Phi}$, there is a connected smooth model $\mathfrak{G}_f$ of G such that $\mathfrak{G}_f(\mathcal{O}_K) = P_f$. Moreover:*

(i). *The schematic closure of T in $\mathfrak{G}_f$ is $\mathfrak{T}_{f(0)}$.*

(ii). *For each $a \in \Phi$, the schematic closure of U_a in $\mathfrak{G}_f$ is $\mathfrak{U}_{a,f(a)}$.*

(iii). *The multiplication morphism (the products can be taken in any order)*

$$\prod_{a\in\Phi_f^+} \mathfrak{U}_{a,f(a)} \cdot \mathfrak{T}_{f(0)} \cdot \prod_{a\in\Phi_f^-} \mathfrak{U}_{a,f(a)} \longrightarrow \mathfrak{G}_f \tag{4.4.2}$$

is an open immersion. If $f(0) > 0$, it induces an isomorphism on special fibers.

Remark. Note that Eq. (4.4.2) actually gives a bijection on $\mathcal{O}_K$-points (and more generally, $\mathcal{O}_K/I$-points for any ideal I) using the Henselian property.

Definition 4.4.3 ([BT-2, 5.2.6])**.** Let $(\mathfrak{T}_k, (\mathfrak{U}_{a,k})_{a\in\Phi})$ be as in 4.4.1 and 4.4.2 and $f = f_F$ for some facet F. The group P_{f_F} is called the *parahoric subgroup* of G. When F is an alcove, it is called an *Iwahori subgroup*. A parahoric subgroup P_{f_F} is also called the *connected stabilizer* of F in the sense that $\mathfrak{G}_f$ is a connected group scheme and P_{f_F} is the largest subgroup of $\widehat{P}_F$ having this property.

Example 4.4.4. In Examples 4.3.6 and 4.3.10, the smooth model $\mathfrak{G}_{f_o}$ can be taken as

$$\mathfrak{G}_{f_o} : R \longmapsto \mathsf{GL}_n(R).$$

Indeed, in this case, we have $\mathfrak{T}_{f(0)}(R) = \mathsf{D}_n(R)$ and

$$\mathfrak{U}_{a_{ij}, f_o(a_{ij})}(R) = \left\{ I_n + rE_{ij} \,\middle|\, r \in R \right\}.$$

Remark. More generally, let (G, T) be a split reductive group and x be a special point in its Bruhat-Tits building. Then the smooth model $\mathfrak{G}_{f_x}$ is the bare bone of the *Chevalley group scheme* [BT-2, §3.2 and 4.6.15; SGA3, XXV]: it says that for the reduced root datum $\mathcal{R}(\mathsf{G}, \mathsf{T})$, there is a smooth affine group scheme $\mathfrak{G}$ over $\mathbb{Z}$ such that for any field F, $\mathfrak{G}_F$ is a split reductive group with root datum $\mathcal{R}(\mathsf{G}, \mathsf{T})$ over F.

Definition 4.4.5. Let $(\mathfrak{T}_k, (\mathfrak{U}_{a,k})_{a\in\Phi})$ be as in 4.4.1 and 4.4.2 and $f = f_F$ for some facet F. The group P_{f_F+r} is called the *Moy-Prasad subgroup* of the *parahoric subgroup* P_{f_F}. When F is a vertex x, we will use $P_{x,r}$ to denote this group.

Chapter 5.

Incidence Geometry of Lattices

In this chapter, we will review the incidence-geometric descriptions of Tits buildings and Bruhat-Tits buildings of classical groups following [Gar97]. In which, a building is determined by the following data:

(i). The set of *vertices* $\mathcal{V}$.

(ii). The *incidence relation* $\sim$ on $\mathcal{V}$.

(iii). The set of *frames* $\mathcal{A}$.

Given the set $\mathcal{V}$ and the relation $\sim$, we can obtain a simplicial complex $\mathcal{F}$, called its *flag complex*, in which simplices are precisely the mutually incident subsets of $\mathcal{V}$. Each $\mathbf{\Lambda} \in \mathcal{A}$ specifies a subset $\mathcal{V}_{\mathbf{\Lambda}}$ of $\mathcal{V}$, and hence a subcomplex $\mathcal{F}_{\mathbf{\Lambda}}$ of $\mathcal{F}$. One can then verify that $\mathcal{F}$ is an abstract building with apartments $(\mathcal{F}_{\mathbf{\Lambda}})_{\mathbf{\Lambda}\in\mathcal{A}}$. We thus use the same notation $\mathcal{A}$ for both the set of frames and the set of apartments. To relate this abstract building to a classical group, one also needs to define how the group acts on it. Finally, a certain geometric realization of above identify this abstract building to the desired Euclidean building.

§ 5.1. Tits buildings of classical groups

We first consider Tits buildings. During this section, K is a field, not necessarily valued.

5.1.1 ([Gar97, chap.9])**.** The Tits building of type A_n is essentially given in § 2.1 and 2.2. Let V be a K-vector space of dimension $n+1$. Consider the following data:

(i). The *vertex* set $\mathcal{V}$ consists of proper, non-trivial vector subspaces of V.

(ii). The *incidence relation* is: $x \sim y$ for $x, y \in \mathcal{V}$ whenever $x \subseteq y$ or $x \supseteq y$.

(iii). A *frame* is an unordered tuple of lines in V

$$\mathbf{\Lambda} = \{\lambda_1, \cdots, \lambda_{n+1}\}$$

such that $\lambda_1 + \cdots + \lambda_{n+1} = \lambda_1 \oplus \cdots \oplus \lambda_{n+1} = V$.

We say $\mathbf{\Lambda}$ *splits* a subspace W of V if W can be expressed as a sum of members of $\mathbf{\Lambda}$. Define the subset $\mathcal{V}_{\mathbf{\Lambda}}$ as

$$\mathcal{V}_{\mathbf{\Lambda}} := \{x \in \mathcal{V} \mid \mathbf{\Lambda} \text{ splits } x\}.$$

We thus obtain an abstract building $\mathcal{F}$. The action of $G = \mathsf{GL}(V, K)$ on it is clear: $g \in G$ maps any proper, non-trivial subspace W of V to a proper, non-trivial subspace $g.W$.

Any frame $\mathbf{\Lambda}$ specifies a maximal torus $\mathsf{T}(\mathbf{\Lambda})$ of $\mathsf{GL}(V)$: it consists of the linear transformations with eigenspaces λ_i. The simplicial complex $\mathcal{F}_{\mathbf{\Lambda}}$ is identified with the complex of facets in ${}^v\!\mathscr{A}(\mathsf{GL}(V), \mathsf{T}(\mathbf{\Lambda}))$ as follows. First, each chamber in $\mathcal{F}_{\mathbf{\Lambda}}$ is a maximal flag of subspaces of V:

$$V = \sum_{i=1}^{n+1} \lambda_{\sigma(i)} \supsetneq \sum_{i=1}^{n} \lambda_{\sigma(i)} \supsetneq \cdots \supsetneq \lambda_{\sigma(1)} \supsetneq 0,$$

where $(\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(n+1)})$ is a labeling of members of $\mathbf{\Lambda}$. Such labeling are indexed by the symmetric group $\mathfrak{S}_{n+1}$, i.e. the linear Weyl group of type A_n. Let $\mathsf{B}(\mathbf{\Lambda}_\sigma)$ be the stabilizer of the above flag. Then $\mathsf{B}(\mathbf{\Lambda}_\sigma)$ is a Borel subgroup of $\mathsf{GL}(V)$ containing $\mathsf{T}(\mathbf{\Lambda})$. Furthermore, for any subflag, its stabilizer is a parabolic subgroup of $\mathsf{GL}(V)$ containing $\mathsf{B}(\mathbf{\Lambda}_\sigma)$. Therefore, by Proposition 3.5.2, we obtain a morphism of simplicial complexes, which turns out to be an isomorphism.

The actions of $g \in G$ on $\mathcal{F}$ and ${}^v\mathscr{B}(\mathsf{GL}(V))$ give us the following commutative diagram of abstract simplicial complexes:

$$\begin{tikzcd}
\mathcal{F}_{\mathbf{\Lambda}} \arrow[r, "\sim"] \arrow[d, "g"'] & {}^v\mathscr{A}(\mathsf{GL}(V), \mathsf{T}(\mathbf{\Lambda})) \arrow[d, "g"] \\
\mathcal{F}_{g.\mathbf{\Lambda}} \arrow[r, "\sim"] & {}^v\mathscr{A}(\mathsf{GL}(V), \mathsf{T}(\mathbf{\Lambda})^g)
\end{tikzcd}$$

Hence the abstract building $\mathcal{F}$ is isomorphic to the Tits building ${}^v\mathscr{B}(\mathsf{GL}(V))$.

Remark. Let D be a division algebra over K. One can replace K by D in the above argument. This gives us the Tits building of non-split reductive groups $\mathsf{GL}_D(V)$.

5.1.2 ([Gar97, chap.10])**.** The Tits building of type C_n (or B_n) can be obtained similarly. Following Example 3.1.7.(ii), let D be a division algebra over K with an involution σ such that K is precisely the σ-fixed-point subfield of the center of D. Let V be a hermitian space over D. Let's fix the following terminology [Gar97, chap.7]:

- Two vectors $v, w \in V$ are *orthogonal* to each other, denoted by $v \perp w$, if $\langle v \mid w \rangle = 0$. Two subspaces W, W' are *orthogonal* to each other, denoted by $W \perp W'$, if $\langle \cdot \mid \cdot \rangle$ vanishes on $W \times W'$. A direct sum of subspaces is an *orthogonal sum* if each summand is orthogonal to others.

- A vector $v \in V$ is *isotropic* if $w \perp w$. A subspace W of V is *totally isotropic* if $W \perp W$.
- A subspace W of V is *anisotropic* if any non-zero vector in W is not isotropic.
- A *hyperbolic plane* in V is a two-dimensional subspace admitting a basis e_+, e_- such that

$$\langle e_+ \mid e_+ \rangle = \langle e_- \mid e_- \rangle = 0 \qquad \text{and} \qquad \langle e_+ \mid e_- \rangle = 1.$$

- A *hyperbolic space* is an orthogonal sum of hyperbolic planes. It turns out that V can be written as an orthogonal sum of a hyperbolic subspace and an anisotropic subspace. Although such a decomposition is not unique, their dimensions are invariants of the space V.

To obtain a building of type C_n, let's assume the dimension of maximal totally isotropic subspace of V is n and exclude the case when V is a hyperbolic space with a symmetric form $\langle \cdot \mid \cdot \rangle$. Consider the following data:

(i). The *vertex* set $\mathcal{V}$ consists of non-trivial totally isotropic subspaces of V.

(ii). The *incidence relation* is: $x \sim y$ for $x, y \in \mathcal{V}$ whenever $x \subseteq y$ or $x \supseteq y$.

(iii). A *(hyperbolic) frame* is an unordered $2n$-tuple of lines in V

$$\mathbf{\Lambda} = \left\{\lambda_1^+, \lambda_1^-, \cdots, \lambda_n^+, \lambda_n^-\right\}$$

which admit grouping into unordered pairs $\left\{\lambda_i^+, \lambda_i^-\right\}$ such that each $\lambda_i^+ + \lambda_i^-$ is a hyperbolic plane H_i in V and that $H_\bullet := H_1 + \cdots + H_n$ is an orthogonal sum. Then the whole space V can be written as an orthogonal sum of $H_\bullet$ and an anisotropic subspace A.

Given a frame $\mathbf{\Lambda}$, one can see that any totally isotropic subspace of V can be expressed as a sum of members of it. Define the subset $\mathcal{V}_{\mathbf{\Lambda}}$ as

$$\mathcal{V}_{\mathbf{\Lambda}} := \{x \in \mathcal{V} \mid \mathbf{\Lambda} \text{ splits } x\}.$$

We thus obtain an abstract building $\mathcal{F}$. The action of $G = \mathsf{O}(V, K)$ on it is clear: $g \in G$ maps any non-trivial totally isotropic subspace W of V to a non-trivial totally isotropic subspace $g.W$.

A hyperbolic frame $\mathbf{\Lambda}$ specifies a maximal torus $\mathsf{T}(\mathbf{\Lambda})$ of $\mathsf{O}(V)$: it is the subgroup of linear transformations with eigenspaces $\lambda_i^{\pm}$ and A. The simplicial complex $\mathcal{F}_{\mathbf{\Lambda}}$ is identified with the complex of facets in ${}^{v}\!\mathscr{A}(\mathsf{O}(V), \mathsf{T}(\mathbf{\Lambda}))$ as follows. First, each chamber in $\mathcal{F}_{\mathbf{\Lambda}}$ is a maximal flag of totally isotropic subspaces of V:

$$\sum_{i=1}^{n} \lambda_{\sigma(i)}^{\epsilon_i} \supsetneq \sum_{i=1}^{n-1} \lambda_{\sigma(i)}^{\epsilon_i} \supsetneq \cdots \supsetneq \lambda_{\sigma(1)}^{\epsilon_1} \supsetneq 0,$$

where $\sigma \in \mathfrak{S}_n$ and each $\epsilon_i \in \{+, -\}$. We should think this is a labeling of members of the frame $\mathbf{\Lambda}$ respecting its grouping. That is to say, we have a labeling $(H_{\sigma(1)}, \cdots, H_{\sigma(n)})$ of the hyperbolic planes, indexed by the symmetric group $\mathfrak{S}_n$, and then in each $H_{\sigma(i)}$, specify $\lambda_{\sigma(i)}^{\epsilon_i}$ from the unordered pair $\left\{\lambda_{\sigma(i)}^{+}, \lambda_{\sigma(i)}^{-}\right\}$. Therefore, the labeling is indexed by $\mathfrak{S}_n \rtimes \{\pm\}^n$, i.e. the linear Weyl group of type C_n. Let $\mathsf{B}(\mathbf{\Lambda}_{\sigma}^{\epsilon})$ be the stabilizer of the above flag. Then $\mathsf{B}(\mathbf{\Lambda}_{\sigma}^{\epsilon})$ is a Borel subgroup of $\mathsf{O}(V)$ containing $\mathsf{T}(\mathbf{\Lambda})$. Furthermore, for any subflag, its stabilizer is a parabolic subgroup of $\mathsf{O}(V)$ containing $\mathsf{B}(\mathbf{\Lambda}_{\sigma}^{\epsilon})$. Therefore, by Proposition 3.5.2, we obtain a morphism of simplicial complexes, which turns out to be an isomorphism.

The actions of $g \in G$ on $\mathcal{F}$ and ${}^{v}\!\mathscr{B}(\mathsf{O}(V))$ give us the following commutative diagram of abstract simplicial complexes:

$$\begin{tikzcd}
\mathcal{F}_{\mathbf{\Lambda}} \arrow[r, "\sim"] \arrow[d, "g"'] & {}^{v}\!\mathscr{A}(\mathsf{O}(V), \mathsf{T}(\mathbf{\Lambda})) \arrow[d, "g"] \\
\mathcal{F}_{g.\mathbf{\Lambda}} \arrow[r, "\sim"] & {}^{v}\!\mathscr{A}(\mathsf{O}(V), \mathsf{T}(\mathbf{\Lambda})^{g})
\end{tikzcd}$$

Hence the abstract building $\mathcal{F}$ is isomorphic to the Tits building ${}^{v}\!\mathscr{B}(\mathsf{O}(V))$.

5.1.3 ([Gar97, chap.11])**.** The Tits building of type D_n needs a specific construction. Let V be a $2n$-dimensional hyperbolic space with a symmetric form $\langle \cdot \mid \cdot \rangle$. Consider the following data:

(i). The *vertex* set $\mathcal{V}$ consists of non-trivial totally isotropic subspaces of V with dimension not $n-1$.

(ii). The *incidence relation* is: $x \sim y$ for $x, y \in \mathcal{V}$ whenever $x \subseteq y$ or $x \supseteq y$ or both x, y are n-dimensional and $x \cap y$ has dimension $n-1$.

(iii). A *(hyperbolic) frame* is an unordered $2n$-tuple of lines in V

$$\mathbf{\Lambda} = \left\{\lambda_1^+, \lambda_1^-, \cdots, \lambda_n^+, \lambda_n^-\right\}$$

which admit grouping into unordered pairs $\left\{\lambda_i^+, \lambda_i^-\right\}$ such that each $\lambda_i^+ + \lambda_i^-$ is a hyperbolic plane H_i in V and that $H_1 + \cdots + H_n = V$ is an orthogonal sum.

Defining $\mathcal{V}_{\mathbf{\Lambda}}$ as before, we obtain an abstract building $\mathcal{F}$. The action of $G = \mathsf{SO}(V, K)$ on it is clear: $g \in G$ maps any non-trivial totally isotropic subspace W of V to a non-trivial totally isotropic subspace $g.W$ with the same dimension.

A hyperbolic frame $\mathbf{\Lambda}$ specifies a maximal torus $\mathsf{T}(\mathbf{\Lambda})$ of $\mathsf{SO}(V)$: it is the subgroup of linear transformations with eigenspaces $\lambda_i^{\pm}$. The simplicial complex $\mathcal{F}_{\mathbf{\Lambda}}$ is identified with the complex of facets in ${}^{v}\!\mathscr{A}(\mathsf{O}(V), \mathsf{T}(\mathbf{\Lambda}))$ as follows. First, each chamber in $\mathcal{F}_{\mathbf{\Lambda}}$ is now not a maximal flag but a maximal *oriflamme* of totally isotropic subspaces of V:

$$\sum_{i=1}^{n} \lambda_{\sigma(i)}^{\epsilon_i}, \sum_{i=1}^{n-1} \lambda_{\sigma(i)}^{\epsilon_i} + \lambda_{\sigma(n)}^{-\epsilon_n} \supsetneq \sum_{i=1}^{n-2} \lambda_{\sigma(i)}^{\epsilon_i} \supsetneq \cdots \supsetneq \lambda_{\sigma(1)}^{\epsilon_1} \supsetneq 0,$$

where $\sigma \in \mathfrak{S}_n$ and each $\epsilon_i \in \{+,-\}$. Then point is that there is no priority between the maximal totally isotropic subspaces $\sum_{i=1}^{n} \lambda_{\sigma(i)}^{\epsilon_i}$ and $\sum_{i=1}^{n-1} \lambda_{\sigma(i)}^{\epsilon_i} + \lambda_{\sigma(n)}^{-\epsilon_n}$ and that $\sum_{i=1}^{n-2} \lambda_{\sigma(i)}^{\epsilon_i}$ lives in their intersection, a totally isotropic subspace of dimension $n-1$. Hence, if we want to think such an oriflamme as a labeling of members of the frame $\mathbf{\Lambda}$, we need to exclude the specification of the last $\lambda_{\sigma(n)}^{\epsilon_n}$. Consequently, the labeling is indexed by $\mathfrak{S}_n \times \{\pm\}^{n-1}$, i.e. the linear Weyl group of type D_n. Let $\mathsf{B}(\mathbf{\Lambda}_\sigma^\epsilon)$ be the stabilizer of the above oriflamme. Then $\mathsf{B}(\mathbf{\Lambda}_\sigma^\epsilon)$ is a Borel subgroup of $\mathsf{SO}(V)$ containing $\mathsf{T}(\mathbf{\Lambda})$. Furthermore, for any sub-oriflamme, its stabilizer is a parabolic subgroup of $\mathsf{SO}(V)$ containing $\mathsf{B}(\mathbf{\Lambda}_\sigma^\epsilon)$. Therefore, by Proposition 3.5.2, we obtain a morphism of simplicial complexes, which turns out to be an isomorphism.

The actions of $g \in G$ on $\mathcal{F}$ and ${}^v\mathscr{B}(\mathsf{SO}(V))$ give us the following commutative diagram of abstract simplicial complexes:

$$\begin{tikzcd} \mathcal{F}_{\mathbf{\Lambda}} \arrow[r, "\sim"] \arrow[d, "g"'] & {}^v\mathscr{A}(\mathsf{SO}(V), \mathsf{T}(\mathbf{\Lambda})) \arrow[d, "g"] \\ \mathcal{F}_{g.\mathbf{\Lambda}} \arrow[r, "\sim"] & {}^v\mathscr{A}(\mathsf{SO}(V), \mathsf{T}(\mathbf{\Lambda})^g) \end{tikzcd}$$

Hence the abstract building $\mathcal{F}$ is isomorphic to the Tits building ${}^v\mathscr{B}(\mathsf{SO}(V))$.

§ 5.2. Lattices and norms

In this section, we follow [Gar97, chap.19] to describe the incidence geometry of the Bruhat-Tits building of split type A_n using the language of lattices. Then we relate it to what we have seen in Chapter 4 through the notion of norms.

Definition 5.2.1. Let V be a vector space over K. A *lattice* in V is a finitely generated $\mathcal{O}_K$-submodule of V spanning V. Two lattices are *homothetic* if they are different by a nonzero constant factor.

5.2.2 ([Gar97, chap.19])**.** Consider the following data:

(i). The *vertex* set $\mathcal{V}$ consists of homothety classes of lattices in V.

(ii). The *incidence relation* is: $x \sim y$ for $x, y \in \mathcal{V}$ whenever they admit representatives L and L', respectively, such that $L \supsetneq L' \supsetneq \varpi L$.

(iii). A *frame* is an unordered tuple of lines in V

$$\boldsymbol{\Lambda} = \{\lambda_1, \cdots, \lambda_{n+1}\}$$

such that $\lambda_1 + \cdots + \lambda_{n+1} = \lambda_1 \oplus \cdots \oplus \lambda_{n+1} = V$.

We say a frame $\boldsymbol{\Lambda}$ *splits* a lattice L in V if L can be expressed as a sum $L_1 + \cdots + L_{n+1}$, where each L_i is a lattice in a member λ_i of $\boldsymbol{\Lambda}$. It is clear such a property is maintained through homotheties. We say $\boldsymbol{\Lambda}$ *splits* a vertex $x \in \mathcal{V}$ if it splits a representative L of x. Define the subset $\mathcal{V}_{\boldsymbol{\Lambda}}$ as

$$\mathcal{V}_{\boldsymbol{\Lambda}} := \{x \in \mathcal{V} \mid \boldsymbol{\Lambda} \text{ splits } x\}.$$

We thus obtain an abstract building $\mathcal{F}$. The action of $G = \mathsf{GL}(V, K)$ on it is clear: $g \in G$ maps any homothety classes of lattices $[L]$ in V to another one $[g.L]$.

To see how this incidence geometry is related to the affine geometric description, we need the following description of $\mathscr{B}(\mathsf{GL}(V))$ in [BT-3].

Definition 5.2.3. A *norm* on V is a map $\alpha \colon V \longrightarrow \mathbb{R} \cup \{\infty\}$ such that for any $u, v \in V$ and any $t \in K$,

(i). $\alpha(tu) = \mathrm{val}(t) + \alpha(u)$;

(ii). $\alpha(u + v) \geqslant \inf\{\alpha(u), \alpha(v)\}$;

(iii). $\alpha(u) = \infty$ if and only if $u = 0$.

The set of norms on V is denoted by $\mathcal{N}(V)$. If α is a norm, then so is $\alpha + c$ for any $c \in \mathbb{R}$. Such a norm is said to be *homothetic* to α. The set of homothety classes of norms on V is denoted by $\mathscr{X}(V)$.

Example 5.2.4. Any lattice L in V defines a norm α_L:

$$\alpha_L \colon v \in V \longmapsto \sup\{\mathrm{val}(t) \mid v \in tL\}.$$

It is clear that homothetic lattices define homothetic norms. We thus obtain an injective map from $\mathcal{V}$ to $\mathscr{X}(V)$ mapping the class of lattices $[L]$ to the class of norms $[\alpha_L]$.

Definition 5.2.5. We say a frame $\mathbf{\Lambda}$ *splits* a norm α on V if for any tuple $(v_\lambda)_{\lambda \in \mathbf{\Lambda}}$ of vectors in V, we have

$$\alpha\Big(\sum_{\lambda \in \mathbf{\Lambda}} v_\lambda\Big) = \inf\{\alpha(v_\lambda) \mid \boldsymbol{\lambda} \in \mathbf{\Lambda}\}.$$

Given a frame $\mathbf{\Lambda}$, let $\widetilde{\mathbb{A}}_{\mathbf{\Lambda}}$ denote the set of norms split by $\mathbf{\Lambda}$. It is naturally a real affine space under $\mathbb{R}^{\mathbf{\Lambda}}$: for any $\mathbf{v} = (c_\lambda)_{\lambda \in \mathbf{\Lambda}} \in \mathbb{R}^{\mathbf{\Lambda}}$ and any $\alpha \in \widetilde{\mathbb{A}}_{\mathbf{\Lambda}}$, define $\alpha + \mathbf{v}$ as

$$\alpha + \mathbf{v} \colon \sum_{\lambda \in \mathbf{\Lambda}} v_\lambda \longmapsto \inf\{\alpha(v_\lambda) + c_\lambda \mid \boldsymbol{\lambda} \in \mathbf{\Lambda}\}.$$

One can see that $\widetilde{\mathbb{A}}_{\mathbf{\Lambda}}$ is invariant under homotheties. Its homothety quotient is denoted by $\mathbb{A}_{\mathbf{\Lambda}}$. The above a real affine space structure induces one on $\mathbb{A}_{\mathbf{\Lambda}}$.

Definition 5.2.6. Suppose L and L' are two lattices split by $\mathbf{\Lambda}$. Then we can express them as

$$L = \sum_{\lambda \in \mathbf{\Lambda}} L_\lambda \qquad \text{and} \qquad L' = \sum_{\lambda \in \mathbf{\Lambda}} L'_\lambda,$$

where each L_λ and L'_λ are lattices in the line $\lambda \in \mathbf{\Lambda}$. The *elementary index* of L_λ over L'_λ is the real number

$$[L_\lambda : L'_\lambda] := \sup\left\{\mathrm{val}(t) \,\middle|\, L_\lambda \subseteq tL'_\lambda\right\}.$$

Then the *elementary index* of L over L' is the sequence of real numbers

$$(L : L') = ((L : L')_\lambda)_{\lambda \in \mathbf{\Lambda}} := ([L_\lambda : L'_\lambda])_{\lambda \in \mathbf{\Lambda}}$$

Since we only care about homothety classes of lattices, we allow the sequence to be determined up to a nonzero common factor. We will use $[L : L']$ or $[x : y]$ (where $x = [L]$ and $y = [L']$) to denote the image of the sequence $(L : L')$ in the quotient vector space $\mathbb{R}^{\mathbf{\Lambda}}/\mathbb{R}\mathbf{1}$. One can further spell out that

$$[\alpha_L] = [\alpha_{L'}] - [L : L']. \tag{5.2.1}$$

Definition 5.2.7. A *basis* of a frame $\mathbf{\Lambda}$ is a basis $\boldsymbol{e} = (e_1, \cdots, e_{n+1})$ of V such that each e_i spans a member of $\mathbf{\Lambda}$. Given such a basis $\boldsymbol{e}$ is amount to identify the split reductive groups $(\mathsf{GL}(V), \mathsf{T}(\mathbf{\Lambda}))$ and $(\mathsf{GL}_{n+1}, \mathsf{D}_{n+1})$.

Recall notations from Example 3.4.4. We further identify the coroot space $\mathbb{V}$ of the pair $(\mathsf{GL}_{n+1}, \mathsf{D}_{n+1})$ with the quotient vector space $\mathbb{R}^{\mathbf{\Lambda}}/\mathbb{R}\mathbf{1}$ by identifying the cocharacters λ_i with the members $\boldsymbol{\lambda}_i$ of $\mathbf{\Lambda}$. Then we can assign each vector $\mathbf{v} \in \mathbb{V}$ a norm $\alpha_{\boldsymbol{e},\mathbf{v}}$:

$$\alpha_{\boldsymbol{e},\mathbf{v}} : e_i \longmapsto \langle \chi_i, \mathbf{v} \rangle.$$

Then it is clear that

$$\alpha_{\boldsymbol{e},\mathbf{0}} + \mathbf{v} = \alpha_{\boldsymbol{e},\mathbf{v}},$$

and hence we have $\mathbb{A}_{\boldsymbol{\Lambda}} = \alpha_{\boldsymbol{e},\mathbf{0}} + \mathbb{V}$.

On the other hand, recall in Example 3.3.12, we have the following Chevalley system $(u_{a_{ij}})_{1\leqslant i,j\leqslant n+1}$:

$$u_{a_{ij}}(-) = \xi_{ij}\left(\begin{pmatrix} 1 & - \\ 0 & 1 \end{pmatrix}\right).$$

Let φ be the associated valuation as in Example 4.1.6. Namely, $\varphi_{a_{ij}} = \mathrm{val} \circ u_{a_{ij}}^{-1}$. Then $\varphi + \mathbf{v} \mapsto \alpha_{\boldsymbol{e},\mathbf{v}}$ identifies the real affine spaces $\mathbb{A}$ (defined in 4.1.8) with $\mathbb{A}_{\boldsymbol{\Lambda}}$.

5.2.8. The space $\mathscr{N}(V)$ carries an action of $\mathsf{GL}(V, K)$:

$$g.\alpha := \alpha \circ g^{-1}.$$

One can see that such an action is compatible with homotheties. We thus obtain an action of $\mathsf{GL}(V, K)$ on $\mathscr{X}(V)$. It is then clear that $\mathsf{T}(\boldsymbol{\Lambda})$ acts vectorially on $\mathbb{A}_{\boldsymbol{\Lambda}}$ and its normalizer $\mathsf{N}(\boldsymbol{\Lambda})$ stabilizes $\mathbb{A}_{\boldsymbol{\Lambda}}$. We are thus able to compare the abstract apartment $\mathscr{F}_{\boldsymbol{\Lambda}}$ with the affine apartments

- $\mathscr{A}(\boldsymbol{\Lambda})$, with affine space $\mathbb{A}_{\boldsymbol{\Lambda}}$ and group action as above, and
- $\mathscr{A}(\mathsf{GL}(V), \mathsf{T}(\boldsymbol{\Lambda}))$, with affine space $\mathbb{A}$ and group action following Example 4.1.12.

It is straightforward to verify that the identification in Definition 5.2.7 gives us an isomorphism between above apartments. Moreover, the following commutative diagram of affine apartments is evident.

$$\begin{tikzcd}
\mathcal{F}_{\mathbf{\Lambda}} \arrow[r, "\sim"] \arrow[d, "g"'] & \mathscr{A}(\mathbf{\Lambda}) \arrow[r, "\sim"] \arrow[d, "g"'] & \mathscr{A}(\mathsf{GL}(V), \mathsf{T}(\mathbf{\Lambda})) \arrow[d, "g"] \\
\mathcal{F}_{g.\mathbf{\Lambda}} \arrow[r, "\sim"] & \mathscr{A}(g.\mathbf{\Lambda}) \arrow[r, "\sim"] & \mathscr{A}(\mathsf{GL}(V), \mathsf{T}(\mathbf{\Lambda})^{g})
\end{tikzcd}$$

Hence, the abstract building $\mathcal{F}$ is isomorphic to the Euclidean building $\mathscr{X}(V)$ and is isomorphic to the Bruhat-Tits building $\mathscr{B}(\mathsf{GL}(V))$.

§ 5.3. Primary lattices and maximinorante norms

In this section, $(V, \langle \cdot \mid \cdot \rangle)$ is a hermitian space over K, i.e. we take (D, σ) to (K, id) in Example 3.1.7.(ii). There are three cases:

- (C_n type) $\langle \cdot \mid \cdot \rangle$ is alternative.
- (D_n type) $\langle \cdot \mid \cdot \rangle$ is symmetric and V is hyperbolic.
- (B_n type) $\langle \cdot \mid \cdot \rangle$ is symmetric and V is not hyperbolic.

In any case, we consider the following notion:

Definition 5.3.1 ([Gar97, chap.20])**.** A lattice L in V is *primitive* if $\langle \cdot \mid \cdot \rangle$ is $\mathcal{O}_K$-valued on L and if $\langle \cdot \mid \cdot \rangle \pmod{\varpi}$ is non-degenerate on the κ-vector space $L/\varpi L$.

The *dual lattice* of a lattice L in V is

$$L^* := \{v \in V \mid \langle L \mid v \rangle \in \mathcal{O}_K\}.$$

It is clear that $L^* = L$ if and only if L is primitive.

Remark. The existence of primitive lattices is not always automatic. In all following cases, we assume the existence of primitive lattices.

5.3.2 ([Gar97, §20.1])**.** Assume $\langle \cdot \mid \cdot \rangle$ is alternative. Consider the following data:

(i). The *vertex* set $\mathcal{V}$ consists of homothety classes of lattices in V which admits a representative L with the following property: there is a primitive lattice L_o such that

$$L_o \supseteq L \supseteq \varpi L_o,$$

and that

$$\langle L \mid L \rangle \subseteq \mathfrak{m}_K.$$

In other words, $L/\varpi L_o$ is a totally isotropic subspace of $L_o/\varpi L_o$.

(ii). The *incidence relation* is: $x \sim y$ for $x, y \in \mathcal{V}$ whenever they admit representatives L and L', respectively, with the following property: there is a primitive lattice L_o such that

$$L_o \supseteq L \supseteq \varpi L_o \qquad \text{and} \qquad L_o \supseteq L' \supseteq \varpi L_o,$$

and that either $L \subseteq L'$ or $L \supseteq L'$.

(iii). A *frame* is a hyperbolic frame $\mathbf{\Lambda} = \left\{\lambda_1^+, \lambda_1^-, \cdots, \lambda_n^+, \lambda_n^-\right\}$ in V

Defining $\mathcal{V}_{\mathbf{\Lambda}}$ as before. Namely,

$$\mathcal{V}_{\mathbf{\Lambda}} := \{x \in \mathcal{V} \mid \mathbf{\Lambda} \text{ splits } x\}.$$

We thus obtain an abstract building $\mathcal{F}$. The action of $G = \mathsf{Sp}(V, K)$ on it is clear: $g \in G$ maps any primitive lattices in V primitive ones and hence the properties mentioned above are maintained.

5.3.3 ([Gar97, §20.2])**.** Assume $\langle \cdot \mid \cdot \rangle$ is symmetric and V is hyperbolic of dimension $2n$ ($n \geqslant 4$). Consider the following data:

(i). The *vertex* set $\mathcal{V}$ consists of homothety classes of lattices in V which admits a representative L with the following property: there is a primitive lattice L_o such that

$$L_o \supseteq L \supseteq \varpi L_o,$$

and that $L/\varpi L_o$ is a totally isotropic subspace of $L_o/\varpi L_o$ of dimension other than 1 and $n-1$.

(ii). The *incidence relation* is: $x \sim y$ for $x, y \in \mathcal{V}$ whenever they admit representatives L and L', respectively, with one of the following property: (a) there is a primitive lattice L_o such that

$$L_o \supseteq L \supseteq \varpi L_o \qquad \text{and} \qquad L_o \supseteq L' \supseteq \varpi L_o,$$

and that either $L \subseteq L'$ or $L \supseteq L'$; (b) there is a primitive lattice L_o such that the κ-vector spaces $L/\varpi L_o$ and $L'/\varpi L_o$ are both 0-dimensional or are both n-dimensional, and that all the following κ-vector spaces

$$\frac{L}{L \cap L'}, \qquad \frac{L'}{L \cap L'}, \qquad \frac{L + L'}{L}, \qquad \frac{L + L'}{L'}$$

are one-dimensional.

(iii). A *frame* is a hyperbolic frame $\mathbf{\Lambda} = \left\{\lambda_1^+, \lambda_1^-, \cdots, \lambda_n^+, \lambda_n^-\right\}$ in V

Defining $\mathcal{V}_{\mathbf{\Lambda}}$ as before. Namely,

$$\mathcal{V}_{\mathbf{\Lambda}} := \{x \in \mathcal{V} \mid \mathbf{\Lambda} \text{ splits } x\}.$$

We thus obtain an abstract building $\mathcal{F}$. The action of $G = \mathsf{SO}(V, K)$ on it is clear: $g \in G$ maps any primitive lattices in V to primitive ones and hence the properties mentioned above are maintained.

5.3.4 ([Gar97, §20.3]). Assume $\langle \cdot \mid \cdot \rangle$ is symmetric and V is not hyperbolic. Hence, V is the orthogonal sum of a hyperbolic subspace and a nonzero anisotropic subspace A. Consider the following data:

(i). The *vertex* set $\mathcal{V}$ consists of homothety classes of lattices in V which admits a representative L with the following property: there is a primitive lattice L_o such that

$$L_o \supseteq L \supseteq \varpi L_o,$$

and that $L/\varpi L_o$ is a totally isotropic subspace of $L_o/\varpi L_o$ of dimension other than 1.

(ii). The *incidence relation* is: $x \sim y$ for $x, y \in \mathcal{V}$ whenever they admit representatives L and L', respectively, with one of the following property: (a) there is a primitive lattice L_o such that

$$L_o \supseteq L \supseteq \varpi L_o \qquad \text{and} \qquad L_o \supseteq L' \supseteq \varpi L_o,$$

and that either $L \subseteq L'$ or $L \supseteq L'$; (b) both L and L' are primitive and that all the following κ-vector spaces

$$\frac{L}{L \cap L'}, \qquad \frac{L'}{L \cap L'}, \qquad \frac{L + L'}{L}, \qquad \frac{L + L'}{L'}$$

are one-dimensional.

(iii). A *frame* is a hyperbolic frame $\boldsymbol{\Lambda} = \left\{\lambda_1^+, \lambda_1^-, \cdots, \lambda_n^+, \lambda_n^-\right\}$ in V

Since V is not hyperbolic, the notion of splitting needs to be modified. We say a hyperbolic frame $\boldsymbol{\Lambda}$ *splits* a lattice L in V if L can be expressed as

$$L = \sum_{\lambda \in \boldsymbol{\Lambda}} L_\lambda \oplus A^\circ,$$

where A° is the unique* maximal $\mathscr{O}_K$-valued lattice in A. Define

$$\mathcal{V}_{\mathbf{\Lambda}} := \{x \in \mathcal{V} \mid \mathbf{\Lambda} \text{ splits } x\}.$$

We thus obtain an abstract building $\mathcal{F}$. The action of $G = \mathsf{SO}(V, K)$ on it is clear: $g \in G$ maps any primitive lattices in V to primitive ones and preserves A°. Hence, the properties mentioned above are maintained.

To see how this incidence geometry is related to the affine geometric description, we need the following notions.

Definition 5.3.5 ([BT-4, §2])**.** Let $(V, \langle \cdot \mid \cdot \rangle)$ be a hermitian space. A norm α on V is *minorante* if

$$\alpha(u) + \alpha(y) \leqslant \mathrm{val}(\langle u \mid v \rangle) \qquad \text{for all} \qquad u, v \in V,$$

and that (if $\mathfrak{q}$ is associated with $\langle \cdot \mid \cdot \rangle$)

$$\alpha(v) \leqslant \tfrac{1}{2}\,\mathrm{val}(\mathfrak{q}(v)) \qquad \text{for all} \qquad v \in V.$$

A norm is *maximinorante* if it is a maximal element in the set of minorante norms.

The *dual norm* of a norm α is the norm

$$\alpha^* : u \in V \longmapsto \inf\{\mathrm{val}(\langle u \mid v \rangle) - \alpha(v) \mid v \in V\}.$$

If $\alpha^* = \alpha$, we say the norm α is *self-dual*.

We use $\mathscr{M}(V)$ to denote the subset of $\mathscr{N}(V)$ consisting of maximinorante norms.

Remark. In general, maximinorante norms and self-dual norms are not the same. To simplify discussion, we may assume 2 is invertible in K and then the two notions

*One can try $A^\circ = \{v \in A \mid \langle v \mid v \rangle \in \mathscr{O}_K\}$.

coincide.

Example 5.3.6. One can see

$$\alpha_L^* = \alpha_{L^*}.$$

Hence, for any lattice L, the average

$$\alpha_{L,L^*} := \frac{1}{2}(\alpha_L + \alpha_{L^*})$$

is a maximinorante norm. Furthermore, α_{L,L^*} depends only on the homothety class $[L]$. We thus obtain an injective map from $\mathcal{V}$ to $\mathscr{M}(V)$.

Definition 5.3.7. For each hyperbolic frame $\boldsymbol{\Lambda} = \{\lambda_1^+, \lambda_1^-, \cdots, \lambda_n^+, \lambda_n^-\}$, let $\mathbb{A}_{\boldsymbol{\Lambda}}$ denote the subset of $\mathscr{M}(V)$ consisting of norms split by $\boldsymbol{\Lambda}$.

Let I be an index set for the hyperbolic planes $H_i = \lambda_i^+ + \lambda_i^-$. Then $\mathbb{A}_{\boldsymbol{\Lambda}}$ is naturally a real affine space under $\mathbb{R}^I$: for any vector $\mathbf{v} = (c_i)_{i\in I} \in \mathbb{R}^I$ and any $\alpha \in \widetilde{\mathbb{A}}_{\boldsymbol{\Lambda}}$, define $\alpha + \mathbf{v}$ as follows: for each $i \in I$,

$$v \in \lambda_i^+ \longmapsto \alpha(v) + c_i \qquad \text{and} \qquad v \in \lambda_i^- \longmapsto \alpha(v) - c_i,$$

and on A, $\alpha + \mathbf{v}$ agrees with α.

Suppose L and L' are two lattices split by $\boldsymbol{\Lambda}$. Then Eq. (5.2.1) implies that

$$\alpha_{L,L^*} = \alpha_{L',L'^*} - [L : L']_{\pm}, \tag{5.3.1}$$

where the real vector $[L : L']^{\pm}$ is built from $[L : L']$ as follows: its i-th component is

$$[L : L']_i^{\pm} := \frac{1}{2}\left([L : L']_{\lambda_i^+} - [L : L']_{\lambda_i^-}\right). \tag{5.3.2}$$

We call it the *hyperbolic index* of L over L'.

Definition 5.3.8. A *hyperbolic basis* of a hyperbolic frame $\mathbf{\Lambda}$ is a basis

$$\boldsymbol{e} = (e_1, e_{-1}, \cdots, e_n, e_{-n})$$

of the frame $\mathbf{\Lambda}$ of $H_\bullet$ such that $\langle e_i \,|\, e_{-i}\rangle = 1$. Given such a basis $\boldsymbol{e}$ is amount to identify the semisimple group $(\mathsf{SO}(V), \mathsf{T}(\mathbf{\Lambda}))$ with one of the followings*: $(\mathsf{Sp}_{2n}, \mathsf{D}_{2n} \cap \mathsf{Sp}_{2n})$, $(\mathsf{SO}_{2n}, \mathsf{D}_{2n} \cap \mathsf{SO}_{2n})$, and $(\mathsf{SO}_{2n+1}, \mathsf{D}_{2n+1} \cap \mathsf{SO}_{2n+1})$.

Recall notations from Examples 3.4.7 and 3.4.8. We can further identify the coroot space $\mathbb{V}$ with the quotient vector space $\mathbb{R}^I$ through the assignment $i \in I \mapsto \lambda_{\pm i}$. Then we can assign each vector $\mathbf{v} \in \mathbb{V}$ a norm $\alpha_{\boldsymbol{e},\mathbf{v}}$:

$$e_i \longmapsto \langle \chi_{\pm i}, \mathbf{v}\rangle \qquad \text{and} \qquad e_{-i} \longmapsto -\langle \chi_{\pm i}, \mathbf{v}\rangle, \qquad \text{for all} \qquad i \in I.$$

Then it is clear that

$$\alpha_{\boldsymbol{e},\mathbf{0}} + \mathbf{v} = \alpha_{\boldsymbol{e},\mathbf{v}},$$

and hence we have $\mathbb{A}_{\mathbf{\Lambda}} = \alpha_{\boldsymbol{e},\mathbf{0}} + \mathbb{V}$.

5.3.9 ([BT-1, 10.1.2, 10.1.13])**.** Suppose $\boldsymbol{e} = (e_1, e_{-1}, \cdots, e_n, e_{-n})$ is a hyperbolic basis. We extend the index set I to $\overline{I} := I \cup (-I)$ by introducing $\epsilon(i)$ as

$$\epsilon(i) = \begin{cases} 1 & \text{if } i \in I, \\ \epsilon & \text{if } -i \in I. \end{cases}$$

Let's introduce the argument space**

$$\mathcal{Z} := \{(z, x) \in \mathbb{W}(A) \times \mathbb{G}_{\mathrm{a}} \mid (1+\epsilon)(\mathfrak{q}(z) - x) = 0, (1-\epsilon)z = 0\}.$$

*More precisely, we only identify a split semisimple subgroup of $\mathsf{SO}(V)$, which acts as $\mathbb{G}_{\mathrm{m}}$ on the anisotropic subspace A (if it is non-trivial).

**Following previous footnote, such an argument space is essentially $\mathbb{G}_{\mathrm{a}}$.

Consider the following linear transformations:

- for each $i \in \bar{I}$ and $(z, x) \in \mathcal{Z}$:

$$u_i(z,x)\colon \begin{cases} v \mapsto v - \epsilon(i)\langle z \mid v\rangle e_{-i} & \text{if } v \in \mathbb{W}(A) \\ e_i \mapsto z + e_i - \epsilon(i) x e_{-i} & \\ e_j \mapsto e_j & \text{if } j \neq i \end{cases}$$

- for any $i, j \in \bar{I}$ such that $j \neq \pm i$ and each $x \in \mathbb{G}_a$:

$$u_{ij}(x)\colon \begin{cases} v \mapsto v & \text{if } v \in \mathbb{W}(A) \\ e_i \mapsto e_i + \epsilon(-j) x e_{-j} & \\ e_j \mapsto e_j - \epsilon(i) x e_{-i} & \\ e_k \mapsto e_k & \text{if } k \neq i, j \end{cases}$$

Then they give the following Chevalley systems:

- (C_n type) For $\Phi = \left\{\pm\chi_{\pm i} \pm \chi_{\pm j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\} \cup \{\pm 2\chi_{\pm i} \mid 1 \leqslant i \leqslant n\}$:

$$\pm 2\chi_{\pm i} \longmapsto u_{\pm i}(0, -) \qquad \text{and} \qquad \pm\chi_{\pm i} \pm \chi_{\pm j} \longmapsto u_{\pm i \pm j}(-).$$

- (B_n type) For $\Phi = \left\{\pm\chi_{\pm i} \pm \chi_{\pm j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\} \cup \{\pm\chi_{\pm i} \mid 1 \leqslant i \leqslant n\}$:

$$\pm\chi_{\pm i} \longmapsto u_{\pm i}(-, -) \qquad \text{and} \qquad \pm\chi_{\pm i} \pm \chi_{\pm j} \longmapsto u_{\pm i \pm j}(-).$$

- (D_n type) For $\Phi = \left\{\pm\chi_{\pm i} \pm \chi_{\pm j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$:

$$\pm\chi_{\pm i} \pm \chi_{\pm j} \longmapsto u_{\pm i \pm j}(-).$$

The associated valuations φ following Example 4.1.7 are:

- (C_n type) For $\Phi = \left\{\pm\chi_{\pm i} \pm \chi_{\pm j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\} \cup \{\pm 2\chi_{\pm i} \mid 1 \leqslant i \leqslant n\}$:

$$\varphi_{\pm 2\chi_{\pm i}} \colon u_{\pm i}(0, x) \longmapsto \mathrm{val}(x) \qquad \text{and} \qquad \varphi_{\pm\chi_{\pm i}\pm\chi_{\pm j}} \colon u_{\pm i \pm j}(x) \longmapsto \mathrm{val}(x).$$

- (B_n type) For $\Phi = \left\{\pm\chi_{\pm i} \pm \chi_{\pm j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\} \cup \{\pm\chi_{\pm i} \mid 1 \leqslant i \leqslant n\}$:

$$\varphi_{\pm\chi_{\pm i}} \colon u_{\pm i}(z, x) \longmapsto \frac{1}{2}\,\mathrm{val}(x) \qquad \text{and} \qquad \varphi_{\pm\chi_{\pm i}\pm\chi_{\pm j}} \colon u_{\pm i \pm j}(x) \longmapsto \mathrm{val}(x).$$

- (D_n type) For $\Phi = \left\{\pm\chi_{\pm i} \pm \chi_{\pm j} \,\middle|\, 1 \leqslant i \neq j \leqslant n\right\}$:

$$\varphi_{\pm\chi_{\pm i}\pm\chi_{\pm j}} \colon u_{\pm i \pm j}(x) \longmapsto \mathrm{val}(x).$$

Then one can verify that $\varphi + \mathbf{v} \mapsto \alpha_{\boldsymbol{e},\mathbf{v}}$ identifies the real affine spaces $\mathbb{A}$ (defined in 4.1.8) with $\mathbb{A}_{\boldsymbol{\Lambda}}$.

5.3.10. The space $\mathscr{M}(V)$ carries an action of $\mathsf{O}(V, K)$:

$$g.\alpha := \alpha \circ g^{-1}.$$

It is then clear that $\mathsf{T}(\boldsymbol{\Lambda})$ acts vectorially on $\mathbb{A}_{\boldsymbol{\Lambda}}$ and its normalizer $\mathsf{N}(\boldsymbol{\Lambda})$ stabilizes $\mathbb{A}_{\boldsymbol{\Lambda}}$. We are thus able to compare the abstract apartment $\mathcal{F}_{\boldsymbol{\Lambda}}$ with the affine apartments

- $\mathscr{A}(\boldsymbol{\Lambda})$, with affine space $\mathbb{A}_{\boldsymbol{\Lambda}}$ and group action as above, and
- $\mathscr{A}(\mathsf{SO}(V), \mathsf{T}(\boldsymbol{\Lambda}))$ given by the valuation φ (see 4.1.14).

It is straightforward to verify that the identification in Definition 5.3.8 gives us an isomorphism between above apartments. Moreover, the following commutative diagram of affine apartments is evident.

$$\begin{tikzcd}
\mathcal{F}_{\mathbf{\Lambda}} \arrow[r, "\sim"] \arrow[d, "g"'] & \mathscr{A}(\mathbf{\Lambda}) \arrow[r, "\sim"] \arrow[d, "g"'] & \mathscr{A}(\mathsf{SO}(V), \mathsf{T}(\mathbf{\Lambda})) \arrow[d, "g"] \\
\mathcal{F}_{g.\mathbf{\Lambda}} \arrow[r, "\sim"] & \mathscr{A}(g.\mathbf{\Lambda}) \arrow[r, "\sim"] & \mathscr{A}(\mathsf{SO}(V), \mathsf{T}(\mathbf{\Lambda})^g)
\end{tikzcd}$$

Hence, the abstract building $\mathcal{F}$ is isomorphic to the Euclidean building $\mathscr{M}(V)$ and is isomorphic to the Bruhat-Tits building $\mathscr{B}(\mathsf{SO}(V))$.

Part II.

Simplicial Distance

Chapter 6.

Vertices and Simplicial Distance in Apartments

The purpose of this chapter is to characterize the simplicial distance:

Theorem 6.1. *In an irreducible Bruhat-Tits building of split classical type, two vertices x and y have simplicial distance at most d if and only if they are separated by at most $d-1$ parallel walls. In particular, fixing a fixed special vertex o as the reference point, for any vertex x, we have*

$$d(x,o) \leqslant d \iff a_0(x-o) \leqslant d, \tag{6.1}$$

where a_0 is the highest root relative to a Weyl chamber covering x.

In § 6.1, we will see that problems in this dissertation can be reduced to irreducible cases. This justifies why the above theorem only mention irreducible buildings. Then, in § 6.2, a general framework will be established. Follow which, in § 6.3 to 6.6, we give explicit descriptions of vertices and then characterize the simplicial distance when the Bruhat-Tits building $\mathscr{B}$ is of split classical type.

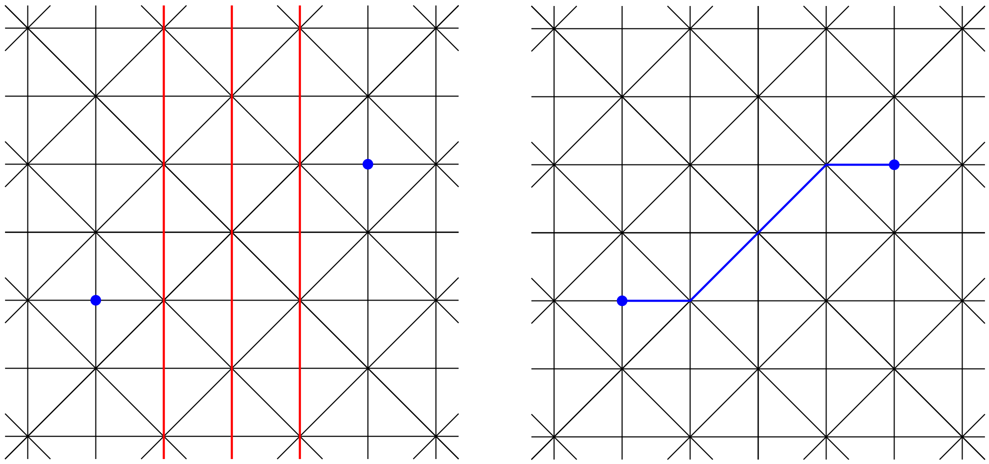


Figure 6.1. The two blues vertices are separated by at most 3 parallel walls (such as the red ones) and have simplicial distance 4 (the right-hand side displays a path of length 4)

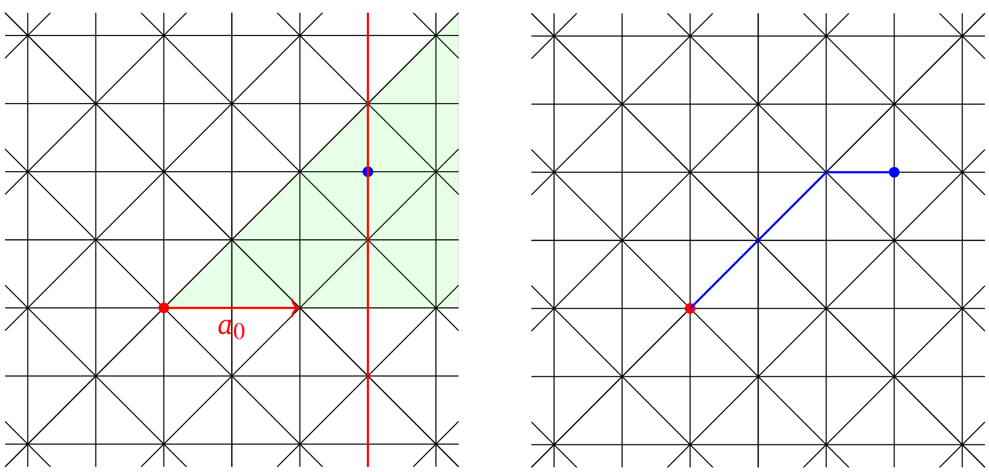


Figure 6.2. The blue vertex has value 3 under the highest root a_0 relative to the Weyl chamber (the green cone) and is of simplicial distance 3 from the red origin (the right-hand side displays a path of length 3)

§ 6.1. Reduce to irreducible ones

The following lemma is essential for reduction purpose.

Lemma 6.1.1. *Suppose $\mathscr{B} = \mathscr{B}_1 \times \mathscr{B}_2$ is a decomposition of Bruhat-Tits buildings. Let $d_i(\,\cdot\,,\,\cdot\,)$ ($i = 1, 2$) be the simplicial distance on $\mathscr{B}_i$ and pr_i the canonical projection from $\mathscr{B}$ to $\mathscr{B}_i$. Then we have*

$$d(x, y) = d_1(\mathrm{pr}_1(x), \mathrm{pr}_1(y)) + d_2(\mathrm{pr}_2(x), \mathrm{pr}_2(y)). \tag{6.1.1}$$

Proof. First, the left-hand side is no larger than the right-hand side by triangle inequality. To show the equality, we only need to show that if x, y are adjacent, then $\mathrm{pr}_i(x) = \mathrm{pr}_i(y)$ holds either for $i = 1$ or 2. Indeed, x, y are adjacent means that the segment $[x, y]$ contains no vertex inside it and the set

$$\Psi = \{a \in \Phi \mid [x, y] \subseteq \partial\alpha \text{ for some } \alpha \in \Sigma\}$$

has rank one less than Φ. Then, we must have $\Psi \cap \Phi_i = \Phi_i$ for either $i = 1$ or 2. Suppose $\Psi \cap \Phi_1 = \Phi_1$. For any $(a, \alpha) \in \mathscr{E}$, $\mathrm{pr}_1(\partial\alpha)$ is either a wall in $\mathscr{B}_1$ (if $a \in \Phi_1$) or the entire building (if $a \notin \Phi_1$). Therefore, $\mathrm{pr}_1([x, y])$ is a vertex and hence $\mathrm{pr}_1(x) = \mathrm{pr}_1(y)$. □

Hence, the discussion of simplicial distance can be reduced to irreducible cases. The following corollary is an example.

Corollary 6.1.2. *Suppose $\mathscr{B}$ is decomposed into irreducible ones:*

$$\mathscr{B} = \mathscr{B}_1 \times \cdots \times \mathscr{B}_m\,.$$

Let $\mathrm{SSA}_i(\,\cdot\,)$ *(*$1 \leqslant i \leqslant m$*) be the simplicial surface area in* $\mathscr{B}_i$*. Then, we have*

$$\mathrm{SSA}(r) = \sum_{r_1+\cdots+r_m=r} \mathrm{SSA}_1(r_1)\cdots\mathrm{SSA}_m(r_m).$$

Proof. Let $\partial_i(r)$ ($1 \leqslant i \leqslant m$) be the simplicial sphere in $\mathscr{B}_i$, then we need to show:

$$\partial(r) = \bigsqcup_{r_1+\cdots+r_m=r} \partial^1(r_1)\times\cdots\times\partial^m(r_m).$$

This follows from Lemma 6.1.1. □

In particular, in order to compute the simplicial surface area in general, it suffices to do that for irreducible ones. Since $\mathrm{SSA}(r) = \mathrm{SV}(r) - \mathrm{SV}(r-1)$, the same holds for the simplicial volume.

§ 6.2. Generality on vertices and simplicial distance

Lemma 6.1.1 suggests that, in order to understand the simplicial distance on general Bruhat-Tits buildings, we only need to do so on irreducible ones. Now, suppose $\mathscr{B}$ is an irreducible Bruhat-Tits building of split type X_n. Then we can deduce explicit characterization of the simplicial distance as follows.

We should first notice that, any two vertices are contained in a common apartment. Therefore, instead of simplicial distance on a ***building***, we may consider simplicial distance on an ***apartment***. Throughout this section, we temporarily forget the background of classical groups and focus on the affine apartment $\mathscr{A}(\Phi)$ of split type Φ.

(i). We first fix a realization of $\mathscr{A}(\Phi)$ We start with the Euclidean space $\mathbb{R}^m$. Then the underlying Euclidean space $\mathbb{V}$ of Φ is a certain subspace of $\mathbb{R}^m$. We use $(\mathbf{e}_1,\cdots,\mathbf{e}_m)$ to denote the standard basis of $\mathbb{R}^m$ and $(\chi_1,\cdots,\chi_m)$ the dual basis in $(\mathbb{R}^m)^*$. By an abuse of notation, we do not distinguish χ_i from its restriction to $\mathbb{V}$. The standard inner product on $\mathbb{R}^m$ is denoted by $(\cdot\,,\,\cdot)$. Then the underlying Euclidean affine space of $\mathscr{A}(\Phi)$ can be written as $\mathbb{A} = o + \mathbb{V}$. We keep the convention that any linear function on $\mathbb{V}$ is also viewed as an affine function on $\mathbb{A}$ by taking o as the reference point.

The root system Φ can be written in terms of linear functions on $\mathbb{V}$, and we thus obtain a concrete description of the *coroot lattice* $\mathcal{Q}^\vee$ in $\mathbb{V}$. This gives the translation group of the apartment $\mathscr{A}(\Phi)$. On the other hand, the action of the Weyl group ${}^vW$ has a concrete geometric interpretation on $\mathbb{V}$. Then the affine Weyl group W is obtained as the semi-product of them.

(ii). Next, we choose a Weyl chamber ${}^vC$ and describe the following data:

- Φ^+, the associated system of positive roots;
- $\Delta = \{a_1, \cdots, a_n\}$, the system of simple roots;
- a_0, the highest root,
- 2ρ, the sum of positive roots.

Note that $\overline{o + {}^vC}$ is a fundamental domain under the Weyl group ${}^vW$. We will use $\mathcal{D}({}^vC)$ to denote it.

(iii). The *fundamental coweights* $\omega_1, \cdots, \omega_n$ relative to Δ are the vectors in $\mathbb{V}$ such that

$$a_i(\omega_j) = \delta_{ij} \qquad \text{for all} \qquad 1 \leqslant i, j \leqslant n. \tag{6.2.1}$$

They form a basis of the *coweight lattice* $\mathcal{P}^\vee$ in $\mathbb{V}$. Recall that special vertices in $\mathscr{A}(\Phi)$ are the points $x \in \mathbb{A}$ such that $a(x) \in \Gamma$ for all $a \in \Phi$. Hence, the set of special vertices in $\mathscr{A}(\Phi)$ are precisely $o + \mathcal{P}^\vee \otimes_{\mathbb{Z}} \Gamma$.

The above can be found in [Bourbaki, chap.VI, §4, no. 5-9].

(iv). The simple roots $a_1, \cdots, a_n$, together with the highest root a_0, give rise to a basis $\widetilde{\Delta} = \{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ of Σ as in 2.4.9. To simplify notations, we assume that $\mathrm{val}(\,\cdot\,)$ is *normalized* in the sense that $\Gamma = \mathbb{Z}$. Then we have $\alpha_0 = \{x \in \mathbb{A} \mid -a_0(x) + 1 \geqslant 0\}$. Hence, the fundamental alcove C associated to $\widetilde{\Delta}$ can be expressed as

$$C := \left\{ o + \mathbf{v} \,\middle|\, \mathbf{v} \in {}^{v}C, a_0(\mathbf{v}) < 1 \right\}. \tag{6.2.2}$$

Let $v_0 = o, v_1, \cdots, v_n$ be its extreme points, where each v_i is opposite to the wall $\partial\alpha_i$. Then we have (recall Convention 2.4.11 for h_i)

$$a_j(v_i) = h_i^{-1}\delta_{ij}, \qquad \text{for all} \qquad 1 \leqslant j \leqslant n \tag{6.2.3}$$

Therefore, $v_i = o + h_i^{-1}\omega_i$.

(v). Following Convention 2.4.11, a vertex has *color* i if it is conjugated to v_i by the affine Weyl group W. Let $\mathcal{V}_i$ be the sets of vertices in $\mathscr{A}(\Phi)$ having color i. Since W is the semi-product of $W_o \cong {}^{v}W$ and $Q^\vee$, we have

$$\mathcal{V}_i = W_o.v_i + Q^\vee\,. \tag{6.2.4}$$

Note that for any $1 \leqslant j \leqslant n$, we have

$$r_{\alpha_j}(v_i) = v_i + a_j(v_i)a_j^\vee = v_i + h_i^{-1}\delta_{ij}a_j^\vee\,. \tag{6.2.5}$$

Since W_o is generated by $\left\{r_{\alpha_j} \,\middle|\, 1 \leqslant j \leqslant n\right\}$, we see that

$$v_i + Q^\vee \subseteq \mathcal{V}_i \subseteq v_i + h_i^{-1}\,Q^\vee\,. \tag{6.2.6}$$

In particular, if v_i is a special vertex, then $\mathcal{V}_i = v_i + Q^\vee$. In general, $\mathcal{V}_i$ can be obtained by computing $W_o.v_i$. The set $\mathcal{V}$ of vertices in $\mathscr{A}(\Phi)$ is then the disjoint union of $\mathcal{V}_i$ for

$0 \leqslant i \leqslant n$. Let h be the maximum of $h_1, \cdots, h_n$. Then we have

$$o + \mathcal{P}^\vee \subseteq \mathcal{V} \subseteq o + \tfrac{1}{h}\mathcal{P}^\vee \,. \tag{6.2.7}$$

The next steps are to apply previous results to characterize the simplicial distance. The goal is to prove Theorem 6.1.

Definition 6.2.1. A hyperplane H is *strictly between* two points x and y if its intersection with the open geodesic (x, y) is non-empty. For each $a \in \Phi$, let $a(x, y)$ denote the number of walls that are parallel to $H_a := \mathrm{Ker}(a)$ and are strictly between x and y.

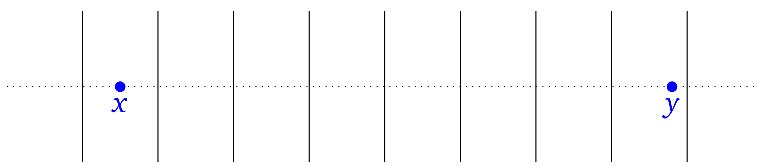


Figure 6.3. *Seven* parallel walls between two points x and y

Since any edge intersects with a wall by a vertex, we must have

$$d(x, y) \geqslant a(x, y) + 1 \qquad \text{for all} \qquad a \in \Phi. \tag{6.2.8}$$

Therefore, to prove Theorem 6.1 amounts to *construct* a path between x and y whose length is

$$\max\{a(x, y) + 1 \mid a \in \Phi\}.$$

Note that $a(x, y) = -a(x, y)$ by definition. Hence, we may only focus on positive roots.

Note that the following is a parabolic subset of Φ.

$$\Psi_{x \geqslant y} := \{a \in \Phi \mid a(x - y) \geqslant 0\}. \tag{6.2.9}$$

Hence, we may choose a system of positive roots Φ^+ contained in $\Psi_{x\geqslant y}$. Then we have

$$a(x,y) = \max\{\lceil a(x)\rceil - \lfloor a(y)\rfloor - 1, 0\} \tag{6.2.10}$$

for all $a \in \Phi^+$. Since we only consider the classical root systems, we always have $h \leqslant 2$ and hence $a(x) \in \frac{1}{2}\mathbb{Z}$. Consequently, $-\lfloor a(y)\rfloor = \lceil a(-y)\rceil$.

We separate the discussion into three cases:

(i). Both x and y are special.

(ii). One of x and y is special.

(iii). None of x and y is special.

For the first case, we need the following technical lemma.

Lemma 6.2.2. *Suppose $h_{i_0} \leqslant 2$. Let x, y be two vertices such that $a_i(x) = a_i(y) = k_i \in \mathbb{Z}$ for all $1 \leqslant i \leqslant n$ except $i = i_0$ and that either $a_{i_0}(x) \in \mathbb{Z}$ or $a_{i_0}(y) \in \mathbb{Z}$. Then there is a path of length $m = \left|a_{i_0}(x) - a_{i_0}(y)\right| h_{i_0}$ between them.*

Proof. We may assume that $a_{i_0}(y) \in \mathbb{Z}$ and that $a_{i_0}(x) - a_{i_0}(y) > 0$. Consider the sequence $x_j = \frac{j}{m}x + (1 - \frac{j}{m})y$ $(0 \leqslant j \leqslant m)$. Then we have $a_i(x_j) = k_i$ for all $1 \leqslant i \leqslant n$ except $i = i_0$. Moreover, since $a_{i_0}(x_j) = a_{i_0}(y) + \frac{j}{h_{i_0}}$, we have $a_0(x_j) \in \mathbb{Z}$.

Now, we need to show that the segment $[x_{j-1}, x_j]$ is an edge for all $1 \leqslant j \leqslant m$. Since this segment already lies in the intersection of the walls $\partial\alpha_{a_i - k_i}$ $(i \neq i_0)$, it remains to show that there is no vertex inside it. Suppose $x_t := tx_{j-1} + (1-t)x_j$ $(0 < t < 1)$ is a vertex inside the segment $[x_{j-1}, x_j]$. Then there must be another root $a = c_1a_1 + \cdots + c_na_n$ linearly independent of a_i $(i \neq i_0)$ such that $a(x_t) \in \mathbb{Z}$. Then $c_{i_0}a_{i_0}(x_t)$ has to be a nonzero integer. Now, we have

$$\begin{aligned}
c_{i_0}a_{i_0}(x_t) &= c_{i_0}a_{i_0}\left(tx_{j-1} + (1-t)x_j\right) \\
&= c_{i_0}a_{i_0}\left(y + \frac{t(j-1)+(1-t)j}{m}(x-y)\right) \\
&= c_{i_0}a_{i_0}(y) + \frac{c_{i_0}}{h_{i_0}}(j-t).
\end{aligned}$$

Therefore, we have $\frac{c_{i_0}}{h_{i_0}}(j-t) \in \mathbb{Z}$. By the basic property of the highest root, we must have $1 \leqslant c_{i_0} \leqslant h_{i_0}$. If $c_{i_0} = h_{i_0}$, then $t \in \mathbb{Z}$, a contradiction. If $c_{i_0} < h_{i_0}$, since we have assumed that $h_{i_0} \leqslant 2$, we must have $c_{i_0} = 1$. Then $\frac{1}{h_{i_0}}(j-t) \in \mathbb{Z}$ implies that $t \in \mathbb{Z}$, a contradiction. □

Now, we can prove the following result:

Lemma 6.2.3. *Suppose $h \leqslant 2$. Then, for any special vertices x and y, we have*

$$d(x,y) \leqslant d \iff a_0(x-y) \leqslant d, \tag{6.2.11}$$

where a_0 is the highest root relative to a system of positive roots Φ^+ contained in $\Psi_{x \geqslant y}$.

Proof. Consider the sequence $x_i = x_{i-1} + a_i(x-y)\omega_i$ $(1 \leqslant i \leqslant n)$ with $x_0 = y$. Then all x_i are special vertices and for the successive vertices x_{i-1} and x_i have the property that $a_j(x_{i-1}) = a_j(x_i)$ for all $1 \leqslant j \leqslant n$ except $j = i$. Hence, by Lemma 6.2.2, there is a path of length $h_i a_i(x-y)$ between them. In this way, we obtain a path from o to x of length

$$h_1 a_1(x-y) + \cdots + h_n a_n(x-y) = a_0(x-y).$$

This proves the lemma. □

6.2.4. Next, let's assume y is special. Then we may attach the Weyl chamber ${}^{v}C$ to y by choosing y as the reference point o. Now we have

$$a(x, y) = \max\{\lceil a(x)\rceil - 1, 0\} \qquad \text{for all} \qquad a \in \Phi^{+}. \tag{6.2.12}$$

In particular, $\max a(x, y) + 1 = \lceil a_0(x)\rceil$. Then, what we need to do is to *construct* a suitable path from x to a special vertex $x_0 \in \mathcal{D}\left({}^{v}C\right)$, verifying that

$$d(x, x_0) \leqslant \lceil a_0(x)\rceil - a_0(x_0). \tag{6.2.13}$$

Indeed, if such a path exists, by Lemma 6.2.3, we have

$$d(x, y) \leqslant d(x, x_0) + d(x_0, y) \leqslant \lceil a_0(x)\rceil - a_0(x_0) + a_0(x_0) = \lceil a_0(x)\rceil. \tag{6.2.14}$$

There is no uniform way to construct such a path. Hence, we have to do it case by case.

Remark. In the actual construction of such a path, one may find that we do not need x being contained in $\mathcal{D}\left({}^{v}C\right)$.

(vi). Under the assumptions in 6.2.4, construct the desired path explicitly. The following technical lemma will be used in our constructions.

Lemma 6.2.5. *Suppose* $\left\{a_{[i]} \,\middle|\, 1 \leqslant i \leqslant n\right\}$ *is a linearly independent set of roots. Let* x, y *be two vertices such that* $a_{[i]}(x) = a_{[i]}(y) = k_i \in \mathbb{Z}$ *for all* $1 \leqslant i \leqslant n$ *expect* $i = i_0$ *and that* $|a_{[i_0]}(x) - a_{[i_0]}(y)| = h^{-1}$. *Then* x *and* y *are adjacent.*

Proof. We need to show the segment $[x, y]$ is an edge. Since it already lies in the intersection of the walls $\partial\alpha_{a_{[i]}-k_i}$ $(i \neq i_0)$, it remains to show that there is no vertex inside this segment. Suppose $x_t := tx + (1-t)y$ $(0 < t < 1)$ is a vertex inside the segment $[x, y]$. By Eq. (6.2.7), we have $ha_{[i_0]}(x_t)$ and $ha_{[i_0]}(y) \in \mathbb{Z}$. But

$$ha_{[i_0]}(x_t) = ha_{[i_0]}(y) + th(a_{[i_0]}(x) - a_{[i_0]}(y)) = ha_{[i_0]}(y) \pm t.$$

Therefore, $t \in \mathbb{Z}$, which is a contradiction. □

Finally, suppose none of x and y is special. Then they are either adjacent to each other or separated by at least one special point s in the sense that $a(x) \geqslant a(s) \geqslant a(y)$. Choosing s as our reference point o, we have

- both x and $-y$ are in $\mathcal{D}({}^{v}C)$, and
- $a(x, y) = \max\{\lceil a(x)\rceil + \lceil a(-y)\rceil - 1, 0\}$.

Therefore, Eq. (6.2.14) implies:

$$d(x, y) \leqslant d(x, o) + d(o, y) = d(x, o) + d(o, -y) \leqslant \lceil a_0(x)\rceil + \lceil a_0(-y)\rceil$$

Hence, the desired path can be obtained by taking a shortest path from x to o and then from o to y, each of which is in the case 6.2.4.

§ 6.3. Vertices in the apartment $\mathscr{A}(A_n)$

(i). The underlying Euclidean vector space is the following:

(6.3.1) $$\mathbb{V} := \left\{\mathbf{v} \in \mathbb{R}^{n+1} \,\middle|\, \chi_1(\mathbf{v}) + \cdots + \chi_{n+1}(\mathbf{v}) = 0\right\}.$$

Its dual space is $\mathbb{V}^* = (\mathbb{R}\chi_1 \oplus \cdots \oplus \mathbb{R}\chi_{n+1})/\mathbb{R}(\chi_1 + \cdots + \chi_{n+1})$. After identifying each χ_i with its restriction to $\mathbb{V}$, the root system can be written as follows:

(6.3.2) $$\Phi := \left\{\chi_i - \chi_j \,\middle|\, 1 \leqslant i \neq j \leqslant n+1\right\}.$$

Then the coroot lattice $Q^\vee$ is the restriction of the standard lattice $\mathbb{Z}^{n+1}$ in $\mathbb{R}^{n+1}$ to $\mathbb{V}$, and the Weyl group ${}^vW$ acts on $\mathbb{V}$ as permutations of coordinates.

(ii). We can choose the following Weyl chamber:

(6.3.3)
$$ {}^vC := \left\{ \mathbf{v} \in \mathbb{V} \,\middle|\, \chi_i(\mathbf{v}) > \chi_j(\mathbf{v}) \text{ for all } 1 \leqslant i < j \leqslant n+1 \right\}. $$

Then the system of positive roots associated to ${}^vC$ is the following:

(6.3.4)
$$ \Phi^+ := \left\{ \chi_i - \chi_j \,\middle|\, 1 \leqslant i < j \leqslant n+1 \right\}. $$

Among them, the simple roots are the following:

(6.3.5)
$$ a_i := \chi_i - \chi_{i+1}. \quad (1 \leqslant i \leqslant n) $$

Using the basis $\Delta = \{a_1, \cdots, a_n\}$, the positive roots can be written as follows:

(6.3.6)
$$ \chi_i - \chi_j = a_i + \cdots + a_{j-1}. \quad (1 \leqslant i < j \leqslant n+1) $$

Among them, the highest root a_0 relative to Δ is

(6.3.7)
$$ a_0 := \chi_1 - \chi_{n+1} = a_1 + \cdots + a_n. $$

Moreover, the sum of positive roots is

(6.3.8)
$$ 2\rho = \sum_{i=1}^{n} i(n+1-i)a_i. $$

(iii). The fundamental coweights relative to Δ are the following:

(6.3.9)
$$ \omega_i := (\mathbf{e}_1 + \cdots + \mathbf{e}_i) - \tfrac{i}{n+1}(\mathbf{e}_1 + \cdots + \mathbf{e}_{n+1}). \quad (1 \leqslant i \leqslant n) $$

Then the coweight lattice $\mathcal{P}^\vee$ is $\mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$.

(iv). The fundamental alcove associated to Δ can be expressed as follows:

$$C := \{x \in \mathbb{A} \mid \chi_1(x) > \cdots > \chi_{n+1}(x), \chi_1(x) - \chi_{n+1}(x) < 1\}. \tag{6.3.10}$$

The extreme points of C other than $v_0 = o$ are the following:

$$v_i := o + \omega_i = o + (\mathbf{e}_1 + \cdots + \mathbf{e}_i) - \tfrac{i}{n+1}(\mathbf{e}_1 + \cdots + \mathbf{e}_{n+1}). \quad (1 \leqslant i \leqslant n) \tag{6.3.11}$$

Note that all of them are special vertices.

(v). Since each v_i is a special vertex, we have

$$\mathcal{V}_i = v_i + Q^\vee = \left\{x \in \mathbb{A} \,\middle|\, \chi_j(x) + \tfrac{i}{n+1} \in \mathbb{Z} \text{ for all } 1 \leqslant j \leqslant n+1\right\}. \tag{6.3.12}$$

In particular, all vertices are special and hence $\mathcal{V} = o + \mathcal{P}^\vee$.

(vi). Since all vertices are special, the A_n case of Theorem 6.1 follows from Lemma 6.2.3.

§ 6.4. Vertices in the apartment $\mathscr{A}(C_n)$ $(n \geqslant 2)$

(i). The underlying Euclidean vector space $\mathbb{V}$ is the entire $\mathbb{R}^n$, and its dual space $\mathbb{V}^*$ is thus spanned by the coordinate functions $\chi_1, \cdots, \chi_n$. Then the root system can be written as follows:

$$\Phi := \left\{\pm\chi_i \pm \chi_j \,\middle|\, 1 \leqslant i < j \leqslant n\right\} \cup \{\pm 2\chi_i \mid 1 \leqslant i \leqslant n\}. \tag{6.4.1}$$

Then its coroot lattice $Q^\vee$ is precisely the standard lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$, and the Weyl group ${}^vW$ acts on $\mathbb{V}$ as permutations and sign changes of coordinates.

(ii). We can choose the following Weyl chamber:

(6.4.2) $$ {}^{v}C := \{\mathbf{v} \in \mathbb{V} \,|\, \chi_i(\mathbf{v}) > \chi_j(\mathbf{v}) > 0 \text{ for all } 1 \leqslant i < j \leqslant n\}. $$

Then the system of positive roots associated to ${}^{v}C$ is the following:

(6.4.3) $$ \Phi^+ := \{\chi_i \pm \chi_j \,|\, 1 \leqslant i < j \leqslant n\} \cup \{2\chi_i \,|\, 1 \leqslant i \leqslant n\}. $$

Among them, the simple roots are the following:

(6.4.4) $$ a_i := \chi_i - \chi_{i+1} (1 \leqslant i \leqslant n-1) \qquad \text{and} \qquad a_n := 2\chi_n. $$

Using the basis $\Delta = \{a_1, \cdots, a_n\}$, the positive roots can be written as follows:

(6.4.5) $$ \begin{aligned} \chi_i - \chi_j &= a_i + \cdots + a_{j-1}, && (1 \leqslant i < j \leqslant n) \\ \chi_i + \chi_j &= a_i + \cdots + a_{j-1} + 2a_j + \cdots + 2a_{n-1} + a_n, && (1 \leqslant i < j \leqslant n) \\ 2\chi_i &= 2a_i + \cdots + 2a_{n-1} + a_n. && (1 \leqslant i \leqslant n) \end{aligned} $$

Among them, the highest root a_0 relative to Δ is

(6.4.6) $$ a_0 := 2\chi_1 = 2a_1 + \cdots + 2a_{n-1} + a_n. $$

Moreover, the sum of positive roots is

(6.4.7) $$ 2\rho = \sum_{i=1}^{n-1} i(2n+1-i)a_i + \binom{n+1}{2} a_n. $$

(iii). The fundamental coweights relative to Δ are the following:

(6.4.8) $$ \begin{aligned} \omega_i &:= \mathbf{e}_1 + \cdots + \mathbf{e}_i, && (1 \leqslant i \leqslant n-1) \\ \omega_n &:= \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n). && \end{aligned} $$

Hence, the coweight lattice $\mathcal{P}^\vee$ is $\mathbb{Z}^n + \mathbb{Z}\frac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n)$.

(iv). The fundamental alcove associated to Δ can be expressed as follows:

$$C := \left\{x \in \mathbb{A} \,\middle|\, \tfrac{1}{2} > \chi_1(x) > \cdots > \chi_n(x) > 0\right\}. \tag{6.4.9}$$

The extreme points of C other than $v_0 = o$ are the following:

$$\begin{aligned} v_i &:= o + \tfrac{1}{2}\omega_i = o + \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_i), \quad (1 \leqslant i \leqslant n-1) \\ v_n &:= o + \omega_n = o + \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n). \end{aligned} \tag{6.4.10}$$

Note that v_n is a special vertex, while v_i $(1 \leqslant i \leqslant n-1)$ are not special.

(v). For each i, by (i), $W_o.v_i$ consists of the points $x \in \mathbb{A}$ whose coordinates are either 0 or $\pm\frac{1}{2}$ and exactly i of them are nonzero. Then we have

$$\mathcal{V}_i = \left\{ x \in \mathbb{A} \;\middle|\; \begin{array}{c} \chi_1(x), \cdots, \chi_n(x) \in \frac{1}{2}\mathbb{Z} \\ \text{and exactly } i \text{ of them are non-integers} \end{array} \right\}. \tag{6.4.11}$$

Hence, we have $\mathcal{V} = o + \frac{1}{2}\mathbb{Z}^n$. In particular, we have $a_i(x) \in \frac{1}{2}\mathbb{Z}$ for all $1 \leqslant i \leqslant n-1$ and $a_n(x) \in \mathbb{Z}$. Conversely, if $a_i(x) \in \frac{1}{2}\mathbb{Z}$ for all $1 \leqslant i \leqslant n-1$ and $a_n(x) \in \mathbb{Z}$, then we can see that $x - o \in \frac{1}{2}\mathbb{Z}^n$. Let ω_i' denote $h_i^{-1}\omega_i$. Then we have

$$\mathcal{V} = o + \mathbb{Z}\omega_1' \oplus \cdots \oplus \mathbb{Z}\omega_n'. \tag{6.4.12}$$

(vi). Not every vertex is special. We thus need the following notion:

Definition 6.4.1. Let $x \in \mathbb{A}$ be a point. Then an index $j \in \{1, \cdots, n\}$ is called a *jump* if $a_j(x) \notin \mathbb{Z}$. The set of jumps of x is denoted by J_x.

Let x be a vertex in $\mathcal{D}({}^vC)$ with jumps $j_1, \cdots, j_s$, ordered from smallest to largest.

Note that we must have $j_s < n$. Let $x_i = x - \frac{1}{2}(\omega_{j_1} + \cdots + \omega_{j_i})$ for $1 \leqslant i \leqslant s$. Then the following lemma tells us that x_i and x_{i+1} are adjacent vertices.

Lemma 6.4.2. *Let $x \in \mathbb{A}$ be a vertex and j_1 its smallest jump. Then $y = x - \frac{1}{2}\omega_{j_1}$ is a vertex in $\mathcal{D}({}^vC)$ adjacent to x.*

Proof. First note that, by (v), we have $a_j(x) \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$ for all $j \in J_x$. Hence, $J_y = J_x \backslash \{j_1\}$. We define the roots $a_{[i]}$ $(1 \leqslant i \leqslant n)$ as follows. If $i \in J_y$, let

$$a_{[i]} = 2a_i + \cdots + 2a_{n-1} + a_n.$$

Otherwise, let $a_{[i]} = a_i$. Then $a_{[1]}, \cdots, a_{[n]}$ are linearly independent positive roots, and $a_{[1]}(y), \cdots, a_{[n]}(y)$ are non-negative integers. Hence, y is a vertex in $\mathcal{D}({}^vC)$. Since $x - y = \frac{1}{2}\omega_{j_1}$, we have $a_{[i]}(x) = a_{[i]}(y)$ for all i except $i = j_1$ and $a_{[j_1]}(x) - a_{[j_1]}(y) = \frac{1}{2}$. Then Lemma 6.2.5 applies to the roots $a_{[i]}$ $(1 \leqslant i \leqslant n)$ and the vertices x and y. □

Then the sequence $(x, x_1, \cdots, x_s)$ forms a path from x to x_s of length s in $\mathcal{D}({}^vC)$. Since x_s has no jumps, it is a special vertex. Moreover, we have

$$a_0(x) - a_0(x_s) = \tfrac{1}{2}a_0(\omega_{j_1} + \cdots + \omega_{j_s}) = s.$$

Then this x_s is the expected x_0 verifying Eq. (6.2.13). Thus, Eq. (6.1) is proved.

§ 6.5. Vertices in the apartment $\mathscr{A}(B_n)$ $(n \geqslant 3)$

(i). The underlying Euclidean vector space $\mathbb{V}$ is the entire $\mathbb{R}^n$, and its dual space $\mathbb{V}^*$ is thus spanned by the coordinate functions $\chi_1, \cdots, \chi_n$. Then the root system can be written as follows:

$$\Phi := \left\{\pm\chi_i \pm \chi_j \,\middle|\, 1 \leqslant i < j \leqslant n\right\} \cup \{\pm\chi_i \mid 1 \leqslant i \leqslant n\}. \tag{6.5.1}$$

Then its coroot lattice is the following sublattice of the standard lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$:

$$Q^\vee := \{\mathbf{v} \in \mathbb{Z}^n \mid (\mathbf{v}, \mathbf{v}) \in 2\mathbb{Z}\}. \tag{6.5.2}$$

The Weyl group ${}^vW$ acts on $\mathbb{V}$ as permutations and sign changes of coordinates.

(ii). We can choose the following Weyl chamber:

$${}^vC := \left\{\mathbf{v} \in \mathbb{V} \,\middle|\, \chi_i(\mathbf{v}) > \chi_j(\mathbf{v}) > 0 \text{ for all } 1 \leqslant i < j \leqslant n\right\}. \tag{6.5.3}$$

Then the system of positive roots associated to ${}^vC$ is the following:

$$\Phi^+ := \left\{\chi_i \pm \chi_j \,\middle|\, 1 \leqslant i < j \leqslant n\right\} \cup \{\chi_i \mid 1 \leqslant i \leqslant n\}. \tag{6.5.4}$$

Among them, the simple roots are the following:

$$\begin{aligned} a_i &:= \chi_i - \chi_{i+1}, \quad (1 \leqslant i \leqslant n-1) \\ a_n &:= \chi_n. \end{aligned} \tag{6.5.5}$$

Using the basis $\Delta = \{a_1, \cdots, a_n\}$, the positive roots can be written as follows:

$$\begin{aligned} \chi_i - \chi_j &= a_i + \cdots + a_{j-1}, & (1 \leqslant i < j \leqslant n) \\ \chi_i + \chi_j &= a_i + \cdots + a_{j-1} + 2a_j + \cdots + 2a_n, & (1 \leqslant i < j \leqslant n) \\ \chi_i &= a_i + \cdots + a_n. & (1 \leqslant i \leqslant n) \end{aligned} \tag{6.5.6}$$

Among them, the highest root a_0 relative to Δ is

$$a_0 := \chi_1 + \chi_2 = a_1 + 2a_2 + \cdots + 2a_n. \tag{6.5.7}$$

Moreover, the sum of positive roots is

$$2\rho = \sum_{i=1}^{n} i(2n-i)a_i. \tag{6.5.8}$$

(iii). The fundamental coweights relative to Δ are the following:

$$\omega_i = \mathbf{e}_1 + \cdots + \mathbf{e}_i. \quad (1 \leqslant i \leqslant n) \tag{6.5.9}$$

Hence, the coweight lattice $\mathcal{P}^\vee$ is precisely the standard lattice $\mathbb{Z}^n$.

(iv). The fundamental alcove associated to Δ can be expressed as follows:

$$C = \left\{ x \in \mathbb{A} \;\middle|\; \begin{array}{c} \chi_1(x) > \cdots > \chi_n(x) > 0, \\ \chi_1(x) + \chi_2(x) < 1 \end{array} \right\}. \tag{6.5.10}$$

The extreme points of C other than $v_0 = o$ are the following:

$$\begin{aligned} v_1 &= o + \omega_1 = o + \mathbf{e}_1, \\ v_i &= o + \tfrac{1}{2}\omega_i = o + \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_i). \quad (2 \leqslant i \leqslant n) \end{aligned} \tag{6.5.11}$$

Note that v_1 is a special vertex, while v_i $(2 \leqslant i \leqslant n)$ are not special.

(v). First, apply the affine Weyl group W to v_0, we have

$$\mathcal{V}_0 = \left\{ x \in \mathbb{A} \;\middle|\; \begin{array}{c} \chi_1(x), \cdots, \chi_n(x) \in \mathbb{Z}, \\ \chi_1(x) + \cdots + \chi_n(x) \text{ is even} \end{array} \right\}. \tag{6.5.12}$$

Applying W_o to v_1, we see that $W_o.v_1$ consists of the points $x \in \mathbb{A}$ having one coordinate being 1 or -1 and all others are 0. Then we have:

$$\mathcal{V}_1 = \left\{ x \in \mathbb{A} \;\middle|\; \begin{array}{c} \chi_1(x), \cdots, \chi_n(x) \in \mathbb{Z}, \\ \chi_1(x) + \cdots + \chi_n(x) \text{ is odd} \end{array} \right\} \tag{6.5.13}$$

For each $i > 1$, $W_o.v_i$ consists of the points $x \in \mathbb{A}$ whose coordinates are either 0 or $\pm\frac{1}{2}$ and exactly i of them are nonzero. Then we have:

$$\mathcal{V}_i = \left\{ x \in \mathbb{A} \;\middle|\; \begin{array}{c} \chi_1(x), \cdots, \chi_n(x) \in \frac{1}{2}\mathbb{Z} \\ \text{and exactly } i \text{ of them are non-integers} \end{array} \right\}. \tag{6.5.14}$$

Hence, the vertices are all the point $x \in \mathbb{A}$ such that $\chi_j(x) \in \frac{1}{2}\mathbb{Z}$ for all j and the number of non-integer coordinates is not 1. In particular, we have

$$\mathcal{V} \subseteq o + \tfrac{1}{2}\mathbb{Z}^n = o + \tfrac{1}{2}\mathcal{P}^\vee . \tag{6.5.15}$$

However, the equality doesn't hold. For instance, the point $o + \frac{1}{2}\omega_1$ is clearly not a vertex. Another example is $o + \frac{1}{2}(\omega_{i-1} + \omega_i)$ where all $\chi_j(x)$ are integers except $j = i$.

To better describe the vertices, we need the notion introduced in Definition 6.4.1. Then the complement of $\mathcal{V}$ in $o + \frac{1}{2}\mathcal{P}^\vee$ can be described as follows.

Lemma 6.5.1. *A point $x \in o + \frac{1}{2}\mathcal{P}^\vee$ belongs to the complement if and only if either $J_x = \{j_1, j_2\}$ and $j_2 - j_1 = 1$, or $J_x = \{1\}$.*

We will use Ξ to denote the set of points $x \in \mathbb{A}$ having the property in the lemma.

Proof. First, points in Ξ cannot be vertices. If $J_x = \{1\}$, then x is not a vertex since all $\chi_j(x)$ are integers except $j = 1$. If $J_x = \{j_1, j_2\}$ and $j_2 - j_1 = 1$, then x is not a vertex since all $\chi_j(x)$ are integers except $j = j_2$.

Conversely, suppose $x \in o + \frac{1}{2}\mathcal{P}^\vee$ and $x \notin \Xi$, then there are four cases:

(i). $|J_x| \geqslant 3$. Then at least two coordinates of x are non-integers.

(ii). $J_x = \{j_1, j_2\}$ and $j_2 - j_1 > 1$. Then $\chi_{j_1+1}(x), \cdots, \chi_{j_2}(x)$ are non-integers.

(iii). $J_x = \{j_1\}$ and $j_1 > 1$. Then $\chi_1(x), \cdots, \chi_{j_1}(x)$ are non-integers.

(iv). x has no jumps. Then it is a special vertex.

In any of above cases, x is a vertex by our characterization. $\square$

We illustrate the related structures of $\mathcal{V}$ by the following diagram, where $\dagger$ denotes "being special".

$$\begin{tikzcd}
 & & o+\frac{1}{2}\mathcal{P}^\vee \arrow[dl, dash] \arrow[dr, dash] & & \Xi\cap o+\frac{1}{2}\mathcal{P}^\vee \arrow[dl, dash] \\
 & \mathcal{V} \arrow[dl, dash] \arrow[dr, dash] & & \Xi\cap o+\frac{1}{2}\mathcal{P}^\vee & \\
\mathcal{V}_\dagger & & \bigcup\limits_{i=2,\cdots,n} \mathcal{V}_i & &
\end{tikzcd}$$

Figure 6.4. The set $\mathcal{V}$ of vertices and related structures in $\mathscr{A}(B_n)$.

(vi). Let $x \in \mathcal{D}({}^vC)$ be a vertex having jumps $j_1, \cdots, j_s$, ordered from smallest to largest. To construct a path between x and a special vertex x_0 in $\mathcal{D}({}^vC)$ verifying Eq. (6.2.13), we need the following lemmas.

Lemma 6.5.2. *Suppose either* $j_s - j_{s-1} > 1$ *or* $s > 3$*. Then* $y = x - \frac{1}{2}\omega_{j_1}$ *is a vertex in* $\mathcal{D}({}^vC)$ *adjacent to* x*.*

Proof. Since $\mathcal{V} \subseteq o+\frac{1}{2}\mathcal{P}^\vee$, we have $J_y = J_x \setminus \{j_1\}$. We define the roots $a_{[j]}$ $(1 \leqslant j \leqslant n)$ as follows. First, let

$$a_{[j_s]} = \begin{cases} a_{j_s-1} + 2a_{j_s} + \cdots + 2a_n & \text{if } j_s - j_{s-1} > 1, \\ a_{j_{s-2}} + \cdots + a_{j_s-1} + 2a_{j_s} + \cdots + 2a_n & \text{otherwise.} \end{cases}$$

For $2 \leqslant i \leqslant s-1$, let

$$a_{[j_i]} = a_{j_i} + \cdots + a_{j_{i+1}}.$$

Finally, if $j \notin J_y$, let $a_{[j]} = a_j$. Then $a_{[1]}, \cdots, a_{[n]}$ are linearly independent positive roots, and $a_{[1]}(y), \cdots, a_{[n]}(y)$ are non-negative integers. Hence, y is a vertex in $\mathcal{D}({}^{v}C)$. Since $x - y = \frac{1}{2}\omega_{j_1}$, we have $a_{[j]}(x) = a_{[j]}(y)$ for all j except $j = j_1$ and $a_{[j_1]}(x) - a_{[j_1]}(y) = \frac{1}{2}$. Then Lemma 6.2.5 applies to the roots $a_{[j]}$ $(1 \leqslant j \leqslant n)$ and the vertices x and y. $\square$

Let $y = x - \frac{1}{2}\omega_{j_1}$ be as in Lemma 6.5.2. Then we must have $a_0(y) \in \mathbb{Z}$ and

$$\lceil a_0(x) \rceil - a_0(y) = \begin{cases} a_0(x) - a_0(y) & \text{if } j_1 > 1, \\ a_0(x) + \frac{1}{2} - a_0(y) & \text{if } j_1 = 1; \end{cases}$$

$$a_0(x) - a_0(y) = \tfrac{1}{2}a_0(\omega_{j_1}) = \begin{cases} 1 & \text{if } j_1 > 1, \\ \frac{1}{2} & \text{if } j_1 = 1. \end{cases}$$

Hence, $\lceil a_0(x) \rceil - a_0(y) = 1$. By repeating using Lemma 6.5.2, we can reduce our problem to the case where $s = 3$, or further $s = 1$ if we start with $j_s - j_{s-1} > 1$.

Now, we may assume either $s \leqslant 3$ with $j_s - j_{s-1} = 1$ or $s = 1$.

Lemma 6.5.3. *Suppose $s = 3$ and $j_1 > 1$. Then $x_0 = x - \frac{1}{2}(\omega_{j_1} - \omega_{j_2} + \omega_{j_3})$ is a special vertex in $\mathcal{D}({}^{v}C)$ adjacent to x and verifying Eq. (6.2.13).*

Proof. First note that $a_1(x_0), \cdots, a_n(x_0)$ are non-negative integers. Hence, x_0 is a special vertex in $\mathcal{D}({}^{v}C)$. Since $j_1 > 1$, we have $a_0(x) \in \mathbb{Z}$ and

$$a_0(x) - a_0(x_0) = \tfrac{1}{2}a_0(\omega_{j_1} - \omega_{j_2} + \omega_{j_3}) = 1.$$

Hence, it remains to show that x_0 is adjacent to x.

To do this, we define the roots $a_{[j]}$ $(1 \leqslant j \leqslant n)$ as follows. If $j \neq j_2, j_3$, let $a_{[j]} = a_j$. Otherwise, let

$$a_{[j_2]} = a_{j_1} + \cdots + a_{j_2},$$
$$a_{[j_3]} = a_{j_2} + \cdots + a_{j_3}.$$

Then $a_{[1]}, \cdots, a_{[n]}$ are linearly independent. Since $x - x_0 = \frac{1}{2}(\omega_{j_1} - \omega_{j_2} + \omega_{j_3})$, we have $a_{[j]}(x) = a_{[j]}(x_0)$ for all j except $j = j_1$ and $a_{[j_1]}(x) - a_{[j_1]}(x_0) = \frac{1}{2}$. Then Lemma 6.2.5 applies to the roots $a_{[j]}$ $(1 \leqslant j \leqslant n)$ and the vertices x and x_0. □

Lemma 6.5.4. *Suppose $s = 3$, $j_3 - j_2 = 1$ and $j_1 = 1$, then $y = x - \frac{1}{2}(-\omega_{j_1} + \omega_{j_2})$ is a vertex in $\mathcal{D}({}^{v}C)$ adjacent to x.*

Proof. We define the roots $a_{[j]}$ $(1 \leqslant j \leqslant n)$ as follows. If $j \neq j_2, j_3$, let $a_{[j]} = a_j$. Otherwise, let

$$a_{[j_2]} = a_{j_1} + \cdots + a_{j_2},$$
$$a_{[j_3]} = a_{j_1} + \cdots + a_{j_2} + 2a_{j_3} + \cdots + 2a_n.$$

Then $a_{[1]}, \cdots, a_{[n]}$ are linearly independent positive roots, and $a_{[1]}(y), \cdots, a_{[n]}(y)$ are non-negative integers. Hence, y is a vertex in $\mathcal{D}({}^{v}C)$. Since $x - y = \frac{1}{2}(-\omega_{j_1} + \omega_{j_2})$, we have $a_{[j]}(x) = a_{[j]}(y)$ for all j except $j = j_1$ and $a_{[j_1]}(x) - a_{[j_1]}(y) = -\frac{1}{2}$. Then Lemma 6.2.5 applies to the roots $a_{[j]}$ $(1 \leqslant j \leqslant n)$ and the vertices x and y. □

Let $y = x - \frac{1}{2}(-\omega_{j_1} + \omega_{j_2})$ be as in Lemma 6.5.4. Then the only jump of y is $j_3 > 1$. Hence, $a_0(y) \in \mathbb{Z}$ and $\lceil a_0(x) \rceil - a_0(y) = 1$. Therefore, Lemma 6.5.4 reduces our problem to the case where $s = 1$.

Note that, by Lemma 6.5.1, $s = 2$ and $j_s - j_{s-1} = 1$ contradict to each other. Therefore, we may assume $s = 1$ now. By Lemma 6.5.1 again, we must have $j_1 > 1$. Let $x_0 = x - \frac{1}{2}\omega_{j_1}$. Then it is a special vertex in $\mathcal{D}({}^{v}C)$ since $a_1(x_0), \cdots, a_n(x_0)$ are non-negative integers. Applying Lemma 6.2.5 to the simple roots $a_1, \cdots, a_n$ and the vertices x and x_0, we see that they are adjacent. Moreover, we have $a_0(x) - a_0(x_0) = 1$ verifying Eq. (6.2.13). This finishes the proof of Theorem 6.1.

§ 6.6. Vertices in the apartment $\mathscr{A}(D_n)$ ($n \geqslant 4$)

(i). The underlying Euclidean vector space $\mathbb{V}$ is the entire $\mathbb{R}^n$, and its dual space $\mathbb{V}^*$ is thus spanned by the coordinate functions $\chi_1, \cdots, \chi_n$. Then the root system can be written as follows:

$$\Phi = \left\{\pm\chi_i \pm \chi_j \,\middle|\, 1 \leqslant i < j \leqslant n\right\}. \tag{6.6.1}$$

Then its coroot lattice is the following sublattice of the standard lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$:

$$Q^\vee := \{\mathbf{v} \in \mathbb{Z}^n \mid (\mathbf{v}, \mathbf{v}) \in 2\mathbb{Z}\}. \tag{6.6.2}$$

The Weyl group ${}^{v}W$ acts on $\mathbb{V}$ as permutations and even number of sign changes of coordinates.

(ii). We can choose the following Weyl chamber:

$${}^{v}C = \left\{\mathbf{v} \in \mathbb{V} \,\middle|\, \chi_i(\mathbf{v}) > \left|\chi_j(\mathbf{v})\right| \text{ for all } 1 \leqslant i < j \leqslant n\right\}. \tag{6.6.3}$$

Then the system of positive roots Φ^+ associated to ${}^{v}C$ is the following:

$$\Phi^+ := \left\{\chi_i \pm \chi_j \,\middle|\, 1 \leqslant i < j \leqslant n\right\}. \tag{6.6.4}$$

Among them, the simple roots are the following:

$$a_i := \chi_i - \chi_{i+1}, \quad (1 \leqslant i \leqslant n-1) \tag{6.6.5}$$
$$a_n := \chi_{n-1} + \chi_n.$$

Using the basis $\Delta = \{a_1, \cdots, a_n\}$, the positive roots can be written as follows:

$$\begin{aligned} \chi_i - \chi_j &= a_i + \cdots + a_{j-1}, && (1 \leqslant i < j \leqslant n) \\ \chi_i + \chi_n &= a_i + \cdots + a_{n-2} + a_n, && (1 \leqslant i \leqslant n-1) \\ \chi_i + \chi_{n-1} &= a_i + \cdots + a_{n-2} + a_{n-1} + a_n, && (1 \leqslant i \leqslant n-2) \\ \chi_i + \chi_j &= \begin{matrix} a_i + \cdots + a_{j-1} \\ +2a_j + \cdots + 2a_{n-2} + a_{n-1} + a_n. \end{matrix} && (1 \leqslant i < j \leqslant n-2) \end{aligned} \tag{6.6.6}$$

Among them, the highest root a_0 relative to Δ is

$$a_0 := \chi_1 + \chi_2 = a_1 + 2a_2 + \cdots + 2a_{n-2} + a_{n-1} + a_n. \tag{6.6.7}$$

Moreover, the sum of positive roots is

$$2\rho = \sum_{i=1}^{n-2} i(2n-1-i)a_i + \binom{n}{2}(a_{n-1} + a_n). \tag{6.6.8}$$

(iii). The fundamental coweights relative to Δ are the following:

$$\begin{aligned} \omega_i &= \mathbf{e}_1 + \cdots + \mathbf{e}_i, && (1 \leqslant i \leqslant n-2) \\ \omega_{n-1} &= \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_{n-1} - \mathbf{e}_n), \\ \omega_n &= \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n). \end{aligned} \tag{6.6.9}$$

Hence, the coweight lattice $\mathcal{P}^\vee$ is $\mathbb{Z}^n + \mathbb{Z}\frac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n)$.

(iv). The fundamental alcove associated to Δ can be expressed as follows:

$$C = \left\{ x \in \mathbb{A} \;\middle|\; \begin{array}{c} \chi_1(x) > \cdots > \chi_{n-1}(x) > |\chi_n(x)|, \\ \chi_1(x) + \chi_2(x) < 1 \end{array} \right\}. \tag{6.6.10}$$

The extreme points of C other than $v_0 = o$ are the following:

$$\begin{aligned} v_1 &= o + \omega_1 = o + \mathbf{e}_1, \\ v_i &= o + \tfrac{1}{2}\omega_i = o + \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_i), \qquad (2 \leqslant i \leqslant n-2) \\ v_{n-1} &= o + \omega_{n-1} = o + \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_{n-1} - \mathbf{e}_n), \\ v_n &= o + \omega_n = o + \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n). \end{aligned} \tag{6.6.11}$$

Note that v_1, v_{n-1}, v_n are special vertices, while v_i $(2 \leqslant i \leqslant n-2)$ are not special.

(v). First, apply the affine Weyl group W to v_0, we have

$$\mathcal{V}_0 = \left\{ x \in \mathbb{A} \;\middle|\; \begin{array}{c} \chi_1(x), \cdots, \chi_n(x) \in \mathbb{Z}, \\ \chi_1(x) + \cdots + \chi_n(x) \text{ is even} \end{array} \right\}. \tag{6.6.12}$$

Applying W_o to v_1, we see that $W_o.v_1$ consists of the points $x \in \mathbb{A}$ having one coordinate being 1 or -1 and all others are 0. Then we have:

$$\mathcal{V}_1 = \left\{ x \in \mathbb{A} \;\middle|\; \begin{array}{c} \chi_1(x), \cdots, \chi_n(x) \in \mathbb{Z}, \\ \chi_1(x) + \cdots + \chi_n(x) \text{ is odd} \end{array} \right\} \tag{6.6.13}$$

For each $1 < i < n-1$, $W_o.v_i$ consists of the points $x \in \mathbb{A}$ whose coordinates are either 0 or $\pm\frac{1}{2}$ and exactly i of them are nonzero. Then we have:

$$\mathcal{V}_i = \left\{ x \in \mathbb{A} \;\middle|\; \begin{array}{c} \chi_1(x), \cdots, \chi_n(x) \in \tfrac{1}{2}\mathbb{Z} \\ \text{and exactly } i \text{ of them are non-integers} \end{array} \right\}. \tag{6.6.14}$$

Then $W_o.v_{n-1}$ consists of the points $x \in \mathbb{A}$ whose coordinates are $\pm\frac{1}{2}$ and odd numbers of them are negative. Hence, we have:

$$\mathcal{V}_{n-1} = \left\{ x \in \mathbb{A} \;\middle|\; \begin{array}{c} \chi_1(x), \cdots, \chi_n(x) \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \\ \chi_1(x) + \cdots + \chi_n(x) - \frac{n}{2} \text{ is odd} \end{array} \right\} \tag{6.6.15}$$

Then $W_o.v_n$ consists of the points $x \in \mathbb{A}$ whose coordinates are $\pm\frac{1}{2}$ and even numbers of them are negative. Then we have:

$$\mathcal{V}_{n} = \left\{ x \in \mathbb{A} \;\middle|\; \begin{array}{c} \chi_1(x), \cdots, \chi_n(x) \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \\ \chi_1(x) + \cdots + \chi_n(x) - \frac{n}{2} \text{ is even} \end{array} \right\} \tag{6.6.16}$$

Hence, the vertices are all the point $x \in \mathbb{A}$ such that $\chi_j(x) \in \frac{1}{2}\mathbb{Z}$ for all j and the number of non-integer coordinates is neither 1 nor $n-1$. In particular, we have

$$\mathcal{V} \subsetneq o + \tfrac{1}{2}\mathbb{Z}^n \subsetneq o + \tfrac{1}{2}\mathcal{P}^\vee . \tag{6.6.17}$$

Let $\omega_i' = h_i^{-1}\omega_i$. Consider the following sets:

$$\begin{aligned} \mathcal{X}^{00} &:= o + \mathbb{Z}\omega_1' + \cdots + \mathbb{Z}\omega_n', \\ \mathcal{X}^{01} &:= \mathcal{X}^{00} - \tfrac{1}{2}(\omega_{n-1} + \omega_n), \\ \mathcal{X}^{10} &:= \mathcal{X}^{00} - \tfrac{1}{2}\omega_1, \\ \mathcal{X}^{11} &:= \mathcal{X}^{00} - \tfrac{1}{2}(\omega_1 + \omega_{n-1} + \omega_n), \\ \mathcal{X}^{(0)} &:= \mathcal{X}^{00} \cup \mathcal{X}^{10}, \\ \mathcal{X}^{(1)} &:= \mathcal{X}^{01} \cup \mathcal{X}^{11} . \end{aligned} \tag{6.6.18}$$

Lemma 6.6.1. *We have* $\mathcal{X}^{(0)} \cup \mathcal{X}^{(1)} = o + \frac{1}{2}\mathbb{Z}^n$.

Proof. It is clear that $\mathcal{X}^{(0)} \cup \mathcal{X}^{(1)} \subseteq o + \frac{1}{2}\mathbb{Z}^n$. Conversely, we have

$$\begin{aligned}
\tfrac{1}{2}\mathbf{e}_1 &= \tfrac{1}{2}\omega_1, \\
\tfrac{1}{2}\mathbf{e}_2 &= \omega_2' - \tfrac{1}{2}\omega_1, \\
&\vdots \\
\tfrac{1}{2}\mathbf{e}_i &= \omega_i' - \omega_{i-1}', \\
&\vdots \\
\tfrac{1}{2}\mathbf{e}_{n-1} &= \tfrac{1}{2}(\omega_{n-1} + \omega_n) - \omega_{n-2}', \\
\tfrac{1}{2}\mathbf{e}_n &= \omega_n - \tfrac{1}{2}(\omega_{n-1} + \omega_n).
\end{aligned}$$

Then the statement follows. □

To better describe the vertices, we need the notion introduced in Definition 6.4.1. Then the complement of $\mathcal{V}$ in $\mathcal{X}^{(0)} \cup \mathcal{X}^{(1)}$ can be described as follows.

Lemma 6.6.2. *A point $x \in \mathcal{X}^{(0)}$ is not a vertex if and only if either $J_x = \{j_1, j_2\}$ and $j_2 - j_1 = 1$, or $J_x = \{1\}$.*

We will use $\Xi^{(0)}$ to denote the set of points $x \in \mathbb{A}$ having the property in the lemma.

Proof. First, points in $\Xi^{(0)}$ cannot be vertices. If $J_x = \{1\}$, then x is not a vertex since either all $\chi_j(x)$ are integers except $j = 1$ (when $a_{n-1}(x) + a_n(x)$ is even), or all $\chi_j(x)$ are non-integers except $j = 1$ (when $a_{n-1}(x) + a_n(x)$ is odd). Next, suppose $J_x = \{j_1, j_2\}$ and $j_2 - j_1 = 1$. If $j_2 < n - 1$, then x is not a vertex since either all $\chi_j(x)$ are integers except $j = j_2$ (when $a_{n-1}(x) + a_n(x)$ is even), or all $\chi_j(x)$ are non-integers except $j = j_2$ (when $a_{n-1}(x) + a_n(x)$ is odd). If $j_2 = n - 1$, then x is not a vertex since $\chi_n(x) \notin \frac{1}{2}\mathbb{Z}$. If $j_2 = n$, then $J_x = \{n-1, n\}$ and we leave this situation in Lemma 6.6.3.

Conversely, suppose $x \in \mathcal{X}^{(0)}$ and $x \notin \Xi^{(0)}$. Then there are four cases:

(i). $|J_x| \geqslant 3$. Then among the coordinates of x, at least two of them are integers and two of them are non-integers.

(ii). $J_x = \{j_1, j_2\}$ and $1 < j_2 - j_1 < n-2$. Then $\chi_{j_1+1}(x), \cdots, \chi_{j_2}(x)$ are either all the integer coordinates of x, or all the non-integer coordinates of x.

(iii). $J_x = \{j_1\}$ and $1 < j_1 < n-1$. Then $\chi_1(x), \cdots, \chi_{j_1}(x)$ are either all the integer coordinates of x, or all the non-integer coordinates of x.

(iv). x has no jumps. Then it is a special vertex.

In any of above cases, x is a vertex by our characterization. □

Lemma 6.6.3. *A point $x \in \mathcal{X}^{(1)}$ is not a vertex if and only if $J_x \subseteq \{n-2, n-1, n\}$.*

Note that $x \in \mathcal{X}^{(1)}$ implies that $\{n-1, n\} \subseteq J_x$. We will use $\Xi^{(1)}$ to denote the set of points $x \in \mathbb{A}$ having the property that $\{n-1, n\} \subseteq J_x \subseteq \{n-2, n-1, n\}$.

Proof. First, points in $\Xi^{(1)}$ cannot be vertices. Indeed, if $x \in \Xi^{(1)}$, then $a_j(x) \in \mathbb{Z}$ for all $j < n-2$. Hence, $\chi_1(x), \cdots, \chi_{n-2}(x)$ are either all integers or all non-integers. Hence, by Eq. (6.6.17), for x to be a vertex, we must have that $2\chi_{n-1}(x)$ and $2\chi_n(x)$ are integers in the same parity. But $\{n-1, n\} \subseteq J_x$ implies that they are not.

Conversely, suppose $x \in \mathcal{X}^{(1)}$ and $x \notin \Xi^{(1)}$. Then $\{n-1, n\} \subseteq J_x$ implies that exactly one of $\chi_{n-1}(x)$ and $\chi_n(x)$ is an integer. If $j < n-2$ is an index in J_x, then exactly one of $\chi_j(x)$ and $\chi_{j+1}(x)$ is an integer. Hence, among the coordinates of x, at least two of them are integers and two of them are non-integers. □

We illustrate the related structures of $\mathcal{V}$ by the following diagram, where † denotes "being special".

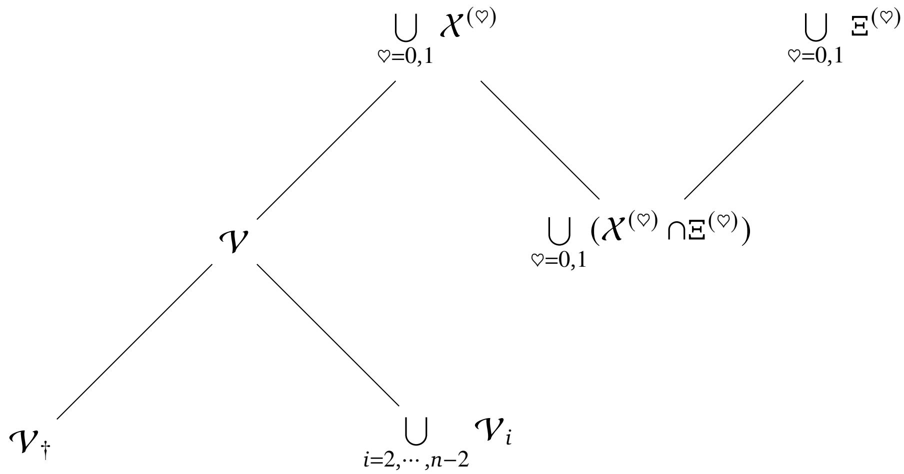


Figure 6.5. The set $\mathcal{V}$ of vertices and related structures in $\mathscr{A}(D_n)$.

(vi). Let $x \in \mathcal{D}({}^{v}C)$ be a vertex. We divide into two cases: $x \in \mathcal{X}^{(0)}$ or $x \in \mathcal{X}^{(1)}$.

First, let us assume $x \in \mathcal{X}^{(0)}$ and suppose x has jumps $j_1, \cdots, j_s$, ordered from smallest to largest. To construct a path between x and a special vertex x_0 in $\mathcal{D}({}^{v}C)$ verifying Eq. (6.2.13), we need the following lemmas.

Lemma 6.6.4. *Suppose either $j_s - j_{s-1} > 1$ or $s > 3$. Then $y = x - \frac{1}{2}\omega_{j_1}$ is a vertex in $\mathcal{D}({}^{v}C) \cap \mathcal{X}^{(0)}$ adjacent to x.*

Proof. It is clear that $y \in \mathcal{X}^{(0)}$. Then the proof is similar to Lemma 6.5.2 except that the root $a_{[j_s]}$ is defined as follows:

$$a_{[j_s]} = \begin{cases} a_{j_s-1} + 2a_{j_s} + \cdots + 2a_{n-2} + a_{n-1} + a_n & \text{if } j_s - j_{s-1} > 1, \\ a_{j_{s-2}} + \cdots + a_{j_s-1} + 2a_{j_s} + \cdots + 2a_{n-2} + a_{n-1} + a_n & \text{otherwise,} \end{cases}$$

Hence, we omit the proof here. □

Note that $a_0(y) \in \mathbb{Z}$ and $\lceil a_0(x) \rceil - a_0(x) = 1$. Hence, by repeating using Lemma 6.6.4, we can reduce our problem to the case where $s = 3$, or further $s = 1$ if we start with

$j_s - j_{s-1} > 1$.

Now, we may assume either $s \leqslant 3$ with $j_s - j_{s-1} = 1$ or $s = 1$.

Lemma 6.6.5. *Suppose $s = 3$ and $j_1 > 1$. Then $x_0 = x - \frac{1}{2}(\omega_{j_1} - \omega_{j_2} + \omega_{j_3})$ is a special vertex in $\mathcal{D}({}^{v}C)$ adjacent to x and verifying Eq. (6.2.13).*

Proof. The proof is the same as Lemma 6.5.3. □

Lemma 6.6.6. *Suppose $s = 3$, $j_3 - j_2 = 1$ and $j_1 = 1$, then $y = x - \frac{1}{2}(-\omega_{j_1} + \omega_{j_2})$ is a vertex in $\mathcal{D}({}^{v}C) \cap \mathcal{X}^{(0)}$ adjacent to x.*

Proof. It is clear that $y \in \mathcal{X}^{(0)}$. Then the proof is similar to Lemma 6.5.4 except that the root $a_{[j_s]}$ is defined as follows:

$$a_{[j_s]} = a_{j_1} + \cdots + a_{j_2} + 2a_{j_3} + \cdots + 2a_{n-2} + a_{n-1} + a_n.$$

Hence, we omit the proof here. □

Note that the only jump of y is $j_3 > 1$. Hence, $a_0(y) \in \mathbb{Z}$ and $\lceil a_0(x) \rceil - a_0(y) = 1$. Therefore, Lemma 6.6.6 reduces our problem to the case where $s = 1$.

Note that, by Lemma 6.6.2, $s = 2$ and $j_s - j_{s-1} = 1$ contradict to each other. Therefore, we may assume $s = 1$ now. By Lemma 6.6.2 again, we must have $j_1 > 1$. Let $x_0 = x - \frac{1}{2}\omega_{j_1}$. Then it is in a special vertex in $\mathcal{D}({}^{v}C)$ since $a_1(x_0), \cdots, a_n(x_0)$ are non-negative integers. Applying Lemma 6.2.5 to the simple roots $a_1, \cdots, a_n$ and the vertices x and x_0, we see that they are adjacent. Moreover, we have $a_0(x) - a_0(x_0) = 1$ verifying Eq. (6.2.13). This finishes the proof of Theorem 6.1 when $x \in \mathcal{X}^{(0)}$.

Next, let us assume $x \in \mathcal{X}^{(1)}$. Then we must have $\{n-1, n\} \subseteq J_x$. Suppose x has jumps $j_1, \cdots, j_s, n-1, n$, ordered from smallest to largest. To construct a path between x and a special vertex x_0 in $\mathcal{D}({}^{v}C)$ verifying Eq. (6.2.13), we need the following lemmas.

Lemma 6.6.7. *Suppose either $j_s - j_{s-1} > 1$ or $s > 2$. Then $y = x - \frac{1}{2}\omega_{j_1}$ is a vertex in $\mathcal{D}({}^{v}C) \cap \mathcal{X}^{(1)}$ adjacent to x.*

Proof. It is clear that $y \in \mathcal{X}^{(1)}$. When $j_s - j_{s-1} > 1$ or $s > 3$, the proof is similar to Lemma 6.6.4 except that we have to define $a_{[n-1]}$ and $a_{[n]}$ as follows:

$$a_{[n-1]} = a_{j_s} + \cdots + a_{n-2} + a_{n-1},$$

$$a_{[n]} = a_{j_s} + \cdots + a_{n-2} + a_n.$$

When $s = 3$ and $j_3 - j_2 = 1$, the proof still works if we define $a_{[j_s]}$ as follows:

$$a_{[j_3]} = a_{j_2} + \cdots + a_n. \qquad \square$$

Note that $a_0(y) \in \mathbb{Z}$ and $\lceil a_0(x) \rceil - a_0(x) = 1$. Hence, by repeating using Lemma 6.6.7, we can reduce our problem to the case where $s = 2$, or further $s = 1$ if we start with $j_s - j_{s-1} > 1$.

Now, we may assume either $s \leqslant 2$ with $j_s - j_{s-1} = 1$ or $s = 1$.

Lemma 6.6.8. *Suppose $s = 2$ and $j_1 > 1$. Then $x_0 = x - \frac{1}{2}(\omega_{j_1} - \omega_{j_2} + \omega_{n-1} + \omega_n)$ is a special vertex in $\mathcal{D}({}^{v}C)$ adjacent to x and verifying Eq. (6.2.13).*

Proof. First note that $a_0(x) \in \mathbb{Z}$ and

$$a_0(x) - a_0(x_0) = \tfrac{1}{2}a_0(\omega_{j_1} - \omega_{j_2} + \omega_{n-1} + \omega_n) = 1.$$

Then the proof is similar to Lemma 6.5.3 except that there is no j_3 and that we need to define $a_{[n-1]}$ and $a_{[n]}$ as follows:

$$a_{[n-1]} = a_{j_2} + \cdots + a_{n-2} + a_{n-1},$$

$$a_{[n]} = a_{j_2} + \cdots + a_{n-2} + a_n.$$

Then the statement follows. □

Lemma 6.6.9. *Suppose* $s = 2$ *and* $j_1 = 1$*, then* $x_0 = x - \frac{1}{2}(-\omega_{j_1} + \omega_{j_2} - \omega_{n-1} + \omega_n)$ *is a special vertex in* $\mathcal{D}({}^{v}C)$ *adjacent to* x *and verifying Eq. (6.2.13).*

Proof. First note that $a_0(x) \in \frac{1}{2} + \mathbb{Z}$ and

$$a_0(x) - a_0(x_0) = \tfrac{1}{2}a_0(-\omega_{j_1} + \omega_{j_2} - \omega_{n-1} + \omega_n) = \tfrac{1}{2}.$$

Then the proof is similar to Lemma 6.6.8 except that $a_{[n]}$ is defined as follows:

$$a_{[n]} = a_1 + \cdots + a_n.$$

Then the statement follows. □

Lemma 6.6.10. *Suppose* $s = 1$*. Then* $x_0 = x - \frac{1}{2}(\omega_{j_1} - \omega_{n-1} + \omega_n)$ *is a special vertex in* $\mathcal{D}({}^{v}C)$ *adjacent to* x *and verifying Eq. (6.2.13).*

Proof. First note that

$$a_0(x) - a_0(x_0) = \tfrac{1}{2}a_0(\omega_{j_1} - \omega_{n-1} + \omega_n) = \tfrac{1}{2}a_0(\omega_{j_1}).$$

Hence, $\lceil a_0(x) \rceil - a_0(x_0) = 1$. Then the proof is similar to Lemma 6.6.8 or Lemma 6.6.9 except that there is no j_2 and that the root $a_{[n]}$ is defined as follows:

$$a_{[n]} = a_{n-2} + a_{n-1} + a_n.$$

Here we need $j_1 < n - 2$, which is guaranteed by Lemma 6.6.3. $\square$

This finishes the proof of Theorem 6.1.

Chapter 7.

Simplicial Distance and Simplicial Balls

In this chapter, we continue to discuss the simplicial distance. We will derive some consequences of Theorem 6.1.

§ 7.1. Lattice descriptions of the simplicial distance

We begin with the following characterization.

Theorem 7.1. *In the Bruhat-Tits building $\mathscr{B}(\mathsf{GL}(V))$, two vertices x and y have simplicial distance at most d if and only if they admit representatives $L \in x$ and $L' \in y$, respectively, such that*

$$L \supseteq L' \supseteq \varpi^d L. \tag{7.1.1}$$

Our characterization Theorem 6.1 allows us to prove the above and further to seek its siblings in the Bruhat-Tits buildings of other classical groups.

To better clarify the relation between Eq. (7.1.1) and the simplicial distance, let's define

$$d_l(x,y) := \min\left\{d \,\middle|\, \exists L \in x, L' \in y \text{ s.t. } L \supseteq L' \supseteq \varpi^d L\right\} \tag{7.1.2}$$

First note that (notations are as in § 5.2 and 5.3).

Lemma 7.1.1. *For vertices x and y contained in a common apartment $\mathscr{A}(\mathbf{\Lambda})$, we have*

$$d_l(x,y) = \max_{\lambda \in \mathbf{\Lambda}} [x : y]_\lambda - \min_{\lambda \in \mathbf{\Lambda}} [x : y]_\lambda. \tag{7.1.3}$$

Proof. Let L and L' be two lattices in V, that are split by a common frame $\mathbf{\Lambda}$. Then we have $L \supseteq L'$ if and only if

$$(L : L')_\lambda \leqslant 0 \qquad \text{for all} \qquad \lambda \in \mathbf{\Lambda}. \tag{7.1.4}$$

Hence, Eq. (7.1.1) holds if and only if

$$-d \leqslant (L : L')_\lambda \leqslant 0 \qquad \text{for all} \qquad \lambda \in \mathbf{\Lambda}. \tag{7.1.5}$$

In particular,

$$\max_{\lambda \in \mathbf{\Lambda}} (L : L')_\lambda - \min_{\lambda \in \mathbf{\Lambda}} (L : L')_\lambda \leqslant d.$$

This implies that

$$\max_{\lambda \in \mathbf{\Lambda}} [x : y]_\lambda - \min_{\lambda \in \mathbf{\Lambda}} [x : y]_\lambda \leqslant d. \tag{7.1.6}$$

Conversely, we can always choose representatives $L \in x$ and $L' \in y$ such that

$$\max_{\lambda \in \mathbf{\Lambda}} (L : L')_\lambda = 0.$$

Then Eq. (7.1.4) follows and Eq. (7.1.6) is thus equivalent to Eq. (7.1.5). □

Now, let's prove Theorem 7.1

Proof. Notations are as in § 5.2. Let x and y be two vertices in it, contained in a common apartment, saying $\mathscr{A}(\mathbf{\Lambda})$. Then by Eq. (5.2.1), the elementary index $[x:y]$ is identified with the vector $y-x$. In particular,

$$[x:y]_i - [x:y]_j = \chi_j(x-y) - \chi_i(x-y). \tag{7.1.7}$$

On the other hand, by Theorem 6.1, Eq. (6.3.2), and the fact that all vertices are special, we have

$$\begin{aligned} d(x,y) &= \max\left\{(\chi_i - \chi_j)(x-y) \,\middle|\, 1 \leqslant i \neq j \leqslant n+1\right\} \\ &= \max_i \chi_i(x-y) - \min_i \chi_i(x-y). \end{aligned}$$

Therefore, by Eq. (7.1.7), we have

$$d(x,y) = \max_i [x:y]_i - \min_i [x:y]_i.$$

$$d(x,y) = \max\left\{(\chi_i - \chi_j)([x:y]) \,\middle|\, 1 \leqslant i \neq j \leqslant n+1\right\}. \tag{7.1.8}$$

Then Theorem 7.1 follows from Lemma 7.1.1. □

Unfortunately, Eq. (7.1.1) does not characterize the simplicial distance in the Bruhat-Tits building of a general classical group. Following § 5.3, assume $(V, \langle\cdot \mid \cdot\rangle)$ is a hermitian space over K. Let's first get a characterization of d_l.

Theorem 7.2. *In the Bruhat-Tits building $\mathscr{B}(\mathsf{SO}(V))$, two vertices x and y admit representatives $L \in x$ and $L' \in y$ satisfying*

$$L \supseteq L' \supseteq \varpi^d L,$$

if and only if $|2\chi_i(x-y)| \leqslant d$ *for all the coordinates* χ_i.

Proof. Notations are as in § 5.3. Let x and y be two vertices in it, contained in a common apartment, saying $\mathscr{A}(\mathbf{\Lambda})$. By Eq. (5.3.2), the hyperbolic index $[x:y]^{\pm}$ has components

$$[x:y]_i^{\pm} = \frac{1}{2}\left([x:y]_{\lambda_i^+} - [x:y]_{\lambda_i^-}\right). \tag{7.1.9}$$

Let $L \in x$ and $L' \in y$ be two representatives. Then we have

$$(L:L')_{\lambda_i^+} = [x:y]_i^{\pm} + c \qquad \text{and} \qquad (L:L')_{\lambda_i^-} = -[x:y]_i^{\pm} + c$$

for some $c \in \mathbb{R}$. Then

$$\begin{aligned} \max_{\lambda\in\mathbf{\Lambda}}(L:L')_{\lambda} &= \max_i \left|[x:y]_i^{\pm}\right| + c, \\ \min_{\lambda\in\mathbf{\Lambda}}(L:L')_{\lambda} &= -\max_i \left|[x:y]_i^{\pm}\right| + c. \end{aligned}$$

Therefore, by Lemma 7.1.1, we have

$$d_l(x,y) = 2\max_i \left|[x:y]_i^{\pm}\right|. \tag{7.1.10}$$

By Eq. (5.3.1), the hyperbolic index $[x:y]^{\pm}$ is identified with the vector $y - x \in {}^{v}\mathscr{A}(\mathbf{\Lambda})$. Hence, we have $d_l(x,y) = \max_i |2\chi_i(y-x)|$. $\square$

We see that there are two issues to prevent $d_l(x,y) = d(x,y)$:

(i). $2\chi_i$ may not be a root, and

(ii). $\max\{\lceil a(x)\rceil - \lfloor a(y)\rfloor \mid a \in \Phi\}$ may not be exactly $a(x-y)$ for some root a.

However, we have

Theorem 7.3. *In the Bruhat-Tits building $\mathscr{B}(\mathsf{Sp}(V))$, a vertex x has simplicial distance at most d from the origin if and only if it admits representative $L \in x$ such that*

$$L_o \supseteq L \supseteq \varpi^d L_o, \tag{7.1.11}$$

where L_o is the primitive lattice belonging to o.

Proof. This is because $d(x, o) = \max_{a\in\Phi}\lceil a(x)\rceil = \max_i |2\chi_i(x)|$. □

Theorem 7.4. *In the Bruhat-Tits building $\mathscr{B}(\mathsf{SO}(V))$, a vertex x has simplicial distance at most d from the origin if and only if in the elementary index $[x : o]$,*

$$|[x : o]_\lambda - [x : o]_{\lambda'}| \leqslant d \tag{7.1.12}$$

where $\lambda, \lambda' \in \mathbf{\Lambda}$ satisfying

$$\langle \lambda \,|\, \lambda' \rangle + \langle \lambda' \,|\, \lambda \rangle = 0. \tag{7.1.13}$$

Remark. Recall that, by Lemma 7.1.1, if we drop the requirement Eq. (7.1.13), then Eq. (7.1.12) is equivalent to Eq. (7.1.11). In the case $\langle \cdot \mid \cdot \rangle$ is alternative, the condition Eq. (7.1.13) is satisfied trivially, and the statement becomes Theorem 7.3. Hence, in the proof below, we assume $\langle \cdot \mid \cdot \rangle$ is symmetric. Namely, the Bruhat-Tits building is of split type B_n or D_n.

Proof. Choose ${}^vC$ to be a Weyl chamber such that $x \in \mathscr{D}({}^vC)$. By Theorem 6.1 and Eq. (6.5.7) (or Eq. (6.6.7)), we have

$$d(x, o) = \lceil(\chi_1 + \chi_2)(x - o)\rceil$$

By Eq. (5.3.1), the vector $x - o$ is identified with the hyperbolic index $[x : o]^{\pm}$. Hence,

$$d(x, o) = \left\lceil [x : o]^{\pm}_1 + [x : o]^{\pm}_2 \right\rceil. \tag{7.1.14}$$

To see what does the above measure in terms of lattices, we choose $L \in x$ to be the standard representative in 5.3.3 and 5.3.4. That is to say, there is a primitive lattice L_0 (do not be confused with the primitive lattice L_o representing o) such that

$$L_0 \supseteq L \supseteq \varpi L_0 \qquad \text{and} \qquad \langle L \,|\, L \rangle \subseteq \mathfrak{m}_K.$$

Then we have

$$-1 \leqslant (L : L_0)_\lambda \leqslant 0 \qquad \text{and} \qquad (L : L_0)_{\lambda_i^+} + (L : L_0)_{\lambda_i^-} \leqslant -1.$$

Hence, by Eq. (7.1.9), we have

$$(L : L_0)_{\lambda_i^+} = \left\lceil [x : [L_0]]^{\pm}_i \right\rceil - 1 \qquad \text{and} \qquad (L : L_0)_{\lambda_i^-} = \left\lceil -[x : [L_0]]^{\pm}_i \right\rceil - 1.$$

Since both L_0 and L_o are primitive, we thus have

$$(L : L_o)_{\lambda_i^+} = \left\lceil [x : o]^{\pm}_i \right\rceil - 1 \qquad \text{and} \qquad (L : L_o)_{\lambda_i^-} = \left\lceil -[x : o]^{\pm}_i \right\rceil - 1.$$

Therefore, by Eq. (7.1.14), we have

$$d(x, o) = \max\left\{ (L : L_o)_{\lambda_1^+} - (L : L_o)_{\lambda_2^-}, (L : L_o)_{\lambda_2^+} - (L : L_o)_{\lambda_1^-} \right\}. \tag{7.1.15}$$

By $x \in \mathscr{D}({}^{v}C)$, we have

$$[x : o]^{\pm}_1 \geqslant \cdots \geqslant [x : o]^{\pm}_{n-1} \geqslant \pm[x : o]^{\pm}_n \geqslant -[x : o]^{\pm}_{n-1} \geqslant \cdots \geqslant -[x : o]^{\pm}_1.$$

Hence, Eq. (7.1.15) amounts to say

$$d(x,o) = \max\{(L : L_o)_\lambda - (L : L_o)_{\lambda'} \mid \lambda, \lambda' \in \mathbf{\Lambda}, \langle \lambda \mid \lambda' \rangle = 0\}.$$

Here, the condition $\langle \lambda \mid \lambda' \rangle = 0$ excludes $\{\lambda, \lambda'\} = \left\{\lambda_1^+, \lambda_1^-\right\}$. □

§ 7.2. Simplicial balls as fixed-point sets

In the rest of this dissertation, we will focus on the *simplicial ball*

$$B(r) := \{x \text{ is a vertex in } \mathscr{B} \mid d(x,o) \leqslant r\}.$$

This section devotes to prove the following theorem.

Theorem 7.5. *In an irreducible Bruhat-Tits building of split classical type, the simplicial ball $B(r)$ is precisely the set of fixed-vertices under the action of the Moy-Prasad subgroup** $P_{o,r}$.*

Proof. We first assume x belongs to the apartment $\mathscr{A}(\varphi)$ we start to construct the building in § 4.2. Then the Moy-Prasad subgroup $P_{o,r}$ is contained in the parahoric subgroup P_x if and only if

$$f'_x \leqslant f_o + r. \tag{7.2.1}$$

But this is equivalent to

$$\lceil -a(x) \rceil \leqslant r \qquad \text{for all} \qquad a \in \Phi.$$

By Theorem 6.1, this amounts to say $d(x,o) \leqslant r$.

*Recall its definition in Definition 4.4.5.

Next, for a general x, there is a $g \in P_o$ such that $g.x \in \mathscr{A}(\varphi)$. Hence, x is fixed by $P_{o,r}$ if and only if $g.x$ is fixed by $\mathrm{inn}(g).P_{o,r}$. But $P_{o,r}$ is a normal subgroup of P_o, hence $\mathrm{inn}(g).P_{o,r} = P_{o,r}$. We have seen that it fixes $g.x \in \mathscr{A}(\varphi)$ if and only if $d(g.x, o) \leqslant d$. Since g maps a path to a path of the same length, we have $d(x, o) = d(g.x, g.o) = d(g.x, o)$. Therefore, x is fixed by $P_{o,r}$ if and only if $d(x, o) \leqslant r$. $\square$

§ 7.3. Simplicial distance along extensions

Let E/K be a *totally ramified extension*. Namely,

(i). the *inertia degree* is 1, i.e. the residue field $\mathscr{O}_E/\mathfrak{m}_E$ equals κ; and

(ii). the *ramification index* e is precisely $[E : K]$.

But the effect we care about is:

(iii). the valuation $\mathrm{val}(\,\cdot\,)$ extends to E with $[\mathrm{val}(E^\times) : \Gamma] = e$.

The base change $\mathsf{G} \mapsto \mathsf{G}_E$ gives us an embedding of Euclidean buildings:

$$j_{E/K} \colon \mathscr{B}(\mathsf{G}) \longhookrightarrow \mathscr{B}(\mathsf{G}_E).$$

We may omit $j_{E/K}$ and identify points in $\mathscr{B}(\mathsf{G})$ with their images. In particular, we may ask how does the simplicial distance change along the extension E/K.

To compare the simplicial distances in $\mathscr{B}(\mathsf{G})$ and $\mathscr{B}(\mathsf{G}_E)$, we need to denormalize the valuation. In this section,

- $\lceil \cdot \rceil_E$ and $\lceil \cdot \rceil_K$ will denote the *ceiling functions* with respect to the valuation group $\mathrm{val}(E^\times)$ and $\mathrm{val}(K^\times)$ respectively, following Definition 4.3.1;
- γ_E and γ_K will denote the valuations of the uniformizers in E and K respectively;

- $d_E(x, y)$ and $d_K(x, y)$ will denote the simplicial distance between x and y in $\mathscr{B}(\mathsf{G})$ and $\mathscr{B}(\mathsf{G}_E)$ respectively.

Theorem 7.6. *Let $\mathscr{B}$ be a Bruhat-Tits building of split classical type, E/K a totally ramified extension, and $\mathscr{B}_E$ its base change along E/K. Suppose either both x, y are special vertices in $\mathscr{B}$ or the ramification index e is odd. Then the simplicial distances in $\mathscr{B}$ and $\mathscr{B}_E$ have the following relation:*

$$d_E(x, y) = e \cdot d_K(x, y). \tag{7.3.1}$$

Proof. By Corollary 6.1.2, we may assume $\mathscr{B}$ is irreducible. Then Theorem 6.1 tells us that (after denormalize the valuation)

$$d_E(x, y) = \frac{1}{\gamma_E} \max_{a \in \Phi}(\lceil a(x) \rceil_E + \lceil a(-y) \rceil_E),$$
$$d_K(x, y) = \frac{1}{\gamma_K} \max_{a \in \Phi}(\lceil a(x) \rceil_K + \lceil a(-y) \rceil_K).$$

Since $\gamma_E = \frac{1}{e}\gamma_K$, when both x, y are special vertices, Eq. (7.3.1) holds. However, Eq. (7.3.1) is not always true: there may be some $a \in \Phi$ such that either $a(x)$ or $a(y)$ belongs Γ_E while not in Γ_K. Since we always have $a(x) \in \frac{1}{2}\Gamma_K$ for all roots $a \in \Phi$, that situation happens only if $2 \mid e$. We thus finish proving the theorem. □

Chapter 8.

Formula of the Simplicial Volume

In this chapter, we will deduce the following formulas for the simplicial volume and simplicity surface area.

Theorem 8.1. *Let $\mathscr{B}$ be a Bruhat-Tits building of split type Φ over a local field K with residue cardinality q. Then the simplicial volume $\mathrm{SV}(\,\cdot\,)$ and the simplicial surface area $\mathrm{SSA}(\,\cdot\,)$ in it can be computed by the following formulas:*

$$\mathrm{SV}(r) = \sum_{I \subseteq \Delta} \frac{\mathscr{P}_{\Phi;I}(q)}{q^{\deg(\mathscr{P}_{\Phi;I})}} \sum_{x \in B(r, {}^{v}C, I)} \prod_{a(x)>0} q^{\lceil a(x) \rceil},$$

$$\mathrm{SSA}(r) = \sum_{I \subseteq \Delta} \frac{\mathscr{P}_{\Phi;I}(q)}{q^{\deg(\mathscr{P}_{\Phi;I})}} \sum_{x \in \partial(r, {}^{v}C, I)} \prod_{a(x)>0} q^{\lceil a(x) \rceil},$$

where

- *$\lceil\,\cdot\,\rceil$ is the ceiling function,*
- *Δ is a basis of the root system Φ,*
- *$\mathscr{P}_{\Phi;I}$ is the Poincaré polynomial associated to the pair (Φ, I),*
- *${}^{v}C$ is a Weyl chamber of Φ,*
- *and the index sets $B(r, {}^{v}C, I)$ (resp. $\partial(r, {}^{v}C, I)$) consists of the vertices in $\overline{o + {}^{v}C}$*

having type I with simplicial distance at most r (resp. exactly r) from o.

The strategy is to employ a *strongly transitive* and *type-preserving* automorphism group. Let $\mathscr{B}$ be a Bruhat-Tits building of split type Φ. Then we can realize it as the Bruhat-Tits building of a simply-connected splittable semisimple group G having root system Φ over the ground local field K. The group G of K-points of G is such an automorphism group.

From now on, we fix a special vertex o in $\mathscr{B}$ and choose a split maximal torus T in G such that the apartment $\mathscr{A}$ associated to (G, T) contains o. We will follow the notations and conventions in Chapter 4. In particular, o is the reference point of the underlying Euclidean affine space $\mathbb{A}$ of $\mathscr{A}$.

For any vertex x in $\mathscr{B}$, it is clear that a type-preserving automorphism $\phi \in G$ will map a path from o to x to a path from $\phi(o)$ to $\phi(x)$. Hence, G preserves the simplicial distance. Therefore, we have

$$\mathrm{SV}(r) \text{ (resp. } \mathrm{SSA}(r)) = \sum_{x} \left[P_o : P_{o,x}\right], \tag{8.0.1}$$

where

- P_o is the stabilizer of o in G,
- $P_{o,x}$ is the stabilizer of x in P_o, and
- the summation is taking over the intersection of $B(r)$ (resp. $\partial(r)$) with a fundamental domain $\mathscr{D}$ of the action of P_o.

The computation will be done as follows. In § 8.1, we break the index $\left[P_o : P_{o,x}\right]$ into two factors. In § 8.2, we will see that the first factor can be given by Poincaré polynomials. In § 8.3, we will compute the second factor using the theory of concave functions. In § 8.4, we will describe a fundamental domain of the action of P_o and finally prove Theorem 8.1.

§ 8.1. Parahoric reduction

The goal of this section is to break the index $\left[P_o : P_{o,x}\right]$ into two factors, one of which is a power of q. For this purpose, we need some facts about concave functions, recalling § 4.4.

8.1.1. Let f be a concave function on Φ. Let $\overline{\mathfrak{R}}_f$ and $\overline{\mathfrak{G}}_f$ denote the unipotent radical and the reductive quotient of $(\mathfrak{G}_f)_\kappa$ respectively. Since κ is a finite field, we have $\overline{\mathfrak{G}}_f(\kappa) = \mathfrak{G}_f(\kappa)/\overline{\mathfrak{R}}_f(\kappa)$. Note that, by Theorem 4.2, we have [BT-2, 4.6.4]

(i). $(\mathfrak{T}_0)_\kappa$ is the centralizer of itself in $(\mathfrak{G}_f)_\kappa$.

(ii). Φ is the root system of the pair $\left((\mathfrak{G}_f)_\kappa, (\mathfrak{T}_0)_\kappa\right)$ and for any $a \in \Phi$, $(\mathfrak{U}_{a,f(a)})_\kappa$ is the root subgroup associated to it.

Let f^* be defined as in 4.3.8. Using the filtrations in 4.4.1 and 4.4.2, we have:

(iii). The unipotent radical of $(\mathfrak{T}_0)_\kappa$ is the image of $(\mathfrak{T}_{0+})_\kappa$ in it.

(iv). [BT-2, 4.6.10] The intersection of the unipotent radical $\overline{\mathfrak{R}}_f$ and the root subgroup $(\mathfrak{U}_{a,f(a)})_\kappa$ is the image of $(\mathfrak{U}_{a,f^*(a)})_\kappa$ in $(\mathfrak{G}_f)_\kappa$.

Remark. However, in our case, what are in the unipotent radical is clear: the congruence property in 4.4.1 implies that $\mathfrak{T}_{0+}$ maps to 0 in $(\mathfrak{T}_0)_\kappa$; then 4.4.2 plus the fact that $\mathfrak{U}_{a,f(a)}$ is one-dimensional vectorial group imply that the intersection $\overline{\mathfrak{R}}_f \cap (\mathfrak{U}_{a,f(a)})_\kappa$ is either trivial or the entire $(\mathfrak{U}_{a,f(a)})_\kappa$.

(v). [BT-2, 1.1.11] The multiplication morphism

$$\prod_{a\in\Phi_f^+}\left(\overline{\mathfrak{R}}_f\cap(\mathfrak{U}_{a,f(a)})_\kappa\right)\cdot\mathscr{R}_u((\mathfrak{T}_0)_\kappa)\cdot\prod_{a\in\Phi_f^-}\left(\overline{\mathfrak{R}}_f\cap(\mathfrak{U}_{a,f(a)})_\kappa\right)\longrightarrow\overline{\mathfrak{R}}_f$$

is an isomorphism.

Therefore, we have exact sequence

$$P_{f^*}\hookrightarrow\mathfrak{G}_f(\mathcal{O}_K)\twoheadrightarrow\overline{\mathfrak{G}}_f(\kappa).$$

Moreover, let $\overline{\mathfrak{T}}_f$ (resp. $\overline{\mathfrak{U}}_{f;a}$) denote the image of $\mathfrak{T}_0$ (resp. $\mathfrak{U}_{a,f(a)}$) in $\overline{\mathfrak{G}}_f$. Then, $(\overline{\mathfrak{G}}_f,\overline{\mathfrak{T}}_f)$ is a split reductive group with root system Φ_f and root subgroups $(\overline{\mathfrak{U}}_{f;a})_{a\in\Phi_f}$.

Remark. Note that P_{f^*} is a *pro-unipotent group* in the following sense. First, we have a projective system of groups

$$\cdots\twoheadrightarrow\mathfrak{G}_f\left(\mathcal{O}_K/\varpi^{i+1}\right)\twoheadrightarrow\mathfrak{G}_f\left(\mathcal{O}_K/\varpi^{i}\right)\twoheadrightarrow\cdots\twoheadrightarrow\mathfrak{G}_f(\kappa).$$

Then, by the theory of Greenberg functors, we have a projective system of algebraic groups over κ:

$$\cdots\twoheadrightarrow\mathscr{F}_{\mathcal{O}_K/\varpi^{i+1}}\left(\mathfrak{G}_f\right)\twoheadrightarrow\mathscr{F}_{\mathcal{O}_K/\varpi^{i}}\left(\mathfrak{G}_f\right)\twoheadrightarrow\cdots\twoheadrightarrow(\mathfrak{G}_f)_\kappa.$$

It induces a projective system of their unipotent radicals and for each $\mathscr{F}_{\mathcal{O}_K/\varpi^{i}}\left(\mathfrak{G}_f\right)$, its unipotent radical is precisely the preimage of $\overline{\mathfrak{R}}_f$. Therefore, P_{f^*} is the limit of a projective system of groups of κ-points of unipotent algebraic groups over κ.

8.1.2. Let f,g be two concave functions on Φ with $g\geqslant f$. Then $P_g\subseteq P_f$ extends to a morphism of group schemes [BT-2, 6.4.24]

$$\mathfrak{G}_g\longrightarrow\mathfrak{G}_f\,.$$

Since $g^* \geqslant f^*$, the image of

$$\mathscr{R}_u\big((\mathfrak{G}_g)_\kappa\big) \subset (\mathfrak{G}_g)_\kappa \longrightarrow (\mathfrak{G}_f)_\kappa$$

is contained in $\mathscr{R}_u\big((\mathfrak{G}_f)_\kappa\big)$ by 8.1.1.

Now, suppose for any $a \in \Phi$, either $f(a) = g(a)$ or $f(-a) = g(-a)$. Then

$$\Psi_{f,g} := \{a \in \Phi \mid f(a) = g(a)\}$$

is a parabolic subset. The image of $(\mathfrak{U}_{a,g(a)})_\kappa$ in $(\mathfrak{G}_f)_\kappa$ is either the entire $(\mathfrak{U}_{a,f(a)})_\kappa$ if $a \in \Psi_{f,g}$ or contained in $\mathscr{R}_u\big((\mathfrak{G}_f)_\kappa\big)$ if $a \notin \Psi_{f,g}$. So the image of $(\mathfrak{G}_g)_\kappa$ in $\overline{\mathfrak{G}}_f$ is generated by $\overline{\mathsf{T}}$ and $\overline{\mathsf{U}}_a$ for all $a \in \Psi$. This shows that the image $\overline{\mathsf{P}}_{f,g}$ is a parabolic subgroup of $\overline{\mathfrak{G}}_f$ with parabolic subset $\Psi_{f,g}$.

Example 8.1.3. Let Ω be a set in an apartment containing a special point x. Then we have $f_\Omega \geqslant f_x$ and for any $a \in \Phi$, either $f_\Omega(a) = f_x(a)$ or $f_\Omega(-a) = f_x(-a)$. Hence, above applies, and we get a parabolic subgroup of $\overline{\mathfrak{G}}_{f_x}$.

First note that P_o is a *parahoric subgroup* of G: it is indeed P_{f_o} using the notations in Example 4.3.4. Then we have a generating root group datum $(\overline{T}_o, (\overline{U}_{o,a})_{a\in\Phi})$ of type Φ in the quotient $P_o/P_{f_o^*}$, following 4.3.8. Moreover, using Theorem 4.2, we can see that this datum arises from a split reductive group over κ.

Lemma 8.1.4. *Let f be a concave function on Φ. Denote the unipotent radical and the reductive quotient of $(\mathfrak{G}_f)_\kappa$ by $\overline{\mathfrak{R}}_f$ and $\overline{\mathfrak{G}}_f$ respectively. Let $\overline{\mathfrak{T}}_f$ (resp. $\overline{\mathfrak{U}}_{f,a}$) be the image of $(\mathfrak{T}_0)_\kappa$ (resp. $(\mathfrak{U}_{a,f(a)})_\kappa$) in $\overline{\mathfrak{G}}_f$. Then $(\overline{\mathfrak{G}}_f, \overline{\mathfrak{T}}_f)$ is a split reductive group with root system Φ_f and root subgroups $(\overline{\mathfrak{U}}_{f,a})_{a\in\Phi_f}$. Moreover, the generating root group datum $(\overline{T}_f, (\overline{U}_{f,a})_{a\in\Phi_f})$ associated to $(\overline{\mathfrak{G}}_f, \overline{\mathfrak{T}}_f)$ is the same as in 4.3.8.*

Proof. Applying Theorem 4.2 to f, we see that:

(i). $(\mathfrak{T}_0)_\kappa$ is a split maximal torus in $(\mathfrak{G}_f)_\kappa$, the pair $\big((\mathfrak{G}_f)_\kappa, (\mathfrak{T}_0)_\kappa\big)$ has root system Φ, and for any $a \in \Phi$, $(\mathfrak{U}_{a,f(a)})_\kappa$ is the root subgroup associated to it.

We also refer to [BT-2, 4.6.4] for a direct proof.

By [BT-2, 1.1.11], the multiplication morphism

$$(*) \qquad \prod_{a\in\Phi_f^-}\Big(\overline{\mathfrak{R}}_f \cap (\mathfrak{U}_{a,f(a)})_\kappa\Big)\cdot \mathscr{R}_u((\mathfrak{T}_0)_\kappa)\cdot \prod_{a\in\Phi_f^+}\Big(\overline{\mathfrak{R}}_f \cap (\mathfrak{U}_{a,f(a)})_\kappa\Big) \longrightarrow \overline{\mathfrak{R}}_f$$

is an isomorphism. Hence, (i) implies that first statement of this lemma except that the root system is Φ_f.

Next, applying Theorem 4.2 to both f and f^*, and using the inductive systems in 4.4.1 and 4.4.2, we see that the inclusion $P_{f^*} \subseteq P_f$ extends to a homomorphism $\mathfrak{G}_{f_*} \to \mathfrak{G}_f$ so that

(ii). through the homomorphism $(\mathfrak{G}_{f_*})_\kappa \to (\mathfrak{G}_f)_\kappa$, the unipotent group $(\mathfrak{T}_{0+})_\kappa$ is mapped onto the unipotent radical $\mathscr{R}_u((\mathfrak{T}_0)_\kappa)$ of $(\mathfrak{T}_0)_\kappa$, and for any $a \in \Phi$, $(\mathfrak{U}_{a,f^*(a)})_\kappa$ is mapped onto the intersection of the unipotent radical $\overline{\mathfrak{R}}_f$ and the root subgroup $(\mathfrak{U}_{a,f(a)})_\kappa$.

We also refer to [BT-2, 4.6.10] for another proof.

Now, (ii) tells us that, through the reduction $P_{f^*} \subseteq P_f \twoheadrightarrow \mathfrak{G}_f(\kappa)$, T_{0+} (resp. $U_{a,f^*(a)}$) is mapped to the group of κ-points of $\mathscr{R}_u((\mathfrak{T}_0)_\kappa)$ (resp. $\overline{\mathfrak{R}}_f \cap (\mathfrak{U}_{a,f(a)})_\kappa$). Note that, $\overline{\mathfrak{R}}_f \cap (\mathfrak{U}_{a,f(a)})_\kappa$ is the entire $(\mathfrak{U}_{a,f(a)})_\kappa$ if and only if $f(a) = f^*(a)$. Then, using the isomorphism Eq. $(*)$, the second statement of the lemma follows, and we see that the root system of $(\overline{\mathfrak{G}}_f, \overline{\mathfrak{T}}_f)$ is Φ_f. $\square$

Now, back to our situation. Let $\overline{P}_o$ denote the quotient $P_o/P_{f_o^*}$ and $\overline{P}_{o,x}$ the image of $P_{o,x}$ in it. Then we claim that $\overline{P}_{o,x}$ is the group of κ-points of a parabolic subgroup $\overline{\mathfrak{P}}_{o,x}$ of $\overline{\mathfrak{G}}_o$, the reductive quotient of $(\mathfrak{G}_{f_o})_\kappa$. To see this, first note that $P_{o,x}$ is the group $P_{f_{\{o,x\}}}$ defined in Example 4.3.4. Then we consider the following lemma.

Lemma 8.1.5. *Let f, g be two concave functions on Φ with $g \geqslant f$. Suppose*

$$\Psi_{f,g} := \{a \in \Phi \mid f(a) = g(a)\}$$

is a parabolic subset of Φ. Then the image of $(\mathfrak{G}_g)_\kappa$ in $\overline{\mathfrak{G}}_f$ is a parabolic subgroup $\overline{\mathfrak{P}}_{f,g}$ containing $\overline{\mathfrak{T}}_f$, corresponding to the parabolic subset $\Psi_{f,g}$.

Proof. Since $g \geqslant f$, we have $P_g \subseteq P_f$, which extends to a homomorphism $\mathfrak{G}_g \to \mathfrak{G}_f$. By the proof of Lemma 8.1.4, we have the follows. First, since $g^* \geqslant f^*$, the image of $\mathscr{R}_u\big((\mathfrak{G}_g)_\kappa\big)$ in $(\mathfrak{G}_f)_\kappa$ is contained in $\mathscr{R}_u\big((\mathfrak{G}_f)_\kappa\big)$. Then, for each $a \in \Phi$, the image of $(\mathfrak{U}_{a,g(a)})_\kappa$ in $(\mathfrak{G}_f)_\kappa$ is either the entire $(\mathfrak{U}_{a,f(a)})_\kappa$ if $g(a) = f(a)$ or is contained in $\mathscr{R}_u\big((\mathfrak{G}_f)_\kappa\big)$ if $g(a) \geqslant f^*(a)$. Therefore, the image of $(\mathfrak{G}_g)_\kappa$ in $\overline{\mathfrak{G}}_f$ is generated by $\overline{\mathfrak{T}}_f$ and $\overline{\mathfrak{U}}_{f,a}$ for all $a \in \Psi$. Then the statement follows. □

The parabolic subset of Φ corresponding to $\overline{\mathfrak{P}}_{o,x}$ is

$$\Psi_{o,x} := \{a \in \Phi \mid a(x) \geqslant 0\}.$$

Now, we choose a Weyl chamber ${}^v C$ such that $x \in \overline{o + {}^v C}$. Let Φ^+ be the system of positive roots corresponding to ${}^v C$. Then we have $\Psi_{o,x} = \Phi^+ \cup \Phi_{o,x}$, where $\Phi_{o,x}$ is the root subsystem associated to the concave function $f_{\{o,x\}}$:

$$\Phi_{o,x} := \left\{a \in \Phi \,\middle|\, f_{\{o,x\}}(a) \in \Gamma, f_{\{o,x\}}(a) + f_{\{o,x\}}(-a) = 0\right\} = \{a \in \Phi \mid a(x) = 0\}.$$

The simple roots in $\Phi^+ \cap \Phi_{o,x}$ form a *type $I_{o,x}$* (see Conventions 2.4.5 and 3.5.3).

Convention 8.1.6. Fix a choice of ${}^{v}C$. We say a point $x \in \overline{o + {}^{v}C}$ has *type* I if $I_{o,x} = I$.

At this stage, we have

$$[P_o : P_{o,x}] = [\overline{P}_o : \overline{P}_{o,x}] \cdot [P_{f_o^*} : P_{f_o^*} \cap P_{o,x}], \tag{8.1.1}$$

where $\overline{P}_o$ is (the group of κ-points of) a splittable reductive group with root system Φ, $\overline{P}_{o,x}$ is (the group of κ-points of) a parabolic subgroup of the former having type $I_{o,x}$, and $P_{f_o^*}$ is a *pro-unipotent group* in the sense that it is a projective limit of groups, each of them is the group of κ-points of an unipotent group over κ.

§ 8.2. Poincaré polynomials of parabolic subgroups

This section treats the first factor $[\overline{P}_o : \overline{P}_{o,x}]$ in Eq. (8.1.1). That is the index of (the group of κ-points of) a parabolic subgroup in a splittable reductive group over κ.

Let $(\overline{\mathsf{G}}, \overline{\mathsf{T}})$ be a split reductive group over κ and $(\overline{\mathsf{P}}, {}^{v}F, \Psi, I)$ a quadruple as in Proposition 3.5.2. Then $\overline{\mathsf{G}}(\kappa)$ acts strongly transitively and type-preserving on the Tits building ${}^{v}\mathscr{B}$ of $\overline{\mathsf{G}}$, and $\overline{\mathsf{P}}(\kappa)$ is the stabilizer of ${}^{v}F$. Hence, the quotient $\overline{\mathsf{G}}(\kappa)/\overline{\mathsf{P}}(\kappa)$ counts the facets in ${}^{v}\mathscr{B}$ having type I. Note that $\overline{\mathsf{G}}(\kappa)/\overline{\mathsf{P}}(\kappa) \cong \overline{\mathsf{G}}/\overline{\mathsf{P}}(\kappa)$ according to *Lang's theorem* (see, e.g. [Mil17, 17.98]).

Let ${}^{v}W$ be the Weyl group of $(\overline{\mathsf{G}}, \overline{\mathsf{T}})$. Then the *generalized Bruhat decomposition* (see, e.g. [Mil17, 21.h and 21.i]) says that

$$\overline{\mathsf{G}}/\overline{\mathsf{P}} = \bigsqcup_{\bar{w} \in {}^{v}W/{}^{v}W_I} \mathsf{C}(\bar{w}) \cong \bigsqcup_{w \in {}^{v}W^I} \mathsf{C}(w),$$

where each $\mathsf{C}(\bar{w})$ (as well as $\mathsf{C}(w)$) is an affine space of dimension $\ell(w)$ (the *length* of w in ${}^{v}W$), called the *Schubert cell* of w, ${}^{v}W_I$ is the subgroup of ${}^{v}W$ generated by reflections with respect to the simple roots in I, and ${}^{v}W^I$ is a system of representatives.

Definition 8.2.1. The *Poincaré polynomial* of the pair (Φ, I) is the following:

$$\mathscr{P}_{\Phi;I}(z) := \sum_{w \in {}^{v}W^I} z^{\ell(w)}. \tag{8.2.1}$$

When $I = \emptyset$, it is denoted by $\mathscr{P}_\Phi$ and called the *Poincaré polynomial* of Φ.

Then we have $\overline{\mathsf{G}}/\overline{\mathsf{P}}(\kappa) = \mathscr{P}_{\Phi;I}(q)$. Note that the image of $\prod_{a\in\Phi\setminus\Psi} \overline{\mathsf{U}}_a \to \overline{\mathsf{G}}/\overline{\mathsf{P}}$ is the *big cell*. Hence, $\deg\left(\mathscr{P}_{\Phi;I}\right) = |\Phi \setminus \Psi|$.

At this stage, we already know that:

Lemma 8.2.2. *The index $\left[\overline{P}_o : \overline{P}_{o,x}\right]$ is computed by an integral polynomial $\mathscr{P}_{\Phi;I}$ of degree $|\Phi \setminus \Psi|$.*

The rest of this section aims to deduce $\mathscr{P}_{\Phi;I}$ from the information of (Φ, I).

Lemma 8.2.3. *$\mathscr{P}_{\Phi;I} = \mathscr{P}_\Phi/\mathscr{P}_{\Phi_I}$, where Φ_I is the root subsystem of Φ generated by I.*

Proof. Let $\overline{\mathsf{B}}$ be a Borel subgroup of $\overline{\mathsf{G}}$ contained in $\overline{\mathsf{P}}$. Then we have

$$\left[\overline{G} : \overline{P}\right] = \left[\overline{G} : \overline{B}\right] \cdot \left[\overline{P} : \overline{B}\right]. \tag{$*$}$$

Let $\overline{\mathsf{L}}$ be the Levi subgroup of $\overline{\mathsf{P}}$. Then $(\overline{\mathsf{L}}, \overline{\mathsf{T}})$ is a split reductive group with root system Φ_I. Moreover, $\overline{\mathsf{B}} \cap \overline{\mathsf{L}}$ is a Borel subgroup of $\overline{\mathsf{L}}$. Then we have

$$\overline{\mathsf{P}}/\overline{\mathsf{B}} = \overline{\mathsf{B}}\,\overline{\mathsf{L}}/\overline{\mathsf{B}} = \overline{\mathsf{L}}/\overline{\mathsf{B}} \cap \overline{\mathsf{L}}.$$

Applying this to Eq. $(*)$, the statement follows. $\square$

Lemma 8.2.4. *Suppose Φ can be decomposed into root subsystems $\Phi_1, \cdots, \Phi_s$. Then we have $\mathscr{P}_\Phi(z) = \mathscr{P}_{\Phi_1} \cdots \mathscr{P}_{\Phi_s}$.*

Proof. This is because the decomposition of split reductive groups corresponds to the decomposition of Weyl groups and root systems. □

Hence, it suffices to know the Poincaré polynomials of irreducible root systems. When Φ is irreducible of type X_n, we will denote its Poincaré polynomial by $\mathscr{P}_{X_n}$.

Lemma 8.2.5. *Let Φ be a reduced root system of rank n. Then there are positive integers $d_1, \cdots, d_n$ depending only on the Weyl group W of Φ, such that*

$$\mathscr{P}_\Phi(z) = \prod_{i=1}^{n} [d_i](z),$$

where $[d](z) := 1 + z + \cdots + z^{d-1}$.

Proof. Let G be a complex semisimple group, T a maximal torus in it, and B a Borel subgroup of G containing T. Let Φ be the associated root system, ${}^{v}W$ the Weyl group, and $\mathbb{V}$ be the complexification of the coroot space. Then we have:

(i). The complex singular cohomology ring of G/B vanishes at odd degree and has a basis dual to the Schubert cells (see [BGG73]).

(ii). The *Borel's theorem* (see [Bor53]) says that, after dividing degree by two, the complex singular cohomology ring of G/B is isomorphic to the *coinvariant algebra* $\mathbb{C}[\mathbb{V}]_{{}^{v}W}$, which is $\mathbb{C}[\mathbb{V}] \otimes_{\mathbb{C}[\mathbb{V}]^{{}^{v}W}} \mathbb{C}$, where $\mathbb{C}[\mathbb{V}]$ is the ring of complex polynomial functions on $\mathbb{V}$ and $\mathbb{C}[\mathbb{V}]^{{}^{v}W}$ is the subalgebra of invariant.

(iii). The *Chevalley-Shephard-Todd theorem* (see [Bourbaki, chap.VI, §3 no.3 thm.3]) says that $\mathbb{C}[\mathbb{V}]^{{}^{v}W}$ is a polynomial algebra generated by homogeneous polynomials on $\mathbb{V}$. Let $d_1, \cdots, d_n$ be the degrees of them.

Recall that the *Hilbert-Poincaré series* of a graded commutative $\mathbb{C}$-algebra $S_\bullet$ is defined to be $\sum_d \dim_{\mathbb{C}}(S_d)z^d$. Now, considering the Hilbert-Poincaré series of above graded algebras, the statement follows. □

The numbers $d_1, \cdots, d_n$ are called the *degrees* of ${}^vW$ (and of Φ). When Φ is irreducible, they can be found in [Bourbaki, chap.VI, §4].

For irreducible root systems of type A_n, B_n, C_n, and D_n, the explicit formulas for their Poincaré polynomials $\mathscr{P}_{X_n;I}$ with various types I are listed in Appendix A.

§ 8.3. Concave functions

This section treats the second factor $\left[P_{f_o^*} : P_{f_o^*} \cap P_{o,x}\right]$ in Eq. (8.1.1). It has to be a power of q since $P_{f_o^*}$ is a pro-unipotent group.

First, let f_{o^*x} be the following concave function:

$$f_{o^*x}\colon a \in \widetilde{\Phi} \longmapsto \max\left\{f_o^*(a), f_{\{o,x\}}(a)\right\} = \max\{0+, -a(x)\}.$$

Then, from the definition of P_f, we have $P_{f_{o^*x}} = P_{f_o^*} \cap P_{o,x}$. Note that both f_{o^*x} and f_o^* take the value $0+$ at $0 \in \widetilde{\Phi}$.

Lemma 8.3.1. *Let f, g be two concave functions on $\widetilde{\Phi}$ such that $f(0) = g(0) > 0$ and $g \geqslant f$. Then we have*

$$\left[P_f : P_g\right] = \prod_{a \in \Phi} \left|\varphi_a^{-1}[f(a), g(a)]\right|, \tag{8.3.1}$$

where $\varphi = (\varphi_a)_{a\in\Phi}$ is the valuation corresponding to the reference point o.

Proof. There are two ways to show this. By 4.4.1 and 4.4.2, we can extend the decomposition Eq. (3.3.1) to obtain a Lie algebra version of Eq. (8.3.1). Hence, if the characteristic of K is 0, the statement follows from the bijective exponential maps of unipotent groups.

In general case, we can consider the morphism $\mathfrak{G}_g \to \mathfrak{G}_f$ obtained by extending the inclusion $P_g \subseteq P_f$. Then the multiplicative morphism Eq. (4.4.2) induces the following commutative diagram for all positive integer i.

$$\begin{tikzcd}
\prod_{a\in\Phi^-} \mathfrak{U}_{a,g(a)}\left(K^\circ/\varpi^i\right) \cdot \mathfrak{T}_{g(0)}\left(K^\circ/\varpi^i\right) \cdot \prod_{a\in\Phi^+} \mathfrak{U}_{a,g(a)}\left(K^\circ/\varpi^i\right) \arrow[r] \arrow[d] & \mathfrak{G}_g\left(K^\circ/\varpi^i\right) \arrow[d] \\
\prod_{a\in\Phi^-} \mathfrak{U}_{a,f(a)}\left(K^\circ/\varpi^i\right) \cdot \mathfrak{T}_{f(0)}\left(K^\circ/\varpi^i\right) \cdot \prod_{a\in\Phi^+} \mathfrak{U}_{a,f(a)}\left(K^\circ/\varpi^i\right) \arrow[r] & \mathfrak{G}_f\left(K^\circ/\varpi^i\right)
\end{tikzcd}$$

By Theorem 4.2.(iii), since $f(0) > 0$ and $g(0) > 0$, the horizontals are isomorphisms. Since $f(0) = g(0)$, at the level of K°/ϖ^i, we have

$$\begin{aligned}
&\operatorname{Coker}\Big(\mathfrak{G}_g\left(K^\circ/\varpi^i\right) \to \mathfrak{G}_f\left(K^\circ/\varpi^i\right)\Big) \\
&\qquad \cong \prod_{a\in\Phi} \operatorname{Coker}\Big(\mathfrak{U}_{a,g(a)}\left(K^\circ/\varpi^i\right) \to \mathfrak{U}_{a,f(a)}\left(K^\circ/\varpi^i\right)\Big).
\end{aligned}$$

By 4.4.2, for each $a \in \Phi$, we have

$$\begin{aligned}
&\operatorname{Coker}\Big(\mathfrak{U}_{a,g(a)}\left(K^\circ/\varpi^i\right) \to \mathfrak{U}_{a,f(a)}\left(K^\circ/\varpi^i\right)\Big) \\
&\qquad \cong \operatorname{Coker}\Big(U_{a,g(a)} \otimes_{K^\circ} K^\circ/\varpi^i \to U_{a,f(a)} \otimes_{K^\circ} K^\circ/\varpi^i\Big) \\
&\qquad = U_{a,f(a)}/U_{a,g(a)} \otimes_{K^\circ} K^\circ/\varpi^i = \varphi_a^{-1}[f(a), g(a)] \otimes_{K^\circ} K^\circ/\varpi^i,
\end{aligned}$$

which equals $\varphi_a^{-1}[f(a), g(a)]$ if $i \cdot \operatorname{val}(\varpi) > g(a) - f(a)$ (see Example 4.1.7).

Now, we pass to the limit of the following projective system of homomorphisms.

$$\begin{tikzcd}
\cdots \arrow[r, two heads] & \mathfrak{G}_g\left(K^\circ/\varpi^{i+1}\right) \arrow[r, two heads] \arrow[d] & \mathfrak{G}_g\left(K^\circ/\varpi^{i}\right) \arrow[r, two heads] \arrow[d] & \cdots \arrow[r, two heads] & \mathfrak{G}_g(\kappa) \arrow[d] \\
\cdots \arrow[r, two heads] & \mathfrak{G}_f\left(K^\circ/\varpi^{i+1}\right) \arrow[r, two heads] & \mathfrak{G}_f\left(K^\circ/\varpi^{i}\right) \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathfrak{G}_f(\kappa)
\end{tikzcd}$$

Then we have

$$\begin{aligned}
P_f/P_g &= \varprojlim_i \operatorname{Coker}\left(\mathfrak{G}_g\left(K^\circ/\varpi^i\right) \to \mathfrak{G}_f\left(K^\circ/\varpi^i\right)\right) \\
&= \varprojlim_i \prod_{a\in\Phi} \varphi_a^{-1}[f(a), g(a)] \otimes_{K^\circ} K^\circ/\varpi^i = \prod_{a\in\Phi} \varphi_a^{-1}[f(a), g(a)].
\end{aligned}$$

Then Eq. (8.3.1) follows. □

Applying Eq. (8.3.1) to f_{o^*x} and f_o^*, we have

$$\begin{aligned}
\left[P_{f_o^*} : P_{f_o^*} \cap P_{o,x}\right] &= \prod_{a\in\Phi} \varphi_a^{-1}[0+, \max\{0+, -a(x)\}] \\
&= \prod_{a\in\Phi} \varphi_{-a}^{-1}[0+, \max\{0+, a(x)\}].
\end{aligned}$$

Then by the definition of φ (see Example 4.1.7), we have

$$\begin{aligned}
\left[P_{f_o^*} : P_{f_o^*} \cap P_{o,x}\right] &= \prod_{a\in\Phi} \exp_q\left(\frac{\lceil\max\{0+, a(x)\}\rceil - \lceil 0+\rceil}{\operatorname{val}(\varpi)}\right) \\
&= \prod_{a(x)>0} \exp_q\left(\frac{\lceil a(x)\rceil - \lceil 0+\rceil}{\operatorname{val}(\varpi)}\right),
\end{aligned} \tag{8.3.2}$$

where $\exp_q(\,\cdot\,)$ is the exponent function with base q.

§ 8.4. Fundamental domain and the proof of Theorem 1.4

The following lemma gives us a fundamental domain of P_o in $\mathscr{B}$.

Lemma 8.4.1. *The convex cone $\overline{o + {}^vC}$ is a fundamental domain of P_o.*

Proof. Let x be any point in $\mathscr{B}$. We need to show that there exists some $g_x \in P_o$ mapping x into $\overline{o + {}^vC}$. First, let $g.\mathbb{A}$ be an apartment containing both o and x. More precisely, suppose $o = [g, o+\mathbf{v}_0]$ and $x = [g, o+\mathbf{v}]$. Then, from the equivalence relation in Definition 4.2.9, there is an $n \in N$, such that $o + \mathbf{v}_0 = \nu(n).o$ and $gn \in P_o$. Let $\mathbf{v}_1 \in \mathbb{V}$ be the vector $\nu(n)^{-1}.(o + \mathbf{v}) - o$. Since $\overline{{}^vC}$ is the fundamental domain of ${}^vW$ in $\mathbb{V}$, there is a $w \in {}^vW$ such that $w.\mathbf{v}_1 \in \overline{{}^vC}$. Now, let n_1 be a preimage of w under $N_o \longrightarrow W_o \cong {}^vW$. Then $n_1 n^{-1} g^{-1} \in P_o$ and it maps x into $\overline{o + {}^vC}$.

On the other hand, if there are two points $x, y \in \overline{o + {}^vC}$ such that $y = g.x$ for some $g \in P_o$. Then, by the *vectorial Bruhat decomposition* [BT-1, 7.3.4], we have

$$g = h_1 n h_2,$$

where $h_1, h_2 \in B_{o,{}^vC}$ and $n \in N$. Therefore, $n \in N_o$, which implies $x = y$ since $\overline{o + {}^vC}$ is the fundamental domain of W_o. □

We will denote $\overline{o + {}^vC}$ by $\mathscr{D}({}^vC)$ to emphasize that it is a fundamental domain.

Applying Lemma 8.4.1 to Eq. (8.0.1) and using Eq. (8.1.1), we have

$$\mathrm{SV}(r) = \sum_{x \in B(r) \cap \mathscr{D}\left({}^{v}C\right)} \left[\overline{P}_o : \overline{P}_{o,x}\right] \cdot \left[P_{f_o^*} : P_{f_o^*} \cap P_{o,x}\right], \tag{8.4.1}$$

$$\mathrm{SSA}(r) = \sum_{x \in \partial(r) \cap \mathscr{D}\left({}^{v}C\right)} \left[\overline{P}_o : \overline{P}_{o,x}\right] \cdot \left[P_{f_o^*} : P_{f_o^*} \cap P_{o,x}\right]. \tag{8.4.2}$$

By Lemma 8.2.2, the first factor $\left[\overline{P}_o : \overline{P}_{o,x}\right]$ is computed by the Poincaré polynomial $\mathscr{P}_{\Phi;I_{o,x}}(q)$, which depends only on the *type* of x (see Convention 8.1.6). Hence, we can decompose the index sets $B(r) \cap \mathscr{D}\left({}^{v}C\right)$ and $\partial(r) \cap \mathscr{D}\left({}^{v}C\right)$ according to the types:

$$B(r, {}^{v}C, I) := \left\{x \in B(r) \cap \mathscr{D}\left({}^{v}C\right) \;\middle|\; x \text{ has type } I\right\}, \tag{8.4.3}$$

$$\partial(r, {}^{v}C, I) := \left\{x \in \partial(r) \cap \mathscr{D}\left({}^{v}C\right) \;\middle|\; x \text{ has type } I\right\}. \tag{8.4.4}$$

Then Eqs. (8.4.1) and (8.4.2) become the following ones:

$$\mathrm{SV}(r) = \sum_{I \subseteq \Delta} \mathscr{P}_{\Phi;I}(q) \sum_{x \in B(r, {}^{v}C, I)} \left[P_{f_o^*} : P_{f_o^*} \cap P_{o,x}\right],$$

$$\mathrm{SSA}(r) = \sum_{I \subseteq \Delta} \mathscr{P}_{\Phi;I}(q) \sum_{x \in \partial(r, {}^{v}C, I)} \left[P_{f_o^*} : P_{f_o^*} \cap P_{o,x}\right].$$

Applying Eq. (8.3.2) to the above, we have

$$\mathrm{SV}(r) = \sum_{I \subseteq \Delta} \mathscr{P}_{\Phi;I}(q) \sum_{x \in B(r, {}^{v}C, I)} \prod_{a(x) > 0} \exp_q\left(\frac{\lceil a(x) \rceil - \lceil 0+ \rceil}{\mathrm{val}(\varpi)}\right),$$

$$\mathrm{SSA}(r) = \sum_{I \subseteq \Delta} \mathscr{P}_{\Phi;I}(q) \sum_{x \in \partial(r, {}^{v}C, I)} \prod_{a(x) > 0} \exp_q\left(\frac{\lceil a(x) \rceil - \lceil 0+ \rceil}{\mathrm{val}(\varpi)}\right).$$

Note that, the ceiling function $\lceil \cdot \rceil$ used here follows Definition 4.3.1. If we use the usual ceiling function instead and note that $\deg\left(\mathscr{P}_{\Phi;I_{o,x}}\right) = \left|\Phi \setminus \Psi_{I_{o,x}}\right|$ equals the number of roots $a \in \Phi$ such that $a(x) > 0$, then we obtain the following formulas:

$$\mathrm{SV}(r) = \sum_{I \subseteq \Delta} \frac{\mathscr{P}_{\Phi;I}(q)}{q^{\deg(\mathscr{P}_{\Phi;I})}} \sum_{x \in B(r, {}^{v}C, I)} \prod_{a(x)>0} q^{\lceil a(x) \rceil}, \tag{8.4.5}$$

$$\mathrm{SSA}(r) = \sum_{I \subseteq \Delta} \frac{\mathscr{P}_{\Phi;I}(q)}{q^{\deg(\mathscr{P}_{\Phi;I})}} \sum_{x \in \partial(r, {}^{v}C, I)} \prod_{a(x)>0} q^{\lceil a(x) \rceil}. \tag{8.4.6}$$

This proves Theorem 8.1.

Remark. If the valuation $\mathrm{val}(\,\cdot\,)$ is normalized, namely $\mathrm{val}(\varpi) = 1$ and hence Γ equals the additive group of integers $\mathbb{Z}$, then the two versions of ceiling functions $\lceil\,\cdot\,\rceil$ coincide and the formulas Eqs. (8.4.5) and (8.4.6) can be understood in either way.

Convention 8.4.2. From now on, we assume the valuation $\mathrm{val}(\,\cdot\,)$ is normalized.

§ 8.5. Variants of the simplicial volume

Let $\tau\colon \mathcal{V} \to \mathfrak{I}$ be a function factoring through the type function $x \mapsto I_{x,o}$. Then we can define the *τ-variants* of the simplicial volume $\mathrm{SV}(\,\cdot\,)$ and the simplicial surface area $\mathrm{SSA}(\,\cdot\,)$ as follows. For any $\dagger \in \mathfrak{I}$, the quantities $\mathrm{SV}_{\dagger}(r)$ and $\mathrm{SSA}_{\dagger}(r)$ count the following sets respectively:

$$B_{\dagger}(r) := \{x \in B(r) \mid \tau(x) = \dagger\},$$

$$\partial_{\dagger}(r) := \{x \in \partial(r) \mid \tau(x) = \dagger\}.$$

Following Eqs. (8.4.3) and (8.4.4), we can introduce the following subsets:

$$B_{\dagger}(r, {}^{v}C, I) := \left\{x \in B_{\dagger}(r) \cap \mathscr{D}({}^{v}C) \;\middle|\; x \text{ has type } I \right\},$$

$$\partial_{\dagger}(r, {}^{v}C, I) := \left\{x \in \partial_{\dagger}(r) \cap \mathscr{D}({}^{v}C) \;\middle|\; x \text{ has type } I \right\}.$$

Then the same argument for Theorem 8.1 works and gives us the following formulas:

$$\mathrm{SV}_{\dagger}(r) = \sum_{I \subseteq \Delta} \frac{\mathscr{P}_{\Phi;I}(q)}{q^{\deg(\mathscr{P}_{\Phi;I})}} \sum_{x \in B_{\dagger}(r, {}^{v}C, I)} \prod_{a(x)>0} q^{\lceil a(x) \rceil}, \tag{8.5.1}$$

$$\mathrm{SSA}_{\dagger}(r) = \sum_{I \subseteq \Delta} \frac{\mathscr{P}_{\Phi;I}(q)}{q^{\deg(\mathscr{P}_{\Phi;I})}} \sum_{x \in \partial_{\dagger}(r, {}^{v}C, I)} \prod_{a(x)>0} q^{\lceil a(x) \rceil}. \tag{8.5.2}$$

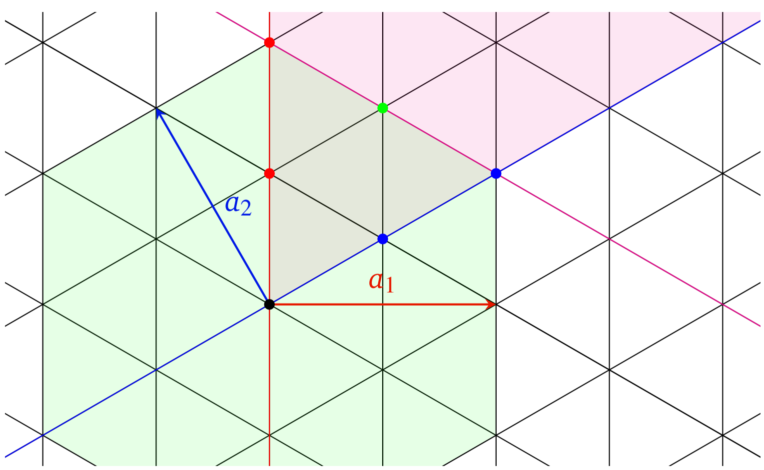
Such variants may be interesting when we need to focus on certain types of vertices (for instance, when not all vertices are special). In this dissertation, we will consider † = "being special", although other variants are also worth considering.

Chapter 9.

The Index Sets

The purpose of this chapter is to deduce explicit descriptions of the index sets $B(r, {}^{v}C, I)$ and $\partial(r, {}^{v}C, I)$ in Theorem 8.1 from a careful study of the vertices in the affine apartments of split type A_n, C_n, B_n, and D_n respectively.

The following example may give the reader some intuition.



$B(2, \blacksquare, a_1) = \{\bullet, \bullet\}$, $\quad B(2, \blacksquare, a_2) = \{\bullet, \bullet\}$,

$B(2, \blacksquare, \Delta) = \{\bullet\}$, $\quad B(2, \blacksquare, \emptyset) = \{\bullet\}$.

Figure 9.1. Examples of index sets $B(r, {}^{v}C, I)$ in $\mathscr{A}(A_2)$.

§ 9.1. Generalities

We will keep notations and conventions in § 6.2. Then we can describe the index set $B(r, {}^vC, I)$ and $\partial(r, {}^vC, I)$ as follows.

First, by Theorem 6.1, we see that

$$B(r) \cap \mathcal{D}({}^vC) = \mathcal{D}({}^vC) \cap \mathcal{V} \cap \alpha_{-a_0+r},$$
$$\partial(r) \cap \mathcal{D}({}^vC) = \mathcal{D}({}^vC) \cap \mathcal{V} \cap (\alpha_{-a_0+r} \setminus \alpha_{-a_0+r-1}).$$

For a type I, let ${}^vC_I$ denote the face of ${}^vC$ having type I:

$${}^vC_I := \{\mathbf{v} \in \mathbb{V} \mid \forall a \in I, a(\mathbf{v}) = 0; \forall a \in \Delta \setminus I, a(\mathbf{v}) > 0\}.$$

Recall that for any point x, its type is $I_{o,x} = \{a \in \Delta \mid a(x) = 0\}$. Hence, a point $x \in \mathcal{D}({}^vC)$ has type I if and only if $x \in o + {}^vC_I$. Then we have

$$B(r, {}^vC, I) = (o + {}^vC_I) \cap \mathcal{V} \cap \alpha_{-a_0+r} \tag{9.1.1}$$

and

$$\partial(r, {}^vC, I) = (o + {}^vC_I) \cap \mathcal{V} \cap (\alpha_{-a_0+r} \setminus \alpha_{-a_0+r-1}). \tag{9.1.2}$$

Similar descriptions for the †-variants can be obtained similarly.

By Convention 2.4.5, a point x has type I if and only if it is of the form

$$x = o + c_1\omega_{\ell_1} + \cdots + c_t\omega_{\ell_t}.$$

Then the condition $x \in o + {}^vC_I$ can be interpreted as "$c_1, \cdots, c_t > 0$". Next, x is a *special* vertex if and only if $c_1, \cdots, c_t \in \mathbb{Z}$. Let † = "being special". Then, by Eqs. (9.1.1) and (9.1.2), we have the following explicit descriptions:

$$B_\dagger(r, {}^vC, I) = \left\{ x = o + c_1\omega_{\ell_1} + \cdots + c_t\omega_{\ell_t} \;\middle|\; \begin{array}{c} c_1, \cdots, c_t \in \mathbb{Z}_{>0}, \\ h_{\ell_1}c_1 + \cdots + h_{\ell_t}c_t \leqslant r \end{array} \right\}, \tag{9.1.3}$$

$$\partial_\dagger(r, {}^vC, I) = \left\{ x = o + c_1\omega_{\ell_1} + \cdots + c_t\omega_{\ell_t} \;\middle|\; \begin{array}{c} c_1, \cdots, c_t \in \mathbb{Z}_{>0}, \\ h_{\ell_1}c_1 + \cdots + h_{\ell_t}c_t = r \end{array} \right\}, \tag{9.1.4}$$

where $\mathbb{Z}_{>0}$ denotes the set of positive integers.

§ 9.2. Index sets in $\mathscr{B}(A_n)$

Let I be a type and follow Convention 2.4.5. Since all vertices are special, by Eqs. (9.1.3) and (9.1.4), we have the following explicit descriptions:

$$B(r, {}^vC, I) = \left\{ x = o + c_1\omega_{\ell_1} + \cdots + c_t\omega_{\ell_t} \;\middle|\; \begin{array}{c} c_1, \cdots, c_t \in \mathbb{Z}_{>0}, \\ c_1 + \cdots + c_t \leqslant r \end{array} \right\}, \tag{9.2.1}$$

$$\partial(r, {}^vC, I) = \left\{ x = o + c_1\omega_{\ell_1} + \cdots + c_t\omega_{\ell_t} \;\middle|\; \begin{array}{c} c_1, \cdots, c_t \in \mathbb{Z}_{>0}, \\ c_1 + \cdots + c_t = r \end{array} \right\}. \tag{9.2.2}$$

§ 9.3. Index sets in $\mathscr{A}(C_n)$ $(n \geqslant 2)$

Let I be a type and follow Convention 2.4.5. By introducing $\omega'_i = h_i^{-1}\omega_i$, we can write a point x having type I as follows:

$$x = o + c_1\omega'_{\ell_1} + \cdots + c_t\omega'_{\ell_t}.$$

By Eq. (6.4.12), such an x is a vertex if and only if $c_1, \cdots, c_t \in \mathbb{Z}$. Therefore, by Eqs. (9.1.1) and (9.1.2), we have the following explicit description:

$$B(r, {}^{v}C, I) = \left\{ x = o + c_1\omega'_{\ell_1} + \cdots + c_t\omega'_{\ell_t} \;\middle|\; \begin{array}{c} c_1, \cdots, c_t \in \mathbb{Z}_{>0}, \\ c_1 + \cdots + c_t \leqslant r \end{array} \right\}, \tag{9.3.1}$$

$$\partial(r, {}^{v}C, I) = \left\{ x = o + c_1\omega'_{\ell_1} + \cdots + c_t\omega'_{\ell_t} \;\middle|\; \begin{array}{c} c_1, \cdots, c_t \in \mathbb{Z}_{>0}, \\ c_1 + \cdots + c_t = r \end{array} \right\}. \tag{9.3.2}$$

§ 9.4. Index sets in $\mathscr{A}(B_n)$ $(n \geqslant 3)$

Let I be a type and follow Convention 2.4.5. For any set $\mathcal{X}$ of points, we introduce the following subsets:

$$\begin{aligned} \mathcal{X}(I) &:= (o + {}^{v}C_I) \cap \mathcal{X}, \\ \mathcal{X}(I, r) &:= (o + {}^{v}C_I) \cap \mathcal{X} \cap (\alpha_{-a_0+r} \setminus \alpha_{-a_0+r-1}). \end{aligned} \tag{9.4.1}$$

Then we have $\mathcal{V}(I, r) = \partial(r, {}^{v}C, I)$ by Eqs. (9.1.1) and (9.1.2).

9.4.1. By introducing $\omega'_i = h_i^{-1}\omega_i$, we can write a point x having type I as follows:

$$x = o + c_1\omega'_{\ell_1} + \cdots + c_t\omega'_{\ell_t}.$$

Then $a_0(x) \leqslant r$ if and only if $c_1 + \cdots + c_t \leqslant r$.

9.4.2. Consider the set $o + \frac{1}{2}\mathcal{P}^\vee$ and recall that $h_1 = 1$ while $h_2 = \cdots = h_n = 2$. Let $\mathcal{X}^0$ be the set $o + \mathbb{Z}\omega'_1 + \cdots + \mathbb{Z}\omega'_n$ and $\mathcal{X}^1 = \mathcal{X}^0 - \frac{1}{2}\omega_1$. Then we have

$$o + \tfrac{1}{2}\mathcal{P}^\vee = \mathcal{X}^0 \cup \mathcal{X}^1.$$

This gives a superset of the $\mathcal{V}$.

By Lemma 6.5.1, the complement of $\mathcal{V}$ in $o + \frac{1}{2}\mathcal{P}^\vee$ is the restriction of Ξ. Inspired by this and Definition 6.4.1, we can consider the following sets for each $J \subseteq \{1, \cdots, n\}$:

$$\Xi_J := \{x \in \mathbb{A} \mid J_x = J\} \qquad \text{and} \qquad \mathcal{X}_J := \Xi_J \cap o + \tfrac{1}{2}\mathcal{P}^\vee .$$

Then we have

$$o + \tfrac{1}{2}\mathcal{P}^\vee \setminus \mathcal{V} = \mathcal{X}_{\{1\}} \cup \mathcal{X}_{\{1,2\}} \cup \cdots \cup \mathcal{X}_{\{n-1,n\}} .$$

Note that, for any J, we have

$$\mathcal{X}_J = \mathcal{X}_\emptyset - \sum_{j \in J} \tfrac{1}{2}\omega_j. \tag{9.4.2}$$

Moreover, it is clear that $\mathcal{X}_\emptyset$ is precisely $o + \mathcal{P}^\vee$, the set of special vertices.

9.4.3. Next, we consider $\mathcal{X}(I)$ for above sets. First, if $\ell_1 > 1$, then $\mathcal{V}(I) \subseteq \mathcal{X}^0(I)$. Otherwise, $\mathcal{V}(I) \cap \mathcal{X}^1(I) \neq \emptyset$. For any J, it is clear that $\mathcal{X}_J(I) \neq \emptyset$ if and only if $I \cap J = \emptyset$. If this is the case, we have the following refinement of Eq. (9.4.2):

$$\mathcal{X}_J(I) = \mathcal{X}_\emptyset(I) - \sum_{j \in J} \tfrac{1}{2}\omega_j. \tag{9.4.3}$$

9.4.4. Finally, we consider $\mathcal{X}(I, r)$. First, it is clear that

$$\mathcal{X}^0(I, r) = \left\{ x = o + c_1\omega'_{\ell_1} + \cdots + c_t\omega'_{\ell_t} \;\middle|\; \begin{array}{c} c_1, \cdots, c_t > 0, \\ c_1, \cdots, c_t \in \mathbb{Z}_{>0}, \\ c_1 + \cdots + c_t = r \end{array} \right\}. \tag{9.4.4}$$

Also note that

$$\mathcal{X}^1(I, r) = \mathcal{X}^0(I, r) - \tfrac{1}{2}\omega_1. \tag{9.4.5}$$

Then we need to work out $\mathcal{X}_J(I, r)$. For $\mathcal{X}_\emptyset(I, r)$, an explicit description is given in

Eqs. (9.1.3) and (9.1.4). For general J, we have the following refinement of Eq. (9.4.3):

Lemma 9.4.5. *Suppose $I \cap J = \emptyset$. Then we have*

$$\mathcal{X}_J(I, r) = \mathcal{X}_\emptyset(I, r + |J| - \delta(J)) - \sum_{j \in J} \tfrac{1}{2}\omega_j,$$

where $\delta(J)$ is defined as follows:

$$\delta(J) = \begin{cases} 1 & \text{if } 1 \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Eq. (9.4.3), it suffices to show that for any $x \in \mathcal{X}_\emptyset(I, r)$,

$$(*) \qquad \left\lceil a_0\left(x - \sum_{j \in J} \tfrac{1}{2}\omega_j\right)\right\rceil = a_0(x) - |J| + \delta(J).$$

Note that $a_0(x) \in \mathbb{Z}$ and that

$$a_0\left(\sum_{j \in J} \tfrac{1}{2}\omega_j\right) = \begin{cases} |J| - \tfrac{1}{2} & \text{if } 1 \in J, \\ |J| & \text{otherwise.} \end{cases}$$

Then Eq. (∗) follows. ◻

We illustrate above discussions by the following diagrams.

$$\begin{tikzcd}
& \mathcal{X}^0(I,r) \arrow[dl, dash] \arrow[dr, dash] & & \bigcup\limits_{J=\{2,3\},\cdots,\{n-1,n\}} \Xi_J \arrow[dl, dash] \\
\mathcal{V}(I,r) \arrow[d, dash] & & \bigcup\limits_{J=\{2,3\},\cdots,\{n-1,n\}} \mathcal{X}_J(I,r) & \\
\mathcal{V}_\dagger(I,r) = \mathcal{X}_\emptyset(I,r) & & &
\end{tikzcd}$$

Figure 9.2. Vertices of type I in $\mathscr{A}(B_n)$ $(\ell_1 > 1)$

$$\begin{tikzcd}
& \bigcup\limits_{\square=0,1} \mathcal{X}^\square(I,r) \arrow[dl, dash] \arrow[dr, dash] & & \bigcup\limits_{J=\{1\},\{1,2\},\cdots,\{n-1,n\}} \Xi_J \arrow[dl, dash] \\
\mathcal{V}(I,r) \arrow[d, dash] & & \bigcup\limits_{J=\{1\},\{1,2\},\cdots,\{n-1,n\}} \mathcal{X}_J(I,r) & \\
\mathcal{V}_\dagger(I,r) = \mathcal{X}_\emptyset(I,r) & & &
\end{tikzcd}$$

Figure 9.3. Vertices of type I in $\mathscr{A}(B_n)$ $(\ell_1 = 1)$

§ 9.5. Index sets in $\mathscr{A}(D_n)$ $(n \geqslant 4)$

Let I be a type and follow Convention 2.4.5. Using the notations introduced in Eq. (9.4.1), we have $\mathcal{V}(I, r) = \partial(r, {}^{v}C, I)$.

9.5.1. We also have introduced the sets

$$\mathcal{X}^{00}, \mathcal{X}^{10}, \mathcal{X}^{01}, \mathcal{X}^{11}, \mathcal{X}^{(0)}, \qquad \text{and} \qquad \mathcal{X}^{(1)}$$

in Eq. (6.6.18). Inspired Lemmas 6.6.1 to 6.6.3 and Definition 6.4.1, we consider the following sets for each $J \subseteq \{1, \cdots, n\}$:

$$\Xi_J := \{x \in \mathbb{A} \mid J_x = J\} \qquad \text{and} \qquad \mathcal{X}_J := \Xi_J \cap o + \tfrac{1}{2}\mathbb{Z}^n.$$

Note that $o + \frac{1}{2}\mathbb{Z}^n = \mathcal{X}^{(0)} \cup \mathcal{X}^{(1)}$. Then we have

$$\begin{aligned} o + \tfrac{1}{2}\mathbb{Z}^n \setminus \mathcal{V} & \\ &= \mathcal{X}_{\{1\}} \cup \mathcal{X}_{\{1,2\}} \cup \cdots \cup \mathcal{X}_{\{n-3,n-2\}} \\ &\qquad \cup \mathcal{X}_{\{n-1,n\}} \cup \mathcal{X}_{\{n-2,n-1,n\}} \,. \end{aligned}$$

Note that, for any J, we have

$$\mathcal{X}_J = \mathcal{X}_\emptyset - \sum_{j \in J} \tfrac{1}{2}\omega_j. \tag{9.5.1}$$

Moreover, it is clear that $\mathcal{X}_\emptyset$ is precisely $o + \mathcal{P}^\vee$, the set of special vertices.

9.5.2. Next, we consider $\mathcal{X}(I)$ for above sets.

First, if $\{n-1, n\} \cap I \neq \emptyset$ then $\mathcal{V}(I) \subseteq \mathcal{X}^{(0)}(I)$. Otherwise, $\mathcal{V}(I) \cap \mathcal{X}^{(1)}(I) \neq \emptyset$. In each case, we have

$$\mathcal{V}(I) \cap (\mathcal{X}^{10}(I) \cup \mathcal{X}^{11}(I)) = \emptyset \iff \ell_1 > 1.$$

For any J, it is clear that

$$\mathcal{X}_J(I) \neq \emptyset \iff I \cap J = \emptyset.$$

If this is the case, we have the following refinement of Eq. (9.5.1):

$$\mathcal{X}_J(I) = \mathcal{X}_\emptyset(I) - \sum_{j \in J} \tfrac{1}{2}\omega_j. \tag{9.5.2}$$

9.5.3. Finally, we consider $\mathcal{X}(I, r)$.

First, it is clear that

$$\mathcal{X}^{00}(I, r) = \left\{ x = o + c_1\omega'_{\ell_1} + \cdots + c_t\omega'_{\ell_t} \;\middle|\; \begin{array}{c} c_1, \cdots, c_t > 0, \\ c_1, \cdots, c_t \in \mathbb{Z}_{>0}, \\ c_1 + \cdots + c_t = r \end{array} \right\}. \tag{9.5.3}$$

Then the followings follow from Eq. (6.6.18):

$$\mathcal{X}^{01}(I, r) = \mathcal{X}^{00}(I, r+1) - \tfrac{1}{2}(\omega_{n-1} + \omega_n), \tag{9.5.4}$$

$$\mathcal{X}^{10}(I, r) = \mathcal{X}^{00}(I, r) - \tfrac{1}{2}\omega_1, \tag{9.5.5}$$

$$\mathcal{X}^{11}(I, r) = \mathcal{X}^{00}(I, r+1) - \tfrac{1}{2}(\omega_1 + \omega_{n-1} + \omega_n). \tag{9.5.6}$$

Then we need to work out $\mathcal{X}_J(I, r)$:

- For $\mathcal{X}_\emptyset(I, r)$, an explicit description is given in Eqs. (9.1.3) and (9.1.4).
- For general J, we have the following refinement of Eq. (9.5.2).

Lemma 9.5.4. *Suppose $I \cap J = \emptyset$. Then we have*

$$\mathcal{X}_J(I, r) = \mathcal{X}_\emptyset(I, r + |J| - \delta(J)) - \sum_{j \in J} \tfrac{1}{2}\omega_j,$$

where $\delta(J)$ is defined as follows:

$$\delta(J) = \begin{cases} 2 & \textit{if } \{1, n-1, n\} \subseteq J, \\ 0 & \textit{if } \{1, n-1, n\} \cap J = \emptyset, \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. By Eq. (9.4.3), it suffices to show that for any $x \in \mathcal{X}_\emptyset(I, r)$,

$$(*) \qquad \left\lceil a_0\left(x - \sum_{j \in J} \tfrac{1}{2}\omega_j\right)\right\rceil = a_0(x) - |J| + \delta(J).$$

Note that $a_0(x) \in \mathbb{Z}$ and that

$$a_0\left(\sum_{j \in J} \tfrac{1}{2}\omega_j\right) = \begin{cases} |J| - \frac{3}{2} & \text{if } \{1, n-1, n\} \subseteq J, \\ |J| - 1 & \text{if } 1 \notin J \text{ but } \{n-1, n\} \subseteq J, \\ |J| - \frac{1}{2} & \text{if } 1 \in J \text{ but } \{n-1, n\} \cap J = \emptyset, \\ |J| & \text{if } \{1, n-1, n\} \cap J = \emptyset. \end{cases}$$

Then Eq. $(*)$ follows. □

We illustrate above discussions by the following diagrams.

$$\begin{tikzcd}[column sep=small]
 & \mathcal{X}^{00}(I,r) \arrow[dl, dash] \arrow[dr, dash] & & \bigcup\limits_{J=\{2,3\},\cdots,\{n-3,n-2\}} \Xi_J \arrow[dl, dash] \\
\mathcal{V}(I,r) \arrow[d, dash] & & \bigcup\limits_{J=\{2,3\},\cdots,\{n-3,n-2\}} \mathcal{X}_J(I,r) & \\
\mathcal{V}_{\dagger}(I,r) = \mathcal{X}_{\emptyset}(I,r) & & &
\end{tikzcd}$$

Figure 9.4. Vertices of type I in $\mathscr{A}(D_n)$ $(1 \in I$ and $\{n-1,n\} \cap I \neq \emptyset)$

$$\begin{tikzcd}[column sep=small]
 & \bigcup_{\square=0,1} \mathcal{X}^{\square 0}(I,r) \arrow[dl, dash] \arrow[dr, dash] & & \bigcup\limits_{J=\{1\},\{1,2\},\cdots,\{n-3,n-2\}} \Xi_J \arrow[dl, dash] \\
\mathcal{V}(I,r) \arrow[d, dash] & & \bigcup\limits_{J=\{1\},\{1,2\},\cdots,\{n-3,n-2\}} \mathcal{X}_J(I,r) & \\
\mathcal{V}_{\dagger}(I,r) = \mathcal{X}_{\emptyset}(I,r) & & &
\end{tikzcd}$$

Figure 9.5. Vertices of type I in $\mathscr{A}(B_n)$ $(1 \notin I$ and $\{n-1,n\} \cap I \neq \emptyset)$

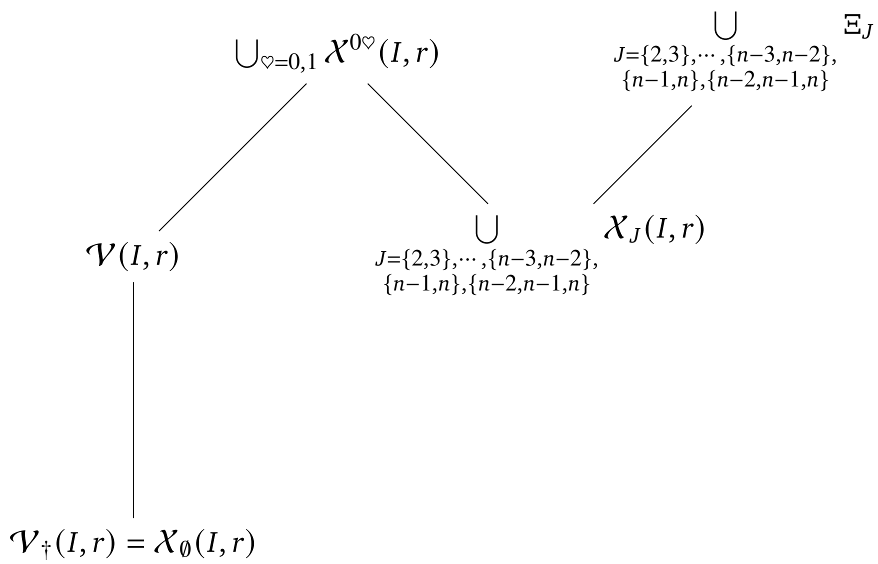


Figure 9.6. Vertices of type I in $\mathscr{A}(D_n)$ ($1 \in I$ and $\{n-1,n\} \cap I = \emptyset$)

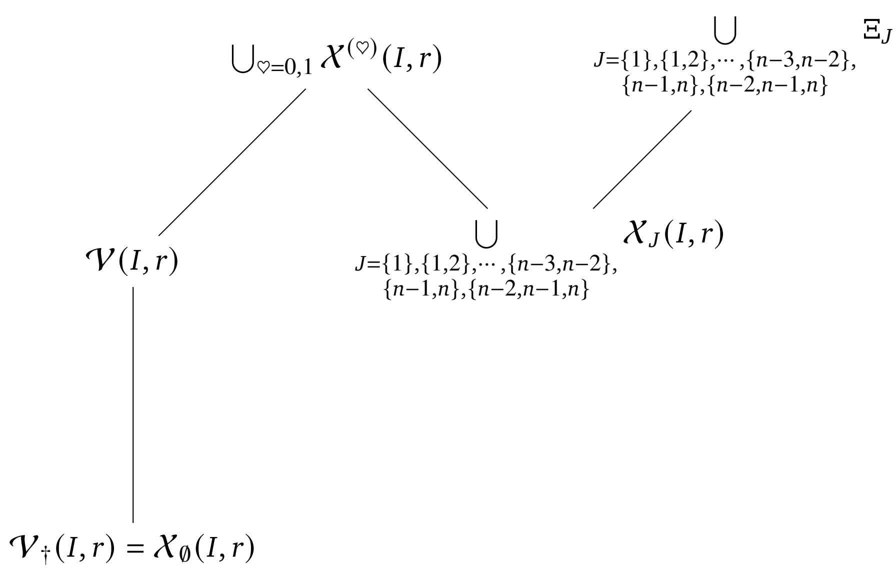


Figure 9.7. Vertices of type I in $\mathscr{A}(B_n)$ ($1 \notin I$ and $\{n-1,n\} \cap I = \emptyset$)

Part III.

Asymptotic study of the simplicial volume

Chapter 10.

Asymptotic analysis

This chapter aims to provide tools to analyze the asymptotic behavior of the simplicial volume and the simplicial surface area.

We have already seen that $\mathrm{SV}(r) \asymp \mathrm{SSA}(r)$ in Chapter 1, the introduction. Hence, in order to prove Theorem 1.3, it suffices to prove the simplicial surface area part. Likewise, the simplicial volume part of Theorem 1.5 can be deduced from the simplicial surface area part, either by Lemma 10.1.9 or direct computation. Therefore, it suffices to consider the asymptotic analysis of the simplicial surface area $\mathrm{SSA}(\,\cdot\,)$ only.

In the formulas Eqs. (8.4.5) and (8.4.6), there are only finitely many types $I \subseteq \Delta$ and each $\mathscr{P}_{\Phi;I}(q)$ is an integral polynomial. Hence, the asymptotic study of $\mathrm{SSA}(\,\cdot\,)$ can be reduced to summations of the following form:

$$\mathrm{S}_{\mathcal{X}(I)}(r) := \sum_{x \in \mathcal{X}(I,r)} \prod_{\alpha \in \Phi^+} q^{\lceil \alpha(x) \rceil}, \tag{10.1}$$

where $\mathcal{X}$ is a set of points, and the notations $\mathcal{X}(I)$ and $\mathcal{X}(I,r)$ follow Eq. (9.4.1).

The growth of $\mathrm{S}_{\mathcal{X}(I)}(r)$ varies for different types I. For the purpose of asymptotic analysis, only the dominant ones are relevant. To better analyze their growth, we introduce the following auxiliary functions:

$$\mathrm{S}^{\asymp}_{\mathcal{X}(I)}(r) := \sum_{x \in \mathcal{X}(I,r)} q^{2\rho(x)}, \tag{10.2}$$

where 2ρ is the sum of positive roots. Note that

$$2\rho(x) = \sum_{\alpha \in \Phi^+} \alpha(x) \leqslant \sum_{\alpha \in \Phi^+} \lceil \alpha(x) \rceil \leqslant \sum_{\alpha \in \Phi^+} (\alpha(x) + 1) = 2\rho(x) + \deg\big(\mathscr{P}_{\Phi;I}\big).$$

Hence, we have $\mathrm{S}_{\mathcal{X}(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{X}(I)}(r)$. But the later one is easier to study.

This chapter is structured as follows. In § 10.1, we will introduce *q-numbers* and *q-functions* and discuss the *discrete calculus* on them. We will then only focus on the q-functions defined by *(super) q-exponential polynomials*. To study them algebraically, we will review gradings and filtrations in § 10.2. Then in § 10.3 and 10.4, we will introduce *(super) q-exponential polynomials* and study the asymptotic properties of the q-functions defined by them. Finally, with those notions in hand, we will study the asymptotic growth of multi-summations in § 10.5 and 10.6.

§ 10.1. Discrete calculus of q-functions

It is often more convenient to treat q as a formal variable when we apply algebraic operations to $\mathrm{S}_{\mathcal{X}(I)}(r)$ and $\mathrm{S}^{\asymp}_{\mathcal{X}(I)}(r)$. But to carry out the asymptotic analysis, we need to view q as a real number. Inspired by this, we have the following definition.

Definition 10.1.1. Let q be a formal variable and h a positive integer. Then a *q-number* (of *level* h) is a rational function of $q^{1/h}$ over $\mathbb{Q}$ having no poles on the half real line $\mathbb{R}_{>1} := \{r \in \mathbb{R} \mid r > 1\}$. Let $\mathbb{Q}(q;h)$ denote the ring of q-numbers of level h. Then a *q-function* (of *level* h) is a function defined for sufficiently large integers and valued in

$\mathbb{Q}(q;h)$.

Example 10.1.2. Let h be a positive integer larger than 1. Then $(q^{1/h}-1)^{-1}$ is a q-number of level h, while $(q-h)^{-1}$ is not a q-number.

Remark. A rational function of $q^{1/h}$ is in particular an algebraic function of q and hence we can talk about its poles. On the half real line $\mathbb{R}_{>1}$, the function $q^{1/h}$ has a unique real-valued branch. This allows us to treat q-numbers as real-valued continuous functions on $\mathbb{R}_{>1}$.

Each $\mathbb{Q}(q;h)$ is a principal ideal domain. When the level h varies, they form an inductive system. Let $\mathbb{Q}(q;-)$ denote the inductive limit. We will view it as the ring of all q-numbers. On this principal ideal domain, we will consider the *pointwise topology* inheriting from the algebra $\mathscr{C}(\mathbb{R}_{>1})$ of real-valued continuous functions on $\mathbb{R}_{>1}$. In particular, if f is a q-function, then the *limit* of $f(z)$ as $z\to\infty$ is defined pointwise:

$$\lim_{z\to\infty} f(z) = \left(\lim_{z\to\infty} f(z)(q)\right)_{q>1}.$$

Then we can view each q-function f as a family of discrete functions $(f_q)_{q>1}$ indexed by the half real line $\mathbb{R}_{>1}$, where $f_q(z) := f(z)(q)$.

Definition 10.1.3. Let f and g be two q-functions. We say that they are *asymptotically equal* and that f has *asymptotic growth* g, denoted by $f(z)\sim g(z)$, if

$$\lim_{z\to\infty}\frac{f(z)}{g(z)} = 1.$$

We also need asymptotic dominant relations of q-functions. Like the topology, these notions are defined *pointwise*.

Definition 10.1.4. A q-number C is said to be *positive* (resp. *non-negative*) if for all $q > 1$, $C(q)$ is a positive (resp. non-negative) real number. A q-function f is said to be *eventually positive* (resp. *eventually non-negative*) if for all $q > 1$, f_q is an eventually positive (resp. eventually non-negative) function, namely: $f_q(z) > 0$ (resp. $f_q(z) \geqslant 0$) for sufficiently large z.

Definition 10.1.5. Let f and g be two q-functions. We say that f *dominates* g, denoted by $f(z) \gg g(z)$, if there exists a positive q-number C such that $|f| - C \cdot |g|$ is an eventually non-negative q-function. We will denote $f(z) \asymp g(z)$ if both $f(z) \gg g(z)$ and $g(z) \gg f(z)$.

Then we can consider the *discrete calculus* on q-numbers.

Definition 10.1.6. Let f be a q-function. Its *difference* Δf is the following q-function:

$$\Delta f(z) := f(z+1) - f(z).$$

The *difference operator* Δ is $\mathbb{Q}(q; -)$-linear and satisfies the *Leibniz rule*:

$$\Delta(fg) = f \cdot \Delta g + g \cdot \Delta f + \Delta f \cdot \Delta g. \tag{10.1.1}$$

Definition 10.1.7. A q-function f is said to be *eventually strictly increasing* if for all $q > 1$, f_q is an eventually strictly increasing function.

Clearly, f is eventually strictly increasing if and only if Δf is eventually positive.

Definition 10.1.8. A q-function f is said to be *unbounded* if for all q-number C, the q-function $|f| - C$ is eventually positive.

Lemma 10.1.9. *Let f and g be two eventually strictly increasing unbounded q-functions. Then we have $f(z) \sim g(z)$ if and only if $\Delta f(z) \sim \Delta g(z)$.*

Proof. Apply *Stolz-Cesàro theorem* (see e.g. [CN14, theorem 2.7.2]) to f_q and g_q for all $q > 1$. Then the statement follows. □

We also need the discrete version of integrals.

Definition 10.1.10. Let f be a q-function. Then an *anti-difference* of f is a q-function g such that $\Delta g = f$. Since $\mathrm{Ker}(\Delta)$ consists of constant q-functions, we see that the anti-difference is not unique but unique up to a constant q-function. By an abuse of notation, we will use Σf to denote an anti-difference of f. Let a be an integer in the domain of f. Then the *anti-difference* of f with *anchor* a, denoted by $\Sigma_a f$, is defined as follows:

$$\Sigma_a f(z) := (\Sigma f)(z) - (\Sigma f)(a).$$

Note that $\Sigma_a f$ is well-defined although Σf is not.

Note that, if a, b are two integers, then we have the following summation formula:

$$\sum_{z=a}^{b-1} f(z) = (\Sigma f)(b) - (\Sigma f)(a) = (\Sigma_a f)(b). \tag{10.1.2}$$

We will consider the following notions of q-numbers and q-functions.

Definition 10.1.11. A q-number is said to be *primary* if it is of level one. Then a q-function is said to be *primary* if its values are primary q-numbers.

§ 10.2. Weakly graded algebras

Before moving on, let's review gradings and then the filtrations induced by them. In the study of (super) q-exponential polynomials, it is the filtration induced by a grading, rather than the grading itself, will play an essential role.

Throughout this section, R is a commutative ring and Γ is an additive monoid. In the applications later, Γ could be $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{F}_2$, or products of them.

Definition 10.2.1. A Γ-*grading* on an R-module M is a decomposition into a direct sum

$$M = \bigoplus_{g\in\Gamma} M_g,$$

where each M_g is an R-submodule, called the *homogeneous component* of *grade* g. Elements of M_g are said to be *homogeneous* of *grade* g. A general element m of M is decomposed into homogeneous elements m_g ($g \in \Gamma$), each m_g is called its *homogeneous component* of *grade* g. An R-module equipped with a Γ-*grading* is called a Γ-*graded module* over R.

Definition 10.2.2. Let M be a Γ-graded module over R and $h \in \Gamma$. Then an operator T on M is said to *respect the grading* if $T(M_g) \subseteq M_g$ for all $g \in \Gamma$ and *shift the grading homogeneously by h* if $T(M_g) \subseteq M_{g+h}$ for all $g \in \Gamma$.

Lemma 10.2.3. *Suppose Γ is a group and $h \in \Gamma$. Let M be a Γ-graded projective module over R and T a surjective R-linear operator T on M shifting the grading homogeneously by h. Then there is a section of T shifts the grading homogeneously by $-h$.*

Proof. The conditions on T imply that its restriction to each M_g is surjective onto M_{g+h}. Since M_{g+h} is projective, $T|_{M_g}$ admits a section $S_g : M_{g+h} \to M_g$. Then the desired section of T is the direct sum of the sections S_g. $\square$

Remark. The *Grothendieck group $\mathcal{G}(\Gamma)$* of Γ is the universal Abelian group under Γ. If M is a Γ-graded module, then we will treat it as a $\mathcal{G}(\Gamma)$-graded module by defining $M_g = \{0\}$ if $g \in \mathcal{G}(\Gamma) \setminus \Gamma$. With this convention, Lemma 10.2.3 holds even without assuming that Γ is a group.

Corollary 10.2.4. *Let M be a Γ-graded projective module over R and T a surjective R-linear operator T on M respecting the grading. Then there is a section of T respects the grading.*

Definition 10.2.5. A Γ-*graded algebra* over R is an R-algebra A equipped with a Γ-grading such that

$$A_g A_h \subseteq A_{g+h}, \qquad \text{for all} \qquad g, h \in \Gamma.$$

Note that A_0 is a subalgebra and each A_g ($g \in \Gamma$) is an A_0-bimodule. The subalgebra A_0 is called the *subalgebra of grade* 0.

If A is a Γ-graded algebra over R, then a Γ-*graded (left) A-module* is a Γ-graded module M over R equipped with a (left) A-action such that

$$A_g M_h \subseteq M_{g+h}, \qquad \text{for all} \qquad g, h \in \Gamma.$$

Definition 10.2.6. Let A be an R-algebra A equipped with a Γ-grading. Suppose that A_0 is a subalgebra of A and that each homogeneous component A_g ($g \in \Gamma$) is a free A_0-module of rank one. Then the grading can be written as follows:

$$A = \bigoplus_{g \in \Gamma} A_0 e_g,$$

where each $e_g \in A_g$ is a generator of the A_0-module A_g and $e_0 = 1 \in A_0$. The family $(e_g)_{g \in \Gamma}$ is called a *homogeneous basis* of A over A_0. Note that a homogeneous basis of A determines the Γ-grading on it. Let $a \in A$. Then its homogeneous component of grade g is of the form $a_g = c_g e_g$ with $c_g \in A_0$. The element c_g is called the *coefficient* of a_g and is said to be a *coefficient* of a *attached* to e_g.

Remark. The assumption in Definition 10.2.6 does not require A to be a Γ-graded algebra.

In the rest of this section, Γ is a totally ordered additive monoid (for instance, Γ is a submonoid of $(\mathbb{R}, +)$).

Definition 10.2.7. A Γ-*filtration* on an R-module M is a family of R-submodules $(M_{\leqslant g})_{g\in\Gamma}$ of M such that $M_{\leqslant g} \subseteq M_{\leqslant h}$ whenever $g \leqslant h$ and that $\bigcup_{g\in\Gamma} M_{\leqslant g} = M$. We will use $M_{\approx g}$ to denote the set of elements in $M_{\leqslant g}$ but not in any $M_{\leqslant h}$ with $h < g$.

A Γ-grading on an R-module M induces a Γ-filtration as follows:

$$M_{\leqslant g} := \bigoplus_{h \leqslant g} M_h. \qquad (g \in \Gamma)$$

Then we say an element $m \in M$ is of *grade* g if $m \in M_{\approx g}$. If an element is of grade g, then its homogeneous component of grade g will be called its *leading term*.

Note that a nonzero homogeneous element of grade g is of grade g. For a general nonzero element $m \in M$, its grade is the largest g such that the homogeneous component of grade g of m is nonzero.

Convention 10.2.8. If M is $\mathbb{N}$-graded, we will say that $0 \in M$ has grade -1. If M is $\mathbb{Q}$-graded, we will say that $0 \in M$ has grade $-\infty$.

Definition 10.2.9. Let M be a Γ-graded module over R and $h \in \Gamma$. Then an operator T on M is said to *respect the filtration strictly* if $T(M_{\approx g}) \subseteq M_{\approx g}$ for all $g \in \Gamma$ and *shift the grading by h* if $T(M_{\approx g}) \subseteq M_{\approx g+h}$ for all $g \in \Gamma$.

Lemma 10.2.10. *Suppose Γ is a totally ordered group and $h \in \Gamma$. Let M be a Γ-graded module over R and T a surjective R-linear operator T on M shifting the grading by h. Then any section of T shifts the grading by $-h$.*

Proof. Since $M_{\approx g} \cap M_{\approx g'} = \emptyset$ whenever $g \neq g'$, the conditions on T imply that the preimage of $M_{\approx g+h}$ under T is precisely $M_{\approx g}$. Hence, if S is a section of T, we have $S(M_{\approx g+h}) \subseteq M_{\approx g}$ as expected. □

Remark. The total order on Γ can be canonically extended to its Grothendieck group $\mathcal{G}(\Gamma)$. Then Lemma 10.2.10 holds without assuming that Γ is a group.

Corollary 10.2.11. *Let M be a Γ-graded module over R and T a surjective R-linear operator T on M respecting the filtration strictly. Then any section of T respects the filtration strictly.*

Definition 10.2.12. A *weakly Γ-graded algebra* over R is an R-algebra A equipped with a Γ-grading such that

$$A_{\approx g} A_{\approx h} \subseteq A_{\approx g+h} \qquad \text{for all} \qquad g, h \in \Gamma.$$

If A is a weakly Γ-graded algebra over R, then a *weakly Γ-graded (left) A-module* is a Γ-graded module M over R equipped with a (left) A-action such that

$$A_{\approx g} M_{\approx h} \subseteq M_{\approx g+h} \qquad \text{for all} \qquad g, h \in \Gamma.$$

Remark. If A is a weakly Γ-graded algebra over R. Then a free (left) module over A is naturally a weakly Γ-graded (left) A-module.

Example 10.2.13. The ring of polynomials $R[z]$ over R is naturally a $\mathbb{N}$-graded algebra, where the monomials $(z^n)_{n\in\mathbb{N}}$ forms a homogeneous basis of it. However, if the characteristic of R is 0, then the induced $\mathbb{N}$-filtration on $R[z]$, namely the *degree filtration*, can also be induced from the following alternative grading:

$$R[z] = \bigoplus_{n\in\mathbb{N}} R\binom{z}{n},$$

where

$$\binom{z}{n} := \frac{1}{n!}z(z-1)\cdots(z-n+1).$$

In this dissertation, this grading will be called the *degree*. Note that this convention is different from the usual one. In particular, this grading does not make $R[z]$ into a $\mathbb{N}$-graded algebra over R, only a weakly $\mathbb{N}$-graded algebra.

§ 10.3. q-exponential polynomials

In this section and the next, we will introduce *(super) q-exponential polynomials* and study their interaction with anti-difference operators.

Definition 10.3.1. A *q-polynomial* is a polynomial with q-number coefficients. Following the usual notation, we will denote the ring of q-polynomials by $\mathbb{Q}(q;-)[z]$.

Definition 10.3.2. A *q-exponential polynomial* is a finite formal sum

$$f(z) = \sum_{\nu} f_\nu(z)q^{\nu z}, \tag{10.3.1}$$

where $\nu \in \mathbb{Q}$ and each $f_\nu(z)$ is a q-polynomial. The ring of q-exponential polynomials will be denoted by $\mathbb{Q}(q;-)[z]q^{\mathbb{Q}z}$.

Definition 10.3.3. Following Example 10.2.13, we will consider the following grading on $\mathbb{Q}(q;-)[z]$ and call it the *degree*:

$$\mathbb{Q}(q;-)[z] = \bigoplus_{n\in\mathbb{N}} \mathbb{Q}(q;-)\binom{z}{n}.$$

That is to say, an element of grade n in the sense of Definition 10.2.7 will be said to be of *degree* n. However, note that this grading only makes $\mathbb{Q}(q;-)[z]$ a weakly $\mathbb{N}$-graded algebra over $\mathbb{Q}(q;-)$. Let f be a q-polynomial. We will use $\deg(f)$ to denote its *degree*. The *leading coefficient* $\operatorname{lead}(f)$ of f is defined to be the coefficient of its leading term, namely the q-number attached to $\binom{z}{\deg(f)}$ in f.

Definition 10.3.4. The following grading on $\mathbb{Q}(q;-)[z]q^{\mathbb{Q}z}$ will be called the *order*:

$$\mathbb{Q}(q;-)[z]q^{\mathbb{Q}z} = \bigoplus_{\nu\in\mathbb{Q}} \mathbb{Q}(q;-)[z]q^{\nu z}.$$

That is to say, an element of grade ν in the sense of Definition 10.2.7 will be said to be of *order* ν. Note that this grading makes $\mathbb{Q}(q;-)[z]q^{\mathbb{Q}z}$ a $\mathbb{Q}$-graded algebra over $\mathbb{Q}(q;-)[z]$. We will use $\operatorname{ord}(f)$ to denote the *order* of a q-exponential polynomial f.

Each homogeneous component $\mathbb{Q}(q;-)[z]q^{\nu z}$ is a free module of rank one over the weakly $\mathbb{N}$-graded algebra $\mathbb{Q}(q;-)[z]$ and thus naturally a weakly $\mathbb{N}$-graded module. The *degree* $\deg(f)$ and the *leading coefficient* $\operatorname{lead}(f)$ of a q-exponential polynomial f are defined to be the degree and the leading coefficient of its leading term in the order grading.

Example 10.3.5. Let f be a q-exponential polynomial as in Eq. (10.3.1). Then its order is the largest $\nu \in \mathbb{Q}$ such that $f_\nu \neq 0$, its degree and leading coefficient is the degree and the leading coefficient of the q-polynomial $f_{\operatorname{ord}(f)}$.

Remark. Every q-polynomial will be viewed as a q-exponential polynomial which is homogeneous of order 0.

It is clear that a q-exponential polynomial f defines a q-function. We will use the same notion to denote this q-function. Then it has the following asymptotic growth:

$$f(z) \sim \operatorname{lead}(f)\binom{z}{\deg(f)}q^{\operatorname{ord}(f)z}. \tag{10.3.2}$$

We thus introduce the following convention.

Convention 10.3.6. Let S be a q-function and f a q-exponential polynomial. If S can be expressed as a q-exponential polynomial whose leading term is the same as $f(z)$, then we will say that $\mathrm{S}(z)$ has *asymptotic growth* $f(z)$ and write

$$\mathrm{S}(z) \sim f(z)$$

by an abuse of language. Note that this implies that S is asymptotically equal to the q-function defined by f.

Now, we turn to the discrete calculus.

Definition 10.3.7. The *difference operator* Δ on q-polynomials is the $\mathbb{Q}(q;-)$-linear operator vanishing on constant q-polynomials and satisfying the following:

$$\Delta\binom{z}{n} = \binom{z}{n-1}. \qquad (n \geqslant 1) \tag{10.3.3}$$

This operator extends to q-exponential polynomials as follows:

$$\Delta\left(\binom{z}{n}q^{\nu z}\right) := \left((q^{\nu}-1)\binom{z}{n} + q^{\nu}\binom{z}{n-1}\right)q^{\nu z}. \qquad (n \geqslant 1, \nu \neq 0) \tag{10.3.4}$$

It is straightforward to verify that the difference operator Δ satisfies the *Leibniz rule* Eq. (10.1.1). For a q-exponential polynomial f, the q-function defined by Δf is precisely the *difference* of the q-function defined by f.

The following lemma follows from the definition.

Lemma 10.3.8. *The linear operator* Δ *respects the order grading on* $\mathbb{Q}(q;-)[z]q^{\mathbb{Q}z}$. *On each homogeneous component* $\mathbb{Q}(q;-)[z]q^{\nu z}$ *(*$\nu \neq 0$*), the operator* Δ *respects the degree filtration strictly. On the subalgebra* $\mathbb{Q}(q;-)[z]$*, the operator* Δ *shifts the degree homogeneously by* -1.

Next, we will introduce the anti-difference operators.

Lemma 10.3.9. *The linear operators* Δ *admits a section* Σ *such that*

(i). *it respects the order;*

(ii). *on each homogeneous component of order* $\nu \neq 0$*, it respects the degree filtration strictly;*

(iii). *on the subalgebra of order* 0*, it shifts the degree homogeneously by* 1.

Moreover, if f *is a q-exponential polynomial, then we have the following formula:*

$$\operatorname{lead}(\Sigma f) = \begin{cases} \left(q^{\operatorname{ord}(f)} - 1\right)^{-1} \operatorname{lead}(f) & \text{if } \operatorname{ord}(f) \neq 0, \\ \operatorname{lead}(f) & \text{if } \operatorname{ord}(f) = 0. \end{cases} \tag{10.3.5}$$

Proof. First note that the restriction of the linear operator Δ to each homogeneous component $\mathbb{Q}(q;-)[z]q^{\nu z}$ ($\nu \neq 0$) is bijective, while its restriction to the subalgebra $\mathbb{Q}(q;-)[z]$ is surjective. In particular, the linear operator Δ itself is surjective. Since $\mathbb{Q}(q;-)[z]q^{\mathbb{Q}z}$ is a free module over $\mathbb{Q}(q;-)$, the statements on orders and degrees follows from Lemma 10.3.8 by applying Corollaries 10.2.4 and 10.2.11 and Lemma 10.2.3 to the operator Δ. As for the leading coefficients, $\operatorname{ord}(f) = 0$ case follows from Eq. (10.3.3). If $\operatorname{ord}(f) > 0$, by Eq. (10.3.4), we have

$$\operatorname{lead}(\Delta f) = \left(q^{\operatorname{ord}(f)} - 1\right) \operatorname{lead}(f).$$

Replacing f by Σf, Eq. (10.3.5) follows. □

Definition 10.3.10. The linear operator Σ in Lemma 10.3.9 is called the *free anti-difference operator*. Let a be an integer. The *anti-difference operator with anchor a*, denoted by Σ_a, is the linear operator $\Sigma - \mathrm{ev}_a \circ \Sigma$, where ev_a evaluates a q-exponential polynomial $f(z)$ at $z = a$.

For a q-exponential polynomial f, the q-function defined by $\Sigma_a f$ is the *anti-difference with anchor a* of the q-function defined by f.

We end this section with discussions of primarity.

Definition 10.3.11. Let f be a q-exponential polynomial as in Eq. (10.3.1). Then f is said to be *primary* if its coefficients are primary q-numbers and $f_\nu = 0$ for all $\nu \notin \mathbb{Z}$.

Clearly, primary q-exponential polynomials define primary q-functions.

Lemma 10.3.12. *Let f be a q-exponential polynomial. If $\operatorname{ord}(f) \geqslant 0$, then the q-functions defined by Σf for all $a \in \mathbb{Z}$ are asymptotically equal to the q-function defined by Σf. If f is a primary, then so are Δf, Σf, and $\Sigma_a f$ $(a \in \mathbb{Z})$.*

Proof. We have $\Sigma_a f - \Sigma f \in \mathbb{Q}(q; -)$. Note that elements of $\mathbb{Q}(q; -)$ have order 0 and degree 0, while Σf has nonzero order or degree by Lemma 10.3.9. Hence, we have $\Sigma_a f \sim \Sigma f$. The last statement is evident. □

§ 10.4. Super q-exponential polynomials

Definition 10.4.1. A *parity function* is a function which is defined on integers and factors through the projection $\mathbb{Z} \to \mathbb{F}_2$. A parity function valued in q-numbers is called a *parity q-function*.

Definition 10.4.2. A *super q-polynomial* is a polynomial with coefficients in parity q-functions. A *super q-exponential polynomial* is a finite formal sum

$$f(z) = \sum_{\nu} f_{\nu}(z) q^{\nu z}, \tag{10.4.1}$$

where $\nu \in \mathbb{Q}$ and each $f_{\nu}(z)$ is a super q-polynomial.

To better understand the structure of the ring of super q-exponential polynomials, we recall the following notions.

Definition 10.4.3. A *superalgebra* over a commutative ring R is a $\mathbb{F}_2$-graded algebra A over R. The $\mathbb{F}_2$-grading $A = A_0 \oplus A_1$ is called the *parity*. In particular, the subalgebra A_0 is called the *even part* and the A_0-module A_1 is called the *odd part*. For any element $a \in A$, its homogeneous component of parity 0 (resp. 1) is called its *even part* (resp. *odd part*). If A is a superalgebra, then an A-*supermodule* is a $\mathbb{F}_2$-graded A-module.

Convention 10.4.4. We will use $(-1)^z$ to denote the parity function mapping even numbers to 1 and odd numbers to -1.

Then the following lemma is easy to verify.

Lemma 10.4.5. *Let $e(z)$ be a parity function. Then we have*

$$e(z) = \frac{1}{2}(e(0) + e(1)) + \frac{1}{2}(e(0) - e(1))(-1)^z.$$

By this lemma, we have the following.

Corollary 10.4.6. *The ring of parity q-functions with formal variable z is precisely the $\mathbb{Q}(q; -)$-algebra generated by $(-1)^z$ and is a superalgebra decomposed into even and* *odd parts as follows:*

$$\mathbb{Q}(q;-)[(-1)^z] = \mathbb{Q}(q;-) \oplus \mathbb{Q}(q;-)(-1)^z.$$

Then the ring of super q-polynomials can be denoted by $\mathbb{Q}(q;-)[(-1)^z, z]$.

Corollary 10.4.7. *The ring of super q-polynomials is a superalgebra decomposed into even and odd parts as follows:*

$$\mathbb{Q}(q;-)[(-1)^z, z] = \mathbb{Q}(q;-)[z] \oplus \mathbb{Q}(q;-)[z](-1)^z.$$

Definition 10.4.8. The *degree* grading on $\mathbb{Q}(q;-)[(-1)^z, z]$ is defined similarly to that on $\mathbb{Q}(q;-)[z]$ in Definition 10.3.3 and makes it a weakly $\mathbb{N}$-graded algebra over $\mathbb{Q}(q;-)[(-1)^z]$. Since this weakly graded algebra is also a superalgebra, the degree grading induces a grading on its even part and one on its odd parts. They are called the *even degree* and the *odd degree*. Note that the even degree is precisely the degree grading on $\mathbb{Q}(q;-)[z]$ defined in Definition 10.3.3, and the odd degree is precisely the natural grading on a free module of rank one over $\mathbb{Q}(q;-)[z]$.

Let f be a super q-polynomial. We will use $\deg(f)$ to denote its *degree*. Its *even degree* $\deg_0(f)$ (resp. *odd degree* $\deg_1(f)$) is the even degree (resp. odd degree) of its even part (resp. odd part). The *leading coefficient* $\mathrm{lead}(f)$ of f is defined to be the coefficient of its leading term, namely the parity q-function attached to $\binom{z}{\deg(f)}$ in f. Its *even leading coefficient* $\mathrm{lead}_0(f)$ (resp. *odd leading coefficient* $\mathrm{lead}_1(f)$) is the leading coefficient of its even part (resp. odd part).

Example 10.4.9. Let f be a super q-polynomial as follows:

$$f(z) = f_0(z) + f_1(z)(-1)^z,$$

where f_0 and f_1 are q-polynomials. Then the even degree of f is $\deg(f_0)$, the odd degree of f is f is $\deg(f_1)$, and the degree of f is the larger one of them. If $\deg(f_0) > \deg(f_1)$, then the leading coefficient of f is precisely its even leading coefficient, which is $\operatorname{lead}(f_0)$. If $\deg(f_0) < \deg(f_1)$, then the leading coefficient of f is precisely its odd leading coefficient multiplied by $(-1)^z$, which is $\operatorname{lead}(f_1)(-1)^z$. If $\deg(f_0) = \deg(f_1)$, then the leading coefficient of f is the parity q-function $\operatorname{lead}(f_0) + \operatorname{lead}(f_1)(-1)^z$.

Similarly to Definition 10.3.2, the ring of super q-exponential polynomials will be denoted by $\mathbb{Q}(q;-)[(-1)^z, z]q^{\mathbb{Q}z}$.

Definition 10.4.10. The *order* grading on $\mathbb{Q}(q;-)[(-1)^z, z]q^{\mathbb{Q}z}$ is defined similarly to that on $\mathbb{Q}(q;-)[z]q^{\mathbb{Q}z}$ in Definition 10.3.4 and makes it a $\mathbb{Q}$-graded algebra over the superalgebra $\mathbb{Q}(q;-)[(-1)^z, z]$. We will use $\operatorname{ord}(f)$ to denote the *order* of a super q-exponential polynomial f.

Each homogeneous component $\mathbb{Q}(q;-)[(-1)^z, z]q^{\nu z}$ is a free supermodule of rank one over the weakly $\mathbb{N}$-graded superalgebra $\mathbb{Q}(q;-)[(-1)^z, z]$ and thus naturally a weakly $\mathbb{N}$-graded supermodule. The *degree* $\deg(f)$, the *even degree* $\deg_0(f)$, the *odd degree* $\deg_1(f)$, the *leading coefficient* $\operatorname{lead}(f)$, the *even leading coefficient* $\operatorname{lead}_0(f)$, and the *odd leading coefficient* $\operatorname{lead}_1(f)$ of a super q-exponential polynomial f are defined to be the degree, the even degree, the odd degree, the leading coefficient, the even leading coefficient, and the odd leading coefficient of its leading term in the order grading.

Example 10.4.11. Let f be a q-exponential polynomial as in Eq. (10.4.1). Then the order of f is the largest $\nu \in \mathbb{Q}$ such that $f_\nu \neq 0$ and its leading term is the product of the super q-polynomial $f_{\operatorname{ord}(f)}$ and $q^{\operatorname{ord}(f)z}$.

Remark. Every super q-polynomial will be viewed as a super q-exponential polynomial which is homogeneous of order 0.

It is clear that a super q-exponential polynomial f defines a q-function. We will use the same notion to denote this q-function. Then we have the following asymptotic equalities:

$$\begin{aligned} f(z) &\sim \left(\mathrm{lead}_0(f)\binom{z}{\deg_0(f)} + \mathrm{lead}_1(f)\binom{z}{\deg_1(f)}(-1)^z\right) q^{\mathrm{ord}(f)z} \\ &\sim \mathrm{lead}(f)\binom{z}{\deg(f)} q^{\mathrm{ord}(f)z}. \end{aligned} \tag{10.4.2}$$

Note that the leading coefficient $\mathrm{lead}(f)$ is a parity q-function rather than a q-number. In particular, the asymptotic behaviors of $f(z)$ along even integers and odd integers are different if $\deg_0(f) = \deg_1(f)$.

Convention 10.4.12. Let S be a q-function and f a super q-exponential polynomial. If S can be expressed as a super q-exponential polynomial whose leading term is the same as $f(z)$, then we will say that S has *asymptotic growth* f and write

$$\mathrm{S}(z) \sim f(z)$$

by an abuse of language. Note that this implies that S is asymptotically equal to the q-function defined by f.

Now, we turn to the discrete calculus.

Definition 10.4.13. The *difference operator* Δ on super q-exponential polynomials is the extension of the difference operator defined in Definition 10.3.7 satisfying the Leibniz rule Eq. (10.1.1) and acts on parity q-functions as in Definition 10.1.6.

Remark. By Lemma 10.4.5, the action of Δ on parity q-functions is determined by its action on $(-1)^z$. Note that $\Delta(-1)^z = -2(-1)^z$. Hence, the action of Δ on the superalgebra of super q-exponential polynomials respects the parity.

For a super q-exponential polynomial f, the q-function defined by Δf is precisely the *difference* of the q-function defined by f.

The following lemma follows from the definition.

Lemma 10.4.14. *The linear operator Δ respects the parity and the order grading on $\mathbb{Q}(q;-)[(-1)^z, z]q^{\mathbb{Q}z}$. On each homogeneous component $\mathbb{Q}(q;-)[(-1)^z, z]q^{\nu z}$ ($\nu \neq 0$), the operator Δ respects the degree filtration, the even degree filtration on its even part, and the odd degree filtration on its odd parts strictly. On the subalgebra $\mathbb{Q}(q;-)[(-1)^z, z]$, the operator Δ shifts the even degree homogeneously by -1 and respects the odd degree filtration strictly.*

Next, we will introduce the anti-difference operators.

Lemma 10.4.15. *The linear operators Δ admits a section Σ such that*

(i). *it respects the parity and the order;*

(ii). *on each homogeneous component of order $\nu \neq 0$, it respects the degree filtration, the even degree filtration on its even part, and odd degree filtration on its odd part strictly;*

(iii). *on the subalgebra of order 0, it shifts the even degree homogeneously by 1 and respects the odd degree filtration strictly.*

Moreover, if f is a q-exponential polynomial, then we have the following formulas:

$$
\begin{aligned}
\operatorname{lead}_0(\Sigma f) &= \begin{cases} \left(q^{\operatorname{ord}(f)} - 1\right)^{-1} \operatorname{lead}_0(f) & \text{if } \operatorname{ord}(f) \neq 0, \\ \operatorname{lead}_0(f) & \text{if } \operatorname{ord}(f) = 0, \end{cases} \\
\operatorname{lead}_1(\Sigma f) &= -\left(q^{\operatorname{ord}(f)} + 1\right)^{-1} \operatorname{lead}_1(f).
\end{aligned} \tag{10.4.3}
$$

Proof. First note that the restriction of the linear operator Δ to each homogeneous component $\mathbb{Q}(q;-)[(-1)^z, z]q^{\nu z}$ ($\nu \neq 0$) is bijective, while its restriction to the subalgebra $\mathbb{Q}(q;-)[(-1)^z, z]$ is surjective. In particular, the linear operator Δ itself is surjective. Since $\mathbb{Q}(q;-)[(-1)^z, z]q^{\mathbb{Q}z}$ is a free module over $\mathbb{Q}(q;-)$, the statements on orders and degrees follows from Lemma 10.4.14 by applying Corollaries 10.2.4 and 10.2.11 and Lemma 10.2.3 to the operator Δ. The statements on even leading coefficients follows from Lemma 10.3.9. As for the odd leading coefficients, first note that for each $\nu \in \mathbb{Q}$, we have

$$\Delta\left((-1)^z \binom{z}{n} q^{\nu z}\right) = -\left(q^\nu + 1\right)(-1)^z \binom{z}{n} q^{\nu z} - q^\nu (-1)^z \binom{z}{n-1} q^{\nu z}.$$

Therefore, we have

$$\operatorname{lead}_1(\Delta f) = -\left(q^{\operatorname{ord}(f)} + 1\right) \operatorname{lead}_1(f).$$

Replacing f by Σf, Eq. (10.3.5) follows. ◻

Definition 10.4.16. The linear operator Σ in Lemma 10.4.15 is called the *free anti-difference operator*. Let a be an integer. The *anti-difference operator with anchor* a, denoted by Σ_a, is the linear operator $\Sigma - \operatorname{ev}_a \circ \Sigma$, where ev_a evaluates a super q-exponential polynomial $f(z)$ at $z = a$.

For a super q-exponential polynomial f, the q-function defined by $\Sigma_a f$ is the *anti-*

difference with anchor a of the q-function defined by f.

We end this section with discussions of primarity.

Definition 10.4.17. Let f be a super q-exponential polynomial as in Eq. (10.4.1). Then f is said to be *primary* if its coefficients are primary q-numbers and $f_\nu = 0$ for all $\nu \notin \mathbb{Z}$.

Clearly, primary super q-exponential polynomials define primary q-functions.

Lemma 10.4.18. *Let f be a q-exponential polynomial. If* $\operatorname{ord}(f) > 0$ *or* $\operatorname{ord}(f) = 0$ *with either* $\deg_0(f) \geqslant 0$ *or* $\deg_1(f) > 0$*, then the q-functions defined by Σf for all $a \in \mathbb{Z}$ are asymptotically equal to the q-function defined by Σf. If f is primary, then so are Δf, Σf, and $\Sigma_a f$ ($a \in \mathbb{Z}$).*

Proof. We have $\Sigma_a f - \Sigma f \in \mathbb{Q}(q; -)$. Elements of $\mathbb{Q}(q; -)$ have order 0 and even degree 0, while the assumption on f implies that Σf has nonzero order or degree by Lemma 10.4.15. Hence, we have $\Sigma_a f \sim \Sigma f$. The last statement is evident. □

§ 10.5. Asymptotic growth of multi-summations

To analyze the growth of the q-functions $\mathrm{S}_{\mathcal{X}(I)}(r)$ and $\mathrm{S}^{\asymp}_{\mathcal{X}(I)}(r)$, we need to write them as q-exponential polynomials. This can be done by considering multi-summations of homogeneous (super) q-exponential polynomials. In this section, we give some general results.

The strategy is: we will inductively construct a sequence of (super) q-exponential polynomials where the final one defines the desired q-function, and then we will compare the orders, the degrees, and the leading coefficients of them.

To better describe the results, let's introduce some conventions.

Convention 10.5.1. Let $\mathfrak{i}$ be an index set. The set of functions from $\mathfrak{i}$ to another set X will be denoted by $X^{\mathfrak{i}}$. Such a function $\mathbf{c}$ will be identified with a sequence $(c_i)_{i\in\mathfrak{i}}$ indexed by $\mathfrak{i}$, where $c_i = \mathbf{c}(i)$. The constant sequence mapping all $i \in \mathfrak{i}$ to 1 will be denoted by $\mathbf{1}$. If $\boldsymbol{\mu}$ and $\mathbf{c}$ are two sequences of real numbers indexed by $\mathfrak{i}$, then $\boldsymbol{\mu}\cdot\mathbf{c}$ denotes their dot product, namely $\sum_{i\in\mathfrak{i}} \mu_i c_i$.

Lemma 10.5.2. *Let* S *be the q-function defined by the following multi-summation*

$$\mathrm{S}(z) = \sum_{\mathbf{c}\in\mathbb{Z}^{\mathfrak{i}}_{>0}:\ \mathbf{1}\cdot\mathbf{c}=z} q^{\boldsymbol{\mu}\cdot\mathbf{c}},$$

where $\boldsymbol{\mu}$ *is a sequence of non-negative rational numbers. Define the following notations:*

- $\mu_{\max}$ *is the maximum of* $\boldsymbol{\mu}$*;*
- $\mathfrak{i}_{\max}$ *is the set of indices* $i \in \mathfrak{i}$ *such that* $\mu_i = \mu_{\max}$.

Then S *can be expressed as a q-exponential polynomial so that*

$$\mathrm{S}(z) \sim \prod_{i\notin\mathfrak{i}_{\max}} (q^{\mu_{\max}-\mu_i} - 1)^{-1} \cdot \binom{z}{|\mathfrak{i}_{\max}| - 1} q^{\mu_{\max} z}.$$

Moreover, if $\boldsymbol{\mu}$ *takes integral values, then the q-exponential polynomial is primary.*

Remark. Note that the q-function S is eventually positive since the leading coefficient of the q-exponential polynomial defining it is positive.

Proof. First note that the condition on the sequence $\mathbf{c}$ of variables is stable under reindexing. Hence, we may assume $\mathfrak{i} = \{1, \cdots, t\}$ and $\mathfrak{i}_{\max} = \{1, \cdots, i_0\}$ by reindexing the sequence $\boldsymbol{\mu}$ if necessary. We change the variables from $\mathbf{c}$ to $\mathbf{b}$ as follows:

$$b_i := c_1 + \cdots + c_i. \qquad (1 \leqslant i \leqslant t)$$

Then we can write $\mathrm{S}(z)$ as follows:

$$\mathrm{S}(z) = q^{\nu_t z} \sum_{b_{t-1}=t-1}^{z-1} q^{\nu_{t-1} b_{t-1}} \cdots \sum_{b_1=1}^{b_2-1} q^{\nu_1 b_1},$$

where $\nu_i = \mu_i - \mu_{i+1}$ for $1 \leqslant i \leqslant t-1$ and $\nu_t = \mu_t$.

To analyze the growth of $\mathrm{S}(z)$, we define $f_1, \cdots, f_t$ inductively as follows:

$$\begin{aligned} f_1(z) &= q^{\nu_1 z}, \\ f_i(z) &= q^{\nu_i z} \Sigma_{i-1} f_{i-1}(z). && (1 < i \leqslant t) \end{aligned}$$

Then each f_i is a q-exponential polynomial, and we can analyze them by induction. In particular, f_t defines the q-function S by Eq. (10.1.2). Moreover, if $\mu_1, \cdots, \mu_t$ are integers, then every f_i is primary by Lemma 10.3.12.

For $1 < i \leqslant i_0$, repeatedly applying Lemma 10.3.9.(i), we have

$$\operatorname{ord}(f_i) = \nu_i + \operatorname{ord}(f_{i-1}) = \nu_i = \begin{cases} 0 & \text{if } i < i_0, \\ \mu_{i_0} - \mu_{i_0+1} & \text{if } i = i_0. \end{cases}$$

By Lemma 10.3.9.(iii), we have the following recurrence relations:

$$\deg(f_i) = \deg(f_{i-1}) + 1, \qquad \operatorname{lead}(f_i) = \operatorname{lead}(f_{i-1}).$$

In particular, we have $\operatorname{ord}(f_{i_0}) = \mu_{i_0} - \mu_{i_0+1}$, $\deg(f_{i_0}) = i_0 - 1$, and $\operatorname{lead}(f_{i_0}) = 1$.

For $i_0 < i \leqslant t$, repeatedly applying Lemma 10.3.9.(i), we have

$$\operatorname{ord}(f_i) = \nu_i + \operatorname{ord}(f_{i-1}) = \nu_i + \mu_{i_0} - \mu_i = \begin{cases} \mu_{i_0} - \mu_{i+1} & \text{if } i < t, \\ \mu_{i_0} & \text{if } i = t. \end{cases}$$

In particular, they are positive. Then by Lemma 10.3.9.(ii), we have the following recurrence relations:

$$\deg(f_i) = \deg(f_{i-1}), \qquad \operatorname{lead}(f_i) = (q^{\mu_{i_0}-\mu_i} - 1)^{-1} \operatorname{lead}(f_{i-1}).$$

In particular, we have $\operatorname{ord}(f_t) = \mu_{i_0}$, $\deg(f_t) = i_0 - 1$, and $\operatorname{lead}(f_t)$ equals the product of $(q^{\mu_{i_0}-\mu_i} - 1)^{-1}$ for $i_0 + 1 \leqslant i \leqslant t$.

Therefore, we have the following:

$$\mathrm{S}(z) \sim \prod_{i=i_0+1}^{t} (q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \binom{z}{i_0 - 1} q^{\mu_{i_0} z}.$$

This proves the lemma. □

In the rest of this section, we will consider multi-summations involving parity functions. We first extend Definition 10.4.1 to the following definition.

Definition 10.5.3. A *multivariable parity function* (indexed by $\mathfrak{i}$) is a function defined on $\mathbb{Z}^{\mathfrak{i}}$ factoring through the projection $\mathbb{Z}^{\mathfrak{i}} \to \mathbb{F}_2^{\mathfrak{i}}$.

Convention 10.5.4. By an abuse of notation, we will use the same notation to denote a multivariable parity function indexed by $\mathfrak{i}$ and a function on defined on $\mathbb{F}_2^{\mathfrak{i}}$. In other words, we will treat any sequence in $\mathbb{F}_2$ as a sequence in $\mathbb{Z}$ by viewing $0 \in \mathbb{F}_2$ and $1 \in \mathbb{F}_2$ as their standard representatives $0 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$.

Lemma 10.5.5. *Let* S *be the q-function defined by the following multi-summation*

$$\mathrm{S}(z) = \sum_{\mathbf{c} \in \mathbb{Z}_{>0}^{\mathfrak{i}} :\, \mathbf{1} \cdot \mathbf{c} = z} q^{\boldsymbol{\mu} \cdot \mathbf{c} + e(\mathbf{c})},$$

where $\boldsymbol{\mu}$ *is a sequence of non-negative rational numbers and* e *is a multivariable parity function. Define the following notations:*

- $\mu_{\max}$ *is the maximum of* $\boldsymbol{\mu}$*;*
- $\mathfrak{i}_{\max}$ *is the set of indices* $i \in \mathfrak{i}$ *such that* $\mu_i = \mu_{\max}$.

Then S *can be expressed as a super q-exponential polynomial so that*

$$\mathrm{S}(z) \sim \left(C_{\boldsymbol{\mu},e,0} + C_{\boldsymbol{\mu},e,1}(-1)^z\right) \cdot \binom{z}{|\mathfrak{i}_{\max}| - 1} q^{\mu_{\max} z},$$

where the constants $C_{\boldsymbol{\mu},e,0}$ *and* $C_{\boldsymbol{\mu},e,1}$ *are defined as follows:*

$$C_{\boldsymbol{\mu},e,0} := C_{\boldsymbol{\mu}} \cdot \sum_{\mathbf{s} \in \mathbb{F}_2^{\mathfrak{i}}} q^{e(\mathbf{s}) + (\mu_{\max} - \boldsymbol{\mu}) \cdot \mathbf{s}}, \qquad C_{\boldsymbol{\mu},e,1} := C_{\boldsymbol{\mu}} \cdot \sum_{\mathbf{s} \in \mathbb{F}_2^{\mathfrak{i}}} (-1)^{\mathbf{1} \cdot \mathbf{s}} q^{e(\mathbf{s}) + (\mu_{\max} - \boldsymbol{\mu}) \cdot \mathbf{s}},$$

where $\mu_{\max} - \boldsymbol{\mu}$ *denotes the sequence* $(\mu_{\max} - \mu_i)_{i \in \mathfrak{i}}$ *and* $C_{\boldsymbol{\mu}}$ *the following the constant:*

$$C_{\boldsymbol{\mu}} := \frac{1}{2^{|\mathfrak{i}_{\max}|}} \prod_{i \notin \mathfrak{i}_{\max}} \left(q^{2(\mu_{\max} - \mu_i)} - 1\right)^{-1}.$$

Moreover, if $\boldsymbol{\mu}$ *and* e *take integral values, then the super q-exponential polynomial is primary.*

Remark. Note that the even leading coefficient $C_{\boldsymbol{\mu},e,0}$ is positive and the odd leading coefficient $C_{\boldsymbol{\mu},e,1}$ satisfies $\left|C_{\boldsymbol{\mu},e,1}\right| < C_{\boldsymbol{\mu},e,0}$. Hence, the q-function S is eventually positive. Note that $C_{\boldsymbol{\mu},e,1}$ could be 0, in which case the asymptotic growth of $\mathrm{S}(z)$ along even integers and odd integers coincide.

To prove Lemma 10.5.5, we begin with some special cases.

Lemma 10.5.6. *Let* S *be the q-function defined by the following multi-summation*

$$\mathrm{S}(z) = \sum_{\mathbf{c} \in \mathbb{Z}_{>0}^{\mathfrak{i}} :\ \mathbf{1} \cdot \mathbf{c} = z} (-1)^{\mathbf{s} \cdot \mathbf{c}},$$

where $\mathbf{s}$ *is a sequence of integers. Define the following notations:*

- $\mathfrak{i}^0$ *(resp.* $\mathfrak{i}^1$*) is the set of indices* $i \in \mathfrak{i}$ *such that* s_i *is even (resp. odd).*

Then S *can be expressed as a primary super* q*-polynomial so that*

$$\mathrm{S}(z) \sim \left(-\tfrac{1}{2}\right)^{|\mathfrak{i}^1|}\binom{z}{|\mathfrak{i}^0|-1} + \left(-\tfrac{1}{2}\right)^{|\mathfrak{i}^0|}\binom{z}{|\mathfrak{i}^1|-1}(-1)^z.$$

Proof. First, if either $\mathbf{s}$ contains no even numbers or no odd numbers, then the statement follows from Lemma 10.5.2. We may assume that the sequence $\mathbf{s}$ contains at least one even number and one odd number. Since the condition on the sequence $\mathbf{c}$ of variables is stable under reindexing, we may assume $\mathfrak{i} = \{1, \cdots, t\}$ and $\mathfrak{i}^\circ = \{2, \cdots, i_0 + 1\}$ by reindexing the sequence $\mathbf{s}$ if necessary. We change the variables from $\mathbf{c}$ to $\mathbf{b}$ as follows:

$$b_i := c_1 + \cdots + c_i. \qquad (1 \leqslant i \leqslant t)$$

Then we can write $\mathrm{S}(z)$ as follows:

$$\mathrm{S}(z) = (-1)^{r_t z} \sum_{b_{t-1}=t-1}^{z-1} (-1)^{r_{t-1} b_{t-1}} \cdots \sum_{b_1=1}^{b_2-1} (-1)^{r_1 b_1},$$

where $r_i = s_i - s_{i+1}$ for $1 \leqslant i \leqslant t-1$ and $r_t = s_t$. Then our assumption implies that r_i is even whenever $i \notin \{1, i_0+1, t\}$.

To analyze the growth of $\mathrm{S}(z)$, we can define $f_1, \cdots, f_t$ inductively as follows:

$$f_1(z) = (-1)^{r_1 z},$$
$$f_i(z) = (-1)^{r_i z}\, \Sigma_{i-1} f_{i-1}(z). \qquad (1 < i \leqslant t)$$

Note that f_1 fails the condition of Lemma 10.4.18. But we can compute $\Sigma_1 f_1$ directly:

$$\Sigma_1 f_1(z) = -\tfrac{1}{2} - \tfrac{1}{2}(-1)^z.$$

Then each f_i is a primary super q-polynomial by Lemma 10.4.18. Moreover, f_t defines the q-function S by Eq. (10.1.2).

For $i \notin \{1, i_0 + 1, t\}$, since r_i is even, by Lemma 10.4.15.(iii), we have the following recurrence relations:

$$\deg_0(f_i) = \deg_0(f_{i-1}) + 1, \qquad \deg_1(f_i) = \deg_1(f_{i-1}),$$
$$\mathrm{lead}_0(f_i) = \mathrm{lead}_0(f_{i-1}), \qquad \mathrm{lead}_1(f_i) = -\tfrac{1}{2}\,\mathrm{lead}_1(f_{i-1}).$$

On the other hand, when r_i is odd, we have

$$\deg_0(f_i) = \deg_1(f_{i-1}), \qquad \deg_1(f_i) = \deg_0(f_{i-1}) + 1,$$
$$\mathrm{lead}_0(f_i) = -\tfrac{1}{2}\,\mathrm{lead}_1(f_{i-1}), \qquad \mathrm{lead}_1(f_i) = \mathrm{lead}_0(f_{i-1}).$$

If $i_0 = t - 1$, then $r_t = r_{i_0+1} = s_t$ is also even, and we have

$$\begin{aligned}
\deg_0(f_t) &= \deg_0(f_{t-1}) + 1 = \cdots = \deg_0(f_2) + t - 2 = t - 2,\\
\deg_1(f_t) &= \deg_1(f_{t-1}) = \cdots = \deg_1(f_2) = 0,\\
\mathrm{lead}_0(f_t) &= \mathrm{lead}_0(f_{t-1}) = \cdots = \mathrm{lead}_0(f_2) = -\tfrac{1}{2},\\
\mathrm{lead}_1(f_t) &= -\tfrac{1}{2}\,\mathrm{lead}_1(f_{t-1}) = \cdots = \left(-\tfrac{1}{2}\right)^{t-2}\mathrm{lead}_1(f_2) = \left(-\tfrac{1}{2}\right)^{t-1}.
\end{aligned}$$

Otherwise, both r_t and r_{i_0+1} are odd, and we have

$$\begin{aligned}
\deg_0(f_t) &= \deg_1(f_{t-1}) = \cdots = \deg_1\left(f_{i_0+1}\right) \\
&= \deg_0\left(f_{i_0}\right) + 1 = \cdots = \deg_0(f_2) + i_0 - 2 + 1 = i_0 - 1, \\
\deg_1(f_t) &= \deg_0(f_{t-1}) + 1 = \cdots = \deg_0\left(f_{i_0+1}\right) + t - i_0 - 2 + 1 \\
&= \deg_1\left(f_{i_0}\right) + t - i_0 - 1 = \cdots = \deg_1(f_2) + t - i_0 - 1 = t - i_0 - 1, \\
\mathrm{lead}_0(f_t) &= -\tfrac{1}{2}\,\mathrm{lead}_1(f_{t-1}) = \cdots = -\tfrac{1}{2}\left(-\tfrac{1}{2}\right)^{t-i_0-2} \mathrm{lead}_1\left(f_{i_0+1}\right) \\
&= \left(-\tfrac{1}{2}\right)^{t-i_0-1} \mathrm{lead}_0\left(f_{i_0}\right) = \cdots = \left(-\tfrac{1}{2}\right)^{t-i_0-1} \mathrm{lead}_0(f_2) = \left(-\tfrac{1}{2}\right)^{t-i_0}, \\
\mathrm{lead}_1(f_t) &= \mathrm{lead}_0(f_{t-1}) = \cdots = \mathrm{lead}_0\left(f_{i_0+1}\right) \\
&= -\tfrac{1}{2}\,\mathrm{lead}_1\left(f_{i_0}\right) = \cdots = -\tfrac{1}{2}\left(-\tfrac{1}{2}\right)^{i_0-2} \mathrm{lead}_1(f_2) = \left(-\tfrac{1}{2}\right)^{i_0}.
\end{aligned}$$

Then the lemma follows. ◻

Next, we consider the following situation.

Lemma 10.5.7. *Let* S *be the q-function defined by the following multi-summation*

$$\mathrm{S}(z) = \sum_{\mathbf{c}\in\mathbb{Z}^{\mathfrak{i}}_{>0}\colon\, \mathbf{1}\cdot\mathbf{c}=z} (-1)^{\mathbf{s}\cdot\mathbf{c}} q^{\boldsymbol{\mu}\cdot\mathbf{c}},$$

where $\mathbf{s}$ *is a sequence of integers and* $\boldsymbol{\mu}$ *is a sequence of non-negative rational numbers. Define the following notations:*

- $\mu_{\max}$ *is the maximum of* $\boldsymbol{\mu}$*;*
- $\mathfrak{i}_{\max}$ *is the set of indices* $i \in \mathfrak{i}$ *such that* $\mu_i = \mu_{\max}$*;*
- $\mathfrak{i}^0_{\max}$ *(resp.* $\mathfrak{i}^1_{\max}$*) is the set of indices* $i \in \mathfrak{i}_{\max}$ *such that* s_i *is even (resp. odd);*
- $\mathfrak{i}^0$ *(resp.* $\mathfrak{i}^1$*) is the set of indices* $i \in \mathfrak{i} \setminus \mathfrak{i}_{\max}$ *such that* s_i *is even (resp. odd).*

Then S *can be expressed as a super q-exponential polynomial so that*

$$\mathrm{S}(z) \sim \left(f_{\mathbf{s},\boldsymbol{\mu},0}(z) + f_{\mathbf{s},\boldsymbol{\mu},1}(z)(-1)^{z}\right) q^{\mu_{\max} z},$$

where $f_{\mathbf{s},\boldsymbol{\mu},\square}(z)$ *(*$\square = 0, 1$*) is the following q-polynomial:*

$$f_{\mathbf{s},\boldsymbol{\mu},\square}(z) := \prod_{i \notin \mathfrak{i}_{\max}} \left((-1)^{\square + s_i} q^{\mu_{\max} - \mu_i} - 1\right)^{-1} \cdot \left(-\frac{1}{2}\right)^{\left|\mathfrak{i}_{\max}^{1-\square}\right|} \binom{z}{\left|\mathfrak{i}_{\max}^{\square}\right| - 1}.$$

Moreover, if $\boldsymbol{\mu}$ *takes integral values, then the super q-exponential polynomial is primary.*

Remark. Note that the leading coefficient may be negative. However, if $\mathbf{s}$ contains no even numbers (resp. odd numbers), then the even (resp. odd) leading coefficient is zero and the odd (resp. even) leading coefficient is positive.

Proof. First note that the condition on the sequence $\mathbf{c}$ of variables is stable under reindexing. Hence, we may assume $\mathfrak{i} = \{1, \cdots, t\}$, $\mathfrak{i}_{\max} = \{1, \cdots, i_0\}$, and $\mathfrak{i}^0 = \{i_0 + 1, \cdots, i_1\}$ by reindexing the sequences $\boldsymbol{\mu}$ and $\mathbf{s}$ if necessary. We change the variables from $\mathbf{c}$ to $\mathbf{b}$ as follows:

$$b_i := c_1 + \cdots + c_i. \qquad (i_0 \leqslant i \leqslant t)$$

Then we can write $\mathrm{S}(z)$ as follows:

$$\mathrm{S}(z) = (-1)^{r_t z} q^{\nu_t z} \sum_{b_{t-1}=t-1}^{z-1} (-1)^{r_{t-1} b_{t-1}} q^{\nu_{t-1} b_{t-1}} \cdots \sum_{b_{i_0}=i_0}^{b_{i_0+1}-1} (-1)^{-s_{i_0+1} b_{i_0}} q^{\nu_{i_0} b_{i_0}} \mathrm{S}_{\mathbf{s}|_{\mathfrak{i}_{\max}}}(b_{i_0}),$$

where

- $r_i = s_i - s_{i+1}$ for $i_0 + 1 \leqslant i \leqslant t - 1$ and $r_t = s_t$;
- $\nu_i = \mu_i - \mu_{i+1}$ for $i_0 \leqslant i \leqslant t - 1$ and $\nu_t = \mu_t$;

- $\mathbf{s}|_{\mathfrak{i}_{\max}}$ is the subsequence of $\mathbf{s}$ indexed by $\mathfrak{i}_{\max}$ and the q-function $\mathrm{S}_{\mathbf{s}|_{\mathfrak{i}_{\max}}}$ is defined as follows:

$$\mathrm{S}_{\mathbf{s}|_{\mathfrak{i}_{\max}}}(z) := \sum_{\mathbf{c}\in\mathbb{Z}_{>0}^{\mathfrak{i}_{\max}}\,:\,\mathbf{1}\cdot\mathbf{c}=z} (-1)^{\mathbf{s}|_{\mathfrak{i}_{\max}}\cdot\mathbf{c}}.$$

By Lemma 10.5.6, the q-function $\mathrm{S}_{\mathbf{s}|_{\mathfrak{i}_{\max}}}$ can be expressed as a primary super q-polynomial f, for which we have

$$\deg_0(f) = \left|\mathfrak{i}^0_{\max}\right| - 1, \qquad \deg_1(f) = \left|\mathfrak{i}^1_{\max}\right| - 1,$$
$$\mathrm{lead}_0(f) = \left(-\tfrac{1}{2}\right)^{\left|\mathfrak{i}^1_{\max}\right|}, \qquad \mathrm{lead}_1(f) = \left(-\tfrac{1}{2}\right)^{\left|\mathfrak{i}^0_{\max}\right|}.$$

To analyze the growth of $\mathrm{S}(z)$, we can define $f_{i_0}, \cdots, f_t$ inductively as follows:

$$f_{i_0}(z) = (-1)^{-s_{i_0+1}z} q^{\nu_{i_0}z} f(z),$$
$$f_i(z) = (-1)^{r_i z} q^{\nu_i z}\, \Sigma_{i-1}\, f_{i-1}(z). \qquad (i_0 < i \leqslant t)$$

Then each f_i is a super q-exponential polynomial, and we can analyze them by induction. In particular, f_t defines the q-function S by Eq. (10.1.2). Moreover, if $\mu_1, \cdots, \mu_t$ are integers, then every f_i is primary by Lemma 10.4.18.

For each $i_0 < i \leqslant t$, repeatedly applying Lemma 10.4.15.(i), we have

$$\mathrm{ord}(f_i) = \nu_i + \mathrm{ord}(f_{i-1}) = \nu_i + \mu_{i_0} - \mu_i = \begin{cases} \mu_{i_0} - \mu_{i+1} & \text{if } i < t, \\ \mu_{i_0} & \text{if } i = t. \end{cases}$$

When $i \notin \{i_1, t\}$, since r_i is even, by Lemma 10.4.15.(ii), we have the following recurrence relations:

$$\deg_0(f_i) = \deg_0(f_{i-1}), \qquad \deg_1(f_i) = \deg_1(f_{i-1}),$$
$$\mathrm{lead}_0(f_i) = (q^{\mu_{i_0}-\mu_i} - 1)^{-1}\,\mathrm{lead}_0(f_{i-1}), \quad \mathrm{lead}_1(f_i) = (-q^{\mu_{i_0}-\mu_i} - 1)^{-1}\,\mathrm{lead}_1(f_{i-1}).$$

On the other hand, when r_i is odd, we have

$$\deg_0(f_i) = \deg_1(f_{i-1}), \qquad \deg_1(f_i) = \deg_0(f_{i-1}),$$
$$\operatorname{lead}_0(f_i) = (-q^{\mu_{i_0}-\mu_i} - 1)^{-1} \operatorname{lead}_1(f_{i-1}), \quad \operatorname{lead}_1(f_i) = (q^{\mu_{i_0}-\mu_i} - 1)^{-1} \operatorname{lead}_0(f_{i-1}).$$

If $i_1 = t$, namely $\mathfrak{i}^1 = \emptyset$, then both s_{i_0+1} and $r_t = s_t$ are even, and we have

$$\begin{aligned}
\deg_0(f_t) &= \deg_0(f_{t-1}) = \cdots = \deg_0\big(f_{i_0}\big) = \left|\mathfrak{i}^0_{\max}\right| - 1 = t - 1,\\
\deg_1(f_t) &= \deg_1(f_{t-1}) = \cdots = \deg_1\big(f_{i_0}\big) = \left|\mathfrak{i}^1_{\max}\right| - 1 = -1,\\
\operatorname{lead}_0(f_t) &= (q^{\mu_{i_0}-\mu_t} - 1)^{-1} \cdot \operatorname{lead}_0(f_{t-1}) = \cdots\\
&= \prod_{i=i_0+1}^{t} (q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \operatorname{lead}_0\big(f_{i_0}\big) = \prod_{i=i_0+1}^{t} (q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \left(-\tfrac{1}{2}\right)^{\left|\mathfrak{i}^1_{\max}\right|}.
\end{aligned}$$

If $i_1 = i_0$, namely $\mathfrak{i}^0 = \emptyset$, then both s_{i_0+1} and $r_t = s_t$ are odd, and we have

$$\begin{aligned}
\deg_0(f_t) &= \deg_1(f_{t-1}) = \cdots = \deg_1\big(f_{i_0}\big) = \left|\mathfrak{i}^0_{\max}\right| - 1 = -1,\\
\deg_1(f_t) &= \deg_0(f_{t-1}) = \cdots = \deg_0\big(f_{i_0}\big) = \left|\mathfrak{i}^1_{\max}\right| - 1 = t - 1,\\
\operatorname{lead}_1(f_t) &= (q^{\mu_{i_0}-\mu_t} - 1)^{-1} \cdot \operatorname{lead}_0(f_{t-1}) = \cdots\\
&= \prod_{i=i_0+1}^{t} (q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \operatorname{lead}_0\big(f_{i_0}\big) = \prod_{i=i_0+1}^{t} (q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \left(-\tfrac{1}{2}\right)^{\left|\mathfrak{i}^0_{\max}\right|}.
\end{aligned}$$

If $i_0 < i_1 < t$, then s_{i_0+1} is even while r_{i_1} and $r_t = s_t$ are odd. We thus have

$$\begin{aligned}
\deg_0(f_t) &= \deg_1(f_{t-1}) = \cdots = \deg_1\big(f_{i_1}\big) \\
&= \deg_0\big(f_{i_1-1}\big) = \cdots = \deg_0\big(f_{i_0}\big) = \big|\mathfrak{i}^0_{\max}\big| - 1 \geqslant 0, \\
\deg_1(f_t) &= \deg_0(f_{t-1}) = \cdots = \deg_0\big(f_{i_1}\big) \\
&= \deg_0\big(f_{i_1-1}\big) = \cdots = \deg_0\big(f_{i_0}\big) = \big|\mathfrak{i}^1_{\max}\big| - 1 \geqslant 0, \\
\operatorname{lead}_0(f_t) &= (-q^{\mu_{i_0}-\mu_t} - 1)^{-1} \cdot \operatorname{lead}_1(f_{t-1}) = \cdots = \prod_{i=i_1+1}^{t} (-q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \operatorname{lead}_1\big(f_{i_1}\big) \\
&= \prod_{i=i_1+1}^{t} (-q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot (q^{\mu_{i_0}-\mu_{i_1}} - 1)^{-1} \cdot \operatorname{lead}_0\big(f_{i_1-1}\big) = \cdots \\
&= \prod_{i=i_1+1}^{t} (-q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \prod_{i=i_0+1}^{i_1} (q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \operatorname{lead}_0\big(f_{i_0}\big) \\
&= \prod_{i=i_1+1}^{t} (-q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \prod_{i=i_0+1}^{i_1} (q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \left(-\tfrac{1}{2}\right)^{|\mathfrak{i}^1_{\max}|}, \\
\operatorname{lead}_1(f_t) &= (q^{\mu_{i_0}-\mu_t} - 1)^{-1} \cdot \operatorname{lead}_0(f_{t-1}) = \cdots = \prod_{i=i_1+1}^{t} (q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \operatorname{lead}_0\big(f_{i_1}\big) \\
&= \prod_{i=i_1+1}^{t} (q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot (-q^{\mu_{i_0}-\mu_{i_1}} - 1)^{-1} \cdot \operatorname{lead}_1\big(f_{i_1-1}\big) = \cdots \\
&= \prod_{i=i_1+1}^{t} (q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \prod_{i=i_0+1}^{i_1} (-q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \operatorname{lead}_1\big(f_{i_0}\big) \\
&= \prod_{i=i_1+1}^{t} (q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \prod_{i=i_0+1}^{i_1} (-q^{\mu_{i_0}-\mu_i} - 1)^{-1} \cdot \left(-\tfrac{1}{2}\right)^{|\mathfrak{i}^0_{\max}|}.
\end{aligned}$$

Then the lemma follows. □

To deduce Lemma 10.5.5 from Lemma 10.5.7, we need the following notions.

Definition 10.5.8. Let e be a multivariable parity function indexed by $\mathfrak{i}$. Then its *Fourier* *transform* $\hat{e}$ is the following:

$$\hat{e}(-) := \sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}}} e(\mathbf{s})(-1)^{\mathbf{s}\cdot -}.$$

Convention 10.5.9. It is often the case that the index set $\mathfrak{i}$ admits a partition $\mathfrak{i} = \mathfrak{i}_1 \sqcup \mathfrak{i}_2$. For any sequence $\mathbf{c}$ indexed by $\mathfrak{i}$, we will use $\mathbf{c}_1$ and $\mathbf{c}_2$ to denote the subsequences of $\mathbf{c}$ indexed by $\mathfrak{i}_1$ and $\mathfrak{i}_2$ respectively. Conversely, if $\mathbf{c}_1$ and $\mathbf{c}_2$ are two sequences indexed by $\mathfrak{i}_1$ and $\mathfrak{i}_2$ respectively, then we will use $\mathbf{c}_1 \sqcup \mathbf{c}_2$ to denote the sequence indexed by $\mathfrak{i}$ obtained from them.

We have the following multivariable version of Lemma 10.4.5.

Lemma 10.5.10. *Let e be a multivariable parity function indexed by $\mathfrak{i}$. Then we have*

$$e(\mathbf{c}) = \frac{1}{2^{|\mathfrak{i}|}} \sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}}} \hat{e}(\mathbf{s})(-1)^{\mathbf{s}\cdot\mathbf{c}}.$$

Moreover, if $\mathfrak{i}$ admits a partition $\mathfrak{i} = \mathfrak{i}_1 \sqcup \mathfrak{i}_2$, then we have

$$\sum_{\mathbf{s}_1\in\mathbb{F}_2^{\mathfrak{i}_1}} e(\mathbf{s}_1 \sqcup \mathbf{c}_2)(-1)^{\mathbf{s}_1\cdot\mathbf{c}_1} = \frac{1}{2^{|\mathfrak{i}_2|}} \sum_{\mathbf{s}_2\in\mathbb{F}_2^{\mathfrak{i}_2}} \hat{e}(\mathbf{c}_1 \sqcup \mathbf{s}_2)(-1)^{\mathbf{s}_2\cdot\mathbf{c}_2}.$$

Proof. This follows from the general theory of Fourier transforms on finite Abelian groups (see e.g. [Lang, chap. XVIII, §5 and §6]). To verify the lemma directly, note that for any index set $\mathfrak{i}$ and any sequence $\mathbf{c}$ indexed by $\mathfrak{i}$, we have

$$\sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}}} (-1)^{\mathbf{s}\cdot\mathbf{c}} = \begin{cases} 2^{|\mathfrak{i}|} & \text{if } \bar{\mathbf{c}} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then the statement follows by straightforward computations. ◻

We are now able to prove Lemma 10.5.5.

Proof of Lemma 10.5.5. By Lemma 10.5.10, we can write S as follows:

$$\mathrm{S}(z) = \frac{1}{2^{|\mathfrak{i}|}} \sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}}} \widehat{q^{e(-)}}(\mathbf{s})\, \mathrm{S}_{\mathbf{s}}(z),$$

where each $\mathrm{S}_{\mathbf{s}}$ is defined as follows:

$$\mathrm{S}_{\mathbf{s}}(z) = \sum_{\mathbf{c}\in\mathbb{Z}_{>0}^{\mathfrak{i}}:\ \mathbf{1}\cdot\mathbf{c}=z} (-1)^{\mathbf{s}\cdot\mathbf{c}} q^{\boldsymbol{\mu}\cdot\mathbf{c}}.$$

By Lemma 10.5.7, each $\mathrm{S}_{\mathbf{s}}$ can be expressed as a super q-exponential polynomial of order $\mu_{\max}$, even degree $\left|\mathfrak{i}^0_{\max}\right| - 1$, and odd degree $\left|\mathfrak{i}^1_{\max}\right| - 1$ (see there for the notations). Moreover, if $\boldsymbol{\mu}$ takes integral values, then these super q-exponential polynomials are primary.

Note that the degree of $\mathrm{S}_{\mathbf{s}}$ achieves its maximum $|\mathfrak{i}_{\max}| - 1$ if and only if $\mathbf{s}$ contains no odd numbers or no even numbers. Furthermore, in that case, both the even and odd leading coefficients of $\mathrm{S}_{\mathbf{s}}$ are non-negative (indeed, one is zero and another is positive).

Therefore, S can be expressed as a super q-exponential polynomial of order $\mu_{\max}$ and degree $|\mathfrak{i}_{\max}| - 1$, and if $\boldsymbol{\mu}$ and e take integral values, then this super q-exponential polynomial is primary. Moreover, we have

$$(*) \qquad \mathrm{S}(z) \sim \frac{1}{2^{|\mathfrak{i}|}}\, \mathrm{E}(z) \binom{z}{|\mathfrak{i}_{\max}| - 1} q^{\mu_{\max} z},$$

where the parity q-function E is given as follows:

$$\begin{aligned} \mathrm{E}(z) := &\sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}\setminus\mathfrak{i}_{\max}}} \widehat{q^{e(-)}}(\mathbf{0}\sqcup\mathbf{s}) \prod_{i\notin\mathfrak{i}_{\max}} \left((-1)^{s_i} q^{\mu_{\max}-\mu_i} - 1\right)^{-1} \\ &+ \sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}\setminus\mathfrak{i}_{\max}}} \widehat{q^{e(-)}}(\mathbf{1}\sqcup\mathbf{s}) \prod_{i\notin\mathfrak{i}_{\max}} \left((-1)^{1+s_i} q^{\mu_{\max}-\mu_i} - 1\right)^{-1} \cdot (-1)^z. \end{aligned}$$

To deduce the formula in Lemma 10.5.5 from above one, note that

$$\prod_{i\notin\mathbf{i}_{\max}}\left(q^{2(\mu_{\max}-\mu_i)}-1\right)\prod_{i\notin\mathbf{i}_{\max}}((-1)^{s_i}q^{\mu_{\max}-\mu_i}-1)^{-1}$$
$$=\prod_{i\notin\mathbf{i}_{\max}}((-1)^{s_i}q^{\mu_{\max}-\mu_i}+1)=\sum_{\mathbf{s}'\in\mathbb{F}_2^{\mathbf{i}\setminus\mathbf{i}_{\max}}}(-1)^{\mathbf{s}\cdot\mathbf{s}'}q^{(\mu_{\max}-\boldsymbol{\mu})\cdot\mathbf{s}'},$$

and similarly

$$\prod_{i\notin\mathbf{i}_{\max}}\left(q^{2(\mu_{\max}-\mu_i)}-1\right)\prod_{i\notin\mathbf{i}_{\max}}\left((-1)^{1+s_i}q^{\mu_{\max}-\mu_i}-1\right)^{-1}=\sum_{\mathbf{s}'\in\mathbb{F}_2^{\mathbf{i}\setminus\mathbf{i}_{\max}}}(-1)^{(\mathbf{1}+\mathbf{s})\cdot\mathbf{s}'}q^{(\mu_{\max}-\boldsymbol{\mu})\cdot\mathbf{s}'}.$$

Therefore, we have

$$\prod_{i\notin\mathbf{i}_{\max}}\left(q^{2(\mu_{\max}-\mu_i)}-1\right)\cdot\mathrm{E}(z)$$
$$=\sum_{\mathbf{s}\in\mathbb{F}_2^{\mathbf{i}\setminus\mathbf{i}_{\max}}}\widehat{q^{e(-)}}(\mathbf{0}\sqcup\mathbf{s})\sum_{\mathbf{s}'\in\mathbb{F}_2^{\mathbf{i}\setminus\mathbf{i}_{\max}}}(-1)^{\mathbf{s}\cdot\mathbf{s}'}q^{(\mu_{\max}-\boldsymbol{\mu})\cdot\mathbf{s}'}$$
$$+\sum_{\mathbf{s}\in\mathbb{F}_2^{\mathbf{i}\setminus\mathbf{i}_{\max}}}\widehat{q^{e(-)}}(\mathbf{1}\sqcup\mathbf{s})\sum_{\mathbf{s}'\in\mathbb{F}_2^{\mathbf{i}\setminus\mathbf{i}_{\max}}}(-1)^{(\mathbf{1}+\mathbf{s})\cdot\mathbf{s}'}q^{(\mu_{\max}-\boldsymbol{\mu})\cdot\mathbf{s}'}\cdot(-1)^{z}$$
$$=\sum_{\mathbf{s}'\in\mathbb{F}_2^{\mathbf{i}\setminus\mathbf{i}_{\max}}}\left(\sum_{\mathbf{s}\in\mathbb{F}_2^{\mathbf{i}\setminus\mathbf{i}_{\max}}}\widehat{q^{e(-)}}(\mathbf{0}\sqcup\mathbf{s})(-1)^{\mathbf{s}\cdot\mathbf{s}'}\right)q^{(\mu_{\max}-\boldsymbol{\mu})\cdot\mathbf{s}'}$$
$$+\sum_{\mathbf{s}'\in\mathbb{F}_2^{\mathbf{i}\setminus\mathbf{i}_{\max}}}\left(\sum_{\mathbf{s}\in\mathbb{F}_2^{\mathbf{i}\setminus\mathbf{i}_{\max}}}\widehat{q^{e(-)}}(\mathbf{1}\sqcup\mathbf{s})(-1)^{\mathbf{s}\cdot\mathbf{s}'}\right)q^{(\mu_{\max}-\boldsymbol{\mu})\cdot\mathbf{s}'}\cdot(-1)^{z+\mathbf{1}\cdot\mathbf{s}'}.$$

By Lemma 10.5.10, we have

$$\begin{aligned}
&\frac{1}{2^{|\mathfrak{i}\setminus\mathfrak{i}_{\max}|}}\cdot\prod_{i\notin\mathfrak{i}_{\max}}\left(q^{2(\mu_{\max}-\mu_i)}-1\right)\cdot\mathrm{E}(z)\\
&=\sum_{\mathbf{s}'\in\mathbb{F}_2^{\mathfrak{i}\setminus\mathfrak{i}_{\max}}}\left(\sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}_{\max}}}q^{e(\mathbf{s}\sqcup\mathbf{s}')}(-1)^{\mathbf{s}\cdot\mathbf{0}}\right)q^{(\mu_{\max}-\boldsymbol{\mu})\cdot\mathbf{s}'}\\
&\qquad+\sum_{\mathbf{s}'\in\mathbb{F}_2^{\mathfrak{i}\setminus\mathfrak{i}_{\max}}}\left(\sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}\setminus\mathfrak{i}_{\max}}}q^{e(\mathbf{s}\sqcup\mathbf{s}')}(-1)^{\mathbf{s}\cdot\mathbf{1}}\right)q^{(\mu_{\max}-\boldsymbol{\mu})\cdot\mathbf{s}'}\cdot(-1)^{z+\mathbf{1}\cdot\mathbf{s}'}\\
&=\sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}}}q^{e(\mathbf{s})+(\mu_{\max}-\boldsymbol{\mu})\cdot\mathbf{s}}\left(1+(-1)^{z+\mathbf{1}\cdot\mathbf{s}}\right).
\end{aligned}$$

Apply this to Eq. (∗), then Lemma 10.5.5 follows. □

§ 10.6. Asymptotic growth of non-balanced multi-summations

This section aims to apply the results in § 10.5 to get asymptotic growth of multi-summations which are *non-balanced* in the sense that the summation condition of variables is no longer $\mathbf{1}\cdot\mathbf{c}=z$. In our applications, the coefficients in the summation condition can only be either 1 or 2, see § 9.2 to 9.5. Hence, we will assume that the index set $\mathfrak{i}$ admits a partition $\mathfrak{i}=\mathfrak{i}_1\sqcup\mathfrak{i}_2$ and then follow Convention 10.5.9.

Lemma 10.6.1. *Let* S *be the q-function defined by the following multi-summation*

$$\mathrm{S}(z)=\sum_{\mathbf{c}\in\mathbb{Z}_{>0}^{\mathfrak{i}}:\ \mathbf{1}\cdot\mathbf{c}_1+2(\mathbf{1}\cdot\mathbf{c}_2)=z}q^{\boldsymbol{\mu}\cdot\mathbf{c}},$$

where $\boldsymbol{\mu}$ *is a sequence of non-negative rational numbers. For* $\square=1,2$*, define the* *following notations:*

- $\mu_{\square\,\max}$ *is the maximum of* $\boldsymbol{\mu}_\square$*;*
- $\mathfrak{i}_{\square\,\max}$ *is the set of indices* $i \in \mathfrak{i}_\square$ *such that* $\mu_i = \mu_{\square\,\max}$.

Then S *can be expressed as a super q-exponential polynomial.*

(i). *If* $2\mu_{1\,\max} > \mu_{2\,\max}$*, then we have*

$$\mathrm{S}(z) \sim C_{\boldsymbol{\mu}} \cdot \sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}_1 \setminus \mathfrak{i}_{1\,\max}}} q^{(\mu_{1\,\max} - \boldsymbol{\mu}|_{\mathfrak{i}_1 \setminus \mathfrak{i}_{1\,\max}})\cdot\mathbf{s}} \cdot \binom{z}{|\mathfrak{i}_{1\,\max}| - 1} q^{\mu_{1\,\max} z},$$

where $\mu_{1\,\max} - \boldsymbol{\mu}|_{\mathfrak{i}_1 \setminus \mathfrak{i}_{1\,\max}}$ *denotes the sequence* $(\mu_{1\,\max} - \mu_i)_{i\in\mathfrak{i}_1 \setminus \mathfrak{i}_{1\,\max}}$ *and the constant* $C_{\boldsymbol{\mu}}$ *is defined as follows:*

$$C_{\boldsymbol{\mu}} := \prod_{i\in\mathfrak{i}_1 \setminus \mathfrak{i}_{1\,\max}} \left(q^{2\mu_{1\,\max} - 2\mu_i} - 1\right)^{-1} \prod_{i\in\mathfrak{i}_2} \left(q^{2\mu_{1\,\max} - \mu_i} - 1\right)^{-1}.$$

(ii). *If* $2\mu_{1\,\max} < \mu_{2\,\max}$*, then we have*

$$\mathrm{S}(z) \sim \left(C_{\boldsymbol{\mu},0} + C_{\boldsymbol{\mu},1}(-1)^z\right) \cdot \binom{z}{|\mathfrak{i}_{2\,\max}| - 1} q^{\frac{1}{2}\mu_{2\,\max} z},$$

where the constants $C_{\boldsymbol{\mu},0}$ *and* $C_{\boldsymbol{\mu},1}$ *are defined as follows:*

$$C_{\boldsymbol{\mu},0} = C_{\boldsymbol{\mu}} \cdot \sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}_1}} q^{(\frac{1}{2}\mu_{2\,\max} - \boldsymbol{\mu}_1)\cdot\mathbf{s}}, \qquad C_{\boldsymbol{\mu},1} = C_{\boldsymbol{\mu}} \cdot \sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}_1}} (-1)^{\mathbf{1}\cdot\mathbf{s}} q^{(\frac{1}{2}\mu_{2\,\max} - \boldsymbol{\mu}_1)\cdot\mathbf{s}},$$

where $\frac{1}{2}\mu_{2\,\max} - \boldsymbol{\mu}_1$ *denotes the sequence* $(\frac{1}{2}\mu_{2\,\max} - \mu_i)_{i\in\mathfrak{i}_1}$ *and the constant* $C_{\boldsymbol{\mu}}$ *is defined as follows:*

$$C_{\boldsymbol{\mu}} := \frac{1}{2^{|\mathfrak{i}_{2\,\max}|}} \prod_{i\in\mathfrak{i}_1} \left(q^{\mu_{2\,\max} - 2\mu_i} - 1\right)^{-1} \prod_{i\in\mathfrak{i}_2 \setminus \mathfrak{i}_{2\,\max}} \left(q^{\mu_{2\,\max} - \mu_i} - 1\right)^{-1}.$$

(iii). *If* $2\mu_{1\,\max} = \mu_{2\,\max}$*, then we have*

$$\mathrm{S}(z) \sim C_{\boldsymbol{\mu}} \cdot \sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}_1\setminus\mathfrak{i}_{1\,\max}}} q^{(\mu_{1\,\max}-\boldsymbol{\mu}|_{\mathfrak{i}_1\setminus\mathfrak{i}_{1\,\max}})\cdot\mathbf{s}} \cdot \binom{z}{|\mathfrak{i}_{1\,\max}|+|\mathfrak{i}_{2\,\max}|-1} q^{\mu_{1\,\max}z},$$

where $\mu_{1\,\max} - \boldsymbol{\mu}|_{\mathfrak{i}_1\setminus\mathfrak{i}_{1\,\max}}$ *denotes the sequence* $(\mu_{1\,\max} - \mu_i)_{i\in\mathfrak{i}_1\setminus\mathfrak{i}_{1\,\max}}$ *and the constant* $C_{\boldsymbol{\mu}}$ *is defined as follows:*

$$C_{\boldsymbol{\mu}} := \frac{1}{2^{|\mathfrak{i}_{2\,\max}|}} \prod_{i\in\mathfrak{i}_1\setminus\mathfrak{i}_{1\,\max}} \left(q^{2\mu_{1\,\max}-2\mu_i}-1\right)^{-1} \prod_{i\in\mathfrak{i}_2\setminus\mathfrak{i}_{2\,\max}} \left(q^{2\mu_{1\,\max}-\mu_i}-1\right)^{-1}.$$

Moreover, if both $\boldsymbol{\mu}_1$ *and* $\frac{1}{2}\boldsymbol{\mu}_2$ *take integral values, then the super* q*-exponential polynomial is primary.*

Remark. Note that the even leading coefficient $C_{\boldsymbol{\mu},0}$ is positive and the odd leading coefficient $C_{\boldsymbol{\mu},1}$ satisfies $\left|C_{\boldsymbol{\mu},1}\right| < C_{\boldsymbol{\mu},0}$. Hence, the q-function S is eventually positive. Note that $C_{\boldsymbol{\mu},1}$ could be 0, in which case the asymptotic growth of $\mathrm{S}(z)$ along even integers and odd integers coincide.

In the proof of above lemma and its many applications, a fundamental trick is to extend the domain of a q-function to include non-integers. If this q-function is defined by a (super) q-exponential polynomial, then it is clear how to do this: simply evaluate this (super) q-exponential polynomial. On the other hand, when the q-function is given by a (multi-)summation, it is natural to define its value at non-integer points being zero. However, keep these two conventions may cause confusions especially when an asymptotic equality connecting a (multi-)summation and a (super) q-exponential polynomial is provided. Hence, we will abandon the second convention and use the following one instead.

Convention 10.6.2. Suppose S is a q-function such that it can be expressed as a (super)

q-exponential polynomial f. When we write $\mathrm{S}\left(\frac{1}{2}n\right)$, where $n \in \mathbb{Z}$, we actually mean the evaluation of $f(z)$ at $z = \frac{1}{2}n$. Note that this may cause S having nonzero value at a half-integer even though S may be given by a (multi-)summation.

Then by Lemma 10.4.5, the q-function that gives $\mathrm{S}\left(\frac{1}{2}z\right)$ when z is even and 0 when z is odd is the following one:

$$\tfrac{1}{2}(1 + (-1)^z)\,\mathrm{S}\left(\tfrac{1}{2}z\right).$$

Proof of Lemma 10.6.1. By introducing new variables $\mathbf{s} \in \mathbb{F}_2^{\mathfrak{i}}$ and replacing $\mathbf{c}_1$ by $2\mathbf{c}_1 - \mathbf{s}$, we can write the q-function S as follows:

$$\mathrm{S}(z) = \sum_{\substack{\mathbf{c}\in\mathbb{Z}_{>0}^{\mathfrak{i}},\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}_1}\\ 2(\mathbf{1}\cdot\mathbf{c})=z+\mathbf{1}\cdot\mathbf{s}}} q^{-\boldsymbol{\mu}_1\cdot\mathbf{s}+(2\boldsymbol{\mu}_1\sqcup\boldsymbol{\mu}_2)\cdot\mathbf{c}}.$$

Consider the following q-function:

$$\mathrm{S}'(z) := \sum_{\mathbf{c}\in\mathbb{Z}_{>0}^{\mathfrak{i}}\,:\,\mathbf{1}\cdot\mathbf{c}=z} q^{\boldsymbol{\mu}'\cdot\mathbf{c}},$$

where $\boldsymbol{\mu}'$ is the sequence $2\boldsymbol{\mu}_1 \sqcup \boldsymbol{\mu}_2$. Then we have

$$\mathrm{S}(z) = \sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}_1}\,:\,z+\mathbf{1}\cdot\mathbf{s}\in2\mathbb{Z}} q^{-\boldsymbol{\mu}_1\cdot\mathbf{s}}\,\mathrm{S}'\left(\tfrac{1}{2}(z+\mathbf{1}\cdot\mathbf{s})\right).$$

Note that the summation only takes over those sequence $\mathbf{s} \in \mathbb{F}_2^{\mathfrak{i}_1}$ satisfying $z + \mathbf{1}\cdot\mathbf{s} \in 2\mathbb{Z}$. Hence, following Convention 10.6.2, we have

$$(*) \qquad \mathrm{S}(z) = \sum_{\mathbf{s}\in\mathbb{F}_2^{\mathfrak{i}_1}} \tfrac{1}{2}\left(1 + (-1)^{z+\mathbf{1}\cdot\mathbf{s}}\right) q^{-\boldsymbol{\mu}_1\cdot\mathbf{s}}\,\mathrm{S}'\left(\tfrac{1}{2}(z+\mathbf{1}\cdot\mathbf{s})\right).$$

By Lemma 10.5.2, the q-function S′ can be expressed as a q-exponential polynomial

f which is primary when $\boldsymbol{\mu}'$ only contains integers. Moreover, we have

$$\mathrm{S}'(z) \sim \prod_{i \notin \mathfrak{i}'_{\max}} \left(q^{\mu'_{\max}-\mu'_i}-1\right)^{-1} \cdot \binom{z}{|\mathfrak{i}'_{\max}|-1} q^{\mu'_{\max} z},$$

where $\mu'_{\max}$ is the maximum of the sequence $\boldsymbol{\mu}'$ and $\mathfrak{i}'_{\max}$ is the set of indices $i \in \mathfrak{i}$ such that μ'_i achieves this maximum. Applying this to Eq. $(*)$ and noticing that

$$\binom{\frac{1}{2}(z+\mathbf{1}\cdot\mathbf{s})}{|\mathfrak{i}'_{\max}|-1} q^{\mu'_{\max}\frac{1}{2}(z+\mathbf{1}\cdot\mathbf{s})} \sim \left(\frac{1}{2}\right)^{|\mathfrak{i}'_{\max}|-1} q^{\frac{1}{2}\mu'_{\max}\mathbf{1}\cdot\mathbf{s}} \binom{z}{|\mathfrak{i}'_{\max}|-1} q^{\frac{1}{2}\mu'_{\max} z},$$

we obtain the following asymptotic equality:

$$\begin{aligned} \mathrm{S}(z) \sim \frac{1}{2^{|\mathfrak{i}'_{\max}|}} & \prod_{i \notin \mathfrak{i}'_{\max}} \left(q^{\mu'_{\max}-\mu'_i}-1\right)^{-1} \\ & \cdot \sum_{\mathbf{s} \in \mathbb{F}_2^{\mathfrak{i}_1}} \left(1+(-1)^{z+\mathbf{1}\cdot\mathbf{s}}\right) q^{(\frac{1}{2}\mu'_{\max}-\boldsymbol{\mu}_1)\cdot\mathbf{s}} \cdot \binom{z}{|\mathfrak{i}'_{\max}|-1} q^{\frac{1}{2}\mu'_{\max} z}. \end{aligned}$$

In the sequence $\boldsymbol{\mu}'$, the maximum $\mu'_{\max}$ is $\max\{2\mu_{1\max}, \mu_{2\max}\}$, and we have

$$\mathfrak{i}'_{\max} = \begin{cases} \mathfrak{i}_{1\max} & \text{if } 2\mu_{1\max} > \mu_{2\max}, \\ \mathfrak{i}_{2\max} & \text{if } 2\mu_{1\max} < \mu_{2\max}, \\ \mathfrak{i}_{1\max} \sqcup \mathfrak{i}_{2\max} & \text{if } 2\mu_{1\max} = \mu_{2\max}. \end{cases}$$

In the first and the third case, the sequence $\frac{1}{2}\mu'_{\max} - \boldsymbol{\mu}_1$ contains a zero. Hence, we have

$$\sum_{\mathbf{s} \in \mathbb{F}_2^{\mathfrak{i}_1}} (-1)^{z+\mathbf{1}\cdot\mathbf{s}} q^{(\frac{1}{2}\mu'_{\max}-\boldsymbol{\mu}_1)\cdot\mathbf{s}} = 0.$$

Then the asymptotic relations in the lemma follows. Note that the proof of Lemma 10.5.2 also shows that $f(\frac{1}{2}z)$ is a primary q-exponential polynomial if $\boldsymbol{\mu}'$ only contains even integers. Then the last statement follows. □

The following lemma will not be used in this dissertation. It can be deduced from Lemma 10.5.5 similarly to Lemma 10.6.1

Lemma 10.6.3. *Let* S *be the q-function defined by the following multi-summation*

$$\mathrm{S}(z) = \sum_{\mathbf{c}\in\mathbb{Z}_{>0}^{\mathfrak{i}}:\ \mathbf{1}\cdot\mathbf{c}_1+2(\mathbf{1}\cdot\mathbf{c}_2)=z} q^{\boldsymbol{\mu}\cdot\mathbf{c}+e(\mathbf{c})},$$

where $\boldsymbol{\mu}$ *is a sequence of non-negative rational numbers and* e *is a multivariable parity function. For* $\square = 1, 2$*, define the following notations:*

- $\mu_{\square\max}$ *is the maximum of* $\boldsymbol{\mu}_\square$*;*
- $\mathfrak{i}_{\square\max}$ *is the set of indices* $i \in \mathfrak{i}_\square$ *such that* $\mu_i = \mu_{\square\max}$.

Then S *can be expressed as a super q-exponential polynomial.*

(i). *If* $2\mu_{1\max} > \mu_{2\max}$*, then we have*

$$\mathrm{S}(z) \sim \left(C_{\boldsymbol{\mu},e,0} + C_{\boldsymbol{\mu},e,1}(-1)^z\right) \cdot \binom{z}{|\mathfrak{i}_{1\max}| - 1} q^{\mu_{1\max} z},$$

where the constants $C_{\boldsymbol{\mu},e,0}$ *and* $C_{\boldsymbol{\mu},e,1}$ *are defined as follows:*

$$\begin{aligned}
C_{\boldsymbol{\mu},e,0} = C_{\boldsymbol{\mu}} \cdot &\sum_{\mathbf{s}_1\in\mathbb{F}_2^{\mathfrak{i}_1\setminus\mathfrak{i}_{1\max}}} q^{2(\mu_{1\max}-\boldsymbol{\mu}|_{\mathfrak{i}_1\setminus\mathfrak{i}_{1\max}})\cdot\mathbf{s}_1} \\
&\cdot \sum_{\mathbf{s}_0\in\mathbb{F}_2^{\mathfrak{i}_1},\mathbf{s}_2\in\mathbb{F}_2^{\mathfrak{i}_2}} q^{e(\mathbf{s}_0\sqcup\mathbf{s}_2)+(\mu_{1\max}-\boldsymbol{\mu}_1)\cdot\mathbf{s}_0+(2\mu_{1\max}-\boldsymbol{\mu}_2)\cdot\mathbf{s}_2}, \\
C_{\boldsymbol{\mu},e,1} = C_{\boldsymbol{\mu}} \cdot &\sum_{\mathbf{s}_1\in\mathbb{F}_2^{\mathfrak{i}_1\setminus\mathfrak{i}_{1\max}}} q^{2(\mu_{1\max}-\boldsymbol{\mu}|_{\mathfrak{i}_1\setminus\mathfrak{i}_{1\max}})\cdot\mathbf{s}_1} \\
&\cdot \sum_{\mathbf{s}_0\in\mathbb{F}_2^{\mathfrak{i}_1},\mathbf{s}_2\in\mathbb{F}_2^{\mathfrak{i}_2}} (-1)^{\mathbf{1}\cdot\mathbf{s}_0} q^{e(\mathbf{s}_0\sqcup\mathbf{s}_2)+(\mu_{1\max}-\boldsymbol{\mu}_1)\cdot\mathbf{s}_0+(2\mu_{1\max}-\boldsymbol{\mu}_2)\cdot\mathbf{s}_2},
\end{aligned}$$

where $\mu_{1\,\max} - \boldsymbol{\mu}|_{\mathfrak{i}_1 \setminus \mathfrak{i}_{1\,\max}}$ *denotes the sequence* $(\mu_{1\,\max} - \mu_i)_{i \in \mathfrak{i}_1 \setminus \mathfrak{i}_{1\,\max}}$, $\mu_{1\,\max} - \boldsymbol{\mu}_1$ *the sequence* $(\mu_{1\,\max} - \mu_i)_{i \in \mathfrak{i}_1}$, $2\mu_{1\,\max} - \boldsymbol{\mu}_2$ *the sequence* $(2\mu_{1\,\max} - \mu_i)_{i \in \mathfrak{i}_2}$, *and* $C_{\boldsymbol{\mu}}$ *the following constant:*

$$C_{\boldsymbol{\mu}} := \frac{1}{2^{|\mathfrak{i}_{1\,\max}|}} \prod_{i \in \mathfrak{i}_1 \setminus \mathfrak{i}_{1\,\max}} \left(q^{4\mu_{1\,\max} - 4\mu_i} - 1\right)^{-1} \prod_{i \in \mathfrak{i}_2} \left(q^{4\mu_{1\,\max} - 2\mu_i} - 1\right)^{-1}.$$

(ii). *If* $2\mu_{1\,\max} < \mu_{2\,\max}$, *then we have*

$$\mathrm{S}(z) \sim \left(C_{\boldsymbol{\mu},e,0} + C_{\boldsymbol{\mu},e,1}(-1)^z\right) \cdot \binom{z}{|\mathfrak{i}_{2\,\max}| - 1} q^{\frac{1}{2}\mu_{2\,\max} z},$$

where the constants $C_{\boldsymbol{\mu},e,0}$ *and* $C_{\boldsymbol{\mu},e,1}$ *are defined as follows:*

$$C_{\boldsymbol{\mu},e,0} = C_{\boldsymbol{\mu}} \cdot \sum_{\mathbf{s}_0, \mathbf{s}_1 \in \mathbb{F}_2^{\mathfrak{i}_1}, \mathbf{s}_2 \in \mathbb{F}_2^{\mathfrak{i}_2}} \left(1 + (-1)^{\mathbf{1}\cdot\mathbf{s}_0 + \mathbf{1}\cdot\mathbf{s}_1 + \mathbf{1}\cdot\mathbf{s}_2}\right) \Big(q^{e(\mathbf{s}_0 \sqcup \mathbf{s}_2) + (\frac{1}{2}\mu_{2\,\max} - \boldsymbol{\mu}_1)\cdot\mathbf{s}_0 + 2(\frac{1}{2}\mu_{2\,\max} - \boldsymbol{\mu}_1)\cdot\mathbf{s}_1 + (\mu_{2\,\max} - \boldsymbol{\mu}_2)\cdot\mathbf{s}_2} \Big),$$

$$C_{\boldsymbol{\mu},e,1} = C_{\boldsymbol{\mu}} \cdot \sum_{\mathbf{s}_0, \mathbf{s}_1 \in \mathbb{F}_2^{\mathfrak{i}_1}, \mathbf{s}_2 \in \mathbb{F}_2^{\mathfrak{i}_2}} \left((-1)^{\mathbf{1}\cdot\mathbf{s}_0} + (-1)^{\mathbf{1}\cdot\mathbf{s}_1 + \mathbf{1}\cdot\mathbf{s}_2}\right) \Big(q^{e(\mathbf{s}_0 \sqcup \mathbf{s}_2) + (\frac{1}{2}\mu_{2\,\max} - \boldsymbol{\mu}_1)\cdot\mathbf{s}_0 + 2(\frac{1}{2}\mu_{2\,\max} - \boldsymbol{\mu}_1)\cdot\mathbf{s}_1 + (\mu_{2\,\max} - \boldsymbol{\mu}_2)\cdot\mathbf{s}_2} \Big),$$

where $\frac{1}{2}\mu_{2\,\max} - \boldsymbol{\mu}_1$ *denotes the sequence* $(\frac{1}{2}\mu_{2\,\max} - \mu_i)_{i \in \mathfrak{i}_1}$, $\mu_{2\,\max} - \boldsymbol{\mu}_2$ *the sequence* $(\mu_{2\,\max} - \mu_i)_{i \in \mathfrak{i}_2}$, *and* $C_{\boldsymbol{\mu}}$ *the following the constant:*

$$C_{\boldsymbol{\mu}} := \frac{1}{2^{2|\mathfrak{i}_{2\,\max}|}} \prod_{i \in \mathfrak{i}_1} \left(q^{2\mu_{2\,\max} - 4\mu_i} - 1\right)^{-1} \prod_{i \in \mathfrak{i}_2 \setminus \mathfrak{i}_{2\,\max}} \left(q^{2\mu_{2\,\max} - 2\mu_i} - 1\right)^{-1}.$$

(iii). *If* $2\mu_{1\,\max} = \mu_{2\,\max}$, *then we have*

$$\mathrm{S}(z) \sim \left(C_{\boldsymbol{\mu},e,0} + C_{\boldsymbol{\mu},e,1}(-1)^z\right) \cdot \binom{z}{|\mathfrak{i}_{1\,\max}| + |\mathfrak{i}_{2\,\max}| - 1} q^{\mu_{1\,\max} z},$$

where the constants $C_{\boldsymbol{\mu},e,0}$ and $C_{\boldsymbol{\mu},e,1}$ are defined as follows:

$$\begin{aligned} C_{\boldsymbol{\mu},e,0} = C_{\boldsymbol{\mu}} \cdot & \sum_{\mathbf{s}_1 \in \mathbb{F}_2^{\mathfrak{i}_1 \setminus \mathfrak{i}_{1\max}}} q^{2(\mu_{1\max} - \boldsymbol{\mu}|_{\mathfrak{i}_1 \setminus \mathfrak{i}_{1\max}}) \cdot \mathbf{s}_1} \\ & \cdot \sum_{\mathbf{s}_0 \in \mathbb{F}_2^{\mathfrak{i}_1}, \mathbf{s}_2 \in \mathbb{F}_2^{\mathfrak{i}_2}} q^{e(\mathbf{s}_0 \sqcup \mathbf{s}_2) + (\mu_{1\max} - \boldsymbol{\mu}_1) \cdot \mathbf{s}_0 + (\mu_{2\max} - \boldsymbol{\mu}_2) \cdot \mathbf{s}_2}, \\ C_{\boldsymbol{\mu},e,1} = C_{\boldsymbol{\mu}} \cdot & \sum_{\mathbf{s}_1 \in \mathbb{F}_2^{\mathfrak{i}_1 \setminus \mathfrak{i}_{1\max}}} q^{2(\mu_{1\max} - \boldsymbol{\mu}|_{\mathfrak{i}_1 \setminus \mathfrak{i}_{1\max}}) \cdot \mathbf{s}_1} \\ & \cdot \sum_{\mathbf{s}_0 \in \mathbb{F}_2^{\mathfrak{i}_1}, \mathbf{s}_2 \in \mathbb{F}_2^{\mathfrak{i}_2}} (-1)^{\mathbf{1} \cdot \mathbf{s}_0} q^{e(\mathbf{s}_0 \sqcup \mathbf{s}_2) + (\mu_{1\max} - \boldsymbol{\mu}_1) \cdot \mathbf{s}_0 + (2\mu_{1\max} - \boldsymbol{\mu}_2) \cdot \mathbf{s}_2}, \end{aligned}$$

where $\mu_{1\max} - \boldsymbol{\mu}|_{\mathfrak{i}_1 \setminus \mathfrak{i}_{1\max}}$ denotes the sequence $(\mu_{1\max} - \mu_i)_{i \in \mathfrak{i}_1 \setminus \mathfrak{i}_{1\max}}$, $\mu_{1\max} - \boldsymbol{\mu}_1$ the sequence $(\mu_{1\max} - \mu_i)_{i \in \mathfrak{i}_1}$, $\mu_{2\max} - \boldsymbol{\mu}_2$ the sequence $(\mu_{2\max} - \mu_i)_{i \in \mathfrak{i}_2}$, and $C_{\boldsymbol{\mu}}$ the following constant:

$$C_{\boldsymbol{\mu}} := \frac{1}{2^{|\mathfrak{i}_{1\max}| + 2|\mathfrak{i}_{2\max}|}} \prod_{i \in \mathfrak{i}_1 \setminus \mathfrak{i}_{1\max}} \left(q^{4\mu_{1\max} - 4\mu_i} - 1 \right)^{-1} \prod_{i \in \mathfrak{i}_2 \setminus \mathfrak{i}_{2\max}} \left(q^{2\mu_{2\max} - 2\mu_i} - 1 \right)^{-1}.$$

Moreover, if both $\boldsymbol{\mu}_1$ and $\frac{1}{2}\boldsymbol{\mu}_2$ take integral values, then the super q-exponential polynomial is primary.

Remark. Note that the even leading coefficient $C_{\boldsymbol{\mu},e,0}$ is positive and the odd leading coefficient $C_{\boldsymbol{\mu},e,1}$ satisfies $\left|C_{\boldsymbol{\mu},e,1}\right| < C_{\boldsymbol{\mu},e,0}$. Hence, the q-function S is eventually positive. Note that $C_{\boldsymbol{\mu},e,1}$ could be 0, in which case the asymptotic growth of $\mathrm{S}(z)$ along even integers and odd integers coincide.

Chapter 11.

Simplicial volume in buildings of A_n type

In this chapter, we will prove the A_n part of Theorems 1.3 and 1.5. More precisely, we will prove the following stronger theorem.

Theorem 11.1. *Let $\mathscr{B}$ be a Bruhat-Tits building of split classical type A_n over a local field K with residue cardinality q. Then the simplicial volume* $\mathrm{SV}(\,\cdot\,)$ *and the simplicial surface area* $\mathrm{SSA}(\,\cdot\,)$ *in it can be defined by primary q-exponential polynomials whose leading terms are of the form:*

$$\mathrm{SV}(r) \sim \tilde{C}(n) \cdot r^{\varepsilon(n)} q^{\pi(n) r}, \qquad \mathrm{SSA}(r) \sim C(n) \cdot r^{\varepsilon(n)} q^{\pi(n) r},$$

where $\varepsilon(n) = 0$ and $\pi(n) = (\frac{n+1}{2})^2$ if n is odd, while $\varepsilon(n) = 1$ and $\pi(n) = \frac{n}{2}(\frac{n}{2}+1)$ if n is even.

Moreover, we will obtain explicit formulas for the constants $\tilde{C}(n)$ and $C(n)$.

By the discussion at the beginning of Chapter 10, this can be done as follows. First, we will compute the asymptotic growth of $\mathrm{S}_{\mathcal{V}(I)}(r)$ for each type $I \subseteq \Delta$. This allows us to find the dominant ones. On the other hand, by Eq. (8.4.6), we have

$$\mathrm{SSA}(r) = \sum_{I \subseteq \Delta} \frac{\mathscr{P}_{A_n;I}(q)}{q^{\deg(\mathscr{P}_{A_n;I})}} \mathrm{S}_{\mathcal{V}(I)}(r). \tag{11.0.1}$$

Then we can obtain the asymptotic growths of $\mathrm{SSA}(r)$ and $\mathrm{SV}(r)$.

§ 11.1. Asymptotic growth of $\mathrm{S}_{\mathcal{V}(I)}(r)$

Now, let I be a type and follow Convention 2.4.5. We are going to compute the asymptotic growth of $\mathrm{S}_{\mathcal{V}(I)}(r)$. Since all vertices are special, we have

$$\mathrm{S}_{\mathcal{V}(I)}(r) = \mathrm{S}^{\asymp}_{\mathcal{V}(I)}(r) = \sum_{x \in \mathcal{V}(I,r)} q^{2\rho(x)}.$$

Then by Eqs. (6.3.8) and (9.2.2), we have

$$\mathrm{S}_{\mathcal{V}(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ c_1 + \cdots + c_t = r}} q^{\sum_{i=1}^{t} \ell_i (n+1-\ell_i) c_i}.$$

Now, we apply Lemma 10.5.2 to above summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \ell_i(n+1-\ell_i). \qquad (1 \leqslant i \leqslant t)$$

Since all members of $\boldsymbol{\mu}$ are integers, $\mathrm{S}_{\mathcal{V}(I)}$ can be defined by a primary q-exponential polynomial. Note that, by Eq. (11.0.1), this already implies that $\mathrm{SV}(\,\cdot\,)$ and $\mathrm{SSA}(\,\cdot\,)$ can be defined by primary q-exponential polynomials.

The knowledge of quadratic function shows that either $\mathfrak{i}_{\max}$ is a singleton $\{i_0\}$ or it consists of two consecutive indices $\{i_0, i_0+1\}$.

If $\mathfrak{i}_{\max} = \{i_0\}$. Then $\mu_{\max} = \ell_{i_0}(n+1-\ell_{i_0})$, $\mu_{\max} - \mu_i = (\ell_{i_0} - \ell_i)(n+1-\ell_{i_0}-\ell_i)$, and we have

$$S_{\mathcal{V}(I)}(r) \sim \prod_{i\neq i_0} \left(q^{(\ell_{i_0}-\ell_i)(n+1-\ell_{i_0}-\ell_i)} - 1\right)^{-1} \cdot q^{\ell_{i_0}(n+1-\ell_{i_0})r}. \tag{11.1.1}$$

In particular, $S_{\mathcal{V}(I)}$ has order $\ell_{i_0}(n+1-\ell_{i_0})$ and degree 0.

If $\mathfrak{i}_{\max} = \{i_0, i_0+1\}$. Then $\mu_{\max} = \ell_{i_0}(n+1-\ell_{i_0})$, $\mu_{\max} - \mu_i = (\ell_{i_0}-\ell_i)(n+1-\ell_{i_0}-\ell_i)$, and we have

$$S_{\mathcal{V}(I)}(r) \sim \prod_{i\neq i_0, i_0+1} \left(q^{(\ell_{i_0}-\ell_i)(n+1-\ell_{i_0}-\ell_i)} - 1\right)^{-1} \cdot rq^{\ell_{i_0}(n+1-\ell_{i_0})r}. \tag{11.1.2}$$

In particular, $S_{\mathcal{V}(I)}$ has order $\ell_{i_0}(n+1-\ell_{i_0})$ and degree 1.

§ 11.2. Dominant types

Now, we are able to figure out for which type I, the q-function $S_{\mathcal{V}(I)}$ is dominant among its siblings. We will say that such a type is a .

When n is odd, we have

$$\mu_{\max} = \ell_{i_0}(n+1-\ell_{i_0}) \leqslant (\frac{n+1}{2})^2.$$

The equality achieves exactly when $\ell_{i_0} = \frac{n+1}{2}$. Therefore, $S_{\mathcal{V}(I)}$ is dominant exactly when $\frac{n+1}{2} \notin I$. In this case, we have $\mathfrak{i}_{\max} = \{i_0\}$ and $\ell_{i_0} = \frac{n+1}{2}$.

When n is even, we have

$$\mu_{\max} = \ell_{i_0}(n+1-\ell_{i_0}) \leqslant \frac{n}{2}\left(\frac{n}{2}+1\right).$$

The equality achieves exactly when $\ell_{i_0} = \frac{n}{2}$ or $\frac{n}{2}+1$. Therefore, $S_{\mathcal{V}(I)}$ is dominant only if $\left\{\frac{n}{2}, \frac{n}{2}+1\right\} \not\subset I$. There are three cases: if $\left\{\frac{n}{2}, \frac{n}{2}+1\right\} \cap I = \left\{\frac{n}{2}+1\right\}$, we have

$\mathfrak{i}_{\max} = \{i_0\}$ and $\ell_{i_0} = \frac{n}{2}$; if $\left\{\frac{n}{2}, \frac{n}{2}+1\right\} \cap I = \left\{\frac{n}{2}\right\}$, we have $\mathfrak{i}_{\max} = \{i_0\}$ and $\ell_{i_0} = \frac{n}{2}+1$; if $\left\{\frac{n}{2}, \frac{n}{2}+1\right\} \cap I = \emptyset$, we have $\mathfrak{i}_{\max} = \{i_0, i_0+1\}$, $\ell_{i_0} = \frac{n}{2}$, and $\ell_{i_0+1} = \frac{n}{2}+1$. Among them, the last one gives the dominant type since $\mathrm{S}_{\mathcal{V}(I)}$ has degree 1 in that case while it has degree 0 in the first two cases.

§ 11.3. Asymptotic growths of $\mathrm{SSA}(r)$ and $\mathrm{SV}(r)$

We are now going to obtain the asymptotic growth of $\mathrm{SSA}(r)$. By Eq. (11.0.1), we have

$$\mathrm{SSA}(r) \sim \sum_{I \text{ is dominant}} \frac{\mathscr{P}_{A_n;I}(q)}{q^{\deg\left(\mathscr{P}_{A_n;I}\right)}} \mathrm{S}_{\mathcal{V}(I)}(r).$$

When n is odd, by § 11.2 and Eq. (11.1.1), we see that $\mathrm{SSA}(\,\cdot\,)$ can be defined by a primary q-exponential polynomial so that

$$\mathrm{SSA}(r) \sim C(n) \cdot q^{(\frac{n+1}{2})^2 r}, \tag{11.3.1}$$

where the constant $C(n)$ is defined as follows:

$$C(n) := \sum_{\substack{I \subseteq \Delta \\ \frac{n+1}{2} \notin I}} \frac{\mathscr{P}_{A_n;I}(q)}{q^{\deg\left(\mathscr{P}_{A_n;I}\right)}} \prod_{\substack{1 \leqslant i \leqslant t_I \\ \ell_i(I) \neq \frac{n+1}{2}}} \left(q^{\left(\ell_i(I) - \frac{n+1}{2}\right)^2} - 1\right)^{-1}. \tag{11.3.2}$$

As a consequence, we see that $\mathrm{SV}(\,\cdot\,)$ can be defined by a primary q-exponential polynomial so that

$$\mathrm{SV}(r) = \sum_{z=0}^{r} \mathrm{SSA}(z) \sim \frac{q^{(\frac{n+1}{2})^2}}{q^{(\frac{n+1}{2})^2} - 1} C(n) \cdot q^{(\frac{n+1}{2})^2 r}, \tag{11.3.3}$$

When n is even, by § 11.2 and Eq. (11.1.2), we see that $\mathrm{SSA}(\,\cdot\,)$ can be defined by a primary q-exponential polynomial so that

$$\mathrm{SSA}(r) \sim C(n) \cdot r q^{\frac{n}{2}(\frac{n}{2}+1)r}, \tag{11.3.4}$$

where the constant $C(n)$ is defined as follows:

$$C(n) := \sum_{\substack{I \subseteq \Delta \\ \frac{n}{2}, \frac{n}{2}+1 \notin I}} \frac{\mathscr{P}_{A_n;I}(q)}{q^{\deg(\mathscr{P}_{A_n;I})}} \prod_{\substack{1 \leqslant i \leqslant t_I \\ \ell_i(I) \neq \frac{n}{2}, \frac{n}{2}+1}} \left(q^{(\ell_i(I)-\frac{n}{2})(\ell_i(I)-\frac{n}{2}-1)} - 1\right)^{-1}. \tag{11.3.5}$$

As a consequence, we see that SV$(\,\cdot\,)$ can be defined by a primary q-exponential polynomial so that

$$\mathrm{SV}(r) = \sum_{z=0}^{r} \mathrm{SSA}(z) \sim \frac{q^{\frac{n}{2}(\frac{n}{2}+1)}}{q^{\frac{n}{2}(\frac{n}{2}+1)} - 1} C(n) \cdot q^{\frac{n}{2}(\frac{n}{2}+1)r}, \tag{11.3.6}$$

Since all vertices are special, we have $\mathrm{SSA}_{\dagger}(r) = \mathrm{SSA}(r)$ and $\mathrm{SV}_{\dagger}(r) = \mathrm{SV}(r)$, where $\dagger$ denotes "being special".

By Eqs. (11.3.1) to (11.3.6), we have proved Theorem 11.1. Moreover, by Eq. (A.1.4), we have the following explicit formulas for the first factor of $C(n)$:

$$\mathscr{P}_{A_n;I}(q) = \begin{bmatrix} n+1 \\ \ell_1(I), \ell_2(I)-\ell_1(I), \cdots, \ell_t(I)-\ell_{t-1}(I), n+1-\ell_t(I) \end{bmatrix}(q),$$

$$q^{\deg(\mathscr{P}_{A_n;I})} = \frac{q^{\binom{n+1}{2}}}{q^{\binom{\ell_1(I)}{2}} q^{\binom{\ell_2(I)-\ell_1(I)}{2}} \cdots q^{\binom{\ell_t(I)-\ell_{t-1}(I)}{2}} q^{\binom{n+1-\ell_t(I)}{2}}}.$$

See Eq. (A.1.3) for the definition of the symbol $\begin{bmatrix} \cdot \\ \cdot, \cdots, \cdot \end{bmatrix}$.

Chapter 12.

Simplicial volume in buildings of C_n type

In this chapter, we will prove the C_n part of Theorems 1.3 and 1.5. More precisely, we will prove the following stronger theorem.

Theorem 12.1. *Let $\mathscr{B}$ be a Bruhat-Tits building of split classical type C_n over a local field K with residue cardinality q. Then the simplicial volume* $\mathrm{SV}(\,\cdot\,)$ *and the simplicial surface area* $\mathrm{SSA}(\,\cdot\,)$ *in it can be defined by primary super q-exponential polynomials whose leading terms are of the form:*

$$\mathrm{SV}(r) \sim \tilde{C}(n) \cdot q^{\frac{n(n+1)}{2}r}, \qquad \mathrm{SSA}(r) \sim C(n) \cdot q^{\frac{n(n+1)}{2}r},$$

where $\tilde{C}(n)$ and $C(n)$ are primary q-numbers, not just parity q-functions.

Moreover, we will obtain explicit formulas for the constants $\tilde{C}(n)$ and $C(n)$.

By the discussion at the beginning of Chapter 10, this can be done as follows. First, we estimate the asymptotic growth of $\mathrm{S}_{\mathcal{V}(I)}(r)$ for each type $I \subseteq \Delta$ using the auxiliary function $\mathrm{S}^{\asymp}_{\mathcal{V}(I)}$ in § 12.1. This allows us to find the dominant ones in § 12.3. Then we can compute the leading coefficient of $\mathrm{S}_{\mathcal{V}(I)}(r)$ for dominant ones in § 12.5. Finally, by

Eq. (8.4.6), we have

$$\mathrm{SSA}(r) = \sum_{I\subseteq\Delta} \frac{\mathscr{P}_{C_n;I}(q)}{q^{\deg(\mathscr{P}_{C_n;I})}}\,\mathrm{S}_{\mathcal{V}(I)}(r) \sim \sum_{I\text{ is dominant}} \frac{\mathscr{P}_{C_n;I}(q)}{q^{\deg(\mathscr{P}_{C_n;I})}}\,\mathrm{S}_{\mathcal{V}(I)}(r). \tag{12.0.1}$$

Then we can obtain the asymptotic growths of $\mathrm{SSA}(r)$ and $\mathrm{SV}(r)$.

Along the discussion, we will also consider the asymptotic growths of $\mathrm{SSA}_\dagger(r)$ and $\mathrm{SV}_\dagger(r)$, where $\dagger$ denotes "being special". Namely, we will also prove the following theorem (in § 12.2 and 12.4).

Theorem 12.2. *Let $\mathscr{B}$ be a Bruhat-Tits building of split classical type C_n over a local field K with residue cardinality q. Then the special simplicial volume $\mathrm{SV}_\dagger(\,\cdot\,)$ and the special simplicial surface area $\mathrm{SSA}_\dagger(\,\cdot\,)$ in it can be defined by primary super q-exponential polynomials whose leading terms are of the form:*

$$\mathrm{SV}_\dagger(r) \sim \tilde{C}_\dagger(n)\cdot q^{\frac{n(n+1)}{2}r}, \qquad \mathrm{SSA}_\dagger(r) \sim C_\dagger(n)\cdot q^{\frac{n(n+1)}{2}r},$$

where $\tilde{C}_\dagger(n)$ and $C_\dagger(n)$ are primary q-numbers, not just parity q-functions.

Moreover, we will obtain explicit formulas for the constants $\tilde{C}_\dagger(n)$ and $C_\dagger(n)$.

§ 12.1. Asymptotic growth of $\mathrm{S}^{\asymp}_{\mathcal{V}(I)}(r)$

Now, let I be a type and follow Convention 2.4.5. We are going to estimate the asymptotic growth of $\mathrm{S}_{\mathcal{V}(I)}(r)$ up to the leading coefficient.

By Eqs. (6.4.7) and (9.3.2), we have

$$\mathrm{S}^{\asymp}_{\mathcal{V}(I)}(r) = \sum_{\substack{c_i\in\mathbb{Z}_{>0}\\ c_1+\cdots+c_t=r}} q^{\sum_{i=1}^{t}\frac{1}{2}\ell_i(2n+1-\ell_i)c_i}.$$

Now, we apply Lemma 10.5.2 to above summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \tfrac{1}{2}\ell_i(2n+1-\ell_i). \qquad (1 \leqslant i \leqslant t)$$

Since all members of $\boldsymbol{\mu}$ are integers, $\mathrm{S}^{\asymp}_{\mathcal{V}(I)}$ can be defined by a primary q-exponential polynomial. The knowledge of quadratic function shows that $\mathfrak{i}_{\max} = \{t\}$ with $\mu_{\max} = \frac{1}{2}\ell_t(2n+1-\ell_t)$. Then we have

$$\mathrm{S}^{\asymp}_{\mathcal{V}(I)}(r) \sim \prod_{i=1}^{t-1}\left(q^{\frac{1}{2}(\ell_t-\ell_i)(2n+1-\ell_t-\ell_i)} - 1\right)^{-1} \cdot q^{\frac{1}{2}\ell_t(2n+1-\ell_t)r}.$$

Since $\mathrm{S}_{\mathcal{V}(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{V}(I)}(r)$, we see that $\mathrm{S}_{\mathcal{V}(I)}$ has order $\frac{1}{2}\ell_t(2n+1-\ell_t)$ and degree 0.

§ 12.2. Asymptotic growth of $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r)$

Next, we are going to compute the asymptotic growth of $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r)$.

If $\ell_t < n$, then by Eqs. (6.4.6), (6.4.7), and (9.1.4), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ 2c_1+\cdots+2c_t=r}} q^{\sum_{i=1}^{t}\ell_i(2n+1-\ell_i)c_i}.$$

Now, we apply Lemma 10.6.1 to this summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$, the partition $\mathfrak{i} = \mathfrak{i}_1 \sqcup \mathfrak{i}_2$ is $\{1, \cdots, t\} = \emptyset \sqcup \{1, \cdots, t\}$, and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \ell_i(2n+1-\ell_i). \qquad (1 \leqslant i \leqslant t)$$

Since all members of $\boldsymbol{\mu}$ are even integers, $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}$ can be defined by a primary super q-exponential polynomial. The knowledge of quadratic function shows that $\mathfrak{i}_{\max} =$

$\mathfrak{i}_{2\max} = \{t\}$ with $\mu_{\max} = \mu_{2\max} = \ell_t(2n+1-\ell_t)$. Then by Lemma 10.6.1.(ii), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) \sim \prod_{i=1}^{t-1} \left(q^{(\ell_t-\ell_i)(2n+1-\ell_t-\ell_i)} - 1\right)^{-1} \cdot \tfrac{1}{2}\left(1+(-1)^r\right) \cdot q^{\frac{1}{2}\ell_t(2n+1-\ell_t)r}.$$

In particular, it has order $\frac{1}{2}\ell_t(2n+1-\ell_t)$ and degree 0.

If $\ell_t = n$, then by Eqs. (6.4.6), (6.4.7), and (9.1.4), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ 2c_1+\cdots+2c_{t-1}+c_t = r}} q^{\sum_{i=1}^{t-1} \ell_i(2n+1-\ell_i)c_i + \frac{n(n+1)}{2}c_t}.$$

Now, we apply Lemma 10.6.1 to this summation, where the index set $\mathfrak{i}$ is $\{1,\cdots,t\}$, the partition $\mathfrak{i} = \mathfrak{i}_1 \sqcup \mathfrak{i}_2$ is $\{1,\cdots,t\} = \{t\} \sqcup \{1,\cdots,t-1\}$, and the sequence $\boldsymbol{\mu}$ is

$$\begin{aligned} \mu_i &= \ell_i(2n+1-\ell_i), && (1 \leqslant i < t) \\ \mu_t &= \tfrac{n(n+1)}{2}. \end{aligned}$$

Since μ_t is an integer and all members of $\boldsymbol{\mu}_2$ are even integers, $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}$ can be defined by a primary super q-exponential polynomial. The knowledge of quadratic function shows that $\mathfrak{i}_{2\max} = \{t-1\}$, $\mu_{2\max} = \ell_{t-1}(2n+1-\ell_{t-1})$, and $2\mu_{1\max} > \mu_{2\max}$. Then by Lemma 10.6.1.(i), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) \sim \prod_{i=1}^{t-1} \left(q^{(n-\ell_i)(n+1-\ell_i)} - 1\right)^{-1} \cdot q^{\frac{n(n+1)}{2}r}. \tag{12.2.1}$$

In particular, it has order $\frac{n(n+1)}{2}$ and degree 0.

§ 12.3. Dominant types

Now, we are able to figure out which type is *dominant*.

We begin with $\mathrm{S}_{\mathcal{V}(I)}$, since $\ell_t \leqslant n$, we have

$$\mu_{\max} = \tfrac{1}{2}\ell_t(2n+1-\ell_t) \leqslant \tfrac{n(n+1)}{2}.$$

The equality achieves exactly when $\ell_t = n$. Therefore, $\mathrm{S}_{\mathcal{V}(I)}$ is dominant exactly when $n \notin I$. In this case, we have $\mathfrak{i}_{\max} = \{t\}$ and $\ell_t = n$.

Next, we consider $\mathrm{S}_{\mathcal{V}_\dagger(I)}$. Then similar argument shows that it is dominant exactly when $n \notin I$.

§ 12.4. Asymptotic growths of $\mathrm{SSA}_\dagger(r)$ and $\mathrm{SV}_\dagger(r)$

We are now able to obtain the asymptotic growths of $\mathrm{SSA}_\dagger(r)$ and $\mathrm{SV}_\dagger(r)$.

By Eq. (12.0.1), Eq. (12.2.1), and § 12.3, we see that $\mathrm{SSA}_\dagger(\,\cdot\,)$ can be defined by a primary super q-exponential polynomial so that

$$\mathrm{SSA}_\dagger(r) \sim C_\dagger(n) \cdot q^{\frac{n(n+1)}{2}r}, \tag{12.4.1}$$

where the constant $C_\dagger(n)$ is defined as follows:

$$C_\dagger(n) := \sum_{n\notin I} \frac{\mathscr{P}_{C_n;I}(q)}{q^{\deg(\mathscr{P}_{C_n;I})}} \prod_{i=1}^{t_I-1} \left(q^{(n-\ell_i(I))(n+1-\ell_i(I))} - 1\right)^{-1}. \tag{12.4.2}$$

As a consequence, we see that $\mathrm{SV}_\dagger(\,\cdot\,)$ can be defined by a primary super q-exponential polynomial so that

$$\mathrm{SV}_\dagger(r) = \sum_{z=0}^{r} \mathrm{SSA}_\dagger(z) \sim \frac{q^{\frac{n(n+1)}{2}}}{q^{\frac{n(n+1)}{2}} - 1} C_\dagger(n) \cdot q^{\frac{n(n+1)}{2}r}. \tag{12.4.3}$$

§ 12.5. Asymptotic growth of dominant $\mathrm{S}_{\mathcal{V}(I)}(r)$

Let I be a type and follow Convention 2.4.5. We are going to compute the asymptotic growth of $\mathrm{S}_{\mathcal{V}(I)}(r)$ when I is dominant.

First, we need to write $\mathrm{S}_{\mathcal{V}(I)}(r)$ into a multi-summation. To do this, we pick an arbitrary $x \in \mathcal{V}(I)$ and investigate the difference between $2\rho(x)$ and the sum of $\lceil a(x) \rceil$ for $a(x) > 0$. To better describe this sum, we introduce the following conventions.

Convention 12.5.1. For any $j \in \{1, \cdots, n\}$, we will use $\ell^{-1}(j)$ to denote the index $i \in \{1, \cdots, t+1\}$ such that $\ell_{i-1} < j \leqslant \ell_i$, where $\ell_{t+1} = n$.

Convention 12.5.2. We will use $\overline{z}$ to denote *standard parity function* mapping even numbers to 0 and odd numbers to 1. Note that $\lceil \frac{1}{2} z \rceil = \frac{1}{2}(z + \overline{z})$.

Convention 12.5.3. The summation $c_i + \cdots + c_j$ is read to be 0 when $i > j$.

Now, suppose $x = o + c_1\omega'_{\ell_1} + \cdots + c_t\omega'_{\ell_t} \in \mathcal{V}(I)$. By Eq. (6.4.5), we have

$$(\chi_j - \chi_{j'})(x) = \frac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}\right), \qquad (1 \leqslant j < j' \leqslant n)$$

$$(\chi_j + \chi_{j'})(x) = \frac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}\right) + c_{\ell^{-1}(j')} + \cdots + c_t, \qquad (1 \leqslant j < j' \leqslant n)$$

$$2\chi_j(x) = c_{\ell^{-1}(j)} + \cdots + c_t. \qquad (1 \leqslant j \leqslant n)$$

Therefore, we have

$$\begin{aligned}\sum_{a \in \Phi^+} \lceil a(x) \rceil &= \sum_{1 \leqslant j < j' \leqslant n} \left(\lceil (\chi_j - \chi_{j'})(x) \rceil + \lceil (\chi_j + \chi_{j'})(x) \rceil\right) + \sum_{j=1}^{n} \lceil 2\chi_j(x) \rceil \\ &= 2\rho(x) + \sum_{1 \leqslant j < j' \leqslant n} \overline{c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}}.\end{aligned}$$

From above analysis, we can define the parity function e_I as follows:

$$e_I(c_1, \cdots, c_t) := \sum_{1 \leqslant i < i' \leqslant t+1} (\ell_i - \ell_{i-1})(\ell_{i'} - \ell_{i'-1})\overline{c_i + \cdots + c_{i'-1}}. \tag{12.5.1}$$

Then we have

$$\sum_{a \in \Phi^+} \lceil a(x) \rceil = 2\rho(x) + e_I(c_1, \cdots, c_t).$$

Now, we apply Lemma 10.5.5 to the following summation.

$$\mathrm{S}_{\mathcal{V}(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ c_1 + \cdots + c_t = r}} q^{\sum_{i=1}^{t} \frac{1}{2}\ell_i(2n+1-\ell_i)c_i + e_I(c_1, \cdots, c_t)}.$$

Note that the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \tfrac{1}{2}\ell_i(2n + 1 - \ell_i). \qquad (1 \leqslant i \leqslant t)$$

Since all members of $\boldsymbol{\mu}$ are integers and e_I is valued in integers, $\mathrm{S}_{\mathcal{V}(I)}$ can be defined by a primary super q-exponential polynomial. Note that, by Eq. (12.0.1), this already implies that $\mathrm{SV}(\,\cdot\,)$ and $\mathrm{SSA}(\,\cdot\,)$ can be defined by primary super q-exponential polynomials.

Now, let I be a dominant type, namely $n \notin I$. Then we have $\mathfrak{i}_{\max} = \{t\}$, $\ell_t = n$, and $\mu_{\max} = \frac{n(n+1)}{2}$. Therefore,

$$\mathrm{S}_{\mathcal{V}(I)}(r) \sim \left(C_{I,0} + C_{I,1}(-1)^r\right) \cdot q^{\frac{n(n+1)}{2} r},$$

where the constants $C_{I,0}$ and $C_{I,1}$ are defined as follows:

$$C_{I,0} := C_I \cdot \sum_{\mathbf{s} \in \mathbb{F}_2^t} \mathrm{E}_I(\mathbf{s}), \qquad C_{I,1} := C_I \cdot \sum_{\mathbf{s} \in \mathbb{F}_2^t} (-1)^{\mathbf{1} \cdot \mathbf{s}}\, \mathrm{E}_I(\mathbf{s}),$$

where the constant C_I and the function $\mathrm{E}_I \colon \mathbb{F}_2^t \to \mathbb{Q}(q;-)$ are defined as follows:

$$C_I := \frac{1}{2}\prod_{i=1}^{t-1}\left(q^{(n-\ell_i)(n+1-\ell_i)} - 1\right)^{-1}, \qquad \mathrm{E}_I(\mathbf{s}) := q^{e_I(\mathbf{s})+\sum_{i=1}^{t-1}\frac{1}{2}(n-\ell_i)(n+1-\ell_i)s_i}. \tag{12.5.2}$$

From the definition Eq. (12.5.1) of e_I, it is clear that

$$\mathrm{E}_I(s_1,\cdots,s_{t-1},0) = \mathrm{E}_I(s_1,\cdots,s_{t-1},1).$$

Therefore, $C_{I,1} = 0$ and we thus have

$$\mathrm{S}_{\mathcal{V}(I)}(r) \sim C_I \cdot \sum_{\mathbf{s}\in\mathbb{F}_2^{t-1}} 2\,\mathrm{E}_I(\mathbf{s}\sqcup 0)\cdot q^{\frac{n(n+1)}{2}r}, \tag{12.5.3}$$

where $\mathbf{s}\sqcup 0$ is the sequence $s_1,\cdots,s_{t-1},0$.

§ 12.6. Asymptotic growths of SSA(r) and SV(r)

We are now going to obtain the asymptotic growth of SSA(r).

By Eq. (12.0.1), § 12.3, and Eqs. (12.5.1) to (12.5.3), we see that SSA$(\,\cdot\,)$ can be defined by a primary super q-exponential polynomial so that

$$\mathrm{SSA}(r) \sim C(n)\cdot q^{\frac{n(n+1)}{2}r}, \tag{12.6.1}$$

where the constant $C(n)$ is defined as follows:

$$C(n) := \sum_{n\notin I}\left(\frac{\mathscr{P}_{C_n;I}(q)}{q^{\deg\left(\mathscr{P}_{C_n;I}\right)}}\prod_{i=1}^{t_I-1}\left(q^{(n-\ell_i(I))(n+1-\ell_i(I))} - 1\right)^{-1}\sum_{\mathbf{s}\in\mathbb{F}_2^{t_I-1}}\mathrm{E}_{C_n;I}(\mathbf{s})\right), \tag{12.6.2}$$

where the function $\mathrm{E}_{C_n;I}\colon \mathbb{F}_2^{t_I-1} \longrightarrow \mathbb{Q}(q;-)$ is defined as follows:

$$\mathrm{E}_{C_n;I} := q^{\sum\limits_{1\leqslant i<i'\leqslant t_I}(\ell_i(I)-\ell_{i-1}(I))(\ell_{i'}(I)-\ell_{i'-1}(I))\overline{s_i+\cdots+s_{i'-1}}} \cdot q^{\sum\limits_{i=1}^{t_I-1}\frac{1}{2}(n-\ell_i(I))(n+1-\ell_i(I))s_i}. \tag{12.6.3}$$

As a consequence, we see that $\mathrm{SV}(\,\cdot\,)$ can be defined by a primary super q-exponential polynomial so that

$$\mathrm{SV}(r) = \sum_{z=0}^{r}\mathrm{SSA}(z) \sim \frac{q^{\frac{n(n+1)}{2}}}{q^{\frac{n(n+1)}{2}}-1}C(n)\cdot q^{\frac{n(n+1)}{2}r}. \tag{12.6.4}$$

By Eqs. (12.6.1) to (12.6.4), we have proved Theorem 12.1. Moreover, by Eq. (A.2.2), we have the following explicit formulas for the first factor of $C(n)$:

$$\mathscr{P}_{C_n;I}(q) = \frac{[2n]!!(q)}{\prod\limits_{i=1}^{t}[\ell_i(I)-\ell_{i-1}(I)]!(q)}, \qquad q^{\deg(\mathscr{P}_{C_n;I})} = \frac{q^{n^2}}{\prod\limits_{i=1}^{t}q^{\binom{\ell_i(I)-\ell_{i-1}(I)}{2}}}.$$

See Eqs. (A.1.2) and (A.2.1) for the definitions of the symbols $[\,\cdot\,]!$ and $[2\cdot\,]!!$.

Chapter 13.

Simplicial volume in buildings of B_n type

In this chapter, we will prove the B_n part of Theorems 1.3 and 1.5. More precisely, we will prove the following stronger theorem.

Theorem 13.1. *Let $\mathscr{B}$ be a Bruhat-Tits building of split classical type B_n over a local field K with residue cardinality q. Then the simplicial volume* $\mathrm{SV}(\,\cdot\,)$ *and the simplicial surface area* $\mathrm{SSA}(\,\cdot\,)$ *in it can be defined by super q-exponential polynomials whose leading terms are of the form:*[*]

$$\mathrm{SV}(r) \sim \tilde{C}(n) \cdot q^{\pi(n)r}, \qquad\qquad \mathrm{SSA}(r) \sim C(n) \cdot q^{\pi(n)r},$$

where $\pi(n) = \frac{n^2}{2}$ when $n \geqslant 4$ and $\pi(3) = 5$. The leading coefficients $\tilde{C}(3)$ and $C(3)$ are primary q-numbers, not just parity q-functions. Moreover, the four q-functions $\mathrm{SV}(2\,\cdot\,)$, $\mathrm{SV}(2\,\cdot\,+1)$, $\mathrm{SSA}(2\,\cdot\,)$, *and* $\mathrm{SSA}(2\,\cdot\,+1)$ *can be defined by primary q-exponential*

[*]The leading terms may give an impression that these q-functions can be defined by primary q-exponential polynomials when $n = 3$. However, we will see this is false in § 13.1.

polynomials whose leading terms are of the form:

$$\mathrm{SV}(2r) \sim \tilde{C}_0(n) \cdot q^{2\pi(n)r}, \qquad \mathrm{SSA}(2r) \sim C_0(n) \cdot q^{2\pi(n)r},$$
$$\mathrm{SV}(2r+1) \sim \tilde{C}_1(n) \cdot q^{2\pi(n)r}, \qquad \mathrm{SSA}(2r+1) \sim C_1(n) \cdot q^{2\pi(n)r}.$$

We will obtain explicit formulas for the parity functions $\tilde{C}(n)$ and $C(n)$, and the constants $\tilde{C}_0(n)$, $C_0(n)$, $\tilde{C}_1(n)$, and $C_1(n)$.

But before proving Theorem 13.1, we will first analyze the asymptotic growths of $\mathrm{SSA}_\dagger(r)$ and $\mathrm{SV}_\dagger(r)$, where † denotes "being special". We will prove the following.

Theorem 13.2. *Let $\mathscr{B}$ be a Bruhat-Tits building of split classical type B_n over a local field K with residue cardinality q. Then the special simplicial volume $\mathrm{SV}_\dagger(\,\cdot\,)$ and the special simplicial surface area $\mathrm{SSA}_\dagger(\,\cdot\,)$ in it can be defined by super q-exponential polynomials whose leading terms are of the form:**

$$\mathrm{SV}_\dagger(r) \sim \tilde{C}_\dagger(n) \cdot q^{\pi(n)r}, \qquad \mathrm{SSA}_\dagger(r) \sim C_\dagger(n) \cdot q^{\pi(n)r},$$

where $\pi(n) = \frac{n^2}{2}$ when $n \geqslant 4$ and $\pi(3) = 5$. The leading coefficients $\tilde{C}_\dagger(3)$ and $C_\dagger(3)$ are primary q-numbers, not just parity q-functions. Moreover, the four q-functions $\mathrm{SV}_\dagger(2\,\cdot\,)$, $\mathrm{SV}_\dagger(2\,\cdot\,+1)$, $\mathrm{SSA}_\dagger(2\,\cdot\,)$, and $\mathrm{SSA}_\dagger(2\,\cdot\,+1)$ can be defined by primary q-exponential polynomials whose leading terms are of the form:

$$\mathrm{SV}_\dagger(2r) \sim \tilde{C}_{\dagger 0}(n) \cdot q^{2\pi(n)r}, \qquad \mathrm{SSA}_\dagger(2r) \sim C_{\dagger 0}(n) \cdot q^{2\pi(n)r},$$
$$\mathrm{SV}_\dagger(2r+1) \sim \tilde{C}_{\dagger 1}(n) \cdot q^{2\pi(n)r}, \qquad \mathrm{SSA}_\dagger(2r+1) \sim C_{\dagger 1}(n) \cdot q^{2\pi(n)r}.$$

We will also give explicit formulas for the parity functions $\tilde{C}_\dagger(n)$ and $C_\dagger(n)$, and the constants $\tilde{C}_{\dagger 0}(n)$, $C_{\dagger 0}(n)$, $\tilde{C}_{\dagger 1}(n)$, and $C_{\dagger 1}(n)$. The proof of Theorem 13.2 turns out to play an essential role in the study of $\mathrm{SSA}(r)$ and $\mathrm{SV}(r)$.

This chapter is structured as follows. In § 13.1, we will compute the asymptotic growth of $\mathrm{S}_{\mathcal{V}_\dagger(I)}(r)$ for each type $I \subseteq \Delta$. This allows use to find the dominant ones of $\mathrm{S}_{\mathcal{V}_\dagger(I)}(r)$, which will be done in § 13.2. Then in § 13.3, we will obtain the asymptotic growths of $\mathrm{SSA}_\dagger(r)$ and $\mathrm{SV}_\dagger(r)$. In § 13.4, we will analyze $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2r)$ and $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2r+1)$. Combine § 13.3 and 13.4, we finish proving Theorem 13.2. After that, in § 13.5, we will estimate the asymptotic growths of $\mathrm{S}_{\mathcal{X}^0(I)}(r)$ and $\mathrm{S}_{\mathcal{X}^1(I)}(r)$ using the auxiliary functions $\mathrm{S}^{\asymp}_{\mathcal{X}^0(I)}$ and $\mathrm{S}^{\asymp}_{\mathcal{X}^1(I)}$. Note that $\mathcal{V}$ is between $\mathcal{V}_\dagger$ and $\mathcal{X}^0 \cup \mathcal{X}^1$. Therefore, we can combine § 13.1 and § 13.5 to estimate the asymptotic growth of each $\mathrm{S}_{\mathcal{V}(I)}(r)$ and find the dominant ones, which will be done in § 13.6. Once we found the dominant types, we can proceed to compute the asymptotic growth of dominant $\mathrm{S}_{\mathcal{V}(I)}(r)$. This will be done in three steps: in § 13.7, we will compute the asymptotic growths of $\mathrm{S}_{\mathcal{X}^0(I)}(r)$ and $\mathrm{S}_{\mathcal{X}^1(I)}(r)$; in § 13.8, we will deduce the asymptotic growth of $\mathrm{S}_{\mathcal{X}_J(I)}(r)$ from that of $\mathrm{S}_{\mathcal{V}_\dagger(I)}(r)$; finally in § 13.9, the asymptotic growth of $\mathrm{S}_{\mathcal{V}(I)}(r)$ will be deduced from them. Then in § 13.10, we will obtain the asymptotic growths of $\mathrm{SSA}(r)$ and $\mathrm{SV}(r)$. In § 13.11, we will analyze $\mathrm{S}_{\mathcal{X}_J(I)}(2r)$ and $\mathrm{S}_{\mathcal{X}_J(I)}(2r+1)$. Combine § 13.10 and 13.11, we finish proving Theorem 13.1.

Throughout this chapter, we will heavily use the various index sets $\mathcal{V}$, $\mathcal{V}_\dagger$, $\mathcal{X}^0$, $\mathcal{X}^1$, and $\mathcal{X}_J$. We refer to Figs. 9.2 and 9.3 for the structure of them.

§ 13.1. Asymptotic growth of $\mathrm{S}_{\mathcal{V}_\dagger(I)}(r)$

Now, let I be a type and follow Convention 2.4.5. We are going to compute the asymptotic growth of $\mathrm{S}_{\mathcal{V}_\dagger(I)}(r)$. We will separate the discussion into two cases: (i) $\ell_1 > 1$ and (ii) $\ell_1 = 1$.

(i). If $\ell_1 > 1$, then by Eqs. (6.5.7), (6.5.8), and (9.1.4), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ 2c_1+\cdots+2c_t=r}} q^{\sum_{i=1}^{t} \ell_i(2n-\ell_i)c_i}.$$

Now, we apply Lemma 10.6.1 to this summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$, the partition $\mathfrak{i} = \mathfrak{i}_1 \sqcup \mathfrak{i}_2$ is $\{1, \cdots, t\} = \emptyset \sqcup \{1, \cdots, t\}$, and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \ell_i(2n - \ell_i). \qquad (1 \leqslant i \leqslant t)$$

The knowledge of quadratic function shows that $\mathfrak{i}_{\max} = \mathfrak{i}_{2\max} = \{t\}$ with $\mu_{\max} = \mu_{2\max} = \ell_t(2n - \ell_t)$. Then by Lemma 10.6.1.(ii), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) \sim \prod_{i=1}^{t-1} \left(q^{(\ell_t-\ell_i)(2n-\ell_t-\ell_i)} - 1\right)^{-1} \cdot \tfrac{1}{2}\left(1 + (-1)^r\right) \cdot q^{\frac{1}{2}\ell_t(2n-\ell_t)r}. \tag{13.1.1}$$

In particular, it has order $\frac{1}{2}\ell_t(2n - \ell_t)$ and degree 0.

(ii). If $\ell_1 = 1$, then by Eqs. (6.5.7), (6.5.8), and (9.1.4), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ c_1+2c_2+\cdots+2c_t=r}} q^{\sum_{i=1}^{t} \ell_i(2n-\ell_i)c_i}.$$

Now, we apply Lemma 10.6.1 to this summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$, the partition $\mathfrak{i} = \mathfrak{i}_1 \sqcup \mathfrak{i}_2$ is $\{1, \cdots, t\} = \{1\} \sqcup \{2, \cdots, t\}$, and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \ell_i(2n - \ell_i). \qquad (1 \leqslant i \leqslant t)$$

The knowledge of quadratic function shows that $\mathfrak{i}_{2\max} = \{t\}$ and $\mu_{2\max} = \ell_t(2n - \ell_t)$.

Depending on n and ℓ_t, there are two possibilities: $2(2n-1) > \ell_t(2n-\ell_t)$ and $2(2n-1) < \ell_t(2n-\ell_t)$. Note that $2(2n-1) = \ell_t(2n-\ell_t)$ is impossible since the left-hand side has remainder 2 modulo 4 while the right-hand side is either odd or a multiple of 4.

If $2(2n-1) > \ell_t(2n-\ell_t)$, then by Lemma 10.6.1.(i), we have

$$S_{\mathcal{V}_\dagger(I)}(r) \sim \prod_{i=2}^{t} \left(q^{2(2n-1)-\ell_i(2n-\ell_i)} - 1\right)^{-1} \cdot q^{(2n-1)r}. \tag{13.1.2}$$

In particular, it has order $2n-1$ and degree 0.

If $2(2n-1) < \ell_t(2n-\ell_t)$, then by Lemma 10.6.1.(ii), we have

$$\begin{aligned} S_{\mathcal{V}_\dagger(I)}(r) \sim & \left(q^{\ell_t(2n-\ell_t)-2(2n-1)} - 1\right)^{-1} \prod_{i=2}^{t-1} \left(q^{(\ell_t-\ell_i)(2n-\ell_t-\ell_i)} - 1\right)^{-1} \\ & \cdot \frac{1}{2}\left(\left(1 + q^{\frac{1}{2}\ell_t(2n-\ell_t)-(2n-1)}\right) + \left(1 - q^{\frac{1}{2}\ell_t(2n-\ell_t)-(2n-1)}\right)(-1)^r\right) \\ & \cdot q^{\frac{1}{2}\ell_t(2n-\ell_t)r}. \end{aligned} \tag{13.1.3}$$

In particular, it has order $\frac{1}{2}\ell_t(2n-\ell_t)$ and degree 0.

§ 13.2. Dominant types for $S_{\mathcal{V}_\dagger(I)}(r)$

Now, we are able to figure out for which type I, $S_{\mathcal{V}_\dagger(I)}(r)$ is dominant.

When $n = 3$, we have $\ell_t(I) \leqslant 3$ for all I. Therefore, $2(2n-1) > \ell_t(I)(2n-\ell_t(I))$. Hence, $S_{\mathcal{V}_\dagger(I)}(r)$ is dominant exactly when $1 \notin I$. Note that, such a type I must be one of the following: $\{2,3\}$, $\{2\}$, $\{3\}$, and $\emptyset$. Using Eq. (13.1.2), we can deduce the asymptotic growth of dominant $\mathrm{S}_{\mathcal{V}_\dagger(I)}(r)$ as follows.

$$\mathrm{S}_{\mathcal{V}_\dagger(\{2,3\})}(r) = q^{(2n-1)r} = q^{5r}, \tag{13.2.1}$$

$$\begin{aligned} \mathrm{S}_{\mathcal{V}_\dagger(\{2\})}(r) &\sim \left(q^{2(2n-1)-\ell_2(2n-\ell_2)} - 1\right)^{-1} \cdot q^{(2n-1)r} \\ &= \frac{1}{(q-1)} \cdot q^{5r}, \end{aligned} \tag{13.2.2}$$

$$\begin{aligned} \mathrm{S}_{\mathcal{V}_\dagger(\{3\})}(r) &\sim \left(q^{2(2n-1)-\ell_2(2n-\ell_2)} - 1\right)^{-1} \cdot q^{(2n-1)r} \\ &= \frac{1}{\left(q^2-1\right)} \cdot q^{5r}, \end{aligned} \tag{13.2.3}$$

$$\begin{aligned} \mathrm{S}_{\mathcal{V}_\dagger(\emptyset)}(r) &\sim \prod_{i=2}^{3} \left(q^{2(2n-1)-\ell_i(2n-\ell_i)} - 1\right)^{-1} \cdot q^{(2n-1)r} \\ &= \frac{1}{(q-1)\left(q^2-1\right)} \cdot q^{5r}. \end{aligned} \tag{13.2.4}$$

If $n \geqslant 4$, then we have $2(2n-1) < \ell_t(I)(2n-\ell_t(I))$ when $\ell_t(I) \geqslant 4$. On the other hand, since $\ell_t(I) \leqslant n$, we have

$$\mu_{2\max} = \ell_t(I)(2n-\ell_t(I)) \leqslant n^2.$$

The equality achieves when $\ell_t(I) = n$. Hence, $\mathrm{S}_{\mathcal{V}_\dagger(I)}(r)$ is dominant exactly when $n \notin I$. In that case, its asymptotic growth is given by Eq. (13.1.3).

§ 13.3. Asymptotic growths of $\mathrm{SSA}_\dagger(r)$ and $\mathrm{SV}_\dagger(r)$

We are now able to obtain the asymptotic growth of $\mathrm{SSA}_\dagger(r)$. By Eq. (8.5.2), we have

$$\mathrm{SSA}_\dagger(r) = \sum_{I\subseteq\Delta} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg\left(\mathscr{P}_{B_n;I}\right)}} \mathrm{S}_{\mathcal{V}_\dagger(I)}(r) \sim \sum_{I \text{ is dominant}} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg\left(\mathscr{P}_{B_n;I}\right)}} \mathrm{S}_{\mathcal{V}_\dagger(I)}(r). \tag{13.3.1}$$

Then by the discussion in § 13.2, we see that

$$\mathrm{SSA}_\dagger(r) \sim \begin{cases} C_\dagger(3)\cdot q^{5r} & \text{if } n=3, \\ C_\dagger(n)\cdot q^{\frac{n^2}{2}r} & \text{if } n \geqslant 4. \end{cases} \tag{13.3.2}$$

When $n = 3$, by Eqs. (13.2.1) to (13.2.4), the constant $C_\dagger(3)$ is defined as follows:

$$C_\dagger(3) := \frac{\mathscr{P}_{B_3;\{2,3\}}(q)}{q^{\deg\left(\mathscr{P}_{B_3;\{2,3\}}\right)}} + \frac{\mathscr{P}_{B_3;\{2\}}(q)}{(q-1)\,q^{\deg\left(\mathscr{P}_{B_3;\{2\}}\right)}} + \frac{\mathscr{P}_{B_3;\{3\}}(q)}{\left(q^2-1\right)q^{\deg\left(\mathscr{P}_{B_3;\{3\}}\right)}} + \frac{\mathscr{P}_{B_3;\emptyset}(q)}{(q-1)\left(q^2-1\right)q^{\deg\left(\mathscr{P}_{B_3;\emptyset}\right)}}.$$

Moreover, by Eq. (A.2.2), we have

$$\begin{aligned} C_\dagger(3) &= \frac{\left(q^6-1\right)}{(q-1)\,q^5} + \frac{\left(q^6-1\right)\left(q^4-1\right)}{\left(q^2-1\right)(q-1)^2\,q^7} + \frac{\left(q^6-1\right)\left(q^4-1\right)}{(q-1)^2\left(q^2-1\right)q^8} \\ &\quad + \frac{\left(q^6-1\right)\left(q^4-1\right)\left(q^2-1\right)}{(q-1)^4\left(q^2-1\right)q^9} \\ &= \frac{\left(q^2+q+1\right)\left(q^2-q+1\right)(q+1)}{(q-1)^2q^9}\left(q^6-q^5+q^4+q^3+q^2+1\right). \end{aligned} \tag{13.3.3}$$

As a consequence, we have

$$\mathrm{SV}_\dagger(r) = \sum_{z=0}^{r} \mathrm{SSA}_\dagger(z) \sim \frac{q^5}{q^5-1} C_\dagger(3)\cdot q^{5r}. \tag{13.3.4}$$

When $n \geqslant 4$, by Eqs. (13.1.1) and (13.1.3), $C_\dagger(n)$ is a parity q-function defined as follows:

$$C_\dagger(n)(\text{even}) := \sum_{1,n\notin I} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg(\mathscr{P}_{B_n;I})}} \prod_{i=2}^{t-1} \left(q^{(n-\ell_i(I))^2} - 1\right)^{-1} \cdot \frac{1}{q^{n^2-2(2n-1)} - 1} + \sum_{1\in I,n\notin I} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg(\mathscr{P}_{B_n;I})}} \prod_{i=1}^{t-1} \left(q^{(n-\ell_i(I))^2} - 1\right)^{-1}, \tag{13.3.5}$$

$$C_\dagger(n)(\text{odd}) := \sum_{1,n\notin I} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg(\mathscr{P}_{B_n;I})}} \prod_{i=2}^{t-1} \left(q^{(n-\ell_i(I))^2} - 1\right)^{-1} \cdot \frac{q^{\frac{n^2}{2}-(2n-1)}}{q^{n^2-2(2n-1)} - 1}. \tag{13.3.6}$$

As a consequence, we have

$$\mathrm{SV}_\dagger(r) = \sum_{z=0}^{r} \mathrm{SSA}_\dagger(z) \sim \tilde{C}_\dagger(n) q^{\frac{n^2}{2}r}, \tag{13.3.7}$$

where the parity q-function $\tilde{C}_\dagger(n)$ is defined as follows:

(13.3.8)

$$\tilde{C}_\dagger(n)(\text{even}) := \sum_{1,n\notin I} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg(\mathscr{P}_{B_n;I})}} \prod_{i=2}^{t-1} \left(q^{(n-\ell_i(I))^2} - 1\right)^{-1} \cdot \frac{q^{n^2-(2n-1)} + q^{n^2}}{\left(q^{n^2} - 1\right)\left(q^{n^2-2(2n-1)} - 1\right)} + \sum_{1\in I,n\notin I} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg(\mathscr{P}_{B_n;I})}} \prod_{i=1}^{t-1} \left(q^{(n-\ell_i(I))^2} - 1\right)^{-1} \cdot \frac{q^{n^2}}{q^{n^2} - 1},$$

(13.3.9)

$$\tilde{C}_\dagger(n)(\text{odd}) := \sum_{1,n\notin I} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg(\mathscr{P}_{B_n;I})}} \prod_{i=2}^{t-1} \left(q^{(n-\ell_i(I))^2} - 1\right)^{-1} \cdot \frac{q^{\frac{3n^2}{2}-(2n-1)} + q^{\frac{n^2}{2}}}{\left(q^{n^2} - 1\right)\left(q^{n^2-2(2n-1)} - 1\right)} + \sum_{1\in I,n\notin I} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg(\mathscr{P}_{B_n;I})}} \prod_{i=1}^{t-1} \left(q^{(n-\ell_i(I))^2} - 1\right)^{-1} \cdot \frac{q^{\frac{n^2}{2}}}{q^{n^2} - 1}.$$

By Eqs. (13.3.2) to (13.3.9), we have proved the asymptotic relations in Theorem 13.2, where

$$C_{\dagger 0}(n) = C_\dagger(n)(\text{even}), \qquad C_{\dagger 1}(n) = C_\dagger(n)(\text{odd}) \cdot q^{\pi(n)},$$
$$\tilde{C}_{\dagger 0}(n) = \tilde{C}_\dagger(n)(\text{even}), \qquad \tilde{C}_{\dagger 1}(n) = \tilde{C}_\dagger(n)(\text{odd}) \cdot q^{\pi(n)}.$$

Moreover, by Eq. (A.2.2), we have the following explicit formulas:

$$\mathscr{P}_{B_n;I}(q) = \frac{[2n]!!(q)}{\prod_{i=1}^{t}[\ell_i(I) - \ell_{i-1}(I)]!(q)}, \qquad q^{\deg(\mathscr{P}_{B_n;I})} = \frac{q^{n^2}}{\prod_{i=1}^{t} q^{\binom{\ell_i(I)-\ell_{i-1}(I)}{2}}}.$$

See Eqs. (A.1.2) and (A.2.1) for the definitions of the symbols $[\,\cdot\,]!$ and $[2\cdot\,]!!$.

§ 13.4. Analysis of $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2r)$ and $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2r+1)$

Now, let I be a type and follow Convention 2.4.5. We are going to show that $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2\cdot\,)$ and $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2\cdot+1)$ can be defined by primary q-exponential polynomials. We will separate the discussion into two cases: (i) $\ell_1 > 1$ and (ii) $\ell_1 = 1$.

(i). If $\ell_1 > 1$, then by Eqs. (6.5.7), (6.5.8), and (9.1.4), we have

$$\mathrm{S}_{\mathcal{V}_\dagger(I)}(2r) = \sum_{\substack{c_i\in\mathbb{Z}_{>0}\\ 2c_1+\cdots+2c_t=2r}} q^{\sum_{i=1}^{t}\ell_i(2n-\ell_i)c_i} = \sum_{\substack{c_i\in\mathbb{Z}_{>0}\\ c_1+\cdots+c_t=r}} q^{\sum_{i=1}^{t}\ell_i(2n-\ell_i)c_i},$$
$$\mathrm{S}_{\mathcal{V}_\dagger(I)}(2r+1) = \sum_{\substack{c_i\in\mathbb{Z}_{>0}\\ 2c_1+\cdots+2c_t=2r+1}} q^{\sum_{i=1}^{t}\ell_i(2n-\ell_i)c_i} = 0.$$

Now, we apply Lemma 10.5.2 to $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2r)$, where the index set $\mathfrak{i}$ is $\{1,\cdots,t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \ell_i(2n-\ell_i). \qquad (1 \leqslant i \leqslant t)$$

Since all members of $\boldsymbol{\mu}$ are integers, $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2\,\cdot\,)$ can be defined by a primary super q-exponential polynomial.

(ii). If $\ell_1 = 1$, then by Eqs. (6.5.7), (6.5.8), and (9.1.4), we have (noticing the involved change of variables)

$$\mathrm{S}_{\mathcal{V}_\dagger(I)}(2r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ c_1+c_2+\cdots+c_t=r}} q^{2(2n-1)c_1+\sum_{i=2}^{t}\ell_i(2n-\ell_i)c_i},$$

$$\mathrm{S}_{\mathcal{V}_\dagger(I)}(2r+1) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ c_1+c_2+\cdots+c_t=r+1}} q^{(2n-1)(2c_1-1)+\sum_{i=2}^{t}\ell_i(2n-\ell_i)c_i}.$$

Now, we apply Lemma 10.5.2 to these summations, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\mu_1 = 2(2n-1),$$

$$\mu_i = \ell_i(2n-\ell_i). \qquad (2 \leqslant i \leqslant t)$$

Since all members of $\boldsymbol{\mu}$ are integers, the q-functions $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2\,\cdot\,)$ and $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2\,\cdot\,+1)$ can be defined by primary super q-exponential polynomials.

By Eq. (13.3.1), the q-functions

$$\mathrm{SV}_\dagger(2\,\cdot\,), \mathrm{SV}_\dagger(2\,\cdot\,+1), \mathrm{SSA}_\dagger(2\,\cdot\,), \qquad \text{and} \qquad \mathrm{SSA}_\dagger(2\,\cdot\,+1)$$

are $\mathbb{Q}(q;1)$-combinations of $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2\,\cdot\,)$ and $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2\,\cdot\,+1)$. We thus finish proving Theorem 13.2.

§ 13.5. Asymptotic growths of $\mathrm{S}^{\asymp}_{\mathcal{X}^0(I)}(r)$ and $\mathrm{S}^{\asymp}_{\mathcal{X}^1(I)}(r)$

Now, let I be a type and follow Convention 2.4.5. We are going to estimate the asymptotic growths of $\mathrm{S}_{\mathcal{X}^0(I)}(r)$ and $\mathrm{S}_{\mathcal{X}^1(I)}(r)$ up to the leading coefficient. We will separate the discussion into two cases: (i) $\ell_1 > 1$ and (ii) $\ell_1 = 1$.

(i). If $\ell_1 > 1$, then by Eqs. (6.5.8) and (9.4.4), we have

$$\mathrm{S}^{\asymp}_{\mathcal{X}^0(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ c_1+\cdots+c_t=r}} q^{\sum_{i=1}^{t} \frac{1}{2}\ell_i(2n-\ell_i)c_i}.$$

Now, we apply Lemma 10.5.2 to above summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \tfrac{1}{2}\ell_i(2n-\ell_i). \qquad (1 \leqslant i \leqslant t)$$

The knowledge of quadratic function shows that $\mathfrak{i}_{\max} = \{t\}$ with $\mu_{\max} = \frac{1}{2}\ell_t(2n-\ell_t)$. Then we have

$$\mathrm{S}^{\asymp}_{\mathcal{X}^0(I)}(r) \sim \prod_{i=1}^{t-1}\left(q^{\frac{1}{2}(\ell_t-\ell_i)(2n-\ell_t-\ell_i)} - 1\right)^{-1} \cdot q^{\frac{1}{2}\ell_t(2n-\ell_t)r}.$$

Since $\mathrm{S}_{\mathcal{X}^0(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{X}^0(I)}(r)$, we see that $\mathrm{S}_{\mathcal{X}^0(I)}$ has order $\frac{1}{2}\ell_t(2n-\ell_t)$ and degree 0.

(ii). If $\ell_1 = 1$, then by Eqs. (6.5.8), (9.4.4), and (9.4.5), we have

$$\mathrm{S}^{\asymp}_{\mathcal{X}^0(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ c_1+\cdots+c_t=r}} q^{(2n-1)c_1+\sum_{i=2}^{t} \frac{1}{2}\ell_i(2n-\ell_i)c_i}, \qquad \mathrm{S}^{\asymp}_{\mathcal{X}^1(I)}(r) = q^{-\frac{1}{2}(2n-1)}\,\mathrm{S}^{\asymp}_{\mathcal{X}^0(I)}(r).$$

Now, we apply Lemma 10.5.2 to above summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$

and the sequence $\boldsymbol{\mu}$ is

$$\mu_1 = 2n-1,$$
$$\mu_i = \tfrac{1}{2}\ell_i(2n-\ell_i). \qquad (1 < i \leqslant t)$$

Depending on n and ℓ_t, there are two possibilities: $2n-1 > \frac{1}{2}\ell_t(2n-\ell_t)$ and $2n-1 < \frac{1}{2}\ell_t(2n-\ell_t)$. If $2n-1 > \frac{1}{2}\ell_t(2n-\ell_t)$, then we have $\mathfrak{i}_{\max} = \{1\}$, $\mu_{\max} = 2n-1$, and

$$\mathrm{S}^{\asymp}_{\mathcal{X}^0(I)}(r) \sim \prod_{i=2}^{t}\left(q^{(2n-1)-\frac{1}{2}\ell_i(2n-\ell_i)}-1\right)^{-1}\cdot q^{(2n-1)r}.$$

If $2n-1 < \frac{1}{2}\ell_t(2n-\ell_t)$, then we have $\mathfrak{i}_{\max} = \{t\}$, $\mu_{\max} = \frac{1}{2}\ell_t(2n-\ell_t)$, and

$$\mathrm{S}^{\asymp}_{\mathcal{X}^0(I)}(r) \sim \left(q^{\frac{1}{2}\ell_t(2n-\ell_t)-(2n-1)}-1\right)^{-1}\prod_{i=2}^{t-1}\left(q^{\frac{1}{2}(\ell_t-\ell_i)(2n-\ell_t-\ell_i)}-1\right)^{-1}\cdot q^{\frac{1}{2}\ell_t(2n-\ell_t)r}.$$

Since $\mathrm{S}_{\mathcal{X}^0(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{X}^0(I)}(r)$, $\mathrm{S}_{\mathcal{X}^1(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{X}^1(I)}(r)$, and $\mathrm{S}^{\asymp}_{\mathcal{X}^1(I)}(r) = q^{-\frac{1}{2}(2n-1)}\,\mathrm{S}^{\asymp}_{\mathcal{X}^0(I)}(r)$, we see that $\mathrm{S}_{(\mathcal{X}^0\cup\mathcal{X}^1)(I)}$ has order $\max\left\{2n-1, \frac{1}{2}\ell_t(2n-\ell_t)\right\}$ and degree 0.

§ 13.6. Dominant types for $\mathrm{S}_{\mathcal{V}(I)}(r)$

We are going to estimate the asymptotic growth of each $\mathrm{S}_{\mathcal{V}(I)}(r)$ and figure out the *dominant types*, namely the types for which $\mathrm{S}_{\mathcal{V}(I)}(r)$ is dominant.

Let I be a type and follow Convention 2.4.5. If $\ell_1 > 1$, then $\mathcal{V}(I)$ is between $\mathcal{V}_{\dagger}(I)$ and $\mathcal{X}^0(I)$ by Fig. 9.2. Therefore,

$$\mathrm{S}_{\mathcal{X}^0(I)}(r) \gg \mathrm{S}_{\mathcal{V}(I)}(r) \gg \mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r).$$

From § 13.1 and 5.0.(i), we see that both $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}$ and $\mathrm{S}_{\mathcal{X}^0(I)}$ have order $\frac{1}{2}\ell_t(2n-\ell_t)$ and degree 0. Note that

$$\tfrac{1}{2}\ell_t(2n-\ell_t) \leqslant \tfrac{n^2}{2},$$

where the equality holds exactly when $\ell_t = n$. We thus see that I is dominant among those satisfying $\ell_1(I) > 1$ if and only if $n \notin I$. In that case, $\mathrm{S}_{\mathcal{V}(I)}(r)$ has order $\frac{n^2}{2}$ and degree 0.

If $\ell_1 = 1$, then $\mathcal{V}(I)$ is between $\mathcal{V}_{\dagger}(I)$ and $\mathcal{X}^0(I) \cup \mathcal{X}^1(I)$ by Fig. 9.3. Therefore,

$$\mathrm{S}_{(\mathcal{X}^0\cup\mathcal{X}^1)(I)}(r) \gg \mathrm{S}_{\mathcal{V}(I)}(r) \gg \mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r).$$

Depending on n and ℓ_t, there are two there are two possibilities: $2n-1 > \frac{1}{2}\ell_t(2n-\ell_t)$ and $2n-1 < \frac{1}{2}\ell_t(2n-\ell_t)$. From § 13.1 and 5.0.(ii), we see that $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}$, $\mathrm{S}_{\mathcal{X}^0(I)}$, and $\mathrm{S}_{\mathcal{X}^1(I)}$ have the same order and degree. Hence, $\mathrm{S}_{\mathcal{V}(I)}(r)$ has the same order and degree with them. If $n = 3$, then we must have $\ell_t \leqslant 3$ and hence $2n-1 = 5 > \frac{1}{2}\ell_t(2n-\ell_t)$. Then $\mathrm{S}_{\mathcal{V}(I)}(r)$ has order 5 and degree 0. In this case, all types I satisfying $\ell_1(I) = 1$ are dominant. If $n \geqslant 4$, then we may have $2n-1 < \frac{1}{2}\ell_t(2n-\ell_t)$. Note that

$$\tfrac{1}{2}\ell_t(2n-\ell_t) \leqslant \tfrac{n^2}{2},$$

where the equality holds exactly when $\ell_t = n$. We thus see that I is dominant if and only if $n \notin I$. In that case, $\mathrm{S}_{\mathcal{V}(I)}(r)$ has order $\frac{n^2}{2}$ and degree 0.

To summarize, when $n = 3$, a type I is dominant if and only if $1 \notin I$; when $n \geqslant 4$, a type I is dominant if and only if $n \notin I$.

§ 13.7. Asymptotic growth of dominant $\mathrm{S}_{\mathcal{X}^0(I)}(r)$ and $\mathrm{S}_{\mathcal{X}^1(I)}(r)$

Now, let I be a type and follow Convention 2.4.5. We are going to compute the asymptotic growths of $\mathrm{S}_{\mathcal{X}^0(I)}(r)$ and $\mathrm{S}_{\mathcal{X}^1(I)}(r)$ when I is dominant.

To do this, we pick an arbitrary $x \in \mathcal{X}^0(I)$ (or $x \in (\mathcal{X}^0 \cup \mathcal{X}^1)(I)$ if $\ell_1 = 1$) and investigate the difference between $2\rho(x)$ and the sum of $\lceil a(x) \rceil$ for $a(x) > 0$. To better describe these sums, we follow Conventions 12.5.1 to 12.5.3. We will separate the discussion into two cases: (i) $\ell_1 > 1$ and (ii) $\ell_1 = 1$.

(i). We begin with the $\ell_1 > 1$ case. Suppose $x = o + c_1 \cdot \frac{1}{2}\omega_{\ell_1} + \cdots + c_t \cdot \frac{1}{2}\omega_{\ell_t} \in \mathcal{X}^0(I)$. By Eq. (6.5.6), we have

$$
\begin{aligned}
(\chi_j - \chi_{j'})(x) &= \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}\right), && (1 \leqslant j < j' \leqslant n)\\
(\chi_j + \chi_{j'})(x) &= \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}\right) && (1 \leqslant j < j' \leqslant n)\\
&\quad + c_{\ell^{-1}(j')} + \cdots + c_t,\\
\chi_j(x) &= \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_t\right). && (1 \leqslant j \leqslant n)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\sum_{a \in \Phi^+} \lceil a(x) \rceil &= \sum_{1 \leqslant j < j' \leqslant n} \left(\lceil (\chi_j - \chi_{j'})(x) \rceil + \lceil (\chi_j + \chi_{j'})(x) \rceil\right) + \sum_{j=1}^{n} \lceil \chi_j(x) \rceil\\
&= 2\rho(x) + \sum_{1 \leqslant j < j' \leqslant n} \overline{c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}} + \sum_{j=1}^{n} \tfrac{1}{2}\overline{c_{\ell^{-1}(j)} + \cdots + c_t}.
\end{aligned}
$$

From above analysis, we can define the parity function $e_{\mathcal{X}^0(I)}$ as follows:

$$e_{\mathcal{X}^0(I)}(c_1,\cdots,c_t) := \sum_{1\leqslant i<i'\leqslant t+1} (\ell_i-\ell_{i-1})(\ell_{i'}-\ell_{i'-1})\overline{c_i+\cdots+c_{i'-1}} + \sum_{i=1}^{t} \tfrac{1}{2}(\ell_i-\ell_{i-1})\overline{c_i+\cdots+c_t}. \tag{13.7.1}$$

Then we have

$$\sum_{a\in\Phi^+} \lceil a(x)\rceil = 2\rho(x) + e_{\mathcal{X}^0(I)}(c_1,\cdots,c_t).$$

Now, we apply Lemma 10.5.5 to the following summation.

$$\mathrm{S}_{\mathcal{X}^0(I)}(r) = \sum_{\substack{c_i\in\mathbb{Z}_{>0}\\ c_1+\cdots+c_t=r}} q^{\sum_{i=1}^{t}\frac{1}{2}\ell_i(2n-\ell_i)c_i + e_{\mathcal{X}^0(I)}(c_1,\cdots,c_t)}.$$

Note that the index set $\mathfrak{i}$ is $\{1,\cdots,t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \tfrac{1}{2}\ell_i(2n-\ell_i). \qquad (1\leqslant i\leqslant t)$$

Now, let I be a dominant type, namely $n\notin I$. Then we have $\mathfrak{i}_{\max}=\{t\}$, $\ell_t=n$, and $\mu_{\max}=\frac{n^2}{2}$. Therefore,

$$\mathrm{S}_{\mathcal{X}^0(I)}(r) \sim C_{\mathcal{X}^0(I)}\cdot\left(\left(\sum_{\mathbf{s}\in\mathbb{F}_2^t}\mathrm{E}_{\mathcal{X}^0(I)}(\mathbf{s})\right)+\left(\sum_{\mathbf{s}\in\mathbb{F}_2^t}(-1)^{\mathbf{1}\cdot\mathbf{s}}\,\mathrm{E}_{\mathcal{X}^0(I)}(\mathbf{s})\right)(-1)^r\right)\cdot q^{\frac{n^2}{2}r}, \tag{13.7.2}$$

where the constant $C_{\mathcal{X}^0(I)}$ and the function $\mathrm{E}_{\mathcal{X}^0(I)}\colon\mathbb{F}_2^t\to\mathbb{Q}(q;-)$ are defined as follows:

$$C_{\mathcal{X}^0(I)} := \frac{1}{2}\prod_{i=1}^{t-1}\left(q^{(n-\ell_i)^2}-1\right)^{-1}, \qquad \mathrm{E}_{\mathcal{X}^0(I)}(\mathbf{s}) := q^{e_{\mathcal{X}^0(I)}(\mathbf{s})+\sum_{i=1}^{t-1}\frac{1}{2}(n-\ell_i)^2 s_i}. \tag{13.7.3}$$

(ii). Now, we turn to $\ell_1=1$ case. Let $\square$ be either 0 or 1. Suppose

$$x = o + (c_1-\tfrac{1}{2}\cdot\square)\cdot\omega_1 + c_2\cdot\tfrac{1}{2}\omega_{\ell_2}+\cdots+c_t\cdot\tfrac{1}{2}\omega_{\ell_t}\in\mathcal{X}^{\square}(I),$$

By Eq. (6.5.6), we have

$$(\chi_1 - \chi_j)(x) = (c_1 - \tfrac{1}{2}\cdot\square) + \tfrac{1}{2}\left(c_2 + \cdots + c_{\ell^{-1}(j)-1}\right), \qquad (1 < j \leqslant n)$$

$$(\chi_j - \chi_{j'})(x) = \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}\right), \qquad (1 < j < j' \leqslant n)$$

$$\begin{aligned}(\chi_1 + \chi_j)(x) & \qquad (1 < j \leqslant n)\\ &= (c_1 - \tfrac{1}{2}\cdot\square) + \tfrac{1}{2}\left(c_2 + \cdots + c_{\ell^{-1}(j)-1}\right) + c_{\ell^{-1}(j)} + \cdots + c_t,\end{aligned}$$

$$\begin{aligned}(\chi_j + \chi_{j'})(x) & \qquad (1 < j < j' \leqslant n)\\ &= \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}\right) + c_{\ell^{-1}(j')} + \cdots + c_t,\end{aligned}$$

$$\chi_1(x) = (c_1 - \tfrac{1}{2}\cdot\square) + \tfrac{1}{2}\left(c_2 + \cdots + c_t\right),$$

$$\chi_j(x) = \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_t\right). \qquad (1 < j \leqslant n)$$

Therefore, we have

$$\begin{aligned}\sum_{a\in\Phi^+}\lceil a(x)\rceil = 2\rho(x) + \sum_{j=2}^{n}\overline{c_2 + \cdots + c_{\ell^{-1}(j)-1} - \square} + \sum_{2\leqslant j<j'\leqslant n}\overline{c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}}\\ + \tfrac{1}{2}\overline{c_2 + \cdots + c_t - \square} + \sum_{j=2}^{n}\tfrac{1}{2}\overline{c_{\ell^{-1}(j)} + \cdots + c_t}.\end{aligned}$$

From above analysis, we can define the parity function $e_{\chi^\square(I)}$ ($\square = 0, 1$) as follows:

$$\begin{aligned}e_{\chi^\square(I)}(c_1, \cdots, c_t) := &\sum_{i=1}^{t}(\ell_{i+1} - \ell_i)\overline{c_2 + \cdots + c_i - \square}\\ &+ \sum_{2\leqslant i<i'\leqslant t+1}(\ell_i - \ell_{i-1})(\ell_{i'} - \ell_{i'-1})\overline{c_i + \cdots + c_{i'-1}}\\ &+ \tfrac{1}{2}\overline{c_2 + \cdots + c_t - \square} + \sum_{i=2}^{t}\tfrac{1}{2}(\ell_i - \ell_{i-1})\overline{c_i + \cdots + c_t}.\end{aligned} \tag{13.7.4}$$

Then we have

$$\sum_{a\in\Phi^+}\lceil a(x)\rceil = 2\rho(x) + e_{\chi^\square(I)}(c_1,\cdots,c_t).$$

Now, we apply Lemma 10.5.5 to the following summation ($\square = 0, 1$).

$$\mathrm{S}_{\chi^\square(I)}(r) = \sum_{\substack{c_i\in\mathbb{Z}_{>0}\\ c_1+\cdots+c_t=r}} q^{(2n-1)(c_1-\frac{1}{2}\cdot\square)+\sum_{i=2}^{t}\frac{1}{2}\ell_i(2n-\ell_i)c_i+e_{\chi^\square(I)}(c_1,\cdots,c_t)}.$$

Note that the index set $\mathbf{i}$ is $\{1,\cdots,t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\mu_1 = 2n-1,$$
$$\mu_i = \tfrac{1}{2}\ell_i(2n-\ell_i). \qquad (1 < i \leqslant t)$$

Now, let I be a dominant type. Depending on n, there are two cases: (ii-a) $n = 3$ and (ii-b) $n \geqslant 4$.

(ii-a). When $n = 3$, this means $1 \notin I$. Then we have $\mathbf{i}_{\max} = \{1\}$ and $\mu_{\max} = 5$. Therefore, for $\square = 0, 1$, we have

$$\mathrm{S}_{\chi^\square(I)}(r) \sim C_{\chi^\square(I)}\cdot\left(\left(\sum_{\mathbf{s}\in\mathbb{F}_2^t}\mathrm{E}_{\chi^\square(I)}(\mathbf{s})\right)+\left(\sum_{\mathbf{s}\in\mathbb{F}_2^t}(-1)^{\mathbf{1}\cdot\mathbf{s}}\,\mathrm{E}_{\chi^\square(I)}(\mathbf{s})\right)(-1)^r\right)\cdot q^{5r},$$

where the constant $C_{\chi^\square(I)}$ and the function $\mathrm{E}_{\chi^\square(I)}\colon \mathbb{F}_2^t \to \mathbb{Q}(q;-)$ are defined as follows:

$$C_{\chi^\square(I)} := \frac{1}{2}q^{-\frac{5}{2}\cdot\square}\prod_{i=2}^{t}\left(q^{(10-\ell_i(6-\ell_i))}-1\right)^{-1}, \tag{13.7.5}$$

$$\mathrm{E}_{\chi^\square(I)}(\mathbf{s}) := q^{e_{\chi^\square(I)}(\mathbf{s})+\sum_{i=2}^{t}\left(5-\frac{1}{2}\ell_i(6-\ell_i)\right)s_i}. \tag{13.7.6}$$

From the definition Eq. (13.7.4) of $e_{\chi^\square(I)}$, we have

$$\mathrm{E}_{\chi^\square(I)}(0,s_2,\cdots,s_t) = \mathrm{E}_{\chi^\square(I)}(1,s_2,\cdots,s_t).$$

Therefore, we have

$$\mathrm{S}_{\mathcal{X}^{\square}(I)}(r) \sim C_{\mathcal{X}^{\square}(I)} \cdot \left(\sum_{\mathbf{s}\in\mathbb{F}_2^t} \mathrm{E}_{\mathcal{X}^{\square}(I)}(\mathbf{s}) \right) \cdot q^{5r}, \tag{13.7.7}$$

(ii-b). When $n \geqslant 4$, I is dominant means $n \notin I$. Then we have $\mathfrak{i}_{\max} = \{t\}$, $\ell_t = n$, and $\mu_{\max} = \frac{n^2}{2}$. Therefore, for $\square = 0, 1$, we have

$$\mathrm{S}_{\mathcal{X}^{\square}(I)}(r) \sim C_{\mathcal{X}^{\square}(I)} \cdot \left(\left(\sum_{\mathbf{s}\in\mathbb{F}_2^t} \mathrm{E}_{\mathcal{X}^{\square}(I)}(\mathbf{s}) \right) + \left(\sum_{\mathbf{s}\in\mathbb{F}_2^t} (-1)^{\mathbf{1}\cdot\mathbf{s}}\, \mathrm{E}_{\mathcal{X}^{\square}(I)}(\mathbf{s}) \right) (-1)^r \right) \cdot q^{\frac{n^2}{2} r}, \tag{13.7.8}$$

where the constant $C_{\mathcal{X}^{\square}(I)}$ and the function $\mathrm{E}_{\mathcal{X}^{\square}(I)} \colon \mathbb{F}_2^t \to \mathbb{Q}(q;-)$ are defined as follows:

$$C_{\mathcal{X}^{\square}(I)} := \frac{1}{2} q^{-\frac{1}{2}(2n-1)\cdot\square} \left(q^{n^2-2(2n-1)} - 1 \right)^{-1} \prod_{i=2}^{t-1} \left(q^{(n-\ell_i)^2} - 1 \right)^{-1}, \tag{13.7.9}$$

$$\mathrm{E}_{\mathcal{X}^{\square}(I)}(\mathbf{s}) := q^{e_{\mathcal{X}^{\square}(I)}(\mathbf{s}) + \left(\frac{n^2}{2} - (2n-1)\right) s_1 + \sum_{i=2}^{t-1} \frac{1}{2}(n-\ell_i)^2 s_i}. \tag{13.7.10}$$

§ 13.8. Asymptotic growth of dominant $\mathrm{S}_{\mathcal{X}_J(I)}(r)$

Now, let I be a type and follow Convention 2.4.5. We are going to analyze $\mathrm{S}_{\mathcal{X}_J(I)}(r)$.

Suppose $x \in \mathcal{X}_J(I, r)$, where $I \cap J = \emptyset$. Since $\mathcal{X}_\emptyset = \mathcal{V}_\dagger$, by Lemma 9.4.5, we can write x as $x_0 - \sum_{j\in J} \frac{1}{2}\omega_j$, where $x_0 \in \mathcal{V}_\dagger(I, r + |J| - \delta(J))$. Then we have

$$\sum_{a\in\Phi^+} \lceil a(x) \rceil = 2\rho(x_0) + \sum_{a\in\Phi^+} \left\lceil - \sum_{j\in J} a(\tfrac{1}{2}\omega_j) \right\rceil.$$

Note that the last summation gives an integral constant. Then we have

$$\mathrm{S}_{\mathcal{X}_J(I)}(r) = q^{\sum_{a\in\Phi^+} \left\lceil - \sum_{j\in J} a(\frac{1}{2}\omega_j) \right\rceil} \mathrm{S}_{\mathcal{V}_\dagger(I)}(r + |J| - \delta(J)). \tag{13.8.1}$$

Now, we assume I is dominant. We will separate the discussion into two cases: (i) $n = 3$ and (ii) $n \geqslant 4$.

(i). When $n = 3$, this means $\ell_1 = 1$. Then the following J appears in Fig. 9.3: $\{1\}, \{1,2\}, \{2,3\}$. In those cases, by Eq. (6.5.6), we have

$$|\{1\}| - \delta(\{1\}) = 0, \qquad \sum_{a\in\Phi^+} \lceil -a(\tfrac{1}{2}\omega_1) \rceil = 0,$$

$$|\{1,2\}| - \delta(\{1,2\}) = 1, \qquad \sum_{a\in\Phi^+} \lceil -a(\tfrac{1}{2}\omega_1 + \tfrac{1}{2}\omega_2) \rceil = -4,$$

$$|\{2,3\}| - \delta(\{2,3\}) = 2, \qquad \sum_{a\in\Phi^+} \lceil -a(\tfrac{1}{2}\omega_2 + \tfrac{1}{2}\omega_3) \rceil = -6.$$

Then by Eqs. (13.1.2) and (13.8.1), we have

$$\mathrm{S}_{X_{\{1\}}(I)}(r) \sim \prod_{i=2}^{t} \left(q^{10-\ell_i(6-\ell_i)} - 1\right)^{-1} \cdot q^{5r}, \tag{13.8.2}$$

$$\mathrm{S}_{X_{\{1,2\}}(I)}(r) \sim q \cdot \prod_{i=2}^{t} \left(q^{10-\ell_i(6-\ell_i)} - 1\right)^{-1} \cdot q^{5r}, \tag{13.8.3}$$

$$\mathrm{S}_{X_{\{2,3\}}(I)}(r) \sim q^4 \cdot \prod_{i=2}^{t} \left(q^{10-\ell_i(6-\ell_i)} - 1\right)^{-1} \cdot q^{5r}. \tag{13.8.4}$$

(ii). Now, we assume $n \geqslant 4$. Then I is dominant means $\ell_t = n$. Depending on ℓ_1, there are two cases: (ii-a) $\ell_1 > 1$ and (ii-b) $\ell_1 = 1$.

(ii-a). If $\ell_1 > 1$, then the following J appears in Fig. 9.2: $\{2,3\}, \cdots, \{n-1, n\}$. In those cases, we have $|J| - \delta(J) = 2$ and by Eq. (6.5.6),

$$\sum_{a\in\Phi^+} \lceil -a(\tfrac{1}{2}\omega_j + \tfrac{1}{2}\omega_{j+1}) \rceil = -j(2n-1-j). \qquad (1 < j < n) \tag{13.8.5}$$

Then by Eqs. (13.1.1) and (13.8.1), we have ($1 < j < n$)

$$\mathrm{S}_{\mathcal{X}_{\{j,j+1\}}(I)}(r) \sim q^{n^2-j(2n-1-j)} \cdot \prod_{i=1}^{t-1}\left(q^{(n-\ell_i)^2}-1\right)^{-1} \cdot \tfrac{1}{2}\left(1+(-1)^r\right) \cdot q^{\frac{n^2}{2}r}. \tag{13.8.6}$$

(ii-b). If $\ell_1 = 1$, then the following J appears in Fig. 9.3: $\{1\}$, $\{1,2\}$, $\cdots$, $\{n-1,n\}$.

When $J = \{1\}$, we have $|\{1\}| - \delta(\{1\}) = 0$ and

$$\sum_{a\in\Phi^+}\left\lceil -a(\tfrac{1}{2}\omega_1)\right\rceil = 0.$$

Then by Eqs. (13.1.3) and (13.8.1), we have

$$\begin{aligned}\mathrm{S}_{\mathcal{X}_{\{1\}}(I)}(r) \sim &\left(q^{n^2-2(2n-1)}-1\right)^{-1}\prod_{i=2}^{t-1}\left(q^{(n-\ell_i)^2}-1\right)^{-1} \\ &\cdot \tfrac{1}{2}\left(\left(1+q^{\frac{n^2}{2}-(2n-1)}\right)+\left(1-q^{\frac{n^2}{2}-(2n-1)}\right)(-1)^r\right)\cdot q^{\frac{n^2}{2}r}.\end{aligned} \tag{13.8.7}$$

When $J = \{1,2\}$, we have $|\{1,2\}| - \delta(\{1,2\}) = 1$ and

$$\sum_{a\in\Phi^+}\left\lceil -a(\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_2)\right\rceil = -(2n-2).$$

Then by Eqs. (13.1.3) and (13.8.1), we have

$$\begin{aligned}\mathrm{S}_{\mathcal{X}_{\{1,2\}}(I)}(r) \sim &\, q^{\frac{n^2}{2}-(2n-2)}\cdot\left(q^{n^2-2(2n-1)}-1\right)^{-1}\prod_{i=2}^{t-1}\left(q^{(n-\ell_i)^2}-1\right)^{-1} \\ &\cdot \tfrac{1}{2}\left(\left(1+q^{\frac{n^2}{2}-(2n-1)}\right)+\left(1-q^{\frac{n^2}{2}-(2n-1)}\right)(-1)^r\right)\cdot q^{\frac{n^2}{2}r}.\end{aligned} \tag{13.8.8}$$

When $J = \{j, j+1\}$ ($1 < j < n$), we have $|J| - \delta(J) = 2$ and Eq. (13.8.5). Then by

Eqs. (13.1.3) and (13.8.1), we have

$$\mathrm{S}_{\mathcal{X}_{\{j,j+1\}}(I)}(r) \sim q^{n^2-j(2n-1-j)} \cdot \left(q^{n^2-2(2n-1)}-1\right)^{-1} \prod_{i=2}^{t-1} \left(q^{(n-\ell_i)^2}-1\right)^{-1} \cdot \tfrac{1}{2}\left(\left(1+q^{\frac{n^2}{2}-(2n-1)}\right)+\left(1-q^{\frac{n^2}{2}-(2n-1)}\right)(-1)^r\right)\cdot q^{\frac{n^2}{2}r}. \tag{13.8.9}$$

§ 13.9. Asymptotic growth of dominant $\mathrm{S}_{\mathcal{V}(I)}(r)$

We are now able to compute the asymptotic growth of $\mathrm{S}_{\mathcal{V}(I)}(r)$ when I is dominant. We will separate the discussion into two cases: (i) $n = 3$ and (ii) $n \geqslant 4$.

(i). When $n = 3$, the dominant types are $\{2,3\}$, $\{2\}$, $\{3\}$, and $\emptyset$. By Fig. 9.3, we have (where zero summations are omitted)

$$\begin{aligned}
\mathrm{S}_{\mathcal{V}(\{2,3\})}(r) &= \mathrm{S}_{\mathcal{X}^0(\{2,3\})}(r) + \mathrm{S}_{\mathcal{X}^1(\{2,3\})}(r) - \mathrm{S}_{\mathcal{X}_{\{1\}}(\{2,3\})}(r),\\
\mathrm{S}_{\mathcal{V}(\{2\})}(r) &= \mathrm{S}_{\mathcal{X}^0(\{2\})}(r) + \mathrm{S}_{\mathcal{X}^1(\{2\})}(r) - \mathrm{S}_{\mathcal{X}_{\{1\}}(\{2\})}(r),\\
\mathrm{S}_{\mathcal{V}(\{3\})}(r) &= \mathrm{S}_{\mathcal{X}^0(\{3\})}(r) + \mathrm{S}_{\mathcal{X}^1(\{3\})}(r) - \mathrm{S}_{\mathcal{X}_{\{1\}}(\{3\})}(r) - \mathrm{S}_{\mathcal{X}_{\{1,2\}}(\{3\})}(r),\\
\mathrm{S}_{\mathcal{V}(\emptyset)}(r) &= \mathrm{S}_{\mathcal{X}^0(\emptyset)}(r) + \mathrm{S}_{\mathcal{X}^1(\emptyset)}(r) - \mathrm{S}_{\mathcal{X}_{\{1\}}(\emptyset)}(r) - \mathrm{S}_{\mathcal{X}_{\{1,2\}}(\emptyset)}(r) - \mathrm{S}_{\mathcal{X}_{\{2,3\}}(\emptyset)}(r).
\end{aligned}$$

Therefore, by Eqs. (13.7.7) and (13.8.2) to (13.8.4), we have

$$\mathrm{S}_{\mathcal{V}(\{2,3\})}(r) \sim (1+1-1)\,q^{5r} = q^{5r}, \tag{13.9.1}$$

$$\mathrm{S}_{\mathcal{V}(\{2\})}(r) \sim \frac{(q^2+1)+(q+1)-1}{q-1}q^{5r} = \frac{q^2+q+1}{q-1}q^{5r}, \tag{13.9.2}$$

$$\mathrm{S}_{\mathcal{V}(\{3\})}(r) \sim \frac{(q^4+1)+(q+1)-1-q}{q^2-1}q^{5r} = \frac{q^4+1}{q^2-1}q^{5r}, \tag{13.9.3}$$

$$\begin{aligned}\mathrm{S}_{\mathcal{V}(\emptyset)}(r) &\sim \frac{(2q^4+q^2+1)+(q^2+2q+1)-1-q-q^4}{(q-1)(q^2-1)}q^{5r} \\ &= \frac{q^4+2q^2+q+1}{(q-1)(q^2-1)}q^{5r}.\end{aligned} \tag{13.9.4}$$

(ii). Now, we assume $n \geqslant 4$. Then I is dominant exactly when $n \notin I$. Depending on ℓ_1, there are two cases: (ii-a) $\ell_1 > 1$ and (ii-b) $\ell_1 = 1$.

(ii-a). If $\ell_1 > 1$, then by Fig. 9.2, we have (including the zero summations)

$$\mathrm{S}_{\mathcal{V}(I)}(r) = \mathrm{S}_{\mathcal{X}^0(I)}(r) - \sum_{j=2}^{n-1} \mathrm{S}_{\mathcal{X}_{\{j,j+1\}}(I)}(r).$$

Therefore, by Eqs. (13.7.2), (13.7.3), and (13.8.6), we have

$$\mathrm{S}_{\mathcal{V}(I)}(r) \sim \frac{1}{2}\prod_{i=1}^{t-1}\left(q^{(n-\ell_i)^2}-1\right)^{-1}\cdot\left(C_{I,0}+C_{I,1}(-1)^r\right)\cdot q^{\frac{n^2}{2}r}, \tag{13.9.5}$$

where the constants $C_{I,0}$ and $C_{I,1}$ are defined as follows:

$$C_{I,0} := \sum_{\mathbf{s}\in\mathbb{F}_2^t} q^{e_{\mathcal{X}^0(I)}(\mathbf{s})+\sum_{i=1}^{t-1}\frac{1}{2}(n-\ell_i)^2 s_i} - \sum_{\substack{1<j<n\\ \{j,j+1\}\cap I=\emptyset}} q^{n^2-j(2n-1-j)}, \tag{13.9.6}$$

$$C_{I,1} := \sum_{\mathbf{s}\in\mathbb{F}_2^t} (-1)^{\mathbf{1}\cdot\mathbf{s}} q^{e_{\mathcal{X}^0(I)}(\mathbf{s})+\sum_{i=1}^{t-1}\frac{1}{2}(n-\ell_i)^2 s_i} - \sum_{\substack{1<j<n\\ \{j,j+1\}\cap I=\emptyset}} q^{n^2-j(2n-1-j)}. \tag{13.9.7}$$

Note that the multivariable parity function $e_{\mathcal{X}^0(I)}$ is defined in Eq. (13.7.1).

(ii-b). If $\ell_1 = 1$, then by Fig. 9.3, we have (including the zero summations)

$$\mathrm{S}_{\mathcal{V}(I)}(r) = \mathrm{S}_{\mathcal{X}^0(I)}(r) + \mathrm{S}_{\mathcal{X}^0(I)}(r) - \mathrm{S}_{\mathcal{X}_{\{1\}}(I)}(r) - \sum_{j=1}^{n-1} \mathrm{S}_{\mathcal{X}_{\{j,j+1\}}(I)}(r).$$

Therefore, by Eqs. (13.7.8) to (13.7.10) and (13.8.7) to (13.8.9), we have

$$\text{(13.9.8)}\quad \mathrm{S}_{\mathcal{V}(I)}(r) \sim \frac{1}{2}\left(q^{n^2-2(2n-1)} - 1\right)^{-1} \prod_{i=2}^{t-1} \left(q^{(n-\ell_i)^2} - 1\right)^{-1} \cdot \left(C_{I,0} + C_{I,1}(-1)^r\right) \cdot q^{\frac{n^2}{2}r},$$

where the constants $C_{I,0}$ and $C_{I,1}$ are defined as follows:

$$\text{(13.9.9)}\quad C_{I,0} := \sum_{\square=0,1} \sum_{\mathbf{s}\in\mathbb{F}_2^t} q^{e_{\mathcal{X}^\square(I)}(\mathbf{s}) - \frac{1}{2}(2n-1)\cdot\square + \left(\frac{n^2}{2}-(2n-1)\right)s_1 + \sum_{i=2}^{t-1} \frac{1}{2}(n-\ell_i)^2 s_i}$$
$$- \left(1 + \delta_I(2) q^{\frac{n^2}{2}-(2n-2)} + \sum_{\substack{1<j<n \\ \{j,j+1\}\cap I=\emptyset}} q^{n^2-j(2n-1-j)}\right)\left(1 + q^{\frac{n^2}{2}-(2n-1)}\right),$$

(13.9.10)

$$C_{I,1} := \sum_{\square=0,1} \sum_{\mathbf{s}\in\mathbb{F}_2^t} (-1)^{\mathbf{1}\cdot\mathbf{s}} q^{e_{\mathcal{X}^\square(I)}(\mathbf{s}) - \frac{1}{2}(2n-1)\cdot\square + \left(\frac{n^2}{2}-(2n-1)\right)s_1 + \sum_{i=2}^{t-1} \frac{1}{2}(n-\ell_i)^2 s_i}$$
$$- \left(1 + \delta_I(2) q^{\frac{n^2}{2}-(2n-2)} + \sum_{\substack{1<j<n \\ \{j,j+1\}\cap I=\emptyset}} q^{n^2-j(2n-1-j)}\right)\left(1 - q^{\frac{n^2}{2}-(2n-1)}\right),$$

where $\delta_I(i) = 0$ if $1 \in I$ and 1 if not. Note that the multivariable parity functions $e_{\mathcal{X}^\square(I)}$ ($\square = 0, 1$) are defined in Eq. (13.7.4).

§ 13.10. Asymptotic growths of $\mathrm{SSA}(r)$ and $\mathrm{SV}(r)$

We are now able to obtain the asymptotic growth of $\mathrm{SSA}(r)$. By Eq. (8.4.6), we have

$$\mathrm{SSA}(r) = \sum_{I\subseteq\Delta} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg\left(\mathscr{P}_{B_n;I}\right)}} \mathrm{S}_{\mathcal{V}(I)}(r) \sim \sum_{I \text{ is dominant}} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg\left(\mathscr{P}_{B_n;I}\right)}} \mathrm{S}_{\mathcal{V}(I)}(r). \tag{13.10.1}$$

What remains is to plug in the asymptotic growth of dominant $\mathrm{S}_{\mathcal{V}(I)}(r)$. We will separate the discussion into two cases: (i) $n = 3$ and (ii) $n \geqslant 4$.

(i). When $n = 3$, the dominant types are $\{2,3\}$, $\{2\}$, $\{3\}$, and $\emptyset$. By Eqs. (13.9.1) to (13.9.4) and (A.2.2), we have

$$\mathrm{SSA}(r) \sim C(3) \cdot q^{5r}, \tag{13.10.2}$$

where the constant $C(3)$ is defined as follows:

(13.10.3)
$$\begin{aligned}
C(3) :=& \frac{\mathscr{P}_{B_3;\{2,3\}}(q)}{q^{\deg\left(\mathscr{P}_{B_3;\{2,3\}}\right)}} + \frac{\mathscr{P}_{B_3;\{2\}}(q)}{q^{\deg\left(\mathscr{P}_{B_3;\{2\}}\right)}} \frac{q^2+q+1}{q-1} + \frac{\mathscr{P}_{B_3;\{3\}}(q)}{q^{\deg\left(\mathscr{P}_{B_3;\{3\}}\right)}} \frac{q^4+1}{q^2-1} \\
&+ \frac{\mathscr{P}_{B_3;\emptyset}(q)}{q^{\deg\left(\mathscr{P}_{B_3;\emptyset}\right)}} \frac{q^4+2q^2+q+1}{(q-1)(q^2-1)} \\
=& \frac{\left(q^6-1\right)}{(q-1)\, q^5} + \frac{\left(q^6-1\right)\left(q^4-1\right)\left(q^2+q+1\right)}{\left(q^2-1\right)(q-1)^2\, q^7} + \frac{\left(q^6-1\right)\left(q^4-1\right)\left(q^4+1\right)}{(q-1)^2\left(q^2-1\right) q^8} \\
&+ \frac{\left(q^6-1\right)\left(q^4-1\right)\left(q^2-1\right)\left(q^4+2q^2+q+1\right)}{(q-1)^4\left(q^2-1\right) q^9} \\
=& \frac{\left(q^2+q+1\right)\left(q^2-q+1\right)(q+1)}{(q-1)^2 q^9} \\
&\cdot \left(q^8+q^7+3q^6+q^5+5q^4+3q^3+4q^2+q+1\right).
\end{aligned}$$

As a consequence, we have

$$\mathrm{SV}(r) = \sum_{z=0}^{r} \mathrm{SSA}(z) \sim \frac{q^5}{q^5 - 1} C(3) \cdot q^{5r}. \tag{13.10.4}$$

(ii). Now, we assume $n \geqslant 4$. Then I is dominant exactly when $n \notin I$. By Eqs. (13.9.5) and (13.9.8), we have

$$\mathrm{SSA}(r) \sim C(n) \cdot q^{\frac{n^2}{2} r}, \tag{13.10.5}$$

where the parity q-function $C(n)$ is defined as follows:

(13.10.6)

$$\begin{aligned} C(n)(r) := & \sum_{1, n \notin I} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg(\mathscr{P}_{B_n;I})}} \prod_{i=2}^{t-1} \left(q^{(n-\ell_i(I))^2} - 1 \right)^{-1} \cdot \frac{\frac{1}{2}\left(C_{I,0} + C_{I,1}(-1)^r \right)}{q^{n^2 - 2(2n-1)} - 1} \\ & + \sum_{1 \in I, n \notin I} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg(\mathscr{P}_{B_n;I})}} \prod_{i=1}^{t-1} \left(q^{(n-\ell_i(I))^2} - 1 \right)^{-1} \cdot \frac{1}{2}\left(C_{I,0} + C_{I,1}(-1)^r \right). \end{aligned}$$

As a consequence, we have

$$\mathrm{SV}(r) = \sum_{z=0}^{r} \mathrm{SSA}(z) \sim \tilde{C}(n) q^{\frac{n^2}{2} r}, \tag{13.10.7}$$

where the parity q-function $\tilde{C}(n)$ is defined as follows:

$$\tilde{C}(n)(r) := \sum_{1,n\notin I} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg\left(\mathscr{P}_{B_n;I}\right)}} \prod_{i=2}^{t-1} \left(q^{(n-\ell_i(I))^2} - 1\right)^{-1} \cdot \frac{\frac{1}{2}\left((1+q^{\frac{n^2}{2}})C_{I,0} + (1-q^{\frac{n^2}{2}})C_{I,1}(-1)^r\right)}{\left(q^{n^2}-1\right)\left(q^{n^2-2(2n-1)}-1\right)} + \sum_{1\in I,n\notin I} \frac{\mathscr{P}_{B_n;I}(q)}{q^{\deg\left(\mathscr{P}_{B_n;I}\right)}} \prod_{i=1}^{t-1} \left(q^{(n-\ell_i(I))^2} - 1\right)^{-1} \cdot \frac{\frac{1}{2}\left((1+q^{\frac{n^2}{2}})C_{I,0} + (1-q^{\frac{n^2}{2}})C_{I,1}(-1)^r\right)}{q^{n^2}-1}. \tag{13.10.8}$$

Remark. Note that the constants $C_{I,\square}$ ($\square = 0, 1$) depends on I. When $1 \in I$ and $n \notin I$, they are defined in Eqs. (13.9.6) and (13.9.7). When $1, n \notin I$, they are defined in Eqs. (13.9.9) and (13.9.10).

By Eqs. (13.10.2) to (13.10.8) we have proved the asymptotic relations in Theorem 13.1, where

$$C_0(n) = C(n)(\text{even}), \qquad C_1(n) = C(n)(\text{odd}) \cdot q^{\pi(n)},$$
$$\tilde{C}_0(n) = \tilde{C}(n)(\text{even}), \qquad \tilde{C}_1(n) = \tilde{C}(n)(\text{odd}) \cdot q^{\pi(n)}.$$

One can see they are primary q-numbers by either § 13.11 or direct verification using Eqs. (13.10.6) and (13.10.8). Moreover, by Eq. (A.2.2), we have the following explicit formulas:

$$\mathscr{P}_{B_n;I}(q) = \frac{[2n]!!(q)}{\prod_{i=1}^{t}[\ell_i(I) - \ell_{i-1}(I)]!(q)}, \qquad q^{\deg\left(\mathscr{P}_{B_n;I}\right)} = \frac{q^{n^2}}{\prod_{i=1}^{t} q^{\binom{\ell_i(I)-\ell_{i-1}(I)}{2}}}. \tag{13.10.9}$$

See Eqs. (A.1.2) and (A.2.1) for the definitions of the symbols $[\,\cdot\,]!$ and $[2\cdot\,]!!$.

§ 13.11. Analysis of $\mathrm{S}_{\mathcal{X}_J(I)}(2r)$ and $\mathrm{S}_{\mathcal{X}_J(I)}(2r+1)$

Now, let I be a general type and follow Convention 2.4.5. We are going to show that $\mathrm{S}_{\mathcal{X}_J(I)}(2r)$ and $\mathrm{S}_{\mathcal{X}_J(I)}(2r+1)$ can be defined by primary q-exponential polynomials.

Suppose $I \cap J = \emptyset$. By Eq. (13.8.1), we have ($\square = 0, 1$)

$$\mathrm{S}_{\mathcal{X}_J(I)}(2r+\square) = q^{\sum\limits_{\alpha\in\Phi^+}\left\lceil -\sum\limits_{j\in J}\alpha(\frac{1}{2}\omega_j)\right\rceil} \mathrm{S}_{\mathcal{V}_\dagger(I)}(2r+\square+|J|-\delta(J)).$$

We have seen that the q-functions $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2\,\cdot\,)$ and $\mathrm{S}_{\mathcal{V}_\dagger(I)}(2\,\cdot\,+1)$ can be defined by primary super q-exponential polynomials in § 13.4. The exponent $\sum\limits_{\alpha\in\Phi^+}\left\lceil -\sum\limits_{j\in J}\alpha(\frac{1}{2}\omega_j)\right\rceil$ is an integer. Therefore, $\mathrm{S}_{\mathcal{X}_J(I)}(2\,\cdot\,+\square)$ can be defined by a primary super q-exponential polynomial.

Note that the proof of Lemma 6.5.1 implies

$$\mathcal{V}(I,r) = \bigcup_{J\neq\{1\},\{1,2\},\cdots,\{n-1,n\}} \mathcal{X}_J(I,r).$$

Hence, the q-function $\mathrm{S}_{\mathcal{V}(I)}(2\,\cdot\,+\square)$ ($\square = 0, 1$) is clearly a $\mathbb{Q}(q;1)$-combination of $\mathrm{S}_{\mathcal{X}_J(I)}(2\,\cdot\,+\square)$. On the other hand, by Eq. (13.10.1), the q-functions $\mathrm{SV}(2\,\cdot\,)$, $\mathrm{SV}(2\,\cdot\,+1)$, $\mathrm{SSA}(2\,\cdot\,)$, and $\mathrm{SSA}(2\,\cdot\,+1)$ are $\mathbb{Q}(q;1)$-combinations of $\mathrm{S}_{\mathcal{V}(I)}(2\,\cdot\,)$ and $\mathrm{S}_{\mathcal{V}(I)}(2\,\cdot\,+1)$. We thus finish proving Theorem 13.1.

Chapter 14.

Simplicial volume in buildings of D_n type

In this chapter, we will prove the D_n part of Theorems 1.3 and 1.5. More precisely, we will prove the following stronger theorem.

Theorem 14.1. *Let $\mathscr{B}$ be a Bruhat-Tits building of split classical type D_n over a local field K with residue cardinality q. Then the simplicial volume* $\mathrm{SV}(\,\cdot\,)$ *and the simplicial surface area* $\mathrm{SSA}(\,\cdot\,)$ *in it can be defined by primary super q-exponential polynomials whose leading terms are of the form:*

$$\mathrm{SV}(r) \sim \tilde{C}(n) \cdot \binom{r}{\varepsilon(n)} q^{\pi(n)r}, \qquad \mathrm{SSA}(r) \sim C(n) \cdot \binom{r}{\varepsilon(n)} q^{\pi(n)r},$$

where $\varepsilon(n) = 1$ and $\pi(n) = \frac{n(n-1)}{2}$ when $n \geqslant 5$, while $\varepsilon(4) = 2$ and $\pi(4) = 6$. The leading coefficients $\tilde{C}(n)$ and $C(n)$ are primary q-numbers, not just parity q-functions.

We will obtain explicit formulas for the parity functions $\tilde{C}(n)$ and $C(n)$.

But before proving Theorem 14.1, we will first analyze the asymptotic growths of $\mathrm{SSA}_\dagger(r)$ and $\mathrm{SV}_\dagger(r)$, where † denotes "being special". We will prove the following.

Theorem 14.2. *Let $\mathscr{B}$ be a Bruhat-Tits building of split classical type D_n over a local field K with residue cardinality q. Then the special simplicial volume* $\mathrm{SV}_\dagger(\,\cdot\,)$ *and* *the special simplicial surface area* $\mathrm{SSA}_{\dagger}(\,\cdot\,)$ *in it can be defined by primary super q-exponential polynomials whose leading terms are of the form:*

$$\mathrm{SV}_{\dagger}(r) \sim \tilde{C}_{\dagger}(n) \cdot \binom{r}{\varepsilon(n)} q^{\pi(n)r}, \qquad \mathrm{SSA}_{\dagger}(r) \sim C_{\dagger}(n) \cdot \binom{r}{\varepsilon(n)} q^{\pi(n)r},$$

where $\varepsilon(n) = 1$ *and* $\pi(n) = \frac{n(n-1)}{2}$ *when* $n \geqslant 5$, *while* $\varepsilon(4) = 2$ *and* $\pi(4) = 6$. *The leading coefficients* $\tilde{C}_{\dagger}(n)$ *and* $C_{\dagger}(n)$ *are primary q-numbers, not just parity q-functions.*

We will also give explicit formulas for the constants $\tilde{C}_{\dagger}(n)$ and $C_{\dagger}(n)$. The proof of Theorem 14.2 will play an essential role in the study of $\mathrm{SSA}(r)$ and $\mathrm{SV}(r)$.

This chapter is structured as follows. In § 14.1, we will compute the asymptotic growth of $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r)$ for each type $I \subseteq \Delta$. This allows use to find the dominant ones of $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r)$, which will be done in § 14.2. Then in § 14.3, we will obtain the asymptotic growths of $\mathrm{SSA}_{\dagger}(r)$ and $\mathrm{SV}_{\dagger}(r)$. After that, in § 14.4, we will estimate the asymptotic growth of $\mathrm{S}_{\mathcal{X}^{\square\heartsuit}(I)}(r)$ ($\square, \heartsuit$ being 0 or 1) using the auxiliary function $\mathrm{S}^{\asymp}_{\mathcal{X}^{\square\heartsuit}(I)}$. Note that $\mathcal{V}$ is between $\mathcal{V}_{\dagger}$ and $\mathcal{X}^{00} \cup \mathcal{X}^{10} \cup \mathcal{X}^{01} \cup \mathcal{X}^{11}$. Therefore, we can combine § 14.1 and § 14.4 to estimate the asymptotic growth of each $\mathrm{S}_{\mathcal{V}(I)}(r)$ and find the dominant ones, which will be done in § 14.5. Once we found the dominant types, we can proceed to compute the asymptotic growth of dominant $\mathrm{S}_{\mathcal{V}(I)}(r)$. This will be done in three steps: in § 14.6, we will compute the asymptotic growth of each $\mathrm{S}_{\mathcal{X}^{\square\heartsuit}(I)}(r)$; in § 14.7, we will deduce the asymptotic growth of $\mathrm{S}_{\mathcal{X}_J(I)}(r)$ from that of $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r)$; then in § 14.8, the asymptotic growth of $\mathrm{S}_{\mathcal{V}(I)}(r)$ will be deduced from them. Finally, in § 14.9, we will obtain the asymptotic growths of $\mathrm{SSA}(r)$ and $\mathrm{SV}(r)$.

Throughout this chapter, we will heavily use the various index sets $\mathcal{V}$, $\mathcal{V}_{\dagger}$, $\mathcal{X}^{00}$, $\mathcal{X}^{10}$, $\mathcal{X}^{01}$, $\mathcal{X}^{11}$, and $\mathcal{X}_J$. We refer to Figs. 9.4 to 9.7 for the structure of them.

§ 14.1. Asymptotic growth of $\mathrm{S}_{\mathcal{V}_\dagger(I)}(r)$

Now, let I be a type and follow Convention 2.4.5. We are going to compute the asymptotic growth of $\mathrm{S}_{\mathcal{V}_\dagger(I)}(r)$. We will separate the discussion into the following six cases:

	$\{n-1,n\} \subseteq I$	$\lvert\{n-1,n\} \cap I\rvert = 1$	$\{n-1,n\} \cap I = \emptyset$
$1 \in I$	(i)	(ii)	(iii)
$1 \notin I$	(iv)	(v)	(vi)

(i). Suppose $1 \in I$ and $\{n-1,n\} \subseteq I$. By Eqs. (6.6.7), (6.6.8), and (9.1.4), we have

$$\mathrm{S}_{\mathcal{V}_\dagger(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ 2c_1+\cdots+2c_t=r}} q^{\sum\limits_{i=1}^{t} \ell_i(2n-1-\ell_i)c_i}.$$

Now, we apply Lemma 10.6.1 to this summation, where the index set $\mathfrak{i}$ is $\{1,\cdots,t\}$, the partition $\mathfrak{i} = \mathfrak{i}_1 \sqcup \mathfrak{i}_2$ is $\{1,\cdots,t\} = \emptyset \sqcup \{1,\cdots,t\}$, and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \ell_i(2n-1-\ell_i). \qquad (1 \leqslant i \leqslant t)$$

Since all members of $\boldsymbol{\mu}$ are integers, $\mathrm{S}_{\mathcal{V}_\dagger(I)}$ can be defined by a primary super q-exponential polynomial. The knowledge of quadratic function shows that $\mathfrak{i}_{\max} = \mathfrak{i}_{2\max} = \{t\}$ with $\mu_{\max} = \mu_{2\max} = \ell_t(2n-1-\ell_t)$. Then by Lemma 10.6.1.(ii), we have

$$\mathrm{S}_{\mathcal{V}_\dagger(I)}(r) \sim \prod_{i=1}^{t-1} \left(q^{(\ell_t-\ell_i)(2n-1-\ell_t-\ell_i)} - 1\right)^{-1} \cdot \tfrac{1}{2}\left(1+(-1)^r\right) \cdot q^{\frac{1}{2}\ell_t(2n-1-\ell_t)r}.$$

In particular, it has order $\frac{1}{2}\ell_t(2n-1-\ell_t)$ and degree 0.

(ii). Suppose $1 \in I$ and $\{n-1,n\} \cap I$ is a singleton. By Eqs. (6.6.7), (6.6.8), and (9.1.4),

we have

$$\mathrm{S}_{\mathcal{V}_\dagger(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ 2c_1+\cdots+2c_{t-1}+c_t=r}} q^{\sum_{i=1}^{t-1} \ell_i(2n-1-\ell_i)c_i + \frac{n(n-1)}{2}c_t}.$$

Now, we apply Lemma 10.6.1 to this summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$, the partition $\mathfrak{i} = \mathfrak{i}_1 \sqcup \mathfrak{i}_2$ is $\{1, \cdots, t\} = \{t\} \sqcup \{1, \cdots, t-1\}$, and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \ell_i(2n-1-\ell_i), \qquad (1 \leqslant i \leqslant t-1)$$
$$\mu_t = \tfrac{n(n-1)}{2}.$$

Since all members of $\boldsymbol{\mu}$ are integers, $\mathrm{S}_{\mathcal{V}_\dagger(I)}$ can be defined by a primary super q-exponential polynomial. The knowledge of quadratic function shows that $\mathfrak{i}_{2\,\max} = \{t-1\}$, $\mu_{2\,\max} = \ell_{t-1}(2n-1-\ell_{t-1})$, and $2\mu_{1\,\max} > \mu_{2\,\max}$. Then by Lemma 10.6.1.(i), we have

$$\mathrm{S}_{\mathcal{V}_\dagger(I)}(r) \sim \prod_{i=1}^{t-1} \left(q^{(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1} \cdot q^{\frac{n(n-1)}{2}r}.$$

In particular, it has order $\frac{n(n-1)}{2}$ and degree 0.

(iii). Suppose $1 \in I$ and $\{n-1, n\} \cap I = \emptyset$. By Eqs. (6.6.7), (6.6.8), and (9.1.4), we have

$$\mathrm{S}_{\mathcal{V}_\dagger(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ 2c_1+\cdots+2c_{t-2}+c_{t-1}+c_t=r}} q^{\sum_{i=1}^{t-2} \ell_i(2n-1-\ell_i)c_i + \frac{n(n-1)}{2}(c_{t-1}+c_t)}.$$

Now, we apply Lemma 10.6.1 to this summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$, the partition $\mathfrak{i} = \mathfrak{i}_1 \sqcup \mathfrak{i}_2$ is $\{1, \cdots, t\} = \{t-1, t\} \sqcup \{1, \cdots, t-2\}$, and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \ell_i(2n-1-\ell_i), \qquad (1 \leqslant i \leqslant t-2)$$
$$\mu_{t-1} = \tfrac{n(n-1)}{2},$$
$$\mu_t = \tfrac{n(n-1)}{2}.$$

Since all members of $\boldsymbol{\mu}$ are integers, $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}$ can be defined by a primary super q-exponential polynomial. The knowledge of quadratic function shows that $\mathfrak{i}_{2\max} = \{t-2\}$, $\mu_{2\max} = \ell_{t-2}(2n-1-\ell_{t-2})$, and $2\mu_{1\max} > \mu_{2\max}$. Then by Lemma 10.6.1.(i), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) \sim \prod_{i=1}^{t-2} \left(q^{(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1} \cdot rq^{\frac{n(n-1)}{2}r}. \tag{14.1.1}$$

In particular, it has order $\frac{n(n-1)}{2}$ and degree 1.

(iv). Suppose $1 \notin I$ and $\{n-1, n\} \subseteq I$. By Eqs. (6.6.7), (6.6.8), and (9.1.4), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ c_1+2c_2+\cdots+2c_t=r}} q^{\sum\limits_{i=1}^{t} \ell_i(2n-1-\ell_i)c_i}.$$

Now, we apply Lemma 10.6.1 to this summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$, the partition $\mathfrak{i} = \mathfrak{i}_1 \sqcup \mathfrak{i}_2$ is $\{1, \cdots, t\} = \{1\} \sqcup \{2, \cdots, t\}$, and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \ell_i(2n-1-\ell_i). \qquad (1 \leqslant i \leqslant t)$$

Since all members of $\boldsymbol{\mu}$ are integers, $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}$ can be defined by a primary super q-exponential polynomial. The knowledge of quadratic function shows that $\mathfrak{i}_{2\max} = \{t\}$ and $\mu_{2\max} = \ell_t(2n-1-\ell_t)$. On the other side $\mu_{1\max} = (2n-2)$.

Depending on n and ℓ_t, there are three possibilities.

If $2\mu_{1\max} > \mu_{2\max}$, then by Lemma 10.6.1.(i), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) \sim \prod_{i=2}^{t} \left(q^{2(2n-2)-\ell_i(2n-1-\ell_i)} - 1\right)^{-1} \cdot q^{(2n-2)r}.$$

In particular, it has order $2n-2$ and degree 0.

If $2\mu_{1\max} < \mu_{2\max}$, then by Lemma 10.6.1.(ii), we have

$$\begin{aligned}\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) \sim &\left(q^{\ell_t(2n-1-\ell_t)-2(2n-2)}-1\right)^{-1}\prod_{i=2}^{t-1}\left(q^{(\ell_t-\ell_i)(2n-1-\ell_t-\ell_i)}-1\right)^{-1}\\ &\cdot\frac{1}{2}\left(\left(1+q^{\frac{1}{2}\ell_t(2n-1-\ell_t)-(2n-2)}\right)+\left(1-q^{\frac{1}{2}\ell_t(2n-1-\ell_t)-(2n-2)}\right)(-1)^r\right)\\ &\cdot q^{\frac{1}{2}\ell_t(2n-1-\ell_t)r}.\end{aligned}$$

In particular, it has order $\frac{1}{2}\ell_t(2n-1-\ell_t)$ and degree 0.

If $2\mu_{1\max} = \mu_{2\max}$, then by Lemma 10.6.1.(iii), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) \sim \frac{1}{2}\prod_{i=2}^{t-1}\left(q^{(\ell_t-\ell_i)(2n-1-\ell_t-\ell_i)}-1\right)^{-1}\cdot rq^{(2n-2)r}.$$

In particular, it has order $2n-2$ and degree 1.

(v). Suppose $1 \notin I$ and $\{n-1, n\} \cap I$ is a singleton. By Eqs. (6.6.7), (6.6.8), and (9.1.4), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0}\\ c_1+2c_2+\cdots+2c_{t-1}+c_t=r}} q^{\sum_{i=1}^{t-1}\ell_i(2n-1-\ell_i)c_i+\frac{n(n-1)}{2}c_t}.$$

Now, we apply Lemma 10.6.1 to this summation, where the index set $\mathfrak{i}$ is $\{1,\cdots,t\}$, the partition $\mathfrak{i} = \mathfrak{i}_1 \sqcup \mathfrak{i}_2$ is $\{1,\cdots,t\} = \{1,t\} \sqcup \{2,\cdots,t-1\}$, and the sequence $\boldsymbol{\mu}$ is

$$\begin{aligned}\mu_i &= \ell_i(2n-1-\ell_i), &&(1 \leqslant i \leqslant t-1)\\ \mu_t &= \tfrac{n(n-1)}{2}.\end{aligned}$$

Since all members of $\boldsymbol{\mu}$ are integers, $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}$ can be defined by a primary super q-exponential polynomial. The knowledge of quadratic function shows that $\mathfrak{i}_{2\max} = \{t-1\}$ with $\mu_{2\max} = \ell_{t-1}(2n-1-\ell_{t-1})$ and that $t \in \mathfrak{i}_{1\max}$ with $2\mu_{1\max} = n(n-1) > \mu_{2\max}$.

Depending on n, there are two possibilities.

If $n = 4$, then $\mu_{1\,\max} = (2n-2)$ and hence $\mathfrak{i}_{1\,\max} = \{1, t\}$. By Lemma 10.6.1.(i), we have

$$\mathrm{S}_{\mathcal{V}_\dagger(I)}(r) \sim \prod_{i=2}^{t-1} \left(q^{(4-\ell_i)(3-\ell_i)} - 1\right)^{-1} \cdot rq^{6r}.$$

In particular, it has order 6 and degree 1.

If $n \geqslant 5$, then $\mu_{1\,\max} > (2n-2)$ and hence $\mathfrak{i}_{1\,\max} = \{t\}$. By Lemma 10.6.1.(i), we have

$$\mathrm{S}_{\mathcal{V}_\dagger(I)}(r) \sim \prod_{i=2}^{t-1} \left(q^{(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1} \cdot \frac{1 + q^{\frac{n(n-1)}{2} - (2n-2)}}{q^{n(n-1)-2(2n-2)} - 1} \cdot q^{\frac{n(n-1)}{2} r}.$$

In particular, it has order $\frac{n(n-1)}{2}$ and degree 0.

(vi). Suppose $1 \notin I$ and $\{n-1, n\} \cap I = \emptyset$. By Eqs. (6.6.7), (6.6.8), and (9.1.4), we have

$$\mathrm{S}_{\mathcal{V}_\dagger(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ c_1 + 2c_2 + \cdots + 2c_{t-2} + c_{t-1} + c_t = r}} q^{\sum_{i=1}^{t-2} \ell_i(2n-1-\ell_i)c_i + \frac{n(n-1)}{2}(c_{t-1} + c_t)}.$$

Now, we apply Lemma 10.6.1 to this summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$, the partition $\mathfrak{i} = \mathfrak{i}_1 \sqcup \mathfrak{i}_2$ is $\{1, \cdots, t\} = \{1, t-1, t\} \sqcup \{2, \cdots, t-2\}$, and the sequence $\boldsymbol{\mu}$ is

$$\begin{aligned} \mu_i &= \ell_i(2n-1-\ell_i), && (1 \leqslant i \leqslant t-2) \\ \mu_{t-1} &= \tfrac{n(n-1)}{2}, \\ \mu_t &= \tfrac{n(n-1)}{2}. \end{aligned}$$

Since all members of $\boldsymbol{\mu}$ are integers, $\mathrm{S}_{\mathcal{V}_\dagger(I)}$ can be defined by a primary super q-exponential polynomial. The knowledge of quadratic function shows that $\mathfrak{i}_{2\,\max} = \{t-2\}$ with $\mu_{2\,\max} = \ell_{t-2}(2n-1-\ell_{t-2})$ and that $\{t-1, t\} \subseteq \mathfrak{i}_{1\,\max}$ with $2\mu_{1\,\max} =$

$n(n-1) > \mu_{2\max}$.

Depending on n, there are two possibilities.

If $n = 4$, then $\mu_{1\max} = (2n-2)$ and hence $\mathfrak{i}_{1\max} = \{1, t-1, t\}$. By Lemma 10.6.1.(i), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) \sim \prod_{i=2}^{t-2} \left(q^{(4-\ell_i)(3-\ell_i)} - 1\right)^{-1} \cdot \binom{r}{2} q^{6r}. \tag{14.1.2}$$

In particular, it has order 6 and degree 2.

If $n \geqslant 5$, then $\mu_{1\max} > (2n-2)$ and hence $\mathfrak{i}_{1\max} = \{t-1, t\}$. By Lemma 10.6.1.(i), we have

$$\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r) \sim \prod_{i=2}^{t-2} \left(q^{(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1} \cdot \frac{1 + q^{\frac{n(n-1)}{2} - (2n-2)}}{q^{n(n-1)-2(2n-2)} - 1} \cdot r q^{\frac{n(n-1)}{2} r}. \tag{14.1.3}$$

In particular, it has order $\frac{n(n-1)}{2}$ and degree 1.

Note that, in all cases, $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}$ can be defined by a primary super q-exponential polynomial. Then by Eqs. (8.5.1) and (8.5.2), we see that $\mathrm{SV}_{\dagger}(\,\cdot\,)$ and $\mathrm{SSA}_{\dagger}(\,\cdot\,)$ can be defined by primary super q-exponential polynomials.

§ 14.2. Dominant types for $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r)$

Now, we are able to figure out for which type I, $\mathrm{S}_{\mathcal{V}_{\dagger}(I)}(r)$ is dominant. First, we summarize the asymptotic results in § 14.1 as follows.

	$\{n-1,n\} \subseteq I$	$\lvert\{n-1,n\} \cap I\rvert = 1$	$\{n-1,n\} \cap I = \emptyset$
$1 \in I$	$\left(\frac{1}{2}\ell_t(2n-1-\ell_t), 0\right)$	$\left(\frac{n(n-1)}{2}, 0\right)$	$\left(\frac{n(n-1)}{2}, 1\right)$
$1 \notin I$	$(2n-2, 0)$ $\left(\frac{1}{2}\ell_t(2n-1-\ell_t), 0\right)$ $(2n-2, 1)$	$(6,1)$ $\left(\frac{n(n-1)}{2}, 0\right)$	$(6,2)$ $\left(\frac{n(n-1)}{2}, 1\right)$

In the table, the pair in each cell tells us the possible order and degree of $\mathrm{S}_{\mathcal{V}_\dagger(I)}$.

When $n = 4$, we have $\ell_t(I) \leqslant 4$ for all I. Therefore,

$$(2n-2) \geqslant \tfrac{1}{2}\ell_t(I)(2n-1-\ell_t(I)).$$

Hence, $\mathrm{S}_{\mathcal{V}_\dagger(I)}(r)$ is dominant exactly when $1 \notin I$ and $\{n-1,n\} \cap I = \emptyset$. Note that, such a type I must be either $\{2\}$ or $\emptyset$. By Eq. (14.1.2), the asymptotic growth of dominant $\mathrm{S}_{\mathcal{V}_\dagger(I)}(r)$ are as follows:

$$\mathrm{S}_{\mathcal{V}_\dagger(\{2\})}(r) \sim \binom{r}{2} q^{6r} \tag{14.2.1}$$

$$\mathrm{S}_{\mathcal{V}_\dagger(\emptyset)}(r) \sim \left(q^{(4-\ell_2)(3-\ell_2)} - 1\right)^{-1} \cdot \binom{r}{2} q^{6r} = \frac{1}{q^2-1} \cdot \binom{r}{2} q^{6r} \tag{14.2.2}$$

If $n \geqslant 4$, then we have $(2n-2)$ is no longer the highest order. When $\{n-1,n\} \subseteq I$, we have $\ell_t(I) < n-1$ and thus

$$\tfrac{1}{2}\ell_t(I)(2n-1-\ell_t(I)) < \tfrac{n(n-1)}{2}.$$

Therefore, $\mathrm{S}_{\mathcal{V}_\dagger(I)}(r)$ is dominant exactly when $\{n-1,n\} \cap I = \emptyset$. In that case, its asymptotic growth is given by Eqs. (14.1.1) and (14.1.3).

§ 14.3. Asymptotic growths of $\mathrm{SSA}_\dagger(r)$ and $\mathrm{SV}_\dagger(r)$

We are now able to obtain the asymptotic growth of $\mathrm{SSA}_\dagger(r)$. By Eq. (8.5.2), we have

$$\mathrm{SSA}_\dagger(r) = \sum_{I\subseteq\Delta} \frac{\mathscr{P}_{D_n;I}(q)}{q^{\deg\left(\mathscr{P}_{D_n;I}\right)}} \mathrm{S}_{\mathcal{V}_\dagger(I)}(r) \sim \sum_{I \text{ is dominant}} \frac{\mathscr{P}_{D_n;I}(q)}{q^{\deg\left(\mathscr{P}_{D_n;I}\right)}} \mathrm{S}_{\mathcal{V}_\dagger(I)}(r). \tag{14.3.1}$$

Then by the discussion in § 14.2, we see that

$$\mathrm{SSA}_\dagger(r) \sim \begin{cases} C_\dagger(4)\cdot\binom{r}{2}q^{6r} & \text{if } n=4,\\ C_\dagger(n)\cdot rq^{\frac{n(n-1)}{2}r} & \text{if } n\geqslant 5.\end{cases} \tag{14.3.2}$$

When $n=4$, by Eqs. (14.2.1) and (14.2.2), the constant $C_\dagger(4)$ is defined as follows:

$$C_\dagger(4) = \frac{\mathscr{P}_{D_4;\{2\}}(q)}{q^{\deg\left(\mathscr{P}_{D_4;\{2\}}\right)}} + \frac{\mathscr{P}_{D_4;\emptyset}(q)}{\left(q^2-1\right)q^{\deg\left(\mathscr{P}_{D_4;\emptyset}\right)}}.$$

Moreover, by Eq. (A.3.2), we have

$$\begin{aligned} C_\dagger(4) &= \frac{\left(q^6-1\right)\left(q^4-1\right)^2}{(q-1)^3q^{11}} + \frac{\left(q^6-1\right)\left(q^4-1\right)^2}{(q-1)^4q^{12}}\\ &= \frac{\left(q^2+q+1\right)\left(q^2-q+1\right)^2\left(q^2+1\right)^2(q+1)^3}{(q-1)q^{12}}. \end{aligned} \tag{14.3.3}$$

As a consequence, we have

$$\mathrm{SV}_\dagger(r) = \sum_{z=0}^{r}\mathrm{SSA}_\dagger(z) \sim \frac{q^6}{q^6-1}C_\dagger(4)\cdot\binom{r}{2}q^{6r}. \tag{14.3.4}$$

When $n \geqslant 5$, by Eqs. (14.1.1) and (14.1.3), the constant $C_{\dagger}(n)$ is defined as follows:

(14.3.5)

$$C_{\dagger}(n) := \sum_{1,n-1,n\notin I} \frac{\mathscr{P}_{D_n;I}(q)}{q^{\deg(\mathscr{P}_{D_n;I})}} \prod_{i=2}^{t-2} \left(q^{(n-\ell_i(I))(n-1-\ell_i(I))} - 1\right)^{-1} \cdot \frac{1+q^{\frac{n(n-1)}{2}-(2n-2)}}{q^{n(n-1)-2(2n-2)}-1} + \sum_{1\in I,n-1,n\notin I} \frac{\mathscr{P}_{D_n;I}(q)}{q^{\deg(\mathscr{P}_{D_n;I})}} \prod_{i=1}^{t-2} \left(q^{(n-\ell_i(I))(n-1-\ell_i(I))} - 1\right)^{-1}.$$

As a consequence, we have

$$\mathrm{SV}_{\dagger}(r) = \sum_{z=0}^{r} \mathrm{SSA}_{\dagger}(z) \sim \frac{q^{\frac{n(n-1)}{2}}}{q^{\frac{n(n-1)}{2}}-1} C_{\dagger}(n) \cdot r q^{\frac{n(n-1)}{2}r}. \tag{14.3.6}$$

By Eqs. (14.3.2) to (14.3.6), we have proved Theorem 14.2. Moreover, by Eq. (A.3.3), we have the following explicit formulas:

$$\mathscr{P}_{D_n;I}(q) = \frac{[2(n-1)]!!(z) \cdot [n](z)}{\prod_{i=1}^{t-1} [\ell_i(I) - \ell_{i-1}(I)]!(z)}, \qquad q^{\deg(\mathscr{P}_{D_n;I})} = \frac{q^{n(n-1)}}{\prod_{i=1}^{t-1} q^{\binom{\ell_i(I)-\ell_{i-1}(I)}{2}}}.$$

See Lemma 8.2.5 and Eqs. (A.1.2) and (A.2.1) for the definitions of the symbols $[\,\cdot\,]$, $[\,\cdot\,]!$, and $[2\cdot\,]!!$.

§ 14.4. Asymptotic growths of $\mathrm{S}^{\asymp}_{\mathcal{X}^{\square\heartsuit}(I)}(r)$

Now, let I be a type and follow Convention 2.4.5. We are going to estimate the asymptotic growth of $\mathrm{S}_{\mathcal{X}^{\square\heartsuit}(I)}(r)$ ($\square, \heartsuit$ being 0 or 1) up to the leading coefficient. We will separate the discussion into the following six cases:

	$\{n-1,n\}\subseteq I$	$\lvert\{n-1,n\}\cap I\rvert=1$	$\{n-1,n\}\cap I=\emptyset$
$1\in I$	(i)	(ii)	(iii)
$1\notin I$	(iv)	(v)	(vi)

(i). Suppose $1\in I$ and $\{n-1,n\}\subseteq I$. By Fig. 9.4, we only need to consider $\mathcal{X}^{00}(I)$. By Eqs. (6.6.8) and (9.5.3), we have

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r)=\sum_{\substack{c_i\in\mathbb{Z}_{>0}\\ c_1+\cdots+c_t=r}} q^{\sum_{i=1}^{t}\frac{1}{2}\ell_i(2n-1-\ell_i)c_i}.$$

Now, we apply Lemma 10.5.2 to above summation, where the index set $\mathfrak{i}$ is $\{1,\cdots,t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\mu_i=\tfrac{1}{2}\ell_i(2n-1-\ell_i). \qquad (1\leqslant i\leqslant t)$$

The knowledge of quadratic function shows that $\mathfrak{i}_{\max}=\{t\}$ with $\mu_{\max}=\frac{1}{2}\ell_t(2n-1-\ell_t)$. Then we have

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r)\sim\prod_{i=1}^{t-1}\left(q^{\frac{1}{2}(\ell_t-\ell_i)(2n-1-\ell_t-\ell_i)}-1\right)^{-1}\cdot q^{\frac{1}{2}\ell_t(2n-1-\ell_t)r}.$$

Since $\mathrm{S}_{\mathcal{X}^{00}(I)}(r)\asymp\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r)$, it has order $\frac{1}{2}\ell_t(2n-1-\ell_t)$ and degree 0.

(ii). Suppose $1\in I$ and $\{n-1,n\}\cap I$ is a singleton. By Fig. 9.4, we only need to consider $\mathcal{X}^{00}(I)$. By Eqs. (6.6.8) and (9.5.3), we have

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r)=\sum_{\substack{c_i\in\mathbb{Z}_{>0}\\ c_1+\cdots+c_t=r}} q^{\sum_{i=1}^{t-1}\frac{1}{2}\ell_i(2n-1-\ell_i)c_i+\frac{n(n-1)}{2}c_t}.$$

Now, we apply Lemma 10.5.2 to above summation, where the index set $\mathfrak{i}$ is $\{1,\cdots,t\}$

and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \tfrac{1}{2}\ell_i(2n-1-\ell_i), \qquad (1 \leqslant i \leqslant t-1)$$
$$\mu_t = \tfrac{n(n-1)}{2}.$$

The knowledge of quadratic function shows that $\mathfrak{i}_{\max} = \{t\}$ with $\mu_{\max} = \frac{n(n-1)}{2}$. Then we have

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \sim \prod_{i=1}^{t-1}\left(q^{\frac{1}{2}(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1} \cdot q^{\frac{n(n-1)}{2}r}.$$

Since $\mathrm{S}_{\mathcal{X}^{00}(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r)$, it has order $\frac{n(n-1)}{2}$ and degree 0.

(iii). Suppose $1 \in I$ and $\{n-1, n\} \cap I = \emptyset$. By Fig. 9.6, we only need to consider $\mathcal{X}^{00}(I)$ and $\mathcal{X}^{01}(I)$. By Eqs. (6.6.8), (9.5.3), and (9.5.4), we have

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ c_1+\cdots+c_t=r}} q^{\sum_{i=1}^{t-2}\frac{1}{2}\ell_i(2n-1-\ell_i)c_i + \frac{n(n-1)}{2}(c_{t-1}+c_t)},$$
$$\mathrm{S}^{\asymp}_{\mathcal{X}^{01}(I)}(r) = q^{-\frac{n(n-1)}{2}}\,\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r).$$

Now, we apply Lemma 10.5.2 to the first summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\mu_i = \tfrac{1}{2}\ell_i(2n-1-\ell_i), \qquad (1 \leqslant i \leqslant t-2)$$
$$\mu_{t-1} = \tfrac{n(n-1)}{2},$$
$$\mu_t = \tfrac{n(n-1)}{2}.$$

The knowledge of quadratic function shows that $\mathfrak{i}_{\max} = \{t-1, t\}$ with $\mu_{\max} = \frac{n(n-1)}{2}$.

Then we have

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \sim \prod_{i=1}^{t-2}\left(q^{\frac{1}{2}(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1} \cdot rq^{\frac{n(n-1)}{2}r}.$$

In particular, it has order $\frac{n(n-1)}{2}$ and degree 1. We then know that $\mathrm{S}^{\asymp}_{\mathcal{X}^{01}(I)}(r)$ also has the same order and degree. Since $\mathrm{S}_{\mathcal{X}^{00}(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r)$ and $\mathrm{S}_{\mathcal{X}^{01}(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{X}^{01}(I)}(r)$, we see that $\mathrm{S}_{(\mathcal{X}^{00}\cup\mathcal{X}^{01})(I)}(r)$ has order $\frac{n(n-1)}{2}$ and degree 1.

(iv). Suppose $1 \notin I$ and $\{n-1, n\} \subseteq I$. By Fig. 9.5, we only need to consider $\mathcal{X}^{00}(I)$ and $\mathcal{X}^{10}(I)$. By Eqs. (6.6.8), (9.5.3), and (9.5.5), we have

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) = \sum_{\substack{c_i\in\mathbb{Z}_{>0}\\ c_1+\cdots+c_t=r}} q^{(2n-2)c_1+\sum_{i=2}^{t}\frac{1}{2}\ell_i(2n-1-\ell_i)c_i},$$

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{10}(I)}(r) = q^{-\frac{1}{2}(2n-2)}\,\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r).$$

Now, we apply Lemma 10.5.2 to the first summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\mu_1 = 2n-2,$$
$$\mu_i = \tfrac{1}{2}\ell_i(2n-1-\ell_i). \qquad (2 \leqslant i \leqslant t)$$

The knowledge of quadratic function shows that $\mathfrak{i}_{\max} \subseteq \{1, t\}$ with

$$\mu_{\max} = \max\left\{2n-2, \tfrac{1}{2}\ell_t(2n-1-\ell_t)\right\}.$$

Depending on n and ℓ_t, there are three possibilities.

If $2n-2 > \frac{1}{2}\ell_t(2n-1-\ell_t)$, then we have $\mathfrak{i}_{\max} = \{1\}$, $\mu_{\max} = 2n-2$, and

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \sim \prod_{i=2}^{t}\left(q^{(2n-2)-\frac{1}{2}\ell_i(2n-1-\ell_i)} - 1\right)^{-1} \cdot q^{(2n-2)r}.$$

Then we can deduce that $\mathrm{S}_{(\mathcal{X}^{00}\cup\mathcal{X}^{10})(I)}(r)$ has order $2n-2$ and degree 0.

If $2n-2 < \frac{1}{2}\ell_t(2n-1-\ell_t)$, then we have $\mathfrak{i}_{\max} = \{t\}$, $\mu_{\max} = \frac{1}{2}\ell_t(2n-1-\ell_t)$, and

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \sim \left(q^{\frac{1}{2}\ell_t(2n-1-\ell_t)-(2n-2)} - 1\right)^{-1} \prod_{i=2}^{t-1} \left(q^{\frac{1}{2}(\ell_t-\ell_i)(2n-1-\ell_t-\ell_i)} - 1\right)^{-1} \cdot q^{\frac{1}{2}\ell_t(2n-1-\ell_t)r}.$$

Then we can deduce that $\mathrm{S}_{(\mathcal{X}^{00}\cup\mathcal{X}^{10})(I)}(r)$ has order $\frac{1}{2}\ell_t(2n-1-\ell_t)$ and degree 0.

If $2n-2 = \frac{1}{2}\ell_t(2n-1-\ell_t)$, then we have $\mathfrak{i}_{\max} = \{1,t\}$ and

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \sim \prod_{i=2}^{t-1} \left(q^{\frac{1}{2}(\ell_t-\ell_i)(2n-1-\ell_t-\ell_i)} - 1\right)^{-1} \cdot rq^{(2n-2)r}.$$

Then we can deduce that $\mathrm{S}_{(\mathcal{X}^{00}\cup\mathcal{X}^{10})(I)}(r)$ has order $2n-2$ and degree 1.

(v). Suppose $1 \notin I$ and $\{n-1,n\}\cap I$ is a singleton. By Fig. 9.5, we only need to consider $\mathcal{X}^{00}(I)$ and $\mathcal{X}^{10}(I)$. By Eqs. (6.6.8), (9.5.3), and (9.5.5), we have

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) = \sum_{\substack{c_i\in\mathbb{Z}_{>0}\\ c_1+\cdots+c_t=r}} q^{(2n-2)c_1+\sum_{i=2}^{t-1}\frac{1}{2}\ell_i(2n-1-\ell_i)c_i+\frac{n(n-1)}{2}c_t},$$

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{10}(I)}(r) = q^{-\frac{1}{2}(2n-2)}\,\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r).$$

Now, we apply Lemma 10.5.2 to the first summation, where the index set $\mathfrak{i}$ is $\{1,\cdots,t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\begin{aligned} \mu_1 &= 2n-2, \\ \mu_i &= \tfrac{1}{2}\ell_i(2n-1-\ell_i), && (2\leqslant i\leqslant t-1) \\ \mu_t &= \tfrac{n(n-1)}{2}. \end{aligned}$$

The knowledge of quadratic function shows that $t\in\mathfrak{i}_{\max}\subseteq\{1,t\}$ with $\mu_{\max} = \frac{n(n-1)}{2}$.

Depending on n, there are two possibilities.

If $n = 4$, then we have $\mathfrak{i}_{\max} = \{1, t\}$ and

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \sim \prod_{i=2}^{t-1} \left(q^{\frac{1}{2}(4-\ell_i)(3-\ell_i)} - 1 \right)^{-1} \cdot rq^{6r}.$$

Then we can deduce that $\mathrm{S}_{(\mathcal{X}^{00} \cup \mathcal{X}^{10})(I)}(r)$ has order 6 and degree 1.

If $n \geqslant 5$, then we have $\mathfrak{i}_{\max} = \{t\}$ and

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \sim \left(q^{\frac{n(n-1)}{2} - (2n-2)} - 1 \right)^{-1} \prod_{i=2}^{t-1} \left(q^{\frac{1}{2}(n-\ell_i)(n-1-\ell_i)} - 1 \right)^{-1} \cdot q^{\frac{n(n-1)}{2} r}.$$

Then we can deduce that $\mathrm{S}_{(\mathcal{X}^{00} \cup \mathcal{X}^{10})(I)}(r)$ has order $\frac{n(n-1)}{2}$ and degree 0.

(vi). Suppose $1 \notin I$ and $\{n-1, n\} \cap I = \emptyset$. By Fig. 9.7, we have to consider all the sets $\mathcal{X}^{00}(I)$, $\mathcal{X}^{01}(I)$, $\mathcal{X}^{10}(I)$, and $\mathcal{X}^{11}(I)$. By Eqs. (6.6.8) and (9.5.3) to (9.5.6), we have

$$\begin{aligned}
\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) &= \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ c_1 + \cdots + c_t = r}} q^{(2n-2)c_1 + \sum_{i=2}^{t-2} \frac{1}{2}\ell_i(2n-1-\ell_i)c_i + \frac{n(n-1)}{2}(c_{t-1}+c_t)}, \\
\mathrm{S}^{\asymp}_{\mathcal{X}^{01}(I)}(r) &= q^{-\frac{n(n-1)}{2}} \mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r), \\
\mathrm{S}^{\asymp}_{\mathcal{X}^{10}(I)}(r) &= q^{-\frac{1}{2}(2n-2)} \mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r), \\
\mathrm{S}^{\asymp}_{\mathcal{X}^{11}(I)}(r) &= q^{-\frac{1}{2}(2n-2) - \frac{n(n-1)}{2}} \mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \asymp \mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r).
\end{aligned}$$

Now, we apply Lemma 10.5.2 to the first summation, where the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\begin{aligned}
\mu_1 &= 2n-2, \\
\mu_i &= \tfrac{1}{2}\ell_i(2n-1-\ell_i), && (2 \leqslant i \leqslant t-2) \\
\mu_{t-1} &= \tfrac{n(n-1)}{2}, \\
\mu_t &= \tfrac{n(n-1)}{2}.
\end{aligned}$$

The knowledge of quadratic function shows that $\{t-1,t\} \subseteq \mathfrak{i}_{\max} \subseteq \{1,t-1,t\}$ with $\mu_{\max} = \frac{n(n-1)}{2}$.

Depending on n, there are two possibilities.

If $n = 4$, then we have $\mathfrak{i}_{\max} = \{1,t-1,t\}$ and

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \sim \prod_{i=2}^{t-2} \left(q^{\frac{1}{2}(4-\ell_i)(3-\ell_i)} - 1\right)^{-1} \cdot \binom{r}{2} q^{6r}.$$

Then we can deduce that $\mathrm{S}_{(\mathcal{X}^{00} \cup \mathcal{X}^{01} \cup \mathcal{X}^{10} \cup \mathcal{X}^{11})(I)}(r)$ has order 6 and degree 2.

If $n \geqslant 5$, then we have $\mathfrak{i}_{\max} = \{t-1,t\}$ and

$$\mathrm{S}^{\asymp}_{\mathcal{X}^{00}(I)}(r) \sim \left(q^{\frac{n(n-1)}{2}-(2n-2)} - 1\right)^{-1} \prod_{i=2}^{t-2} \left(q^{\frac{1}{2}(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1} \cdot rq^{\frac{n(n-1)}{2}r}.$$

Then we can deduce that $\mathrm{S}_{(\mathcal{X}^{00} \cup \mathcal{X}^{01} \cup \mathcal{X}^{10} \cup \mathcal{X}^{11})(I)}(r)$ has order $\frac{n(n-1)}{2}$ and degree 1.

§ 14.5. Dominant types for $\mathrm{S}_{\mathcal{V}(I)}(r)$

We are going to estimate the asymptotic growth of each $\mathrm{S}_{\mathcal{V}(I)}(r)$ and figure out the *dominant types*, namely the types for which $\mathrm{S}_{\mathcal{V}(I)}(r)$ is dominant.

Let I be a type and follow Convention 2.4.5. Depending on I, the set $\mathcal{V}(I)$ is contained in various sets $\mathcal{X}^{\cup}(I)$, where

$$\mathcal{X}^{\cup}(I) = \begin{cases} \mathcal{X}^{00}(I) & \text{if } 1 \in I \text{ and } \{n-1,n\} \cap I \neq \emptyset, \\ \mathcal{X}^{00}(I) \cup \mathcal{X}^{01}(I) & \text{if } 1 \in I \text{ and } \{n-1,n\} \cap I = \emptyset, \\ \mathcal{X}^{00}(I) \cup \mathcal{X}^{10}(I) & \text{if } 1 \notin I \text{ and } \{n-1,n\} \cap I \neq \emptyset, \\ \mathcal{X}^{00}(I) \cup \mathcal{X}^{01}(I) \cup \mathcal{X}^{10}(I) \cup \mathcal{X}^{11}(I) & \text{if } 1 \notin I \text{ and } \{n-1,n\} \cap I = \emptyset. \end{cases}$$

Refer to Figs. 9.4 to 9.7. Then we have

$$S_{\mathcal{X}^{\cup}(I)}(r) \gg S_{\mathcal{V}(I)}(r) \gg S_{\mathcal{V}_{\dagger}(I)}(r).$$

We summarize § 14.4 as follows.

	$\{n-1,n\} \subseteq I$	$\lvert\{n-1,n\} \cap I\rvert = 1$	$\{n-1,n\} \cap I = \emptyset$
$1 \in I$	$\left(\frac{1}{2}\ell_t(2n-1-\ell_t), 0\right)$	$\left(\frac{n(n-1)}{2}, 0\right)$	$\left(\frac{n(n-1)}{2}, 1\right)$
$1 \notin I$	$(2n-2, 0)$ $\left(\frac{1}{2}\ell_t(2n-1-\ell_t), 0\right)$ $(2n-2, 1)$	$(6, 1)$ $\left(\frac{n(n-1)}{2}, 0\right)$	$(6, 2)$ $\left(\frac{n(n-1)}{2}, 1\right)$

In the table, the pair in each cell tells us the possible order and degree of $S_{\mathcal{X}^{\cup}(I)}(r)$.

Comparing this table with the discussion in § 14.2, we see the followings.

(i). When $n = 4$, a type I is dominant if and only if $\{1, n-1, n\} \cap I = \emptyset$. In that case, $S_{\mathcal{V}(I)}$ has order 6 and degree 2.

(ii). When $n \geqslant 5$, a type I is dominant if and only if $\{n-1, n\} \cap I = \emptyset$. In that case, $S_{\mathcal{V}(I)}$ has order $\frac{n(n-1)}{2}$ and degree 1.

§ 14.6. Asymptotic growth of dominant $S_{\mathcal{X}^{\square\heartsuit}(I)}(r)$

Now, let I be a type and follow Convention 2.4.5. We are going to compute the asymptotic growths of $S_{\mathcal{X}^{0}(I)}(r)$ and $S_{\mathcal{X}^{1}(I)}(r)$ when I is dominant. To do this, we pick an arbitrary $x \in \mathcal{X}^{\square\heartsuit}(I)$ and investigate the difference between $2\rho(x)$ and the sum of $\lceil a(x) \rceil$ for $a(x) > 0$. To better describe these sums, we follow Conventions 12.5.1 to 12.5.3.

By § 14.5, a necessary condition for I being dominant is $\{n-1, n\} \cap I = \emptyset$. We will assume that I satisfies this condition. Depending on $1 \in I$ or not, we will separate the discussion into two cases: (i) and (ii).

(i). Suppose $1 \in I$. By Fig. 9.6, we only need to consider $\mathcal{X}^{00}(I)$ and $\mathcal{X}^{01}(I)$. Let $\heartsuit$ be either 0 or 1. Suppose

$$\begin{aligned} x = o + c_1 \cdot \tfrac{1}{2}\omega_{\ell_1} + \cdots + c_{t-2} \cdot \tfrac{1}{2}\omega_{\ell_{t-2}} \\ + (c_{t-1} - \tfrac{1}{2} \cdot \heartsuit) \cdot \omega_{n-1} + (c_t - \tfrac{1}{2} \cdot \heartsuit) \cdot \omega_n \in \mathcal{X}^{0\heartsuit}(I). \end{aligned}$$

By Eq. (6.6.6), we have

$$\begin{aligned} &(\chi_j - \chi_{j'})(x) && (1 \leqslant j < j' \leqslant n-1) \\ &\quad = \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}\right), \\ &(\chi_j + \chi_{j'})(x) && (1 \leqslant j < j' \leqslant n-1) \\ &\quad = \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}\right) + c_{\ell^{-1}(j')} + \cdots + c_t - \heartsuit, \\ &(\chi_j - \chi_n)(x) && (1 \leqslant j \leqslant n-1) \\ &\quad = \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{t-2}\right) + c_{t-1} - \tfrac{1}{2} \cdot \heartsuit, \\ &(\chi_j + \chi_n)(x) && (1 \leqslant j \leqslant n-1) \\ &\quad = \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{t-2}\right) + c_t - \tfrac{1}{2} \cdot \heartsuit. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \sum_{a \in \Phi^+} \lceil a(x) \rceil &= \sum_{1 \leqslant j < j' \leqslant n} \left(\lceil (\chi_j - \chi_{j'})(x) \rceil + \lceil (\chi_j + \chi_{j'})(x) \rceil\right) \\ &= 2\rho(x) + \sum_{1 \leqslant j < j' \leqslant n-1} \overline{c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}} + \sum_{j=1}^{n-1} \overline{c_{\ell^{-1}(j)} + \cdots + c_{t-2} - \heartsuit}. \end{aligned}$$

From above analysis, we can define the parity functions $e_{\chi^{0\heartsuit}(I)}$ ($\heartsuit = 0, 1$) as follows:

$$e_{\chi^{0\heartsuit}(I)}(c_1, \cdots, c_t) := \sum_{1 \leqslant i < i' \leqslant t-1} (\ell_i - \ell_{i-1})(\ell_{i'} - \ell_{i'-1})\overline{c_i + \cdots + c_{i'-1}} + \sum_{i=1}^{t-1} (\ell_i - \ell_{i-1})\overline{c_i + \cdots + c_{t-2} - \heartsuit}. \tag{14.6.1}$$

Then we have

$$\sum_{a \in \Phi^+} \lceil a(x) \rceil = 2\rho(x) + e_{\chi^{0\heartsuit}(I)}(c_1, \cdots, c_t).$$

Now, we apply Lemma 10.5.5 to the following summation ($\heartsuit = 0, 1$).

$$\mathrm{S}_{\chi^{0\heartsuit}(I)}(r) = \sum_{\substack{c_i \in \mathbb{Z}_{>0} \\ c_1 + \cdots + c_t = r + \heartsuit}} q^{\sum_{i=1}^{t-2} \frac{1}{2}\ell_i(2n-1-\ell_i)c_i + \frac{n(n-1)}{2}(c_{t-1} + c_t - \heartsuit) + e_{\chi^{0\heartsuit}(I)}(c_1, \cdots, c_t)}.$$

Note that the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\begin{aligned} \mu_i &= \tfrac{1}{2}\ell_i(2n-1-\ell_i), && (1 \leqslant i \leqslant t-2) \\ \mu_{t-1} &= \tfrac{n(n-1)}{2}, \\ \mu_t &= \tfrac{n(n-1)}{2}. \end{aligned}$$

Since all members of $\boldsymbol{\mu}$ are integers, $\mathrm{S}_{\chi^{0\heartsuit}(I)}(r)$ can be defined by a primary super q-exponential polynomial. The knowledge of quadratic function shows that $\mathfrak{i}_{\max} = \{t-1, t\}$ with $\mu_{\max} = \frac{n(n-1)}{2}$. Therefore, for $\heartsuit = 0, 1$, we have

$$\mathrm{S}_{\chi^{0\heartsuit}(I)}(r) \sim C_{\chi^{0\heartsuit}(I)} \cdot \left(\left(\sum_{\mathbf{s} \in \mathbb{F}_2^t} \mathrm{E}_{\chi^{0\heartsuit}(I)}(\mathbf{s}) \right) + \left(\sum_{\mathbf{s} \in \mathbb{F}_2^t} (-1)^{\mathbf{1} \cdot \mathbf{s}}\, \mathrm{E}_{\chi^{0\heartsuit}(I)}(\mathbf{s}) \right) (-1)^{r+\heartsuit} \right) \cdot rq^{\frac{n(n-1)}{2} r},$$

where the constant $C_{\chi^{0\heartsuit}(I)}$ and the function $\mathrm{E}_{\chi^{0\heartsuit}(I)} \colon \mathbb{F}_2^t \to \mathbb{Q}(q; -)$ are defined as follows:

$$C_{\mathcal{X}^{0\heartsuit}(I)} := \frac{1}{4}\prod_{i=1}^{t-2}\left(q^{(n-\ell_i)(n-1-\ell_i)}-1\right)^{-1}, \tag{14.6.2}$$

$$\mathrm{E}_{\mathcal{X}^{0\heartsuit}(I)}(\mathbf{s}) := q^{e_{\mathcal{X}^{0\heartsuit}(I)}(\mathbf{s})+\sum_{i=1}^{t-2}\frac{1}{2}(n-\ell_i)(n-1-\ell_i)s_i}. \tag{14.6.3}$$

From the definition Eq. (14.6.1) of $e_{\mathcal{X}^{0\heartsuit}(I)}$, we see that $\mathrm{E}_{\mathcal{X}^{0\heartsuit}(I)}(s_1,\cdots,s_t)$ does not depend on s_{t-1} and s_t. Therefore, we have

$$\mathrm{S}_{\mathcal{X}^{0\heartsuit}(I)}(r) \sim C_{\mathcal{X}^{0\heartsuit}(I)}\cdot\left(\sum_{\mathbf{s}\in\mathbb{F}_2^t}\mathrm{E}_{\mathcal{X}^{0\heartsuit}(I)}(\mathbf{s})\right)\cdot rq^{\frac{n(n-1)}{2}r}. \tag{14.6.4}$$

(ii). Suppose $1\in I$. By Fig. 9.7, we have to consider all the sets $\mathcal{X}^{00}(I)$, $\mathcal{X}^{01}(I)$, $\mathcal{X}^{10}(I)$, and $\mathcal{X}^{11}(I)$. Let $\square,\heartsuit$ be either 0 or 1. Suppose

$$\begin{aligned}x = o &+ (c_1-\tfrac{1}{2}\cdot\square)\cdot\omega_1 + c_2\cdot\tfrac{1}{2}\omega_{\ell_2}+\cdots+c_{t-2}\cdot\tfrac{1}{2}\omega_{\ell_{t-2}}\\ &+ (c_{t-1}-\tfrac{1}{2}\cdot\heartsuit)\cdot\omega_{n-1}+(c_t-\tfrac{1}{2}\cdot\heartsuit)\cdot\omega_n \in \mathcal{X}^{\square\heartsuit}(I).\end{aligned}$$

By Eq. (6.6.6), we have

$$
\begin{aligned}
&(\chi_1 - \chi_n)(x) \\
&\qquad = (c_1 - \tfrac{1}{2}\cdot\square) + \tfrac{1}{2}\,(c_2 + \cdots + c_{t-2}) + c_{t-1} - \tfrac{1}{2}\cdot\heartsuit, \\
&(\chi_1 + \chi_n)(x) \\
&\qquad = (c_1 - \tfrac{1}{2}\cdot\square) + \tfrac{1}{2}\,(c_2 + \cdots + c_{t-2}) + c_t - \tfrac{1}{2}\cdot\heartsuit, \\
&(\chi_1 - \chi_j)(x) && (1 < j \leqslant n-1) \\
&\qquad = (c_1 - \tfrac{1}{2}\cdot\square) + \tfrac{1}{2}\left(c_2 + \cdots + c_{\ell^{-1}(j)-1}\right), \\
&(\chi_1 + \chi_j)(x) && (1 < j \leqslant n-1) \\
&\qquad = (c_1 - \tfrac{1}{2}\cdot\square) + \tfrac{1}{2}\left(c_2 + \cdots + c_{\ell^{-1}(j)-1}\right) + c_{\ell^{-1}(j)} + \cdots + c_t - \heartsuit, \\
&(\chi_j - \chi_n)(x) && (1 < j \leqslant n-1) \\
&\qquad = \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{t-2}\right) + c_{t-1} - \tfrac{1}{2}\cdot\heartsuit, \\
&(\chi_j + \chi_n)(x) && (1 < j \leqslant n-1) \\
&\qquad = \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{t-2}\right) + c_t - \tfrac{1}{2}\cdot\heartsuit, \\
&(\chi_j - \chi_{j'})(x) && (1 < j < j' \leqslant n-1) \\
&\qquad = \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}\right), \\
&(\chi_j + \chi_{j'})(x) && (1 < j < j' \leqslant n-1) \\
&\qquad = \tfrac{1}{2}\left(c_{\ell^{-1}(j)} + \cdots + c_{\ell^{-1}(j')-1}\right) + c_{\ell^{-1}(j')} + \cdots + c_t - \heartsuit.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}\sum_{a\in\Phi^+}\lceil a(x)\rceil &= \sum_{1\leqslant j<j'\leqslant n}\big(\lceil(\chi_j-\chi_{j'})(x)\rceil+\lceil(\chi_j+\chi_{j'})(x)\rceil\big)\\ &= 2\rho(x)+\overline{c_2+\cdots+c_{t-2}-\square-\heartsuit}+\sum_{j=2}^{n-1}\overline{c_2+\cdots+c_{\ell^{-1}(j)-1}-\square}\\ &\quad+\sum_{j=2}^{n-1}\overline{c_{\ell^{-1}(j)}+\cdots+c_{t-2}-\heartsuit}+\sum_{1<j<j'\leqslant n-1}\overline{c_{\ell^{-1}(j)}+\cdots+c_{\ell^{-1}(j')-1}}.\end{aligned}$$

From above analysis, we can define the parity functions $e_{\chi^{\square\heartsuit}(I)}$ as follows:

$$\begin{aligned}e_{\chi^{\square\heartsuit}(I)}(c_1,\cdots,c_t) &:= \overline{c_2+\cdots+c_{t-2}-\square-\heartsuit}\\ &\quad+\sum_{i=2}^{t-1}(\ell_i-\ell_{i-1})\overline{c_2+\cdots+c_{i-1}-\square}\\ &\quad+\sum_{i=2}^{t-1}(\ell_i-\ell_{i-1})\overline{c_i+\cdots+c_{t-2}-\heartsuit}\\ &\quad+\sum_{2\leqslant i<i'\leqslant t-1}(\ell_i-\ell_{i-1})(\ell_{i'}-\ell_{i'-1})\overline{c_i+\cdots+c_{i'-1}}.\end{aligned}\tag{14.6.5}$$

Then we have

$$\sum_{a\in\Phi^+}\lceil a(x)\rceil = 2\rho(x)+e_{\chi^{\square\heartsuit}(I)}(c_1,\cdots,c_t).$$

Now, we apply Lemma 10.5.5 to the following summation ($\square,\heartsuit=0,1$).

$$\mathrm{S}_{\chi^{\square\heartsuit}(I)}(r) = \sum_{\substack{c_i\in\mathbb{Z}_{>0}\\ c_1+\cdots+c_t=r+\heartsuit}} q^{(2n-2)(c_1-\frac{1}{2}\cdot\square)\sum_{i=2}^{t-2}\frac{1}{2}\ell_i(2n-1-\ell_i)c_i+\frac{n(n-1)}{2}(c_{t-1}+c_t-\heartsuit)+e_{\chi^{\square\heartsuit}(I)}(c_1,\cdots,c_t)}.$$

Note that the index set $\mathfrak{i}$ is $\{1, \cdots, t\}$ and the sequence $\boldsymbol{\mu}$ is

$$\begin{aligned} \mu_1 &= 2n-2, \\ \mu_i &= \tfrac{1}{2}\ell_i(2n-1-\ell_i), && (2 \leqslant i \leqslant t-2) \\ \mu_{t-1} &= \tfrac{n(n-1)}{2}, \\ \mu_t &= \tfrac{n(n-1)}{2}. \end{aligned}$$

Since all members of $\boldsymbol{\mu}$ are integers, $\mathrm{S}_{\mathcal{X}^{\square\heartsuit}(I)}(r)$ can be defined by a primary super q-exponential polynomial. The knowledge of quadratic function shows that $\{t-1, t\} \subseteq \mathfrak{i}_{\max} \subseteq \{1, t-1, t\}$ with $\mu_{\max} = \frac{n(n-1)}{2}$.

Depending on n, there are two possibilities.

If $n = 4$, then we have $\mathfrak{i}_{\max} = \{1, t-1, t\}$ and ($\square, \heartsuit = 0, 1$)

$$\mathrm{S}_{\mathcal{X}^{\square\heartsuit}(I)}(r) \sim C_{\mathcal{X}^{\square\heartsuit}(I)} \cdot \left(\left(\sum_{\mathbf{s} \in \mathbb{F}_2^t} \mathrm{E}_{\mathcal{X}^{\square\heartsuit}(I)}(\mathbf{s}) \right) + \left(\sum_{\mathbf{s} \in \mathbb{F}_2^t} (-1)^{\mathbf{1} \cdot \mathbf{s}} \, \mathrm{E}_{\mathcal{X}^{\square\heartsuit}(I)}(\mathbf{s}) \right) (-1)^{r+\heartsuit} \right) \cdot \binom{r}{2} q^{6r},$$

where the constant $C_{\mathcal{X}^{\square\heartsuit}(I)}$ and the function $\mathrm{E}_{\mathcal{X}^{\square\heartsuit}(I)} \colon \mathbb{F}_2^t \to \mathbb{Q}(q; -)$ are defined as follows:

$$C_{\mathcal{X}^{\square\heartsuit}(I)} := \frac{1}{8} q^{-3\cdot\square} \prod_{i=2}^{t-2} \left(q^{(4-\ell_i)(3-\ell_i)} - 1 \right)^{-1}, \tag{14.6.6}$$

$$\mathrm{E}_{\mathcal{X}^{\square\heartsuit}(I)}(\mathbf{s}) := q^{e_{\mathcal{X}^{\square\heartsuit}(I)}(\mathbf{s}) + \sum_{i=2}^{t-2} \frac{1}{2}(4-\ell_i)(3-\ell_i)s_i}. \tag{14.6.7}$$

From the definition Eq. (14.6.5) of $e_{\mathcal{X}^{\square\heartsuit}(I)}$, we see that $\mathrm{E}_{\mathcal{X}^{\square\heartsuit}(I)}(s_1, \cdots, s_t)$ does not depend on s_1, s_{t-1}, and s_t. Therefore, we have

$$\mathrm{S}_{\mathcal{X}^{\square\heartsuit}(I)}(r) \sim C_{\mathcal{X}^{\square\heartsuit}(I)} \cdot \left(\sum_{\mathbf{s} \in \mathbb{F}_2^t} \mathrm{E}_{\mathcal{X}^{\square\heartsuit}(I)}(\mathbf{s}) \right) \cdot \binom{r}{2} q^{6r}. \tag{14.6.8}$$

If $n \geqslant 5$, then we have $\mathfrak{i}_{\max} = \{t-1, t\}$ and $(\square, \heartsuit = 0, 1)$

$$\mathrm{S}_{\mathcal{X}^{\square\heartsuit}(I)}(r) \sim C_{\mathcal{X}^{\square\heartsuit}(I)} \cdot \left(\left(\sum_{\mathbf{s}\in\mathbb{F}_2^t} \mathrm{E}_{\mathcal{X}^{\square\heartsuit}(I)}(\mathbf{s})\right) + \left(\sum_{\mathbf{s}\in\mathbb{F}_2^t} (-1)^{\mathbf{1}\cdot\mathbf{s}}\, \mathrm{E}_{\mathcal{X}^{\square\heartsuit}(I)}(\mathbf{s})\right)(-1)^{r+\heartsuit}\right) \cdot rq^{\frac{n(n-1)}{2}r},$$

where the constant $C_{\mathcal{X}^{\square\heartsuit}(I)}$ and the function $\mathrm{E}_{\mathcal{X}^{\square\heartsuit}(I)} \colon \mathbb{F}_2^t \to \mathbb{Q}(q;-)$ are defined as follows:

$$C_{\mathcal{X}^{\square\heartsuit}(I)} := \frac{1}{4} \frac{q^{-\frac{1}{2}(2n-2)\cdot\square}}{q^{n(n-1)-2(2n-2)} - 1} \prod_{i=2}^{t-2} \left(q^{(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1}, \tag{14.6.9}$$

$$\mathrm{E}_{\mathcal{X}^{\square\heartsuit}(I)}(\mathbf{s}) := q^{e_{\mathcal{X}^{\square\heartsuit}(I)}(\mathbf{s}) + \left(\frac{n(n-1)}{2} - (2n-2)\right)s_1 + \sum_{i=2}^{t-2} \frac{1}{2}(n-\ell_i)(n-1-\ell_i)s_i}. \tag{14.6.10}$$

From the definition Eq. (14.6.5) of $e_{\mathcal{X}^{\square\heartsuit}(I)}$, we see that $\mathrm{E}_{\mathcal{X}^{\square\heartsuit}(I)}(s_1, \cdots, s_t)$ does not depend on s_{t-1} and s_t. Therefore, we have

$$\mathrm{S}_{\mathcal{X}^{\square\heartsuit}(I)}(r) \sim C_{\mathcal{X}^{\square\heartsuit}(I)} \cdot \left(\sum_{\mathbf{s}\in\mathbb{F}_2^t} \mathrm{E}_{\mathcal{X}^{\square\heartsuit}(I)}(\mathbf{s})\right) \cdot rq^{\frac{n(n-1)}{2}r}. \tag{14.6.11}$$

§ 14.7. Asymptotic growth of dominant $\mathrm{S}_{\mathcal{X}_J(I)}(r)$

Now, let I be a type and follow Convention 2.4.5. We are going to analyze $\mathrm{S}_{\mathcal{X}_J(I)}(r)$.

Suppose $x \in \mathcal{X}_J(I, r)$, where $I \cap J = \emptyset$. Since $\mathcal{X}_\emptyset = \mathcal{V}_\dagger$, by Lemma 9.5.4, we can write x as $x_0 - \sum_{j\in J} \frac{1}{2}\omega_j$, where $x_0 \in \mathcal{V}_\dagger(I, r + |J| - \delta(J))$. Then we have

$$\sum_{a\in\Phi^+} \lceil a(x) \rceil = 2\rho(x_0) + \sum_{a\in\Phi^+} \left\lceil -\sum_{j\in J} a(\tfrac{1}{2}\omega_j) \right\rceil.$$

Note that the last summation gives an integral constant. Then we have

$$\mathrm{S}_{\mathcal{X}_J(I)}(r) = q^{\sum_{a\in\Phi^+} \left\lceil -\sum_{j\in J} a(\frac{1}{2}\omega_j) \right\rceil} \mathrm{S}_{\mathcal{V}_\dagger(I)}(r + |J| - \delta(J)). \tag{14.7.1}$$

In particular, each $\mathrm{S}_{\mathcal{X}_J(I)}$ can be defined by a primary super q-exponential polynomial. Since $\mathcal{V}(I,r)$ is a disjointed union of various $\mathcal{X}_J(I,r)$, we see that $\mathrm{S}_{\mathcal{V}(I)}$ can be defined by a primary super q-exponential polynomial. Then by Eqs. (8.4.5) and (8.4.6), we see that $\mathrm{SV}(\,\cdot\,)$ and $\mathrm{SSA}(\,\cdot\,)$ can be defined by primary super q-exponential polynomials.

Now we assume that I is dominant. We will separate the discussion into two cases: (i) $n = 4$ and (ii) $n \geqslant 5$.

(i). Suppose $n = 4$. Then we have $\{1,3,4\} \cap I = \emptyset$. The following J appears in Fig. 9.7: $\{1\}$, $\{1,2\}$, $\{3,4\}$, $\{2,3,4\}$. In those cases, by Eq. (6.6.6), we have

$$|\{1\}| - \delta(\{1\}) = 0, \qquad \sum_{a\in\Phi^+}\left\lceil -a(\tfrac{1}{2}\omega_1)\right\rceil = 0,$$

$$|\{1,2\}| - \delta(\{1,2\}) = 1, \qquad \sum_{a\in\Phi^+}\left\lceil -a(\tfrac{1}{2}\omega_1 + \tfrac{1}{2}\omega_2)\right\rceil = -5,$$

$$|\{3,4\}| - \delta(\{3,4\}) = 1, \qquad \sum_{a\in\Phi^+}\left\lceil -a(\tfrac{1}{2}\omega_3 + \tfrac{1}{2}\omega_4)\right\rceil = -3,$$

$$|\{2,3,4\}| - \delta(\{2,3,4\}) = 2, \qquad \sum_{a\in\Phi^+}\left\lceil -a(\tfrac{1}{2}\omega_2 + \tfrac{1}{2}\omega_3 + \tfrac{1}{2}\omega_4)\right\rceil = -8.$$

Then by Eqs. (14.1.2) and (14.7.1), we have

$$\mathrm{S}_{\mathcal{X}_{\{1\}}(I)}(r) = \prod_{i=2}^{t-2}\left(q^{(4-\ell_i)(3-\ell_i)} - 1\right)^{-1} \cdot \binom{r}{2} q^{6r}, \tag{14.7.2}$$

$$\mathrm{S}_{\mathcal{X}_{\{1,2\}}(I)}(r) = q^{6-5} \cdot \prod_{i=2}^{t-2}\left(q^{(4-\ell_i)(3-\ell_i)} - 1\right)^{-1} \cdot \binom{r}{2} q^{6r}, \tag{14.7.3}$$

$$\mathrm{S}_{\mathcal{X}_{\{3,4\}}(I)}(r) = q^{6-3} \cdot \prod_{i=2}^{t-2}\left(q^{(4-\ell_i)(3-\ell_i)} - 1\right)^{-1} \cdot \binom{r}{2} q^{6r}, \tag{14.7.4}$$

$$\mathrm{S}_{\mathcal{X}_{\{2,3,4\}}(I)}(r) = q^{12-8} \cdot \prod_{i=2}^{t-2}\left(q^{(4-\ell_i)(3-\ell_i)} - 1\right)^{-1} \cdot \binom{r}{2} q^{6r}. \tag{14.7.5}$$

(ii). Suppose $n \geqslant 5$. Then we have $\{n-1, n\} \cap I = \emptyset$. Depending on ℓ_1, there are two cases: (ii-a) $\ell_1 > 1$ and (ii-b) $\ell_1 = 1$.

(ii-a). If $\ell_1 > 1$, then we consider Fig. 9.6 and the following J: $\{2,3\}, \cdots, \{n-3, n-2\}$, $\{n-1, n\}$, and $\{n-2, n-1, n\}$. When $J = \{j, j+1\}$, where $2 \leqslant j \leqslant n-3$, we have $|J| - \delta(J) = 2$ and by Eq. (6.6.6),

$$\sum_{a \in \Phi^+} \left\lceil -a\left(\tfrac{1}{2}\omega_j + \tfrac{1}{2}\omega_{j+1}\right) \right\rceil = -j(2n-j).$$

Then by Eqs. (14.1.1) and (14.7.1), we have

$$\mathrm{S}_{\mathcal{X}_{\{j,j+1\}}(I)}(r) = q^{n(n-1)-j(2n-j)} \prod_{i=1}^{t-2} \left(q^{(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1} \cdot rq^{\frac{n(n-1)}{2}r}. \tag{14.7.6}$$

When $J = \{n-1, n\}$, we have $|J| - \delta(J) = 1$ and by Eq. (6.6.6),

$$\sum_{a \in \Phi^+} \left\lceil -a\left(\tfrac{1}{2}\omega_{n-1} + \tfrac{1}{2}\omega_n\right) \right\rceil = -\tfrac{(n-1)(n-2)}{2}.$$

Then by Eqs. (14.1.1) and (14.7.1), we have

$$\mathrm{S}_{\mathcal{X}_{\{n-1,n\}}(I)}(r) = q^{\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2}} \prod_{i=1}^{t-2} \left(q^{(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1} \cdot rq^{\frac{n(n-1)}{2}r}. \tag{14.7.7}$$

When $J = \{n-2, n-1, n\}$, we have $|J| - \delta(J) = 2$ and by Eq. (6.6.6),

$$\sum_{a \in \Phi^+} \left\lceil -a\left(\tfrac{1}{2}\omega_{n-2} + \tfrac{1}{2}\omega_{n-1} + \tfrac{1}{2}\omega_n\right) \right\rceil = -(n-1)^2.$$

Then by Eqs. (14.1.1) and (14.7.1), we have

$$\mathrm{S}_{\mathcal{X}_{\{n-2,n-1,n\}}(I)}(r) = q^{n(n-1)-(n-1)^2} \prod_{i=1}^{t-2} \left(q^{(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1} \cdot rq^{\frac{n(n-1)}{2}r}. \tag{14.7.8}$$

(ii-b). If $\ell_1 = 1$, then we consider Fig. 9.7 and the following J: $\{1\}$, $\{1,2\}$, $\{2,3\}, \cdots$, $\{n-3, n-2\}$, $\{n-1, n\}$, and $\{n-2, n-1, n\}$. When $J = \{1\}$, we have $|J| - \delta(J) = 0$

and by Eq. (6.6.6),

$$\sum_{a\in\Phi^+}\left\lceil -a(\tfrac{1}{2}\omega_1)\right\rceil = 0.$$

Then by Eqs. (14.1.3) and (14.7.1), we have

$$\mathrm{S}_{\mathcal{X}_{\{1\}}(I)}(r) = \prod_{i=2}^{t-2}\left(q^{(n-\ell_i)(n-1-\ell_i)}-1\right)^{-1}\cdot\frac{1+q^{\frac{n(n-1)}{2}-(2n-2)}}{q^{n(n-1)-2(2n-2)}-1}\cdot rq^{\frac{n(n-1)}{2}r}. \tag{14.7.9}$$

When $J=\{1,2\}$, we have $|J|-\delta(J)=1$ and by Eq. (6.6.6),

$$\sum_{a\in\Phi^+}\left\lceil -a(\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_2)\right\rceil = -(2n-3).$$

Then by Eqs. (14.1.3) and (14.7.1), we have

$$\begin{aligned}\mathrm{S}_{\mathcal{X}_{\{1\}}(I)}(r) = q^{\frac{n(n-1)}{2}-(2n-3)}&\prod_{i=2}^{t-2}\left(q^{(n-\ell_i)(n-1-\ell_i)}-1\right)^{-1}\\ &\cdot\frac{1+q^{\frac{n(n-1)}{2}-(2n-2)}}{q^{n(n-1)-2(2n-2)}-1}\cdot rq^{\frac{n(n-1)}{2}r}.\end{aligned} \tag{14.7.10}$$

When $J=\{j,j+1\}$, where $2\leqslant j\leqslant n-3$, we have $|J|-\delta(J)=2$ and by Eq. (6.6.6),

$$\sum_{a\in\Phi^+}\left\lceil -a(\tfrac{1}{2}\omega_j+\tfrac{1}{2}\omega_{j+1})\right\rceil = -j(2n-j).$$

Then by Eqs. (14.1.3) and (14.7.1), we have

$$\begin{aligned}\mathrm{S}_{\mathcal{X}_{\{j,j+1\}}(I)}(r) = q^{n(n-1)-j(2n-j)}&\prod_{i=2}^{t-2}\left(q^{(n-\ell_i)(n-1-\ell_i)}-1\right)^{-1}\\ &\cdot\frac{1+q^{\frac{n(n-1)}{2}-(2n-2)}}{q^{n(n-1)-2(2n-2)}-1}\cdot rq^{\frac{n(n-1)}{2}r}.\end{aligned} \tag{14.7.11}$$

When $J=\{n-1,n\}$, we have $|J|-\delta(J)=1$ and by Eq. (6.6.6),

$$\sum_{a\in\Phi^+}\left\lceil -a(\tfrac{1}{2}\omega_{n-1}+\tfrac{1}{2}\omega_n)\right\rceil = -\tfrac{(n-1)(n-2)}{2}.$$

Then by Eqs. (14.1.3) and (14.7.1), we have

$$\mathrm{S}_{\mathcal{X}_{\{n-1,n\}}(I)}(r) = q^{\frac{n(n-1)}{2}-\frac{(n-1)(n-2)}{2}} \prod_{i=2}^{t-2} \left(q^{(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1} \cdot \frac{1+q^{\frac{n(n-1)}{2}-(2n-2)}}{q^{n(n-1)-2(2n-2)} - 1} \cdot rq^{\frac{n(n-1)}{2}r}. \tag{14.7.12}$$

When $J = \{n-2, n-1, n\}$, we have $|J| - \delta(J) = 2$ and by Eq. (6.6.6),

$$\sum_{\alpha\in\Phi^+} \left\lceil -\alpha\left(\tfrac{1}{2}\omega_{n-2} + \tfrac{1}{2}\omega_{n-1} + \tfrac{1}{2}\omega_n\right)\right\rceil = -(n-1)^2.$$

Then by Eqs. (14.1.3) and (14.7.1), we have

$$\mathrm{S}_{\mathcal{X}_{\{n-2,n-1,n\}}(I)}(r) = q^{n(n-1)-(n-1)^2} \prod_{i=2}^{t-2} \left(q^{(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1} \cdot \frac{1+q^{\frac{n(n-1)}{2}-(2n-2)}}{q^{n(n-1)-2(2n-2)} - 1} \cdot rq^{\frac{n(n-1)}{2}r}. \tag{14.7.13}$$

§ 14.8. Asymptotic growth of dominant $\mathrm{S}_{\mathcal{V}(I)}(r)$

We are now able to compute the asymptotic growth of $\mathrm{S}_{\mathcal{V}(I)}(r)$ when I is dominant. We will separate the discussion into two cases: (i) $n = 4$ and (ii) $n \geqslant 5$.

(i). Suppose $n = 4$. Then the dominant types are $\{2\}$ and $\emptyset$. By Fig. 9.7, we have

$$\begin{aligned}
\mathrm{S}_{\mathcal{V}(\{2\})}(r) &= \mathrm{S}_{\mathcal{X}^{00}(\{2\})}(r) + \mathrm{S}_{\mathcal{X}^{01}(\{2\})}(r) + \mathrm{S}_{\mathcal{X}^{10}(\{2\})}(r) + \mathrm{S}_{\mathcal{X}^{11}(\{2\})}(r) \\
&\quad - \mathrm{S}_{\mathcal{X}_{\{1\}}(\{2\})}(r) - \mathrm{S}_{\mathcal{X}_{\{3,4\}}(\{2\})}(r), \\
\mathrm{S}_{\mathcal{V}(\emptyset)}(r) &= \mathrm{S}_{\mathcal{X}^{00}(\emptyset)}(r) + \mathrm{S}_{\mathcal{X}^{01}(\emptyset)}(r) + \mathrm{S}_{\mathcal{X}^{10}(\emptyset)}(r) + \mathrm{S}_{\mathcal{X}^{11}(\emptyset)}(r) \\
&\quad - \mathrm{S}_{\mathcal{X}_{\{1\}}(\emptyset)}(r) - \mathrm{S}_{\mathcal{X}_{\{1,2\}}(\emptyset)}(r) - \mathrm{S}_{\mathcal{X}_{\{3,4\}}(\emptyset)}(r) - \mathrm{S}_{\mathcal{X}_{\{2,3,4\}}(\emptyset)}(r).
\end{aligned}$$

Therefore, by Eqs. (14.6.5) to (14.6.8) and (14.7.2) to (14.7.5), we have

$$\mathrm{S}_{\mathcal{V}(\{2\})}(r) \sim \left(1+q^3+1+q-1-q^3\right)\cdot\binom{r}{2}q^{6r} = (q+1)\cdot\binom{r}{2}q^{6r}, \tag{14.8.1}$$

$$\begin{aligned}\mathrm{S}_{\mathcal{V}(\emptyset)}(r) \sim &\Big((1+q^5)+(q^3+q^4)+(1+q)+(1+q^2)\\ &-1-q-q^3-q^4\Big)\left(q^2-1\right)^{-1}\cdot\binom{r}{2}q^{6r}\\ =&\ \frac{q^5+q^2+q+1}{q^2-1}\cdot\binom{r}{2}q^{6r}.\end{aligned} \tag{14.8.2}$$

(ii). Now, we assume $n \geqslant 5$. Then I is dominant exactly when $\{n-1,n\}\cap I = \emptyset$. Depending on ℓ_1, there are two cases: (ii-a) $\ell_1 > 1$ and (ii-b) $\ell_1 = 1$.

(ii-a). When $\ell_1 > 1$, by Fig. 9.6, we have (including the zero summations)

$$\begin{aligned}\mathrm{S}_{\mathcal{V}(I)}(r) = &\ \mathrm{S}_{\mathcal{X}^{00}(I)}(r)+\mathrm{S}_{\mathcal{X}^{01}(I)}(r)+\mathrm{S}_{\mathcal{X}^{10}(I)}(r)+\mathrm{S}_{\mathcal{X}^{11}(I)}(r)\\ &-\sum_{j=2}^{n-3}\mathrm{S}_{\mathcal{X}_{\{j,j+1\}}(I)}(r)-\mathrm{S}_{\mathcal{X}_{\{n-1,n\}}(I)}(r)-\mathrm{S}_{\mathcal{X}_{\{n-2,n-1,n\}}(I)}(r).\end{aligned}$$

Therefore, by Eqs. (14.6.2) to (14.6.4) and (14.7.6) to (14.7.8), we have

$$\mathrm{S}_{\mathcal{V}(I)}(r) \sim \prod_{i=1}^{t-2}\left(q^{(n-\ell_i)(n-1-\ell_i)}-1\right)^{-1}\cdot C_I\cdot rq^{\frac{n(n-1)}{2}r}, \tag{14.8.3}$$

where the constant C_I is defined as follows:

$$\begin{aligned}C_I = &\sum_{\heartsuit=0,1}\ \sum_{s_1,\cdots,s_{t-2}\in\mathbb{F}_2} q^{e_{\mathcal{X}^{0\heartsuit}(I)}(s_1,\cdots,s_{t-2},0,0)+\sum\limits_{i=1}^{t-2}\frac{1}{2}(n-\ell_i)(n-1-\ell_i)s_i}\\ &-\sum_{\substack{2\leqslant j\leqslant n-3\\ \{j,j+1\}\cap I=\emptyset}} q^{(n-j)^2-n}-(1+\delta_I(n-2))q^{n-1},\end{aligned} \tag{14.8.4}$$

where $\delta_I(i) = 0$ if $i \in I$ and 1 is not. Note that the definition of the multivariable parity function $e_{\mathcal{X}^{0\heartsuit}(I)}$ is in Eq. (14.6.1).

(ii-b). When $\ell_1 = 1$, by Fig. 9.7, we have (including the zero summations)

$$\begin{aligned} \mathrm{S}_{\mathcal{V}(I)}(r) = {} & \mathrm{S}_{\mathcal{X}^{00}(I)}(r) + \mathrm{S}_{\mathcal{X}^{01}(I)}(r) + \mathrm{S}_{\mathcal{X}^{10}(I)}(r) + \mathrm{S}_{\mathcal{X}^{11}(I)}(r) \\ & - \mathrm{S}_{\mathcal{X}_{\{1\}}(I)}(r) - \mathrm{S}_{\mathcal{X}_{\{1,2\}}(I)}(r) - \sum_{j=2}^{n-3} \mathrm{S}_{\mathcal{X}_{\{j,j+1\}}(I)}(r) \\ & - \mathrm{S}_{\mathcal{X}_{\{n-1,n\}}(I)}(r) - \mathrm{S}_{\mathcal{X}_{\{n-2,n-1,n\}}(I)}(r). \end{aligned}$$

Therefore, by Eqs. (14.6.9) to (14.6.11) and (14.7.9) to (14.7.13), we have

$$\mathrm{S}_{\mathcal{V}(I)}(r) \sim \left(q^{(n-4)(n-1)} - 1\right)^{-1} \prod_{i=2}^{t-2} \left(q^{(n-\ell_i)(n-1-\ell_i)} - 1\right)^{-1} \cdot C_I \cdot r q^{\frac{n(n-1)}{2} r}, \tag{14.8.5}$$

where the constant C_I is defined as follows:

$$\begin{aligned} C_I = {} & \sum_{\square,\heartsuit=0,1} \sum_{s_1,\cdots,s_{t-2}\in\mathbb{F}_2} q^{e_{\mathcal{X}^{\square\heartsuit}(I)}(\mathbf{s})-(n-1)\cdot\square+\frac{(n-4)(n-1)}{2}s_1+\sum_{i=2}^{t-2}\frac{1}{2}(n-\ell_i)(n-1-\ell_i)s_i} \\ & - \Bigg(1 + \delta_I(2) q^{\frac{n(n-1)}{2}-(2n-3)} + \sum_{\substack{2\leqslant j\leqslant n-3 \\ \{j,j+1\}\cap I=\emptyset}} q^{(n-j)^2-n} \\ & + (1+\delta_I(n-2))q^{n-1}\Bigg) \cdot \left(1 + q^{\frac{n(n-1)}{2}-(2n-2)}\right), \end{aligned} \tag{14.8.6}$$

Note that the definition of the multivariable parity function $e_{\mathcal{X}^{\square\heartsuit}(I)}$ is in Eq. (14.6.5).

§ 14.9. Asymptotic growths of SSA(r) and SV(r)

We are now able to obtain the asymptotic growth of SSA(r). By Eq. (8.4.6), we have

$$\mathrm{SSA}(r) = \sum_{I\subseteq\Delta} \frac{\mathscr{P}_{D_n;I}(q)}{q^{\deg(\mathscr{P}_{D_n;I})}} \mathrm{S}_{\mathcal{V}(I)}(r) \sim \sum_{I \text{ is dominant}} \frac{\mathscr{P}_{D_n;I}(q)}{q^{\deg(\mathscr{P}_{D_n;I})}} \mathrm{S}_{\mathcal{V}(I)}(r). \tag{14.9.1}$$

What remains is to plug in the asymptotic growth of dominant $\mathrm{S}_{\mathcal{V}(I)}(r)$. We will separate the discussion into two cases: (i) $n = 4$ and (ii) $n \geqslant 5$.

(i). When $n = 4$, the only dominant types are $\{2\}$ and $\emptyset$. Then by Eqs. (14.8.1), (14.8.2), and (A.3.3), we have

$$\mathrm{SSA}(r) \sim C(4) \cdot \binom{r}{2} q^{6r}, \tag{14.9.2}$$

where the constant $C(4)$ is defined as follows:

$$\begin{aligned} C(4) &:= \frac{\mathscr{P}_{D_4;\{2\}}(q)}{q^{\deg\left(\mathscr{P}_{D_4;\{2\}}\right)}}(q+1) + \frac{\mathscr{P}_{D_4;\emptyset}(q)}{q^{\deg\left(\mathscr{P}_{D_4;\emptyset}\right)}} \frac{q^5+q^2+q+1}{q^2-1} \\ &= \frac{\left(q^6-1\right)\left(q^4-1\right)^2(q+1)}{(q-1)^3 q^{11}} + \frac{\left(q^6-1\right)\left(q^4-1\right)^2\left(q^5+q^2+q+1\right)}{(q-1)^4 q^{12}} \\ &= \frac{\left(q^2+q+1\right)\left(q^2-q+1\right)^2\left(q^2+1\right)^3(q+1)^4}{(q-1)q^{12}}. \end{aligned} \tag{14.9.3}$$

As a consequence, we have

$$\mathrm{SV}(r) = \sum_{z=0}^{r} \mathrm{SSA}(z) \sim \frac{q^6}{q^6-1} C(4) \cdot \binom{r}{2} q^{6r}. \tag{14.9.4}$$

(ii). Now, we assume $n \geqslant 5$. Then I is dominant exactly when $\{n-1, n\} \cap I = \emptyset$. By Eqs. (14.8.3) and (14.8.5), we have

$$\mathrm{SSA}(r) \sim C(n) \cdot r q^{\frac{n(n-1)}{2} r}, \tag{14.9.5}$$

where the constant $C(n)$ is defined as follows:

$$C(n) := \sum_{1,n-1,n\notin I} \frac{\mathscr{P}_{D_n;I}(q)}{q^{\deg(\mathscr{P}_{D_n;I})}} \prod_{i=2}^{t-2} \left(q^{(n-\ell_i(I))(n-1-\ell_i(I))} - 1\right)^{-1} \cdot \frac{C_I}{q^{(n-4)(n-1)} - 1} + \sum_{1\in I,n-1,n\notin I} \frac{\mathscr{P}_{D_n;I}(q)}{q^{\deg(\mathscr{P}_{D_n;I})}} \prod_{i=1}^{t-2} \left(q^{(n-\ell_i(I))(n-1-\ell_i(I))} - 1\right)^{-1} \cdot C_I. \tag{14.9.6}$$

As a consequence, we have

$$\mathrm{SV}(r) = \sum_{z=0}^{r} \mathrm{SSA}(z) \sim \frac{q^{\frac{n(n-1)}{2}}}{q^{\frac{n(n-1)}{2}} - 1} C(n) \cdot q^{\frac{n(n-1)}{2} r}. \tag{14.9.7}$$

Remark. Note that the constant C_I depends on I. When $1 \in I$ and $n-1, n \notin I$, it is defined in Eq. (14.8.4). When $1, n-1, n \notin I$, it is defined in Eq. (14.8.6).

By Eqs. (14.9.2) to (14.9.7) we have proved Theorem 14.1. Moreover, by Eq. (A.3.3), we have the following explicit formulas:

$$\mathscr{P}_{D_n;I}(q) = \frac{[2(n-1)]!!(z) \cdot [n](z)}{\prod_{i=1}^{t-1} [\ell_i(I) - \ell_{i-1}(I)]!(z)}, \qquad q^{\deg(\mathscr{P}_{D_n;I})} = \frac{q^{n(n-1)}}{\prod_{i=1}^{t-1} q^{\binom{\ell_i(I)-\ell_{i-1}(I)}{2}}}. \tag{14.9.8}$$

See Lemma 8.2.5 and Eqs. (A.1.2) and (A.2.1) for the definitions of the symbols $[\,\cdot\,]$, $[\,\cdot\,]!$, and $[2\cdot\,]!!$.

Appendix

A. Poincaré polynomials of irreducible root systems

In this chapter, we will work out a closed formula for the Poincaré polynomial $\mathscr{P}_{X_n;I}$ of each irreducible reduced root system Φ of type X_n and each type I.

§ A.1. Poincaré polynomials of A_n

First, it is clear that $\mathscr{P}_{A_0}(z) = \mathscr{P}_{\emptyset}(z) = 1$. We then assume that $n \geqslant 1$.

Let Φ be a root system of type A_n. Then the Dynkin diagram with the label of simple roots in Φ is the following one:

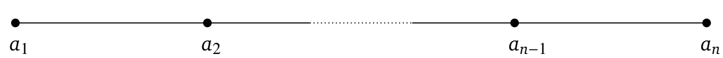


Figure A.1. The Dynkin diagram with the label of simple roots in A_n.

By [Bourbaki, chap.VI, §4, no.7], the degrees of its Weyl group are $d_i = i + 1$. Therefore, by Lemma 8.2.5, we have

$$\mathscr{P}_{A_n}(z) = \prod_{i=1}^{n} [i+1](z). \tag{A.1.1}$$

In particular, $\deg(\mathscr{P}_{A_n}) = \binom{n+1}{2}$. Note that $[1](z)$ is the constant 1. Hence, $\mathscr{P}_{A_n}(z)$

equals to the following *z-factorial* polynomial:

$$[n+1]!(z) := \prod_{i=1}^{n+1} [i](z). \tag{A.1.2}$$

We also need the following *z-multinomial* polynomial:

$$\begin{bmatrix} n \\ n_0, \cdots, n_k \end{bmatrix}(z) := \frac{[n]!(z)}{[n_0]!(z) \cdots [n_k]!(z)}, \tag{A.1.3}$$

where $n_0 + \cdots + n_k = n$ is a partition of n into natural numbers.

Let I be a type and follow Convention 2.4.5. Then the Dynkin diagram of the subsystem Φ_I with labels is the following one:

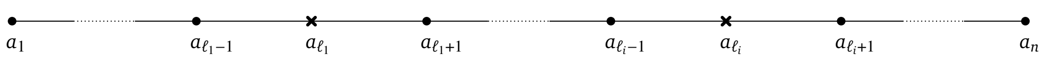


Figure A.2. The Dynkin diagram of a subsystem Φ_I in A_n.

Hence, Φ_I is of type

$$A_{\ell_1-\ell_0-1} \times \cdots \times A_{\ell_{t+1}-\ell_t-1},$$

where ℓ_{t+1} is defined to be $n+1$. Note that

$$(\ell_1 - \ell_0) + \cdots + (\ell_{t+1} - \ell_t) = n + 1.$$

Then we have

$$\mathscr{P}_{A_n,I}(z) = \frac{\mathscr{P}_{A_n}(z)}{\prod_{i=1}^{t+1} \mathscr{P}_{A_{\ell_i-\ell_{i-1}-1}}(z)} = \begin{bmatrix} n+1 \\ \ell_1 - \ell_0, \cdots, \ell_{t+1} - \ell_t \end{bmatrix}(z). \tag{A.1.4}$$

In particular, $\deg(\mathscr{P}_{A_n,I}) = \binom{n+1}{2} - \sum_{i=1}^{t+1} \binom{\ell_i - \ell_{i-1}}{2}$.

§ A.2. Poincaré polynomials of B_n and C_n

First note that the two root systems share the same Weyl group, hence the same Poincaré polynomial. It suffices to only consider one of them. We will consider C_n.

Let Φ be a root system of type C_n. Then the Dynkin diagram with the label of simple roots in Φ is the following one:

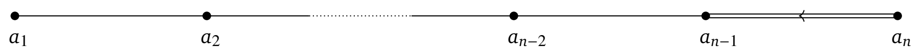


Figure A.3. The Dynkin diagram with the label of simple roots in C_n.

When $n = 0$ or 1, we can see that $C_n = A_n$. We then assume that $n \geqslant 2$. By [Bourbaki, chap.VI, §4, no.5 and no.6], the degrees of its Weyl group are $d_i = 2i$. Therefore, by Lemma 8.2.5, we have

$$\mathscr{P}_{C_n}(z) = \prod_{i=1}^{n} [2i](z). \tag{A.2.1}$$

In particular, $\deg\left(\mathscr{P}_{C_n}\right) = n^2$. We use $[2n]!!(z)$ to denote the right-hand side and use the convention that $[0]!! = 1$.

Let I be a type and follow Convention 2.4.5. Then the Dynkin diagram of the subsystem Φ_I with labels is one of the following three: (focusing on position of ℓ_t)

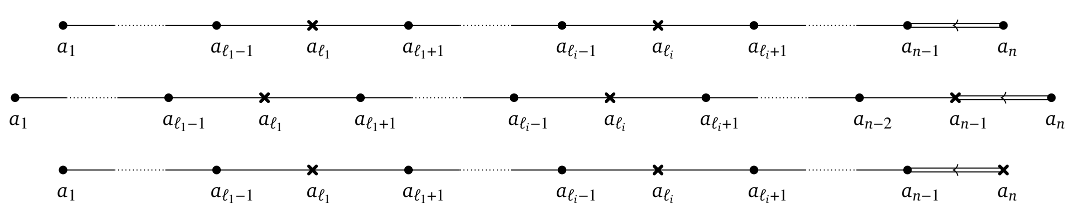


Figure A.4. The three possibilities of the Dynkin diagram of subsystems Φ_I in C_n.

In either case, Φ_I is of type

$$A_{\ell_1-\ell_0-1} \times \cdots \times A_{\ell_t-\ell_{t-1}-1} \times C_{n-\ell_t}$$

(notice that $C_0 = A_0$ and $C_1 = A_1$). Then we have

$$\begin{aligned}\mathscr{P}_{C_n,I}(z) &= \frac{\mathscr{P}_{C_n}(z)}{\prod_{i=1}^{t} \mathscr{P}_{A_{\ell_i-\ell_{i-1}-1}}(z)^{-1} \cdot \mathscr{P}_{C_{n-\ell_t}}(z)} \\ &= \frac{[2n]!!(z)}{\prod_{i=1}^{t} [\ell_i - \ell_{i-1}]!(z) \cdot [2(n-\ell_t)]!!(z)}.\end{aligned} \tag{A.2.2}$$

In particular, $\deg\big(\mathscr{P}_{C_n,I}\big) = n^2 - \sum_{i=1}^{t} \binom{\ell_i-\ell_{i-1}}{2} - (n-\ell_t)^2$.

§ A.3. Poincaré polynomials of D_n

Let Φ be a root system of type D_n. Then the Dynkin diagram with the label of simple roots in Φ is the following one:

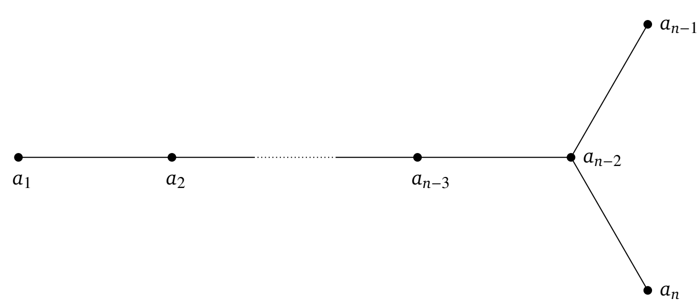


Figure A.5. The Dynkin diagram with the label of simple roots in D_n.

When $n = 0$, 1, or 3, we can see that $D_n = A_n$. When $n = 2$, we have $D_2 = A_1 \times A_1$. We then assume that $n \geqslant 4$. [Bourbaki, chap.VI, §4, no.8], the degrees of W are $d_i = 2i$

for $i < n$ and $d_n = n$. Therefore, by Lemma 8.2.5, we have

$$\mathscr{P}_{D_n}(z) = \prod_{i=1}^{n-1}[2i](z) \cdot [n](z) = [2(n-1)]!!(z) \cdot [n](z). \tag{A.3.1}$$

In particular, $\deg\left(\mathscr{P}_{D_n}\right) = n(n-1)$.

Let I be a type and follow Convention 2.4.5. When $\ell_t < n-2$, the Dynkin diagram of the subsystem Φ_I with labels is one of the followings: (focusing on position of ℓ_t)

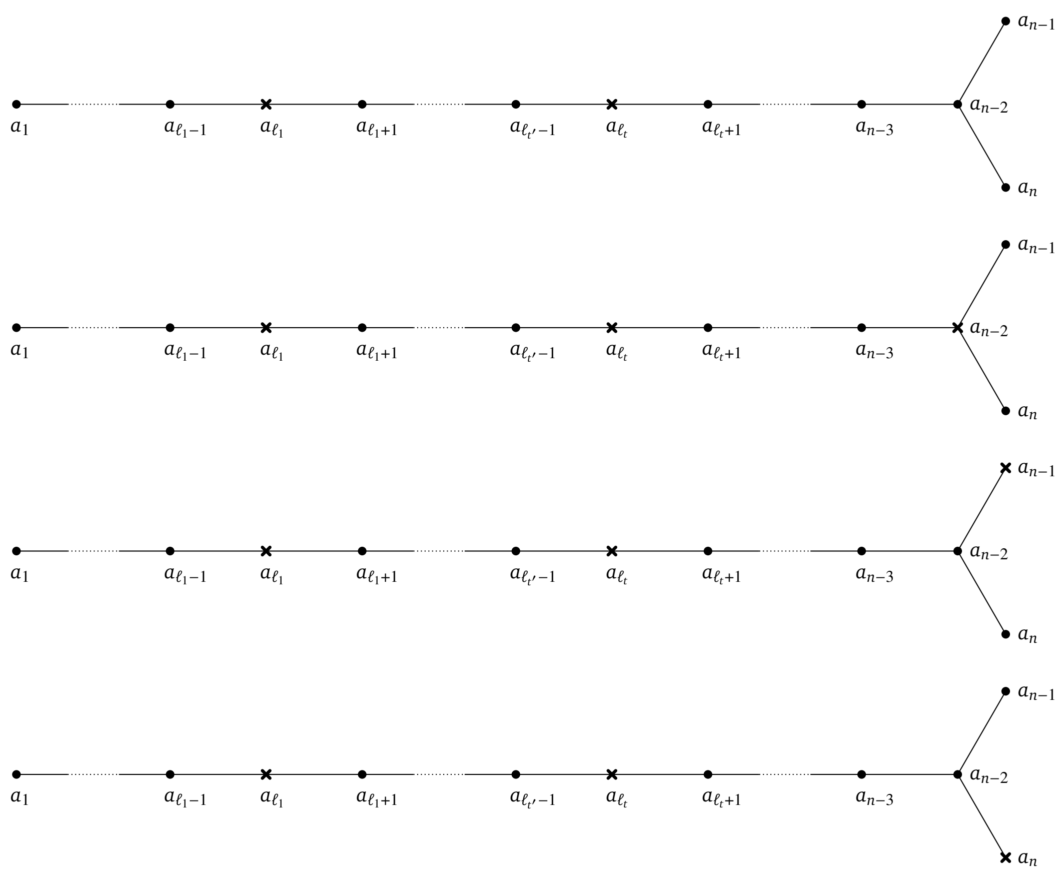


Figure A.6. The four possibilities of the Dynkin diagram of subsystems Φ_I in D_n.

In the first two cases, Φ_I is of the type

$$A_{\ell_1-\ell_0-1}\times\cdots\times A_{\ell_t-\ell_{t-1}-1}\times D_{n-\ell_t}$$

(Noticing that is $D_{n-\ell_t}$ if $n-\ell_t<4$). Therefore, we have

(A.3.2)
$$\begin{aligned}\mathscr{P}_{D_n,I}(z) &= \frac{\mathscr{P}_{D_n}(z)}{\prod_{i=1}^{t}\mathscr{P}_{A_{\ell_i-\ell_{i-1}-1}}(z)^{-1}\cdot\mathscr{P}_{D_{n-\ell_t}}(z)}\\ &= \frac{[2(n-1)]!!(z)\cdot[n](z)}{\prod_{i=1}^{t}[\ell_i-\ell_{i-1}]!(z)\cdot[2(n-\ell_t-1)]!!(z)\cdot[n-\ell_t](z)}.\end{aligned}$$

In particular,

$$\deg\big(\mathscr{P}_{D_n,I}\big)=n(n-1)-\sum_{i=1}^{t}\binom{\ell_i-\ell_{i-1}}{2}-(n-\ell_t)(n-\ell_t-1).$$

In the last two cases, Φ_I is of the type

$$A_{\ell_1-\ell_0-1}\times\cdots\times A_{\ell_{t-1}-\ell_{t-2}-1}\times A_{n-\ell_{t-1}-1}.$$

Therefore, we have

(A.3.3)
$$\begin{aligned}\mathscr{P}_{D_n,I}(z) &= \frac{\mathscr{P}_{D_n}(z)}{\prod_{i=1}^{t-1}\mathscr{P}_{A_{\ell_i-\ell_{i-1}-1}}(z)^{-1}\cdot\mathscr{P}_{A_{n-\ell_{t-1}-1}}(z)}\\ &= \frac{[2(n-1)]!!(z)\cdot[n](z)}{\prod_{i=1}^{t-1}[\ell_i-\ell_{i-1}]!(z)\cdot[n-\ell_{t-1}](z)}.\end{aligned}$$

In particular,

$$\deg\big(\mathscr{P}_{D_n,I}\big)=n(n-1)-\sum_{i=1}^{t-1}\binom{\ell_i-\ell_{i-1}}{2}-\binom{n-\ell_{t-1}}{2}.$$

B. Final Remarks

Here are some possible further research.

Generalizing Theorem 6.1. However, it is already known that this characterization fails for exceptional types. For instance, the following shows such a counterexample in the building $\mathscr{B}(G_2)$.

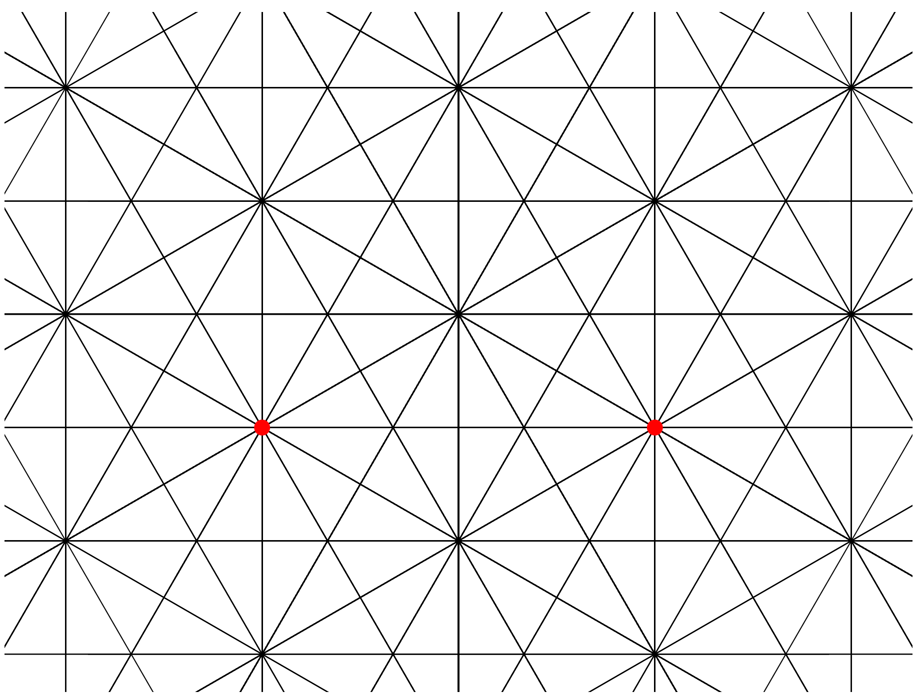


Figure B.1. The two red vertices are separated by at most 3 parallel walls, but the simplicial distance is 4.

Indeed, this is essentially due to the failure of Lemma 6.2.2. Hence, whenever we have $h \leqslant 2$, namely when the root system is of classical type, we should expect the following holds.

Conjecture 1. *In an irreducible Bruhat-Tits building of classical type, two vertices x and y have simplicial distance at most* d *if and only if they are separated by at most $d-1$ parallel walls.*

Concave functions and fixed-point sets. In Theorem 7.5, we have seen that the simplicial ball $B(r)$ is precisely the set of fixed-vertices under the action of the Moy-Prasad subgroup $P_{o,r}$. The key step in the proof is to interpret the statement in terms of concave functions. More generally, we can expect:

Conjecture 2. *The fixed point set of P_f should be determined by f in a direct way.*

More concretely, inspired by the ideas in classical differential geometry, we may think the simplicial balls $B(r)$ measures the *simplicial curvature*. Then we may expect:

Conjecture 3. *The fixed point set S of P_f has simplicial curvature $f(0)$ in the sense that the largest simplicial ball inside S has radius $f(0)$.*

Simplicial volume and (super) q-exponential polynomials. The methods developed in Chapter 10 should contribute to the study of simplicial volume of a general fixed point set S.

Bibliography

[Adl98] J. D. Adler, *Refined anisotropic K-types and supercuspidal representations*, Pacific J. Math. **185** (1998), no. 1, 1–32.

[AD02] J. D. Adler and S. DeBacker, *Some applications of Bruhat-Tits theory to harmonic analysis on the Lie algebra of a reductive p-adic group*, Michigan Math. J. **50** (2002), no. 2, 263–286.

[BM00] D. Barbasch and A. Moy, *A new proof of the Howe conjecture*, J. Amer. Math. Soc. **13** (2000), no. 3, 639–650.

[BGG73] I. N. Bernstein, I. M. Gel'fand, and S. I. Gel'fand, *Schubert cells, and the cohomology of the spaces G/P*, Uspehi Mat. Nauk **28** (1973), no. 3(171), 3–26 (Russian).

[Bourbaki] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990.

[Bor53] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2) **57** (1953), 115–207 (French).

[BS73] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491.

[BS76] ———, *Cohomologie d'immeubles et de groupes S-arithmétiques*, Topology **15** (1976), no. 3, 211–232 (French).

[BT65] A. Borel and J. Tits, *Groupes réductifs*, Publications Mathématiques de l'IHÉS **27** (1965), 55–151.

[BT-1] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local : I. Données radicielles valuées*, Publications Mathématiques de l'IHÉS **41** (1972), 5–251.

[BT-2] ———, *Groupes réductifs sur un corps local : II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Publications Mathématiques de l'IHÉS **60** (1984), 5–184.

[BT-3] ———, *Schémas en groupes et immeubles des groupes classiques sur un corps local*, Bull. Soc. Math. France **112** (1984), no. 2, 259–301.

[BT-4] ———, *Schémas en groupes et immeubles des groupes classiques sur un corps local. II. Groupes unitaires*, Bull. Soc. Math. France **115** (1987), no. 2, 141–195.

[CN14] A. D. R. Choudary and C. P. Niculescu, *Real Analysis on Intervals*, Springer India, 2014.

[CY01] C.-L. Chai and J.-K. Yu, *Congruences of Néron models for tori and the Artin conductor*, Ann. of Math. (2) **154** (2001), no. 2, 347–382. With an appendix by Ehud de Shalit.

[DeB02a] S. DeBacker, *Homogeneity results for invariant distributions of a reductive p-adic group*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 3, 391–422.

[DeB02b] ______, *Some applications of Bruhat-Tits theory to harmonic analysis on a reductive p-adic group*, Michigan Math. J. **50** (2002), no. 2, 241–261.

[DeB02c] ______, *Parametrizing nilpotent orbits via Bruhat-Tits theory*, Ann. of Math. (2) **156** (2002), no. 1, 295–332.

[FR17] J. Fintzen and B. Romano, *Stable vectors in Moy-Prasad filtrations*, Compos. Math. **153** (2017), no. 2, 358–372.

[Fin21a] J. Fintzen, *On the Moy-Prasad filtration*, J. Eur. Math. Soc. (JEMS) **23** (2021), no. 12, 4009–4063.

[Fin21b] ______, *On the construction of tame supercuspidal representations*, Compos. Math. **157** (2021), no. 12, 2733–2746.

[Gar97] P. Garrett, *Buildings and classical groups*, Chapman & Hall, London, 1997.

[GI63] O. Goldman and N. Iwahori, *The space of* $\mathfrak{p}$*-adic norms*, Acta Math. **109** (1963), 137–177.

[GS92] M. Gromov and R. Schoen, *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one*, Inst. Hautes Études Sci. Publ. Math. **76** (1992), 165–246.

[GKP+17] S. S. Gubser, J. Knaute, S. Parikh, A. Samberg, and P. Witaszczyk, *p-adic AdS/CFT*, Comm. Math. Phys. **352** (2017), no. 3, 1019–1059.

[GR06] Y. Guivarc'h and B. Rémy, *Group-theoretic compactification of Bruhat-Tits buildings*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 6, 871–920 (English, with English and French summaries).

[Hij75] H. Hijikata, *On the structure of semi-simple algebraic groups over valuation fields. I*, Japan. J. Math. (N.S.) **1** (1975), no. 2, 225–300.

[IM65] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of* $\mathfrak{p}$*-adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 5–48.

[Ji08] L. Ji, *Large scale geometry, compactifications and the integral Novikov conjectures for arithmetic groups*, Third International Congress of Chinese Mathematicians. Part 1, 2, AMS/IP Stud. Adv. Math., 42, pt. 1, vol. 2, Amer. Math. Soc., Providence, RI, 2008, pp. 317–344.

[Ji12] ______, *Buildings and their applications in geometry and topology*, Differential geometry, Adv. Lect. Math. (ALM), vol. 22, Int. Press, Somerville, MA, 2012, pp. 89–210.

[JSY07] M. Joswig, B. Sturmfels, and J. Yu, *Affine buildings and tropical convexity*, Albanian J. Math. **1** (2007), no. 4, 187–211.

[Lan96] E. Landvogt, *A compactification of the Bruhat-Tits building*, Lecture Notes in Mathematics, vol. 1619, Springer-Verlag, Berlin, 1996.

[Lan00] ______, *Some functorial properties of the Bruhat-Tits building*, J. Reine Angew. Math. **518** (2000), 213–241.

[Lang] S. Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002.

[Lor18] O. Lorscheid, $\mathbb{F}_1$ *for everyone*, Jahresber. Dtsch. Math.-Ver. **120** (2018), no. 2, 83–116.

[Mil17] J. S. Milne, *Algebraic groups*, Cambridge Studies in Advanced Mathematics, vol. 170, Cambridge University Press, Cambridge, 2017.

[MP94] A. Moy and G. Prasad, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), no. 1-3, 393–408.

[MP96] ______, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helv. **71** (1996), no. 1, 98–121.

[Pra20a] G. Prasad, *A new approach to unramified descent in Bruhat-Tits theory*, Amer. J. Math. **142** (2020), no. 1, 215–253.

[Pra20b] ______, *Finite group actions on reductive groups and buildings and tamely-ramified descent in Bruhat-Tits theory*, Amer. J. Math. **142** (2020), no. 4, 1239–1267.

[PY02] G. Prasad and J.-K. Yu, *On finite group actions on reductive groups and buildings*, Invent. Math. **147** (2002), no. 3, 545–560.

[Rag94] M. S. Raghunathan, *Principal bundles admitting a rational section*, Invent. Math. **116** (1994), no. 1-3, 409–423.

[Rou09] G. Rousseau, *Euclidean buildings*, Géométries à courbure négative ou nulle, groupes discrets et rigidités, 2009, pp. 77–116.

[RTW15] B. Rémy, A. Thuillier, and A. Werner, *Bruhat-Tits buildings and analytic geometry*, Berkovich spaces and applications, 2015, pp. 141–202.

[RTW22] ______, *An intrinsic characterization of Bruhat-Tits buildings inside analytic groups*, Michigan Math. J. **72** (2022), 543–557.

[SGA3] M. Demazure and A. Grothendieck, *Schémas en groupes I, II, III, séminaire de géométrie algébrique 3*, Lecture Notes in Math, vol. 151,152,153, Springer-Verlag, Berlin, 1970.

[SS97] P. Schneider and U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. **85** (1997), 97–191.

[Suh21] J. Suh, *Stable lattices in p-adic representations I. regular reduction and schur algebra*, Journal of Algebra **575** (2021), 192–219.

[Suh22] ______, *Stable lattices in p-adic representations II. Irregularity and entropy*, Journal of Algebra **591** (2022), 379-409.

[Tei95] J. Teitelbaum, *The geometry of p-adic symmetric spaces*, Notices Amer. Math. Soc. **42** (1995), no. 10, 1120–1126.

[Tit57] J. Tits, *Sur les analogues algébriques des groupes semi-simples complexes*, Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, 1957, pp. 261–289.

[Tit74] ______, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin-New York, 1974.

[Tit79] ______, *Reductive groups over local fields*, Automorphic forms, representations and L-functions, Part 1, 1979, pp. 29–69.

[Yu01] J.-K. Yu, *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14** (2001), no. 3, 579–622.

[Yu09] ———, *Bruhat-Tits theory and buildings*, Ottawa lectures on admissible representations of reductive p-adic groups, 2009, pp. 53–77.

[Yu15] ———, *Smooth models associated to concave functions in Bruhat-Tits theory*, Autour des schémas en groupes, 2015, pp. 227–258.

[Wer07] A. Werner, *Compactifications of Bruhat-Tits buildings associated to linear representations*, Proc. Lond. Math. Soc. (3) **95** (2007), no. 2, 497–518.

[Wer11] ———, *A tropical view on Bruhat-Tits buildings and their compactifications*, Cent. Eur. J. Math. **9** (2011), no. 2, 390–402.

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