Title

Idealized Objects and Material Diagrams: a Cultural Practice-Approach to Understanding Definition Use in Early Geometrical Problem Solving

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Idealized Objects and Material Diagrams: a Cultural Practice-Approach to Understanding Definition Use in Early Geometrical Problem Solving

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Abstract

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This dissertation targets young students' developing knowledge relevant to a fundamental practice in academic mathematics: the use of diagrams to represent idealized mathematical objects whose properties are established by definition (as in the use of drawn dots to represent zero-dimensional points and drawn lines to represent one-dimensional lines with infinite extent). Initiation into this 'definitional practice' is critical to students' mathematical development. However, the practice is understudied in educational research. It also presents a significant source of confusion for students. Instead of using definitions, students may rely on the appearances of the diagrams and their knowledge of the physical world—an 'empirical' rather than definitional approach.

I have designed three studies to investigate students' developing understanding of the definitional practice, using points and lines in Euclidean geometry as a potentially fruitful mathematical context. Adopting a design research approach that draws on Vygotsky's method of double stimulation, these studies employ a pedagogical strategy that is rarely observed in mathematics classrooms: providing definitions and making explicit the distinction between drawn diagrams and the idealized objects they symbolize, pointing out that the defining features of points and lines are not perfectly embodied in their conventional representations. These studies investigate students' uptake of this support and capture students' developing understanding of the definitional practice.

The first study uses an experimental design to determine whether there are age/grade-related changes in students' uptake of the intervention. Participants include students from a San Francisco Bay Area charter school in fourth grade (n=46), sixth grade (n=53), and eighth grade (n=43). Students are assigned to one of two treatment conditions whose purpose was to manipulate exposure to the definitions of points and lines. In the experimental treatment, students are presented with a sheet that contained mathematical definitions of points and lines, which are unavailable to students in the control condition. Students are then administered a paper-and-pencil assessment consisting of eight multiple choice items, some of which included diagrams of points and lines. Analyses suggest that with age, children shift towards relying on definitions rather than the appearances of the diagrams and knowledge about material objects.

The second study addresses unanswered questions about the general learning trend identified by the first study. Specifically, it uses structured one-on-one interviews to determine whether (a) students are indeed drawing on provided definitions when selecting idealized (rather

than empirical) answers, and (b) whether students are constructing a conceptual differentiation between material diagram and idealized object. Participants include fourth (n=40) and sixth (n=37) grade students.

The third study explores how students make sense of the definitions of points and lines presented in Study 1 and 2. In particular, this study considers how students may be constructing analogies that draw on ideas related to the material world. Results demonstrate that students' materially-based analogies incorporate other ideas typically learned in school to make sense of the definitions of points and lines. For example, students frequently refer to scientific ideas related to the microscopic world of atoms and molecules to explain the zero-dimensionality of a mathematical point. Participants were identical to those of the second study.

The insights generated by these three studies contribute to our understanding of sociocultural processes in cognitive development as well as to mathematics education research. This dissertation also has practical implications for mathematics instruction, suggesting potentially promising pedagogical possibilities for giving students a better 'feel for the game'—an understanding of what academic mathematics is all about.

This is dedicated to my wife, Amelia Bird, and our son, Rowan.
I’m certain they can’t wait to read it.

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INTRODUCTION

I call a sign which stands for something merely because it resembles it, an icon. Icons are so completely substituted for their objects as hardly to be distinguished from them. Such are the diagrams of geometry…[I]n the middle part of our reasonings we forget that abstractness in great measure, and the diagram is for us the very thing. So in contemplating a painting, there is a moment when we lose the consciousness that it is not the thing, the distinction of the real and the copy disappears, and it is for the moment a pure dream….
Peirce, 1885 (pg. 181)

Because of the difficulty of identifying objects which are neither discernible to the sense nor psychological, visible objects have been substituted for them. But this is to forget that these symbols are not what we want to study."
Frege, 1970 (pg.482)

There is arguably a two-sidedness to academic mathematics. Comprising one side is a family of semiotic practices united by an *empirical* orientation. Examples of such practices include quantification, measurement, and modeling—practices that are organized by and in direct reference to objects in the material world, which includes the physicality of constructed mathematical representations. In its second side, academic mathematics is conducted as an axiomatic science, encompassing such practices as stipulation, deduction, and definitional reasoning—practices organized around logical consistency, rules of inference, and adherence to normative definitions. In the context of such practices, mathematical diagrams are taken to be representations of idealized objects, rather than the target of observation or measurement. Consequently, children's development as doers of mathematics is likewise two-sided, as they learn to participate in both kinds of practices, and to switch fluidly back and forth between them, often using them together in the context of the same representation or problem (Laborde, 2005; Schoenfeld, 1983; 1986).

The studies reported here target young students' developing understanding of a core axiomatic practice in academic mathematics—the use of diagrams to represent idealized mathematical objects whose properties are stipulated by definition. In Euclidean geometry, good examples of the practice include using a dot to stand for a zero-dimensional object ("point"), and using a drawn line tipped with arrows to depict a one-dimensional object with infinite extent ("line").[1] Mathematical diagrams are employed in this practice to signify *idealized* entities—that is, entities that constitute forms of "imagined perfection" (G. Lakoff, personal communication, February 11, 2015) and are therefore non-perceptible (Font & Contreras, 2008). This disciplinary practice constitutes a kind of "language game" (Wittgenstein, 1953) involving the coordinated use of two forms of semiotic tools—diagrams and definitions. The 'rules of the game' dictate that 1) diagrams are to be treated as representations of idealized, not material objects, and 2) stipulated definitions ultimately determine the features of mathematical objects, not the appearances of their diagrammatic representations. As I will discuss, this practice is vital to young students' mathematical development, and yet is significantly understudied in

[1] Euclid, the founder of modern geometry, defined a line as "breathless length," and a point as "that which has no part" (Lakoff & Núñez, 2000).

educational research. Moreover, the tension between the inevitably imperfect material diagrams and the idealized objects they represent presents an inherent likelihood of confusion and miscommunication between teachers and students (Laborde, 2005; Font, Godino, Planas, & Acevedo, 2010; Mesquita, 1998), as I will discuss in detail below.

Students' initiation into the practice of using diagrams to represent stipulated mathematical objects is critical to their development as doers of academic mathematics. (Hereafter, I will refer to it as the 'definitional' practice given the prescribed primacy of definitions over the appearances of diagrams.) Reasoning from definitions has been shown to present difficulties for even advanced students of mathematics (Edwards & Ward, 2004; Harel, 1999; Moore, 1994; Tall, 2008; Tall & Vinner, 1981) and has been privileged in ongoing shifts in mathematics education (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). More generally, because this practice in some sense epitomizes the axiomatic facet of academic mathematics, becoming a participant in the practice offers an invaluable "feel for the game" (Bourdieu, 1990)—that is, an orienting and generative understanding of what doing mathematics is all about: the study of idealized entities.

The definitional practice is significantly understudied in educational research. For the most part, researchers have focused on knowledge related to empirical practices, not definitional practices. Well-studied empirical practices include of activities like counting (e.g., Fuson, 1988; Gelman & Gallistel, 1979; Saxe, Guberman, & Gearhart, 1987; Sarnecka, 2013) and measurement (e.g., Carpenter & Lewis, 1976; Hiebert, 1981; Lehrer, 2003; Miller, 1984; Piaget & Inhelder, 1956; Stephan et al, 2003).

Educational researchers have indeed discussed the distinction between representation and mathematical object, and its importance for children's grasp of the discipline, as well as the difficulties this distinction is likely to present for students and teachers (Dörfler, 2000; Duval, 1999; 2006; Font & Contreras, 2008; Lakoff & Núñez, 2000; Mesquita, 1998; Radford, 2002; Sfard, 2000). However, there has been hardly any direct, systematic empirical investigation of students' developing knowledge of and/or engagement with definitional practices (exceptions include Fischbein, 1993). For example, little is known about students' developing understanding that, as depictions of stipulated objects with idealized properties, diagrams are inevitably imperfect but their "imagined perfection" can nonetheless be useful in supporting thinking and communication about mathematical ideas.

The importance and inherent challenges of the definitional practice for young students make its development well worth systematic investigation. The studies reported here use points and lines in Euclidean geometry as a potentially fruitful mathematical context for studying students' developing understanding of the practice, and how specific instructional interventions can support this development.

Theoretical Framework: A cultural practice approach to understanding idealized objects and material diagrams

The notion of a cultural practice—a recurrent activity organized in relation to community norms, values, and conventions—frames the way I conceptualize individuals' use of diagrams to represent idealized objects. The construct of a cultural practice derives from the large body of scholarship on the sociocultural nature of cognitive development

(Cole 1996; Lucariello, 1995; Rogoff, 2003; Wertsch, 1991; Saxe & de Kirby, 2014). Via their participation in cultural practices, individuals perform cognitive work through generating goals, and solving problems (Saxe, 1994; 1999), and in this process, individuals develop novel practice-linked knowledge (Saxe, 2012; Saxe & de Kirby, 2014). Scribner and Cole (1981) offer a more elaborated definition and example of the construct of cultural practice, one that has some resonance with the definitional practice that is of relevance to this inquiry:

> By a practice we mean a recurrent, goal-directed sequence of activities using a particular technology and particular systems of knowledge. …. We can apply this concept to spheres of activity that are predominantly conceptual (for example, the practice of law) as well as to those that are predominantly sensory-motor (for example, the practice of weaving). All practices involve interrelated tasks that share common tools, knowledge base, and skills. But we may construe them more or less broadly to refer to entire domains of activity around a common object (for example, law) or to more specific endeavors within such domains (cross-examination or legal research). Whether defined in broad or narrow terms, practice always refers to socially developed and patterned ways of using technology and knowledge to accomplish tasks. Conversely, tasks that individuals engage in constitute a social practice when they are directed to socially recognized goals and make use of a shared technology and knowledge system.
> Scribner & Cole, 1981 (pg. 236)

The construct of practice has become increasingly influential in mathematics education, leading to a reconceptualization of what should be a principal goal of mathematics instruction. Educators have argued that mathematics education should involve more than acquiring knowledge of particular procedures or concepts. Rather, it should entail learning to participate in forms of activity constitutive of the discipline. For example, Ball and Bass (2003) have argued for the importance of making explicit to students the nature of valued mathematical practices, which are often left implicit, such as the axiomatic practices of proof and definitional reasoning. Further, the Common Core State Standards for Mathematics, an influential policy document adopted by many states in the United States, has elevated the notion of discipline specific practices as being a core target of teaching (Core-State-Standards-Initiative, 2010). The Standards identify several valued mathematical practices, including practices related to mathematical modeling and argumentation.

A practice-based conceptualization of mathematics learning serves to clarify sources of confusion related to the definitional practice that may emerge in classroom instruction. To illustrate, I identify three potential sources of confusions that challenge students' developing understanding that geometric diagrams are representations of idealized objects.

First, the definitional practice contrasts with empirically-oriented mathematical practices rooted in daily life and are explicit targets of instruction. These materially-oriented practices include activities such as fair sharing, counting, and measurement. I show subsequently how not clearly distinguishing the definitional practice from empirical practices may lead to confusion for students.

Second, the potential for students to conflate drawings with the idealized objects is particularly acute in the domain of geometry, because geometric diagrams 'resemble' the mathematical objects they represent (exemplifying Peirce's notion of an "icon" (1885)). For this reason, the very properties that make diagrams so useful in thinking and communicating about idealized objects contribute to the likelihood that students may interpret the diagrams as the mathematical objects themselves (Laborde, 2005; Mesquita, 1998) . On this issue, Dorfler (2000) has pointed to the "two-sidedness" of geometry. On the one hand, students engage in activities involving the construction, measurement, and taxonomic identification of drawings—activities that are chiefly organized by the empirical properties of the material diagrams (i.e., material practices). On the other hand, of course, students also learn to think with formal definitions that include properties that cannot be embodied in drawn diagrams, but which the diagrams symbolize.

Third, a final source of confusion is the fact that formal mathematics instruction often does not make explicit to students the 'rules of the game'—the fact that diagrams are representations, and that the relevant definition is ultimately more important than what the diagram looks like (Laborde, 2005). This echoes the more general observations made by other mathematics education researchers concerning valued mathematical practices (Ball and Bass, 2003)—that core components are often left implicit. Font, Godino, Planas, & Acevedo (2010) provide an excellent summation of this source of confusion regarding the definitional practice:

> Mathematical discourse moves flexibly between representations and the mathematical objects they represent. In mathematical practice it is not always considered necessary to distinguish explicitly representation from the object represented, as this distinction is taken for granted. However, at times it is worth making a clear distinction between the two of them… A basic aim of mathematics education is that students learn to move flexibly between representations and the mathematical objects they represent. This is not easy for students to learn, and it also poses a challenge for teachers because they are not always aware of the complexity….
>
> Godino, Planas, & Acevedo, 2010 (pg. 15)

For the reasons discussed above, it is likely that students will not differentiate[2] diagrams from the idealized objects they symbolize. Below, I provide two brief illustrations from mathematics education. Each demonstrates the utility of a practice-based conceptualization for clarifying potential sources of confusion for children as they begin to learn about and engage with the definitional practice.

Illustration A: Classroom Observation

Consider the following brief classroom observation noted by Saxe (2004), which illustrates how the rules of the game may be left implicit in mathematics instruction. In an upper elementary school classroom, a lesson on fractions is underway. The teacher quickly draws two "cookies" on the board (Figure 1), inadvertently drawing one cookie larger than the other. She asks, if the cookies were split among four people, what fraction of a cookie would each person receive? One student answers "two fourths" and explains that, since the cookie on the right is bigger, each person should get a fourth from each cookie (not a half of any one). Although the student's response is mathematically equal to the intended answer of "one half," the logic underlying the answer reflects an orientation to the diagram that the teacher did not intend.

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[2] My use of the term "differentiation" throughout this dissertation draws on Werner and Kaplan's (1963) orthogenetic principle of development, including the development of symbol use.

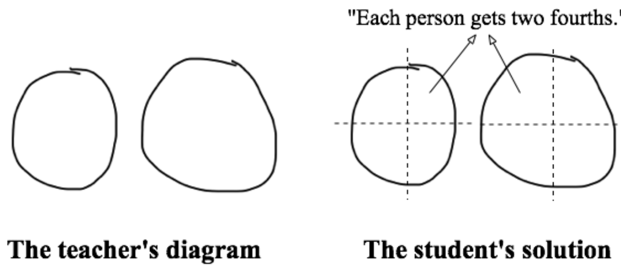

Figure 1. Two “cookies” to be split among four people.

A practice-based conceptualization provides a useful lens with which to capture what is occurring in the classroom observation. In a nutshell, teacher and student are engaging in fundamentally different practices involving the same diagram. For the teacher, the problem was to divide two circles stipulated to be congruent—the definitional practice of using inevitably imperfect diagrams to represent and reason about idealized objects. For the student, the problem was to equally share two cookies whose manifestly unequal size is depicted by diagrams. Conceptualized in terms of this empirical practice, whether it involves diagrams or real cookies, the problem’s initial state and solution is open to empirical verification, rather than established by fiat.

Likewise, a practice-based conceptualization supports the insight that the ‘rules of the game’ can remain implicit in classrooms, as in the one described here. Rarely does a teacher make explicit the distinction between a diagram and what it represents, or discuss the importance of the stipulated definition relative to what the diagram looks like (nothing like this ensued in the episode above). As a result, such miscommunications may very well go unnoticed and unaddressed.

Illustration B: Fischbein’s study of the definitional practice

Fischbein (1993) produced one of the few empirical studies with young students targeting the tension between representational appearances and the definitions of geometric objects.[3] His study, and its limitations, points to the potential utility of a practice-based approach.

Fischbein presented elementary students (grades 2-6) with the task shown in Figure 2, asking if one of two points (Point 1 or Point 2) is bigger or heavier than the other, or if both of the points are the same. Fischbein’s results showed an age-based developmental progression in students’ responses. Young students’ answers were often consistent with the appearances of the representation (e.g., Point 1 is bigger or heavier than Point 2), while older students frequently answered that the two points were the same, consistent with a conception of points as ideal objects that, by definition, always have no size. Students in the middle of the age range often answered in inconsistent or contradictory ways. For Fischbein, this finding captures the progressive development of a “figural concept,” wherein the spatial/perceptual (i.e., “figural”) dimension is increasingly subordinated to the definitional (i.e., “conceptual”) dimension. The task Fischbein used is innovative—the appearances of the diagram and the behavior of

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[3] Fischbein and Nachlieli (1998) have also conducted an empirical study of developmental differences in the relationship between definitions and the appearances of geometric representations in students’ thinking. However, this study targets polygon classification and does not explicitly address the kinds of idealized properties that are of interest here (e.g. missing dimensions and infinite extent).

physical objects suggest a response that conflicts with what is entailed by the definition of a mathematical point. This feature supports Fischbein's inferences about the shifting role of mathematical definitions in students' problem solving.

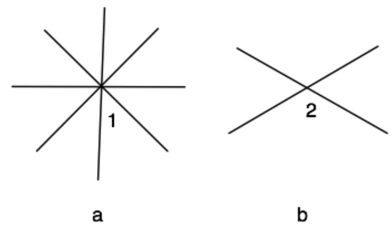


Figure 2. Task used by Fischbein (1993).

However, a practice-based conceptualization provides a useful lens through which to formulate and address important questions that are not posed or remain unanswered in Fischbein's study. Because students' definitional knowledge is not assessed in Fischbein's study, and because Fischbein did not use an experimental manipulation in which some participants had access to definitions and others did not, any inference about developmental differences in the way students use definitional reasoning about mathematical objects is problematic. Thus, the central developmental process in students' engagement with the definitional practice is obscured.

Methodological Approach—The Method of Double Stimulation

To investigate the definitional practice in a way that addresses what I regard as weaknesses in Fischbein's study, I employ a methodological approach that draws on Vygotsky's seminal treatment of the mediated properties of higher psychological functions (Vygotsky, 1978; 1986) (see also Cole, 1996; Saxe, 2012; Wertsch, 1991). Vygotsky argued that humans' characteristic forms of cognitive activity share a common element: they arise from humans' ability to reorganize their own response to the environment through the use of material and semiotic resources. In particular, I make use of Vygotsky's experimental "method of double stimulation." Vygotsky referred to it as the method of "double stimulation" because it incorporated two stimuli: a problem-solving task and some form of auxiliary material provided to participants. The method was designed to reveal developmental differences in participants' use of the auxiliary materials to support their solutions to challenging problems (Vygotsky, 1978, 1986) (see also Cole, 1996; Saxe, 2012; Wertsch, 1991). This contrasts with the dominant approach in psychological research that simply reveals differences in how individuals solve focal problems, with no concern for how individuals transform the task through the use of available resources, thereby regulating their own behavior. In Vygotsky and Luria's own words:

> We do not limit ourselves to the usual method of offering the subject simple stimuli ... to which we can expect a direct response; we simultaneously offer a second series of stimuli which must play a functionally special role, serving as a means by which the subject can organize his own behavior. In this way, we study the process of accomplishing a task by the aid of certain auxiliary means, and ... this way of bringing auxiliary means of behavior to the surface permits the tracing of the entire genesis of

the most complex forms of higher psychological processes.
Vygotsky & Luria, 1994 (pg. 159)

A famous example of the method of double stimulation is the color chip experiment. Vygotsky presented participants with a memory task involving color names, along with physical color chips that could be used as memory aids. Vygotsky found that older individuals appeared to consciously draw upon the auxiliary stimuli (the color chips) to overcome challenges in problem solving, whereas younger children were sometimes confused by the auxiliary stimuli and typically did not make productive use of them. These results frame what is for Vygotsky a central developmental process: drawing upon available resources to, in effect, transform the problem being solved and to generate a solution.

The method of double stimulation has remained significantly underutilized in research on learning and development. While researchers adopting a sociocultural approach to the study of human activity have taken up the method, they have done so largely as a conceptual tool—rather than an empirical technique—in order to investigate and theorize about high level constructs such as agency (e.g., van Oers, 2015; Barma, Lacasse, & Massé-Morneau, 2015; Haapasaari & Kerosuo, 2015; Engeström, Kajamaa, & Nummijoki, 2015). By contrast, there has been little research using the method of double stimulation that—like Vygotsky's own principal work—is both experimental/interventional in design and that focuses on learning and development.

I developed three studies whose design resonates with Vygotsky's method of double stimulation—studies whose purpose is to investigate the development of the definitional practice.[4] In these studies, I present students with definitions of points and lines alongside a set of designed problems that (like Fischbein's) feature potentially misleading diagrams, which may cue students to conceptualize the diagrams in terms of empirical practices and thus to rely on the appearances of the diagrams and their knowledge of the material world rather than the definitions. Reflecting Vygotsky's conceptualization, I regard the provided definitions as a class of 'semiotic artifacts,' a form of auxiliary stimuli that students might draw upon in problem solving. The design of the problems and definitions will be coordinated in the studies, such that the definitions are logically and linguistically consistent with certain answers and therefore can be useful in problem solving.

Following both Vygotsky's and Fischbein's findings, I would expect that younger students with access to definitions would not make use of the definitions as frequently as their older counterparts. As I discussed, however, I would regard this shift as indicating increasing understanding of the practice, rather than simply overall cognitive development or the development of specific mathematical concepts.

The first study uses experimental methods to ask whether there are grade-related changes in the students' definition use to reason about idealized objects (prior to formal instruction in geometry)—specifically, whether access to stipulated definitions might

[4]Contemporary research methods that most closely resemble Vygotsky's are those that engage the neuropsychological construct of executive functioning (Case, 1992; Chan, Shum, Toulopoulou, & Chen, 2008; Zelazo, & Müller, 2002), typically theorized in purely cognitivist terms. Ironically, the construct was pioneered by Alexander Luria (1973), a student of Vygotsky and a proponent of sociocultural approaches to understanding higher cognitive functions.

influence students' idealization of diagrams as representations of zero-dimensional points and one-dimensional lines. Students are assigned to one of two treatment conditions whose purpose was to manipulate exposure to the definitions of points and lines. In the experimental treatment, students are presented with a sheet that contained mathematical definitions of points and lines, which are unavailable to students in the control condition. Students are then administered a paper-and-pencil assessment consisting of eight multiple choice items, some of which include diagrams of points and lines.

The second study uses structured one-on-one interviews in order to directly observe students' definition use. The interviews begin with students solving problems (similar to those used in the first study) without access to definitions and sharing their thinking. Then, students are asked how they think a mathematician would answer these same questions, and they are given an opportunity to select different answers if they choose. Additional items probe students' construction of a conceptual differentiation between idealized object and material diagram—a central understanding in the definitional practice.

The final study investigates students' sense-making of the definitions. It asks students how they would explain the definitions to another student. Specifically, this study seeks to identify the conceptual resources students draw on to make sense of a point with zero-dimensions and a line with infinite extent—two challenging and foreign ideas.

In providing students with definitions of points and lines, these studies incorporate a pedagogical element and therefore constitute a form of design based research. At the core of design based research is the intention of supporting a developmental process in order to study it—a developmental process that is generally not available for study (Cobb et al, 2003; Collins et al, 2009). As mentioned earlier, explicit discussions of the definitional practice appear to be rare in classrooms. It is not often made explicit to students that drawings of points and lines are not themselves the idealized objects to which they refer. It is precisely this pedagogical approach that is deployed in these studies. Investigating how students take up efforts to make explicit the unspoken 'rules of the game' may have useful implications for pedagogy, in addition to shedding light on the development of an important but understudied practice in academic mathematics—one that speaks to the complex epistemology and ontology of mathematical objects.

STUDY 1—A LEARNING TREND IN STUDENTS' DEVELOPING UNDERSTANDING OF THE DEFINITIONAL PRACTICE

Study 1 is an experimental/interventional study whose purpose is to investigate possible developmental shifts in students' developing understanding of the definitional practice. In particular, the study considers whether students draw on definitions of points and lines in challenging problem-solving contexts that feature misleading diagrams.

Overview of Study and Research Questions

The purpose of this experimental study is to explore developmental differences in elementary and middle school students' initial engagement in the definitional practice. This study addresses the limitation of Fischbein's study by employing an experimental design that manipulates students' exposure to definitions, inspired by Vygotsky's method of double stimulation. Recall that Fischbein's study did not manipulate students' exposure to definitions, so no direct inferences can be drawn about the role of definitional knowledge in students' problem-solving on the intersecting lines task. Manipulating students' exposure to definitions, in conjunction with problems that sometimes feature misleading representations (as with Fischbein's task), allows me to draw inferences about students' use of the definitions and the effect of the diagrams on their problem solving. In other words, this design enables me to infer whether students adopt either empirical or definitional approaches, based on the presence of diagrams in the problems and the availability of definitions. Further, I include two types of problems, those featuring potentially misleading diagrams and those without diagrams.

The central expectation is that, with age, there would be an increasing effect of the availability of definitions in students' problem solving, consistent with an increasing understanding of the definitional practice. This expectation is supported by both Vygotsky's treatment of development and Fischbein's findings. Through his "method of double stimulation," Vygotsky found that individuals' propensity to draw on potentially useful auxiliary stimuli (in this case, mathematical definitions) is a core developmental trajectory. In the realm of mathematics—geometry in particular—Fischbein found that older students more frequently provided answers that were consistent with mathematical definitions. Further, it is also expected that younger students would be more influenced by the appearances of the diagrams, resulting in a greater difference in performance for the two types of problems. This expectation is also reflected in Fischbein's finding that younger students' tended to answer in ways that were consistent with the appearances of the diagrams, as well as general findings in development of psychology that younger children tend to rely on surface features of problems (Langer & Strauss, 1972; Piaget, 1948; 1952).

Participants

The participants in the study included students from a San Francisco Bay Area charter school in fourth grade (n=46; F=25, M=21), sixth grade (n=53; F=27, M=26), and eighth grade (n=43; F=16, M=27). Participants had not yet had the standard year-long course in formal geometry. At each grade level, all students in each of two classrooms participated. The ethnic composition of the classrooms varied, although most students were Hispanic (see Table 1*: Ethnic background of participants by grade level*), and 80% of the student body as a whole was classified as English language learners (English Learners (EL) or Reclassified Fluent English Proficient (R-FEP)). A large percentage of the students were from low income families, with 86% eligible for free and reduced priced lunches.

Table 1: Ethnic background of participants by grade level

Grade Level	Hispanic	African American	Caucasian	Asian	Other
8^{th} (n=43)	91%	7%	0%	0%	2%
6^{th} (n=53)	79%	11%	4%	4%	2%
4^{th} (n=46)	67%	25%	2%	4%	2%

Piloting and the Refinement of Geometrical Definitions and Problems

In order to develop age-appropriate definitions to use in the study, I conducted pilot interviews with students in fourth, sixth, seventh, and eighth grade. Based on these interviews, early drafts of definitions were refined and elaborated so they would be comprehensible and most likely to support students in drawing on them in problem solving. In the end, fourth-graders, the youngest age group that participated in the study, could successfully paraphrase the final form of the definitions.

The pilot study also supported the refinement of two classes of multiple choice items that I included in the study. The first class did not include diagrams. Their purpose was to assess whether participants had at least a rudimentary level of "processing" or attention to the definitions (as measured by successful performance on items). The second class, as noted earlier, were intended to confront students with a tension between appearances of diagrams (of points and lines) and relevant definitions in problem solving. For both item types, pilot interviews not only supported development and refinement of multiple choice problem items, but also alternative answer choices that students would find compelling. (See Appendix A for details concerning pilot interviews and the refinement of study materials).

Treatment Conditions

Students were assigned to one of two treatment conditions whose purpose was to manipulate exposure to the definitions of points and lines. (1) In the experimental treatment, students were presented with a sheet that contained mathematical definitions of points and lines. (2) In the control condition, students were presented with a sheet that contained narrative text about the history geometry that was of similar length and reading difficulty to the experimental sheet; this sheet did not contain any mathematical definitions.

The sheets were matched for word count and reading level using the Flesch-Kincaid diagnostic program. Each sheet was written such that it was rated at the fifth-grade level. A fifth-grade reading level was selected to make it maximally accessible to fourth-graders and because a large percentage of the students at each grade were classified as English language learners. A lower reading level was found to be impractical in conveying the necessary ideas.

Experimental treatment

Figure 3 contains the one-page informational sheet used in the experimental treatment condition. To support students' developing understanding of the definitional practice, the sheet contained a 'mini-lesson' on geometric points and lines that included mathematical definitions. The potentially useful information contained on the experimental treatment was designed to serve three purposes.

a. Definitions provided. The sheet presented definitions of points and lines. A point was defined as follows: "In mathematics, a point is a precise location. It is so small that it has no size at all, and it takes up no space." A line was defined as follows: "In mathematics, a line is straight, infinitely long and is so thin that it has no width at all."

b. Representations and Mathematical Objects Distinguished and Compared. The sheet made an explicit distinction between the mathematical objects and the way they are conventionally represented. The sheet pointed out that the defining properties of points and lines are not perfectly embodied in the geometric representations: "You might notice that the drawing of a point is pretty small, but it is still big enough to have some size. You can see it, can't you? *But remember that a true MATHEMATICAL point is so small that it doesn't have any size at all!"*

c. Possibility of Useful Information Reinforced. The sheet underlined the fact that this information might be useful in solving the set of problems that students would be given shortly, and that they could come back to the sheet at any point and as many times as they wished during the assessment.

Points and Lines

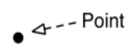
In a few minutes, you will answer some questions about points and lines in geometry. If you have never heard of points and lines, that's okay! If you have already studied them in class, that's okay too!

To help you, here is some IMPORTANT information about what mathematicians mean by points and lines. You can come back to this page anytime you want to help you solve the problems.

Mathematical Point

In mathematics, a point is a precise location. It is so small that it has no size at all, and it takes up no space.

Here is how a point is usually shown in math books. It looks like a dot.



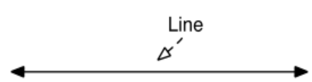
You might notice that the drawing of a point is pretty small, but it is still big enough to have some size. You can see it, can't you?

But remember that a true MATHEMATICAL point is so small that it doesn't have any size at all!

Mathematical Line

In mathematics, a line is straight, infinitely long and is so thin that it has no width at all.

Here is how a line is usually shown in math books.



You might notice that the drawing of a line is pretty long, but the drawing doesn't go on forever.

But remember that a true MATHEMATICAL line is so long that it goes on forever! We show this by putting arrows at the end.

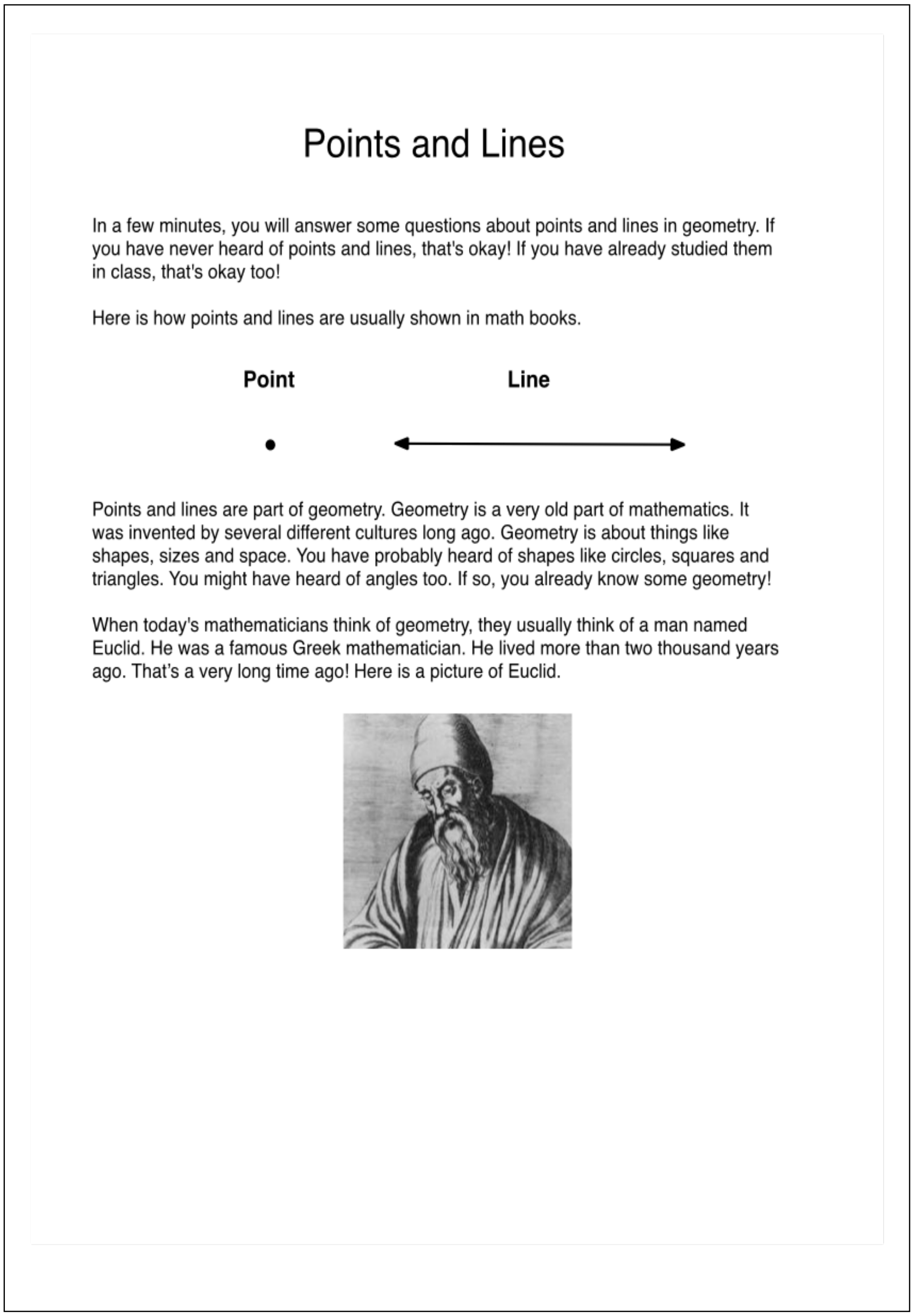
You might also notice that the drawing of a line is pretty skinny, but it is still wide enough to have some thickness.

But remember that a true MATHEMATICAL line is so skinny that it has no thickness at all!

Figure 3. Information sheet (with definitions) used in the experimental condition.

Control treatment

Figure 4 contains the information sheet used in the control condition. As noted, the sheet was matched to the one used in the experimental condition in both word count and reading level. This sheet provided conventional diagrams of points and lines but contained no definitions. Rather, it included information about the disciplinary scope and history of geometry, information that was unlikely to help students solve the problems.



Points and Lines

In a few minutes, you will answer some questions about points and lines in geometry. If you have never heard of points and lines, that's okay! If you have already studied them in class, that's okay too!

Here is how points and lines are usually shown in math books.

Points and lines are part of geometry. Geometry is a very old part of mathematics. It was invented by several different cultures long ago. Geometry is about things like shapes, sizes and space. You have probably heard of shapes like circles, squares and triangles. You might have heard of angles too. If so, you already know some geometry!

When today's mathematicians think of geometry, they usually think of a man named Euclid. He was a famous Greek mathematician. He lived more than two thousand years ago. That's a very long time ago! Here is a picture of Euclid.

Figure 4. Information sheet (without definitions) used in the control condition.

Outcome measure

The outcome measure consisted of eight multiple-choice questions. In keeping with the language used in the experimental prompt, the problems consistently referred to points and lines as “mathematical points” and “mathematical lines.” Each of the eight questions included between three to five answer choices. These answer choices permitted an analysis of students’ responsiveness to the definitions in problem solving. Only one of the choices provided was clearly consistent with the definition of the mathematical object and an idealized interpretation of its representation. The other choices, by contrast, were inconsistent in various ways with the ideal properties of points and lines; in general, the alternative answers attributed materiality to points and lines and reflected a reliance on the appearances of the diagrams or students' knowledge of material objects. As mentioned, these alternative answer choices were developed and refined through a small number of informal pilot interviews with students in 4th, 6th, 7th, and 8th grades with students at the same school. (These students did not participate in the experimental study.)

The eight problems were of two types. The first three problems contained no geometric representations ("*no diagrams*" problems) and the last five problems did include geometric representations ("*diagrams"* problems). I planned to compare students' performance on these two types of items. I considered counterbalancing the order of the two item types to rule out order effects. However, I decided against counterbalancing because I believed the *no diagrams* problems (those without potentially misleading representations) would offer a less demanding entry into the assessment.

No Diagram items

The first three problems were designed to assess the degree to which students minimally processed the definitions presented on the experimental information sheet.

Problem 1: How big is a point? The question asked, "How big is a mathematical point?" Three answer choices were provided: (a) It is pretty small, the size of a small dot; (b) It has no size at all; (c) It depends on how big you draw it.

Problem 2: How long is a line? The question asked, "How long is a mathematical line?" Three answer choices were provided: (a) It depends on how long you draw it; (b) It is usually about as long as a finger; (c) It is infinitely long.

Problem 3: How thick is a line? The question asked, "How thick is a mathematical line?" Again, three answer choices were provided: (a) It has no thickness at all; (b) It is pretty thin, a little bit thicker than a hair, but not as thick as a pencil;(c) It depends on how thick you draw it.

Diagram items

The remaining five problems incorporated geometric representations of points and lines. These problems were designed to provoke a conflict between the appearances of the physical representations and the defining features of the mathematical objects—that the ideal properties of mathematical points and lines are not perfectly embodied in their representations. Selecting the idealized multiple-choice response would indicate whether students drew on the relevant mathematical definition to mediate their problem solving. The five *Diagram* problems—problem numbers four through eight—were as follows:

Problem 4: What is the dimensionality of points that converge on the same spot? The problem asked students to consider the geometrical representation presented in

Figure 5. The diagram and accompanying text presented points in motion that converge on the same spot as well as the accompanying question, “What would happen to these two points?” The student is presented with four answer choices: (a) The two points would turn into one bigger point; (b) The two points would turn into one point, which would still have no size at all; (c) The two points would turn into one point, which would still be the size of a small dot; (d) Even though it might look like there is one point, there would still be two points. One point would be stacked on top of the other.

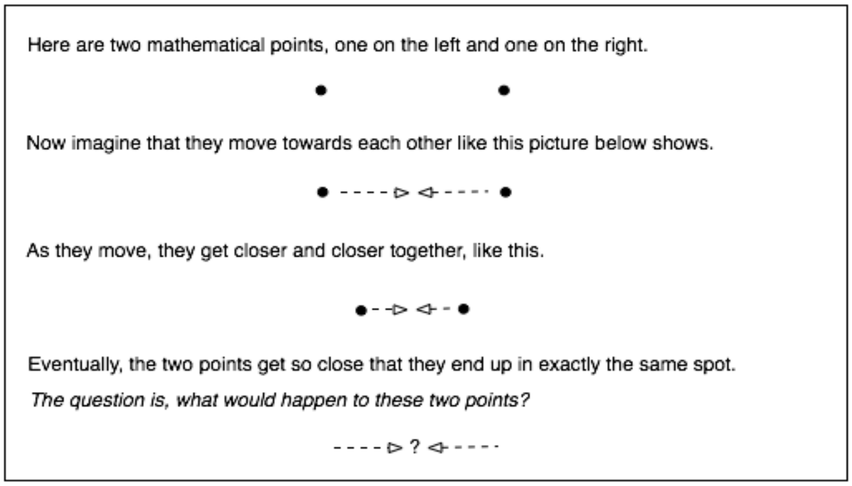


Figure 5. Geometrical representation used for Problem 4: What is the dimensionality of points that converge on the same spot?

Problem 5: What is the dimensionality of lines that converge on the same spot? This problem asked students to consider the problem presented in Figure 6. The diagram and accompanying text presented lines in motion that converge on the same spot as well as the accompanying question, “what would happen to the two lines?” The student was presented with four answer choices: (a) There would be two lines of the same thickness, one on top of the other; (b) There would be one line instead of two, but it would be thicker than the original lines; (c) There would be one line instead of two, and it still would have no thickness at all; (d) There would be one line instead of two, and it would still be a little bit thicker than a hair.

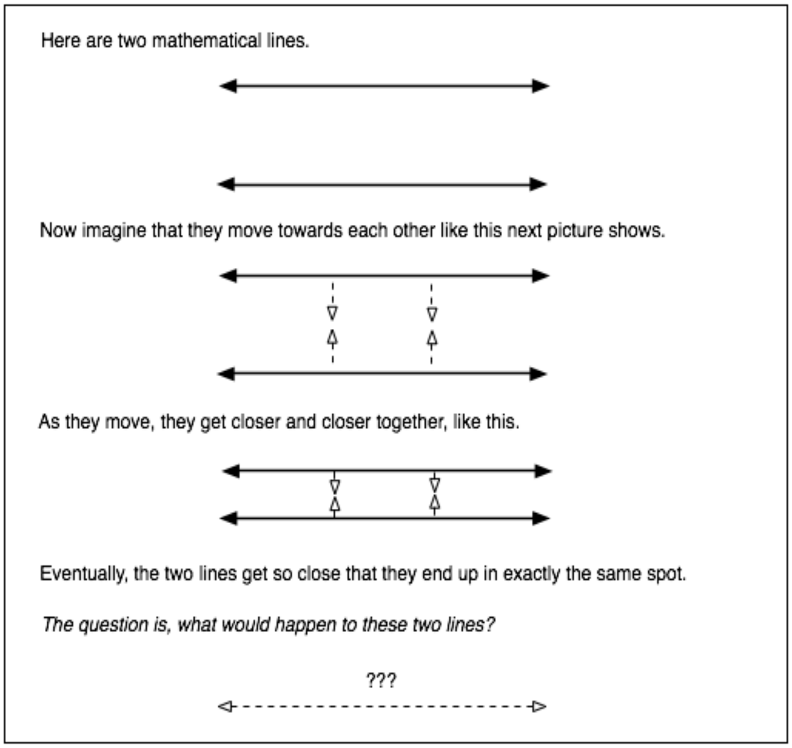


Figure 6. Geometrical representation used for Problem 5: What is the dimensionality of lines that converge on the same spot?

Problem 6: How many points on a line segment? The problem asked students to consider the problem shown in Figure 7. The diagram and accompanying text presented a line segment, a line segment with a point represented, and the accompanying question, "How many more mathematical points would fit on this length?" The student is presented with five answer choices: (a) Between 5 and 20 points; (b) Between 20 and 100 points; (c) Between 100 and 10,000 points; (d) an infinite number of points; (e) It depends on how big you draw the other points.

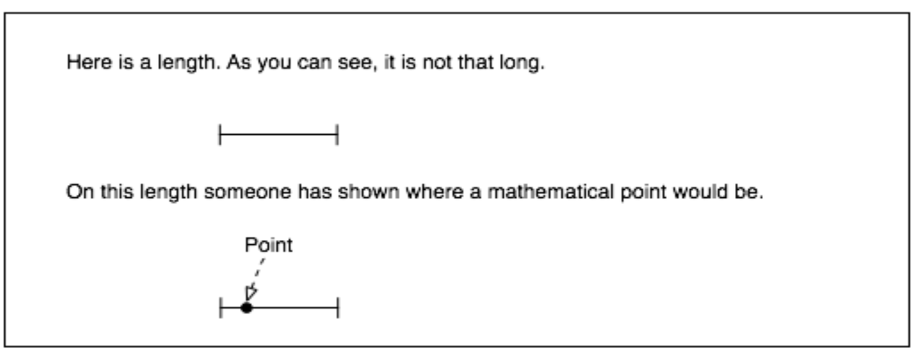


Figure 7. Geometrical representation used for Problem 6: How many points on a line segment?

Problem 7: Relative number of points that can fit on two represented line segments of different lengths? This problem asked students to consider the problem presented in Figure 8. The diagram and accompanying text presented two line segments, one shorter than the other, and asked which of the following three statements is true: (a) More points can fit on the one on the bottom; (b) An infinite number of points can fit on both lengths; (c) It depends on how big you draw the points – that will tell you how many points can fit on each length.

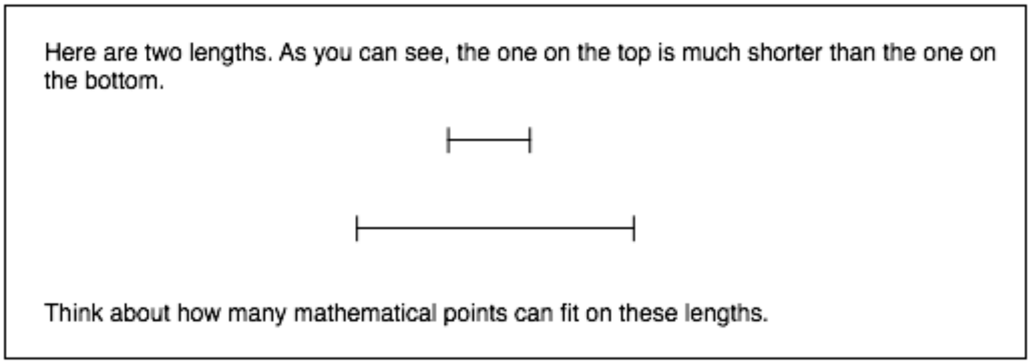
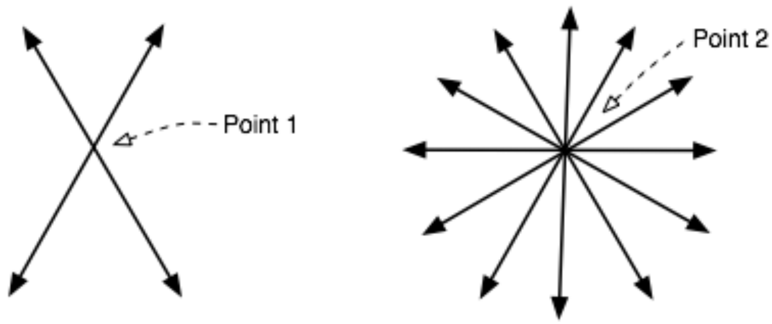


Figure 8. Geometrical representation used for Problem 7: Relative number of points that can fit on two represented line segments of different lengths?

Problem 8: Properties of two points, one constructed by 2 intersecting lines, the other by 6?[5] This problem asked students to consider the problem presented in Figure 9. The diagram and accompanying text presented two images, one of two crossing lines and one of six crossing lines, asking students to consider the two points generated by the intersections (Point 1 and Point 2) and then to indicate which of the following statements are true: (a) Point 1 and 2 are the same because they both have no size; (b) Point 2 is bigger than Point 1 because more lines are crossing; (c) Point 1 is just one point, but Point 2 is actually several points stacked on top of each other.

[5] Problem 8 is adapted from the item used by Fischbein (1993). See Figure 1.

Look at the mathematical lines below, and notice that they cross. On the left, there are two lines that cross. On the right, there are six lines that cross.



On the left, there is a mathematical point at the spot where the two lines cross. We'll call it Point 1.

On the right, there is also a mathematical point. This point is at the spot where the six lines cross. We'll call it Point 2.

Think about how Point 1 and Point 2 might be the same or different from each other.

Figure 9. Geometrical representation used for Problem 8: Properties of two points, one constructed by 2 intersecting lines, the other by 6?

Procedures for administering treatment conditions and outcome measure

The intervention and multiple-choice assessment were administered in three phases—1) sample problem, 2) treatment administration, and 3) outcome measure—which are described below.

Phase 1: Sample problem

In all classrooms, the administration procedure began with the presentation of a sample problem (see Figure 10) displayed on a large poster. This problem was designed to be unproblematic even for fourth-graders. The presentation drew students' attention to the multiple-choice format of each problem.

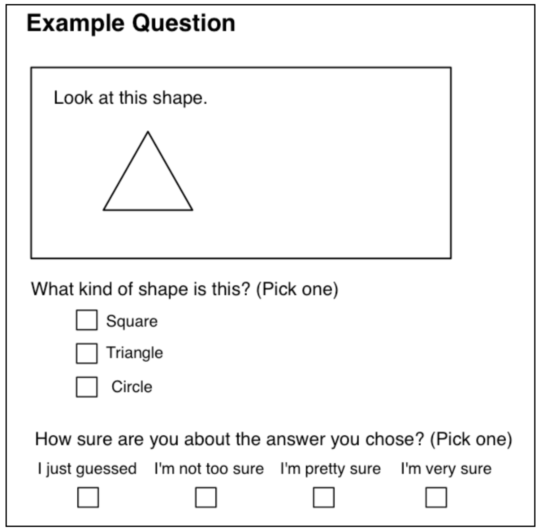


Figure 10. Sample problem displayed on a poster.

Phase 2: Treatment administration

Two different procedures were used to assign participants to treatment conditions, one for sixth and eighth graders, and another for fourth-graders. The purpose of the different procedures was to address the potential confound between reading ability and definition use for the younger participants. Recall that both information sheets were rated at a 5th grade reading level, presumably above the average reading level of the 4^{th} grade participants.

Sixth and eighth grade students: Random assignment of students to treatment conditions within classrooms at each grade level. Recall that two classrooms participated in the study at each grade level. Within each of the two sixth and two eighth grade classrooms, students were randomly assigned to either the control or treatment conditions Random assignment allowed for a true experiment at sixth and eighth grade, with assignment to treatment condition not confounded with classroom. Treatment group assignments were also made such that both groups had an equal proportion of girls and boys, in order to eliminate the potential confound of gender (based on possible gender differences in mathematics proficiency or other relevant variables).

During the study, the information was distributed to students and they were instructed to read the information sheet carefully, as it might help them solve the problems that they would be presented. Students were also reminded that they could reread the information sheet at any point in solving the problems. Students were also informed that their neighbors may get a different sheet, and that this was intentional. Students were asked not to peek at other students' paper, as the goal of the activity was to find out which sheet of paper was more helpful to students. The teacher and two researchers monitored students' activity during the time provided to read the prompt to help ensure that students were reading their prompt and only their prompt.

Fourth grade students: Classroom assignments to treatment conditions. To address the potential confound between reading level and definition use, I administered the treatment to fourth-graders using a different procedure, one that provided additional support for decoding the sheet. To wit, the information sheet was read aloud to the entire class as students followed along with their individual copies of the sheet. This required that all students in one fourth grade classroom were assigned to the experimental condition, and all students in the other fourth grade classroom were assigned to the control condition. The pairing of classroom to treatment condition was random.

While this procedure addressed the potential confound between reading level and definition use, it created another threat to validity: lack of random assignment and the resulting confound of classroom. While there is no way to rule this out because there were only two fourth grade classrooms, the 4th grade teachers were both of the opinion that neither class was noticeably higher performing than the other.

Phase 3: Outcome measure

Students in each participating classroom were presented with the printed assessment materials on points and lines and instructed to provide answers to the problems. The teacher and two researchers roamed the room monitoring students' activity and attention. The assessments were collected after all students had finished.

Results

To examine whether the treatment had a differential influence on the extent of students' idealized responses across grades for the two problem types, I generated two scores. One score was for *no diagram* problems and the other was for *diagram* problems. I computed these scores by assigning students 1 point if they chose the idealized response, and a 0 if they chose an answer that reflected a reliance on the appearance of diagrams. I then summed the points for the no diagram and diagram problems separately and determined the proportion of idealized responses for each problem type, *no diagram* (three problems) and *diagram* (five problems).

Figure 11 shows the proportion of idealized responses for each problem type as a function of grade level and treatment group. The figure reveals that students' choices of idealized answers varied as a function of whether diagrams were present, their grade levels, and whether they participated in the control (no definitions) or experimental (definitions) groups. At Grade 4, in the control group, few students chose idealized answers, regardless of whether diagrams were present or absent in problems; in contrast in the experimental group, many more students selected idealized answers, but predominantly in the absence of diagrams. Like their fourth-grade counterparts, 6th grade students in the control group also selected few idealized answers, and those in the experimental group chose many more idealized answers in the absence of definitions. However, a striking difference between the two grades is that 6th grade students in the experimental group also selected idealized answers when diagrams were present (though not as often as when diagrams were absent). Finally, at 8th grade, students in the experimental group once again selected more idealized responses that those in the control group. Yet, among students in the experimental group, there appears to be less of a difference between the two problem types—diagram versus no diagram.

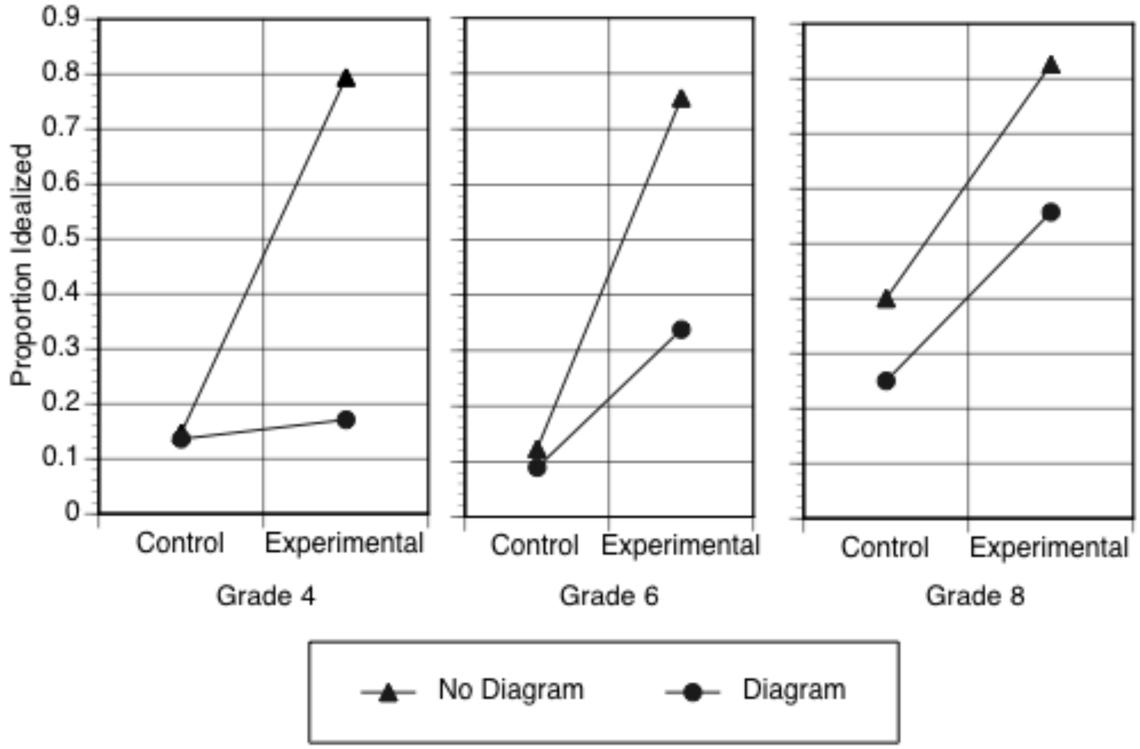


Figure 11. Proportion of idealized answer choices for students in control and experimental groups.

The results of a 3 (GRADE) x 2 (TREATMENT) x 2 (PROBLEM TYPE, repeated measure) MANOVA confirm a three-way interaction (F=7.991, df=136, p<.01). To determine the source of the three-way interaction, I conducted three 2 (PROBLEM) x 2 (TREATMENT) ANOVAs, one at each of the three grade levels.

The central expectation for these analyses was a pronounced interaction effect. This expectation reflects my hypothesis, consistent with Fischbein's findings, that the presence of misleading diagrams would lead younger students to conceptualize problems in terms of empirical practices—that is, to select answers based on appearances of diagrams or the behavior of material objects. As mentioned above, I expected the interaction effect to be less pronounced or absent among older students (6^{th} and 8^{th} graders), indicating that definitions were increasingly used by the older children to solve all problems, including those featuring potentially misleading diagrams (in accordance with the definitional practice). These findings would indicate that with age students have a greater propensity to deliberately draw upon available definitions.

Results were consistent with expectations. The two-way ANOVA at fourth grade revealed a strong PROBLEM TYPE x TREATMENT interaction effect (F=63.831, df=44, p<.01, with a moderate effect size—partial Eta Squared=.592). On the *no diagram* problems, fourth-graders showed selected fewer idealized responses performance when they did not have access to definitions (the control group), and much better performance on these problems when they did have access to definitions (the experimental group). This indicates that fourth-graders did draw on definitions for the *no diagram* problems. On the *diagram* problems, in contrast, fourth-graders showed were less likely to choose idealized answers regardless of treatment condition, confirming my hypothesis that they would select answers that are consistent with the appearances of diagrams or the behavior of physical objects whether or not they had access to definitions. At sixth and eighth grade, I found as expected a diminishing interaction between problem type and treatment. At sixth grade, the interaction effect persisted ($F=23.420$, $df=1$, $p<.01$) but the effect size was reduced compared with fourth-graders (partial Eta Squared=.315 as contrasted with .592). At eighth grade, my analyses revealed no interaction—the presence of definitions shifted participants' solutions to both problem types.[6]

Overall, the results point to a grade-based shift from an empirical to definitional approach, reflecting an increasing understanding of the definitional practice. However, due to the experimental design of this study, students' use of definitions can only be inferred. I discuss this limitation further below.

Summary

This study investigated young students' developing understanding of an important but challenging practice in academic mathematics—the use of geometric diagrams to represent idealized mathematical objects whose properties are stipulated by their definitions, rather than deriving from the appearances of diagrams or the behavior of material objects. Students in fourth (n=46), sixth (n=53), and eighth (n=43) grade were randomly assigned to an experimental or a control group. Students in the experimental group were provided with definitions of Euclidean points and lines, not available to students in the control group. All students were administered assessment items about their understanding of ideal properties of points and lines (e.g. infinite length, zero-dimensionality). Some of the items contained no geometric diagrams (*no diagram* items). Other items incorporated diagrams (*diagram items*) and were constructed to cue a reliance on the appearances of the diagrams and students' knowledge of the physical world, rather than the definitions.

Consistent with expectations, findings indicate that younger students are more likely to rely on the appearances of the representations or knowledge of the physical world, rather than the available definitions. Indeed, fourth-graders, whether in the experimental or the control groups, frequently selected answers on the *diagram* problems that are consistent with figural properties of the representations or the behavior of material objects. Older students, by contrast, are more likely to select definition-consistent answers and are less swayed by the potentially misleading diagrams. With increasing grade, I found that students in the experimental group were more likely to select answers that were consistent with the definitions on the *diagram* problems, compared to students without access to definitions. In summary, Study 1 generates suggestive evidence that younger students are more likely to adopt empirical approaches, and older students more likely to employ idealized approaches, and this difference is most pronounced when problems include diagrams.

A number of questions arise in relation to Study 1's results that are important for understanding the development of the definitional practice. First, is it indeed the case that

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[6] I also found evidence that eighth grade students had more prior knowledge of the definitions of points and lines. Specifically, I found that among students in the control group (those who did not have access to definitions), eighth grade students outperformed six graders and fourth-graders on the *no diagrams* problems, which simply target students' knowledge of the idealized properties of points and lines.

students'—in the absence of available definitions—draw on their knowledge of the physical world and the appearances of the diagrams to reason about points and lines? The design of Study 1 allows me to infer that students are drawing on these conceptual resources when definitions are not available; students overwhelmingly select answers that are consistent with 'empirical' interpretations. However, the experimental design precludes more direct evidence of empirical interpretations, since students cannot be queried about their rationale for selecting empirical answer choices.

Second, are students' indeed drawing on the definitions in their problem solving? Study 1's use of control and experimental conditions allowed me to infer that availability of definitions influenced students' selection of answers, but—once again—it could not provide direct evidence that students were drawing on the definitions when selecting idealized answers.

Third, to what degree are students constructing a conceptual differentiation between material diagram and idealized object? Students may be selecting responses that are consistent with the definition without fully appreciating the distinction between diagrams and the ideas they symbolize. For example, one student who generally selects idealized answers may do so "because the definition says…" while another may also make explicit the understanding that the diagrams in the problems are only representations of points and lines. This question is important because a conceptual differentiation between idealized object and material diagram is central to the definitional practice.

Finally, what specifically accounts for the impact of the experimental condition—is it primarily the provided definitions, or also the additional discussion of the 'rules' of the definitional practice? In other words, what part of the experimental treatment is providing the most support? This question has important implications for pedagogy. It will help determine the possible benefit of an instructional approach that goes beyond simply providing the definitions—an approach that also engages students in explicit discussion of the 'rules of the game,' which distinguishes idealized object from material diagram.

These unanswered questions raised by the results of Study 1 are taken up in the interview/interventional study reported in the following chapter.

STUDY 2—FOLLOW UP INTERVIEW/INTERVENTION STUDY OF THE DEFINITIONAL PRACTICE

Study 1 revealed a developmental trajectory in students' use of definitions in geometric problem-solving, suggesting a growing understanding of the definitional practice. With age and grade, providing students with the definitions increased the likelihood that they would select answers that were consistent with the definitions, rather than the appearances of the diagrams or the behavior of physical objects. Indirectly, this suggests that students are reasoning from the definitions, and subordinating the materiality of the drawn diagrams—which is central to the definitional practice. As discussed above, this finding raises a number of additional questions about students' understanding of the definitional practice and the challenges that the practice presents to students. Using interview and design research methods, Study 2 investigates these unanswered questions, detailed below, and attempts to generate more direct evidence of students' uptake of the definitions.

Study 2 is organized in four phases, each of which addresses a distinct research question. (See Table 2 below for a summary of phases and research questions.) The first phase—Students' Intuitions—investigates students' intuitions about problems involving points and lines when they do not have access to definitions. Recall that the design of Study 1 precluded direct evidence that students were drawing on their knowledge of the physical world and the appearances of the diagrams. By asking students to justify their answer selections, Study 2 will address this question.

The second phase includes a brief intervention, with two conditions to which students will be randomly assigned. One includes the definitions only, and the other includes both definitions and a discussion that distinguishes idealized object from material diagram. This addresses another important question raised by Study 1: what specifically accounts for the impact of the experimental condition—is it primarily the provided definitions, or also the additional discussion of the 'rules' of the definitional practice? The two treatment groups offer a means of assessing the possible utility of an instructional approach that incorporates an explicit discussion of the practice (rather than simply providing the relevant definitions), an approach rarely implemented in classrooms.

The third phase—Drawing on Definitions—considers how students are drawing on the definitions (and the additional support for understanding the practice) to mediate their problem-solving. Recall that Study 1 could not provide direct evidence that students were drawing on the definitions. By querying students on the rationale for their answer choices, this phase addresses whether students are indeed drawing on the definitions in selecting idealized answers.

The fourth phase—Differentiation—investigates how students may be constructing a conceptual differentiation between material diagrams of points and lines and the idealized objects to which the diagrams refer, whose properties are stipulated by the definitions. The additional items in this phase were designed to directly engage the idea that the definitions of mathematical points and lines cannot be perfectly embodied in a drawn diagram. This question was likewise raised by Study 1, which provided only suggestive evidence that students were making such a differentiation when provided with the definitions.

Table 2: Study 2 Phases and Research Questions

Phase	Research Question Addressed
Phase 1: Students' Intuitions	What are students' intuitions about problems involving points and lines, in the absence of definitions? How do students draw on the appearances of the diagrams and their knowledge of the physical world as problem solving resources?
Phase 2: Treatment	What is most helpful for students: providing the definitions, or the addition of an explicit distinction between idealized object and material diagram?
Phase 3: Drawing on Definitions	Do students indeed draw on the definitions, when selecting idealized answers?
Phase 4: Differentiation	How to students draw on definitions and other supports to construct a conceptual differentiation between the material diagrams of points and lines and the idealized objects to which the definitions refer?

Participants

The selection of participants was motivated by the findings of Study 1. Recall that Study 1 showed a marked difference between fourth and sixth-graders when they participated in a condition in which mathematical definitions were made available to them: Sixth-graders were more likely to choose idealized responses regarding points and lines. Study 2 therefore included fourth (n=40) and sixth (n=37) graders to investigate this learning process. Participants came from a San Francisco Bay Area elementary and middle school. Pilot interviews confirmed that students in fourth grade were suitable for the study because they could make sense of the problems and readily select and provide justifications for answer choices.

Table 3: Number of participants in each treatment group and in each grade

Treatment Group	Grade 4	Grade 6
Definitions Only	19	16
Practice Initiation	17	18

*Number of participants does not include students with missing data

Interview Procedure

As described briefly above, the interview procedure was organized in four principal phases. Below, I further describe each phase, focusing on the research question that the phase addressed. Pilot Interviews with students of various grades were used to develop the procedures and materials, and to select participants of suitable grade levels.

Phase 1: Students' Intuitions

The purpose of this phase was to investigate students' intuitions regarding points and lines. This was made possible by the fact that students did not yet have access to definitions during this phase of the interview. These problems were designed to reveal how students may be drawing on the appearances of the diagrams and their knowledge of the physical world as problem solving resources.

I began by presenting participants with eight multiple-choice problems and probing them on their thinking. (e.g., "Why does this answer make the most sense to you?"). As in Study 1, the problems were of two types: those without diagrams (*no diagram* items) and those including diagrams (*diagram* items). With minor variations, all of the problems—with the exception of Problem 7, overlapping lines—were the same as those used in Study 1 (see below for the items themselves).

> Okay, let's look at the first question. [Interviewer reads first problem and answer choices. For each problem, after the student has selected an answer, the interviewer asks:] Why does that answer make the most sense to you? [If the student's answer is at all unclear, interviewer follows up with:] That's interesting—can you tell me more about that to make sure I understand your thinking?

This procedure was designed to support an analysis of grade-related differences in students' propensity to conceptualize diagrams in terms of empirical practices. Students were given an opportunity to generate and justify solutions to the problems, when they do not yet have access to definitions. An analysis of participants' answer choices may provide an opportunity to replicate Study 1's results from students in the control group, who also did not have access to definitions. Participants' justifications or their answer selections has the potential to significantly enrich these findings.

The eight multiple-choice problems included in the Intuitions phase are provided below. Two forms were used to control for order effects. Form A included the *no diagram* problems first (Problem 1-3) followed by the *diagram* problems (Problems 4-8); Form B reversed this order. Students were randomly assigned to the two order conditions.

No Diagram items. Three *No Diagram* problems were included, which queried students about the size and dimensionality of points and lines.

Problem 1: How big is a mathematical point? The question asked, "*How big is a mathematical point?"* Three answer choices were provided: (a) It is pretty small, the size of a small dot; (b) It has no size at all; (c) It depends on how big you draw it.

Problem 2: How long is a mathematical line? The question asked, "How long is a mathematical line?" Three answer choices were provided: (a) It depends on how long you draw it; (b) It is usually about as long as a finger; (c) It is infinitely long, which means it goes on forever.

Problem 3: How thick is a mathematical line? The question asked, "How thick is a mathematical line?" Again, three answer choices were provided: (a) It has no thickness at all; (b) It is pretty thin, a little bit thicker than a hair; (c) It depends on how thick you draw it.

Diagram items. Five *Diagram* problems were included. These problems featured potentially misleading diagrams that were meant to cue a reliance on the appearances of the diagrams or knowledge of material objects.

Problem 4: Converging Points? The problem asked students to consider the geometrical representation presented in Figure 12. The diagram and accompanying text presents points in motion that converge on the same spot as well as the accompanying question, "What would happen to these two points?" The student was presented with four answer choices, with some of the text in bold to help the student distinguish them[7]: (a) There would only be one point, but it would be bigger; (b) There would only be one point, I would still have no size at all; (c) There would only be one point, but it would be darker; (d) There would be two points that are still the size of the small dot, but one point would be in front of the other point.

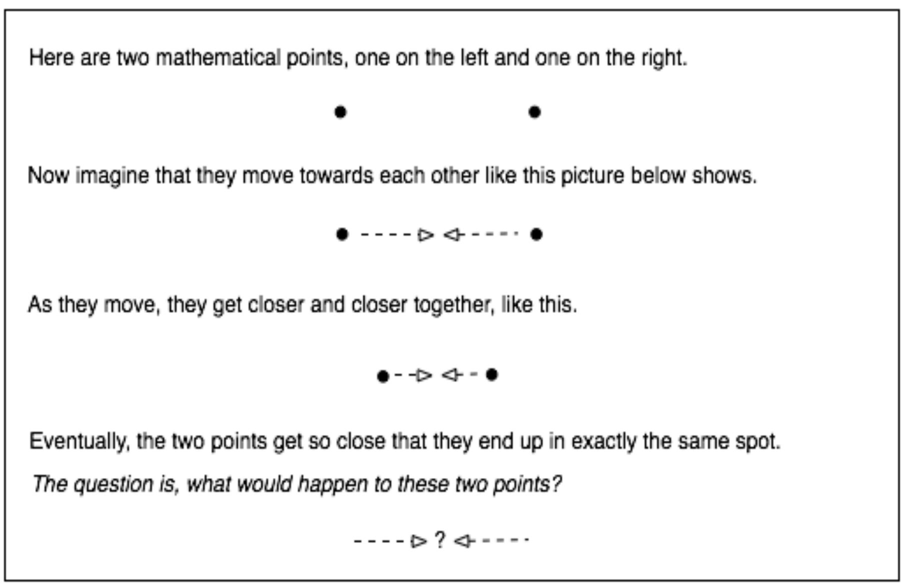


Figure 12. Geometrical representations used for Problem 4: What is the dimensionality of points that converge on the same spot?

Problem 5: Converging Lines? This problem asked students to consider the problem presented in Figure 13. The diagram and accompanying text presented lines in motion that converge on the same spot as well as the accompanying question, "what

[7] This is also done for Problem 2 and 5.

would happen to the two lines?" The student was presented with four answer choices: (a) There would be two lines that are still a little thicker than a hair, but one line would be in front of the other line; (b) There would be only one line, but it would be thicker;(c) There would be only one line, but it would be darker; (d) There would be only one line, and it would still have no thickness at all.

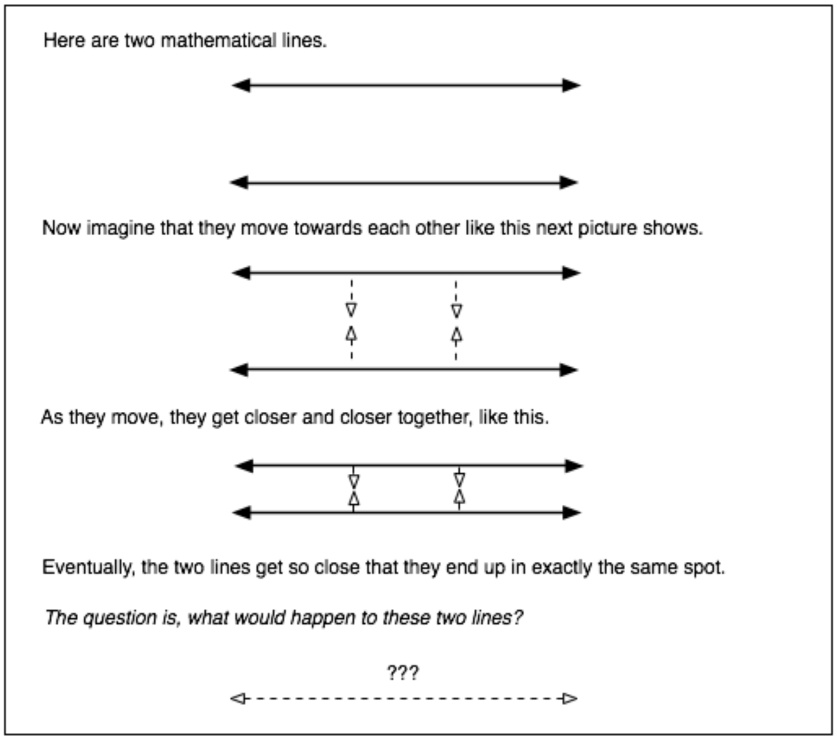


Figure 13. Geometrical representation used for Problem 5: What is the dimensionality of lines that converge on the same spot?

Problem 6: How many points on a line segment? The problem asked students to consider the problem presented in Figure 14. The diagram and accompanying text presented a line segment said to be 3 cm long, and the same line segment with a point represented. The accompanying question was "How many mathematical points would fit on this length?" The problem supplied five answer choices: (a) More than 5 points, but fewer than 30 points; (b) More than 30 points, but fewer than 1,000 points; (d) An infinite number of points; (e) It depends on how big you draw the other points.

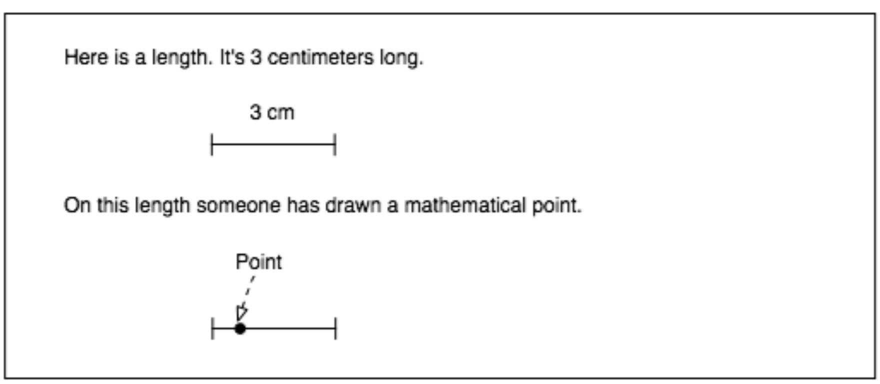
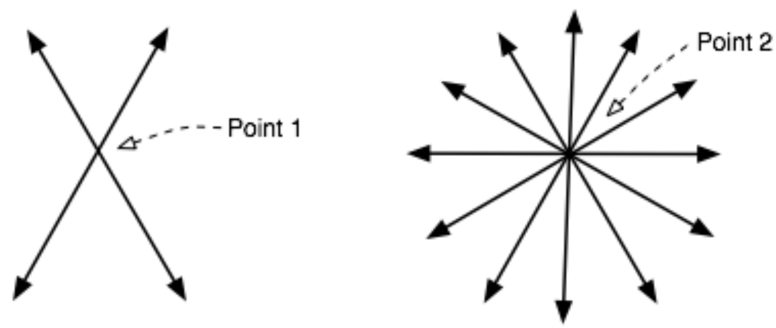


Figure 14. Geometrical representation used for Problem 6: How many points can fit on a line segment?

Problem 7: Properties of two points, one constructed by 2 intersecting lines, the other by 6? This problem asked students to consider the problem presented in Figure 15. The diagram and accompanying text presented two images, one of two crossing lines and one of six crossing lines, asking students to consider the two points generated by the intersections (Point 1 and Point 2) and then to indicate which of the following statements are true: (a) Point 1 and 2 are the same because they both have no size at all; (b) Point 2 is bigger than Point 1; (c) Point 1 is just one point, but Point 2 is actually several points stacked on top of each other, and all of them are the size of a small dot; (d) Point 2 is darker than Point 1.

Look at the mathematical lines below, and notice that they cross. On the left, there are two lines that cross. On the right, there are six lines that cross.



On the left, there is a mathematical point at the spot where the two lines cross. We'll call it Point 1.

On the right, there is also a mathematical point. This point is at the spot where the six lines cross. We'll call it Point 2.

Think about how Point 1 and Point 2 might be the same or different from each other.

Figure 15. Geometrical representation used for Problem 7: Properties of two points, one constructed by 2 intersecting lines, the other by 6?

Problem 8: Placing points on a line segment over and over? This problem asked students to consider the problem shown in Figure 16. The diagram and accompanying text presents a series of line segments with successive points placed on them and asked, “If you kept doing this, over and over, would you ever run out of room?” The following answer choices were given: (a) Yes, you would eventually run out of room, and you couldn’t put any more points on the line; (b) No, you would never run out of room. You could keep putting more points halfway between the last point and the end of the length, and you could keep doing this forever.

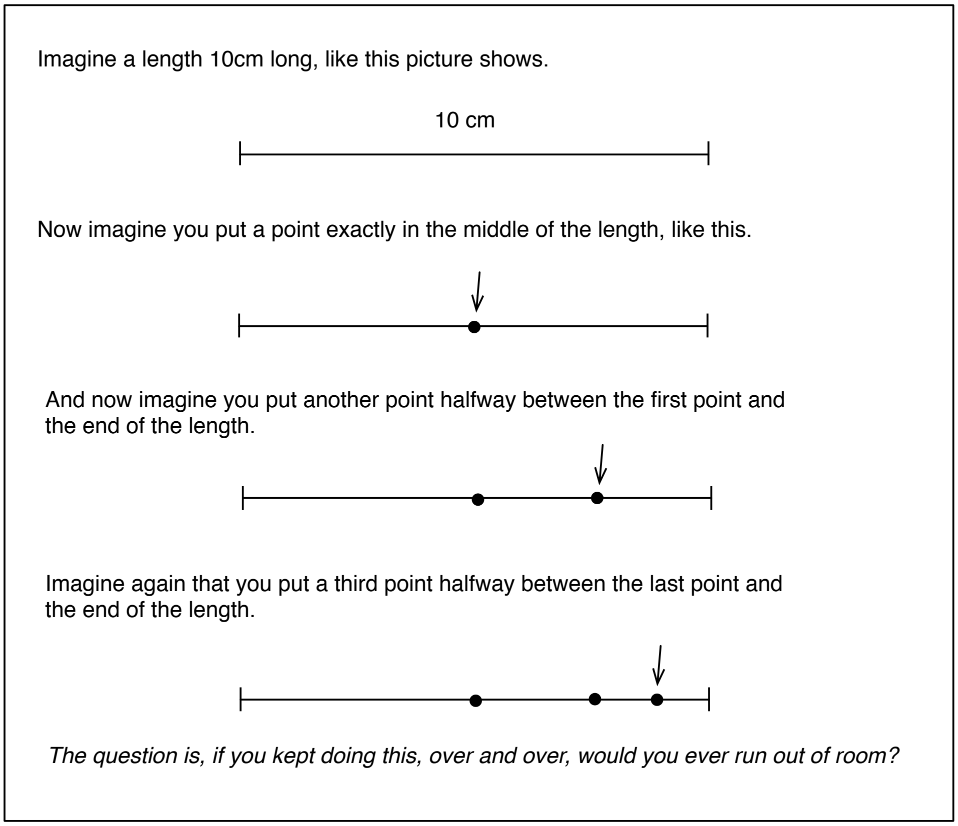


Figure 16. Geometrical representation used for Problem 8: Placing points on a line segment over and over?

Phase 2: Treatment—Definitions Only and Practice Initiation Groups

In the second phase of the interview, students participated in one of two conditions to which students were randomly assigned. In the *Practice Initiation* condition, students received an information sheet that contained core definitions of points and lines that included statements of their idealized properties (e.g. “Mathematicians define a point as being so small that they have no size at all.”). In addition, this sheet included information about the definitional practice—drawing students’ attention to the fact that that conventional diagrams of points and lines are inevitably imperfect in relation to the definitions, but nonetheless useful so long as the definitions are kept in mind. (See Figure 17.)

The Practice Initiation treatment represents a refined version of the text used in Study 1. It has been refined in order to better frame the discussion in terms of the practice of mathematicians (e.g., “Mathematicians use definitions to…”), and in order to incorporate the notion that points and lines are forms of ‘imagined perfection’—imaginary ideas that are too perfect to be shown exactly with drawings. The Practice Initiation treatment also makes more general statements about the definitional practice, beyond geometry or points and lines (e.g., “Drawings still help mathematicians think about the ideas. But mathematicians always remember that what the definition says is more important than what the drawings look like.”).

In the *Definitions Only* condition, students also received an information sheet that contained core definitions of points and lines—the same definitions that appeared on the *Practice Initiation* sheet. The remainder of the text, however, covered the history and scope of geometry—information unlikely to be useful in solving the points and lines problems. (See Figure 18.)

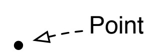
Both treatment texts were rated at a fifth-grade reading level and were of comparable length. Because the treatment texts were read aloud to all students, including fourth-graders, this helps mitigate threat to validity due to requirement of 5^{th} grade reading level for 4^{th} grade participants

Mathematical Points and Lines

Points and lines are ideas. Mathematicians use definitions to say exactly what these ideas are. And they use drawings to show these ideas with a picture.

Mathematicians define a **POINT** as being *so small that it has no size at all. And so small that it takes up no space*.

To show a point, mathematicians use a drawing like this.

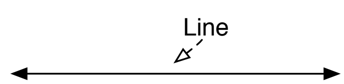


This drawing makes sense because a dot is small. But did you notice that a dot doesn't show the definition perfectly?

Drawing a point perfectly is impossible! We can only imagine a perfect point, we can't draw one. Because the definition says a point has no size at all. And a drawing would always have some size, right?

Mathematicians define a **LINE** as being *so long that it goes on forever*. And so thin that it has *no thickness at all*.

To show a line, mathematicians use a drawing like this.



This drawing makes sense. But did you notice that this isn't a perfect drawing of a mathematical line?

Drawing a line perfectly is impossible! We can only imagine a perfect line, we can't draw one. A drawing could never go on forever, right? Also, it would always have to have some thickness.

So, now you know that drawings can't show the definitions of points and lines perfectly. The ideas of points and lines are too perfect. And that's okay! Drawings still help mathematicians think about the ideas. **But mathematicians always remember that what the definition says is more important that what the drawings look like.**

Figure 17. Practice Initiation treatment.

Mathematical Points and Lines

Geometry is a very old part of mathematics. It was invented by several different cultures long ago. Geometry is about things like shapes, sizes and space.

When today's mathematicians think of geometry, they usually think of a man named Euclid. He was a famous Greek mathematician. He lived more than two thousand years ago.

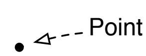
Euclid did not invent geometry. But he is famous for his work. Euclid took many different ideas in geometry and created a system. In Euclid’s system, all the ideas fit together. He also found out that some ideas seem simple, but they are actually hard to define.

Euclid's work is so famous that most of the geometry you learn in school is based on his ideas. Two of these ideas are points and lines. Euclid thought that points and lines were the most important ideas in geometry. He thought that nothing in geometry would make sense without them!

Let’s look at the definitions of points and lines that mathematicians use today.

Mathematicians define a **POINT** as being *so small that it has <u>no size at all.</u> And so small that it <u>takes up no space</u>*.

Here is how mathematicians draw a point.



Mathematicians define a **LINE** as being *so long that it <u>goes on forever</u>*. And so thin that it has *<u>no thickness at all</u>*.

Here is how mathematicians draw a line.

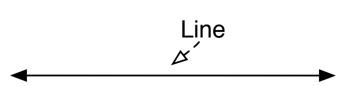


Figure 18. Definitions Only treatment.

Giving students in both the Practice Initiation and Definitions Only group access to the definitions supports an analysis of the effect of what distinguishes the two interventions: in the Practice Initiation group, participants have access to a discussion of the definitional practice as it relates to points and lines, not merely the definitions themselves. As discussed above, the two treatment groups offer a means of assessing the possible utility of an instructional approach that incorporates an explicit discussion of the practice (rather than simply providing the relevant definitions), an approach rarely implemented in classrooms. It may be that students at one grade derive the most benefit from the Practice Initiation treatment, indicating at what grade the instructional approach is most appropriate and effective.

Treatment Administration Procedure. In both treatment groups, the one-page information sheet was read aloud by the interviewer. The sheet was introduced as follows:

> Okay, now that we are done with the questions, I'd like to tell you a little bit about how mathematicians think about points and lines. I'll read it aloud, but it's important that you follow along, okay? [Interviewer reads the text of the treatment group assigned to student.]

Regardless of treatment group, after reading the treatment text aloud, students were asked to repeat verbatim the core definition of points and lines ("Mathematicians define a point as being…") before moving on, to ensure that students have minimally attended to and processed the definitions. The core definitions are glued to squares of cardboard and propped up and placed next to the students. Students are emphatically told that they can look at the definitions whenever they wish during the rest of the interview.

> I'm going to give you the definitions of points and lines that mathematicians use. I'll put them right here [interviewer places core definitions next to student]. You can look at them and read them whenever you want, okay? Before we start, can you read these definitions out loud? Great!"

At the end of this phase, students are provided with the core definitions printed on a note card and propped up to make them clearly visible (See Figure 19). Students are told that they can look at these definitions whenever they want in future problem-solving, if they think it would help them. To ensure all participants are minimally attentive to the definitions, the interviewer asked the student to read the definitions allowed for proceeding with the next phase.

Mathematical Points and Lines

Mathematicians define a point as being *so small that it has no size at all. And so small that it takes up no space*.

Mathematicians define a mathematical line as being *so long that it goes on forever*. And so thin that it has *no thickness at all*.

Figure 19. Definitions made available to participants during Phase 3.

Phase 3: Drawing on Definitions Phase

The purpose of the Drawing on Definitions phase was to investigate students' ability to switch from an empirical to definitional orientation, in different grades and treatment conditions/interventions. This phase queried participants about whether they would revise their answers to the eight problems from Phase 1, after having been exposed to the treatment. Specifically, students were asked how a mathematician who used these definitions of points and lines would answer the questions. This phase allowed for a comparison of the frequency with which fourth and sixth-grade students draw on the definitions, as well as the impact of the different treatment conditions, as described above.

Here, the interviewer began by repeating the procedure in the Intuitions phase, but without asking students to explain their answer choices. Participants are specifically asked how a mathematician using the definitions of point and lines would answer the problems, and this prompt is printed on the items themselves (see below). In most cases, students received abbreviated statements of the problems rather than full re-readings of the problems in their entirety. Again, two different forms were used to control for and investigate order effects of the two problem types—each student receives the same problem ordering they received in Pretest (Form A or Form B). However, note that students selected answers on a second copy of the problems, which included the additional prompt ("How would a mathematician answer this question?). Participants did not have access to their solutions from the Intuitions phase, which ensures equal stimuli during the Drawing on Definitions Phase.

> Now we'll look at the questions again, and I'd like to know how you think a mathematician using these definitions would answer the questions. Sometimes you might pick a different answer than before, and other times you might pick the same answer.

After students have selected answers to all the problems, they are then asked to explain their answer choices about only those problems in which they shifted from a definitions-inconsistent to a definitions-consistent answer. This elicits self-reports concerning whether and how the information sheet led to their shifts in thinking. The purpose for probing students' thinking after they have solved all the problems—instead of probing after each problem—was to ensure that the probing and ensuing conversation does not influence their answers to subsequent problems

> There are a few of your answers that I'd like to ask you about. [Interviewer identifies problems in which students shifted from an empirical to idealized answer. For each problem, interviewer asks:] You said that a mathematician would pick this answer. Can you tell me why a mathematician would pick this answer?

As mentioned, the problems used during Phase 3 were identical to those used during Phase 1. The only difference is that above each problem, the question appears: "How would a mathematician answer this question?" (See Figure 20 for an example.)

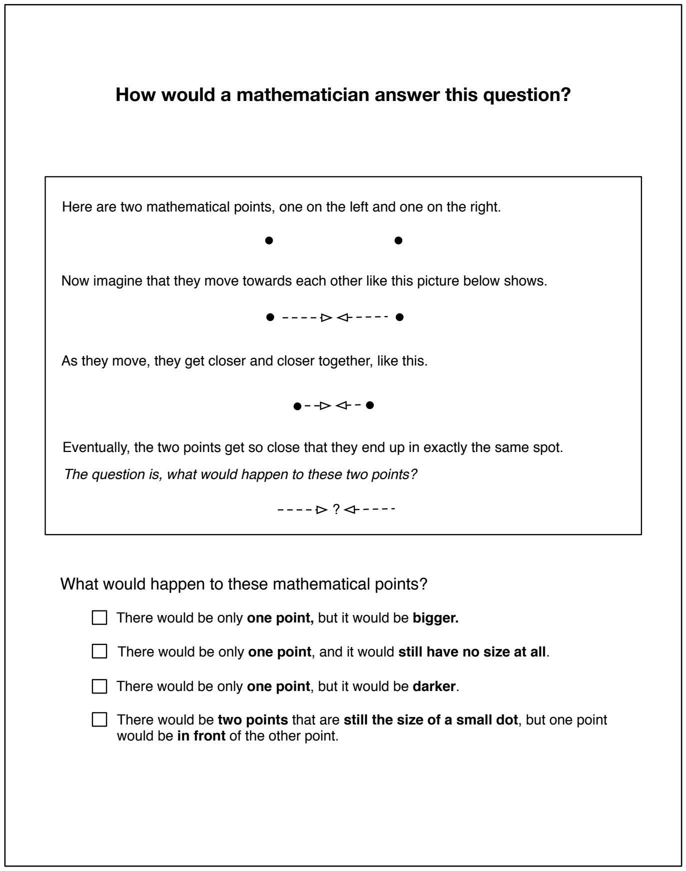


Figure 20. Example of a problem used during Phase 3, all of which includes the prompt, “How would a mathematician answer this question?”

Phase 4: Differentiation Phase

The purpose of this phase was to investigate how students may be drawing on the definitions to construct a conceptual differentiation between material diagram and idealized object. This phase consisted of five multiple-choice questions [yes/no]. For each question, students were asked to explain their thinking.

> Okay, I’d like us to do just a few more questions. I’d like to know how you think mathematician would answer these questions, and I’d love to hear your thinking. [Participants are asked to solve the problems and to explain their thinking after each problem:] Why would a mathematician pick that answer?

These questions directly addressed the idea that conventional diagrams of points and lines are imperfect representations of the imaginary ideas described by the definitions, that producing a perfect representation is impossible, and that—for mathematicians—what matters most is what is stipulated by the definition rather than the appearance of the diagram. Designed for this purpose, these items differed from those in Phase 3; rather than providing students with a problem involving a particular configuration of or scenario involving points and lines (e.g. How many points would fit…?), they ask *explicitly and generally* about the adequacy of diagrams in depicting the idealized objects, and if it is ever possible to perfectly embody the definitions in drawn diagrams. As such, these items allow for an analysis of students' understanding of the relationship between material diagram and idealized object (not simply participants' use of definitions on challenging problems), which may vary by grade and treatment group. The problems are presented below.

Problem 1: Is this a perfect drawing of a mathematical point? The problem asked, "Is this a perfect drawing of a mathematical point?" A diagram of a mathematical point is provided. Two answer choices were provided: 1) yes and 2) no. (See Figure 21.)

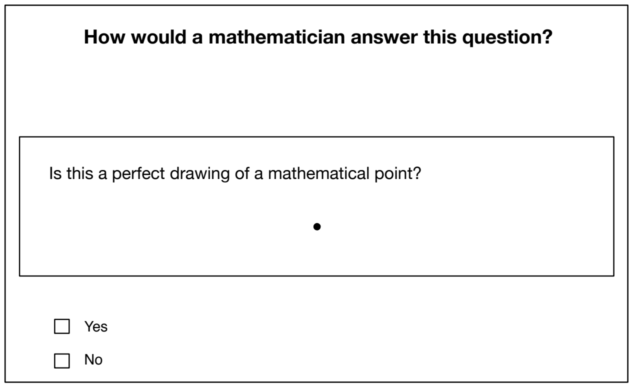


Figure 21. Problem 1: Is this a perfect drawing of a mathematical point?

Problem 2: Is this a perfect drawing of a mathematical lines? The problem asked, "Is this a perfect drawing of a mathematical line?" A diagram of a mathematical line is provided. Two answer choices were provided: 1) yes and 2) no. (See Figure 22.)

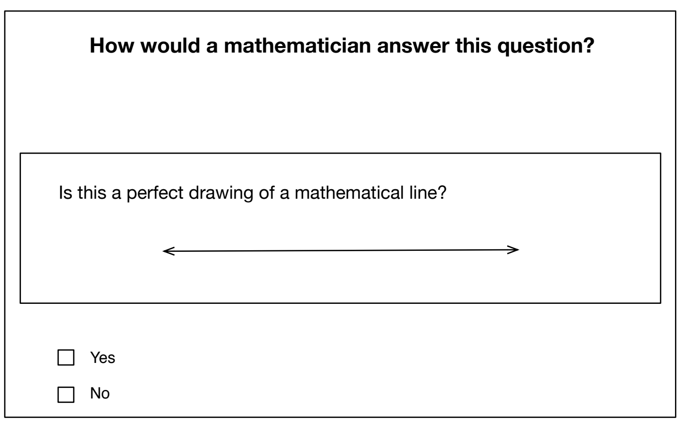


Figure 22. Problem 2: Is this a perfect drawing of a mathematical line?

Problem 3: Is it possible to draw a perfect mathematical point? The problem asked, "Is it possible to draw perfect mathematical point?" Two answer choices were provided: 1) yes and 2) no. (See Figure 23.)

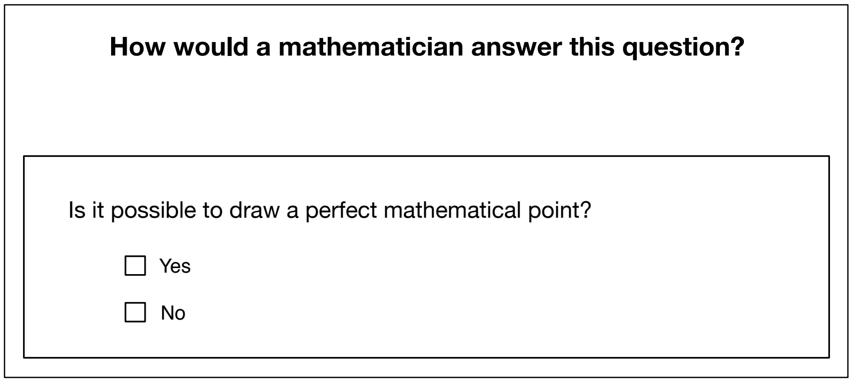


Figure 23. Problem 3: Is it possible to draw a perfect mathematical point?

Problem 4: Is it possible to draw a perfect mathematical lines? The problem asked, “Is it possible to draw perfect mathematical lines?” Two answer choices were provided: 1) yes and 2) no. (See Figure 24.)

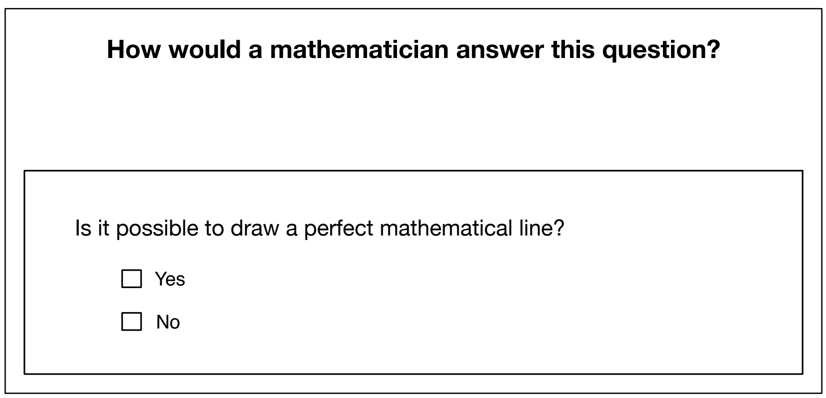


Figure 24. Problem 4: Is it possible to draw a perfect mathematical lines?

Problem 5: which is more important? The problem asked, “Which is more important?” Two answer choices were provided: 1) “What a drawing of a point or line looks like,” and 2) “What the definition of a point or line says.” (See Figure 25.)

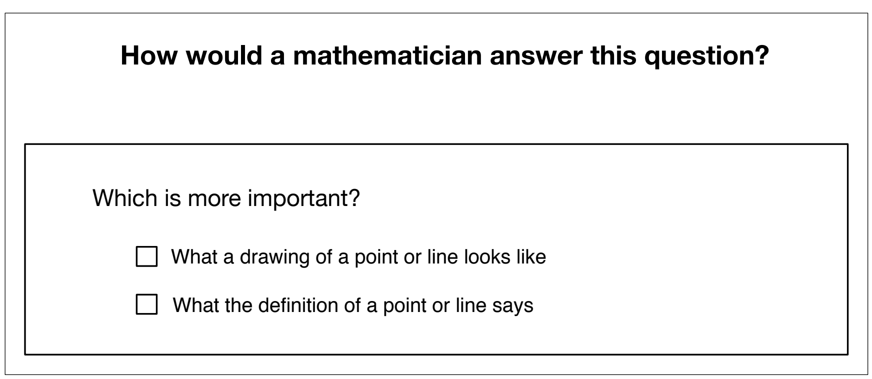


Figure 25. Problem 5: Which is more important?

Results

This section examines how students' answer selections and explanations address the core research questions motivating this study concerning students' developing understanding of the definitional practice. As a summary, Table 4 below provides an overview of the phases, procedures, and associated research questions in Study 2.

This section begins with an analysis of the results from Phase 1 (Students' Intuitions), considering how—in the absence of definitions—students may be drawing on the appearances of the diagrams and their knowledge of the physical world. Recall that the purpose of this phase is to generate direct evidence that students are indeed drawing on these conceptual resources when selecting empirical answer choices—evidence that was not available from Study 1.

Results are then analyzed from Phase 3 (Drawing on Definitions), which considers how the treatments were taken up by students in a second opportunity to solve the problems from Phase 1. An important purpose of this phase is to generate direct evidence of whether students are indeed drawing on the definitions when selecting idealized answers—evidence that, again, was precluded by the design of Study 1.

A second purpose of Phase 3 was to identify possible differences in students' uptake of the 2 interventions. Recall that, while both treatments included definitions of points and lines, the Practice Initiation treatment also included text intended to support the understanding that diagrams are inevitably imperfect representations of points and lines, which are mathematical ideas—forms of 'imagined perfection.' The analysis also considers possible differences based on students' grade level, whether the treatments were taken up in different ways by students of different grade levels.

Next, results are presented from Phase 4 (Differentiation), examining how students may be drawing on the provided definitions to construct a conceptual differentiation between material diagrams and stipulated objects—a central understanding in the definitional practice. This analysis considers how this developmental trajectory may differ by treatment group and grade level.

In addressing the questions motivating Study 2, I consider both the answers that students select, as well as the explanations that they provide for their answer choices. By investigating aspects of students' thinking that are not evident from their answer selections alone, these analyses generate more direct evidence of students' developing understanding of the definitional practice.

Table 4: Overview of Study 2 Design

Phase	Procedure	Research question
Phase 1: Students' Intuitions	Students solve eight multiple-choice questions, 3 without geometric diagrams and 5 with diagrams. Students explain their thinking.	In the absence of definitions, how do students of different grades draw on the appearances of the diagrams and their knowledge of the physical world as problem solving resources?
Phase 2: Treatment	Students are presented with a page of text. For students in the Definitions Only group, the text includes core definitions of points and lines, as well as additional text that is irrelevant to the problems. In the Practice Initiation group, the text includes core definitions, as well as an additional discussion of the definitional practice (e.g. drawings are imperfect).	What is most helpful for students: providing the definitions, or the addition of an explicit distinction between idealized object and material diagram?
Phase 3: Drawing on Definitions	Students solve the same 8 multiple-choice questions again, and are asked what answers a mathematician using the definitions would select. All students are given access to core definitions, which are written on a piece of cardboard that is propped up in clear view. After selecting answers to all the problems, students are asked to share their thinking about only those problems in which they shifted from a definitions-inconsistent (i.e., empirical) answer in the Intuitions Phase to a definitions-consistent (i.e., idealized) answer.	Do students indeed draw on the definitions in selecting idealized answers?
Phase 4: Differentiation	Students solve an additional 5 multiple-choice [yes/no] problems, which are designed to reveal how students may be constructing a conceptual differentiation between diagram and idealized object. Specifically, these items engage students' appreciation that the diagrams of points and lines are inevitably imperfect and that what matters most is the definitions. Students are asked to share their thinking.	Are there grade-related differences in participants' construction of a conceptual differentiation between material diagram and idealized objects—the appreciation that drawings of points and lines are inevitably imperfect and subordinate to the definitions, according to the definitional practice?

Phase 1 Results: Students' Intuitions

Results from the Intuitions Phase indicate that students of both grades overwhelmingly favor "empirical" answer choices, which are consistent with the appearances of the diagrams or the behavior of physical objects. Across both grades, for each problem the answer selected the least frequently was the idealized answer. The mean proportion of idealized responses was .13 for the problems with three answer choices, .08 for problems with four answer choices, and .37 for the problem with two answer choices—which is notably less than the percentage that would be obtained by random guessing (i.e., .33, .25, and .5 respectively). Indeed, Chi-squared analyses of each of the eight pretest items indicate that it is unlikely that students are selecting answers at random ($p<.05$ for all problems).[8] This finding corroborates the results of Study 1, which showed that students in the control condition—who did not have access to definitions of points and lines—overwhelmingly selected empirical answer choices. Recall, however, that Study 1 could only generate indirect evidence that students were drawing on their knowledge of the physical world and the appearances of the diagrams as problem-solving resources. To address this, Study 2 uses interview techniques, querying students on the rationale for their answer selections.

Students' verbal explanations for their answer choices provide direct evidence that students—in the absence of definitions—draw on the appearances of the diagrams and knowledge of the physical world as problem solving resources. Most often students explicitly referenced the appearances of the diagrams in justifying their answer selection. Other times, students offered analogies that reference the behavior of physical objects. Consider an example in which a fourth-grader was solving the Converging Points question (Figure 12). He justified his empirical answer of *There would be only one point, but it would be bigger*, by saying "Because I'm thinking about the points like, I don't know, blobs with jelly or something. They move together and then, squish, there is one blob of jelly, except it's bigger because the 2 small ones went in there." The student is drawing on his knowledge of pliable physical material like jelly, dough, or clay to reason about mathematical points—when two masses combine, the result is a larger mass. Table 5 below provides an illustrative sample of student explanations in which they referenced specific physical objects and their behavior.

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[8] Chi-squared statistics are as follows: Problem 1=53.711; Problem 2=19.132; Problem 3=18.658; Problem 4=42.632; Problem 5=20.316; Problem 6=52.605; Problem 7=18.600; Problem 8=4.813).

Table 5: Appeals to the behavior of physical objects to justify empirical answer selection on problems with diagrams

Grade	Student Explanation
4	Because I'm thinking about the points like, I don't know, **blobs with jelly** or something. They move together and then, squish, there is one blob of jelly, except it's bigger because two small ones went in there.
4	If like two **pieces of paper** move toward each other, then one's going to end up on top of the other. Then one's going to slide right under the other one.
4	Because if you keep pushing something together, it's like these **pens**. [Picks up two pens and holds them in the air, and moves them towards each other]. If you are pushing them together, one would eventually be on top of the other.
4	Well, what are the mathematical points are made of because if it was paint then it would combine, but if it was like two **basketballs** or something then one will go in front of the other.
6	Kind of what I imagine is, because I got to school in a **car**, is two cars driving down the road towards each other and then they get in the middle of the street in the same spot, but one is behind the other one.
6	Yes. I feel…one would be in front of the other…because if they are **objects**, it would be hard to merge together…they couldn't go in the exact same spot.
6	I'm imagining, do you know those little **magnetic balls** that you can make into structures and things? If there's only two balls, and you try pushing them together, the one goes in front of the other. That might be because it's magnetic but also if there're two **boulders**…if they were in a tight situation, if they didn't have to be completely crushed together, if they're just coming together, one would probably force its way above the other one so that they could both be in the same line.
6	Since you're adding **matter** to more matter it just makes…if it's matter then it just makes sense that it would be more, that it would get bigger.

Phase 2 Results: Treatment

The intervention phase proceeded without issue. The students overwhelmingly followed the direction to read along silently while the interviewer read the information sheet. They appeared attentive and engaged. Following the presentation of the information sheet, recall that students were asked to read the definitions aloud, which they did without any apparent struggle. Differences in how students drew on the two interventions (Definitions Only and Practice Initiation) in their problem solving are taken up in the sections below.

Phase 3 Results: Drawing on Definitions

Recall that Study 1 could generate only indirect evidence that students were drawing on the definitions. The results from the Drawing on Definitions phase offer an assessment of how students made use of the provided definitions of points and lines in problems that often feature deliberately misleading diagrams. These problems were designed to cue a reliance on the appearances of the diagrams and students' knowledge of the physical world, which conflicts with the provided definitions of points and lines.

Figure 26 below provides the mean proportion of idealized responses for fourth and sixth-graders in the two treatment groups, and for the two problem types. Note that in the charts, the Practice Initiation treatment is abbreviated as “Def. Plus” (as in “Definitions Plus Practice Initiation”). The results shown in Figure 26 indicate, as predicted, that students in both fourth and sixth grade selected idealized answers more frequently after they were provided the definitions. The data also show that students in both fourth and sixth grade selected idealized answers more frequently on problems without misleading diagrams. (Results for individual problems are provided by charts in Appendix C.)

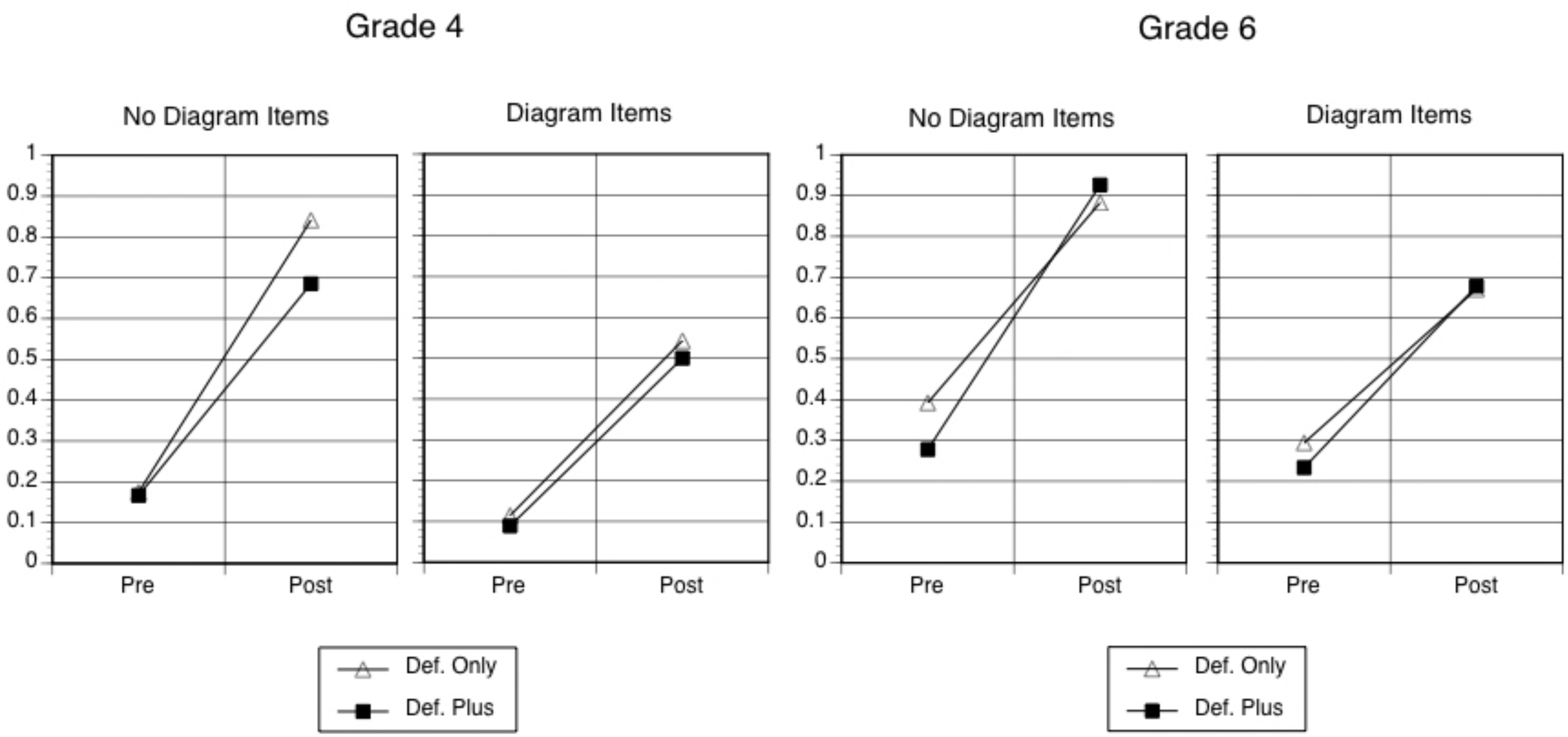

Figure 26. Results for comparing the Intuitions Phase ("Pre" in legend) and Drawing on Definitions Phase ("Post" in legend) in both treatment groups.

The results of a 2 (Treatment) X 2 (Grade) X 2 (Problem Type) X (Pre/Post) ANOVA show a main effect for both Pre/Post (F=243.655, df=1, p<.001, partial eta squared=.787) and Problem Type (F=58.616, df=1, p<.001, partial eta squared=.470), both with large effect sizes. These findings confirm two things. First, the Pre/Post effect confirms that students in both treatment groups—Definitions Only and Practice Initiation—were far more likely to select idealized responses after being exposed to the definitions of points and lines. In other words, students in both treatment groups demonstrated responsiveness to the definitions, regardless of grade. Second, the Problem Type effect confirms that students choose idealized responses far less frequently when the problem includes diagrams. This corroborates the general results of Study 1, indicating that both fourth and sixth-graders are frequently swayed by the misleading appearances of diagrams.

To confirm that students are indeed drawing on the definitions when selecting idealized answers, I now consider students' verbal justifications for their answer selections—specifically, how frequently students make explicit reference to the definitions of points and lines. Recall that during the Drawing on Definitions Phase, students were asked to explain their thinking for each problem in which they initially selected an empirical answer in the Intuitions Phase but shifted to an idealized answer. To analyze students' verbal justifications, I used the video analysis program Studiocode. See Figure 27 below for a screen shot of a coded interview. In the figure, the rows correspond to individual problems, and the coded segments locate the stretch of video in which students justified their answers.

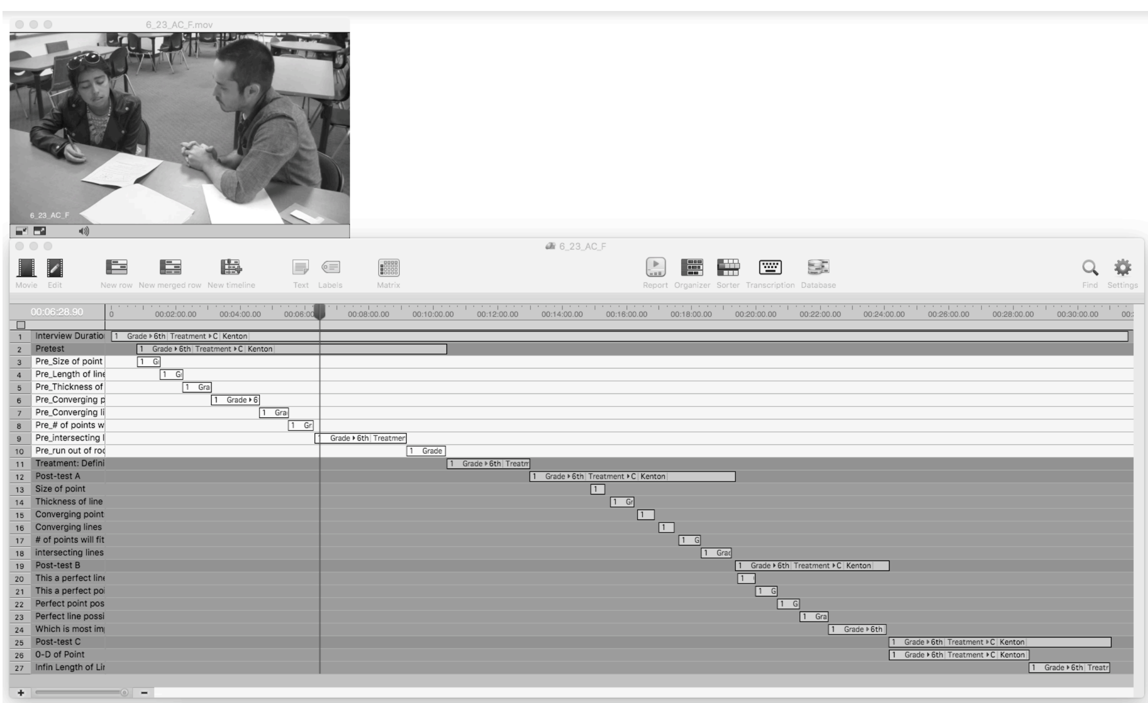

Figure 27. Screen shot of a coded interview using Studiocode.

I developed a code to capture whether students were making explicit reference to definitions in their explanations for their idealized answers—a code called "Explicit Definition Reference." In order to code the data, I first parsed the video into segments corresponding to the various phases of the interview, including individual problems and students' explanations for particular answer selections (see Figure 27). For the Drawing on Definitions problems, I then applied a "Explicit Definition Reference" code to each answer choice that the student justified by making explicit reference to the provided definitions of points and lines, either by A) quoting one of the definitions, in whole or in part, B) paraphrasing one of the definitions in the students' own words, or C) by gesturally pointing to the piece of cardboard on which the definitions were written. As a preliminary to coding, I established high interrater reliability with a second coder (Cohen's Kappa=0.84), based on a 10 percent random sample of the data, stratified by grade.

The frequency of the Explicit Definition Reference code was determined for each item in the Drawing on Definitions Phase, for treatment group and in each grade. Figure 28 below shows the mean proportion of problems in which students made explicit reference to definitions, broken down by treatment group. The chart shows that, at both grades and in both treatment groups, students frequently referenced the definitions when justifying their idealized answer choices.

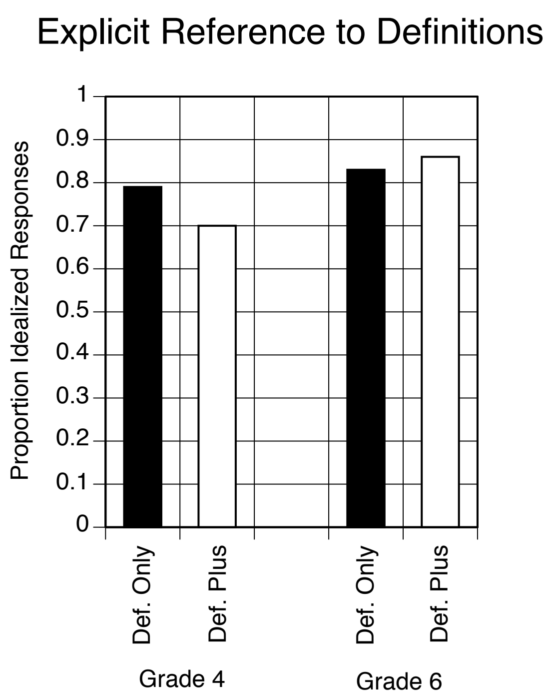

Figure 28. Proportion of items in which students made explicit reference to Definitions during Drawing on Definitions Phase.

To determine if there were grade-based and treatment-based differences in definition reference, median tests of the Explicit Definition Reference code were performed. These tests showed an effect for Grade (p=005), indicating that 6th grade students made explicit reference to the definitions more frequently than 4th graders, but that treatment had no effect. This is consistent with the analysis of students' answer selections, which showed that merely providing fourth-graders and sixth-graders with definitions significantly impacts their selection of idealized answers. There was also no effect for problem type; on problems in which students selected idealized answers, they referenced the definitions with equal frequency on both diagram and no diagram problems.

Phase 4 Results: Differentiation

A central understanding in the definitional practice is the idea that diagrams are not themselves the idealized objects which they (imperfectly) represent—an understanding I refer to as a conceptual differentiation between idealized object and material diagram. The multiple-choice items in Phase 4 offer an assessment of this understanding. Recall that Study 1 provided suggestive evidence that, when provided with definitions, some students may construct a conceptual differentiation, and that they draw on this differentiation when selecting idealized answers. The purpose of Phase 4 is to investigate more directly the possibility of such a differentiation. On these items, the "idealized" answer is logically consistent with the idea that diagrams are not mathematical points and lines, and that definitions of mathematical points and lines could never be drawn perfectly. For example, in the question "Is it possible to draw a perfect mathematical point?" The idealized response is "no."

These items also provide an additional measure of how students take up the different treatments. The purpose of the Practice Initiation treatment was precisely to support a conceptual differentiation between mathematical object and material diagram. It is therefore reasonable to hypothesize that there may be a difference between students' responses in the two treatment groups, only one of which received text that was intended to make explicit the distinction between diagram and idealized object. It may also be that students of different grade levels differed in the understandings that they drew from the Practice Initiation treatment.

Students were given a score of "1" if they selected the idealized answer, and a score of "0" if they selected the empirical answer. Figure 29 below shows mean proportions of idealized responses for fourth- and sixth-graders, separated by treatment groups. Results do indeed suggest that, at both grades, students who received the additional support of the Practice Initiation treatment selected idealized responses on these items more frequently.

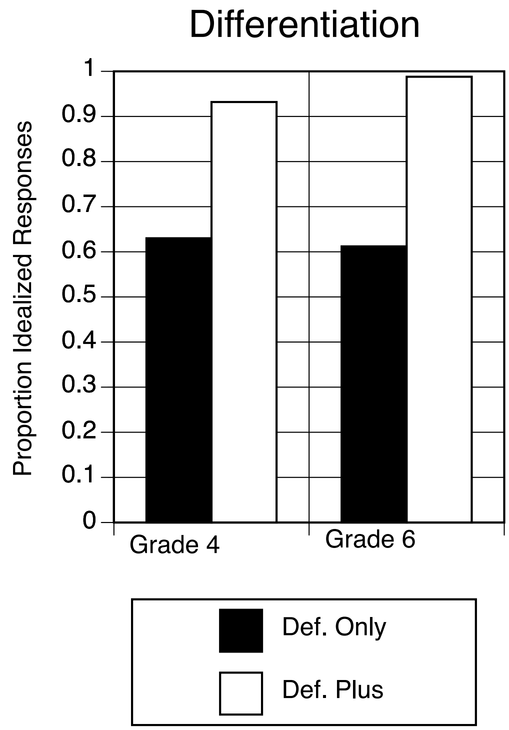

Figure 29. Proportion of items in which students selected idealized responses on Differentiation items.

To identify differences based on grade and treatment group, a 2 (Treatment) X 2 (Grade) ANOVA was performed. Results showed a main effect for Treatment (F=40.827, df=1, p<.01, with a moderate effect size—partial Eta Squared=.372), but no effect for Grade. This confirms that students in both grades who received the Practice Initiation treatment more frequently selected idealized responses on these items, responses which are consistent with the understanding that material diagrams are inevitably imperfect representations of the ideas of points and lines.

To obtain more direct evidence of students' construction of a conceptual differentiation, and how this may differ based on grade and treatment group, I now consider the kinds of justifications students provided for their answer selections during the Differentiation Phase. Because these items have only two choices (yes/no), it may be that students are selecting "idealized" answer choices for reasons that do not reflect a rich conceptual differentiation between symbol and referent in the context of geometric points and lines. With only two choices provided, it is possible—even likely—that students may select the idealized answer for a variety of reasons, only some of which are consistent with an understanding of the definitional practice. This indeed turned out to be the case, as I discuss below.

I found that students frequently provided a predominantly 'empirical' justification for their idealized answer choices during this phase. In other words, with some frequency, students justified their selection of the idealized answer in ways that did not clearly reflect the understanding that diagrams of points and lines are inevitably imperfect representations of idealized objects. In these justifications, students say that points and lines cannot be drawn perfectly, but for reasons that are not clearly connected to the definitions. Rather, they reference properties of the conventional diagrams (like the roundness of points) that are irrelevant to the definitions, or they made sweeping statements about the impossibility of doing anything perfectly. For example, one common empirically-based justification for idealized responses was the idea that, because points and lines can be drawn in any number of ways (e.g., big/small, long/short), there is no single perfect way to draw them—hence, they cannot be drawn perfectly. This contrasts with an understanding rooted in the definition of a mathematical line—the idea that no drawing can perfectly embody the features of infinite length and one dimensionality. Consider an example in which a fourth grade student was responding to the question *Is it possible to draw a perfect mathematical line?*; she responded, "No, because you can draw any kind of line. It can be a squiggly line or a straight line or a line with dots on it." The student is apparently expressing the understanding that it is impossible to draw a mathematical line perfectly because lines can be drawn in any way a person wants, and therefore there is no single perfect drawing of a line.

To determine whether empirical justifications were more common at 4^{th} grade or at 6^{th} grade, I generated a code called "Empirical Justification," which I applied each time a student provided a predominately empirical justification for their answer selection. Examples of this code from students in the Practice Initiation group are provided in Table 6 below. Using ten percent of the data, stratified by grade, two coders obtained high inter-rater reliability for this code (Cohen's Kappa=0.78). To analyze the results, I generated a composite score: each student received a score of 0 or 1 depending on whether or not he or she provided at least one empirical justification for an idealized answer choice during the Differentiation Phase. A Chi-squared test revealed that fourth-graders provided more empirical justifications than sixth-graders (χ^2 (1)=4.15, p<.05). To determine the robustness of this finding, another variable was created with a higher threshold—a score of 1 required at least three materially-based justifications. A chi-squared test once again showed that fourth-graders provided more empirical justifications (χ^2 (1)=5.00, p<.05). In other words, there is a significant difference between the frequency with which fourth and sixth-graders appeared to construct the intended understanding of the Practice Initiation treatment.

More often than their counterparts in sixth grade, fourth-graders in the Practice Initiation group did not seem to coordinate the definitions of points and lines with the idea that points and lines can never be drawn perfectly. That is, although fourth-graders frequently repeated a core idea in the Practice Initiation treatment—that it is impossible to draw points and lines perfectly—the reasons they provided for this impossibility often lacked a clear connection to the stipulated properties of points and lines. Often, fourth-graders seemed to assimilate the statement that it is impossible to draw points and lines perfectly to the idea that there is no one correct way to draw it. To summarize, the Practice Initiation treatment offered more support to sixth-graders in constructing a conceptual differentiation between material diagram and idealized object.

This finding—that fourth-graders did not derive the intended understanding from the Practice Initiation treatment as their sixth-grade counterparts—is also corroborated by a trend in the results from the Drawing on Definitions Phase. While no statistically significant interaction between Grade and Treatment was found, a trend approaching statistical significance was observed. According to this trend, fourth-graders who received the Practice Initiation treatment did not select as many idealized responses as their counterparts in the Definitions Only group, suggesting that the intended support of the Practice Initiation treatment was to some degree counterproductive. In contrast, this trend showed that sixth-graders who received the Practice Initiation treatment more frequently selected idealized responses, consistent with the results of Study 1. Again, this trend did not reach the level of statistical significance. Nonetheless, it resonates with the finding that students in fourth grade did not appear to fully understand the Practice Initiation treatment as often as 6^{th} graders—specifically, they struggled with the reason why it was said that points and lines cannot be drawn perfectly.

Table 6: Examples of Empirical justifications for The Differentiation Phase Items

Grade	Student's explanation
4	Interviewer: Is it possible to draw a perfect mathematical line? Student: Maybe no, because if you draw a perfect mathematical line, it has to be exactly straight. Humans don't have that perfection.
4	Interviewer: What would a mathematician say to this? Is this a perfect drawing of a mathematical line? Student: No, because you can draw any kind of line. It can be a squiggly line or a straight line or a line with dots on it. Interviewer: Is it possible to draw a perfect mathematical point? Student: No. You can draw a star as a point. You can do a face as a point.
4	Interviewer: Is this a perfect drawing of a mathematical line? What do you think a mathematician would say? Student: No…because it would not always be straight. It could be not the thickness that you want.
4	Interviewer: Is this a perfect drawing of a mathematical line? Student: No, it's not. But it's – you can see and it looks pretty much perfect…but it's not. Because from this part it could still be a little bit off, because nothing's like, humans can't draw anything perfect. Like, if you make – like, if you have a machine that makes bottle caps, and if you have two bottle caps, they are not going to be the exact same kind, because they are two, they are not going to be the exact same size. They might look the same, but they are not exactly the same.
4	Interviewer: How would a mathematician answer this question? Is it possible to draw a perfect mathematical point? Student: Nope. Because nothing's perfect in this world.

4	Interviewer: How would a mathematician answer this question? Is this a perfect drawing of a mathematical point, yes or no? Student: No, because some people could make the points bigger or smaller, so none of them are perfect. Interviewer: How would a mathematician answer this question? Is it possible to draw a perfect mathematical line? Student: No, because some lines are longer than others and some people would make them thicker and thinner, so all of them are right, but none of them are like perfect. Interviewer: Okay.
6	Interviewer: Is it possible to draw a perfect mathematical point? Student: No because it's impossible to draw one, because it won't have a perfect size cause' mathematical points don't have a perfect size and it wouldn't have the perfect thickness. It would be too thick or not thick enough maybe. And…the size wouldn't be correct.
6	Interviewer: Is that a perfect drawing of a mathematical point? Student: Because you can only imagine what perfect would be like. But everybody might have a different opinion of what perfect would really be. Really, there's no such thing as a perfect dot because nobody really has the same idea about it.
6	Interviewer: Is it possible to draw a perfect mathematical point? Student: No, it's not possible because… I mean no, it wouldn't look exactly like the one that the machine did. Because if you try to draw it by hand… Interviewer: Got it. Okay. It's impossible to draw perfectly by hand. But if the question was like it's just hypothetically a person need to ask, the question is like, can a computer draw perfect mathematical point? Would the answer be yes? Student: Mm-hmm.

Summary

Recall that Study 1 revealed an apparent learning trend, in which older students and those with access to definitions were more likely to select answers consistent with the idealized properties of points and lines, even when problems included misleading diagrams. However, the design of Study 1 precluded direct evidence of students' uptake of the definitions. The purpose of Study 2 was to address a number of important unanswered questions about this learning trend, in order to generate more direct evidence of the role of the definitions in students' thinking. This study once again sampled students from multiple grade levels in service of a developmental analysis.

Using structured interview methods, Study 2 employed a four-phase design to address the research questions raised by Study 1. The first phase—Students' Intuitions—investigate students' intuitions about problems involving points and lines when they do not have access to definitions. Two different treatments were then used to determine what provided students with the most support: the definitions themselves, or additional discussion about the definitional practice that distinguishes idealized object from (inevitably imperfect) material diagram. The third phase—Drawing on Definitions—considers how students drew on the definitions and additional support to reorganize their problem-solving. The final phase—Differentiation—investigates how students may be drawing on the definitions and additional support to construct a conceptual differentiation between material diagrams of points and lines and the idealized objects.

Results from the Intuitions Phase show that students of both grades overwhelmingly favor "empirical" answer choices, which are consistent with the appearances of the diagrams or the behavior of physical objects. This finding corroborates the results from Study 1. There verbal justifications confirm that students are indeed drawing on the resources of diagrammatic appearances and everyday knowledge of the material world.

In the Drawing on Definitions phase, students in both treatment groups—Definitions Only and Practice Initiation—were far more likely to select idealized responses after being exposed to the definitions of points and lines. In other words, students in both treatment groups demonstrated responsiveness to the definitions, regardless of grade. However, students choose idealized responses far less frequently when the problem includes diagrams. This finding ocean corroborates results from Study 1. Students' verbal justifications included frequent references to the definitions, confirming that they are indeed drawing on the definitions in selecting idealized answers.

During the Differentiation phase, students in both grades who received the Practice Initiation treatment more frequently selected idealized responses than did students who only received the definitions. Recall that on these items, idealized responses are consistent with the understanding that material diagrams are inevitably imperfect representations of the ideas of points and lines. More often than their counterparts in six grade, fourth-graders in the Practice Initiation group did not seem to coordinate the definitions of points and lines with the idea that points and lines can never be drawn perfectly. That is, although fourth-graders frequently expressed the general understanding that it is impossible to draw points and lines perfectly, the reasons they provided for this impossibility often lacked a clear connection to the stipulated properties of points and lines. In summary, findings from the Differentiation phase indicate that the Practice

Initiation treatment had the intended effect—that is, supporting a conceptual differentiation between diagram and idealized object—but it only did so consistently among sixth grade students.

A central concern of Study 1 and 2 is definition use—how students are drawing on the available definitions to solve the problems and reason about points and lines. Neither study investigates how students are making sense of these definitions. Full participation in the definitional practice requires a rich conceptualization of the relevant mathematical objects, a conceptualization that is consistent with the idealized properties stipulated by the definitions.

STUDY 3: HOW STUDENTS MAKE SENSE OF THE DEFINITIONS OF POINTS AND LINES

Authentic participation in the definitional practice requires that participants have a rich understanding of the definitions that stipulate the idealized objects. This brief exploratory study engages this dimension of participation in the definitional practice. The design of this study asks students how they would explain the definitions to another imagined student, in order to reveal the conceptual resources they draw on to support their sense making of these ideas. Put another way, this study considers the sense making processes through which students conceptualize points and lines as forms of "imagined perfection" (G. Lakoff, personal communication, February 11, 2015).

Geometric points and lines are prototypical examples of Vygotsky's notion of "scientific concepts" (Vygotsky, 1986)—they belong to a specialized, highly-structured discourse that is fairly removed from daily life outside of formal schooling. Vygotsky's treatment of scientific concepts engages what is therefore a fundamental conundrum for students—how to draw on available conceptual resources to ground their understanding of such foreign ideas. He argued that students typically draw on "spontaneous concepts"—concepts that develop in the course of daily life that are rooted in particular experiences, rather than explicit definitions that link them with other concepts. For instance, a child's concept of "brother" may be chiefly tied to experiences with the child's own brother, rather than a conceptual map of familial relations in which "brother" is uniquely defined as a male child that shares one's parents. Vygotsky theorizes that the developmental dynamics between spontaneous and scientific concepts are bidirectional. Students draw on familiar concepts to make sense of the novel ones they learn in school, and the explicitness and systematicity of concepts learned in school also shape the further development of familiar concepts. It should be noted that Vygotsky's framework for conceptual development is not intended to establish two mutually exclusive classes of concepts; rather, the intention is to highlight the peculiar nature of conceptual learning in formal schooling, and to illuminate interrelated developmental processes that this learning occasions.

As mentioned, mathematical points and lines are prototypical exemplars of Vygotsky's "scientific concepts." For one, points and lines have explicit definitions and are situated within a broader framework of geometric objects (polygons, angles, etc.). Additionally, idealized objects like points and lines lack a straightforward material existence, and the divergence between students' everyday experience of the material world and the stipulated properties of idealized objects is especially stark when those objects involve properties like zero dimensionality, one dimensionality, or infinite length. After all, no one has ever directly encountered material objects with such properties. So, what kinds of conceptual resources might students be drawing on to make sense of the definitions? What familiar ideas might they appropriate or extend to make sense of zero dimensionality, one dimensionality, and infinite length?

In my analysis, I seek to identify how students draw on a particular category of conceptual resource: ideas related to material objects. This study examines how students construct *material analogies—analogies that reference material concepts*, whether these concepts belong to the very small (atoms, molecules, etc.), the very large (planets, stars, etc.), or anywhere in between (balls, sticks, etc.). This interest in materially-based analogies is motivated by several factors. First, there is good reason to expect that students would draw on ideas related to material objects as sense making resources. This expectation is supported by foundational research in developmental psychology on the importance of sensorimotor experience in cognitive development (Piaget, 1952; Piaget & Inhelder, 1948), as well as cognitive science treatments of mathematics cognition (Lakoff and Núñez, 2000[9]). It is also supported by the empirical findings of this dissertation, which shows that students do indeed have a persistent tendency to draw on knowledge of the material world as problem solving resources.

Capturing students' materially-based analogies exposes the conundrum that students face in making sense of mathematical definitions of idealized objects, like those of points and lines. On the one hand, the conceptual resources that students have available to them are invariably rooted in materiality. On the other hand, the mathematical definitions stipulate *idealized* objects, possessing what might be considered non-material properties like no size and infinite length. Students' materially-based analogies can therefore provide a window into how students address this conundrum, and if they also appreciate the limitations of these analogies in explaining idealized objects.

Methods

Following the procedures of Study 2, I asked students two open-ended questions that probe their understanding of the definitions of points and lines.

> Before we stop, I'd like to ask you about the idea of a point and a line. Let's say I were a student in your class, and I said I didn't really understand the idea of a point. What does it mean that a point is so small that is has no size at all and takes up no place? Can you say that in different words so maybe I'll be able to imagine that better? What about the idea of a line? Let's say I were a student in your class, and I said I didn't really understand the idea of a line. What does it mean that a line is so long it goes on forever? Can you say that in different words so maybe I'll be able to imagine that better?

I elected to use this procedure—asking students how they would explain the ideas to another student—because pilot interviews indicated that this prompt generated more talk than simply asking students' what the definitions mean. I also found that this prompt generated more analogies involving material objects.

Participants

The participants of Study 3 were identical to those of Study 2. They included fourth (n=36) and sixth (n=34) grade students. The data for Study 3 was collected at the conclusion of each interview during Study 2.

[9] Lakoff and Núñez (2000) have provided an influential treatment of how humans are capable of conceiving of idealized mathematical objects with properties like infinite length and less than three dimensions. Their account rests on cognitive processes such as conceptual metaphor, which provides an explanation for how sensorimotor experience can serve as the conceptual bedroom for advanced mathematics. My study, by exploring the specific conceptual resources students draw on in their conceptualization of points and lines, compliments—and potentially corroborates—this important work.

Results

I used an inductive analytic approach to identify material analogies that students referenced to explain the definitions of points and lines. After transcribing all students' explanations, I examined the transcripts and coded every material analogy by looking for references to specific material objects. I found that all of these material analogies involved references to objects that fell into two easily identifiable categories. As I discuss in detail below, students referenced small bits of matter (atom, molecule, etc.) when explaining the zero-dimensionality of a point, and ideas related to outer space (planets, asteroids, etc.) when explaining the infinite length of a line. Notably, both categories encompass concepts from academic science, ideas students most likely learned in school—a finding I will discuss further.

However, I found that majority of students did not make material analogies, and that sixth-graders did so far more frequently than fourth-graders; only 5% of fourth-graders and 18% of sixth graders produced material analogies. The majority of students in both grades adopted the strategy of explaining the definitions by putting them in their own words, saying—for example—that "a line never ends, it just keeps going and going and going."[10] This is unsurprising, as students were not prompted to make analogies involving material objects. Indeed, given the procedure, one might be surprised that so many students did spontaneously reference specific material objects, since they were not prompted to do so. Interestingly, a student who referenced a specific concept to explain one definition was likely to do so when explaining the other definition. Of those students who made a material analogy to explain one definition, roughly half did so when explaining the second definition.

Below I describe the categories of material analogies that students made to explain the definitions of points and lines.

Material analogies for making sense of a mathematical point. Recall that the conceptual resources students often drew on to explain the definitions of points and lines were material ideas they most likely encountered in school—in particular, ideas from science. For the definition of a mathematical point, students referenced small units of matter—like atoms and molecules—that cannot be seen with the naked eye.

For instance, one sixth grader said that a point was small like an atom, and when asked which is smaller, the student responded that a point is smaller, "because an atom is really small, but on a point takes up no size and no space." By comparing a point to an atom, the student is communicating the understanding that a point is very small, but the student is also aware that a point is smaller than even an atom, because even something

[10] While this student is not referencing a specific concept, her response reflects what cognitive linguistics refers to as the cognitive process of fictive motion, wherein a static object is understood as a dynamic process of travel between its endpoints (Talmy 1996). For example, consider the sentence, "The fence runs along the side of the house." Regardless of whether a specific conceptual resource was evident, students' explanations of a mathematical line frequently suggested a conceptualization rooted in fictive motion. This is unsurprising given that the provided definition features fictive motion as well: "A line is so long that it *goes* on forever."

as small as an atom takes up space. This nicely exemplifies the production of the material analogy by a student who also appreciates the necessary limitations of this—or any—material analogy.

Table 7 below provides a few additional illustrative examples of student talk, in which the units of matter they referenced appear in bold. (See Appendix D for a complete list of relevant data extracts.)

Table 7. Material analogies for explaining the definition of a point

Grade	Explanation	
4	Well, you could say that a line or a dot is like nothing. You could say that it's nothing. You can't really see, it's like you can't see **germs** but it's there…It's like that small. It's the tiniest thing. Like you can barely see it, it's like a germ on your hand.	
6	Student:	No size, I think an **atom** or something.
	Interviewer:	So, which is bigger? A mathematical point or an atom?
	Student:	An atom is way bigger.
6	Interviewer:	So, earlier you talked about an atom. I was curious about that. So, an atom is pretty small, right? And it sounds like a point is pretty small too. Which one of those is smaller?
	Student:	A point. Because an **atom** is really small, but a point takes up no size and no space.

Material analogies for making sense of a mathematical line. Again, students' science learning was the most common apparent source of conceptual resources that students appeared to draw on to make sense of a line. In particular, many students reference ideas related to outer space: planets, galaxies, the universe, etc.

For instance, one fourth grader said "It goes on forever so that means it'll go past Mars, it'll go past the asteroid belt, it'll go on forever." The student is referencing two familiar, progressively distant celestial bodies to communicate the idea that a line continues infinitely, mapping an imagined path from the earth to progressively further locations in outer space—a path that never ends. One can plausibly argue that the student is aware of the limitations of the analogy, since he makes explicit that the distances to even Mars and the asteroid belt are insufficient to capture the length of a mathematical line; a mathematical line would continue past these objects.

Table 8 below provides some additional illustrative examples of student talk. (See Appendix D for a complete list of relevant data extracts.)

Table 8: Material analogies for explaining the definition of a line

Grade	Explanation
4	It goes on forever so that means it'll go past **Mars**, it'll go past the **asteroid belt**, it'll go on forever.
6	It'll go on forever. And it'll go on after we all grow old and the **Sun** runs out of… and it explodes and the **Earth** explodes and it'll keep on going forever.
6	Like it goes on throughout the **cosmos**, just goes on until there's no end. I still think about that every day. It's mind boggling.
6	Student: You can only do an infinite line, only if you're in **space** because the **universe** is infinite so there you go. Interviewer: For someone like me who doesn't understand it, you might tell me to imagine **outer space**? Student: Yes.

Confusion or disagreement with definitions. Another notable finding is that a number of students gave voice to either a struggle to understand the definitions, or a disagreement with the definitions. For example, these students asserted that it did not make sense that anything could go on forever or have no size at all. There may have been more students who experienced this confusion—even disagreement—with the definitions, but did not share it aloud.

Consider the following instance of student talk; the student said that a line could "never go on forever because one can never go on forever because… you can't draw something forever." Table 9 below provides additional examples. Such examples of confusion or disagreement with a definition serves to highlight the fact that points and lines are non-intuitive ideas that originate in a particular, specialized academic discourse and diverge from students' everyday embodied experience. They also highlight the persistent tendency among young students to identify points and lines with their material representations, since they cite the impossibility of drawing an infinite line and a zero-dimensional point as a source of confusion or ground for rejecting the definitions.

Table 9: Confusion or disagreement with definitions

Grade	Explanation
4	Well, it can never go on forever because one can never go on forever because… you can't draw something forever.
4	Interviewer: What does that mean—so small that it has no size at all? Student: I didn't get that either. So, it's kind of like I said…everything takes up space, if it has like a—if it has, a solid form. Interviewer: And what about a mathematical line, which is so long that it goes on forever? Student: I'm pretty sure that it ends because technically, the drawing, it ends.
6	Student: I don't think [a line] could go on forever. It could go on like universe but I'm not so good at that, the whole universe and space thing, but I think from what I understand that the universe is always expanding…but if [a line] is going on forever it also will hit the edge maybe and then keep expanding, but I think that this doesn't really count as going on forever because it's still sort of passes stuff…I don't think it's going on forever.

Summary

This study provides a window into how students make sense of the definitions of points and lines; specifically, it considers students' production of analogies rooted in materiality, and how these analogies serve as resources for sense making.

Students were asked how they would explain the definitions of points and lines to another student in their class. In general, students appear to find this a challenging task, and a number voiced confusion or disagreement with the definitions. This puts into relief the fact that mathematical points and lines are non-intuitive ideas from a specialized academic discourse—prototypical examples of Vygotsky's "scientific concepts"—which include challenging ideas like zero-dimensionality and infinite length.

The majority of students did not produce material analogies, and those that did were more often in sixth grade. Rather, they employed the strategy of paraphrasing key ideas from the definitions in their own words. An analysis of the material analogies students did produce uncovered a clear pattern: analogies drew on concepts from academic science, which students likely learned in school. Scientific ideas dealing with the very small, such as molecules and atoms, were frequently referenced to communicate the idea that a point is so small that it has no size at all. To explain the infinite length of a line, students often referred to scientific ideas dealing with the very large, involving outer space and celestial bodies. Interestingly, some students demonstrated an understanding that their analogies were necessarily imperfect—that, for example, a point is smaller than even an atom, and that a line is longer than even the distance from Earth to the asteroid belt.

The fact that students' materially-based analogies draw on concepts from academic science is a theoretically significant result. It is not well captured by Vygotsky's simple framework for understanding the interplay between spontaneous and scientific concepts, in which spontaneous concepts provide an experiential grounding for scientific concepts. This study points to the need to understand developmental dynamics between scientific concepts. Results signal the possibility for unfamiliar scientific concepts (in this case, mathematical points and lines) to enter into trajectories of conceptual development involving somewhat more familiar scientific concepts (in this case, atoms, asteroids, etc.). Whatever resources students are drawing on to make sense of ideas related to outer space and small units of matter, for example, thereby become relevant in their efforts to make sense of mathematical points and lines. Such complex processes linking separate trajectories of conceptual development are little understood and deserving of further study.

CONCLUSION: CONTRIBUTIONS TO SOCIOCOLUTRAL TREATMENTS OF COGNITIVE DEVELOPMENT AND TO MATHEMATICS EDUCATION RESEARCH AND PRACTICE

This dissertation makes theoretical and empirical contributions to our understanding of sociocultural processes in cognitive development. It also contributes to mathematics education research and offers important implications for pedagogy and educational practice. In this chapter, I outline these contributions, and I point to possible directions for future research.

Contributions to Sociocultural Treatments of Learning and Development

At the heart of this dissertation is the construct of a practice—a recognizable, valued form of activity within a cultural community in which individuals draw on artifacts, tools, routines, and semiotic resources to solve problems. Defined in this way, the construct brings into view within a single analytic lens both cognition and cultural-situated activity (such as participation in the discourse of academic mathematics). In adopting this construct, this dissertation contributes to the large body of research on the sociocultural nature of learning and development. It advances this line of research by studying the utility of a practice-based conceptualization for understanding an important but under-researched aspect of mathematics learning.

This work represents a novel extension of Vygotsky's method of double stimulation. Recall that, for Vygotsky, the method of double stimulation was in service of the more fundamental aim of understanding mediation in higher cognitive activity. The notion of mediation points to the insight that humans do not respond automatically to environmental stimuli—rather, they exercise agency in drawing on available resources to reorganize the situation and direct their own action towards a desired end. Despite its elegance and usefulness in capturing the mediated properties of cognitive activity, the method of double stimulation has been underutilized. As mentioned, the few researchers who reference the method in describing their research adopt it largely as a conceptual tool; they do not use it as a structure for study design, nor do they often use it with the purpose of studying development processes in mediational activity.

My use of the method—incorporating an experimental approach that includes potentially useful auxiliary stimuli, with the purpose of capturing a developmental process—shares strong similarities with Vygotsky's reported examples of the method of double stimulation. However, there are notable differences. One of Vygotsky's well known examples involved a simple memory task, with color chips serving as the auxiliary stimuli. One could argue that Vygotsky's example does not capture well the predicament students face in school. Color chips are a far cry from mathematical definitions of a zero-dimensional point and a one-dimensional line with infinite length, for example. Mathematical definitions are verbal, intended to describe idealized entities that diverge from the three-dimensional material objects we encounter in our everyday experience. Participants' uptake of the definitions cannot be ultimately separated from the sense that they make of the definitions' meaning. Indeed, this motivated me to study the auxiliary means (i.e., conceptual resources) that individuals drew on to make sense of the definitions that they had available to them in the other studies. Put another way, in my use of the method of double stimulation, both the problems and the 'auxiliary stimuli' required their own cognitive structuration, and the two needed to be bootstrapped in relation to each other.

My study of students' sense-making of the definitions also has theoretical implications that engage Vygotsky's seminal treatment of conceptual development, as discussed earlier. Among Vygotsky's core insights is that school exposes students to concepts that differ significantly from those they learn in their everyday lives. Academic concepts have explicit definitions and belong to structured discourses in which individual concepts are embedded in a structured network of others concepts. Vygotsky argued that this transforms the process of conceptual development, creating top-down and bottom-up dynamics in which scientific concepts shape the development of spontaneous concepts and vice versa. My work suggests that scientific concepts cross-pollinate in potentially fruitful ways—that students draw on other concepts learned in school to make sense of new ones, sometimes across disciplinary boundaries (as, for example, when students drew on ideas like atoms and molecules to make sense of the infinitesimal size of a mathematical point). In this way, unfamiliar scientific concepts enter into trajectories of conceptual development involving other scientific concepts, with their own histories in students' sense making efforts.

Contributions to Mathematics Education Research and Pedagogy

This dissertation also makes important contributions to mathematics education research. As mentioned, it points to a need to better understand a crucial but under-researched aspect of students' development as doers of mathematics. As other researchers have argued, students cannot not fully understand the discourse of mathematics unless they appreciate what the discourse is about: idealized, imagined objects with specific definitions.

My work engages research on definition use in students' mathematics learning. These studies provide additional evidence that reasoning from definitions can be challenging for young students. This challenge was shown to be most notable in the presence of diagrams, which are likely to cue a reliance on the inevitably inaccurate appearances of the diagrams, or on students' knowledge about the behavior of physical objects.

This dissertation has several implications for the practice of mathematics education. First, these studies suggest interesting possibilities for instructional approaches to supporting the definitional practice. Specifically, they raise the possibility of making the distinction between representation and mathematical object a direct target of instruction and an explicit topic of discussion in elementary and middle school classrooms. This could be done in a way that reflects the general approach taken in the Experimental (Study 1) and Practice Initiation (Study 2) treatments: exposing students to the idea of a breakdown between the defining features of the mathematical object and the manifest properties of the physical representation. Done well, such interventions could help offer more students an invaluable and potentially satisfying 'feel for the game' of academic mathematics.

Second, these studies offer some preliminary indication of at what grade it would be appropriate to implement the general kind of pedagogical approach modeled in these studies—making explicit the ‘rules’ of the definitional practice. The effectiveness of this general instructional approach depends on the extent to which it supports students in 1) drawing on definitions, 2) conceptually differentiating material diagram from idealized object, and 3) making rich sense of the definitions. I consider each below in light of the results of these studies.

Results indicate that students as young as fourth grade can *draw on definitions* in challenging geometrical problems, provided sufficient scaffolding. Study 1 provided little scaffolding, consisting of a paper and pencil assessment administered simultaneously to large groups of students. Results showed that fourth-graders struggled to draw on definitions when problems featured diagrams, which cue a reliance on their appearances and students’ knowledge of the physical world. The design of Study 2, by contrast, provided more scaffolding—a one-on-one interview study that offered more interactional support for the importance and use of definitions. This study showed that fourth-graders can indeed draw upon definitions in such challenging problem-solving contexts. Given the sufficient contextual support, therefore, instruction that makes explicit the rules of the definitional practice may be appropriate for students as young as fourth grade.

However, results also indicate that attempts to make explicit the ‘rules’ of the definitional practice may not be effective in supporting young students to construct a clear *conceptual differentiation* between material diagram and idealized object. Study 2 showed that only sixth-graders consistently gave voice to the understanding that diagrams representing points and lines are not themselves mathematical points and lines. An explicit discussion of the definitional practice may therefore be more confusing than clarifying for many elementary students. That said, it may be that some form of an explicit discussion of the definitional practice could be effective for young students, provided it included more scaffolding than was available in either Study 1 or Study 2.

Finally, results from Study 3 suggest that the definitions of points and lines are indeed challenging ideas for younger students. While there is no clear evidence that fourth-graders were consistently unable to make meaningful sense of the definitions of points and lines, the strongest evidence of rich sense making came from students who made materially-based analogies to explain the zero-dimensionality of points and the infinite length of lines. Recall that these analogies built upon ideas from academic science, such as atoms and outer space—ideas that sixth-graders naturally have had more exposure to. This finding points of the importance of considering available conceptual resources when discussing challenging mathematical definitions like those of points and lines.

Direction for Future Research

This dissertation suggests potentially fruitful directions for future research. Again, these directions advance both research that is specific to mathematics education as well as research on the sociocultural nature of cognitive development.

Within mathematics education, one direction targets the definitional practice specifically. These studies suggest future design-based research related to supporting the definitional practice. The intervention used in these studies was minimal—a one-page information sheet—and students had limited opportunity to engage with it. For instance, students were not given an opportunity to ask clarifying questions. Future design research might develop and test more robust interventions, such as classroom lessons, related to the definitional practice. Such research is needed before making any substantial claims about which students would most benefit from explicit discussion of and engagement with the definitional practice.

Relatedly, future classroom research should study the learning occasioned by instructional approaches developed to support the definitional practice. The studies described here used experimental and interview techniques; as such, they did not investigate the definitional practice "in the wild"—in the context of classroom teaching and learning. Future longitudinal classroom studies may explore teaching-learning dynamics in students' developing participation in the definitional practice over time. Such studies would permit an analysis of classroom discourse—including student-teacher and student-student communication—as a site for students to gain access to (or be excluded from) the definitional practice.

Future research on the definitional practice should also target other idealized mathematical objects. My work adopted points and lines in Euclidean geometry as a potentially fruitful mathematical context for studying the development of the definitional practice. Points and lines are interesting mathematical objects from this standpoint, because they incorporate idealized properties that differ so markedly from the objects that students encounter in daily life, properties that no drawn diagram could possess. Future research on the definitional practice might study other such objects, whether within or outside of geometry. For instance, the Cartesian plane is an excellent candidate. Fluency with the Cartesian plane is critical for success in more advanced mathematics, and—coincidentally—is a complex assemblage of mathematical points and lines.

Another broad direction for future research looks beyond mathematics education. This dissertation targets an important academic practice in which key aspects are typically left implicit, and it explores the possible affordances and limitations of making these aspects an overt target of instruction and topic of discussion. Undoubtedly, there are academic practices in other disciplines, such as science and literacy, in which core aspects are often left implicit. Future studies could adopt the general approach taken here and explore exciting but largely unrealized pedagogical possibilities—with the aim of giving students a richer, more satisfying 'feel for the game.'

This dissertation also points to directions for future research on cognitive development. Future research should explore the untapped potential of the method of double stimulation in order to enrich our understanding of meditational processes in cognitive activities and their development. This work may explore situations in which the 'auxiliary stimulus' requires its own sense making and cognitive structuration, as is the case—for example—with mathematical definitions. This use of the method of double stimulation has the potential illuminate the role of complex cognitive coordinations between individuals' construction of problem-solving goals and their construction of mediational means for solving these problems. Insights resulting from this work may shed further light on characteristic features of human cognition.

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Appendix A: Study 1—Pilot Interviews

Using various iterations of the interview procedure elucidated above, I conducted audio-taped pilot interviews with students in fourth (n=11), fifth (n=3) and sixth (n=3) grade. Pilot interviews were of great help in developing and refining the interview structure, problems, and definitions. Pilot interviews also confirmed the appropriateness of fourth-grade students for the study:

- Fourth-grade students were readily able to select answer choice and, which—during Phase 1—almost always involved appeals to the appearances of the diagrams or knowledge of the physical world.
- Using an early iteration of the interview structure, fourth-graders often did not select idealized answers after discussing the definitions. This finding prompted several changes. First, I decided to place the definitions face up and, for each problem, explicitly ask whether the definitions made students want to change their answers. Second, this finding motivated me to rewrite the elaborated definitions for the Practice Initiation to more explicitly draw the students' attention to the differences between what the diagrams look like and what we have agreed mathematical points and lines are like. Lastly, this finding led me to bold important text in the different answer choices to help students appreciate the differences between them.
- Oftentimes, I asked fourth-graders at the end of the interview if they could remember the definitions. Because many did not, I decided that the interview protocol should require students to repeat or rephrase the definitions before moving on to the problems.

Appendix B: Study 1—Most Frequent Answer Choices, Without Definition Access

The following two tables provide, for students in the control group, the most frequently selected answer choice type for each problem within each grade. Since students in the control group did not receive the informational sheet with the defining properties of points and lines, these students' answer choices reflect answers uninfluenced by the definitional intervention. Among students in the control group, I determined the most frequently selected answer choice and I calculated the percentage of students that selected this answer choice by grade level.

To summarize the results, across all problem types—whether *no diagram* or *diagram*—and all grade levels, the most popular response types were consistent with empirical interpretations of points and lines. Indeed, idealized interpretations were the most popular only at eighth grade and for only two of the eight problems, Problem 2 (50%) and Problem 8 (40%). Further, I found considerable commonality for most popular empirical interpretations across grade levels for all problems. For the *no diagram* problems (Problem 1-3), the most common answer type was "it depends on how you draw the points/lines." For the *diagram* problems (Problem 4-8), the variation in quality of problems and menu of choices led to different choices across problems but considerable uniformity within problems.

Table: Most frequent answer choices by grade for *no diagram* problems among students in the control group (Problem 1-3)

	No Diagram Problems		
Grade Level	P1: How big is a point?	P2: How long is a line?	P3: How thick is a line?
Fourth	(iii) 63%	(iii) 52%	(iii) 40%
Sixth	(iii) 56%	(iii) 56%	(iii) 48%
Eighth	(iii) 50%	(i) 50%	(iii) 40%
Answer choices (abbreviated and re-ordered)	(i) No size at all.* (ii) Small, the size of a dot. (iii) It depends on how big you draw it.	(i) It is infinitely long.* (ii) About as long as a finger. (iii) It depends on how long you draw it.	(i) No thickness at all.* (ii) Thin - a little bit thicker than a hair, but not as thick as a pencil. (iii) It depends on how thick you draw it.

Note. Percentages indicate the proportion of control students who selected the indicated answer choice for each problem at each grade level.
* Indicates the answer that is consistent with the definition

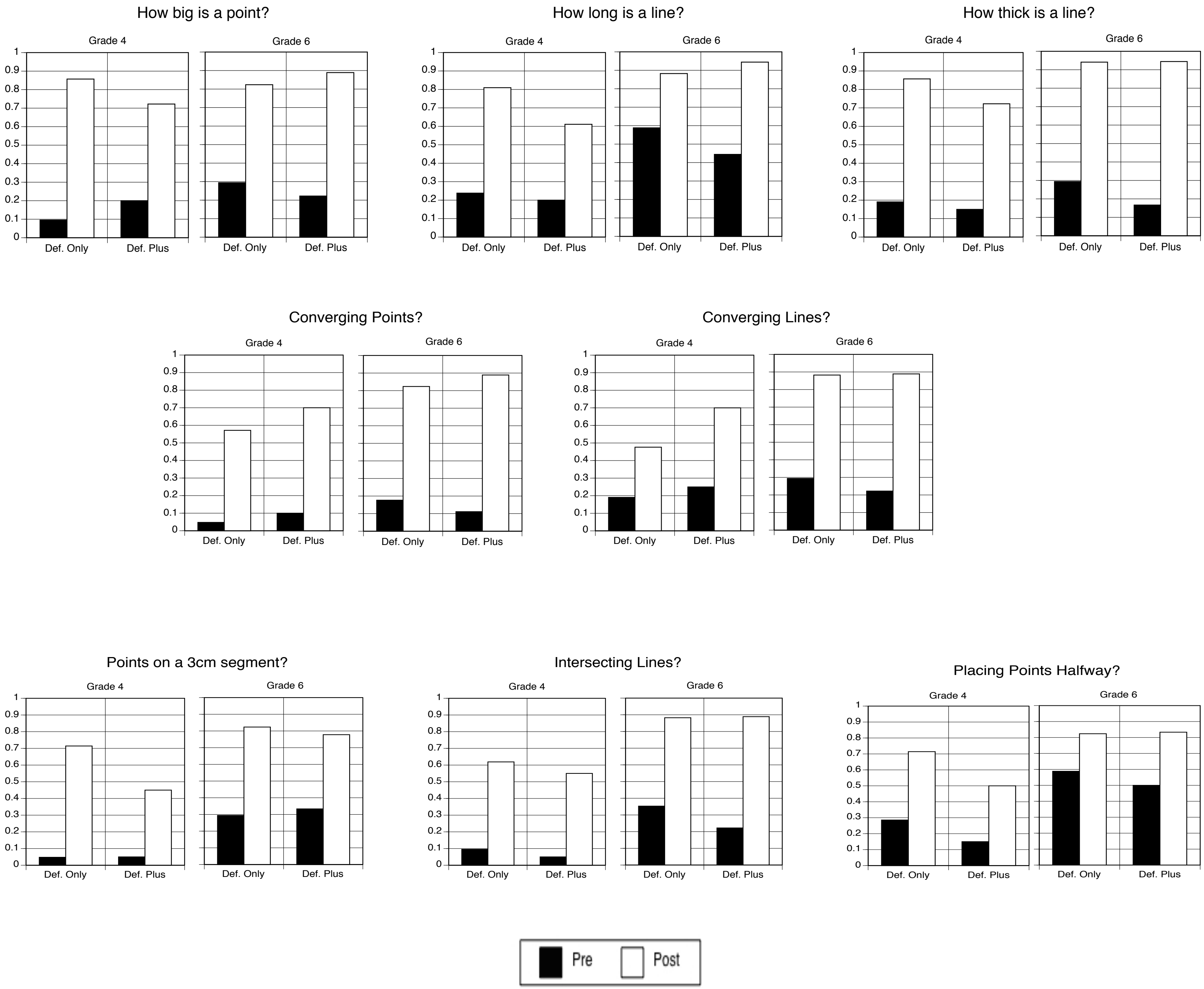
Table: Most frequent answer choices by grade for *diagram* problems among students in the control group (Problem 4-8)

	Diagram Problems				
Grade Level	P4: What is the dimensionality of points that converge on the same spot?	P5: What is the dimensionality of lines that converge on the same spot?	P6: How many points on a line segment?	P7: Relative number of points that can fit on two represented line segments of different lengths?	P8: Properties of two points, one constructed by 2 intersecting lines, the other by 5?
Fourth	(iv) 44%	(ii) 36%	(ii) 72%	(ii) 52%	(ii) 52%
Sixth	(iv) 44%	(iv) 44%	(ii) 50%	(iii) 48%	(ii) 44%
Eighth	(iv) 68%	(iv) 70%	(ii) 42%	(iii) 40%	(i) 40%
Answer choices (abbreviated and re-ordered)	(i) Turn into one point - no size at all* (ii) Turn into one bigger point (iii) One point - still the size of a small dot (iv) Two points, but stacked	(i) One line, no thickness at all* (ii) One line, but thicker (iii) One line, still thicker than a hair (iv) Two lines, but stacked	(i) Infinite number* (ii) 5 to 20 (iii) 20 to 100 (iv) 100 to 10,000 (v) It depends on how big you draw the points	(i) Infinite number can fit on both* (ii) More points can fit on the longer segment (iii) It depends on how big you draw the points	(i) Points are the same – neither has size* (ii) More lines crossing makes bigger point (iii) Two lines makes one point, five lines makes several points

* Indicates the answer that is consistent with the definition

Appendix C: Study 2—Phase 3 Results, Problem by Problem

The following charts show the problem-by-problem results for Phase 3 (following the treatment) based on grade and treatment group. These charts provide the proportion of students in each group who selected the idealized response for each item. On the following page, the first row of charts correspond to the *No Diagram* items, and the remaining charts to the *Diagram* items.


How big is a point?
How long is a line?
How thick is a line?
Converging Points?
Converging Lines?
Points on a 3cm segment?
Intersecting Lines?
Placing Points Halfway?
Grade 4
Grade 6
Def. Only
Def. Plus
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Pre
Post

Appendix D: Study 3—Student Sense-making Extracts

The following tables provide student explanations during Study 3 that included material analogies. The Student ID's are structured as follows: GRADE(4 or 6)_STUDENT NUMBER_PROBLEM ORDER(A or B)/TREATMENT GROUP(E or C)_GENDER(M or F). For treatment group, E refers to Practice Initiation and C refers to definition only.

Mathematical Point: Material Analogies Involving Small Bits of Matter

4_4_AC_F:	Well, you could say that a line or a dot is like nothing. You could say that it's nothing. You can't really see, it's like you can't see germs but it's there…It's like that small. It's the tiniest thing. Like you can barely see it, it's like a germ on your hand.
6_10_BC_F:	I was thinking that in atoms and molecules, like everywhere on Earth, and they're in space I think. There's no atoms, right? And so if it's on Earth they can be talking about something without it being there.
Interviewer:	Mm-hm.
6_10_BC_F:	But it's still there, but in space because it could be something really small, but there's still stuff all around us. I think, and then but in space there's nothing so that wouldn't really make sense there, but here it would because things are everywhere. I think that would – I think that would make sense.
6_23_AC_F	No size, I think an atom or something.
Interviewer:	Uh-huh, okay.
6_23_AC_F:	An atom probably.
Interviewer:	So, which is bigger? A mathematical point or an atom?
6_23_AC_F:	An atom is way bigger.
Interviewer:	It's way bigger?
6_23_AC_F:	It's way bigger.
Interviewer:	It's way bigger, okay.
6_25_AC_F	Like a microscopic tiny, tiny, tiny point that you can't see, and that – I guess it could be in your head because you have to draw something.
6_26_BE_F:	It is there, but it's not physical. It's just like air.

Interviewer:	Mm.
6_26_BE_F:	Like just like the atmosphere's boundary.
Interviewer:	So small it has no size? Had could something be so small?
6_26_BE_F:	Like an atom or –
Interviewer:	Oh okay.
6_26_BE_F:	It can only [inaudible 00:13:42] something.
Interviewer:	Well, which is small. A mathematical point or an atom?
6_26_BE_F:	Probably an atom.
6_38_BE_M:	Well, how can a point take up no space, and no size when it's a point? The only thing that can't take up the space, or size, is air.
Interviewer:	Uh-huh.
6_38_BE_M:	Carbon.
Interviewer:	Uh-huh.
6_38_BE_M:	And more air.
Interviewer:	So, earlier you talked about an atom. I was curious about that. So, an atom is pretty small, right?
6_40_BC_M:	Mm-hm.
Interviewer:	And it sounds like a point is pretty small too. Which one of those is smaller?
6_40_BC_M:	A point.
Interviewer:	Why is a point smaller?
6_40_BC_M:	Because an atom is really small, but on a point it takes no size and no space unless you're really [inaudible 00:18:03] little skittle won't take up any space in your house unless you're seeing it sarcastically [inaudible 00:18:13]

Mathematical Line: Material Analogies Involving Outer Space

4_15_AE_M: Like there could be a line on the ground that goes on for a really, really long time, we could say it goes on forever because it could be super long. But eventually it would have to end so it can't just go on infinitely unless it could be in space and then that's infinite. I don't really know [inaudible 00:00:37.28] space. It is a space as people [inaudible 00:00:43.07] gases, planets and stuff.

Interviewer: So you would say that if a line, you think this definition is, someone said that a line is so long that it goes on forever. I was like, I don't really get that. What would you say to try and help me understand this mathematical definition?

4_15_AE_M: I feel like if you draw a line, this end till there, from one end of this wall to the other end of over there [inaudible 00:01:18.16] If you have a string and you tie it to a light pole on Earth and you fly all the way out to the [inaudible 00:01:35.02] outer space and you tie either like you stick something in the ground [inaudible 00:01:40.14] tie back, you can then say it goes on forever because it goes so far out, straight way to go.

4_32_AE_F: It goes on forever so that means it'll go past Mars, it'll go past [inaudible 00:02:13.07] it would go past-

Interviewer: How far would it go past there?

4_32_AE_F: It would go until, it'll go forever, it will just keep on going, keep on going, keep on going, never stop.

6_01_AE_F: It won't stop here or here at all. I don't know, it goes to outer space and farther.

6_06_BE_F: It's like a circle. A circle goes on forever. It's never ending. And then it's a line, it's just a straight circle. If you un-round the circle, and you kept all of its traits except the fact that it was round, this would be the circle and it would just go on forever. But it's like if you have a pole in the ground, it just goes up and up and it just keeps going up until it hits the Sun and its spots.

6_09_BC_F: It'll go on forever forever. And it'll go on after the [inaudible 00:06:21.25] and the room and the Sun. And it explodes and the Earth explodes and it'll keep on going forever.

6_13_AE_M: Just imagine something that never stops expanding like universe or something. It doesn't stop expanding, it'll go on forever. That's the same with a mathematical line but one goes on in each way forever. It never stops going.
6_26_BE_F: Like it goes on [inaudible 00:12:26.26] the cosmos, just goes on until there's no end. I still think about that every day. It's mind boggling.
6_30_BC_F: Imagine trying to draw [inaudible 00:10:20.15] and never trying to go on forever. Eventually you can [inaudible 00:13:13.13] but eventually you just, if you were drawing forever, and you try to draw around the world, that wouldn't be forever because it would eventually end back here. Interviewer: At the same spot. And that would not be long enough. 6_30_BC_F: And if you kept drawing into space, I think space goes on forever or something but you'd never get to it.
6_38_BE_M: You can only do an infinite line, only if you're in space because the universe is infinite so there you go. Interviewer: For someone like me who doesn't understand it, you might tell me to imagine space? 6_38_BE_M: Yes.