Title
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Journal
Physical Review D, 3(4)

ISSN
0556-2821

Authors
Silverman, D
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Publication Date
1971

DOI
10.1103/PhysRevD.3.991

PHYSICAL REVIEW D VOLUME 3, NUMBER 4 15 FEBRUARY 1971

Scaling Behavior in Multiperipheral Dynamics*†

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(Received 14 September 1970)

We demonstrate the scaling of the single-particle momentum distribution as a general property of all the multiperipheral models which have been proposed. We also show that in these models, pionization is approached as a smooth limit from scaling. The proof is based only on the most general multiperipheral assumption and on Pomeranchuk-pole dominance at high energies. Thus the experimental observation of scaling is required for the validity of any multiperipheral model.

I. INTRODUCTION

USING a great amount of physical insight, Feynman[1] has recently proposed that the longitudinal-momentum distributions in hadron collisions should exhibit certain simple scaling and limiting features. We consider the simplest possible inclusive experiment where only one final particle is detected, and the invariant momentum distribution

$$\frac{d\sigma}{d^3k/2k_0} \equiv F(s,x,k_\perp) \qquad (1)$$

is expressed as a function of the square of the total c.m. energy s, the scaled c.m. longitudinal momentum component of the detected particle $x=k_{\parallel}/k_{\max}$, $k_{\max} \simeq \frac{1}{2}\sqrt{s}$, and the transverse momentum $k_\perp$. For this case, and for small values of $k_\perp{}^2$, where the majority of physical events takes place, Feynman proposed at high energies the scaling property[2]

$$\lim_{s\to\infty} F(s,x.k_\perp) = \tilde{F}(x,k_\perp) \quad \text{for} \quad 0<|x|\leq 1\,, \qquad (2)$$

and the existence of "pionization," i.e., production of low-energy particles in the c.m. system with the simple spectrum

$$\lim_{s\to\infty\,;\,x\to 0} F(s,x,k_\perp) = \bar{F}(k_\perp)\,; \qquad (3)$$

the spectrum is nonzero. It is the purpose of this paper to verify the scaling property as a general property of all multiperipheral models which have been proposed[3,4] and to show that in these models pionization is approached as a smooth limit from scaling

$$\lim_{x\to 0} \tilde{F}(x,k_\perp) = \bar{F}(k_\perp)\,. \qquad (4)$$

Analysis of existing experimental data supports this scaling behavior.[5]

The phenomenon of pionization has been demonstrated with the original ABFST multiperipheral model[3] and the proof can be directly extended to the more general multiperipheral models. The original pionization analysis, however, was performed only for $x\rightarrow 0$, and was not applicable to scaling in what we term the production region (forward production for $x>0$, backward for $x<0$). The present authors[6] have recently demonstrated the scaling phenomena in an analysis of the inclusive single-particle experiment for special multiperipheral models with exponential damping in momentum transfer. We present here a proof applicable to all multiperipheral models so far studied by making use of the CGL model, which includes the ABFST model as a special case. We find that the important requirement for scaling is that the output auxiliary forward amplitude B is dominated, above some finite subenergy, by a Pomeranchon of intercept unity, i.e., that the solution leads to a constant total cross section.

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* Work supported in part by the U. S. Atomic Energy Commission [Report No. NYO 2262-TA-225 (unpublished)].

† Research sponsored in part by the U. S. Air Force Office of Scientific Research under Contract No. AF49(638)-1545.

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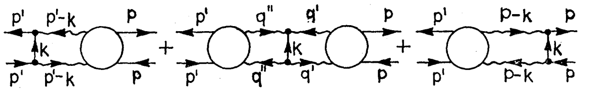

FIG. 1. The left-end, central, and right-end contribution to the single-particle distribution spectrum.

II. INCLUSIVE SINGLE-PARTICLE SPECTRUM

In the CGL multiperipheral model, the single-particle momentum distribution[6–9] is given by

$$\frac{d\sigma}{(d^3k/2k_0)}=\frac{1}{(2\pi)^3\Delta^{1/2}(s,m'^2,m^2)}\times\bigg[\,|G_l((p'-k)^2)|^2B(p',p'-k;p)$$
$$+\frac{2}{(2\pi)^4}\int d^4q'd^4q''\delta^4(q'+q''+k)B(-q',q'';p')$$
$$\times|\beta(q''^2,\omega,q'^2)|^2B(-q'',q';p)$$
$$+|G_r((p-k)^2)|^2B(p,p-k;p')\bigg],\qquad(5)$$

where $B(-q',q'';p')$ is the auxiliary function which satisfies the CGL equation, G is a Reggeon-particle-particle vertex function, and β is a Reggeon-Reggeon-particle vertex function which may include dependence on the Toller angle ω. The first and last terms are the contribution of the left and right end diagrams, as illustrated in Fig. 1. The middle term is the contribution of the central diagram, which dominates for $|x|$ not close to 1. It can be easily seen that the end diagrams directly exhibit scaling,[10] so we shall concentrate on the central diagram.

Our proof proceeds by first transforming integrations to the invariant subenergy and momentum-transfer variables and showing how the Jacobian of this transformation has scaling behavior because of the multiperipheral hypothesis that the momentum transfers are small. We are then able to show that in the important regions of integration, the auxiliary forward amplitudes B exhibit scaling and produce and $s^{\alpha_P(0)}$ behavior to cancel the $1/s$ from the flux. This will complete the proof of scaling for the inclusive single-particle spectrum.

We transform the integrations first to the invariant subenergies $s_l'\equiv p_l^2=(p'+q'')^2$, $s_r'\equiv p_r^2=(p+q')^2$ and momentum transfers $t_l\equiv q''^2$, $t_r\equiv q'^2$, as shown in Fig. 2,

$$\int d^4q'd^4q''\delta^4(q'+q''+k)=\int ds_l'ds_r'dt_ldt_rJ\,.\qquad(6)$$

III. TRANSFORMATION OF VARIABLES AND SCALING OF JACOBIAN

We compute the Jacobian in a useful form by first using the $\delta^4(q''+q'+k)$ to do the q' integration so that the Jacobian is

$$J^{-1}=\det\left[\frac{\partial(s_l',s_r',t_l,t_r)}{\partial(q_0'',q_z'',q_{\perp 1}'',q_{\perp 2}'')}\right],\qquad(7)$$

where $q_{\perp 1}''$ and $q_{\perp 2}''$ are the components of $q_\perp''$ parallel and perpendicular to $k_\perp$, respectively. This may be rearranged to give

$$J^{-1}=8\det\begin{bmatrix}(p'+p)_0 & (p'+p)_z & (p'+p)_{\perp 1} & (p'+p)_{\perp 2}\\ (p'-p)_0 & (p'-p)_z & (p'-p)_{\perp 1} & (p'-p)_{\perp 2}\\ q_0' & q_z' & q_{\perp 1}' & q_{\perp 2}'\\ q_0'' & q_z'' & q_{\perp 1}'' & q_{\perp 2}''\end{bmatrix}.\qquad(8)$$

This is directly evaluated in the c.m. system to be

$$J^{-1}=\left|16(\sqrt{s})p_z'\det\begin{bmatrix}q_{\perp 1}' & q_{\perp 2}'\\ q_{\perp 1}'' & q_{\perp 2}''\end{bmatrix}\right|$$
$$=|16(\sqrt{s})p_z'k_\perp'q_{\perp 2}''|\,.\qquad(9)$$

Now computing $|q_{\perp 2}''|$ and using $p_z'=\Delta^{1/2}(s,m'^2,m^2)/2\sqrt{s}$ gives

$$J=\frac{1}{4\Delta^{1/2}(s,m'^2,m^2)}\frac{\theta(-\Delta(k_\perp^2,q_\perp'^2,q_\perp''^2))}{[-\Delta(k_\perp^2,q_\perp'^2,q_\perp''^2)]^{1/2}}.\qquad(10)$$

We must now express $q_\perp'^2$, $q_\perp''^2$ in terms of the invariant variables, which is simply done by considering as a two-body process $p'+p\rightarrow(p_l+k)+p_r$ and using the Kibble method[11] to calculate $|q_\perp'|$. Since the two-body reaction occurs in a plane, we consider the transverse direction in the plane as that of $q_\perp'$ and define

$$D\equiv\begin{bmatrix}(p'+p)_0 & (p'+p)_z & (p'+p)_\perp\\ (p'-p)_0 & (p'-p)_z & (p'-p)_\perp\\ q_0' & q_z' & |q_\perp'|\end{bmatrix}.$$

In the c.m. system this determinant is evaluated directly as

$$d\equiv|\det D|=2|q_\perp'|p_z'\sqrt{s}=|q_\perp'|\Delta^{1/2}(s,m'^2,m^2)\,.$$

Multiplying D by its transpose, and reversing signs of spatial components, we have

$$\det[DD_-{}^T]=(\det D)(\det D^T)=d^2\,.$$

With minor rearrangement of $[DD_-{}^T]$, we have

$$q_\perp'^2=\frac{4}{\Delta(s,m'^2,m^2)}\det\begin{bmatrix}p^2 & p\cdot p' & p\cdot q'\\ p'\cdot p & p'^2 & p'\cdot q'\\ q'\cdot p & q'\cdot p' & q'^2\end{bmatrix}.\qquad(11)$$

In order to express the determinants in terms of the integration variables, we will use, instead of $k_{\parallel}$ and $k_\perp$,

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[7] L. Caneschi and A. Pignotti, Phys. Rev. Letters 22, 1219 (1969).

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[9] D. Tow, Phys. Rev. D 2, 194 (1970); P. Ting, *ibid.* 2, 2982 (1970).

[10] The end diagram in Ref. 8 exhibits scaling since it only depends on the ratio $s/M^2=(1-x)^{-1}$.

[11] T. W. B. Kibble, Phys. Rev. 117, 135 (1960).

the fixed invariants

$$u_l \equiv (p'-k)^2, \quad u_r \equiv (p-k)^2.$$

We then have the positive invariants which are given, to $O(1)$,

$$\begin{aligned} 2p'\cdot k &= (-u_l+m'^2+\mu^2) = (\surd s)(k_0-k_z) \\ &\quad + \frac{m'^2}{2}\frac{(k_0+k_z)}{k_{\max}} - \frac{m^2}{2}\frac{(k_0-k_z)}{k_{\max}} + O(1/s), \\ 2p\cdot k &= (-u_r+m^2+\mu^2) = (\surd s)(k_0+k_z) \\ &\quad + \frac{m^2}{2}\frac{(k_0-k_z)}{k_{\max}} - \frac{m'^2}{2}\frac{(k_0+k_z)}{k_{\max}} + O(1/s), \end{aligned} \tag{12}$$

and

$$\begin{aligned} \frac{(4)(p'\cdot k)(p\cdot k)}{s} &= k_\perp^2+\mu^2+\left(\frac{m'^2}{2}\right)\left(\frac{k_0+k_z}{k_{\max}}\right)\left(\frac{k_z}{k_{\max}}\right) \\ &\quad +\left(\frac{m^2}{2}\right)\left(\frac{k_0-k_z}{k_{\max}}\right)\left(\frac{-k_z}{k_{\max}}\right)+O(1/s) \\ &= k_\perp^2+\mu^2+m'^2x^2\theta(x)+m^2x^2\theta(-x) \\ &\quad +O(1/s). \end{aligned} \tag{13}$$

In the last formula and later on in computing terms of $O(1)$, we observe that in

$$\left(\frac{k_0}{k_{\max}}\right)\left(\frac{k_z}{k_{\max}}\right) = \left[x^2+\frac{4(k_\perp^2+\mu^2)}{s}\right]^{1/2} x,$$

and

$$\left(\frac{k_0}{k_{\max}}\right)^2 = x^2+\frac{4(k_\perp^2+\mu^2)}{s},$$

we may neglect $(k_\perp^2+\mu^2)/s$ unless $x\sim O(1/\surd s)$, but then these terms will be $O(1/s)$ and negligible, so we may write

$$\left(\frac{k_0}{k_{\max}}\right)\left(\frac{k_z}{k_{\max}}\right) = x|x|+O(1/s),$$

and

$$\left(\frac{k_0}{k_{\max}}\right)^2 = x^2+O(1/s). \tag{14}$$

In terms of these invariants the phase-space regions are as follows.

Pionization: $k_0, k_z = O(1)$; $x \leq O(1/\surd s)$; $-u_l = O(\surd s)$, $-u_r = O(\surd s)$.

Forward production: $k_z = O(\surd s)$ and positive; x a fixed positive fraction; $-u_l = O(1)$, $-u_r = O(s)$.

Backward production: $k_z = O(\surd s)$ and negative; x is a fixed negative fraction; $-u_l = O(s)$, $-u_r = O(1)$.

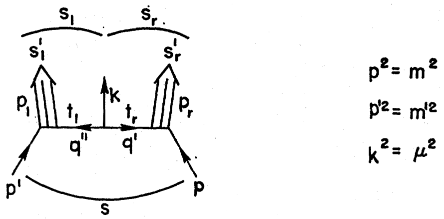

FIG. 2. Kinematics of the central diagram.

The invariants in the Jacobian become

$$\begin{aligned} 2p\cdot p' &= s-m^2-m'^2, \\ 2p\cdot q' &= s_r'-t_r-m^2, \\ 2p'\cdot q' &= -2p'\cdot q''-2p'\cdot k \\ &= -(s_l'-t_l-m'^2)-(-u_l+m'^2+\mu^2). \end{aligned}$$

We now change from the integration variables s_l', s_r' to the scaled integration variables y,z, which will be shown to be of $O(1)$ due to multiperipheralism,

$$\begin{aligned} y &\equiv \frac{p'\cdot q''}{p'\cdot k} = \frac{s_l'-t_l-m'^2}{-u_l+m'^2+\mu^2}, \\ z &\equiv \frac{p\cdot q'}{p\cdot k} = \frac{s_r'-t_r-m^2}{-u_r+m^2+\mu^2}. \end{aligned} \tag{15}$$

In terms of these we have finally the *exact* expression to go into the Jacobian Eq. (10):

$$\begin{aligned} q_\perp'^2 &= -t_r - \frac{(s-m^2-m'^2)}{\Delta(s,m'^2,m^2)} \\ &\quad \times(-u_l+m'^2+\mu^2)(-u_r+m^2+\mu^2)(y+1)z \\ &\quad -(m'^2)\frac{(-u_r+m^2+\mu^2)^2}{\Delta(s,m'^2,m^2)}z^2-(m^2)\frac{(-u_l+m'^2+\mu^2)^2}{\Delta(s,m'^2,m^2)} \\ &\quad \times(y+1)^2. \end{aligned} \tag{16}$$

The expression for $q_\perp''^2$ arises by exchanging $l\leftrightarrow r$, $y\leftrightarrow z$, and $m^2\leftrightarrow m'^2$.

The basic hypothesis of multiperipheralism is now applied by effectively restricting the momentum transfers to $(-t_r)$, $(-t_l) \leq O(1)$. We note that all of the terms in Eq. (16) subtracted from $(-t_r)$ are positive, and that the coefficient of $(y+1)z$ is always $O(1)$ [see Eq. (13)]. The requirements $q_\perp'^2 \geq 0$, $q_\perp''^2 \geq 0$ and the restrictions on $(-t_l)$, $(-t_r)$ lead to the bounds

$$(y+1)z \leq O(1), \quad (z+1)y \leq O(1),$$

and we conclude that

$$y \leq O(1), \quad z \leq O(1). \tag{17}$$

We then write $q_\perp'^2$, $q_\perp''^2$ at large s, keeping all terms of $O(1)$, by using Eqs. (12)–(14):

$$
\begin{aligned}
q_\perp'^2 &= -t_r-(k_\perp^2+\mu^2)(y+1)z-m'^2x^2\theta(x)z(1+y+z) \\
&\quad -m^2x^2\theta(-x)(y+1)(1+y+z)+O(1/s)\,, \\
q_\perp''^2 &= -t_l-(k_\perp^2+\mu^2)(z+1)y-m^2x^2\theta(-x)y(1+y+z) \\
&\quad -m'^2x^2\theta(x)(z+1)(1+y+z)+O(1/s)\,.
\end{aligned}
\tag{18}
$$

The inclusive integration at large s is now expressed in terms of scaled variables with the use of Eqs. (10) and (13):

$$
\begin{aligned}
&\left(\frac{1}{s}\right)\int d^4q'd^4q''\delta^4(q'+q''+k) \\
&= \frac{1}{s}[k_\perp^2+\mu^2+m'^2x^2\theta(x)+m^2x^2\theta(-x)] \\
&\qquad \times\int dydzdt_ldt_r\frac{\theta(-\Delta(k_\perp^2,a_\perp'^2,q_\perp''^2))}{[-\Delta(k_\perp^2,q_\perp'^2,q_\perp''^2)]^{1/2}}.
\end{aligned}
\tag{19}
$$

We have completed calculating the Jacobian at large s using the multiperipheral hypothesis and find from Eq. (18) that it depends only on x and $k_\perp^2$ and integration variables, which proves that the Jacobian scales.

In the pionization region $x\leq O(1/\sqrt{s})$, the terms in x^2 may be dropped and the Jacobian then agrees with that used by ABFST to show independence of x in the pionization region. The terms in x^2 are an essential reason why the production regions have a nontrivial x dependence.

IV. SCALING AND POMERANCHUK-POLE DOMINANCE

The assumption of Pomeranchuk-pole dominance for the auxiliary forward amplitudes B above a finite energy will now be shown to provide an asymptotic behavior $s^{\alpha p(0)}$ to cancel the $1/s$ flux factor, and the remaining dependence in the B's will exhibit scaling by being a function only of x, $k_\perp^2$, and the integration variables. We may express the B's in terms of invariants including the energies

$$
\begin{aligned}
s_l &\equiv (p'-q')^2=(s_l'-t_l-m'^2) \\
&\qquad +(-u_l+m'^2+\mu^2)+t_r+m'^2\,, \\
s_r &\equiv (p-q'')^2=(s_r'-t_r-m^2) \\
&\qquad +(-u_r+m^2+\mu^2)+t_l+m^2\,,
\end{aligned}
\tag{20}
$$

e.g.,

$$
B(-q',q'';p')=B(s_l,s_l',t_l,t_r)\,.
$$

The application of multiperipheralism in Sec. III showing $y\leq O(1)$, $z\leq O(1)$ means that the predominant ranges of the subenergies s_l', s_r' depend upon phase-space regions, as found from Eq. (15), and they must be treated *separately*.

Pionization: $s_l'\sim O(\sqrt{s})$, $s_r'\sim O(\sqrt{s})$.

Forward production: $s_l'\sim O(1)$, $s_r'\sim O(s)$.

Backward production: $s_l'\sim O(s)$, $s_r'\sim O(1)$.

The assumption of Pomeranchuk-pole dominance at large subenergies for the B's will provide the needed $s^{\alpha p(0)}$ by different mechanisms in the three regions, which can be roughly seen as follows.

Pionization: Both B_l and B_r are Pomeranchuk-pole dominated, so that

$$
B_lB_r\propto(s_l')^{\alpha p(0)}(s_r')^{\alpha p(0)}\propto s^{\alpha p(0)}\,.
$$

Forward production: Only B_r is Pomeranchuk-pole dominated because s_l' is small,

$$
B_lB_r\propto(s_r')^{\alpha p(0)}\propto s^{\alpha p(0)}\,.
$$

Backward production: Only B_l is Pomeranchuk-pole dominated because s_r' is small,

$$
B_lB_r\propto(s_l')^{\alpha p(0)}\propto s^{\alpha p(0)}\,.
$$

A. Pionization Region

Since the subenergies become asymptotic in this region, we use the asymptotic relations that follows from the invariance properties of the CGL equation[4] and the hypothesis of Pomeranchuk-pole dominance:

$$
\begin{aligned}
B(s_l,s_l';t_l,t_r) &= (s_l')^{\alpha p(0)}\tilde{B}(s_l/s_l';t_l,t_r)\,, \\
B(s_r,s_r';t_r,t_l) &= (s_r')^{\alpha p(0)}\tilde{B}(s_r/s_r';t_r,t_l)\,.
\end{aligned}
\tag{21}
$$

Using Eqs. (20), (15), and (13) for large values of s_l', s_r', u_l, u_r, and $x\leq O(1/\sqrt{s})$, we convert to scaled variables:

$$
\begin{aligned}
&(s_l')^{\alpha p(0)}\tilde{B}\left(\frac{s_l}{s_l'};t_l,t_r\right)(s_r')^{\alpha p(0)}\tilde{B}\left(\frac{s_r}{s_r'};t_r,t_l\right) \\
&=(u_lu_r)^{\alpha p(0)}\left(\frac{s_l'}{-u_l}\right)^{\alpha p(0)}\left(\frac{s_r'}{-u_r}\right)^{\alpha p(0)}\tilde{B}\left(\frac{s_l'-u_l}{s_l'};t_l,t_r\right) \\
&\qquad\times\tilde{B}\left(\frac{s_r'-u_r}{s_r'};t_r,t_l\right) \\
&=s^{\alpha p(0)}(k_\perp^2+\mu^2)^{\alpha p(0)}y^{\alpha p(0)}z^{\alpha p(0)} \\
&\qquad\times\tilde{B}\left(1+\frac{1}{y},t_l,t_r\right)\tilde{B}\left(1+\frac{1}{z},t_r,t_l\right).
\end{aligned}
\tag{22}
$$

The $s^{\alpha p(0)}$ cancels the $1/s$ from the flux in Eq. (19) and the rest of the dynamical input is seen to exhibit not only scaling but also independence of x. The ω-angle dependence of the coupling $\beta(t_l,\omega,t_r)$ also exhibits scaling, since for large s_l', s_r' it is related to the scaled variables by

$$
\begin{aligned}
\frac{\Delta(\mu^2,t_l,t_r)}{\mu^2-t_l-t_r+2(t_lt_r)^{1/2}\cos\omega} &= \frac{s_ls_r}{s} \\
&=\left(\frac{u_lu_r}{s}\right)\left(1+\frac{s_l'}{-u_l}\right)\left(1+\frac{s_r'}{-u_r}\right) \\
&=(k_\perp^2+\mu^2)(1+y)(1+z)\,.
\end{aligned}
\tag{23}
$$

B. Production Regions

In forward production, $(-u_r)=xs$ and $s_r,s_r'=O(s)$, so we may use Pomeranchuk-pole dominance on

$$\begin{aligned} B(s_r,s_r';t_r,t_l)&=(s_r')^{\alpha p(0)}\tilde{B}(s_r/s_r';t_r,t_l)\\ &=s^{\alpha p(0)}(xz)^{\alpha p(0)}\tilde{B}(1+1/z;t_r,t_l). \end{aligned} \tag{24}$$

Since the subenergies s_l, s_l' are $O(1)$ in this region, we *cannot* use Pomeranchuk-pole dominance or the asymptotic symmetry relation on $B(s_l,s_l';t_l,t_r)$. Instead we show that in this region $(-u_l+m'^2+\mu^2)$, which is $O(1)$, is a function only of x and $k_\perp^2$, and this will lead to scaling of $B(s_l,s_l';t_l,t_r)$. From Eq. (12) we have, for fixed x,

$$\begin{aligned} (-u_l+m'^2+\mu^2)&=\frac{s}{2}\left[\left(x^2+\frac{4(k_\perp^2+\mu^2)}{s}\right)^{\frac{1}{2}}-x\right]\\ &\qquad\qquad+m'^2x+O(1/s)\\ &=(k_\perp^2+\mu^2)/x+m'^2x, \end{aligned} \tag{25}$$

which is independent of s. From Eqs. (15) and (20) we may then convert the subenergies to scaled variables without introducing any dependence on s:

$$\begin{aligned} s_l'&=\left(\frac{k_\perp^2+\mu^2}{x}+m'^2x\right)y+t_l+m'^2,\\ s_l&=\left(\frac{k_\perp^2+\mu^2}{x}+m'^2x\right)(1+y)+t_r+m'^2. \end{aligned} \tag{26}$$

Consequently $B(s_l,s_l';t_l,t_r)$ is a function only of x, $k_\perp^2$, y, z, t_l, and t_r, and it is independent of s. The ω angle in the production region can also be shown to depend only on these variables, but we omit showing this since the calculation is lengthy, though straightforward. In concert with Eq. (24), we have proved scaling in the forward production region. The proof for the backward region follows by similarity.

C. Transition from Production to Pionization

We now show how the scaled momentum distribution in the production region $\tilde{F}(x,k_\perp)$ approaches as $x\to 0$ the pionization result $\bar{F}(k_\perp)$, which was obtained by the limiting procedure in Eq. (3). To do this we consider x to be very small and fixed. Then the terms in the Jacobian in x^2, Eqs. (18) and (19), are negligible and the Jacobian smoothly approaches the pionization limit. Considering again the analysis of the B functions in the forward production region, for very small x, the limit on s_l' given by $y\leq O(1)$ can become very large:

$$s_l'\leq(-u_l)O(1)$$

or

$$s_l'\leq[(k_\perp^2+\mu^2)/x]O(1).$$

For sufficiently small x, the majority of the range of s_l' will be at a sufficiently high subenergy to use Pomeranchuk-pole dominance and

$$\begin{aligned} B(s_l,s_l';t_l,t_r)&=(s_l')^{\alpha p(0)}\tilde{B}(s_l/s_l';t_l,t_r)\\ &=y^{\alpha p(0)}\left(\frac{k_\perp^2+\mu^2}{x}\right)^{\alpha p(0)}\tilde{B}\left(1+\frac{1}{y};t_l,t_r\right). \end{aligned} \tag{27}$$

Multiplying this by the other production result, Eq. (24), we see that the production-region dynamics smoothly approaches the pionization result, Eq. (22). We have thus completed the proof of Eq. (4) and shown that in the multiperipheral model the pionization region is a smooth limit of the scaling behavior in the production region.

V. CONCLUSION

We have presented a proof of scaling in the inclusive single-particle spectrum based only on the most general multiperipheral assumption and on Pomeranchuk-pole dominance at high energies. The experimental observation of scaling is thus a crucial necessity for the validity of any multiperipheral model. However, in order to differentiate between specific multiperipheral models, it is necessary to calculate the detailed dependence of the spectrum on x and $k_\perp^2$ in each of these models. To this end, we have completed an analytical study of the predictions of a simple multiperipheral model which assumes exponential damping in momentum transfers.[6]

Note added in proof. After the submission of our paper we received reports of two studies of single particle distributions in the multiperipheral models from Bali, Pignotti, and Steele, University of Washington report (unpublished), and DeTar, Lawrence Radiation Laboratory report (unpublished). Both studies reach conclusions similar to ours although they base their arguments on specific models. Bali, Pignotti, and Steele use a multi-Regge model with exponential damping in momentum transfers, similar to our previous work in Ref. 6, whereas DeTar uses the Chew-Pignotti model [Phys. Rev. **176**, 2112 (1968)]. The chief advantage of our approach lies in its generality of being applicable to all multiperipheral models. We would also like to add that a detailed treatment of the lower limits of subenergy intergations in Eq. (19) should explicitly exhibit the effect of the mass of the stable particle. Although this does not affect our conclusions, it does have important phenomenological consequences. We would like to thank Professor J. Ball for bringing this matter to our attention.

ACKNOWLEDGMENT

One of the authors (D. S.) wishes to thank the Aspen Center for Physics for their hospitality.