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On the Graded Group $U(1/1)$*

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Abstract

A non-canonoical parametrization for the graded group $U(1/1)$ is introduced similar to the Euler angles for the ordinary group $SU(2)$. Two differential representations for the underlying algebra of $U(1/1)$ are constructed on the full and the resticted, i.e. coset, parameter space, respectively. A space of functions living on the latter is found exhibiting close formal similarities to a Hilbert space. Remarkably, the indices of those functions and thus the orthogonality and completeness relations involve anticommuting variables. Using this Hilbert space a representation of $U(1/1)$ is evaluated which shows analogies to the Wigner functions for $SU(2)$.

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1 Introduction

In a previous paper [1], we have presented the Fourier-Bessel analysis in the space of ordinary and graded 2×2 Hermitean matrices. These investigations revealed the necessity to study the graded group $U(1/1)$ in a way analogous to the common discussion of the ordinary group $SU(2)$ in terms of angular momentum, spherical harmonics and Wigner representation functions, which can be found in Edmond's book [2]. It is the goal of this paper to perform this study. The mathematical literature on those subjects focusses, to the knowledge of the author, more on the graded algebras and their representations. An exception is Berezin [3] who discusses a representation theory for graded groups. However, our point of view seems to be a different one. Motivated by the physical application [1] we try to transfer the concepts developed in the framework of the ordinary group $SU(2)$ to the graded group $U(1/1)$. Our attitude is a more technical one, we introduce a non-canonocal parametrization of $U(1/1)$ in the spirit of the Euler angles for $SU(2)$ and then derive analogies to the ordinary case, i.e. differential representations of the algebra, graded spherical harmonics and graded Wigner representation functions. The discussion of more theoretical questions concerning the representation we construct, especially their reducibility, might be an interesting task for a mathematician.

We tried to write this article self-contained. Although closely related to the preceding paper [1], it can be read independently of it. For the convenience of the reader, we compile some well known results for $SU(2)$ in section 2 before we study the $U(1/1)$ in section 3. Our findings are summarized and discussed in section 4. Some explicit calculations are collected in the appendix.

2 Ordinary Group $SU(2)$

The commutation relations for the generator algebra of the group $SU(2)$ can be written in the form

$$[L_z, L_\pm] = \pm L_\pm \qquad \text{and} \qquad [L_+, L_-] = 2L_z \quad , \tag{1}$$

the vector $\vec{L}$ is the general angular momentum, including half integer quantum numbers, and $L_\pm = L_x \pm iL_y$ are the non-Hermitean shift operators.

The Casimir operator is the squared angular momentum,

$$\vec{L}^2 = \frac{1}{2}\{L_+, L_-\} + L_z^2 \tag{2}$$

where the curly brackets denote the anticommutator. It is often advantageous to choose the non-canonical Euler angles parametrization for the group $SU(2)$, the formal operator expression is given by

$$U(\varphi,\vartheta,\psi) = \exp(i\psi L_z)\exp(i\vartheta L_y)\exp(i\varphi L_z) \quad . \tag{3}$$

Taking the generators from the matrix representation, more specifically one half times the Pauli matrices, one arrives at the familiar result for the $SU(2)$ matrix,

$$U = \begin{bmatrix} \exp(i\psi/2) & 0 \\ 0 & \exp(-i\psi/2) \end{bmatrix} \begin{bmatrix} \cos(\vartheta/2) & \sin(\vartheta/2) \\ -\sin(\vartheta/2) & \cos(\vartheta/2) \end{bmatrix} \begin{bmatrix} \exp(i\varphi/2) & 0 \\ 0 & \exp(-i\varphi/2) \end{bmatrix} \quad . \tag{4}$$

The differential representation of the generators in the Euler angles can be derived straightforwardly,

$$\begin{aligned} L_z &= -i\frac{\partial}{\partial\varphi} \\ L_\pm &= -i\,\exp(\pm i\varphi)\left(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\psi} - \operatorname{cotan}\vartheta\frac{\partial}{\partial\varphi} \pm i\frac{\partial}{\partial\vartheta}\right) \end{aligned} \tag{5}$$

which gives for the Casimir operator

$$\vec{L}^2 = -\frac{\partial^2}{\partial\vartheta^2} - \operatorname{cotan}\vartheta\frac{\partial}{\partial\vartheta} - \frac{1}{\sin^2\vartheta}\left(\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\psi^2} - 2\cos\vartheta\frac{\partial^2}{\partial\varphi\partial\psi}\right) \quad . \tag{6}$$

The group $SU(2)$ considered as a group of transformations of coordinates is associated with a Hilbert space which is spanned by the spherical harmonics Y_{LM}. The $SU(2)$ representation functions, the Wigner functions, are the matrix elements of the operator (3) in this Hilbert space

$$D^{(L)}_{M'M}(U) = \langle LM'|U(\varphi,\vartheta,\psi)|LM\rangle \tag{7}$$

where the generators are now acting on the spherical harmonics. These Wigner functions are eigenfunctions of the Casimir operator (6) in the Euler angles parametrization to the eigenvalue $L(L+1)$, explicit expressions can be found in Edmonds' book [2]. Denoting by U_i an element of $SU(2)$ of the type (4) with parameters $(\varphi_i, \vartheta_i, \psi_i)$ the group operation $U_3 = U_2U_1$ is represented through

$$D^{(L)}_{M'M}(U_3) = \sum_{M''=-L}^{+L} D^{(L)}_{M'M''}(U_2)\, D^{(L)}_{M''M}(U_1) \quad . \tag{8}$$

There is also an orthogonality relation for the Wigner functions,

$$\int D^{(L_1)*}_{M_1'M_1}(U)\, D^{(L_2)}_{M_2'M_2}(U)\, d\mu(U) = \frac{1}{2L_1+1}\, \delta_{L_1L_2}\, \delta_{M_1'M_2'}\, \delta_{M_1M_2} \tag{9}$$

where $d\mu(U) = (8\pi^2)^{-1} d\varphi \sin\vartheta d\vartheta d\psi$ is the normalized invariant Haar measure on $SU(2)$. The character $\chi^{(L)}(U)$ of the representation to the angular momentum L is the trace of the matrix $D^{(L)}(U)$, one finds [4]

$$\chi^{(L)}(U) = \sum_{M=-L}^{+L} D^{(L)}_{MM}(U) = \frac{\sin((L+1/2)\rho)}{\sin(\rho/2)} \tag{10}$$

where $\exp(\pm i\rho/2)$ are the eigenvalues of the $SU(2)$ matrix (4), expressed in the Euler angles one has $\cos(\rho/2) = \cos(\vartheta/2)\cos((\varphi+\psi)/2)$. All results are easily generalized to the group $U(2)$ by introducing an additional phase corresponding to a generator that commutes with all the others.

3 Graded Group $U(1/1)$

In subsection 3.1 we introduce a parametrization for $U(1/1)$, in 3.2 we discuss the generators and the algebra and in 3.3 we construct the group representation. All definitions and conventions concerning the anticommuting variables are the same as in the preceding paper [1].

3.1 Parametrization

For reasons which will become clear later, we do not begin with the discussion of the underlying algebra but with the parametrization of the graded group

$U(1/1)$ itself. It is easily checked that the non-canonical parametrization

$$u = \exp(i\zeta/2) \begin{bmatrix} \exp(i\omega/2) & 0 \\ 0 & \exp(-i\omega/2) \end{bmatrix} \begin{bmatrix} 1+\alpha\alpha^*/2 & \alpha \\ \alpha^* & 1+\alpha^*\alpha/2 \end{bmatrix} \quad (11)$$

with two real commuting angles χ and ω and a complex anticommuting angle α has the required properties. Introducing as bases the Pauli matrices τ_i, $i = 1, 2, 3$ and the 2×2 matrices τ_{ij} with unity in the position (i, j) and zeros elsewhere we can write

$$\begin{aligned} u &= u^C u^A \\ u^C &= \exp\left(i\frac{\zeta}{2}1\right) \exp\left(i\frac{\omega}{2}\tau_3\right) \\ u^A &= \exp(\alpha\tau_{12} + \alpha^*\tau_{21}) \end{aligned} \quad (12)$$

which shows at least formal similarities to the Euler angles parametrization (4) in the ordinary case. The coset u^A is frequently used in applications [5]. It is instructive to evaluate the group multiplication law for the parameters explicitly. We write u_i for a matrix of the type (11) with parameters $(\zeta_i, \omega_i, \alpha_i, \alpha_i^*)$ and solve the set of equations resulting from $u_3 = u_2 u_1$ for the parameters of u_3, we obtain

$$\begin{aligned} \zeta_3 &= \zeta_2 + \zeta_1 - i\exp(-i\omega_1)\,\alpha_2\alpha_1^* - i\exp(+i\omega_1)\,\alpha_2^*\alpha_1 \\ \omega_3 &= \omega_2 + \omega_1 \\ \alpha_3 &= \exp(-i\omega_1)\alpha_2 + \alpha_1 \\ \alpha_3^* &= \exp(+i\omega_1)\alpha_2^* + \alpha_1^* \quad . \end{aligned} \quad (13)$$

Observe that the commuting and anticommuting variables are mixed, they do not appear decoupled. Observe that the angles ω_i satisfy a simple phase shift relation but not the angles ζ_i that actually occur only as phases. This is directly connected to the following consideration.

It is easy to construct a parametrization for the graded subgroup $SU(1/1)$, consisting of those elements of $U(1/1)$ which have graded determinant unity. By using the formula $\operatorname{detg} u = \exp \operatorname{trg} \ln u$ we find immediately from equation (12)

$$\operatorname{detg} u = \exp(i\omega) \quad , \quad (14)$$

hence, the choice $\omega = 0$ restricts the matrices u to $SU(1/1)$. Remarkably, the dependence on the remaining commuting angle χ is only an overall phase. This is of course related to properties of the underlying algebra.

3.2 Generators and Algebra

We construct the generators satisfying the underlying algebra of $U(1/1)$ in the full and in the restricted parameter space of the coset as differential operators. Moreover, we derive a matrix representation.

3.2.1 Full Space

Once the multiplication law (13) has been established, the differential representation of the algebra, i.e. the infinitesimal generators can be evaluated straightforwardly [6]. Here, however, we want to use a technique which does not employ the multiplication law explicitly and seems easier to handle in more complicated cases. Graded invariant Haar measures of various kinds are very efficiently calculated by variation [5, 7, 8]. Naturally, it is possible to extend this method to the evaluation of the generators. We perform this calculation in appendix A. The resulting infinitesimal generators are

$$\begin{aligned}
\Lambda_0 &= -i\frac{\partial}{\partial\zeta} \\
\Lambda_3 &= -i\frac{\partial}{\partial\omega} \\
\Lambda_+ &= \exp(-i\omega)\left(\frac{\partial}{\partial\alpha} + i\alpha^*\frac{\partial}{\partial\zeta}\right) \\
\Lambda_- &= \exp(+i\omega)\left(\frac{\partial}{\partial\alpha^*} + i\alpha\frac{\partial}{\partial\zeta}\right) \quad .
\end{aligned} \tag{15}$$

The first two generators are even, the last two odd. They satisfy the underlying algebra for the group $U(1/1)$,

$$\begin{aligned}
[\Lambda_0, \Lambda_\pm] &= 0 & \{\Lambda_+, \Lambda_-\} &= -2\Lambda_0 \\
[\Lambda_3, \Lambda_\pm] &= \pm\Lambda_\pm & [\Lambda_0, \Lambda_3] &= 0 \quad .
\end{aligned} \tag{16}$$

which has to contain one anticommutator for the two odd generators $\Lambda_\pm$. The first two relations of this algebra build the closed subalgebra for the subgroup $SU(1/1)$. The algebra (16) shows some formal similarities to the algebra (1) for the ordinary group $SU(2)$. The main formal difference is that Λ_0, i.e. the generator commuting with all the others and thus associated with a simple phase, appears in the subalgebra for $SU(1/1)$. In the ordinary case, however, the corresponding generator does not occur in the algebra for $SU(2)$. Using the above algebra it is easily seen that

$$\Lambda^2 = \frac{1}{2}[\Lambda_+, \Lambda_-] + 2\,\Lambda_0\Lambda_3 \tag{17}$$

commutes with all generators and hence plays the role of the Casimir operator. The product $\Lambda_0\Lambda_3$ can be decoupled into a difference of two squared generators by introducing the coordinates $\zeta \pm \omega$. The structure of the Casimir operators (2) and (17) are very similar. Expressed in the differential operators (15) the Casimir operator takes the form

$$\Lambda^2 = \frac{\partial^2}{\partial\alpha\partial\alpha^*} + i\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right)\frac{\partial}{\partial\zeta} + \alpha\alpha^*\frac{\partial^2}{\partial\zeta^2} + 2\frac{\partial^2}{\partial\zeta\partial\omega} \quad . \tag{18}$$

This is the analogy to the operator (6), except for the fact that the latter is restricted to $SU(2)$ whereas the above formula holds for the full group $U(1/1)$.

3.2.2 Restricted Space

In many applications it is sufficient to discuss the restriction of the parameters to some subspace, for example, the Hermitean matrix $U^\dagger x U$ where U is from equation (4) and $x = \mathrm{diag}(x_1, x_2)$ is diagonal does not depend on the Euler angle ψ. The generators $L_\pm$ and L_z on this restricted space (ϑ, φ) have the form well known from the text books on quantum mechanics [2]. In the graded case we face a similar situation [1], the relevant space is restricted to the coset u^A. Hence we need to construct the generators on this space. The calculations are performed in appendix B and yield the operators

$$\begin{aligned} \Lambda_0 &= 0 & \qquad \Lambda_3 &= \alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha} \\ \Lambda_+ &= \frac{\partial}{\partial\alpha} & \qquad \Lambda_- &= \frac{\partial}{\partial\alpha^*} \end{aligned} \tag{19}$$

which of course satisfy the algebra (16). The Casimir operator (17) now reads

$$\Lambda^2 = \frac{\partial^2}{\partial\alpha\partial\alpha^*} \quad . \tag{20}$$

It is precisely this expression that appears as angular part of the full Laplacian on the space of 2×2 graded Hermitean matrices discussed in reference [1].

3.2.3 Matrix Representation

There is also a matrix representation of the generators which follows directly from equation (12),

$$\begin{aligned} \Lambda_0 &= \frac{1}{2}1 & \Lambda_3 &= \frac{1}{2}\tau_3 \\ \Lambda_+ &= \alpha\,\tau_{12} & \Lambda_- &= \alpha^*\,\tau_{21} \quad . \end{aligned} \tag{21}$$

The anticommuting variables α and α^* have to be introduced here as some kind of dummy parameters in order to make the matrices graded. The algebra now reads

$$\begin{aligned} [\Lambda_0, \Lambda_\pm] &= 0 & [\Lambda_+, \Lambda_-] &= -2\alpha^*\alpha\,\Lambda_0 \\ [\Lambda_3, \Lambda_\pm] &= \pm\Lambda_\pm & [\Lambda_0, \Lambda_3] &= 0 \quad . \end{aligned} \tag{22}$$

The anticommutator is replaced by the commutator since the signs are taken care of by the properties of the graded matrices, of course, the dummy parameters appear in the corresponding structure constant.

3.3 Group Representation

After discussing the Hilbert space in some detail we construct the graded Wigner functions, i.e. the representation matrix elements.

3.3.1 Hilbert Space

Since it is the Casimir operator (20) on the restricted space that appears as angular part in the full Laplacian of reference [1] we will use the associated

Hilbert space in order to construct a group representation. As mentioned already [1] the eigenequation for the operator (20) reads

$$\Lambda^2 \, y^{(\pm)}_{\mu\mu^*}(\alpha,\alpha^*) \;=\; \pm\mu\mu^* \, y^{(\pm)}_{\mu\mu^*}(\alpha,\alpha^*) \qquad (23)$$

where μ is an anticommuting variable. The freedom of sign in front of the eigenvalue $\mu\mu^*$ leads to two sets of two eigenfunctions each, we find

$$\begin{aligned} y^{(+)}_{\mu\mu^*}(\alpha,\alpha^*) &= 2\pi \, \exp(+\mu\alpha + \mu^*\alpha^*) \\ \overline{y}^{(+)}_{\mu\mu^*}(\alpha,\alpha^*) &= 2\pi \, \exp(-\mu\alpha - \mu^*\alpha^*) \end{aligned} \qquad (24)$$

in case of the plus sign and

$$\begin{aligned} y^{(-)}_{\mu\mu^*}(\alpha,\alpha^*) &= 2\pi \, \exp(+\mu\alpha - \mu^*\alpha^*) \\ \overline{y}^{(-)}_{\mu\mu^*}(\alpha,\alpha^*) &= 2\pi \, \exp(-\mu\alpha + \mu^*\alpha^*) \end{aligned} \qquad (25)$$

in case of the minus sign. The functions $\overline{y}^{(\pm)}_{\mu\mu^*}$ are the conjugates, in particular we have $\overline{y}^{(-)}_{\mu\mu^*} = y^{(-)*}_{\mu\mu^*}$. For each sign separately, these functions are orthogonal and complete and hence form some kind of Hilbert space. In a self explaining Dirac type of notation [1] we can write

$$\begin{aligned} \langle \pm, \mu\mu^* | \pm, \mu'\mu'^* \rangle &= \delta([\mu] - [\mu']) \\ \int d[\mu] \, |\pm, \mu\mu^*\rangle\langle \pm, \mu\mu^*| &= 1 \end{aligned} \qquad (26)$$

where the $y^{(\pm)}_{\mu\mu^*}$ are in the ket and the conjugates $\overline{y}^{(\pm)}_{\mu\mu^*}$ are in the bra. We used the notation $\delta([\mu] - [\mu']) = \delta(\mu^* - \mu'^*)\delta(\mu - \mu')$. Since both sets (24) and (25) form something like a Hilbert space independent of one another we expire here a duality which has no analogy in the ordinary $SU(2)$. This is why we could restrict the discussion to the set (25) in reference [1]. However, the two sets are not orthogonal, for example we find results like

$$\langle +, \mu\mu^* | -, \mu'\mu'^* \rangle \;=\; \delta(\mu + \mu') \, \delta(\mu^* - \mu'^*) \qquad (27)$$

and similar for the completeness relation. We call these functions graded spherical harmonics.

We now have to study the action of the generators in the representation (19) on this Hilbert space. For the operators $\Lambda_\pm$ we find

$$\begin{aligned} \Lambda_+ \, y^{(\pm)}_{\mu\mu^*} &= -\mu \, y^{(\pm)}_{\mu\mu^*} & \Lambda_+ \, \overline{y}^{(\pm)}_{\mu\mu^*} &= +\mu \, \overline{y}^{(\pm)}_{\mu\mu^*} \\ \Lambda_- \, y^{(\pm)}_{\mu\mu^*} &= \mp\mu^* \, y^{(\pm)}_{\mu\mu^*} & \Lambda_- \, \overline{y}^{(\pm)}_{\mu\mu^*} &= \pm\mu^* \, \overline{y}^{(\pm)}_{\mu\mu^*} \end{aligned} \tag{28}$$

Although the $\Lambda_\pm$ are somehow formally analogous to the shift operators $L_\pm$ they dont shift the eigenfunctions because there is no running integer index to shift, the indices are anticommuting numbers. In other words, the functions (24) and (25) are eigenfunctions of the operators $\Lambda_\pm$ to anticommuting eigenvalues. Thus, Λ^2 and $\Lambda_\pm$ are digonalized simulaneously. Consequently, it is not surprising that these functions are not eigenfunctions of Λ_3, rather then we have

$$\begin{aligned} \Lambda_3 \, y^{(\pm)}_{\mu\mu^*} &= -\frac{1}{2}\left(y^{(\mp)}_{\mu\mu^*} - \overline{y}^{(\mp)}_{\mu\mu^*}\right) \\ \Lambda_3 \, \overline{y}^{(\pm)}_{\mu\mu^*} &= +\frac{1}{2}\left(y^{(\mp)}_{\mu\mu^*} - \overline{y}^{(\mp)}_{\mu\mu^*}\right) \quad . \end{aligned} \tag{29}$$

Remarkably, Λ_3 shifts the set (24) to a linear combination of the set (25) and vice versa. In the ordinary case, the situation is the other way around, the spherical harmonics are eigenfunctions of L_z, but not of $L_\pm$. This also shows that the plus and minus functions are not independent. However, we emphasize once again that each set satisfies the orthogonal and completeness relations (26). This enables us to restrict the further discussion to one of the two sets, as in reference [1] we choose the minus solutions and drop the minus index in our notation.

3.3.2 Graded Wigner Functions

In order to evaluate the matrix elements of the group representation we simply transfer the steps in the ordinary case to the graded. The formal operator expression analogous to equation (3) is given by

$$u(\zeta,\omega,\alpha,\alpha^*) = \exp(i\zeta\Lambda_0)\exp(i\omega\Lambda_3)\exp(\alpha\Lambda_+ + \alpha^*\Lambda_-) \quad . \tag{30}$$

As graded Wigner functions we define the matrix elements taken with the graded spherical harmonics,

$$D_{[\mu'][\mu]}(u) = \langle \mu'\mu'^* | u(\zeta,\omega,\alpha,\alpha^*) | \mu\mu^* \rangle \tag{31}$$

where the generators in the expression (30) are the operators (19) acting on the Hilbert space. The details of the calculation are worked out in appendix C, the result is

$$\begin{aligned} D_{[\mu'][\mu]}(u) &= \exp(+\mu\alpha - \mu^*\alpha^*)\,\delta\left(\mu'^* - \exp(+i\omega)\mu^*\right)\,\delta\left(\mu' - \exp(-i\omega)\mu\right) \\ &= \frac{1}{2\pi}\,y_{\mu\mu^*}(\alpha,\alpha^*)\,\delta\left([\mu'] - [\exp(-i\omega)\mu]\right) \end{aligned} \tag{32}$$

which is independent of the angle ζ as expected.

We now derive some formulae analogous to the ordinary case which show that the graded Wigner functions are really a representation. It is easy to see that the matrix elements (32) are eigenfunctions of the Casimir operator (18) on the full space,

$$\Lambda^2\,D_{[\mu'][\mu]}(u) = -\mu\mu^*\,D_{[\mu'][\mu]}(u) \tag{33}$$

to the eigenvalue $-\mu\mu^*$. According to equation (23) we conclude that the representations of the Casimir operator on the full and the restricted space have the same eigenvalue. This corresponds directly to the ordinary group $SU(2)$. With u_i being an element of $U(1/1)$ of the type (11) with parameters $(\zeta_i,\omega_i,\alpha_i,\alpha_i^*)$ and with the group operation $u_3 = u_2u_1$, the graded version of the fundamental equation (8) is given by

$$D_{[\mu'][\mu]}(u_3) = \int D_{[\mu'][\mu'']}(u_2)\,D_{[\mu''][\mu]}(u_1)\,d[\mu''] \quad . \tag{34}$$

The relation corresponding to equation (9) reads

$$\begin{aligned} \int D^*_{[\mu_1'][\mu_1]}(u)\,D_{[\mu_2'][\mu_2]}(u)\,d\mu(u) &= (2\pi)^2\,\delta([\mu_1]-[\mu_2]) \\ &\quad ((\mu_1'\mu_1'^* + \mu_1\mu_1^*)(\mu_2'\mu_2'^* + \mu_2\mu_2^*) + \mu_1'\mu_1^*\mu_2\mu_2'^* + \mu_1\mu_1'^*\mu_2'\mu_2^*) \end{aligned} \tag{35}$$

where the invariant Haar measure on $U(1/1)$ is simply

$$d\mu(u) = d\zeta\,d\omega\,d\alpha\,d\alpha^* \tag{36}$$

as calculated in appendix A. Details of the derivation of the formulae (34) and (35) are given in appendix D. Remarkably, equation (35) is only an orthogonality relation in the indices μ_1, μ_2 and their complex conjugates. We finally try to evaluate something like the character of our representation. Formally analogous to equation (10) we define the character through

$$\chi(u) = \int D_{[\mu][\mu]}(u)\,d[\mu] \quad . \tag{37}$$

A direct calculation yields

$$\chi(u) = 4 \sin^2 \frac{\omega}{2} \quad , \tag{38}$$

thus, the character depends only on the angle ω. Moreover, due to the nature of our representation and unlike the ordinary character (10), it is independent of any further index.

4 Summary and Discussion

We discussed a non-canonical parametrization of the graded group $U(1/1)$ which is somehow in the spirit of the Euler angles commonly for the ordinary group $SU(2)$. We constructed differential representations of the underlying algebra of $U(1/1)$ on the full parameter space and on the restricted parameter space, i.e. on the coset. Both show close formal similarities to the corresponding representations of the underlying algebra of $SU(2)$, this is also true for the Casimir operators. In order to evaluate the differential representation on the full parameter space we employed a variation technique thus avoiding explicit use of the group multiplication law. In addition, the algebra was represented in graded matrices, this, however, requires the introduction of dummy parameters which then show up in the structure constants. A kind of Hilbert space exists on the coset, i.e. the corresponding functions live on the coset parameter space. This Hilbert space exhibits a duality: we found two sets of functions according to the sign of the eigenvalue of the Casimir operator, each set satisfies an orthogonality and completeness relation and hence seems to form a Hilbert space by its own. We therefore call these functions graded spherical harmonics. Nevertheless, these two sets are not independent, rather then the action of one generator connects the two spaces. This highly unusual situation has no counterpart in an ordinary Hilbert space. Moreover, unlike the ordinary case, the graded spherical harmonics are eigenfunctions of those generators which formally correspond to the raising and lowering operators in the ordinary $SU(2)$. All this must be related to the fact that we can speak of the graded Hilbert space only in the sense of a very formal analogy. The indices in this space are not discrete numbers but anticommuting variables. However, the orthogonality and completeness relations allow the constuction of graded Wigner functions as matrix elements of the operator expression of

$U(1/1)$. It was checked that these graded Wigner functions really provide a representation. Remarkably, they do not obey a full orthogonality relation, this is the only serious difference from the ordinary Wigner functions. This might indicate that the representation we constructed is still in some sense reducible. Since little is known about representations of graded groups and in how far the concept of reducibility of ordinary groups is transferable we feel that a further discussion of these questions is beyond the scope of this paper.

Appendix

The generators on the full and the restricted parameter space of the group are calculated in appendices A and B, respectively. In appendix C we evaluate the representation matrix elements and in appendix D we work out some properties of the representations.

A Generators on the Full Space

In the differential representation the generators create the infinitesimal displacements near the origin. The latter can be considered as the variation parameters in the expansion of a group element around the origin in the matrix representation. Thus we write according to the algebra (21) for such a variation

$$\delta v \;=\; \begin{bmatrix} i(\delta\xi+\delta\eta)/2 & \delta\beta \\ \delta\beta^* & i(\delta\xi-\delta\eta)/2 \end{bmatrix} \quad . \tag{39}$$

On the other hand, moving the variation δu of a group element u somewhere in the parameter space back to the origin by multiplication with $u^{-1} = u^\dagger$ gives also a variation near the origin. Consequently, we can identify both,

$$\delta v \;=\; \delta u\, u^\dagger \quad . \tag{40}$$

The right hand side of this equation is easily calculated in the decomposition (12),

$$\delta u u^\dagger \;=\; \delta u^C u^{C\dagger} + u^C \delta u^A u^{A\dagger} u^{C\dagger} \quad . \tag{41}$$

Explicit evaluation yields the linear mapping M of the differentials in the u parameter space defined in equation (11) onto the v variations, for convenience we write it in the form

$$[\delta\xi\ \delta\eta\ \delta\beta\ \delta\beta^*] \;=\; [d\zeta\ d\omega\ d\alpha\ d\alpha^*]\, M \tag{42}$$

where the matrix M is given by

$$M \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -i\alpha^* & 0 & \exp(+i\omega) & 0 \\ -i\alpha & 0 & 0 & \exp(-i\omega) \end{bmatrix} \quad . \tag{43}$$

The inverse of equation (42) gives the linear mapping in the tangent space,

$$\begin{bmatrix} \delta/\delta\xi \\ \delta/\delta\eta \\ \delta/\delta\beta \\ \delta/\delta\beta^* \end{bmatrix} \;=\; M^{-1} \begin{bmatrix} \partial/\partial\zeta \\ \partial/\partial\omega \\ \partial/\partial\alpha \\ \partial/\partial\alpha^* \end{bmatrix} \quad . \tag{44}$$

By construction, the variation differential operators on the left hand side generate the infinitesimal displacements and can hence be identified with the generators $i\Lambda_0$, $i\Lambda_3$ and $\Lambda_\pm$ where we introduced imaginary units according to equation (39). Thus we only need to invert the matrix M, we find

$$M^{-1} \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ i\alpha^* \exp(-i\omega) & 0 & \exp(-i\omega) & 0 \\ i\alpha \exp(+i\omega) & 0 & 0 & \exp(+i\omega) \end{bmatrix} \quad . \tag{45}$$

Collecting everything yields the desired equations (15). The graded determinant of the linear mapping M gives the invariant Haar measure. This was already used in references [5, 7, 8] in various cases. Here we have $\operatorname{detg} M \;=\; 1$ and thus the graded Haar measure on $U(1/1)$ in the parametrization (12) is given by the expression (36).

B Generators on the Restricted Space

The easiest way to calculate the generators $\Lambda_\pm$ is by using the variation method of appendix A. In the coset space we simply have

$$\begin{aligned} \delta v^A \;&=\; \begin{bmatrix} 0 & \delta\beta \\ \delta\beta^* & 0 \end{bmatrix} \\ \delta u^A\, u^{A\dagger} \;&=\; \begin{bmatrix} (\alpha d\alpha^* + \alpha^* d\alpha)/2 & d\alpha \\ d\alpha^* & (\alpha d\alpha^* + \alpha^* d\alpha)/2 \end{bmatrix} \end{aligned} \tag{46}$$

which means $\delta\beta = d\alpha$ and $\delta\beta^* = d\alpha^*$ in the coset sector and hence

$$\Lambda_+ = \frac{\partial}{\partial\alpha} \quad \text{and} \quad \Lambda_- = \frac{\partial}{\partial\alpha^*} \quad . \tag{47}$$

Consequently, the invariant Haar measure on the coset is simply given by $d\mu(u^A) = d\alpha d\alpha^*$. The diagonal elements of the matrix $\delta u^A u^{A\dagger}$ give no information for the other two generators. Another way to evaluate equations (47) is the coset multiplication law. Writing u_i^A for a matrix of the coset type with parameters (α_i, α_i^*) we find the relation

$$u_2^A u_1^A = u_3^A \left(1 + \frac{1}{2}\alpha_2\alpha_1^*\right)\left(1 + \frac{1}{2}\alpha_2^*\alpha_1\right) \tag{48}$$

for the matrices and

$$\alpha_3 = \alpha_2 + \alpha_1 \quad \text{and} \quad \alpha_3^* = \alpha_2^* + \alpha_1^* \quad . \tag{49}$$

for the parameters. According to equation (48) the coset is a group modulo a factor which can be absorbed into u^C. Hence equation (49) is the relevant coset multiplication law, in this case just a simple translation. This in turn leads immediately to the result (47).

In order to evaluate the expressions for Λ_0 and Λ_3 we have to consider the action of a group element u on a coset matrix [9], say u_1^A, we find

$$u\, u_1^A = u^C u^A u_1^A = \left(u^C u^A u^{C\dagger}\right)\left(u^C u_1^A\right) \quad . \tag{50}$$

Since $u^C u_1^A$ is in the group the relevant part is

$$u^{A\prime} = u^C u^A u^{C\dagger} = \begin{bmatrix} 1 + \alpha\alpha^*/2 & \exp(+i\omega)\alpha \\ \exp(-i\omega)\alpha^* & 1 + \alpha^*\alpha/2 \end{bmatrix} \tag{51}$$

which is again of coset form yielding for the parameters of $u^{A\prime}$ the relations

$$\alpha' = \exp(+i\omega)\,\alpha \quad \text{and} \quad \alpha'^* = \exp(-i\omega)\,\alpha^* \quad . \tag{52}$$

We can now calculate the remaining generators from

$$\begin{aligned} i\Lambda_0 &= -\left(\left.\frac{\partial\alpha'}{\partial\zeta}\right|_{\zeta=0}\frac{\partial}{\partial\alpha} + \left.\frac{\partial\alpha'^*}{\partial\zeta}\right|_{\zeta=0}\frac{\partial}{\partial\alpha^*}\right) \\ i\Lambda_3 &= -\left(\left.\frac{\partial\alpha'}{\partial\omega}\right|_{\omega=0}\frac{\partial}{\partial\alpha} + \left.\frac{\partial\alpha'^*}{\partial\omega}\right|_{\omega=0}\frac{\partial}{\partial\alpha^*}\right) \end{aligned} \tag{53}$$

which is a natural generalization of the formula in the ordinary case [6]. Collecting everything we find the equations (19).

C Calculation of the Matrix Elements

Since $\Lambda_0 = 0$ in the representation (19) of the generators, $D_{[\mu'][\mu]}(u)$ does not depend on the angle ζ. Due to the completeness we can write

$$D_{[\mu'][\mu]}(u) = \int d[\mu'']\, \langle\mu'\mu'^*|\exp(i\omega\Lambda_3)|\mu''\mu''^*\rangle\, \langle\mu''\mu''^*|\exp(\alpha\Lambda_+ + \alpha^*\Lambda_-)|\mu\mu^*\rangle \tag{54}$$

thus splitting the problem into the calculation of two simpler matrix elements. The evaluation of the second matrix element is very easy because the graded spherical harmonics are eigenfunctions of $\Lambda_\pm$. Using the equations (28) we find for all well behaved functions f the result

$$\begin{aligned}\langle\mu''\mu''^*|f(\alpha\Lambda_+ + \alpha^*\Lambda_-)|\mu\mu^*\rangle &= f(\mu\alpha - \mu^*\alpha^*)\,\langle\mu''\mu''^*|\mu\mu^*\rangle \\ &= f(\mu\alpha - \mu^*\alpha^*)\,\delta([\mu''] - [\mu]) \quad . \end{aligned} \tag{55}$$

Here, f is the exponential function yielding the graded spherical harmonic. We can now perform the integral over the double primed indices

$$D_{[\mu'][\mu]}(u) = \frac{1}{2\pi}\, y_{\mu\mu^*}(\alpha, \alpha^*)\, \langle\mu'\mu'^*|\exp(i\omega\Lambda_3)|\mu\mu^*\rangle \tag{56}$$

and are left with the calculation of the remaining matrix element. The action of powers of Λ_3 on the graded spherical harmonics is given by

$$\Lambda_3^n\, y_{\mu\mu^*}(\alpha, \alpha^*) = 2\pi\, (\alpha^*\mu^* - (-1)^n\alpha\mu) \qquad \text{for} \quad n > 0 \tag{57}$$

which gives for the matrix elements

$$\langle\mu'\mu'^*|\Lambda_3^n|\mu\mu^*\rangle = 2\pi\, (\mu'\mu^* - (-1)^n\mu'^*\mu) \qquad \text{for} \quad n > 0 \quad . \tag{58}$$

For $n = 0$ we find of course the orthogonality relation. Expanding now any well behaved function f in a Taylor series and using the above equation yields

$$\begin{aligned}\langle\mu'\mu'^*|f(i\omega\Lambda_3)|\mu\mu^*\rangle &= f(0)\,\delta(\mu'^* - \mu^*)\delta(\mu' - \mu) \\ &\quad + 2\pi\, ((f(+i\omega) - f(0))\mu'\mu^* - (f(-i\omega) - f(0))\mu'^*\mu) \quad . \end{aligned} \tag{59}$$

In our case the function f is the exponential and this equation can be further simplified, we finally arrive at

$$\langle \mu'\mu'^*|\exp(i\omega\Lambda_3)|\mu\mu^*\rangle \;=\; \delta(\mu'^* - \exp(+i\omega)\mu^*)\,\delta(\mu' - \exp(-i\omega)\mu) \qquad (60)$$

which gives in equation (56) the desired result (32).

D Properties of the Representations

Although formula (34) is an immediate consequence of the completeness relation, it is worthwhile to prove it in a direct calculation. The right hand side gives

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int y_{\mu''\mu''^*}(\alpha_2,\alpha_2^*)\delta([\mu'] - [\exp(-i\omega_2)\mu''])\\
&\qquad\qquad y_{\mu\mu^*}(\alpha_1,\alpha_1^*)\delta([\mu''] - [\exp(-i\omega_1)\mu])\,d[\mu'']\\
&= \frac{1}{(2\pi)^2}\, y_{\exp(-i\omega_1)\mu\,\exp(+i\omega_1)\mu^*}(\alpha_2,\alpha_2^*)y_{\mu\mu^*}(\alpha_1,\alpha_1^*)\\
&\qquad\qquad \delta([\mu'] - [\exp(-i(\omega_2+\omega_1))\mu])\\
&= \frac{1}{2\pi}\, y_{\mu\mu^*}(\exp(-i\omega_1)\alpha_2 + \alpha_1, \exp(+i\omega_1)\alpha_2^* + \alpha_1^*)\\
&\qquad\qquad \delta([\mu'] - [\exp(-i(\omega_2+\omega_1))\mu]) \quad . \qquad (61)
\end{aligned}$$

which is, using the group multiplication law (13), indeed the left hand side of equation (34).

Using equations (32) and (36) we find for the left hand side of equation (35)

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\,\langle \mu_1\mu_1^*|\mu_2\mu_2^*\rangle \int_0^{2\pi} d\zeta \int_0^{2\pi} d\omega\, \delta([\mu_1'] - [\exp(-i\omega)\mu_1])\\
&\qquad\qquad \delta([\mu_2'] - [\exp(-i\omega)\mu_2])\\
&= 2\pi\,\delta([\mu_1] - [\mu_2]) \int_0^{2\pi} d\omega\, (\mu_1' - \exp(-i\omega)\mu_1)\,(\mu_1'^* - \exp(+i\omega)\mu_1^*)\\
&\qquad\qquad (\mu_2' - \exp(-i\omega)\mu_2)\,(\mu_2'^* - \exp(+i\omega)\mu_2^*) \quad . \qquad (62)
\end{aligned}$$

Only those terms in the integrand contribute to the integral which do not depend on ω, collecting all these combinations yields the right hand side of equation (35).

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