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Factors Considered by Elementary Teachers When Developing and Modifying
Mathematical Tasks to Support Children's Mathematical Thinking

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Mathematics and Science Education

by

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2015

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University of California, San Diego

San Diego State University

2015

DEDICATION

This dissertation is dedicated to all of those dreamers. Never believe that you are too old to chase them.

To Dr. Randy Philipp whose mentoring has been invaluable: Your dedication to your students and field of mathematics education is unparalleled.

To Dr. Susan Nickerson whose advice and confidence bolstered me during this arduous process: You were wonderful to work with these past years.

To my MSed cohort whose support and encouragement helped make this long journey possible.

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ABSTRACT OF THE DISSERTATION

Factors Considered by Elementary Teachers When Developing and Modifying
Mathematical Tasks to Support Children's Mathematical Thinking

by

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Doctor of Philosophy in Mathematics and Science Education

University of California, San Diego, 2015
San Diego State University, 2015

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The idea that problems and tasks play a pivotal role in a mathematics lesson has a long standing in mathematics education research. Recent calls for teaching reform appeal for training teachers to better understand how students learn mathematics and to employ students' mathematical thinking as the basis for pedagogy (CCSSM, 2010; NCTM, 2000; NRC 1999). The teaching practices of (a) developing a task for a mathematics lesson and, (b) modifying the task for students while enacting the lesson fit within the scope of supporting students' mathematical thinking. Surprisingly, an extensive search of the literature did not yield any research aimed to identify and refine the constituent parts of the aforementioned teaching practices in the manner called for by Grossman and

colleagues (2009). Consequently, my research addresses the two questions: (a) what factors do exemplary elementary teachers consider when developing a task for a mathematics lesson? (b) what factors do they consider when they modify a task for a student when enacting a lesson?

I conducted a multiple case study involving three elementary teachers, each with extensive training in the area of Cognitively Guided Instruction (CGI), as well as several years experience teaching mathematics following the principles of CGI (Carpenter et al., 1999). I recorded video of three mathematics lessons with each participant and after each lesson I conducted a semi-structured stimulated recall interview. A subsequent follow-up clinical interview was conducted soon thereafter to further explore the teacher's thoughts (Ginsberg, 1997). In addition, my methodology included interjecting myself at select times during a lesson to ask the teacher to explain her reasoning.

Qualitative analysis led to a framework that identified four categories of influencing factors and seven categories of supporting objectives for the development of a task. Subsets of these factors and objectives emerged as particularly relevant when the teachers decided to modify a task. Moreover, relationships between and among the various factors were identified. The emergent framework from this study offers insight into decompositions of the two teaching practices of interest, and, in particular, the utility of the number choices made by the teachers.

CHAPTER 1: INTRODUCTION

Rationale: Why Study Mathematical Tasks and Teaching?

Researchers and educators are driven to develop models for effective mathematics teaching, and recent calls for teaching reform appeal for training teachers to better understand how students learn mathematics and to employ students' mathematical thinking as the basis for pedagogy, planning, and instructional decisions (CCSSM, 2010; NCTM, 2000; NRC 1999). However, setting forth revised standards and guidelines for teachers to follow is simply not enough, for example, the Common Core State Standards for Mathematics (2010) direct that the "Standards do not dictate curriculum or teaching methods" (p. 5). Therefore, the issue is not one of striving for a universal guiding agenda for teachers; rather it is one of preparing and enabling teachers to discern and understand their students' mathematical thinking so they may provide responsive instructional actions that support their students' mathematical development.

Motivation for the study

The motivation for this study arose from my experiences working alongside two elementary teachers (first and third grade) and their respective students in classrooms guided by the principles of Cognitively Guided Instruction (CGI). The tenets of CGI assert that children bring to a classroom their unique understandings of arithmetic and number sense as well as their own strategies for solving contextual problems (Carpenter, Fennema, Franke, Levi, & Empson, 1999). The CGI perspective towards children's mathematical development is grounded in empirical data demonstrating that children are not empty vessels when it comes to solving contextual problems involving the operations

of arithmetic. Researchers have identified a taxonomy of problem types involving whole numbers and have related these to specific problem solving strategies (Carpenter et al., 1999). In addition, research in CGI has included algebraic thinking (Carpenter, Franke, & Levi, 2003) and rational numbers (Empson & Levi, 2011). Both researchers and educators have promoted the principles of CGI as the basis for curriculum reform and teacher professional development in the domain of children's mathematical reasoning (Carpenter, Fennema, & Franke, 1996; Fennema et al., 1996; Franke, Carpenter, Levi, & Fennema, 2001; Jacobs, Franke, Carpenter, Levi, & Battey, 2007).

And yet, in spite of my experiences engaging with children's mathematical thinking, I remained surprised by the levels of sophistication often displayed by young children when solving mathematical problems. For example, consider the following problem presented in late May 2013 to the students in the first grade classroom in which I volunteered. The teacher, Ms. F, had taught using the principles of CGI for over a dozen years, and on this particular day she conducted the math lesson centered on the partitive division problem:

The teacher brought _____ snack bars for the field trip, and she did not want to take any of them back to school. If there are _____ groups of children, how many snack bars will each group get?

(10, 5) (30, 3) (25, 5) (15, 2) (20, 4)

This was typical of how Ms. F presented a task to her students. The number choices allowed the students to use various strategies, and the numbers seemed to reflect differing degrees of difficulty. For example, the number pair (15, 2) seemed designed to move students towards thinking about partitioning a whole into equal-sized pieces that could be represented as fractions, in this case, halves.

During this lesson, I was working with David, and he had used a skip counting strategy to solve the problem with the numbers (20, 4). David had written: 4, 8, 12, 16, 20 and presented, 5, as his answer. His strategy was unexpected to me in the sense that he did not ‘fair share’ the snack bars equally among the five groups of students and derived an answer by counting the number of snack bars distributed to each group as most of his classmates had done. Instead, his strategy was to skip count by the number of groups until he reached the total of snack bars, and then he counted how many groups it took, in this case five, and he seemed to understand that this represented the number of snack bars distributed to each group. What followed next was even more remarkable to me.

In an effort to determine if this was David’s normal strategy for this type of problem, I presented him with a new number pair, (12, 4), and he quickly answered, “Three.” I asked him how he got his answer and he explained: “Well, I took two, 4s away from 20 to get 12, so you take two, 1s away from 5 to get 3.” To me, it seemed as if he had employed proportional reasoning by utilizing the previous problem to recognize that taking a set of 4 snack bars from 20 snack bars was the same as taking 1 snack bar away from each group of students. This first grader appeared to have created the unit rate that a set of 4 snack bars was equivalent to 1 snack bar for each group.

It was via encounters such as working with David, and observing Ms. F, and other teachers, over time that piqued my curiosity as to the perceived effect that the problem posed and the number choices seemed to have on how the students thought about the task, and the range of strategies the students employed. Moreover, when I first began working in teachers’ classrooms that emphasized CGI, I observed that they seemed to

make a substantial number of instructional decisions during each math lesson, and that each teaching move seemed tailored to the individual student. The teachers seemed to react to things that I could not appreciate, but I was certain they were reacting to students in a manner that they considered to be in the best interests of each student. To me, it was apparent that these particular teachers were effectively teaching mathematics to young children. These teachers seemed to thoroughly understand the principles of the underlying mathematics, as well as learning trajectories that students might follow when advancing their mathematical knowledge, and they appeared well versed in the pedagogical skills that afforded effective learning in their classrooms. In the next four sections I present an overview of what research suggests concerning the breadth and depth of knowledge necessary to teach mathematics effectively, which I follow by situating the practical work of developing a mathematical task within this knowledge.

Perspectives on Mathematics Teachers Knowledge of Content and Pedagogy

The history of conceptualizing and assessing teachers' mathematical knowledge spans the last two hundred years, and working definitions of metrics for such knowledge have ranged from the number of college courses completed to psychometric measurements (Hill, Sleep, Lewis, & Ball, 2007). Common to all approaches was to agree that it was important to identify teachers with a high degree of mathematical content knowledge who can impact student learning. Additionally, in his influential paper, Shulman (1986) presented the idea of Pedagogical Content Knowledge (PCK) as the basis for an argument that successful teaching requires much more than a thorough knowledge of content. To Shulman, PCK was an amalgam of a teacher's content knowledge with the knowledge of how to teach. In the decades following Shulman's

challenge to the field, a growing number of educators and researchers in the field of mathematics education have adapted and refined his original conception into what is now characterized as *mathematical knowledge for teaching* (MKT), the knowledge necessary for successfully teaching mathematics (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008; Hill et al., 2008; Silverman & Thompson, 2008).

One model of MKT extended PCK by identifying specific domains correlating to what mathematics teachers know and do that advances students' conceptual growth and understanding (Ball, Thames & Phelps, 2008; Hill, Ball & Schilling, 2008). Ball and colleagues (Ball et al., 2008) viewed MKT from two perspectives: subject matter knowledge and pedagogical content knowledge. Ball and colleagues considered subject matter knowledge as the content knowledge relating to the integration of mathematics with the real world and other disciplines, as well as the knowledge associated with teaching from essential conceptual perspectives that are not of immediate interest to students. In this model, the pedagogical knowledge of MKT was described as knowledge of those teaching actions that might assist students in overcoming encountered difficulties (e.g., presenting supporting examples and representations, and orchestrating classroom discussions) along with the understandings as to the manners in which students learn mathematics, and the ways that mathematics is situated within the curriculum (Hill et al., 2008).

Teaching Practices

Over a century ago, Dewey (1904) made the case that professional instruction of teachers should involve the practical work of teaching, and not be based strictly in the theoretical underpinnings of the profession. More recently, Ball and Forzani (2009)

promoted Dewey's position and argued that practice should be at the heart of teacher education and that detailed attention must be paid to the "work of teaching" (p. 497), those core tasks that teachers employ to afford student learning, and the manners in which prospective teachers might be trained to perform that work adroitly. Their argument aligned with those of others in calling for teacher education, and educational research, to concentrate on characterizing and organizing the knowledge and activities that enable teachers to bridge the gulf between theory and practice (Ball & Cohen, 1999; Ball & Forzani, 2009; Ball, Sleep, Boerst, & Bass, 2009; Cohen, Raudenbush, & Ball, 2003; Grossman, Hammerness, & McDonald, 2009; Grossman & McDonald, 2008). This 'work of teaching' was presented by Lampert (2010) as the foundation for different conceptions of the word *practice* used by researchers and educators, and I will explore two of these in greater detail in Chapter 2. For now I concentrate on the idea of teaching as a collection of core practices, those phenomena that teachers habitually carry out (Lampert, 2012), and recent efforts to decompose teaching into its distinct, reproducible and learnable essential skills.

In a comparative case study across the professions of the clergy, clinical psychology, and teaching, Grossman and colleagues (2009) set out to identify the types of pedagogical interventions that professional educators employed to inform novices of the knowledge inherent within their respective profession, and how these interventions might foreground the important features of professional practice. In their findings they presented three main concepts that together form the basis for understanding a professional practice from a pedagogical vantage point: (a) representations, the extent of approaches through which a professional practice was made visible to novices; (b)

decomposition, which entailed fracturing a practice into its constituent parts for teaching and learning; and (c) approximations of practice, those opportunities for novices to emulate the authentic activities of a practice. Of particular interest to me, and this study, was the idea of decomposing mathematics teaching into a set of well-defined clinical practices that form the foundation for pedagogies of enactment through which novice mathematics teachers might be trained (Grossman et al., 2009).

Recent efforts in educational research underscored the importance of decomposing teaching into high-leverage practices (Ball et al., 2009; Kazemi, Franke, & Lampert, 2009; Lampert & Graziani, 2009), that is, those practices “in which the proficient enactment by a teacher is likely to lead to comparatively large advances in student learning” (Ball et al., 2009, p. 460). Examples of high-leverage practices identified in teaching mathematics included eliciting students’ mathematical thinking (Franke et al., 2009; Lobato, Clarke, & Ellis, 2005), orchestrating classroom discussions that highlight student conjectures, justifications and mathematical argumentation (e.g., Forman & Larreamendy-Joerns, 1998; Kazemi & Stipek, 2001; Lau, Singh, & Hwa, 2009), and maintaining the cognitive demand of mathematical tasks (Boston & Smith, 2009; Henningsen & Stein, 1997). I build on this work and argue that developing a mathematical task and selecting appropriate numbers are essential practices for teaching mathematics. The next section supports this argument.

Mathematical tasks and teaching

The Professional Standards for Teaching Mathematics (NCTM, 2000) identified six standards to support the teaching of mathematics, and addressed the importance of engaging students with worthwhile mathematical tasks. Accordingly, an engaging

mathematical task offers students the opportunity to emulate a practitioner of mathematics. Take, for example, the first of eight Mathematical Practices in the more recent and increasingly prevalent Common Core State Standards: Make sense of problems and persevere in solving them (CCSSI, 2010). A challenging task supports this mathematical practice in the sense that one requires discovering an entry point to a solution path among multiple possible solution strategies, and one can promote developing an attitude of perseverance as students struggle to find a successful strategy. A mathematical task can be viewed as one that concentrates students' thinking on a specific mathematical idea (Stein, Grover, & Henningsen, 1996), and mathematical tasks can be neatly categorized as those that elicit a low-level cognitive demand, and those that maintain a high-level of cognitive demand (Boston & Smith, 2009).

However, as the Professional Standards for Teaching Mathematics (NCTM, 2000) point out, a worthwhile task does not guarantee effective teaching. For instance, the task must be launched appropriately (Jackson, Shahan, Gibbons, & Cobb, 2012), followed up with scaffolding questions (Kazemi & Stipek, 2001; Olson & Knott, 2013), and utilized as the theme for student-teacher and student to student discourse (Franke, Kazemi, & Battey, 2007). Therefore, I offer that selecting or developing a mathematical task is an important aspect of the work of teaching, and I feel comfortable classifying developing a mathematical task as a practice of teaching mathematics.

Within this study, I explored types of mathematical tasks of particular interest: the taxonomy of CGI problems identified by Carpenter and colleagues (1999), and more recent work by Land and colleagues (2014). From their research on children's mathematical thinking, Carpenter and colleagues characterized CGI problem types by the

manners through which children tended to view and interpret the problems. For example, a problem such as $42 + \underline{\quad} = 57$, when presented in a story problem with a *joining action* such as--You had 42 rocks and your sister *gave you some more*, and now you have 57 rocks--tended to be interpreted by children as an addition problem, and the strategies they employed tended to reflect this interpretation. In contrast, adults generally view a problem of this nature as a subtraction problem, but a strategy based on this interpretation involves reversing the action of the problem, which Carpenter and colleagues (1999) found to be sophisticated thinking for children. Land and colleagues (2014) emphasized the construct of a *multiple number choice structure*, wherein a CGI problem type is presented with a sequence of number sets, for example:

Sarah has 2 seashells in her collection. While on vacation, she found 3 more seashells. How many seashells does Sarah have in her collection?
(2, 3) (7, 5) (10, 20) (12, 16) (23, 19) (9, 82) (117, 203)

A problem like this can be described as a Join-Result-Unknown problem (Carpenter et al., 1999) and its respective multiple number choice structure.

Developing a mathematical task

As discussed above, a mathematical task can set the stage for students to offer and defend conjectures, and an effective task can assist the teacher in maintaining a high level of cognitive demand in her classroom. Hence, one might posit that a decomposition of the practice would yield significant knowledge for both educators and researchers alike. However, a search of the literature in this area produced limited results. I found no studies purported to decompose the development of effective mathematical tasks. Therefore, as an initial foray into this area, I conducted a small pilot study with a third

grade teacher, Ms. T, who conducted her mathematics lessons aligned with the tenets of CGI using CGI problems with a multiple number choice structure.

The pilot study data to follow below stemmed from (a) video of two classroom mathematics lessons that occurred 8 days apart; (b) video of two short stimulated recall interviews with Ms. T that occurred immediately following the lesson; and, (c) video of two clinical interviews--the first occurred the day following the first lesson, and the second on the same day as the lesson. I chose to conduct the pilot study with Ms. T for several reasons, the most important of which was because she and I had an established professional relationship. In addition, Ms. T had been in the classroom for 16 years, and she started participating in CGI professional development during her third year of teaching, and continued with the PD program for approximately 7 years. Immediately upon beginning the PD, she began running regular math lessons with small groups of students (4 to 5 students per group) using CGI principles. Although she did not use CGI exclusively-she still followed the pacing of a curriculum and textbook, and she presented some material in a traditional more manner during a lesson-she did consider CGI to be the "best part of [her] math instruction".

A typical mathematics lesson in Ms. T's class ran for about 90 minutes. The first 20 to 30 minutes were devoted to a topic or concept with Ms. T at the whiteboard presenting examples, and discussing ideas with entire class. After the whole class session, the students would break into five groups of 4 to 5 students per group and they dispersed to predetermined stations distributed across the classroom. The station of interest to me was the table where Ms. T presented the CGI problem of the day, and interacted with students in a small group as they worked through the problem and its

multiple number choices. After 15-20 minutes, Ms. T rotated the groups to new stations. In this manner, she worked with at least three small groups each lesson at the teaching station, and she utilized the same problem for the following session until all of the students had worked the problem under her facilitation.

With the understanding that the pilot study was limited by issues such as a lack of participants, and an evolving interview protocol that was coupled with an insufficient amount of interview time, analysis of the data did offer several interesting themes. First, evidence suggested that Ms. T seemed to typically develop a CGI problem independent of the math concepts within the curriculum that were currently being investigated by the students. For example, during the first lesson I observed, the whole class discussion centered on the standard algorithm for multiplying multi-digit numbers by single digit numbers, but students were not required to use the algorithm to solve the CGI multiplication problem presented at the main math station, which was:

For the Third grade play, "A Tribute to America", we are setting up ____ rows of chairs in the auditorium. If there are ____ chairs in each row, how many chairs will be set up for the play?
 (10, 18) (20, 30) (43, 6) (126, 9 on the left and 9 on the right) (201, 14) (97, 8 on the left and 8 on the right)

In addition, the following week, the class was presented with a division problem because, as Ms. T reported, "I just like to bounce around with different kinds of things, and I felt we'd do division today." Ms. T also communicated that at the beginning of a school year she does not add addition or subtraction problems because students tend to enter third grade knowing the standard algorithms, and she does not want relying on procedural skills until they develop sound problem solving strategies. In her words, "...in fact I like try and make sure that I do lots of multiplication before I teach the multiplication

algorithm so that they develop their own strategies and understanding before we ever get to the point where they learn the algorithm.”

Another emergent theme was that selecting the numbers for a CGI problem was not an elementary process. For example, Ms. T explained that she typically selected the initial numbers for a problem to offer students a successful entry to point to understanding the problem. For the multiplication problem presented above, she included the first number pair of (10, 18) because “...I was thinking sort of from the onset that I hope they can multiply by 10. I hope they can multiply by multiples of 10.” However, Ms. T did go on to report that she felt her choices for the problem were “bad numbers.” That is, to Ms. T, the numbers she selected did not support an objective of the lesson, which was for the students to use the standard multiplication algorithm as a method to check their work, because many of the number pairs included two, two-digit numbers, (e.g., 20, 30), which seemed to be an obstacle for the students.

The following week, when asked why she selected the particular numbers for the division problem presented that day, Ms. T offered,

“Okay, so after last time’s experience I wanted to put some that I felt would be easy to access and get into the problem. And I was glad to see that they were all able to do the 56 divided by 8 and 36 divided by 6. Most of them didn’t have any struggle with that.”

In addition, she reported that she included number choices to determine if the students could easily see that 12 divided by 24 is a half; whether they would see the relationship between 130 divided by 13 and multiplying 13 by 10; and to see who might use expanded form to break apart 8412 before dividing by 4. Essentially, it seemed that Ms. T selected particular numbers to assess her students’ current understandings, and to promote specific

strategies, such as using expanded form, or the use of number facts within a strategy.

The analysis of the pilot study data suggested that the practice of crafting a CGI problem was a multifaceted process--a process that seemed to work towards objectives such as to foreground the inherent mathematics involved in the problem, and to promote and support the strategies available to students as informed by Ms. T's evolving perceptions of her students' mathematical understandings.

Research Question 1

The finding that developing a mathematical task, specifically a CGI problem, is a complex practice was worthy of further investigation. While intriguing, the aforementioned pilot study data was saddled with limitations, and the study needed refinement. Taken together, these issues led to the development of the following research question:

RQ1) What factors do elementary teachers who have an extensive understanding of children's mathematical thinking consider when developing mathematical tasks in the design phase of mathematics lessons?

To answer this question, the study examined the practice of developing mathematical tasks for classroom use from the point of view of teachers who had considerable teaching experience as well as several years of professional development in the area of CGI.

Following the vernacular of Remillard (1999), and refined by Choppin (2011a), I used the term *design phase* to refer to the work outside of the classroom that teachers allocate to developing mathematical tasks, and the term *enactment phase* to refer to posing and implementing the problem with students in the classroom.

Essentially, the research question aimed to determine what teachers considered when writing a problem to pose during a mathematics lesson. In the case of a CGI problem, the question aimed to identify those factors considered by teachers when deciding what problem type to feature, and then the composition of the problem context and number selection for the problem. Findings of the study offered insight into the practice of developing such problems from the perspective of a set of teachers with years of CGI experience, and extensive knowledge of children's mathematical thinking. To clarify, I defined the term *factors* to be those agents that influenced, or contributed directly to developing a mathematical task.

My conception of a factor mirrored some of the recent work of Land and Drake (2014) with investigating curricular progressions, including number choices made by elementary teachers, and Land and colleagues (2014) emphasis on the importance of number choices when transforming tasks. Although the authors of the respective works did not specifically use the term factor, they seemed to identify agents that would fall under my interpretation of a factor. These included ways in which a multiple number choice structure allowed for a range of mathematical rigor, or a springboard from which to make conjectures about or to assess student thinking. Furthermore, Land and colleagues (2014) pointed out that a multiple number choice structure allowed for alignment to the new Common Core standards. Table 1.1 presents a list of possible factors that teachers might consider when developing a mathematical task along with brief descriptive examples. Evidence for each item in the list emerged from both the pilot study and examination of the literature.

Table 1.1: Possible factors identified from literature and the pilot study

Factor	Example
Mathematical concept	Place value, regrouping, division with remainder
Mathematical rigor and complexity	Provide challenges to students
Accessibility	Entry points to problems
Learning goals short term and long term	Scaffold from direct modeling to counting strategies
Assessment of students' thinking	Check for students' current understandings
Anticipate students' thinking	Hypothesize about students' responses
Curriculum and standards alignment	Meet content standards
Promote particular strategies	Expanded form of multiplication, skip counting
Affective Domain	Teacher or Student Frustration

Noticing and Interactions in the Classroom

In the previous section, I discussed the pilot study and linkages supported by the literature regarding the types of and range of factors that teachers might consider when developing a mathematical task. The next logical step was to examine the same types of phenomena once the task was enacted in the classroom. In other words, it was worthy to investigate the criteria that a teacher might weigh when deciding to modify a task to meet in-the-moment needs of students. Therefore, I focused a portion of the pilot study to investigate the reasoning behind the decisions made by Ms. T when she modified a task for a student during a lesson. However, before I present results from the pilot study, I position this component of the pilot study within two aspects of the enactment of a mathematics lesson: (a) teacher noticing, and (b) teacher-student interactions.

In a mathematics classroom, what the teacher notices shapes her reactions to prevailing situations (Schoenfeld, 2011), and provides the foundation for identifying and interpreting student's mathematical thinking (Goldsmith & Seago, 2011). In the next section I present a brief exploration of the idea of noticing students' mathematical thinking, and the theoretical lens that I adopted for the pilot study and the full investigation. I follow this with a similar exposition on teacher-student interactions, and I finish with a section that offers data from the pilot study that I situate within the adopted noticing and interaction frameworks. These three sections serve as the motivation for, and as a precursor for my second research question, which centered on modifying mathematical tasks when enacting a lesson.

Noticing children's mathematical thinking

To some researchers, noticing was perceived as to where and on what a teacher focused her attention (e.g., Star, Lynch, & Perova, 2011). Others extended the idea of noticing to include identifying, and interpreting key features of student work (Sherin & Van Es, 2008; Goldsmith & Seago, 2011). Jacobs and colleagues (Jacobs, Lamb, & Philipp, 2010) added an additional layer to the concept of noticing by incorporating the decision to respond to a student into a noticing framework, and they presented the *professional noticing of children's mathematical thinking* as a set of three interrelated skills.

In their professional noticing of children's mathematical thinking framework, Jacobs and colleagues (2010) reported that the first interrelated skill was the teacher's ability to attend to a particular child's strategies; that is, the ability to discern the mathematical details, patterns, and nuances in the child's strategies. The second skill was

the teacher's ability to interpret a child's mathematical understandings via reasoning that connected the child's strategies with known research on children's mathematical thinking. The final skill was found in the reasoning employed by the teacher when formulating an intended response to the child. Because of my interests in the ways in which children's mathematical thinking might manifest during instruction, I aligned myself with Jacobs and colleagues (2010) view of the professional noticing of children's thinking.

In my experiences working with children on CGI problems, I, and I believe the teachers I had observed, constantly made decisions based upon the attention paid to details in a student's work, and an interpretation of the student's thought processes. Consequently, the construct of the professional noticing of children's mathematical thinking provided a theoretical lens through which to examine the reasoning behind the in-the-moment decisions that teachers made while working with students. This raised a methodological question as to the manner in which a researcher might access in-the-moment reasoning, and the question is addressed in Chapter 3. For now it is sufficient to presume that the professional noticing of children's mathematical thinking provided the theoretical basis for making an informed assertion of a teacher's reasoning that prompted a decision to respond to a student.

Teacher-student interactions during a mathematics lesson

I presented the concept of the professional noticing of children's mathematical thinking first to establish the phases from which a teacher's decision to respond metamorphoses from internal thought processes to an enacted response. In line with the interests of this study, a teacher's response is characterized as those interactions that

occur between a teacher and a student that center on the students' mathematical thinking within the context of a mathematics lesson. I explicitly ignored teacher-student interactions that could be construed as non-mathematical in nature such as classroom orchestration, or disciplinary actions, for example. In this section I briefly conceptualize teacher-student interactions in the mathematics classroom as viewed through the pertinent literature, and I develop linkages between these interactions and the professional noticing of children's mathematical thinking.

During a mathematics lesson, there occur those singular moments when the confluence of ideas, topics and students' interest form a pedagogical river into which the teacher may or may not dive. Stockero and Van Zoest (2012) described these *pivotal teaching moments* (PTM) as "instance[s] in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding." (p. 127). Because a pivotal teaching moment may go unrealized, a PTM seems to be intimately related to the idea of teacher's professional vision (Sherin & van Es, 2009). That is, for a PTM to realize its pedagogical potential the teacher must focus her attention on the moment at hand and proceed in a manner that logically follows what is perceived. For the purposes of this study I constrained a teacher's decision to respond to the three types of teacher-student interactions presented in the *advancing children's thinking* (ACT) framework (Fraivillig, Murphy, & Fuson, 1999). These were: (a) eliciting students' mathematical thinking, (b) supporting students' mathematical thinking, and (c) extending students' mathematical thinking.

Eliciting, Supporting, Extending Students' Mathematical Thinking

Eliciting students' mathematical thinking provides an opportunity for teacher-student engagement through which the teacher may monitor, assess and interpret students' ideas and understandings (Franke, Kazemi & Battey, 2007; Franke et al., 2009; Lobato, Clarke & Ellis, 2005; Sleep & Boerst, 2012). Furthermore, eliciting and making public students' mathematical thoughts provides a forum for further mathematical discussion as students explain, clarify, and defend their strategies (e.g., Kazemi & Stipek, 2001; Lau et al., 2009), and doing so can lead to the development of student understanding (Franke, Kazemi & Battey, 2007). What became a bit muddled, however, was that an eliciting action might also initiate a supporting, or extending teacher-student interaction, which suggested that the three interactions were not so easily divisible. Therefore, I aligned myself with Cengiz and colleagues' (2011) conception of an eliciting action as one that invited students to express their thinking, and I attached the caveat that such an action might also serve to support or extend students' mathematical thinking.

Building on the idea that eliciting students' to express their thinking might also serve as the initiating action for teachers to interact with students in a supporting or extending capacity, I turned to how other researchers viewed the act of supporting and extending students' mathematical thinking. First, I present the concise delineation between what constituted a supporting action and what defined an extending action as rendered by Jacobs and Ambrose (2008), which I then adopted and utilized for the rest of this exposition. Succinctly, Jacobs and Ambrose identified two overriding teaching actions: (a) supporting actions that were employed before a student produced a correct answer, and (b) extending actions that were used after a student provided a correct

answer. The demarcation provided the means for categorizing a teacher's decision to modify a problem for a student during a mathematics lesson.

To complete this section I present analysis of an episode from the pilot study whereby the teacher, Ms. T, modified the problem by presenting the student a set of new numbers within the context of the problem. The episode occurred during a small group session centered on the multiplication problem:

For the Third grade play, "A Tribute to America", we are setting up 207 rows of chairs in the auditorium. If there are 14 chairs in each row, how many chairs will be set up for the play?

In the episode, the student, Mary was trying to find the product of 207 and 14 using an expanded form strategy, whereby she decomposed 207 into $200 + 0 + 7$, and used the distributed property as the means to multiply 7 by 14, 0 by 14, and 200 by 14. The following is an excerpt from the transcript of the classroom video:

- 1) Mary: Um, I did 207 times 14.
- 2) Ms. T: Okay.
- 3) Mary: So, does 7 times 14 equal 70?
- 4) Ms. T: No, cause 7 times 10 equals 70. You need 4 more 7s.
- 5) Mary: Oh, so it's 74?
- 6) Ms. T: This is hard to do with a 14 and 2 times, this is very hard to do, 14 times
- 7) 200. Do you know what 14 times 2 is?
- 8) Mary: Fourteen times 2 is..28?
- 9) Ms. T: Mm hmm, so if you had 14 times 200?
- 10) Mary: Oh, it would be *inaudible* 2000?
- 11) Ms. T: Fourteen times 200. Two times 200 would be 400, so it would be more than?
- 12) It's really big numbers. Can I give you this strategy with smaller numbers, cause I
- 13) think these numbers are just enormous. Let's keep your two hundred and seven cause
- 14) I like it, but instead of doing a two-digit number, can I give you a one-digit number?
- 15) So, let's only put 9 chairs in a row and try it again. Okay, and use your strategy.

It seemed that Mary was having difficulty with multiplying 7 by 14 (lines 3 - 5) as well as with multiplying 14 times 200 (lines 9 - 10). In lines 12 - 15, Ms. T seemed to come to the decision to respond to Mary with the supporting move of changing the

numbers for Mary to 9 chairs in a row with 14 chairs in each row. In addition, Ms. T also seemed to want to reinforce the strategy that Mary had been employing by presenting her with an easier product to find, that is, 9 times 14, which Mary might expand into $9 \times (10 + 4)$, and then, $(9 \times 10) + (9 \times 4)$. The example demonstrated the utility of the Jacobs and Ambrose framework for categorizing the modification of a problem as a supporting or extending teaching move. The next section presents my second research question.

Research Question 2

In the episode just examined, Ms. T made the decision to change the numbers of the problem that she had developed for that day's lesson when Mary was struggling to multiply 207 by 14. I presented the episode to demonstrate the utility of the Jacobs and Ambrose framework for categorizing the two types of teaching moves that a teacher might make when modifying a problem during the enactment of a lesson. What the framework did not provide was a framework for identifying and categorizing the underlying reasoning behind Ms. T's decision to modify the problem for Mary. Consequently, to examine the phenomenon of a teacher's reasoning behind modifying a mathematical task for a student I formulated my second research question:

RQ2) What factors do elementary teachers who have an extensive understanding of children's mathematical thinking consider when modifying or adapting mathematical tasks during the enactment phase of mathematics lessons?

The research question aimed to determine what teachers considered when they came to a decision to modify the structure of a problem during a classroom teacher-student interaction. For instance, in the prior example, Ms. T changed the numbers for Mary, and one was left to wonder as to why she came to that particular decision. Evidence from the

pilot study seemed to suggest that Ms. T wanted Mary to be successful with her strategy. An excerpt from the stimulated recall conducted with Ms. T immediately following the lesson, and about 45 minutes after the classroom interaction seemed to demonstrate that Ms. T held this objective. In the transcript, Ms. T referenced Mary's work shown in Figure 1.1.

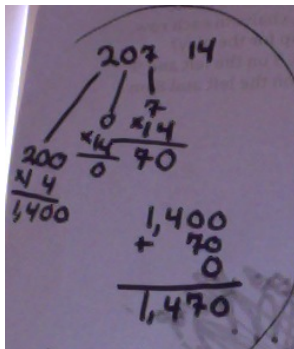


Figure 1.1: Mary's strategy for finding 14×207

An excerpt from the stimulated recall interview follows:

- 1) Ms. T : Okay, pause. It's clear that she's in over her head when she couldn't do,
- 2) whatever, where are we? We're on Mary. *She referenced Mary's work.* So this
- 3) strategy has worked for her with smaller numbers, but the two-digit numbers are
- 4) harder. So here with 7 times 14, where she came to 70, it's like, okay, she's way
- 5) over her head, like how does she not know that 7 times 10 is 70? Like, it would
- 6) be more than that. And then when I came here [*pointed to work in Figure 1.1*]
- 7) so right here I'm thinking I need to do something with a one-digit number where
- 8) she can at least do the math. Because the breaking apart is easy, but I want her to
- 9) have success with her strategy because it's a good strategy. So I want her to be
- 10) able to use her strategy. And then we'll work on, like, the two-digits later.

In line 6, Ms. T indicated that she was attending to Mary's strategy to multiply 207 by 14 whereby Mary separated the multiplication into three separate problems determined by place value (see Figure 1.1). In lines 1 - 2, Ms. T stated that Mary was over her head, which suggested that Ms. T interpreted Mary's work as an indication that Mary was struggling with the task. Ms. T concluded her thoughts (lines 7 - 10) by stating her belief that Mary's strategy was a good strategy, and one that she wanted Mary to continue to employ the strategy with the new numbers.

I acknowledge that my analysis was from a very limited data source, and that it was premature to make any assertions; and yet, at the same time, the analysis illustrated the type of factors (for example, promoting the use of a particular strategy) that a teacher might consider when modifying a problem task during a lesson. A thoroughly formulated answer to the second research question would provide the depth of insight necessary to categorize the factors involved when an in-the-moment decision is made to modify a mathematical task for students.

Significance

The significance of this dissertation study was two-fold. First, I discuss the contribution this study makes to the field regarding the decomposition of a teaching practice. Second, I discuss the potential this study had for supporting the education of prospective teachers as well as for the professional development of practicing teachers.

In order to describe the phenomena of developing and/or modifying mathematical tasks, this study was designed to access the thoughts of experienced elementary teachers to emphasize their perspectives as essential components of the practices. As stated earlier, a review of the literature found no studies intending to decompose the practice of

developing a mathematical task, so alone this study might advance the field in regards to identifying key features of the practice from which future studies may build upon.

Because this study was focused in such concentrated areas, the potential existed to add to the knowledge base of teaching fundamentals from which approximations to specific teaching practices might be developed. It is through such approximations of practice that this study might contribute to the work of educating prospective teachers, and enhancing the abilities of practicing teachers.

Recently the spotlight has turned to research on the role that number choices play when developing and enacting a mathematical task (e.g., Land and Drake, 2014; Tyminski, Land, Drake, Zambak, & Simpson, 2014), and improving the professional noticing of children's mathematical thinking abilities of pre-service teachers (Schack et al., 2013). This study identified and described factors that might contribute to approximations of these practices. For example, when working with pre-service teachers in developing mathematical tasks, the results of this study shed light on the essential factors involved from the viewpoint of experienced teachers that an educator might highlight and foreground in classroom activities and discussion. Or, educators might design activities that allow prospective teachers to emulate the practices of experienced, effective teachers.

CHAPTER 2: LITERATURE REVIEW

This chapter is organized into five main sections. Figure 2.1 outlines the order of presentation and illustrates a simplified conceptualization of the relationships among the major topics. The first section details various perspectives towards the content knowledge and the pedagogical knowledge deemed necessary to effectively teach mathematics. The second section attends to the literature on children's mathematical

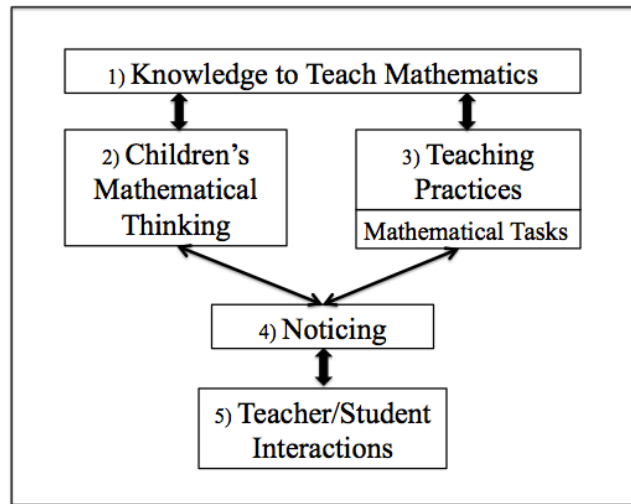


Figure 2.1: Outline of literature review

thinking and learning, emphasizing Cognitively Guided Instruction--the philosophy in which I grounded the investigation. The third section presents a review of teaching practices with a focus on researchers' attention to delineating the essential components of the teaching practice of interest, the practice of developing mathematical tasks and problems. In the fourth section, I draw upon literature in the area of noticing to elaborate the theoretical framework through which I interpreted the process undertaken when a teacher decided to respond to a student by modifying the problem. The fifth section portrays classroom interactions between teachers and students with an emphasis on researchers' views towards those interactions that support or extend students'

mathematical thinking. I close the chapter by briefly re-examining the research questions relative to the literature reviewed.

Knowledge to Teach Mathematics: Content and Pedagogy

Research Question 1 explored the many facets involved when teachers developed mathematical tasks, which in this study were CGI problem types. Any exploration of this scope was subject to a conceptualization of teachers' mathematical content knowledge as well as their pedagogical knowledge towards effectively conveying mathematical topics and concepts to students. This section reviews select conceptualizations of these specific knowledge bases, and I present an argument for the model that I believe best supported my research purposes.

Mathematical content knowledge

As stated in Chapter 1, over the last two centuries researchers and educators have reached the consensus that teachers require a certain degree of specialized subject matter knowledge to be effective mathematics teachers (Ball & Bass, 2000; Ball, Thames, & Phelps, 2008; Ma, 1999; Shulman, 1986; Silverman & Thompson, 2008). In their review of the history of assessing teachers' mathematical knowledge, Hill, Sleep, Lewis, & Ball (2007) presented a progression of what education administrators and researchers have considered to constitute a quality mathematics teacher, and how this knowledge has been measured. Briefly, Hill and colleagues (2007) described the evolution of teacher licensure from first being determined by a set of exams offered by a local school district to licensure commencement upon first completing a professional teacher preparation program typically consisting of courses in educational foundations, psychology, teaching methods, and assessment. In addition, Hill and colleagues illustrated various quantitative

approaches to measuring teachers' mathematical content knowledge that ranged from the number of college courses completed to recently developed psychometric instruments. Therefore, it was not surprising to find that even today a definition of what constitutes mathematical content knowledge had not yet achieved consensus among researchers (Baumert et al., 2010).

What researchers do seem to agree upon is that isolated proficiencies in mathematical facts, concepts, and procedures afford an insufficient set of skills for effectively teaching mathematics (Ball & Bass, 2000; Ball et al., 2008; Borko et al., 1992; Eisenhart et al., 1993; Gess-Newsome, 2002; Hill, Rowan, & Ball, 2005; Ma, 1999; Shulman, 1986). Early research (circa 1970s) emphasized quantitative approaches towards examining the effects of teachers' mathematical content knowledge on student learning and produced statistically insignificant results (Eisenberg, 1977; Mewborn, 2001). For example, Begle (1972, as cited in Eisenberg, 1977) measured 308 teachers' understanding of the real number system and related algebraic concepts using the Algebra Inventory Form B examination and found no correlation between the teachers' knowledge and student performance on a series of pre-and-post academic year Mathematics Inventories exams. Eisenberg (1977) found similar results in a replication of Begle's study, which Eisenberg conducted to eliminate any selection bias in the sample of teachers.

Similarly, small scale studies of a qualitative nature have supported findings of the type presented by Engle and Eisenberg (Borko et al., 1992; Eisenhart et al., 1993; Mewborn, 2001). For example, as part of a larger study that followed eight teacher candidates through their senior year of college and their first year of teaching Borko and

colleagues (1992) focused on a single classroom lesson presented by a teaching candidate, Ms. Daniels. Ms. Daniels had already taken several pure mathematics courses before changing her major to elementary education so by this measure she had the most extensive mathematical background of the participants. In the lesson of interest, Ms. Daniels was reviewing division by fractions by concentrating on the ‘invert and multiply’ rule as the process to follow for the task, $\frac{3}{4}$ divided by $\frac{1}{2}$, when a student, Elise, asked her the question, “I was just wondering why, up there when you go and divide it and down there you multiply it, why do you change over?” (Borko et al., 1992, p. 197). Ms. Daniels attempted to answer the student by presenting an example problem with a related diagram about painting three-quarters of a wall with $\frac{1}{2}$ gallon of paint. However, this example culminated in multiplication of the fractions rather than division, after which, Ms. Daniels was unable to explain her error.

This short episode prompted Borko and colleagues (1992) to ponder the issues: (1) How is it that a pre-service teacher with a solid background in advanced mathematics, and late in her last year of an education program, could not present a conceptual explanation for the invert and multiply algorithm when dividing by a fraction? (2) And more importantly, why did she not later reflect on her error with the intent of developing a suitable explanation for the next time a student asks the same type of question? Borko and colleagues suggested that the answer lay in the interdependence of Ms. Daniels’ mathematical knowledge, her beliefs about teaching mathematics, and the effects of her senior-level methods course. It was through investigations of this ilk that provided the impetus for researchers to develop a more thorough conception of the knowledge base necessary to effectively teach mathematics.

In regard to content knowledge in general, Shulman (1986) offered that such knowledge must go beyond an understanding of the facts and concepts of a subject, and he argued that subject matter knowledge integrates facts and concepts within the structure of the principles of a discipline. For example, a biology teacher should understand the many ways to organize the subject and be fluent in the specific language of biology. In addition, the effective biology teacher should understand the pedagogical reasons behind selecting one perspective of a topic over another when presenting the material (Shulman, 1986). Likewise, in a study of the knowledge of division with fractions held by prospective teachers, Ball (1990) presented a parallel viewpoint wherein she considered three criteria that composed teachers' *substantive knowledge* of mathematics: (a) correct knowledge of concepts and procedures, (b) understanding of underlying principles and meanings, and (c) knowledge about the nature of mathematics and the field in general. For example, Ball presented that teachers must be able to correctly calculate $1\frac{3}{4} \div \frac{1}{2}$; they must understand also what it means for the answer to be $3\frac{1}{2}$; and they should be accomplished at making connections between fractions and division.

Contrasting the two perspectives presented above with those taken by researchers such as Eisenberg (1977), for example, demonstrates a shift from viewing mathematical content knowledge as a solitary ability measurable by a standard instrument (e.g., the Algebra Inventory Form B, as cited in Eisenberg, 1977) to a perspective that views mathematical content knowledge as a complex mixture of aptitudes and abilities. Thus a framework emerged that synthesized mathematical content knowledge with pedagogical knowledge for teaching mathematics.

Mathematical pedagogical knowledge

In his 1985 address to the American Educational Research Association and subsequent publication, Shulman (1986) revolutionized the idea of what constituted the knowledge base for effective teaching. Shulman argued that in addition to possessing a degree of subject matter knowledge (see above), teachers possessed an additional kind of content knowledge, namely, Pedagogical Content Knowledge (PCK). To Shulman, PCK was "...the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman, 1986, p. 7). This included knowledge of highly expedient structures of ideas, pertinent examples, and supporting representations, as well as applicable alternatives of each. Furthermore, Shulman offered that PCK entails knowledge of how students learn certain material, what understandings students of particular ages might possess, and how best to progress them towards higher-level understandings.

Since Shulman (1986) presented his model of the knowledge base required for teaching a discipline, researchers have expanded upon the concept of PCK with arguments to include the clinical aspects of the practice of teaching (Grossman, Hammerness, & McDonald, 2009; Hiebert, Gallimore, & Stigler, 2002), and knowledge of the role of technology in the classroom (Koehler & Mishra, 2009; Niess et al., 2009). In the field of mathematics education, a burgeoning perspective towards teaching mathematics has been depicted as *mathematical knowledge for teaching*, or MKT (Ball, Thames & Phelps, 2008; Hill, Ball, & Schilling, 2008; Hill, Rowan, & Ball, 2005; Hill et al., 2008; Silverman & Thompson, 2008).

As discussed in Chapter 1, the model of MKT that I adopted for this study is that of Hill, Ball and colleagues (Ball et al., 2008; Hill et al., 2008). To reiterate, Hill and others viewed MKT as the subject matter knowledge and pedagogical content knowledge that together support the effective teaching of mathematics. In other words, the MKT framework delineates among the many practices that mathematics teachers are expected to master in the classroom. Specifically, in the model of MKT proposed by Hill, Ball and Schilling (2008) subject matter knowledge was categorized by: (a) common content knowledge (CCK), the knowledge employed in teaching that relates mathematical concepts and topics as they are used in other mathematically rich disciplines such as engineering; (b) specialized content knowledge (SCK), the mathematical knowledge necessary for teaching from essential conceptual perspectives, but which is not of immediate interest to students; and (c) knowledge at the mathematical horizon, an awareness of how mathematical topics are related throughout the expanse of the curriculum. Alternately, pedagogical content knowledge was composed of: (a) knowledge of curriculum, or an understanding of the types of support available to assist teaching mathematics at a particular grade; (b) knowledge of content and teaching (KCT), which can be thought of as knowing what teaching actions might help students overcome an encountered difficulty with mathematics; and (c) knowledge of content and students (KCS), which encompasses how students learn mathematics. In Hill, Ball and Schilling's model subject matter and pedagogical content knowledge meet at the border of SCK, and KCT/KCS. This is demonstrated in Figure 2.2.

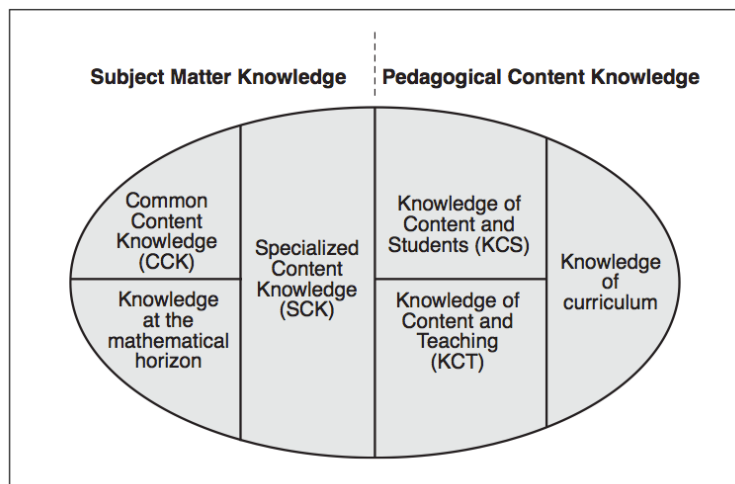


Figure 2.2: Components of MKT (Hill et al., 2008)

In addition to formulating the theoretical underpinnings of MKT, efforts have proven successful in developing measures across the areas of MKT and pairing these measures with student achievement (Baumert et al., 2010; Hill et al., 2005). An example germane to this investigation is Hill's study (2010) that examined connections between elementary teachers' performance on an MKT assessment and their knowledge of numbers and operations, and their teaching within the domains. Hill and colleagues analyzed responses to a previously validated (e.g., Hill et al., 2005) detailed assessment survey of MKT from a representative sample of 625 elementary teachers across the 48 contiguous states. The researchers concluded that instead of mathematical content (e.g., whole versus rational numbers) the MKT domain seemed to be associated with differences in the assessment item difficulties. They also found that assessment tasks falling within the area of Common Content Knowledge proved easier for the participants than those belonging to the SCK and/or the PCK categories. Although the author was upfront in pointing out some limitations of the study (for instance, the low response rate may have inflated the assessment scores) the findings suggested that the MKT framework

is useful in providing a distinct mapping of teachers' mathematical content and pedagogical content knowledge. For this reason, and for the purposes of this dissertation study, this model of MKT supported the goal of identifying and describing the factors involved when elementary teachers develop or modify mathematical tasks.

Children's Mathematical Thinking

As discussed in the previous section, a teacher's Mathematical Knowledge for Teaching (MKT) is a complex assortment of interrelated knowledge bases that are constantly summoned during the practice of teaching. This investigation focused on the decisions that elementary teachers make during the design and enactment phases of mathematics lessons. These types of decisions are dependent upon the breadth and depth of an elementary teachers' MKT, which includes knowledge of children's mathematics and awareness as to how children think about and perceive mathematics. In this section, I discuss the nature of children's mathematical thinking within two domains. First, I discuss the idea of learning trajectories as they relate to mathematics and children, and I elaborate on the framework of Cognitively Guided Instruction relative to this topic. In addition, I address children's mathematical thinking from the teacher's perspective with an emphasis within the CGI framework. Hence, this section is separated into two sections: (a) children's mathematical learning trajectories, and (b) children's mathematical thinking from a teaching perspective.

Children's mathematical learning trajectories

To properly situate the potential learning trajectories of children within the domain of numbers and operations as taken from empirical research focused on CGI (e.g., Carpenter et al., 2003), it was necessary to first examine theoretical

conceptualizations of learning trajectories. Although it is beyond the scope of this document to sufficiently review the entire span of work in this field, a review of the literature following a thread beginning with the formulation of the hypothetical learning trajectory concept (Simon, 1995) through recent work that builds on the construct offers a sufficient basis from which to discuss the learning trajectories pertinent to this study.

In his influential treatise, Simon (1995) introduced the *hypothetical learning trajectory* (HLT) as “the consideration of the learning goal, the learning activities, and the thinking and learning in which students might engage” (p. 133). Essentially, the hypothetical learning trajectory consists of the teacher’s learning goal for the lesson coupled with a working hypothesis as to how learning might occur as related to the teacher’s pedagogical strategies (Simon & Tzur, 2004). Similar to Simon’s concept of the HLT is the Realistic Mathematics Education (RME) movement that originated in the Netherlands.

Realistic Mathematics Education is a theory of instructional design that engages students in mathematical problems that are based within the students’ experiences and uses these types of problems as springboards for students to form more refined mathematical structures (Gravemeijer & Doorman, 1999; Gravemeijer, 1999). Gravemeijer and Doorman (1999) identified these types of problems as context problems (the conditions are real and imaginable to students), and it is through active exploration of these problems that students progress to higher levels of mathematical understanding. Essentially, the teacher promotes problem solving whereby students develop and test hypotheses (Van Den Heuvel-Panhuizen, 2003) with the goal of developing the students’ cognitive abilities. Gravemeijer (1999) described the differences between the RME

model of a *local instruction theory* and Simon's conceptualization as lying within the scope of each model. To Gravemeijer, the HLT model consisted of a small number of instructional activities and pertained to each teacher's individual classroom while a local instruction theory was more general and encompassed an entire instructional sequence.

Since the introduction of the HLT, researchers have added layers of nuance to the complexion of learning trajectories leading some to delineate between student developmental progressions and instructional sequences (Clements & Sarama, 2004; Duschl, Maeng, & Sezen, 2011; Stevens, Delgado, & Krajcik, 2010) while others view instruction as an essential accomplice to students' mathematical development (Wilson, Mojica, & Confrey, 2013; Wilson, Sztajn, Edgington, & Confrey, 2013). It is the latter view that underscores the influence of teachers' pedagogical knowledge to teach on students' learning. Furthermore, Wilson, Mojica, & Confrey (2013) suggested that a mathematics learning trajectory plays a vital role in teachers' developing models of their students' understandings; moreover, as they incorporate students' thinking into their practice they restructure their own knowledge and become better at making sense of students' thinking. In their framework, teachers begin to construct their models of students' thinking from an initial level wherein they describe students' thinking before moving to comparing students' thinking. Teachers restructure their own knowledge as they incorporate the models of students' thinking into their own thinking and a final level is reached where teachers begin to make inferences of student's thinking.

Wilson and colleague's perspective towards a learning trajectory as a source of knowledge as to how students develop mathematical understandings, and that teachers can leverage this knowledge to improve their own teaching practices seemed to align

with the backgrounds and knowledge bases of the participants in this investigation. The participants in this study were all well versed in the typical learning trajectory that emerged from research in the area of CGI (Carpenter et al., 1999), and they all viewed developing models of their students' mathematical thinking as being foundational to their teaching. Thus the CGI research base was fundamental to this investigation.

As mentioned in Chapter 1, the principles of CGI were grounded in empirical data demonstrating that children naturally employ a variety of strategies to solve contextual problems involving whole numbers (Carpenter, Fennema, Franke, Levi, & Empson, 1999). In addition, young children tend to follow a particular learning trajectory within the domain of whole numbers and operations. For example, in the domain of addition problems with whole numbers, Carpenter and colleagues (1981, 1984, 1999) identified particular problem types and the conceptual progression that children typically followed when advancing their mathematical proficiency and fluency. One of the problem types that children naturally tend to associate with the operation of addition is the Join-Change-Unknown problem, wherein the starting and final numbers in a set are given and the child has to find the missing addend. A typical problem might read:

At first, 3 deer were eating in a field when some more deer came out of the woods into the field. Now there are 12 deer in the field. How many deer came out of the woods?

Carpenter and others (1999) described the progression children tended to follow when developing more sophisticated strategies for solving this type of problem.

Typically, children begin by employing *direct modeling* strategies whereby they use physical objects or representations such as blocks, fingers, or tally marks, for example, to represent the initial addend, then add more 'objects' until the sum is reached. Following

the action of a problem, in this case, getting more of something is pivotal to a direct modeling strategy. The answer is found by counting the number of objects added. Thus organizing the addends into discrete groupings is essential so an accurate count can be made. To solve the deer problem, a child might build the initial set using 3 blocks then add blocks one at a time into a separate pile until a count of all the blocks totaled 12. The child would finish by counting the pile with the 9 added blocks.

From a direct modeling strategy, a typical child might move towards using more sophisticated counting strategies. Counting strategies differ from direct modeling strategies in the sense that the child does not have to build each number in the problem. For example, to solve the deer problem, a child might start from 3 and count onto until reaching 12. The child would present the answer as the number of digits, 9, used in the counting sequence. Finally, more experienced children might learn and recall number facts, or derive useful number facts themselves, to assist with solution strategies. For instance, a child might explain her strategy for the deer problem as ‘I added 7 to 3 to make 10, and 2 more made nine.’ It was noteworthy that children might not abandon prior learned strategies when moving along their individual learning trajectories, for example, known number facts may be utilized with direct modeling, and children typically developed a sense of intuition and flexibility with their choice of strategies (Carpenter et al., 1999).

I detailed the above progression in strategies that children generally follow within the CGI framework to illustrate that the empirical findings of Carpenter and colleagues (1999) complemented the ideas of Wilson and colleagues (2013) in the sense that knowledge of a learning trajectory can support teachers in learning about their students’

thinking. I now turn to knowledge of children's mathematical thinking from the teacher's perspective, and how it might inform their teaching practices.

Children's Mathematical Thinking and Teaching

In a comparison study between U. S. and China teachers, Ma (1999) investigated differences between 72 Chinese teachers and 23 teachers from the U. S. in their mathematical content knowledge, and how this knowledge manifested in their teaching. Whereas U. S. teachers tended to exhibit and promote procedural understandings, the Chinese teachers were more inclined to exhibit and promote conceptual understandings. For example, when teaching multiplication U. S. teachers exhibited a lack of place value knowledge and taught lessons that centered on fostering procedural skills. In contrast, Chinese teachers tended to focus on partial products and place value understandings, and were more apt to develop lessons around a *knowledge package* consisting of a web of interconnected mathematical concepts or conceptual focal points (Ma, 1999). Furthermore, Chinese teachers utilized justification and argumentation more often as a teaching strategy than did their U. S. counterparts. Not surprisingly, Ma concluded that subject matter knowledge and teaching strategy appear to be linked, and argued that to be effective, elementary teachers must possess a high level of *profound understanding of fundamental mathematics* (PUFM) whereby the teacher is “not only aware of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics, but is able to teach them to students” (Ma, 1999, p. 24).

The idea that mathematics teachers should possess deep subject matter knowledge as well as applicable pedagogical knowledge has already been presented in this document. I turned to the notion of a knowledge package (Ma, 1999) to set the stage for

examining teaching as it relates to children's mathematical thinking. In Ma's conceptualization, learning sequences were a vital component of a knowledge package, and in China "the teachers believe that these sequences are the main paths through which knowledge and skill about the... topic develop" (Ma, 1999, p.114). It has also been demonstrated that teachers whom observe and question students' strategies seemed to develop more sophisticated models for understanding how students might progress along a learning sequence (Choppin, 2011b; Doerr, 2006). A similar idea exists within the literature relevant to teaching via the principles of CGI.

The learning activities of a CGI classroom are based on problem solving and communicating thinking (Carpenter et al., 1999). A CGI teacher reacts to a research based, and locally conceived, model of each student's mathematical understandings and abilities within the CGI framework for the development of children's mathematical thinking (Carpenter, Fennema, & Franke, 1996; Fennema et al., 1996; Fennema, Franke, Carpenter, & Carey, 1993). In addition, studies indicate that over time those teachers that engage in questioning and reflect upon student thinking tended to generate and embrace new knowledge about children's problem solving (Franke, Carpenter, Levi, & Fennema, 2001; Steinberg, Empson, & Carpenter, 2004). Franke and others (2001) called this type of sustained knowledge acquisition, learning with understanding, and posited that it formed the basis for *generative change*, whereby teachers build upon what they know about students' understandings and use it to generate new knowledge. Consequently, as described by Franke and colleagues, a teacher continues to grow in her practice because she constantly engages with children's mathematical thinking, and the teacher has developed a sense of metacognition to continue learning from children's thinking.

Teaching Practices

In Chapter 1, I presented an argument that teacher education should integrate the pragmatic aspects of teaching within the theoretical foundations of the profession. In doing so I embraced the positions taken by Dewey (1904) and other scholars (Ball & Cohen, 1999; Ball & Forzani, 2009; Ball, Sleep, Boerst, & Bass, 2009; Cohen, Raudenbush, & Ball, 2003; Grossman, Hammerness, & McDonald, 2009; Grossman & McDonald, 2008) who emphasized that practice seeds the core of teacher education. Lampert (2010, 2012) characterized the practice of teaching as the real ‘work of teaching’, and acknowledged that the complexities of this work ranged across the dimensions of subject matter, time, social constraints, and the dynamics within student-teacher relationships and interactions (Lampert, 2001).

In proposing an answer to the question as to what is meant by a teaching practice, and accommodating the ambiguity of the term ‘practice’ (e.g. Schön, 1983), Lampert (2010) presented four conceptions of the construct as supported by researchers and educators: (a) practice in contrast to theory, (b) teaching as a collection of practices, (c) practice for future performance, and (d) the practice of teaching. To situate this investigation within the domain of teaching practices, and to define what I conceived to be teaching practices for this dissertation, I examined the first two of Lambert’s categories in greater detail.

Practice Versus Theory

Dewey (1904) described the dichotomy that some view existed between practice and theory as the difference between practice work, or that of an apprenticeship, and

laboratory work, the knowledge of the principles of the discipline. To Dewey the contrasts between the two perspectives towards teaching were readily apparent:

From one point of view [practical work], the aim is to form and equip the actual teacher; the aim is immediately as well as ultimately practical. From the other point of view [laboratory work], the immediate aim, the way of getting at the ultimate aim, is to supply the intellectual method and material of good workmanship, instead of making on the spot, as it were, an efficient workman. (p. 1).

Dewey's perspective towards theory and practice seemed to mirror the concepts of *knowing-in-action*, and *reflecting-in-action* as perceived by Schön (1983). As defined by Schön (1983) knowing-in-action was reflected in our "actions, recognitions, and judgments which we know how to carry out spontaneously" (p. 54), that is, those understandings that have been internalized of which we are no longer conscious. This idea can be compared to the practical side of teaching as viewed by Dewey. Reflecting-in-action, on the other hand, was displayed by an awareness of what we are doing and manifests during those "situations of uncertainty, instability, uniqueness and value conflict" (Schön, p. 50), which echoed Dewey's perception of the theoretical foundations of teaching.

Other scholars embraced the concept of a cognitive divergence between practice and theory by describing the difference as that between practitioner knowledge and professional knowledge (Hiebert et al., 2002), the perspectives of insider research and outsider research (e.g., Jaworski, 2003), and the generative dance between knowing and knowledge (Cook & Brown, 1999). As stated above, an emerging theme within the literature was a call for bridging the gap between practice and theory and to restructure teacher education to reflect the merging of the two (Ball & Cohen, 1999; Cook & Brown,

1999; Hiebert & Morris, 2012; Kazemi, Franke, & Lampert, 2009; Lampert & Graziani, 2009; McDonald, Kazemi, & Kavanagh, 2013). In line with this point of view is the thought that the practice of teaching is composed of a collection of teaching practices, each of which, in turn, can be fractured into their essential parts to form instructional units for teacher education programs.

Teaching as a Collection of Practices

Researchers in mathematics education have called for identifying and nurturing high-leverage teaching practices within the mathematics classroom (Ball et al., 2009; Kazemi, Franke, & Lampert, 2009; Lampert & Graziani, 2009). Ball et al. (2009) described high-leverage teaching practices as those “in which the proficient enactment by a teacher is likely to lead to comparatively large advances in student learning” (p. 460). For example, teachers eliciting and exploring students’ mathematical thinking and engaging students in mathematical discussions and arguments can be viewed as high-leverage practices (Battey, Ing, Freund, & De, 2007; Cengiz, Kline, & Grant, 2011; Forman & Larreamendy-Joerns, 1998; Franke et al., 2009; Henning, McKenry, Foley, & Balong, 2012; Kazemi & Stipek, 2001; Lau, Singh, & Hwa, 2009; Lobato, Clarke, & Ellis, 2005). Another example of a high-leverage practice is maintaining a high-level of cognitive demand when enacting mathematical tasks in the classroom (Boston & Smith, 2009; Henningsen & Stein, 1997). Some of these examples arose from theory (e.g. Lobato et al., 2005), while others were based in the practical aspects of the classroom (e.g., Kazemi et al., 2009). Thus, a high-leverage teaching practice might be viewed from either ends of the theory-practice spectrum in the sense that some fell within

recognized theories, while others were linked to teachers' execution of particular practices and the effect the practices had on student learning.

As presented in Chapter 1, Grossman and colleagues (2009) investigated the pedagogical interventions that professional educators used to instruct novices and the manners in which these interventions foregrounded the important features of their respective professional practices. Grossman and others (2009) identified three main concepts that together formed the basis for understanding a professional practice from a pedagogical vantage point: (a) representations, the extent of approaches through which a professional practice was made visible to novices; (b) decomposition, which entailed fracturing a practice into its constituent parts for teaching and learning; and (c) approximations of practice, opportunities afforded for novices to emulate the authentic activities of a practice.

As an example of decomposing a teaching practice, I present Sleep's 2012 work of decomposing the practice of teaching to the mathematical point. Sleep (2012) defined *teaching to the mathematical point* as a triplet of (a) articulating the mathematical point, (b) orienting the mathematical point, and (c) steering the instruction. Sleep considered a composition of the first two legs of the triplet as *mathematical purposing*, the work that entailed specifying and coordinating instructional goals, and she argued for the existence of a cyclic relationship between mathematical purposing and steering instruction. In this study, Sleep decomposed the practice of teaching to the mathematical point by identifying subcomponents, or as she stated, the "tasks of steering instruction toward the mathematical point" (p. 938). Essentially, Sleep offered that the practice consisted of identified tasks that maintained instruction on the intended mathematics by unpacking

key mathematical ideas, emphasizing the underlying meaning of the mathematics, and orchestrating classroom activities to keep students' attention on the work at hand. In addition, Sleep couched selecting strategic problems for discussion to support what she called the task of spending instructional time on the intended mathematics. I build on this idea to argue that the work of choosing, or developing a task, or problem, including selecting effective numbers, is an essential practice for teaching mathematics.

Mathematical Tasks and Teaching Practices

The Professional Standards for Teaching Mathematics (NCTM, 2000) and the Common Core State Standards (CCSSI, 2010) both considered mathematical tasks essential for engaging students with mathematics. In a classroom, the mathematical task at hand might serve as the shared object of thought (Kirsh, 2010), provide the intellectual locus for thinking and learning (Kazemi & Hubbard, 2008), and concentrate students' thinking on a specific mathematical idea (Stein, Grover, & Henningsen, 1996). Furthermore, the task at hand serves as a conversational hub through which teacher and students communicate via a common language (Clarke & Clarke, 2004; Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013; Olson & Knott, 2013).

As I argued in Chapter 1, a mathematical task is central to classroom activities, and developing and implementing the task serves a vital role in the work that teachers do; therefore, developing a mathematical task can be classified as an important practice of teaching mathematics. I believe that others support my view. For instance, Smith and Stein (1998) described the process of moving from the theory behind selecting an effective task to the practice of selecting and implementing the task, while Tyminski, Land, Drake, and Zambach (2012) investigated what they called the practice

of posing problems that build on children's mathematical thinking. Thus, central to this investigation are the teaching practices of developing mathematical tasks during the design phase of a mathematics lesson, and the practice of modifying a task during the enactment phase of a lesson.

Noticing Children's Mathematical Thinking

Interpreting students' mathematical thinking involves more than noting surface features of students' work, it entails a fine weft of perceptions as to what the student is doing, why the student is doing it, and what the student might be learning during the process (Sherin, Jacobs, & Philipp, 2011). This practice takes an adept eye, an eye that is unique to the profession of teaching.

Goodwin (1994) characterized professional vision as the systematic way in which a social group perceives and understands events that are of particular interest to that group. As an example, Goodwin offered that where a layman may just see a brown circle of dirt in a field an archeologist "sees" a virtual fence post with surrounding structures. Similarly, in the profession of teaching, Ainley and Luntley (2007) suggested that in addition to their knowledge of content and pedagogy teachers possessed *attention dependent knowledge*, which was invoked when attending to aspects of classroom events. They posited that it is this type of knowledge that allowed teachers to act appropriately in response to situations that arose within the context of a lesson.

Extending these constructs to the mathematics classroom, Sherin and others (Sherin & van Es, 2008; Sherin, Russ, Sherin, & Colestock, 2008) presented their model of professional vision for teachers, whereby they posited that the notable characteristic of such was the merger of *selective attention* (how teachers settled on the focus of their

attention), and *knowledge-based reasoning*, those ways in which teachers used their knowledge and understanding to logically approach what they noticed. For example, when a student introduces a mathematical idea in a class discussion, the teacher either considers it as a worthwhile object of inquiry, or not, (confirming or disconfirming selective attention), and if so, the teacher may or may not reason about the student's idea (confirming or disconfirming knowledge-based reasoning). In the case that the teacher does accept the idea as an object of inquiry and explores it further, the teacher may employ instructional actions that fall within a spectrum of sophistication from simply restating the student's idea to synthesizing across multiple student ideas (Sherin & van Es, 2008).

Related to the aforementioned conceptualizations of teachers' professional vision is the concept of teacher *noticing*, which researchers have construed in numerous forms (Goldsmith & Seago, 2011; Jacobs, Lamb, & Philipp, 2010; Star, Lynch, & Perova, 2011; Talanquer, Tomanek, & Novodvorsky, 2013; van Es & Sherin, 2006). To some, noticing was viewed simply as the focus of a teacher's attention (Star et al., 2011), while others added additional layers of identifying and interpreting student work to the skill (Goldsmith & Seago, 2011; Sherin & van Es, 2008). Mason (2011) proposed that noticing entailed an accumulated wealth of desirable actions and that teachers "need a collection of alternative actions and an awareness of situations in which these actions would be preferable" (p. 38). Mason's idea was that a teacher should be sensitized so as to notice opportunities in the future from which to act freshly rather than automatically out of habit. This suggests that the experienced teacher exhibits a sort of "awareness of

awareness” (Mason, 1998, 2011) in that the experienced teacher’s noticing is subjective via conscious shifts of attention between the moment and history.

Awareness of one’s surroundings is one facet of noticing, another is interpreting events and reasoning about the situation (Jacobs et al., 2010; van Es & Sherin, 2006, 2010). When the locus of attention is on children’s mathematical thinking Jacobs, Lamb, & Philipp (2010) argued that a teacher’s reasoning should be “consistent with the details of the specific children’s strategies and the research on children’s mathematical development” (p. 185). Thus, Jacobs and colleagues posited that a teacher’s skill set should include an interpretive framework of children’s mathematical thinking that makes sense of the details of children’s strategies coupled with how these strategies reflect her students’ mathematical understandings. To capture the essence of the aforementioned skill set Jacobs and others (2010) developed the framework of the *professional noticing of children’s mathematical thinking*.

The Professional Noticing of Children’s Mathematical Thinking

As presented in Chapter 1, for Jacobs and colleagues (2010) the professional noticing of children’s mathematical thinking (hereafter referred to as professional noticing) was a set of three interrelated skills: (a) the ability to attend to a particular child’s strategies, (b) the ability to interpret a child’s mathematical understandings via reasoning that connects the child’s strategies with known research on children’s mathematical thinking, and (c) the ability to respond to a child based on an informed perception of the child’s understandings. Noteworthy is the fact that Jacobs and colleagues found that expertise in professional noticing was a function of professional development, and that the skills did not necessarily improve with teaching experience.

Recently, the framework of professional noticing has gained traction with mathematics education researchers as it provides a basis for gauging teachers' and teaching candidates' expertise across the domains of attending to, interpreting, and deciding to respond to children's thinking. These include studies that focused on analyzing challenging tasks (Choppin, 2011a), student thinking while solving problems (Fernández, Llinares, & Valls, 2013), pre-service teachers writing problems (Tyminski et al., 2014) and discussions in an on-line environment (Fernández, Llinares, & Valls, 2012). In particular, research projects have centered on evaluating pre-service teachers' professional noticing abilities. For example, Schack and colleagues (Schack et al., 2013) employed the framework to measure how well prospective elementary teachers could map videos of children's mathematical thinking to the Stages of Elementary Arithmetic Learning (SEAL) trajectory (Steffe, 1992, as cited in Schack et al., 2013). They found that over a 5-module course designed to improve professional noticing skills that pre-service teachers (PSTs) significantly increased their abilities to properly attend to identified nuances in children's thinking and place the children's understanding within the correlating SEAL developmental level. In addition, the researchers determined that most participants also improved their skills in deciding to respond relative to the children's developmental progression in early numeracy.

Particularly germane to this investigation is the work of Tyminski and others (2014) and their project investigating pre-service teachers (PSTs) writing problems intended to build on children's mathematical thinking. Situated within the context of elementary mathematics methods courses at two university sites, the researchers analyzed 72 PSTs responses to a sequence of three course activities enmeshed with devising tasks

that were designed to scaffold the PSTs proficiency in professional noticing. The progression of activities gradually grew in complexity from the PSTs observing expert teachers designing a task to designing tasks themselves, which in turn evolved from the PSTs addressing a single concept to considering a scope of student understandings. In addition, the PSTs concentrated on the number choices made by an expert teacher, and near the end of the activities sequence they included their own number choices when designing a task.

For example, the third activity in the Tyminski and colleagues' study involved the following, Fishbowl, problem and predetermined number choices:

Sam had ____ fish bowls. He had ____ fish in each bowl. How many fish did Sam have?

A	B	C	D
(2, 10)	(4, 20)	(3, 11)	(4, 12)
(5, 10)	(8, 20)	(6, 11)	(8, 12)

For this activity, the PSTs were first asked to predict the teacher's learning goal for the lesson. Following this the PSTs were presented a description of the teacher's lesson goals and then were asked to examine 14 examples of student work and attend to the details in each student's strategies, and to interpret the strategies in relation to the teacher's learning goals. The last portion of the activity involved the PSTs writing a follow-up problem with number choices together with their rationale. As part of their analysis, Tyminski and colleagues examined the PSTs' responses to the Fishbowl problem in order to classify the professional noticing abilities of the PSTs and to measure their progress across the three activities. The researchers found that the PSTs increased their professional noticing abilities; however, more so within the skills of attending to and interpreting students' thinking than with responding to students' thinking. The PSTs

were much less successful in creating and justifying their number choices in regards to extending student thinking, which demonstrated the difficulty they had in addressing multiple groups of students' learning goals along with the diversity of student thinking.

I chose to elaborate on the study presented above because deciding to respond to a child's mathematical thinking was of particular import to this study. Once a teacher decides to respond to a student, the act of noticing can be transformed into an instructional action, which in turn may trigger a new reaction by the student (which in turn avails itself for noticing by the teacher). Therefore, what a teacher attends to and how it is interpreted is inextricable from the decided upon response (Jacobs et al., 2010). Experienced teachers may even notice details while working with a child in terms of associations distal to that particular interaction (Erickson, 2011). For instance, a child's strategy when using base-ten blocks to solve a problem may mirror that of one or more children with whom the teacher has previously interacted, which may trigger the teacher to decide to respond in a familiar manner.

In addition, in my review of the noticing literature, the Tyminski and colleagues (2014) study was the only one I found that linked professional noticing to the Hill and colleagues' model of mathematical knowledge for teaching (MKT) (Hill et al., 2008). Specifically, Tyminski and colleagues posited that the skills of attending to and interpreting student thinking were integrated within the MKT categories of specialized content knowledge (SCK) and knowledge of content and students (KCS). They argued that teachers utilized their SCK and KCS to simultaneously decipher children's mathematics and the strategies children commonly use. This initial foray into positioning professional noticing within the MKT structure supported the supposition that a teacher

possessing robust professional noticing skills was better equipped to provide effective mathematics instruction to a diverse populace of students.

Teacher-Student Interactions in a Mathematics Classroom

In Chapter 1, I briefly presented the idea of a *pivotal teaching moment* (PTM) to depict those classroom opportunities for the mathematics teacher to broaden students' mathematical understandings (Stockero & Van Zoest, 2012). An example of a PTM is when a student offers an answer or conjecture that is obviously wrong and is based on a misconception or misunderstanding of the mathematical concept being discussed. At this moment, the teacher must decide to act and the action taken holds pedagogical potential; however, this potential is realized only when the teaching action is implemented skillfully to extend students' thinking about the mathematics at hand. How the teacher responds to a PTM, from ignoring it to embracing it as an opportunity to augment the lesson “shapes whether the potential of the PTM is actualized.” (Stockero & Van Zoest, 2012, p. 136). As Stockero and Van Zoest (2012) point out, not every pivotal teaching moment is recognized as such, and for some teachers it can be argued that they lack the skills required to realize the occurrence of all PTMs.

Akin to the idea of a pivotal teaching moment is that of *mathematically significant pedagogical opportunities to build on student thinking*, or MOSTS (Leatham, Peterson, Stockero, & Van Zoest, 2015). MOSTS, as presented by Leatham and colleagues, are those moments during a mathematics lesson wherein occurrences of student thinking offer ideas around which a teacher can frame profitable mathematical discussions. The ideas spring from students, and it is left to the teacher to take advantage of noteworthy ideas by engaging students in discussions centered on sense making and

reasoning, thus building on the mathematical concepts under discussion. The authors offered as an example a student voices that he solved for the angle of depression in a trigonometry problem by finding the angle of elevation. This presents the teacher an opportunity to involve the whole class in a mathematical discussion on the relationship between the angle of elevation and the angle of depression. Succinctly, MOSTS are grounded in students' mathematical thinking, but it is left to the teacher to capitalize on the moments as they arise during a lesson.

I posit that an awareness of PTMs or of MOSTS can be construed as being related to a teacher's professional vision (Sherin & van Es, 2009) in the sense that both are connected to how the teacher focuses her attention and those ways in which the teacher reasons about observed events. For example, Sherin and Van Es (2008) offer that the teacher might respond to a student's conjecture by reasoning about the student's idea, restating the student statement, or by investigating the meaning of the idea and synthesizing across other students' ideas. These examples of teacher responses reflect those presented by Stockero and Van Zoest (2012) as being characteristic of teaching decisions that seem to have a high positive impact on student learning.

However, the manners in which a teacher might react to a student's thinking are virtually uncountable. Thus, I argue to limit the teacher's responses to two types of teacher-student interactions that are closely linked to students' mathematical thinking: (a) supporting students' mathematical thinking, and (b) extending students' mathematical thinking. The first leg of my argument stands on the paradigm of *advancing children's thinking* (ACT) (Fraivillig, Murphy, & Fuson, 1999).

The Advancing Children's Thinking (ACT) Framework (Fraivillig et al., 1999) emerged from a 5-year longitudinal study that followed a cohort of first grade students and their mathematics achievement, and a case study of a teacher whom the researchers deemed as being particularly skillful. In the case study, the researchers identified patterns of teaching practice the teacher employed to access, understand, assist and challenge students' mathematical thinking. These patterns of practice provided the foundation for the ACT framework, which consisted of three overlapping components: (a) eliciting children's solution methods (eliciting), (b) supporting children's conceptual understanding (supporting), and (c) extending children's mathematical thinking (extending). Eliciting was viewed as those teaching actions that provided students an environment whereby they were encouraged to express their unique mathematical ideas. Supporting teaching actions provided assistance to students while they were immersed in familiar and accessible solution strategies, and extending teaching actions involved instructional actions that challenged the students to think beyond their current mathematical understandings.

Eliciting and extending students' thinking are actions that can be exceptionally challenging for teachers to conduct (Cengiz, Kline, & Grant, 2011; Fraivillig, 2001; Franke et al., 2009; Jacobs & Ambrose, 2008; Stein, Engle, Smith, & Hughes, 2008), and the two teaching actions may be ensconced or integrated within broader teaching practices such as ambitious teaching (Kazemi et al., 2009; M. Lampert et al., 2013), and facilitating classroom mathematical discussions (Forman & Larreamendy-Joerns, 1998; Franke et al., 2007; Henning, McKenney, Foley, & Balong, 2012; Reid, 2002; Silver & Smith, 1996; Staples, 2007; Stein et al., 2008). To finish my argument for limiting

teacher-student interactions to those that support, or extend students' mathematical thinking, I review all three components of the ACT framework, and I argue that eliciting can be enfolded within the other two.

Eliciting student thinking

In their expansion of the ACT framework, Cengiz and colleagues (Cengiz et al., 2011) offered a succinct fitting description of the practice of eliciting students' mathematical thinking:

“Eliciting actions provide students with opportunities to express their existing thinking about their mathematical activity or a mathematical phenomenon. They also allow teachers to become knowledgeable about their students' existing thinking so that they can use this information to decide which ideas to pursue.”(p. 362).

Researchers have pinpointed select teaching strategies that might enable teachers to better elicit their students' mathematical thinking. Most prevalent in the literature I reviewed were inviting students to share strategies and encouraging students to elaborate on others' mathematical explanations (e.g., Cengiz et al., 2011; Fraivillig et al., 1999). At the same time it is important for the teacher to promote the norms of attentive listening and conveying an attitude of acceptance toward students' mathematical errors or misjudgments (Fraivillig, 2001).

As stated in Chapter 1, eliciting students' mathematical thinking provides a context through which the teacher can initiate classroom discussions that guides students' thinking (Franke et al., 2009; Lobato, Clarke, & Ellis, 2005; Sleep & Boerst, 2012; Van Zee & Minstrell, 1997), promotes mathematical argumentation and justification (Kazemi & Stipek, 2001; Lau, Singh, & Hwa, 2009; Strom, Kemeny, Lehrer, & Forman, 2001), and may further student understanding (Franke, Kazemi & Battey, 2007). As described

earlier, in their reformulation of telling, Lobato, Clarke, and Ellis (2005) presented the teaching actions of initiating and eliciting, and their model provided an example wherein eliciting was not easily distinguishable from an extending or supporting teacher-student interaction.

A concrete example of the difficulties confronted with separating eliciting from supporting and extending interactions is Franke and colleagues (Franke et al., 2009) analysis of three teachers who were asked to engage their students in working with the equal sign and relational thinking. Franke and colleagues presented evidence that after an initial question to elicit a student's explanation the follow-up questions asked by the teacher significantly influenced the quality of the student's responses. They found that following with a sequence of probing questions frequently supported the student in providing a complete and accurate explanation even when the student had at first provided an inaccurate account. The examples that Franke and colleagues provided seem to blur the lines between eliciting actions, and supporting and extending actions, and I believe that other researchers would tend to agree (see for example, Kazemi & Stipek, 2001, and their conception of pressing for student learning). Hence, I offer that it was more productive to focus on supporting and extending actions, and to accept eliciting as a form of either.

Supporting and extending students' mathematical thinking

In Chapter 1, I outlined the demarcation between supporting and extending students' thinking that I adopted for this study. To reiterate, I embraced the distinction established by Jacobs and Ambrose (2008) who categorized supporting actions as those that occur before a student reaches a correct solution, and extending actions as those a

teacher makes after a student provides a correct solution. After viewing and analyzing more than 1000 video episodes video of teachers interviewing students solving story problems, Jacobs and Ambrose (2008) identified eight categories of teaching moves that were productive in progressing discussion of students' mathematical ideas.

To support children's thinking Jacobs and Ambrose (2008) identified four categories of teaching moves: (a) ensure the child understands the problem, (b) change the mathematics in the problem to match the level of the child's understanding, (c) explore what the child has already done, and (d) remind the child to use other strategies. Similarly, they identified four categories of extending moves: (a) promote reflection on the strategy the child has just completed, (b) encourage the child to explore multiple strategies and their connections, (c) connect the child's thinking to symbolic notation, and (d) generate follow-up problems linked to the problem the child just completed. Thus the lens provided by Jacobs and Ambrose seemed appropriate for examining the reasoning behind a teacher's decision to modify the task for a student during a mathematics lesson.

Research Questions and Literature

In Chapter 1, I posed the following research questions:

- 1) What factors do elementary teachers who have an extensive understanding of children's mathematical thinking consider when developing mathematical tasks in the design phase of mathematics lessons?

- 2) What factors do elementary teachers who have an extensive understanding of children's mathematical thinking consider when modifying or adapting mathematical tasks during the enactment phase of mathematics lessons?

Next, I re-examine the two questions in light of the above literature review.

Research question 1

Research Question 1 investigated the scope of issues that elementary teachers deliberated and reflected upon when developing a mathematical task. The motivation for Research Question 1 rested partly within the call for researchers to examine teaching practices with the intent to decompose the practices into their amalgamated parts from which approximations to the practice may be developed (Grossman et al., 2009). Some of these components were potentially illuminated in the pilot study such as aligning the task with the mathematical topic of the day, and selecting numbers that offered entry points to the problem, and that promoted using particular strategies. For example, Ms. T said, “Okay, so after last time’s experience I wanted to put some that I felt would be easy to access and get into the problem. And I was glad to see that they were all able to do the 56 divided by 8 and 36 divided by 6.”

Adopting a wide purview of the practice of developing a mathematical task offered the potential to identify the constituent building blocks of the practice, as well as to add mortar to the gaps between theory and practice. The intentional concentrated focus of this dissertation was such that subtle disparities between teachers may add nuance to the literature concerning the intellectual commitment involved with developing a mathematical task. Thus, through the concerted observation of and explicit questioning of the specialized cadre of teachers in this study, I anticipated unearthing a narrative that richly described the process of developing a mathematical task.

Research question 2

Research Question 2 centered on the enactment phase of a mathematics lesson and those moments when teachers decide to modify a task to meet the individual needs of students. For this research question the framework of the professional noticing of children's mathematical thinking offered a lens through which to unpack a teacher's in-the-moment thinking within the boundaries of a teacher-student interaction. By investigating practicing teachers during live classroom moments when they decided to modify a task for a student, I hoped to identify and describe the rationale for doing so. To date, teachers' in-the-moment thinking has proven difficult to capture (e.g., Sherin et al., 2008), and in Chapter 3, I elaborate on the methodology that offered the prospect to faithfully reproduce teachers' in-the-moment thoughts.

In addition, this research question foregrounded the importance of number selection within a mathematical problem, and the decision to change numbers during a teacher-student interaction. Consequently, I through this investigation I describe the phenomena of modifying a mathematical task through the eyes of teachers who: (a) had extensive understandings of children's mathematical thinking and learning, (b) invoked pedagogical content knowledge in unique and specific manners, and (c) were proficient in the practice of modifying tasks to meet the individual needs of their student

CHAPTER 3: RESEARCH METHODS

In the first two chapters of this dissertation, I presented arguments with warrants to support the merit of the two research questions. In particular, I demonstrated with evidence from a pilot study that several factors seemed to arise when Ms. T developed a problem for mathematics lesson. The findings from the pilot study, coupled with an apparent lack of literature investigating the teaching practice of developing mathematical tasks, lead to the formulation of Research Question 1. In addition, the literature seemed to confirm that during moments of classroom interactions, the manners in which a teacher might respond to a student was dependent upon the focus of the teacher's attention and the interpretation of the teacher's observations. Therefore, Research Question 2 mirrored the goal of the first question, but as realized during the enactment phase of a mathematics lesson as it sought to identify the factors involved when a teacher modified a mathematical task in response to a student's work.

In this chapter, I present the research design that supported the investigation of my two questions. First, I ground the design in case study theory. Second, I present the research design along with details of the methodology associated with the data collection for each step of the design, and the manner through which each research question is addressed. Third, I elaborate on data analysis, and I close with a section discussing the reliability, validity, and generalization of the proposed study.

Why a Multiple-Case Study?

Since the goal of this study was to identify and describe the essential components of specific teaching practices for the purpose of contributing to the field of teacher education, the study within the spectrum of research called for by Shulman (1986)

wherein he argued for research agendas to contribute to a collection of case literature centered on teacher education. Specifically, Shulman (1986) envisioned developing cases as methods of teacher education instruction that “would involve the careful confrontation of principles with cases, of general rules with concrete documented events--a dialectic of the general with the particular within the limits of the former and the boundaries of the latter are explored’ (p. 13). In line with Shulman’s vision, case study research attempts to examine phenomenon within a real-life context, notably when “the boundaries between phenomenon and context are not clearly evident” (Yin, 1981, p. 59).

The history of mathematics education research is rich with examples of case study. These include seminal studies of single cases such as the interviews conducted between Erlwanger (1973) and a young student, Benny, concerning Benny’s conceptions of his mathematics lessons, as well as important multi-case studies like Boaler’s (1998) three-year study of two high schools with contrasting approaches to teaching mathematics. For this dissertation study, I conducted a multi-case study to generate theory from case-based, empirical research across multiple settings (Eisenhardt & Graebner, 2007; Eisenhardt, 1989; Yin, 1981) in regards to the dynamics present within the teaching practices of developing and modifying mathematical tasks.

The multi-case study approach allowed me, as the researcher, to take an interpretive approach towards understanding the significance and processes of the phenomena of interest within a particular context (Maxwell, 2005; Stake, 1995, 2006). In addition, the design of this proposed study was based on the concept of *replication logic* (Yin, 2003), which was analogous to performing multiple experiments. For this investigation I selected three teachers to serve as the cases who seemed to view the

practice of teaching from similar perspective, but across the three cases there existed contradictory conditions such as their current grade-level for the purpose of conducting comparisons across the cases (Yin, 2003). The selection of the participant teachers is described later in this chapter.

In regard to this investigation, the multi-case study method allowed me to develop assertions in regards to the processes involved with developing and modifying mathematical tasks from the perspectives of three experienced teachers (the individual cases), and to coalesce the emergent assertions into a working theoretical framework through cross-case analysis. The next section details the research design of this dissertation study.

Overview of Research Design

The study adopted a modified version of the 3-Step design presented by Busse and Borromeo Ferri (2003). The 3-Step design of this study linked classroom observations and video, semi-structured stimulated recall interviews, and clinical interviews in a ‘horizontal structure’ of data collection and analysis in the sense that type II data referred to the type I data and type III data to both type I and type II. For example, selected video excerpts and artifacts of student work presented during a clinical interview in Step III served as a source of motivation for inquiry that spanned all three data types. Since the three phases of data collection were inevitably entwined, temporal sequencing was not a concern (Busse & Borromeo Ferri, 2003); hence the clinical interviews included questions related to prior classroom episodes as well as to future events.

In addition, before I collected data via the aforementioned 3-Step Design, I conducted a pre-interview independent with each participant teacher to establish a knowledge base regarding their teaching philosophy and their understandings of children's mathematical thinking, and learning in general. Figure 3.1 illustrates the *modified* 3-Step Design that includes the pre-interview. In addition, Figure 3.1 shows that I conducted three rounds of data collection across the three steps with each participant. The details of the pre-interview and each of the three steps in the research design follow.

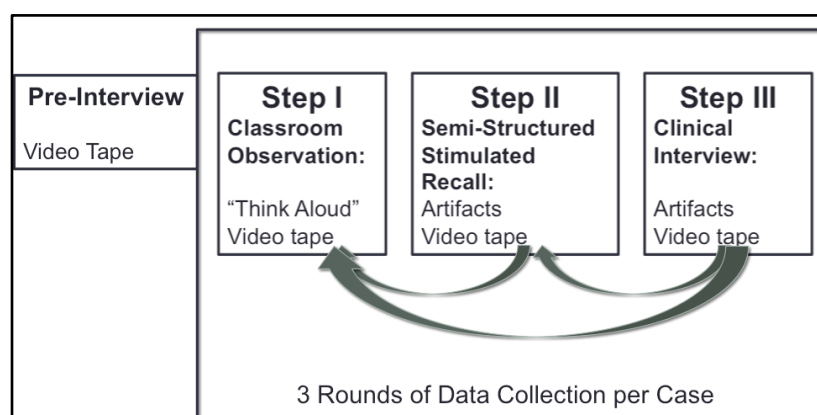


Figure 3.1: Overview of the Modified 3-Step Design

Pre-interview

The purpose of this interview was to collect background information on each participant teacher, and to garner a sense of each participant teacher's views towards teaching and learning. This information provided the foundation for composing a narrative of each case study. In addition to asking the teachers about their years of teaching experience and participation in professional development programs, I asked them questions designed to allow me to make inferences concerning beliefs and attitudes towards the teaching of, and the learning of mathematics. Although, this dissertation

study did not include a specific research question regarding investigating teachers' beliefs and attitudes, data from the pilot study suggested that these affective domains might emerge over the course of this study; therefore, I attended to the affective domain as it emerged across the three stages of data collection and the pre-interview session.

Research design: Step I

In this stage of the research design, I observed and videotaped the participant teachers while they conducted mathematics lessons. In an effort to faithfully reproduce the participant teachers' in-the-moment thoughts when modifying a problem for a student, I added an additional data collection component to Step I that was not in the Busse and Borromeo Ferri (2003) framework. For this, I utilized a short "think-aloud" protocol as described by Ericsson and Simon (1993). In mathematics education research, the think-aloud interview generally consists of a one-on-one interview in which the interviewer asks the subject to explain his or her mathematical thinking as it occurs in the moment, usually while solving a specific problem or task. The purpose of adding the aforementioned think aloud protocol was to specifically address Research Question 2 by focusing on the teachers' in-the-moment reasoning when coming to a decision to modify a problem for a student.

I adapted the technique slightly by asking the teacher to explain her thoughts immediately after modifying a task for a student. The protocol consisted of forms of two questions:

1. What were you thinking there?
2. Why those numbers?

By asking such pointed questions immediately after the teacher modified a problem for a student, I was able to gain some knowledge of the rationale that was behind the teacher's decision to change the numbers or the context of the problem for the student. This techniques seemed to alleviate the most common reliability concerns associated with interviews conducted post occurrence, such as those associated with stimulated recall interviews (discussed in the next section). As Ericsson and Simon contended, when a subject immediately responds to an interviewer's question to explain his or her thinking, the verbalization can be treated as being psychologically consistent, and it can be analyzed under the assumption that it accurately represents the subject's thoughts and cognitive processes at that time. Therefore, the technique seemed to record a faithful reproduction of a teacher's thought processes behind deciding to modify the task in response to a student.

However to be forthright in the methodology employed in this investigation, I did not interrupt the teachers immediately after every instance in which they modified a problem for a student. The reasons for this were twofold. First, at times I made the decision not to interrupt the conversation the teacher was having with a student because to do so would have disrupted the flow of the lesson, and I did not want to impose an additional level of concern for the student or the teacher. Second, there were some modifications made by the teacher such as a quick series of changing the number of groups of tens in a multiplication problem wherein I felt that I could make a strong inference as to the intentions behind the teacher's decision. In either case, I was still able to explore the teacher's thoughts in the subsequent semi-structured stimulated recall interview, which is discussed in the next section.

Research design: Step II

In their conception of the 3-Step design, Busse and Borromeo Ferri (2003) employed stimulated recall interviews during the second stage of the design, but for this dissertation study I decided to modify their design, and I employed what I called semi-structured stimulated recall interviews. Further discussion on this point is warranted so I follow with a brief literature review of stimulated recall interviews, and I follow with evidence from the pilot study that supported my modification.

Stimulated recall interviews

Stimulated recall is a retrospective probing technique used to collect verbal data after completion of the task related processes (Ericsson & Simon, 1993). Essentially, in a stimulated recall interview the interviewee is shown a video of an event of interest, and then stops the video to reflect upon his or her thinking during particular episodes. The method has been found to serve as an indirect means of measure of self-examination in complex interactive settings such as the classroom (Denley & Bishop, 2010; Ericsson & Simon, 1993; Lyle, 2003). However, the methodology has been scrutinized for its effectiveness, particularly the types of research for which it might serve as a valid data collection method, and the affordances and limitations of the procedure (Gass & Mackey, 2012).

In a review of the literature on the use of stimulated recall, Lyle (2003) found that many studies treat the method as non-problematic and often do not make detailed reference to the method itself, which he underscored can lead to doubts as to the validity of a study. Clarke (1997) described how video and interview data might be utilized in a manner designed to be mutually validating. The utmost concern with the stimulated

recall technique was found within the argument that the method does not access the interviewee's short term memory; and thus, any information retrieved would be from long term memory and the data could not serve as a true reproduction of the interviewee's thoughts at a prior moment in time (Lyle, 2003; Yinger, 1986). Denley and Bishop (2010) contended that stimulated recall is a useful methodology but cautioned: If the focus is on the particular event and a reconstruction of the teacher's thinking at the time, the use of stimulated recall becomes problematic. If the focus is on using an event to be in itself the stimulus to reveal more general aspects of teacher thinking, then the technique can be justified. (p. 114)

Lyle (2003) concurred that stimulated recall has a number of limitations, but offered that it remains a valuable mechanism for linking *naturalistic decision making* (such as teaching moves) and cognitive processes, and it provides the opportunity to maintain real-life context. However, care must be given to stimulated recall procedures to meet concerns as those posed by Denley and Bishop (2010) such as: (a) the timing of an event relative to conducting the interview; (b) the amount of the recording to be used, for example, an entire lesson, or researcher selected episodes; and (c) the level of guidance to be given to the interview subject.

Semi-structured stimulated recall

In the course of my pilot study, I identified an issue with the stimulated recall technique whereby, Ms. T would tend to shift into a reflective pose and report her thoughts concerning events that were not related to the classroom moment of interest. Thus, I modified the technique to include subject reflection in addition to recall, and I included those interruptions as opportunities during which I asked follow-up questions.

Essentially, for this dissertation study, I utilized a semi-structured stimulated interview in the sense that I, as the interviewer, was free to introduce unscripted questions as deemed appropriate, but always with the motive of clarifying the subject's reasoning (Ginsburg, 1997).

The purposes of the semi-structured stimulated interview (SSR) conducted during Step II of the research design were to address both of the research questions. The first part of each SSR centered on the participants thinking in regards to developing the problem, for instance, their reasons for selecting the problem type and their choices of numbers.¹ Thus, this part of the interview was focused on collecting data related to the first research question. During the second part of the SSR interview I presented video clips of the teacher-student interactions from the lesson conducted earlier that day to focus on the teacher's reasoning related to her decision to modify the problem for the student in the clip. This part of the SSR interview centered on collecting data related to the second research question. For various reasons I also presented video clips of other teacher-students interactions to stimulate a teacher's thoughts regarding other issues such as the student's thinking during the classroom episode, or as to where teacher had focused her attention during the interaction. This type of data provided insight into composing a case study report concerning each teacher.

Research design: Step III

The third step of the research design employed the clinical interview as the data collection instrument. A one-on-one clinical interview is designed and structured to

¹ In three instances, I was able to collect data of this type immediately prior to the lesson. This is discussed in more detail in the data collection process.

present opportunities for the researcher to observe and make inferences concerning the perceptions and capabilities of the interviewee (Ginsburg, 1997). For example, Ginsburg (1997) posited the method was particularly valuable to investigate the characteristics of a person's reasoning towards and understandings of mathematical concepts, structures, and procedures. A methodological advantage of the clinical interview lay in its flexibility and adaptability because the researcher is free to introduce unscripted questions as deemed appropriate, and to formulate and test hypotheses (Ginsburg, 1997). Thus the clinical interview was an appropriate method for studying both of the research questions in this study.

The third stage of the research design served two purposes. First, the clinical interview allowed me to more thoroughly explore across the dimensions of the first two steps of the research design by asking “deeper going questions concerning the test-subject's reflections and embeddings in his or her personal experiences” (Busse & Borromeo Ferri, 2003, p. 261). In this manner I was able to explore more thoroughly a teacher's perspective towards developing a problem for a lesson, and I could revisit specific teacher-student interactions of interest. Thus, the clinical interviews offered triangulation as it provided cross-examination of data collected during the first two steps. The next section describes the three participants and the reasons for selecting them to serve as the case studies in the investigation.

Participants

In his book on multiple case study analysis, Stake (2006) presented the idea of a *quintain* as the overarching target of an investigation where he defined a quintain as the phenomenon to be studied. In this dissertation study the quintain was composed of those

teachers that develop mathematical problems for their lessons that followed the principles of CGI. This multiple case study investigated the teaching practices of three elementary teachers with extensive knowledge of children's mathematical thinking. The participant teachers were selected using a mixture of theoretical and convenience sampling from a population of teachers that had over 12 years of teaching using the principles of CGI as well as over 6 years of professional development focused on CGI.

Theoretical sampling from the specified population supported building a theory theoretical framework around the teaching practices of interest by constraining extraneous variation among cases, which in turn, enhanced external validity (Eisenhardt & Graebner, 2007; Eisenhardt, 1989). Therefore, to support cross-case analysis I identified and recruited a kindergarten teacher, a first grade teacher, and a second grade teacher from two school districts located approximately 20 miles apart near a large city in the southwestern United States. However, before I began collecting data with the kindergarten teacher she had to withdraw from the study, so I recruited the third grade teacher that was the focus of my pilot study to participate in the main study. This is discussed further in the next section.

Recruitment of participants

The teacher population from which I drew upon was unique and markedly suitable for this the study as there was a cadre of local elementary teachers who participated in the same long-term professional development program centered on CGI, and who conducted their mathematics lessons based on CGI principles. To identify the teachers I utilized existing and former San Diego State University faculty who were familiar with the professional development program mentioned above to assist me in

finding and recruiting the participant teachers for this study. Once potential candidates were identified I contacted several via an email that detailed the study and the selection criteria. The potential candidates were offered \$40 per round of data collection for a total of \$120 to participate in the study.

In addition to the criteria that each participant had at least five years of professional development in CGI, and at least 12 years of experience teaching mathematics via the principle of CGI, I observed at least one lesson in each potential teacher's classroom to insure that each had a tendency to develop their own problems, and that they would frequently modify a problem in response to students' thinking. In addition, I determined that they orchestrated their mathematics in a similar fashion as each interacted with individual students within the setting of students working in small groups. This proved to be beneficial during Step I of the data collection since I was able to query a teacher concerning her decision to modify a task without disturbing the flow of the teacher-students interactions.

As mentioned above the kindergarten teacher withdrew from the study before the data collection process; therefore, I was left to identify and recruit a third participant. I discussed this issue with the chair and other members of the dissertation committee, and it was determined that the participant in the pilot study could serve as a third participant in the main investigation. Hence, the three participants in the study were: (a) Ms. F, a first grade teacher, (b) Ms. S, a second grade teacher, and (c) Ms. T, a third grade teacher. Ms. F and Ms. T happened to teach at the same elementary school, and they could be described as being friends and colleagues. The demographic data for the two elementary

schools is presented in Table 3.1. Ms. F and Ms. T taught at School A, and Ms. S at School B.

Table 3.1: Student demographics of the participant's respective schools

Category	School A Enrollment	School B Enrollment
White	80.6%	72.5%
Hispanic	8.4%	7%
Asian/Pacific Islander	8.2%	15.5%
African American	0.3%	0.5%
Free/Reduced Lunch	1.7%	4.5%

Data Collection

Data were collected for the first case, Ms. F, during May 2014. Data collection for the second case, Ms. S, occurred across the months of November and December 2014, and the beginning of January 2015. For the third case, Ms. T, data were collected during December 2014 and January 2015. For all three cases classroom norms had been established at the respective points in the academic year, and each teacher had demonstrated a thorough knowledge of their students' mathematical understandings and proficiencies. The three participant teachers engaged in the pre-interview session, and three subsequent rounds of the 3-Step design of the study. I chose to collect three rounds of data to provide: (a) a diversity of mathematical tasks to investigate for the first research question, and (b) a sufficient number of classroom episodes involving tasks being modified for the second research question. The amount of data I collected was similar in size to that of an example provided by Stake (2006) of a multi-case study

evaluation of the Veterans Benefits Administration, which Stake reported was about the size of a dissertation study.

The horizontal structure of the 3-Step design allowed for data collection to inform both of the research questions within each of the three stages of the design. In light of this, it seems appropriate to discuss the data collection in terms of each step of the design rather than in terms of the respective research questions. The rest of this section details the methods of the data collection such as camera placement, and the duration and timing of each interview for each of the three steps. A section describing the instruments follows.

Pre-interview

As discussed above, the purpose of the pre-interview was to collect background information on the three participants regarding their experience with teaching CGI, and related professional development. In addition, information concerning each participant's beliefs towards teaching and learning mathematics was collected. During this interview, identical questions were posed to each teacher, but follow-up clarifying and probing questions were tailored to the individual (Ginsburg, 1997). This interview lasted approximately 40 minutes for each participant, and it was conducted prior to the first round of collecting classroom data. The interviews were video taped and an audio record was also captured.

Step I: Data collection

Since the majority of the teachers' one-on-one student interactions occurred working with small groups of students in a limited seating arrangement, a stationary camera was set up to provide an overhead view of the teacher and the small group. This

camera was connected to either a desktop conference microphone, or a wireless microphone worn by the teacher to capture a record of all the teacher-student interactions that occurred with the small groups. In addition, I used a hand held camera to capture different perspectives of the teacher-student interactions, and to record audio of the researcher-teacher interactions. I also used two to four digital audio recorders placed in positions to record the students' voices and the teacher's voice when they worked in the small groups. A typical lesson for Ms. F ran approximately 75 minutes, and the same for Ms. T. Ms. S also conducted a whole-class share out session after she worked with a small group of 5 to 6 students, and only video of the small group sessions were recorded and they tended to last about 30 minutes.

Step II: Data collection

In Step II, the semi-structured stimulated recall interview occurred on the same day as the videotaped lesson. It was fortuitous that each teacher conducted their math lessons the period before lunch, and that they could meet after the school day to be interviewed. Hence, I was able to edit the video files and select clips of interest during this 3-hour period, and each SSR interview occurred within a window of 2-4 hours after the lesson took place. Conducting the SSR interviews within such a time frame fit the criteria for the time lag that researchers consider as acceptable for accessing a subject's short-term memory (e.g., Gass & Mackey, 2012). Each SSR interview was videotaped and an additional audio recording was made for data redundancy purposes. The camera was directed to capture both the teachers' responses as well as the recall stimuli video presented to them on my laptop. Artifacts of student work were also available for referencing.

The length of the SSR interviews varied between 40 minutes and 60 minutes, and they served to collect data regarding both research questions. That is, the first part of an SSR interview focused on investigating the teacher's reasoning behind the development, or selection, of the problem for that day's lesson, and the second part of the SSR interview was focused on the teacher's reasons for deciding to change the numbers or the context of the problem for students during the lesson. As discussed in the overview of the research design there were instances in which I conducted short (approximately 15 minute) *clinical* interviews prior to a lesson to examine the reasons for developing the problem for that day's lesson. These occurred in Rounds 2 and 3 of data collection with Ms. F, and in Round 1 of data collection with Ms. S. The focus of the SSR interviews conducted later on those days focused on classroom events, and instances of the teacher modifying the problem for students.

Step III: Data collection

Data collection during Step III was very similar to that of Step II with the exception that the clinical interviews occurred within a time frame of 1 to 8 days following a lesson. Unfortunately, there were instances in which conducting the clinical interview could not be scheduled promptly with a participant for a variety of reasons. In Round 1 of data collection with Ms. T the lag between Steps I & II and Step III was 8 days, and in Round 2 the lag was 5 days. In Round 1 of data collection with Ms. C, the lag between Steps I & II and Step III was 5 days. In all other instances the lag between steps was between 1 and 3 days.

The time between executing Steps I & II and Step III afforded me the opportunity to pre-analyze data from the previous two steps to inform the clinical interview protocol

in Step III. The camera and digital recorders were employed in the same manner as for Step II. The length of the clinical interviews ranged from 25 minutes (a nearby brush fire caused early evacuation of Ms. F's school one day) to approximately one hour.

Instruments

This section describes the formal interview protocols that were used within the context of each interview for the pre-interview stage, and Steps II and III of the research design. As noted above, during Step I, a teacher was simply asked to explain her in-the-moment reasoning when making a decision to modify a problem. Hence, a formal interview protocol was not necessary for that stage of data collection.

Pre-interview protocol

To reiterate, the purpose of this interview was to collect background information on each participant teacher, and to gain an understanding of each teacher's beliefs and attitudes towards teaching and learning in general, and in particular within the realm of CGI (see *Appendix A* for a copy of the protocol). As discussed above, the one-on-one clinical interview offered the researcher access to a teacher's interpretations and understandings associated with a particular subject of interest (Ginsburg, 1997), for example, her beliefs and attitudes towards the teaching of and the learning of mathematics.

Also discussed was the idea of hypothesis testing (Ginsburg, 1997) so that I spontaneously formulated and tested hypotheses throughout the course of the interview. In the pre-interview phase, this allowed me to refine my interpretations of a teacher's comments, or to clarify her statements that I deemed ambiguous, or lacking in content.

Although it was impossible to predict what each teacher would say during her pre-interview, I attempted to anticipate follow-up questions in the pre-interview protocol.

Step II: Semi-structured stimulated recall instrument

During the first part of a SSR interview (and the 3 instances of a clinical interview held prior to the lesson) I asked a list of scripted foundational questions similar to the questions found in Appendix B. These questions were designed to examine the participants thinking during the development phase of that day's problem with the intent to gather data to support answering Research Question 1. Therefore, each participant was asked a consistent set of questions during each Step II of the study. The foundational questions of each SSR interview referred to the development of the mathematical task presented in Step I of each round of the study, and covered topics such as (a) the choice of problem type, (b) the selection of numbers, (c) short and long-term learning goals for students, and (d) anticipating the students' thinking. I asked follow up clarifying and probing questions when it seemed appropriate during the SSR interview.

In addition, during each SSR interview I presented short video clips of classroom episodes to serve as memory stimuli for the teacher. I selected and edited the clips during the lapse between the lesson and interview. If the video clip was of an interaction between the teacher and a student that resulted in the teacher deciding to modify the problem, then I may or may not have interjected during the lesson to ask the teacher to explain her reasoning for the change. If I did happen to interject during the lesson, then in the SSR interview I asked the teacher to expound on her thinking for the modification. If I did not interject, I asked the teacher to explain her reasoning for the modification. In

both cases, I followed up with scripted and unscripted questions to clarify the teacher's comments, or to probe for more information regarding the teacher's statements.

Due to the semi-structured nature of the SSR interview, I prepared specific questions to ask the teacher based upon the events that occurred during that day's lesson. I composed these questions during the 3-hour or so interval between the lesson and the SSR interview while I was selecting and editing the video clips. For example, during Round 1 of data collection with Ms. T she presented a partitive division problem to her class, and she had a long interaction with a student who was explaining her answer in terms of the quantities and the context within the problem. In the episode of interest, Ms. T changed both the numbers and the context for the student to help guide the student towards an explanation. When editing this particular clip I decided to ask Ms. T how important it was to her that a child could relate her thinking back to the referents, or numbers in the problem so I included the question in that day's SSR interview protocol.

Step III: Clinical interview instrument

The clinical interview in Step III of each round of data collection provided opportunities for me to ask questions relative to the classroom events and the SSR interviews from Steps I and II respectively. Therefore, I did not have a set interview protocol for these interviews before I began the data collection process. However, as I collected and conducted preliminary analysis of the data, particular questions emerged that seemed powerful and, if possible, I asked these questions across cases. Appendix C presents an account of some of these types of questions that were asked of each participant in during Step III of the last round of data collection. As discussed above, the strategy during each step of the research design was to collect data related to both

research questions. Thus, a requirement of the clinical interview protocol during each round of data collection was to provide a list of questions that were designed to reveal data that informed answering each research question respectively.

Data Analysis

The research design of the proposed study fell within the scope of a multiple case study. As advocated by Stake (2006), the analysis was conducted on two levels: (a) within individual cases, and (b) across the three cases. Analyzing the individual cases was important because each brought unique perspectives to the data collection process while cross case analysis allowed me to combine the findings from each case into assertions that provided the foundations for building my theoretical framework (Stake, 2006). In addition, I employed a methodology for analyzing the data as it was collected during each round. The bulk of the data was transcripts of the videos of each interview, of which there were six per case, three SSR interviews (included were the three short interviews conducted prior to a lesson) and three clinical interviews. In addition, transcripts of select classroom interactions between the teachers and students were also analyzed. First, I describe the analysis of data during the data collection process, and second, I discuss the analysis of each single case study and then across the three case studies.

Analysis during data collection

The major goal in analyzing the data as it was being collected was to inform revisions of the interview protocols both within a round of data collection, and prior to the next round of data collection. For example, analysis of the data collected during the SSR interview of a round informed the development of the subsequent clinical interview

protocol for the round. Because of the limited amount of time between the interviews during a round of data collection, the analysis of an SSR interview was not conducted in a formal fashion. Rather, the video, or excerpts of the SSR interview transcript were reviewed for items of interest that seemed to warrant further investigation either for the purpose of clarification, or to examine in greater depth in the subsequent clinical interview of Step III. In a similar fashion, for a single case, analysis of the data collected across a round of data collection informed the development of the interview protocols for the next round, and so forth.

However, because of the extended timeline of data collection, preliminary analysis of the first case study (Ms. F) was done before the beginning of data collection with the next case study, that being Ms. S. The next section describes the analysis of the single case studies, and then the analysis across the cases.

Multiple case study analysis

As discussed in Chapter 1, having identified several potential factors from both the literature and the pilot study data I analyzed the data using a combination of a priori and inductive coding (Miles & Huberman, 1994), wherein the extant potential factors informed the inductive process of open coding within grounded theory (Corbin & Strauss, 1990; Strauss & Corbin, 1994; Strauss, 1987). Following the tenets of the process of open coding, I fractured the transcripts into disjoint segments, which in turn were examined, and categorized by similarities. The process was evolutionary since as more data was collected, the open coding process added more elements to be analyzed and categorized. Thus, my analysis followed the principles of the constant comparative method.

Through the constant comparative method, I refined the resulting categories of factors by evaluating emergent ideas and concepts against each other for similarities and differences (Corbin & Strauss, 1990; Strauss & Corbin, 1994; Strauss, 1987). Thus, I was able to develop and test hypotheses concerning the resulting categories of factors, and about possible relationships between and among the categories throughout the course of the investigation. Next, I describe analysis process for the first case study since this analysis informed the analysis of the second, which taken together, informed analysis of the third.

Analysis of a single case study

Single case study analysis is an intrinsic function of the multiple case analytic processes (Stake, 2006). As described earlier in this chapter, data collection for each of the three cases occurred throughout three rounds of the research design, and a grounded theory approach was employed to become familiar with each case as a separate unit. The first case study involved Ms. F and the grounded theory approach was used to analyze all of the interview transcripts as well as transcripts of select classroom interactions between Ms. F and her students.

The initial analysis of each individual case study involved dividing the transcript data into three main categories to be analyzed separately via open coding. The three segregated categories were: (a) teacher beliefs, (b) problem development, and (c) problem modification. Thus, in the case of Ms. F, I followed the process of open coding for the data corpus related to her beliefs towards the teaching of and the learning of mathematics. In a similar fashion, I coded the data corpus related to her reasoning behind developing a problem for a lesson, and then for the data related to her reasoning for

modifying problems for students. After this initial round of analysis, I examined across the three categories to look for additional patterns. Throughout this process I used a combination of a priori codes that I had identified from the literature and the pilot study, and inductive codes that emerged as the analysis progressed.

Because I completed a preliminary analysis of the first case study before I began analysis of the second case study, I had an incipient structure of a theoretical framework in place. Therefore, it was impossible to analyze the second and third case studies completely independently because I was influenced by the analysis of the initial case study, and the categories of factors that emerged from that analysis influenced the open coding process of the second case. In turn, analysis of the third case was dependent upon the analysis already performed on the first two cases. Nonetheless, the respective data from the single case studies were analyzed as a separate unit before I began to formally analysis across the three cases.

Analysis across the case studies

Once each case had been analyzed to the point that I was comfortable with the patterns that seemed to exist within each case, I began to look for patterns across the three cases. As stated above, it was inevitable that some informal cross case analysis had already influenced the analysis of the second and third cases. Thus, upon completing the analysis of the third case (Ms. T) I had a tentative structure of the final theoretical framework in terms of categories of emergent factors. Since an inherent danger for researchers performing cross-case analysis is the tendency to merge the cases too quickly, thus resulting in premature conclusions (Eisenhardt, 1989; Stake, 2006), I recoded the data corpus of each of the three cases independently using the emergent theoretical

framework. Much of this process mirrored the constant comparative method as I tested my assertions against the data in an iterative manner until I accumulated sufficient evidence from the multiple data sources to present the final theoretical framework that is discussed in Chapters 5 and 6 of this document. The final section in this chapter examines the reliability and validity of the results.

Validity and Reliability

In this section I present an argument to support the quality of the results from this investigation. Due to the structure of my research design, I present my argument concerning the issues of validity, reliability, and generalization along two strands: (a) the grounded theory approach, and (b) the multiple case study approach.

Grounded theory

In qualitative research, the notion of validity can be viewed as a means to direct attention to the credibility and appropriateness of an observation, inference, or interpretation of an episode (Maxwell, 2005). Taken in these terms, the quality of a grounded theory approach is measured directly through examining the process involved in constructing the theory, and a well-constructed grounded theory must fit the real world situation from which it emerged (Strauss & Corbin, 1990).

One measure of the fit, or validity, of a grounded theory is how well it is received by the targeted audiences (Strauss & Corbin, 1990). For this study, I conducted a type of “member check” whereby I gave the three participants a draft of their respective case study narrative to comment upon the veracity of the narrative. With the exception of a few minor corrections to be made, for example, I had incorrectly presented the number of years of teaching experience for Ms. S the narratives were well received.

The reliability of the categories that emerged during the constant comparative method was frequently discussed with the chair of the dissertation committee. He and I met periodically to discuss my emerging framework and his expertise helped shape the categories of the factors that related to answering each of the two research questions respectively. I also had conversations with other mathematics educators concerning my framework and my coding processes. These included discussions with a fellow doctoral candidate who was at a similar stage in her dissertation process. Thus, my data analysis was not conducted in a vacuum; I embraced and built upon constructive criticism from several sources.

In addition, I conducted a reliability check of my coding scheme involving seven of the categories in the final framework. I selected three excerpts of transcripts, one from each case study, that focused on the development of a problem and I gave them to the same doctoral student to code using my definitions and coding scheme. I also attached a list of the definitions of the categories coupled with a short description of each. She independently coded the excerpts and without any specific training her codes matched mine at a reliability of about 75%.

Multiple case studies and validity

Stake (2006) presented two tactics that support the validity and reliability of a multi-case study. The first is to demonstrate adequate concern for alternative interpretations and rival explanations (Stake, 1995, 2006; Yin, 1981, 2003). That is, the researcher should always assume that observed outcomes, emergent themes, or tentative assertions might be the result of other influences. During the course of this study I

viewed all of my assertions through the lens of a “critical friend”, and I compared them against rival explanations.

The second tactic that supports validity and reliability of a multi-case study is to incorporate extant literature. As Suddaby (2006) warned, grounded theory is not an excuse to ignore existing literature. Nor is it an excuse in case study analysis since an “essential feature of theory building is comparison of the emergent concepts, theory, or hypotheses with the extant literature” (Eisenhardt, 1989, p. 544). Thus, throughout the course of this dissertation study, I reviewed contemporary literature to find similar or relevant studies that could be compared and contrasted against my results.

For example, in the proposal phase of this study, I was allowed access to two chapters of a soon to be published book from which I drew upon in the literature review. Near the end of this study the book was published (Land, Drake, Sweeney, Johnson & Franke, 2014). One feature of the book that was relevant to this study were anecdotes from the elementary teachers concerning their reasons for selecting the number sets for a CGI problem type. As an exercise, I coded several of these anecdotes through the lens of my framework, and the results seemed to mirror those from the analysis of the data in this study. Essentially, my framework seemed appropriate for describing the reasoning that the teachers reported.

The next three chapters present the findings of the investigation. In Chapter 4, I present the individual case study reports. These narratives include descriptions of each participant’s teaching experience and background with professional development; their beliefs toward teaching and learning mathematics; the organization of their mathematics lessons; and the manner through which they assess their students’ understandings and

knowledge of mathematics. Chapter 5 follows with the results related to Research Question 1 and the factors involved with developing a problem for a mathematics lesson. The factors include four overarching influencing factors, and related to one of these are seven objectives that serve as benchmarks towards reaching specific learning goals. The final results chapter, Chapter 6, presents the results as they pertain to Research Question 2 and the factors considered by the participants when modifying a problem for students during a lesson. The factors in Chapter 6 compose a subset of five of the seven objectives identified in Chapter 5.

CHAPTER 4: CASE STUDY NARRATIVES

This first chapter of results introduces the three participant teachers through a series of three case study reports to provide information on each that add details to the descriptions of each that were offered in Chapter 3. First, the intent of the individual case study reports was to familiarize the reader with each teacher's backgrounds in teaching and professional development, and the orchestration of their respective mathematics lessons. Second, a description of each teacher's perspective of, and the manner through which they assess their respective students' mathematical growth is provided to gain a sense of the importance that each placed on holding an informed understanding of their students' mathematical thinking when developing, or when modifying a problem. Finally, each case study narrative provides insight into the beliefs of each participant towards the learning of, and the teaching of mathematics to afford the reader an appreciation of the participants' general attitudes and dispositions towards the practice of teaching mathematics.

The chapter is organized into four sections. The case study reports are presented in order by grade level so the report on Ms. F is first, and it is followed by the report on Ms. S, and then the report on Ms. T. The chapter concludes with a summary section that discusses similarities and differences among the case studies and the relevance of each to the study. The summary section also presents a foreshadowing of the two upcoming chapters that specifically focus on the results of this study as they pertain to the factors involved when developing a task, and when modifying a task respectively.

Case Study 1 (Ms. F): First Grade

Background

Ms. F was in her thirty second year of teaching, all of them at the first grade level except for one year with a kinder-first grade mixed class, and one year with a mixed first and second grade class. She has taught at two elementary schools, both in the same school district, and was in her eighth year at her current school. She began emphasizing CGI in her math lessons approximately 16 years ago, which she said is a direct result of her participating in a professional development program that focused on CGI and children's mathematical thinking. As she described it, prior to the professional development program she taught math in a more traditional manner whereby she demonstrated procedures and algorithms for students and then stressed practicing the same as the foundation for learning. Ms. F portrayed that she had become bored with teaching addition and subtraction in this manner so she started doing some small-group math centers, but she still emphasized procedures and algorithms in her instruction. When she heard about a group of teachers participating in a new professional development program that focused on a new way of doing math, she volunteered to participate because she wanted to "know more about math."

Her evolution to teaching via CGI started slowly, first with a monthly lesson based on a CGI task, and by the second year she was doing CGI lessons once a week. It was also during this second year that Ms. F began orchestrating her math lessons in small groups, and conducting math lessons centered on CGI tasks became her sole mathematics curriculum after she moved to her current school. It is worthwhile to describe in more

detail the professional development program in which Ms. F participated, and the climate and structure of her current mathematics classroom.

Professional development program

The PD program was lead by a professor from a university in the southwest, and Ms. F joined the program after the first year of the program. At first, the PD consisted of a group of teachers from different schools spanning grades K-6 meeting once a month for a day to watch and discuss video clips of children solving different types of problems using various strategies. Soon the participants were asked to devote one math lesson each week to solving problems similar to those in the videos, and to bring examples of student work for the group to discuss during the next PD session.

During the second year of PD the teachers were asked to video one of their math lessons for group discussion. In addition, more emphasis was placed on interviewing children in regards to their mathematical thinking, and as part of the PD each teacher made a video clip of themselves interviewing children, and presented the clip to the group for discussion. In subsequent years, as the teachers gained more experience and, as Ms. F said, they “got better,” the PD facilitators began to introduce more advanced CGI concepts such as different problem types, and ways to provide accessibility for students, and to instill mathematical rigor. Ms. F estimated that the professional development program continued for at least 12 years, but that she and some of the participants still continued to meet as a professional learning community.

Typical math lesson

As mentioned above, Ms. F structured her math lessons around five small groups of four to five students per group. She conducted math lessons on a Monday,

Wednesday, Friday schedule, and each was scheduled for approximately 75 minutes.

The focal point of a lesson was a CGI type problem with a multiple number structure.

For example, the first observed lesson during this study focused on the multiplication problem:

Lily has ____ bags of cookies. She has ____ cookies in each bag. How many cookies does Lily have altogether?

(8, 5) (7, 10) (6, 4) (19, 2) (20, 5) (7, 12) (3, 25)

In the main math station, Ms. F sat on the floor amidst a group, and launched the problem by reading it to the students with a number pair that she deemed accessible and friendly to the students. The students typically worked the problem with that number set first, and continued working at their own pace with number sets of their choosing from the provided list. Ms. F used this time to work with each student one-to-one questioning them concerning their strategies, pressing them for explanations, suggesting different approaches or tools, and offering other scaffolding moves, which included modifying the task by changing context or numbers. For example, the following is a transcript from the aforementioned lesson with Ms. F working with a student, Tammy, who had just worked with the number pair, (7, 10). Ms. F quickly changed the number of 10s for Tammy in a series of questions:

- 1) Ms. F: What did you do?
- 2) Tammy: I just used my hands. Like, 10, 20, 30, 40, 50, 60, 70.
- 3) Ms. F: Tammy, can I ask a quick question? What are two 10s?
- 4) Tammy: Two 10s? Twenty.
- 5) Ms. F: What are three 10s?
- 6) Tammy: *pause* Thirty.

- 7) Ms. F: Okay, so you're still counting by 10s. You don't know it automatically
- 8) yet. If I said, what are six 10s, what would you say?
- 9) Tammy: *quickly* Sixty.
- 10) Ms. F: Yeah, you trusted yourself. What are five 10s?
- 11) Tammy: Fifty.

After 15-20 minutes working at the main station, Ms. F would ring a bell and the groups rotated between stations. The group she had just worked with moved to a less cognitively challenging station that typically involved building different geometrical designs with plastic figures, or on a geo-board, and other groups rotated to stations that included: (a) games designed to build place value understandings like Make a Ten, or Quarkle® (geometric attributes); (b) games designed to build social skills like Monopoly® and Sorry®; and (c) a program on iPads called ST Math®, an instructional software designed to increase mathematical comprehension and proficiency. Generally, a student would visit each station during a class period. Sometimes the sessions at the main math station ran long so a group might not have worked with Ms. F that day, but she would work with them first the following lesson with the same problem.

Assessments of students' learning

Ms. F seemed to assess students' learning by noting the increase in the sophistication of strategies they employed during problem solving over time. At the beginning of a school year Ms. F said that she would use a particular problem to assess her new students' mathematical thinking. The problem was a Part-Part-Part-Whole, Whole Unknown with a consistent multiple number choice structure:

There are ____ kids on the play structure, ____ on the grass, and ____ on the blacktop. How many kids are there all together?

(5, 5, 5) (10, 10, 10) (9, 7, 3) (50, 100, 50) (25, 25, 25)

With this problem, said Ms. F, she could assess students for their number sense, specifically their ability to count by 5s, count by 10s, and their ability to put ‘ten’s friends,’ the 7 and 3 together in the third number set, before adding the 9. She also said that she learned whether the students knew that two 50s make 100, and whether they were able to count by 25s. If a child was to use a direct modeling strategy, Ms. F might infer that the child had not had a lot of practice with problem solving. If a child was to select larger numbers and give her immediate answers, she gained insight into the child’s ability to manipulate numbers; although, as she pointed out, “it doesn’t necessarily mean that they’ve been working with CGI, but I can assess where they are.” Furthermore, whenever a new student transferred into her class, Ms. F also used this problem as the baseline diagnostic instrument of the student’s understandings.

To gauge students’ development over the course of a year she kept artifacts of their work, and she would occasionally add anecdotal notes regarding a student’s thinking. For example, a student might have begun the year using a direct modeling strategy and counting by 1s, and later when that same student demonstrated the ability to count by 10s, Ms. F might have notated the milestone on the student’s work that day. In kind, Ms. F employed a unique diagnostic whereby she used the same number sets and problem type (but set in a different context) at different points throughout the school. In this manner, she could pull children’s work to find “where they were then, and at where they are now.” For example, she hoped to see growth in terms of a child moving from a direct modeling strategy to using mental strategies, and known facts like 10 plus 5 equals

15, or a child “tackling numbers that they wouldn’t before.” It was interesting to note that whereas in the past the District employed an end of year standardized mathematics test as the summative assessment for student learning, the practice had been discontinued, and for Ms. F the summative assessment lay in the evidence she had collected of the students’ growing mathematical understandings and knowledge.

Beliefs towards learning and teaching

Ms. F’s teaching practices seemed predicated on the belief that children learn mathematics by doing mathematics, which in her classroom entailed solving problems, struggling with concepts and ideas, working with others, and explaining one’s thinking. To Ms. F, each child was different, and the weight of learning rested on the student. For example, when talking about teaching place value, she said “I don’t [believe] you can successfully teach place value. I think you have to own it, and learn it yourself.” However, she did not seem to believe that a child could learn in a vacuum. As she said, “...and the kids learn so much from each other,” and that her role as the teacher was to “nudge them a little farther from what you see that they already know.” And that in a CGI classroom this was “where you take a child,... and ask ‘em what they know, and build on their own knowledge.” Thus, to Ms. F, teaching seemed to be a cycle of assessing a child’s mathematical thinking, which would lead to instructional moves to build upon the child’s understandings, and would, in turn, inform a restructuring of Ms. F’s model of the child’s knowledge and understandings, hence, renewing the process.

Ms. F also seemed to believe that students should gain conceptual understanding before learning procedures. As she put it:

In first grade I don't want to teach procedures because what happens is the kids internalize the procedure, and forget what they're really doing. And then when they get farther along in those procedures then you see mistakes. In third grade, especially. They can learn the procedures easy in second grade, but do they still understand the place value when they have to go back and start doing expanded notation, and really understanding the concept of place value.

Thus, in her classroom, Ms. F seemed to ground her teaching practice in supporting children in learning concepts such as place value before introducing them to procedures like the traditional algorithms for addition and subtraction. To adopt the framework of Collier (1972) and others (e.g., Roscoe & Sriraman, 2011; Seaman, Szydlik, Szydlik, & Beam, 2005) Ms. F could be characterized as holding *informal* beliefs towards mathematics and teaching mathematics. Informal beliefs of mathematics can be summarized as the viewpoint that investigating mathematics is a creative process that entails actively reasoning through problem solving and exploring conjectures, while ideas are to be communicated and discussed.

In contrast a *formal* belief of mathematics holds procedures, rules and formulas in high regard, and the perspective that memorization of facts and algorithms is the equivalent of mathematical understanding. Along the same lines, informal beliefs of teaching mathematics reinforce student-centered activities to promote students' restructuring their unique mathematical understandings and knowledge, whereas a formal approach to teaching mathematics consists of focusing on teacher centered instruction and the transmission of facts and procedures (Roscoe & Sriraman, 2011). In this regard, Ms. F could be described as invoking her informal beliefs towards teaching mathematics in her classroom.

Case Study 2 (Ms. S): Second Grade

Background

Ms. S was in her sixteenth year of teaching. Over the years she had taught at the kindergarten, first, and second grade levels, and at the time of this study, she was teaching second grade. She had taught at three different schools, in two school districts, this being her second year at her current school. Five years ago she switched to a job share, which is a contract split between two teachers, and she currently teaches Mondays, Tuesdays, and some Wednesdays. On Thursdays and Fridays she frequently conducted staff development sessions that focused on topics in CGI for others in her school district.

During her third year of teaching, Ms. S began what she described as a “really slow process” towards emphasizing CGI in her classroom. Her story mirrored those of Ms. F, and Ms. T, as she learned from a colleague at her district of the same professional development project being lead by a professor at a nearby university, and she decided to visit a session. She recalled seeing videos of children solving two and three-digit subtraction story problems using invented algorithms, which was enlightening to her because she had never before seen children use such strategies. At the time, she was teaching second grade using a more traditional math program, and since her students were struggling with the concept of regrouping, she decided to sign up and learn more about CGI and children’s mathematical thinking. She estimated that it took her about three years to gain enough confidence to gradually shift her practice away from using traditional resources such as MathLand® to grounding her math lessons and instruction completely in CGI. A pivotal moment in the evolution of her practice, she explained, occurred at the end of the second year when her students performed very well on the

District's standardized end of year math assessment, and she was granted full administrative support to continue emphasizing CGI.

Professional development program

As mentioned, Ms. S attended the same professional development program for four years as Ms. F and Ms. T, but she began the program under the guidance of a different lead facilitator, and alongside a different group of teachers. She recalled that the facilitator "would give us homework, and so the first year I would come back, I would try the homework, I would see where it would go, and that would sort of inspire me to try some more problems." In addition, her district had a requirement that she had to find somebody else at her school with whom to attend the PD sessions so she teamed up with a good friend who happened to teach first grade. Ms. S stressed the importance of working in tandem as they would bounce ideas off of each other, brainstorm to write problems, and because they taught different grade levels, they were able to observe and discuss a wide spectrum of strategies.

Ms. S is somewhat unique in that, aside from being a practicing teacher, she also facilitated professional development in CGI, and had done so, on and off, for the past 10 years. These experiences afforded Ms. S both personal and professional growth. She felt that the process of preparing to facilitate PD sessions made her revisit CGI concepts in more depth, and to view activities through a different lens. For instance, she reflected that when she was a participant, her focus was on children's thinking and her own classroom, but as a PD facilitator she focused on the teachers, and how better to assist them in understanding children's mathematical thinking. In essence, she articulated that

preparing for, and implementing a professional development session foregrounded issues that she may not have been in tune with when she was just a participant in the PD.

Typical math lesson

Ms. S conducted her math lessons using a mixture of whole group and small groups. She launched a typical lesson by showing the problems of the day on the overhead projector (usually two, or three with the first considered the main problem), and read them to the whole group as they sat on the floor in front of her. She did this to unpack any unfamiliar vocabulary, or context, and to assist those children that might have had difficulty reading a problem. Interestingly, there was a delicate approach for the numbers she selected to insert when reading a problem. She said she wanted a problem to be accessible, but not easily solvable. If the numbers were too large, or too difficult, some of the kids might not be able to access the problem. If the numbers were too easy, chances are good that someone would quickly solve the problem and blurt out the answer. Thus, at times, she said that she might read a problem with number sets that were not given as part of the multiple number choice structure. After reading a problem, she would follow with questions designed to help the students gain an entry point into the problem. For example, she might ask, “What do we know in the problem, and what do we need to figure out?” The launch usually took 10 to 15 minutes.

After the launch, the children dispersed to their desks that were arranged such that they formed a ‘table’ of four to six desks. These arrangements formed the small groups, of which there were four, with four to six students in each group. Each student was responsible for solving all of the problems, and many were left to work on their own, or with a partner. Ms. S worked with the students at a predetermined table, whatever group

happened to fall within the four-group rotation that particular day. Because she rotated which group she would work with, she was able to interact periodically with each individual in the class.

When Ms. S worked with a small group, she would spend time working one-on-one with each student, or perhaps she might pair them to have them help each other while she worked with someone else. A typical interaction with a student seemed to include Ms. S pressing the student to explain her thinking regarding the strategy involved. This type of interaction is illustrated in the following excerpt from a lesson with a Part-Part-Part-Whole, Whole Unknown problem where the student, Lacy, was adding 23 bumpy, 17 shiny, and 27 smooth rocks. At the end of the interaction, Ms. S offered the student a new set of numbers to work with within the context of the problem. Gestures are indicated by italicized print within a set of brackets, for example, [*counted on fingers*].

- 1) Lacy: So, and add all this together.
- 2) Ms. S: You're going to add it all together. Okay.
- 3) Lacy: And this equals forty.
- 4) Ms. S: Gotcha.
- 5) Lacy: And then fifty. Fifty-seven...58, 59, 60, 61, 62, 63, 64, 65, 66, 67
- 6) [*student counted on fingers*]
- 7) Ms. S: Sixty seven, okay, good.
- 8) ...Ms. S: And you got 67. Fantastic. So what is 60? Let's just oh, okay, thank
- 9) you. What is 67 plus 10? Just curious.
- 10) Lacy: Um, 70, 77.
- 11) Ms. S: Seventy-seven? How did you know 77 like that this time?

- 12) Lacy: Because, 3 more would equal 10? And then you still have to add 7 more.
- 13) Ms. S: And you still have to add 7 more. So are you breaking, you're breaking
- 14) the 10 up into a 3 and a 7? Okay. Very, very smart. So let me see, let's do this,
- 15) for this one, will you please do 47 bumpy, 38 shiny, and 22 smooth? Usually,
- 16) Lacy, you've been using the tools but this time you didn't. So, I'm wondering if
- 17) you could do that same thing, but with these numbers. Okay?

After working with Lacy, Ms. S moved to another student and continued interacting in such a manner for approximately 30 minutes, which seemed to be the norm for a lesson. When the students at the other three tables were done working the problems, they were assigned one of three activities. They were to work in their counting journals, work in their number building journals, or they were to pair up to play a game that focuses on building number sense skills.

When she finished with the students at the main table, Ms. S regrouped the entire class for a share out session of approximately 30 minutes. She selected two students that she had worked with that day to share because “what they did in the strategy that was really, really clever, or it took a lot of hard work.” In her words, the share out sessions were a chance for the students to celebrate:

So we, they share and then the other children, we reflect on it in some way, and that's often different—sometimes it's comparing the strategy, sometimes it's talking about how that strategy was the same as your strategy or different from your strategy. Sometimes the, the children have a compliment for their, for their peer, or a question, or a suggestion. Um, and then, you know, and that becomes kinda celebratory.

What follows is an example of a student, Archie, explaining his work for solving a Join-Change-Unknown problem wherein a person has read 34 pages of a book

containing 148 pages and he had to find the number of pages still to be read. Figure 4.1 shows he used an incrementing strategy in tandem with what Ms. S called a “sort of number line.”

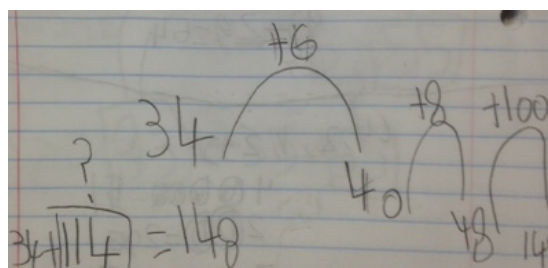


Figure 4.1: Archie's invented number line strategy

The excerpt provided a glimpse of how Ms. S orchestrated a share out session.

- 1) Ms. S: Okay, so you have 6, 8, and 100 across the top. Right?
- 2) Archie: Yes.
- 3) Ms. S: Okay.
- 4) Archie: So, I added 100 plus 8.
- 5) Ms. S: Okay.
- 6) Archie: And 100 plus 8 equals one hundred and eight.
- 7) Ms. S: Equals one hundred and eight. Okay, and then what did you add after
- 8) that?
- 8) Archie: The 6.
- 9) Ms. S: Okay. So what's 108 plus 6? I think you did it in your head when we
- 10) were talking. We'll talk about the number, the hundreds chart in just a
- 11) second. So you said that the hundred and eight plus the 6.
- 12) Archie: Plus the 2 in 6.
- 13) Ms. S: Ooooooh.

- 14) Archie: Is one hundred and ten. Plus the 4, the leftover, is 114.
- 15) Ms. S: Can you say that again cause it's really important and I want Leslie to
- 16) hear it. Okay? So he's trying to add a 6 to one hundred and eight, and that's kind
- 17) of tricky.

In this episode, Ms. S elicited from Archie his thinking in a step-by-step manner, and she also took care to revoice his comments (line 7), and to add clarity to his statements (line 16 - 17). Afterwards, she engaged the rest of the class in discussing Archie's strategy, in particular, his breaking of the 6 into a 2 and a 4 to make, to him, the addition easier.

Assessments of students' learning

Ms. S employed both formal and informal diagnostics to assess her students' learning of mathematics. The principle formal assessments were standardized tests that were developed within the District as part of the District wide shift to a CGI approach to mathematics lesson. The assessments were to be administered during each of the academic year trimesters. Furthermore, the trimester exams were linked to a set of *Proficiency Level Descriptors*, also developed by District personnel, and which, in turn, were linked to the Common Core State Standards. Below is a description of one of the Proficiency Level Descriptors, and the manner in which the students were to be assessed relative to the descriptor. To foreshadow a later section in this chapter, the Proficiency Level Descriptors were important factors that Ms. S considered during the design phase of a task. It was worth noting the difference between the district in which Ms. F and Ms. T taught during this study, and that in which Ms. S taught. Whereas in the former,

student assessment was an informal process, this was not the case for Ms. S since a standardized, formal process was part of her students' assessment.

Each of the Proficiency Level Descriptors was linked to a specific mathematical Domain, for example, the Domain of Operations and Algebraic Thinking, which, in turn, happened to contain three Report Card Line items. The following illustrates the descriptors for the Report Card Line Item: Represents and solves word problems involving addition and subtraction within 100. This item was linked to the CCSS Grade 2 standard, 2.OA.1 (CCSSM, 2010). Included in this descriptor were six levels of proficiency that spanned the academic year, and served as benchmarks for the District. For example, Level 3 was titled "Secure in Trimester 1" with the following two descriptors:

- a) Student uses addition and subtraction within 100 to solve one-step word problems involving JRU, SRU, PPW/Both Addends Unknown. (*Note:* The problems were Join Result Unknown, Separate Result Unknown, and Part-Part-Whole, Whole Unknown)
- b) Student uses a number line diagram to find sums and differences within 100.

To illustrate the District objectives for students' mathematical growth over time, it is helpful to also look at Level 5, "Secure in Trimester 3":

- a) Student uses addition and subtraction within 100 to solve one and two-step word problems involving all problem types.
- b) Student can write an equation that follows their strategy and one that follows the word problem.

As the year progressed the students' growth was to be measured by their ability to use the operations of addition and subtraction within 100, their ability to use the two operations within more sophisticated problem types, and their ability to decontextualize story problems and their strategies. The secure descriptors were assessed by items on the

respective trimester paper and pencil exams, which included Benchmark Assessments (or, performance tasks), and Teacher-designed unit assessments. The Benchmark Assessment items were multiple choice questions designed to mirror those of almost any national standardized assessment, while the Teacher designed unit assessments were word problems written by each teacher from the list of problems found within the secure descriptors. The students were allowed to solve any of the assessment tasks using whatever strategy suits them, and they were also allowed to use tools, and manipulatives like Base-10 blocks. As a formal diagnostic, the exams served as a summative assessment for the respective trimester.

Each Proficiency Level Descriptor also contained a list of informal Formative Assessment Opportunities, which for the aforementioned Report Card line item were: (a) Math Talks, (b) Math Journals, (c) Center Activities, (d) Teacher Observation, and (e) Problem Types: All Problem Types. The informal diagnostics that Ms. S used to measure her students' mathematical growth seemed to be embedded within this list of formative assessments. For example, under the category Math Journals, she used what she called "anecdotal records," or charts that tracked the strategies that each student used to solve a particular problem. She tried to do this once or twice each week by reviewing each child's math journal, and marking on the chart the strategies used, and she might add a note as to what the child needed to work on, or what the child might be ready to explore. For example, the line item for a child solving a two-step Separate Result Unknown problem had check marks under the headings Direct Models By Tens, and Invented Algorithms: Combining Same Units, with an additional note that this child was "*ready to explore i. a.*" that is, the child was ready to explore Invented Algorithms.

Another example of informal assessment fell under the category of Teacher Observation. Here, Ms. S assessed a student's growth based on her interpretation of (a) her interactions with the child, and (b) her interpretation of the child's level of participation during the share out portion of lessons. She explained how each interaction with her students acted as a formative assessment of their understandings:

So, because I'm constantly talking with children, and I'm constantly looking at their work, and building my instruction from where they are? I feel like I have a really good understanding of where my children are in their understanding. So, I look at their, I have a couple of formative, in terms of formative assessment? I'm sort of always refining what I'm doing with my kids based on where they were that day.

Similarly, she evaluated their growth during a share out session by being "really mindful of how many children are engaged in the sharing." For example:

...it might be that I have a child that was able to repeat back another child's strategies really, strategy really articulately when they hadn't been able to really do that before. Or, I have, you know, they were able to talk to their elbow buddy about how their strategy was different from the strategy that they're seeing in the sharing. And maybe that's a measurement of growth.

Thus, her evaluation of a share out session went beyond a simple count of who participated. Ms. S also appeared to make an informal qualitative assessment of the mathematical language communicated by students during the session, which, in turn, contributed to her metric of student growth. That is, it seemed that to Ms. S, children also demonstrated growth in their mathematical understandings as they became more articulate speaking within the *mathematical register*, characterized by the intrinsic terminology, vocabulary, and array of literal symbols inseparable from the field of

mathematics (Gee et al., 1992; Wells, 1996). To her, it wasn't enough that the children participated; she also assigned value to the perceived level of articulation they displayed.

Beliefs towards teaching and learning

Ms. S seemed to believe that children “learn math by...doing and thinking about problems...by making connections,” and that “...students best learn math in context.” She believed that math is not about finding the answer; rather, “that math is about thinking about and visualizing, conceptualizing a problem. And that math should make sense.” She made an interesting analogy to reading in that students learn to read through an organic process whereby they have cueing systems that all work together. She viewed learning math in the same fashion:

Math is sort of the same idea, that you, you have a lot of understanding of math on your own, and you start developing your strategies, and through refinement and conversation, and practice, and talking with your peers, you grow in those strategies.

Succinctly, it appeared Ms. S believed that students learn mathematics by independently solving story problems, and by participating in mathematical discussions. This perspective towards learning seemed in concert with her informal assessments of students' mathematical growth as described in the previous section. It seemed that to Ms. S, mathematical learning was demonstrated by the sophistication of a child's problem solving strategies along with the child's exhibited mathematical understandings, and the level of the child's participation during mathematical discussions.

The classroom practices of Ms. S were grounded in what seemed to be an informal belief system towards the learning of and the teaching of mathematics. Her lessons focused on problem solving rather than procedures, and her practice included

one-on-one instruction that was unique to each child. She valued children explaining their thinking, and she responded to them based on her interpretation of their understandings. When asked about the emphasis that she placed on students explaining their thinking, she offered:

A ton. Yeah, I mean in terms of what they're working with me, I try to put a lot of emphasis on it. I try to start where they are, understand what they were doing so I can understand what their next steps might be. What they're not understanding, what they are understanding, and so, I can be responsive in the way I speak with them.

The importance that Ms. S placed on students' explaining their thinking also influenced how she conducted the whole group share out sessions.

It was observed that during the share out sessions Ms. S did not lecture to the class; that did not appear to be her role. Whereas a more traditional mathematics teacher might have assumed the role as the transmitter of knowledge, Ms. S appeared to view herself as the facilitator of ideas that emerged from the students to be shared communally. In her words, "And then the students are sharing the story problems, and they're teaching each other through their sharing. And I'm sort of facilitating that discussion." She was also observed assuming this role when she paired students in a small group to work together on solving problem, and taking turns explicating their thinking to each other.

Case Study 3 (Ms. T): Third Grade

Background

At the time of the study, Ms. T was in her 18th year of teaching. She had mainly taught second and third grades, but had a combination first and second grade class one year. Her teaching career spanned two elementary schools, both in the same district, and

this was her eighth year at the second school where she has always taught third grade. The story of her shift in practice towards emphasizing CGI in her classroom was very similar to that of Ms. F's, which was not surprising since they have taught at the same schools for the extent of Ms. T's career, and they could be described as being close colleagues and friends. In her first year of teaching Ms. T became aware of the same math study group in which Ms. F was a participant, and Ms. T believed she joined the group in her second, or third year of teaching. She did say that in the year prior to joining the group she was invited to observe Ms. F in her classroom, which resulted in Ms. T exploring teaching through CGI before she joined the study group. During that period, other teachers in the group gave Ms. T problems to use for her math lessons.

Ms. T began emphasizing CGI in her classroom during her second year of teaching; thus, she only taught out of a textbook for a single year. She still continued to work with a textbook, but she used it to supplement CGI problem solving, which Ms. T considered to be the core of her mathematics instruction. Ms. T offered an interesting anecdote about being nervous her first year of emphasizing CGI and spending less time on the textbook knowing that the children were to be tested at the end of the year. When her students did as good as or better than they had in the past, she just stopped worrying, and she bought into the idea of CGI completely because it felt very natural to her.

Professional development program

Once Ms. T joined the professional development program and became a full participant in the math study group, her experiences paralleled those of Ms. F so it would be redundant to provide many of the details here. However, Ms. T did add some finer points to the group's practice of sharing student interview videos. As Ms. T recalled, the

teacher whose video was presented would ask the group to deliberate over difficulties that the teacher encountered during the interview. For instance, the teacher might have had difficulty helping the child get started, or the child may have solved a task very quickly, and the teacher did not know how to respond to the child with a challenging follow up problem. The other group members would project themselves into the situation and discuss and debate how they might have responded to the child. The same type of discussion would occur the following meeting, but concentrated on another participant's video selection.

Typical math lesson

Ms. T conducted her math lessons in a manner similar to Ms. F, and she acknowledged that Ms. F's classroom orchestration served as the model for her own. However, there were differences. For instance, Ms. T typically began a math lesson by demonstrating a principle or concept on the board, and then she would facilitate a whole class discussion on the topic. During one lesson she was observed presenting the standard algorithm for addition, and emphasizing place value concepts and vocabulary with the students. In addition, as mentioned above, she included the use of a textbook as a supplement to instruction, and as a resource for students.

Normally, Ms. T conducted math lessons four days per week, and each lesson ran for approximately 90 minutes. Like Ms. F, the structure of Ms. T's classroom consisted of a main math station, where she worked on CGI problem solving with a small group of 4 or 5 students for approximately 25 minutes, and four other stations that involved various mathematical topics. For example, in one observed lesson, the problem was the multiplication problem with a multiple number structure:

The students at FCC collected ____ toys for the Toys for Tots program. If there are ____ families with children in the hospital, who need help with toys this season, how many toys can each family receive?
 (30, 6) (27, 9) (100, 25) (1000, 100) (4690, 2345) (555, 111)
 (120, 15) (99, 11) (1,296; 3) (4,590; 90) (8,400; 200) (126, 19)

Over the course of two consecutive lessons the students rotated with their small groups among the stations so that each student was insured of working with Ms. T on each CGI task. While working with a group at the main station, Ms. T would ask a student questions, suggest the use of a tool, or have one student explain her work to another student. The following transcript is from the lesson with the above problem. The student had been working on the problem with the number pair, (100, 25):

- 1) Ms. T: Do it with dots. See what I mean when you're looking at these numbers?
- 2) They were getting kinda big? So what if I give you one, see if you can use a
- 3) multiplication fact? Like you did here. To solve it. Okay, so what if I had, um,
- 4) 40 families, and no 40 presents and 5 families.
- 5) Student: And you can just count it by 5s. And see...
- 6) Ms. T: Okay. Uh huh. And that would be your multiplication fact. Right? Okay.

Usually, the other math stations were devoted to activities such as working on specific pages in the textbook (for practice with procedural fluency, and other problem types, or topics), math games such as 24®, the STMath® software program on iPads, and a predetermined exercise that was usually overseen by a parent volunteer. An example of a parent led activity was found in the attentiveness Ms. T gave to fact fluency over the second half of the school year. According to Ms. T, after January she designed homework to practice facts like multiplication through 12, and every week she would have a parent come in to test the students.

Assessments of students' learning

Ms. T measured students' learning by their individual growth, and to Ms. T, the idea of 'growth' had different connotations. For example, many students that entered her class had to adjust to and learn the problem solving process because they were used to being told what to do, and they tended to get frustrated and distraught at the beginning of the year. For these students, Ms. T measured success by a growth in their confidence.

As she said:

So, when you have a kid like that and they come to the end and they sit down and they just get started and they, they're able to approach a problem that they have no idea how to do, but with the confidence that they'll figure it out? That's a huge success for that kid.

She also seemed to assess each student's learning by the growth in the sophistication of the student's problem solving strategies. During the school year in which this study was situated, each student was keeping a problem-solving journal wherein they would record their work during their time spent at the main math station, and Ms. T would look back through each student's journal to gauge the progression of higher levels of thinking. She also admitted that she is constantly evaluating students in her head, and that she could immediately, from memory, provide a list of students that still tended to use direct modeling strategies as well as those students that were using invented algorithms.

When asked how she assessed a child's mathematical thinking and knowledge, she answered simply, "Sit down with them and do a problem." Although she acknowledged that different issues arose with different problems, she said that she could quickly get a good idea of what a child did and did not understand, for instance, his or her understandings of place value, or the level of confidence the child possessed. As for

standardized assessments of student learning, the District used to have benchmark assessments, but no longer subscribed to these measures. Hence, District teachers were left to develop their own benchmarks, and at Ms. T's school she and other teachers were considering developing benchmark exams that aligned with the Smarter Balance® test, or it's equivalent. However, these had yet to be developed within the time frame of this study, so Ms. T's students would be given the Smarter Balance® test as an end of the year assessment for the District. However, for Ms. T, it seemed better left to her words to describe her personal end of the year assessment of students' growth:

“You know, so it's individual for every kid. Like, did they overcome the hurdles that they needed? To overcome, did they build the understandings that they needed to build? Did they have growth? Do they have confidence? Did they come out with a positive attitude?”

Beliefs towards learning and teaching

The beliefs of Ms. T towards the learning of and the teaching of mathematics could be characterized that she held an informal belief structure towards both (e.g, Collier, 1972). It seemed that to Ms. T, children gained mathematical understandings by facing challenges, and overcoming cognitive hurdles; thus, responsibility lay with the teacher to “push them to the next level of understanding.” As she described it, children are only going to have confidence in their abilities if the mathematics makes sense to them, and they won't learn the math if it doesn't make sense to them. As she said:

You know, so that it makes sense to them. They're only gonna have confidence if it makes sense to them. Like, I can tell them a hundred times, they're not gonna learn it if it doesn't make sense to them.

To Ms. T, the CGI approach to mathematics seemed to provide a context for understanding and thinking. Ms. T expected her students to find an entry point to a

problem, develop their own strategies for solving the problem, and then to be forthcoming in explaining their thinking. She contrasted this against a more formal belief towards learning mathematics, wherein emphasis is placed on memorizing facts and procedures, through an analogy of learning to read: “It’s like sight words in reading. You know if we taught reading and language the way we teach math, all we would teach is sight words, and we’d never let kids read a book.”

Ms. T’s instructional practices seemed influenced and guided by her beliefs as to how children learn mathematics. For example, when she designed a problem, she wanted the numbers to be accessible to the students, but not so straightforward that the problem could be solved using a memorized fact because, to her, this “stops the thinking.” Ms. T accepted that fact fluency was an important skill to possess, as she said, “...but when they get into fourth grade I know that they need to have those facts because now they’re doing long division with huge numbers.” But it seemed that she first wanted her students to grapple with a problem:

I could’ve come up with, like, 10 divided by 2, or 10 divided by 5, or something that clearly is a memorized fact. But then it stops the thinking cause they know it, it’s a memorized fact. And they’re just gonna tell you. So it needs to be more than a memorized fact, but it needs to be accessible.

Finally, Ms. T suggested that she believed in employing the Socratic method to inspire and motivate her students’ mathematical thinking. She talked of how this became ingrained in her mind growing up when her father, a lawyer, would lead stimulating and thoughtful discussions at the table. Again, in contrast to a more formal belief system towards teaching mathematics, Ms. T said she did not like to “tell.” She preferred to ask students questions and to have conversations with them “and to be able to talk with them

and to actually have that immediate feedback on what they know, and how they know it.”

In addition, at times she would have students work together. For example, when two boys worked on the same problem but reached different answers, Ms. T did not tell one that he was wrong, and the other he was right; rather, she had them discuss and debate as to whom was correct.

Summary

There were many similarities among the three participants that taken together contributed to the emergent story appertaining to the factors that were involved in developing a problem for a mathematics lesson and the factors that were considered by the teachers when they modified a problem for a student during a lesson. For example, the participants’ extensive training in the principles of CGI was the source of their understandings of the CGI problem types and the perspectives and strategies that children might use to approach the different problem types, and they seemed to draw on this knowledge base when composing a CGI problem. The coupling of their training and their many years of experience teaching via CGI seemed to form the basis for their continuous assessment of their student’s mathematical knowledge and understandings, a driving force behind writing a problem as well as when modifying a problem.

In addition their common system of informal beliefs towards the learning of and the teaching of mathematics was demonstrated by the common teaching practices of focusing lessons on problem solving and eliciting students’ mathematical thinking, both of which seemed instrumental for developing and modifying problems. Moreover, each teacher used the classroom strategy of dividing students into small groups to gain favorable circumstances for interacting with students on a one-to-one basis. This practice

offered the teachers singular opportunities for modifying a problem based on interactions with individual students.

However, the case studies also pointed out differences between the teachers that also contributed to the findings of this study. For example, it was demonstrated that Ms. F and Ms. T were not bound to a set of district standards in the same manner as was Ms. S. Therefore, in the next chapter it will be reported that Ms. S paid much more attention to the goals and objectives listed in the standards of her district when developing a problem for a lesson than did either Ms. F, or Ms. T. This was also the case for the differences presented above in the formal assessments of students that were more constraining for Ms. S than for Ms. F and Ms. T. Essentially, the formalized standards and assessments of her district afforded Ms. S a set of guidelines from which she could backwards plan to assist her with writing a problem for a lesson.

The next chapter is devoted to the results that emerged from the analysis of the data associated with the development of a problem for a lesson. To portend the results, the analysis suggested that the factors the teachers appeared to consider when developing a problem could be categorized as (a) an *influencing factor*, or (b) as a *design-objective*. An influencing factor was defined to be a broad overarching consideration within the process of developing a task, one not be easily measured, or obtainable, and one that did not seem assignable to a specific role in the development of a task process. For example, two long-term learning goals held by the teachers emerged and they were categorized under the influencing factor, learning goals, because a learning goal is expansive, and not easily reached. For example, one of two identified learning goals was for, students to gain understanding of place value. Understanding place value is a very broad

mathematical concept, and the idea is not easily measured; that is, it is difficult to determine when a student actually achieves the goal completely understanding place value.

In contrast, a design-objective (of which there were seven) was defined to be a specific, measurable benchmark towards reaching a learning goal that could also be linked to a specific means through which the objective might be achieved. For example, an objective, or benchmark, towards understanding place value would be to achieve an understanding of regrouping ones into tens. Thus, the design-objective of foregrounding a specific mathematical concept such as regrouping ones into tens is measurable--a student can demonstrate the capacity to do so--and it can be linked to a specific means for working towards the objective, for instance, presenting students with the numbers, 27 and 8, to add within the context of a Join-Change-Unknown problem. The structure of the number choices, 27 and 8, relative to the operation of addition (i.e., adding 7 ones and 8 ones), invites students to regroup 15 ones into 1 ten and 5 ones, thus, providing the means for moving students towards understanding regrouping across the tens place. The ideas of influencing factors and design-objectives with associated means are discussed in detail in Chapter 5.

The final results chapter, Chapter 6, is devoted to the results that emerged from the analysis of the data associated with modifying a problem for students during a lesson. Understandably, the results illustrated direct linkages to the seven emergent design-objectives considered when developing a task since any modification to a problem could only be done after the problem had been developed. In chapter 6, it will be demonstrated

that the five *mod-objectives* considered when modifying a problem were a subset of the seven design-objectives presented in Chapter 5.

CHAPTER 5: FACTORS CONSIDERED WHEN DEVELOPING A MATHEMATICAL TASK

In this chapter, I present my findings from my analysis of the interview data, which addressed the first research question:

RQ1) What factors do elementary teachers who have an extensive understanding of children's mathematical thinking consider when developing mathematical tasks in the design phase of mathematics lessons?

I offer evidence to support the claim that there were four overarching factors that seemed to influence the development of a task, and I categorized these as *influencing factors*.

The four identified *influencing factors* were: (a) problem library, the collection of problems that a participant kept for use as templates, or references, (b) academic calendar, the timing in which a lesson occurred relative to the school year, (c) teaching practices, which refer to a set of identified practices related to the process of developing a task influential; and (d) learning goals, specifically, for students to gain an understanding of place value, and to show growth in sophistication in problem solving strategies.

The emergent influencing factors seemed pertinent within the process of developing a task, but they appeared to serve a supporting role in the process rather than a specific purpose. For example, a teacher might consider the timing of a lesson relative to the academic calendar to guide her to select a particular problem type, and it might influence the magnitude of the numbers she includes. For example, all of the participants introduced multiplication problems early in the year with smaller more friendly numbers for students. Yet, the specific number choices within a problem seemed to originate from other factors involved in the process, and these were categorized as *design-objectives*.

Furthermore, the seven design-objectives appeared to serve as benchmarks towards reaching one, or both, of the aforementioned learning goals. Figure 5.1 illustrates the hierarchy of the influencing factors and the design objectives, and Table 5.1 delineates the seven categories of design-objectives.

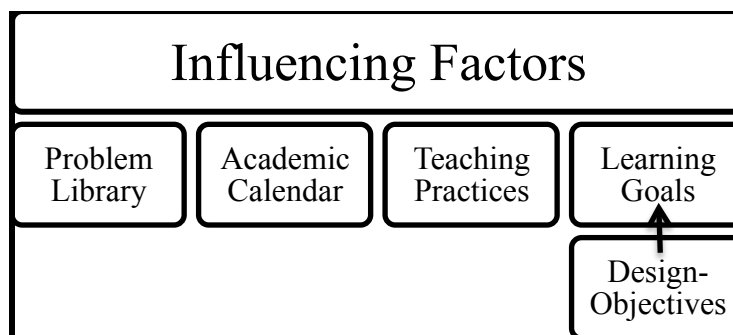


Figure 5.1: Hierarchy of influencing factors and design-objectives

Table 5.1: Definitions of identified design-objectives

Design-Objective	Definition
1) Assess Students' Thinking	The intent to assess students' mathematical thinking and understandings.
2) Promote/Reinforce Strategy	The intent to support the development of a particular strategy(s)
3) Provide Mathematical Access/Rigor	The intent to match the problem to a variety of students' mathematical understandings.
4) Problem Makes Sense	The intent to insure that students can make sense of the task
5) Foreground Mathematical Concept	The intent to focus on a specific mathematical concept, or topic
6) Meet District Standards	The intent to link the problem to a particular content standard
7) Promote Fact Fluency	The intent to support the development of fact fluency within the basic arithmetic operations.

Goals/objectives/means triad

In this document, I use the terms goal and objective in a manner whereby I suggest the two words are not exactly synonymous. Therefore, before I begin to present evidence for the claims made above, it behooves me to insert a short exposition to clarify my conceptions of a goal and objective, as well as the idea of a *means*. To define my interpretations of (a) a goal, (b) an objective, and (c) a means, and the relationships among the ideas, I present an illustrative analogy.

I consider a goal to be an ambitious, well-conceived target, or ambition that is very difficult to achieve because of its capacious scope. For example, suppose a person held the goal to live a healthy lifestyle. We can certainly discuss and describe what encapsulates a healthy lifestyle, and we can dissect a person's lifestyle to conclude whether or not he or she is working towards a healthy style of living. Yet, it seems unlikely that we would conclude that the person had reached the ultimate goal of living a healthy lifestyle because of all that is involved. Thus, to help quantify progress towards reaching a goal I turn to an objective which I view as being a well defined, and measureable benchmark along the way to reaching the desired goal

To continue the example, an objective for living a healthy lifestyle might be to lose 10 pounds. This is easily measured, and one can report whether or not the benchmark has been reached. In the model an objective can be linked to an activity, or means, designed to support reaching the objective. For instance, to reach the objective of losing 10 pounds, one might keep a food journal to count calories, or one might begin an exercise regimen. Because an activity (or set of activities) is directly coupled to an objective, a formative assessment of the benchmark might result in an evaluation of, and

the realignment of, the linked activity. For example, if one found that over the course of a month that weight had been gained, it might prove beneficial in the future to cut out cheesecake as a nightly dessert.

In the above example, the goal/objective/means triplet was--live a healthy lifestyle/lose 10 pounds/count calories. The same idea can be applied to a mathematics lesson. As presented above, a goal for a math teacher might be to have all of her students understand place value. A realizable objective under the goal of developing place value understanding in students would be that they understand how to regroup tens into hundreds. This benchmark could be measured by assessing their strategies and through their explanations of their thinking, and their written work (e.g., Billy grouped together 13 ten-sticks and exchanges 10 of them for a hundred flat). A means to help students reach the desired objective might be a multiplication problem that included products involving groups of tens that extend beyond 100. For instance, consider the problem: There are 15 packages of gum with 10 sticks in each package, how many pieces of gum are there total? This problem might support students in thinking about 15 groups of tens, which, in turn, might promote them thinking about grouping 10 of the tens into a single 100.

The analogy can also be extended to influencing factors for living a healthy lifestyle, and for writing a problem for a mathematics lesson. For example, the weather can serve as an influencing factor towards living a healthy lifestyle since a person might be less likely to go for a jog on a cold wet day. In the same manner, the academic calendar might play a role in influencing a teacher's choice of problem type, or the magnitude of the number choices as the teacher begins to develop a problem for a lesson.

Chapter 5 Organization

For each of the aforementioned four influencing factors and seven design-objectives I present evidence from interviews with each of the three participants to support the claim that the identified factors seemed characteristic of the considerations made by teachers when developing a task for a mathematics lesson. Although not specifically addressed by the research question, evidence is also presented to demonstrate that relationships seemed to exist between and among the factors. Together the conclusions underscored the complex nature of the practice of developing a task. The rest of the chapter is organized into four sections: (a) a discussion of the four influencing factors, (b) a discussion of the seven design-objectives, (c) a discussion on emergent relationships among the factors, and (d) a closing summary.

Influencing Factors

During the analysis portion of this study there emerged four factors that seemed to influence the participant teachers' decisions when developing a task during the design phase of a mathematics lesson, but which did not meet the definition of a design-objective. These influencing factors appeared to be broad in scope, yet at times each seemed to be considered during the task development process. The four factors were: (a) problem library, (b) academic calendar, (c) teaching practices, and (d) learning goals. In this section, the four factors are described, and evidence is presented to support the inclusion of each as influential elements when developing a mathematical task for the participants in this study.

Problem library

Over their years of teaching via CGI, the participant teachers have designed and written hundreds of CGI type problems, and each has kept copies of many of their problems in various forms. Therefore, most of the development of a task for a lesson might have already been completed, that is, the task was composed prior to the current academic year in which this study occurred. The main reason offered by the participants for keeping a library of problems from which to select for any given lesson was to save time. For example, Ms. F said, “It’s easier than sitting down three times a week and writing a problem...I don’t want to create more work for myself.” In the same vein Ms. S offered, “Anything to make it a little bit faster,” and Ms. T, “Some days when I’m more rushed I look through my folder and see if there’s one that will suffice from previous years.” However, it became apparent that when one of the teachers did decide to select a problem from their library, it was not a simple matter of making a copy and distributing it to their respective students. There appeared to be other considerations, and each of the participants spoke to the process of problem selection.

Of the three participants, Ms. F appeared to utilize her library of problems more often than did either Ms. S, or Ms. T. In fact, during each round of classroom data collection, she had selected the problem for that particular day from her repertoire of problems. Despite the fact that she did not compose each of the problems under consideration for a specific lesson, she did explain throughout the interviews that there was more to the process than choosing a problem. In particular, Ms. F reviewed the appropriateness of the number choices for her current class:

- 1) R: Why this problem today?

- 2) Ms. S: When I was looking through, I copied a lot of different problems out of a
- 3) book that I keep, and then sometimes I have to adjust the numbers before I give
- 4) them. Um, haven't done a multiplication for a long time, and I wanted to revisit it
- 5) with numbers that were really powerful.

So, it appeared that one of the considerations is the number sets in the problem, and whether or not the number choices fit her students (Lines 3-5). As she elaborated:

- 6) And I've taught first grade for so long I kind of know where I want to go with the
- 7) number choices, and when the child breaks out of the mold, then I will add
- 8) numbers, or take numbers away...
- 9) ...when I looked through my collection of problems I look at the numbers that I
- 10) have chosen, and I see the, where I think the kids are struggling, and see if I have,
- 11) um, problems with situations and numbers that work for my kids.

And in a later interview she reiterated the importance that she placed on the number choices:

- 12) ...because from year to year I've like the same numbers with first graders, they
- 13) tend to be doing the same kind of things at the same time of year. And so when I
- 14) look through my repertoire of problems, I decide if, do these numbers fit this
- 15) group? Or do I have to take some out, or do I have to know I'm gonna have to
- 16) change on the spot? Or, will these work? And so the context fit, and the numbers
- 17) fit for this group.

Thus, for Ms. F, it seemed that when she selected a problem, the process was to inspect the number choices to match her perceptions of her students' mathematical

understandings (Lines 9-11 & Lines 13-17), and then she would decide if the context needed alteration (Lines 11 & 16).

In contrast to Ms. F, Ms. S reported that although she did keep copies of older problems, she did not often use the same problems from previous years.

- 1) Ms. S: I keep 'em, I have them in files. Part-Part-Whole, Whole-Unknown and
- 2) there's a whole file there, you know. I keep them in files by problem type. I
- 3) don't often go and use, use the exact same problems? But, you know, I go and I
- 4) look, and I see, oh, what'd I do, what did I do before? What numbers did I choose
- 5) before? And I kinda tweak it here or there. Steal the clip art from last year.

To Ms. S, it seemed that she could save time by reviewing what she had done before, revise the problem if she chose to use it again, or to reuse the clip art from the previous year. However, Ms. S did go one record stating that she wrote new problems for her class everyday.

- 6) R: Okay. Do you write your own problems for each lesson?
- 7) Ms. S: I do. Yeah.
- 8) R: Everyday?
- 9) Ms. S: Mm hmm. Yeah, I do.

Thus, it appeared that Ms. S did not tend to use exact copies of problems from her library, as was the case with Ms. F. It seemed that Ms. S used her files of problem types more as a reference source than as templates for lessons.

Ms. T was similar to Ms. F in how she utilized her library of problems. The following excerpt is from a clinical interview when she was asked if she kept copies of

old problems. Ms. T tendered that she did not organize them, but she did look through them to find a problem that might be useful for a particular day's lesson (Lines 4-8).

- 1) R: So you keep a repository?
- 2) Ms. T: I throw them in a file, yeah.
- 3) R: How do you organize them?
- 4) Ms. T: I don't. I just throw them in there, and then I look through it. That's, you
- 5) know, I'm sorry. I wish I could say I had them divided by problem type, of
- 6) course, and then I pull out the ones that I'm suppose to be using. But that
- 7) wouldn't be me so I would be lying. Um, so sometimes I look through and go, oh
- 8) yeah, that would be fun today, or whatever. Um, but I do have to be careful
- 9) about, like, well, did I write those numbers for the end of the year, or the
- 10) beginning of the year because sometimes I need more transition than I have. Or
- 11) more, you know, sort of approachable numbers, lower numbers. Um, and then
- 12) once I pick I start putting numbers in, and I try and think about the numbers.

Here, Ms. T reported that she, akin to Ms. F, took care to examine the number choices, and that she might alter the number sets based upon other factors such as where the class stood relative to the academic calendar (Lines 8-12). Thus, like Ms. F, a problem from the library of Ms. T might serve as a template for a day's task. The next subsection reports on the importance that each of the participants placed on developing, or writing, their own problems.

Writing own problems

For Ms. F, I stated that the development of a task might be an ongoing process, but that most of the design had occurred at an earlier date. The story would not be

complete without reiterating the importance that each of the teachers placed on writing their own tasks. Hence, if one of the participants had selected a problem from her respective library, she was certain that she had written it, and that she could modify it to fit the needs of her current students. For example, Ms. F when asked about the importance of writing her own problems replied:

- 1) Ms. F: ... Well, you, you write your own numbers to see where kids are, and what
- 2) they know. What strategies, are you looking for? Um, I think I've talked before I
- 3) like to make sure that assessment at the beginning of the year is to see what kids
- 4) know. Um, I like to, yeah, yeah, you wanna make some numbers that are
- 5) accessible to all the kids, at first, therefore, the 3 and the 8. Something like that to
- 6) start with, so they have some good understanding. And then you'd wanna put
- 7) numbers in that would challenge kids as well.

So, to Ms. F, it seemed important that she write her own problems so that she could assess her students' current mathematical knowledge and understandings (Line 1-2), and so she could provide access to the problem as well as to challenge kids with number sets of her choice (Lines 4-7). Ms. S voiced similar sentiments during an interview:

- 1) Ms. S: Well, I mean, people ask me sometimes 'wouldn't it just be faster to use
- 2) the prepared problems?' So, sometimes, like, you know, we have Investigations®
- 3) now, and they have prepared problems. Um, and it's really not faster for me
- 4) because, I'm looking for, I'm wanting to develop a problem with certain
- 5) things? Based on what I, um, seeing my children understand, and what I think
- 6) that they don't understand yet? And so, um, I need to have the flexibility of being

- 7) able to choose the problem type and the numbers? And with a prepared problem
- 8) you can't always, you can't do that. And then I also like to be able to put their
- 9) names in it, and to have the context that we've been working on in class. Just in
- 10) terms of making everything a little bit, um, making them having, having them
- 11) have more ownership.

To Ms. S, it seemed important that she retain the flexibility to choose the problem type and numbers (Lines 6-7) to insure that she was meeting the needs of her students (Lines 5-6). In addition, by writing her own problems she could relate the problem back to the classroom so that her students could have more ownership of the task (Lines 8-11).

Ms. T reiterated the importance of the number sets when writing a problem:

- 1) Ms. T: I mean, it's mostly about the numbers for me. And so, I always try and
- 2) start with numbers that are approachable that, like, even my direct modelers can
- 3) draw out and do and make sense of the problem. And then I start thinking about,
- 4) it depends on what kind of math it is, like, if it's multiplication, well, I gotta throw
- 5) in some numbers that are round-able. I have to throw in some numbers that kids
- 6) can break apart. I have to throw in some numbers, um, you know, like, with
- 7) money. Or whatever, things that I think that they can approach with mental math,
- 8) without algorithms.... While I'm working on this too, and my goal really usually
- 9) is to be able to pull a variety of strategies. I mean I'm thinking about what
- 10) strategies the kids will use, but I'm thinking about the numbers I want to put in.

By writing her own problems and choosing her own numbers it seemed that Ms. T could, among other things, insure access to a problem for her students (Lines 1-3), and she could inspire the kids to try different strategies when solving a problem (Lines 4-8),

which seemed of great import to her (Lines 8-10). The common theme that emerged across the three cases was that by designing their own problems, the teachers were able to act in response to where they conceived the mathematical understandings of their students to be at that moment. That is, the flexibility involved in writing their own problems, in particular the flexibility in choosing numbers, seemed to allow them to better build upon their respective students' understandings. Or, as Ms. S said succinctly, "...always trying to extend or support the students with, with my number choice."

Number choices

In the previous section each participant emphasized the importance of choosing her own numbers when writing a problem. In addition, the upcoming sections of the seven categories of design-objectives demonstrate the efficacy of the number choices in serving as the means through which the objectives might be met. Hence, it seemed opportune to define terms that I will use throughout the rest of this document. First, I use the term *number structure(s)* to typify the characteristics of the number choices, but also to align with the CCSSM Mathematical Practice #7: *Look for and make use of structure*. The alignment seemed appropriate since "to discern a pattern or structure" (CCSSM, 2010, pg. 8) appeared to be distinctive of the strategies the participants promoted in their lessons.

Consider the utility of the structure of the number pair (7, 12) within the context of a multiplication problem. The two-digit number, 12, can be decomposed into $10 + 2$, which when multiplied by the single digit multiplier, 7, produces patterns related to counting by 10s and 2s respectively. I also use terms related to the idea of number

structure that seemed familiar to the practice of teaching mathematics, or that were colloquial to this study (e.g., “ten-ness”). These include:

- Benchmark numbers, such as 10, 50, 100, and their proximity to number choices,
- Decade numbers, 10, 20, 30, ...,
- Friendly numbers, that is, those that are easy to add, subtract, or multiply,
- “Tens-friends,” or numbers that combine to make a 10,
- “Ten-ness,” or the advantages offered through combinations of 10s within whole number operations, and
- “Break apart”, to decompose numbers, for example, breaking apart 15 into $10 + 5$.

The above list of terms describing number structures provides a common language for the transcripts and related discussions that are provided as evidence in the rest of this and the following chapters. The next section returns to the discussion of the four influencing factors, specifically, the academic calendar.

Academic calendar

When a particular lesson might occur during the academic year seemed to serve as an influencing factor in the design phase of a lesson, and the development of a task for all of the participants. In the case of Ms. S, the academic calendar was formally divided into trimesters, and as presented in her case study report, her students’ learning was assessed through a District diagnostic at the end of each trimester. Thus, when Ms. S designed a task, she chose the problem type and some of the number sets relative to the requirements associated with each trimester (this is examined in more detail in the upcoming section regarding the design-objective: Meet District Standards). In more general terms, she talked about introducing easier problem types early in the year to help

support students with their problem solving abilities. For example, when asked when she might introduce more sophisticated problem types, she answered:

Well, there's some problem types I wouldn't start with, so like a start unknown problem type? Um, is very difficult for children, and so I wouldn't start there? I tend to start more with the easier problem types, um, and get them, uh, feeling confident about their problem solving abilities.

Thus, Ms. S selected certain problem types early in the year so that her students could build confidence in their problem solving abilities. Later in the school year, Ms. S might choose a problem type or numbers to support her students in exploring more focused mathematical ideas. In an interview conducted in November she offered:

And then right now I'm also working really hard at sort of on the idea of, um, place value understanding. So I've been doing multiplication problem types...a lot. And also, just in terms of wanting to develop some fluency with facts, so that's why I've been, um, when I do like a Join-Result-Unknown, or a Part-Part-Whole, Whole-Unknown I'm trying to choose numbers that help to develop fact fluency as well.

Whereas Ms. S was influenced by the requirements of formalized District trimester standards, Ms. T, and Ms. F were not bound by such constraints. However, the timing of a lesson relative to the academic calendar was a consideration for both. For example, Ms. T would not give a new class of third graders addition or subtraction problems because she has learned that students typically come to her from second grade relying on the standard algorithms as their only problem solving strategy. Instead, she would give a multiplication or division problem to break the habit of kids "waiting to be told what to do" in favor of them assuming the problem solving process, and finding their own solution strategy. In her words:

Um, I, I tend to steer clear of adding and subtracting at the beginning of the year, and, um, give a lot more multiplication and division. Just

because the kids have the algorithms already for addition and subtraction, and I don't like them to get caught up in the 'that's the way I'm supposed to do it so I don't know what else to do?'

Additionally, in a manner similar to Ms. S, Ms. T might concentrate on more specific mathematical ideas at a point later in the academic year. The following excerpt is from an interview in January, and in it Ms. S explained that at that point in the year she wanted to concentrate on place value understandings and fact families. As she said:

They're all at this point of the year, um, have some strategies to get started. Um, so at this point of the year I would like them to have a good sense of place value. It's not as strong as I would like in a lot of my kids. So that they have the facility of breaking numbers apart and doing different things with them, and that's, that's hard for some of them.

In the case of Ms. F, where the class stood relative to the academic calendar seemed to influence her selection of problem type and number sets in terms of assessing her students' mathematical growth. For example, she said, "Early in the year I'm looking to see if kids can count by 5s and 10s. If they've learned that already in kindergarten per se, or at home, if that comes easy for them." And when talking about later in the year, from an interview conducted in May of the school year, she added, "But at this point in the year, I wanted to see if they could do some adjusting, and I think it was successful with many of the kids."

An interesting theme emerged in terms of the types of problems that all three participants would give their students at the beginning of a school year. In contrast to a more traditional progression of problem types that might introduce, in order, the operations of addition, subtraction, multiplication and division, the participants seemed to

prefer introducing multiplication earlier. As Ms. F explained, “Because I think it’s easier for kids that are using stuff to make piles and then they simply add it. So, right away, probably in the first month I’ll do a multiplication problem.” Similarly, when Ms. S was asked when she introduced multiplication problems she said:

We start at the beginning of the year...But we don’t have a standard for multiplication, but it really supports our place value standards. So we’ve been starting at the beginning of the year with multiplication with groups of ten.

As for Ms. T, it was already noted that she did not start with addition and subtraction problems preferring instead to present multiplication and division problems early in the year to her students.

Teaching practices

Another theme that emerged was the influence that some identified teaching practices seemed to bear on developing a task. I am not claiming that the teaching practices identified in this study form an exhaustive list, but they seemed to be teaching practices in which the teachers engaged during the process of developing a problem. The rest of this section describes three specific teaching practices common to each participant that at times seemed to surface as influencing factors in the design phase of a lesson. These were: (a) anticipating students’ thinking, (b) anticipating a response to students’ thinking, and (c) reflecting upon a lesson.

Anticipating students’ thinking

As described in the three case study reports, each of the teachers placed great emphasis on their students’ mathematical thinking during the enactment of a lesson; therefore, it was not surprising to find that each seemed to anticipate some of the

strategies that their respective students might employ during the lesson. For example, during the semi-structured stimulated recall (SSR) interview after a lesson centered on a multiplication problem, Ms. F described the work of one student, Sandy, when multiplying 7 and 12, and her strategy of breaking apart the 12 and doubling the two 7s:

...and then you had, Sandy, who knew that it was fourteen. She knows her doubles, and so she just added that 10 really easily, and there were, was the extra 4. So, that was the kind of thing that I was looking for.

It appeared that Ms. F was anticipating how the student, Sandy, might approach multiplying a particular number pair based on the model that Ms. F had of Sandy's mathematical knowledge. Ms. F expressed that Sandy, "knows her doubles," which seemed to inform her anticipating the strategy that Sandy might use to multiply 12 by 7, albeit in a retrospective fashion.

In a similar type of interview, Ms. S talked about the strategies that certain students might use to solve a Part-Part-Part-Whole, Whole-Unknown problem in an upcoming lesson:

- 1) I think that Jay will increment. I think that Max will probably combine tens and
- 2) ones. Um, I think, I'm trying to remember whom I'm working with. Sasha will
- 3) probably combine tens and ones. Um, I imagine, uh, both Lacy and Sally will
- 4) direct model with tens. And Sally's getting really, really close to thinking about
- 5) doing some invented algorithms as well.

Again, Ms. S seemed to anticipate the strategies that specific students might use based upon her interpretations of the students' understandings at that particular moment in time. Of those students with whom she would be working with during the lesson, she appeared to know which might use similar strategies (Max and Sasha, Lines 1- 3, and Lacy and

Sally, Lines 3 - 4), and she implied that she might look for Sally to use a new strategy that went beyond direct modeling with tens.

To finish, Ms. T, described, in general terms, that she anticipated the strategies students might use when writing a problem. During a clinical interview, she said, “I mean I’m thinking about what strategies the kids will use, but I’m thinking about the numbers I want to put in.” And during a SSR interview she also described student thinking in general that she had anticipated after conducting a lesson on multiplication²

I’ve been playing with the 11s a lot [*indicated the pair (99, 11)*] and they’re still not getting them. So, like, I try to sort of throw things in that I thought that they would be able to see with place value and so forth too, um, and just with breaking apart. So I tried to put in a variety of things that I thought they could access cause I thought they were all pretty friendly.

During a subsequent round of data collection, Ms. T described her surprise that some students did not use the distributive property as she had anticipated them doing when solving a multi-step multiplication problem:

- 1) R: And then you grouped the 12 and the 15, in your explanation because of the
- 2) distributive..
- 3) Ms. T: Because of the distributive property, yeah. Although it was very
- 4) interesting to see the kids who were, like John, who were using the distributive
- 5) property, who were breaking numbers apart, were breaking it into things that
- 6) weren’t what I was anticipating, but more just like, well, I know what 3 times 4
- 7) is...So, or whatever it was, so, I’m gonna do 4 and 4 and 2, or whatever it was to
- 8) get to the 10, or whatever. And they weren’t seeing the 10.

² Gestures are denoted by the use of italicized phrases within a pair of italicized brackets.

In this excerpt, Ms. T described her surprise that some students did not break apart the 12 into 10 and 2 before multiplying, as she thought that they would (Lines 3 - 6). Moreover, she expressed surprise that some of the students factored the 12 into 3 times 4 (Line 6), while some decomposed 10 into 4 plus 4 plus 2 (Lines 7 - 8). It appeared that she had selected the number sets with the anticipation that students that might employ a “break apart” strategy would decompose the 12 into 10 and 2, and the 15 into 10 and 5.

In the examples presented above, each of the participants described that they anticipated their students’ thinking as it related to selecting or designing a particular task, or when designing a task in general. However, although anticipating students’ mathematical thinking seemed to play an influencing role in the design of a task, it did not meet the constraints for what I considered to be a design-objective. That is, anticipating student thinking was a very broad concept, and it was very difficult to measure. Thus, I categorized the teaching practice of anticipating students’ thinking as a guiding, or influencing factor in the design phase of a task.

Anticipating responses to students’ thinking

Another factor that seemed to be considered by the participants when selecting or developing a task, and one that appeared to be directly linked to anticipating students’ thinking, was anticipating responses to students thinking that the teachers might enact during a lesson. As was the case for the factor, Anticipating Students’ Thinking, the anticipated responses to students’ thinking that participants’ described composed a very small sampling of the range of responses that the participants were observed enacting during the classroom visits of this study. Examples of the anticipated responses

described by the participants included responding to students' thinking through certain conversations with students, or with particular lines of questioning, and these are examined below. However, particularly germane to this study was the idea that the participants might anticipate responding to a student by modifying the number and I discuss this anticipated response next.

Consider the following excerpt from a clinical interview with Ms. F that occurred prior to a lesson that focused on the two-step problem:

Sally had ____ sticks of licorice. On her way home she ate _____. The rest she shared with Rick. How many sticks did they each get?
(7, 3) (20, 2) (35, 5) (43, 7) (90, 4) (115, 25) (206, 8)

- 1) Ms. F: Um, because it's a two-step problem, I wanted them, it to be even and not
- 2) a fraction of a piece of licorice. Because I think that it's gonna be difficult for
- 3) some of the kids to do a two-step problem. But when I looked at these numbers, I
- 4) have this one in mind [*indicated (115, 25)*] and I debated whether to make this a
- 5) 27, and I decided I was gonna wait and hold off to see which kids were flying
- 6) through these and then change it on the spot.
- 7) R: So, right now, let's see, one fifteen, is 90, so 27, why would that be so much
- 8) different to you?
- 9) Ms. F: Just because I think the kids that can take the 5 away first, and then easily
- 10) take the 20 out of a hundred and ten, it would just make it, tweak it just enough.
- 11) Make it more exciting.
- 12) R: Okay, so the subtraction would be a little more difficult.
- 13) Ms. F: Yes, right. The division I don't think would be more difficult just the
- 14) subtraction.

In lines 2 - 3, Ms. F appeared to anticipate that the two-step problem would prove to be difficult for some of her students, but she also appeared to think that some of her higher thinking students might solve the problem quickly (lines 5 - 6). In anticipation of those students that might find the problem to be relatively easy, she had already thought of changing the 25 to a 27 (lines 3 - 6) when they finished with the number pair (115, 25). Her rationale was that she thought some kids would solve the problem quickly using an incrementing strategy (lines 9 - 10) so she was anticipating changing the numbers to make the problem more challenging for those students.

Ms. S demonstrated that at times she might be inclined to engage students in a conversation about certain number combinations when she was talking about a lesson that had occurred earlier that day. The problem to which she referred was a two-step problem involving first a Part-Part-Whole, Whole-Unknown, and then a Separate-Result-Unknown problem:

The old lady in the Gingerbread Man makes ____ girl gingerbread cookies, and ____ boy gingerbread cookies. ____ of the cookies get burnt and ruined in the oven. How many cookies are left?
(36, 46, 20) (44, 47, 30) (56, 37, 25) (146, 267, 20)

In this short excerpt, Ms. S explained her rationale for the number choices:

- 1) Ms. S: Um, here, there's an easy 10 to pull out [*indicated (36, 46, 20)*]. Here
- 2) [*indicated (44, 47, 30)*] there might be a conversation about 4 and 7, and that
- 3) there's a 3 inside of the, that there's a 3 inside of the 4 that you could put with the
- 4) 7? Or something like that.

In lines 2 - 4, Ms. S indicated that she anticipated having a conversation with students about adding 44 and 47 by first adding the digits in the ones place using a pair of 'tens-

friends,' which would entail summing $3 + 7$ to get 10, and then adding the remaining 1 to make 11. It seemed that this type of response to students' thinking could support or extend students' understandings of place value, a long-term learning goal held by Ms. S for her class.

Ms. T demonstrated a similar capacity for anticipating questions that she might ask students when they were working with particular number sets. For example, in speaking about the ways in which she hoped to move her students along in their thinking, she offered, "So, I mean I think when I'm reaching levels is always by numbers, but it's also by questions, is where I push to the next level." This quote supported the following excerpt from a SSR interview that followed a lesson with a multiplication problem. In the excerpt, Ms. T explained her rationale for including the number set (99, 7) which in context were 99 reindeer given 7 carrots each:

- 1) Ms. T: ...But I put in, uh, 99 and 7 specifically cause I wanted to see kids round,
- 2) and be able to know that then they had to subtract because there was an extra
- 3) reindeer...
- 4) R: Right the round and adjust strategy.
- 5) Ms. T: Right. So, the round and adjust, yeah. Um, when I put in like a one-digit
- 6) by a three-digit, or four-digit, I want them to break the numbers apart. And if
- 7) they go, 'I don't know what to do,' I'll say something like, 'Well, could you,
- 8) would expanded form help you?' Or whatever, um, which I know is sort of
- 9) pushing but I want them to try different things, you know?

Here it appeared that Ms. T anticipated students using a round and adjust strategy when multiplying 99 by 7 (lines 1-3 & lines 5 - 6), and if they didn't know how to start, she

already had a supporting question for them in mind (lines 6 - 8). This was another example of a link between anticipating students' thinking and an anticipated response.

Reflecting upon a lesson

The final teaching practice that seemed to emerge from this study that appeared to influence the development (or redesign) of a task was the teachers' indication that they might reflect on a lesson as to what seemed to work well, or what did not seem to work well. For example, Ms. F might go so far as to make notes concerning specific students to aide her in responding to her perceived needs of the students in the future:

I might keep number choices in my head to try with kids, from what I've seen with their work. I've made little notes, for instance, when the kids were adding tens. The kids that were having really difficult time doing that, it was subtracting tens, and I wrote their names down, and so I, I made problems for them so I could see if we could get past that hurdle.

As Ms. F explained, what occurred in a prior lesson could influence the problem types and number choices she would use in the future to meet a specific need of a group of students.

Ms. S told a similar story regarding her number choices for a two-step problem:

So like the other day they had, and they really struggled with it, the other day they had what I kinda consider a two-step, they had this is how much money and he spends this much, so, they had to find out how much money that was all together, and spend the amount...so this was a, because it was two steps, I tried to make it pretty easy...so I made the initial two numbers pretty easy, like they're all, you know, underneath a hundred.

It seemed that Ms. S composed the two-step problem for that day based upon what her students had done previously. Thus, it could be argued that it was her reflection on the

previous lesson that influenced her to choose easier number sets for the next two-step problem type she wrote.

Ms. T also offered thoughts that supported the claim that her reflection on lessons might influence the design of a future task. She referred to the lesson on adding fractions that she had conducted earlier in the day, and changes she might make to the problem:

But it would have been good to have a few more like, if I were to go back and do this again, and I probably should keep one of these [*indicated a copy of the problem*] and write on it after we finish talking. But I would not start with fifths. I would like to start with something that's much easier to cut a pie into...I would like to do several that are just adding fractions that all have like denominators before I go playing around with equivalent fractions. That might have been a nice transition.

Essentially, Ms. T redesigned the problem during the interview, and she did indeed modify the problem at the conclusion of the interview. This was perhaps the most compelling evidence that reflection on a lesson might influence the participants to revise an existing problem, or influence, say, their choice of numbers within a future problem. This episode is revisited in the later section that examines relationships among factors in the design phase of a lesson.

Learning goals

Because of the constraints imposed by my definitions that goals are ambitious targets that are difficult to achieve, and objectives are well defined, measureable benchmarks along the path to reaching a goal, there is a distinction between a learning goal and a learning objective that deserves attention. Consider, for example, the following excerpt from the CCSSM second grade standard, 2.NBT: Number and Operations in Base Ten:

Understand Place Value

1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:
 - a. 100 can be thought of as a bundle of ten tens--called a "hundred."
 - b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones) (CCSSM, 2010, pg. 19).

Under my interpretation, the header, *Understand Place Value*, would be defined as a goal; it is broad in scope, and difficult to achieve. It may even be called a long-term goal as it spans grades one and two in the CCSSM. In contrast, each of the bulleted items would fall under my definition of an objective. They are well defined, and they are measureable--a student could demonstrate that she thinks of 100 as a bundle of 10 tens by exchanging a hundred flat for 10, ten-sticks. In addition, a teacher could design activities, or tasks, to assist students in reaching the bulleted items, which would fall under my definition of means towards meeting an objective. Hence, in this section, and the upcoming design-objective section titled Meet District Standards, I follow the distinction between definitions.

As was discussed in the section on Teaching Practices, the learning goals that were identified in this study do not compose a comprehensive list of those held by the participants. Nor, do I claim that they are the only learning goals that were considered by the participants during the design phase of a mathematics lesson. Two learning goals did, however, seem to emerge as being particularly relevant for the participants when developing or selecting a problem for a lesson: (a) understand place value, and (b) gain sophistication in strategies.

Understand place value

The major learning goal that emerged among the participants was that they all wanted their students to gain an understanding of place value. Ms. F stated that place value was important when her kids matriculated. As she said, “If they have a good solid place value knowledge they tend to do better in the upper grades,” and later, “I want them to have place value. That’s huge in math.” This was evident throughout the interviews as Ms. F often referred to the idea of “ten-ness,” and that she wanted her students to develop “strong knowledge of that 10 in there, that ten-ness.” When asked to clarify her conception of ten-ness, she answered:

I think it’s a step to place value--is knowing that when they understand these numbers make a 10, when I get to, to put, I get to switch these out for a ten-stick, or I know that that makes a 10.

So, for Ms. F, it seemed as if she was highly influenced by the goal of her students developing understanding of place value when she wrote a problem. A specific example of this was found when she was discussing the rationale behind selecting numbers for a Separate-Result-Unknown problem. She was talking about the number pair, (102, 73), when she said, “It helps a lot to get to that ‘ten-ness’ when you’re estimating your answer. They’ll need that later on. So, they might, same with this one? It’s very close to a hundred.” Here, she was alluding to students that might round the 102 to 100, which to her, was an indication of some semblance of place value understanding, in particular the hundreds place.

Throughout the interviews, Ms. S also brought up the goal of developing her students’ place value understandings. For example, when asked what math concepts she had been emphasizing, she answered, “Well, we’ve been working really hard on place

value.” And when talking about standards that she thought that she might be meeting with a Separate-Change-Unknown problem, she said, “...you know, all of what I consider the important ones in terms of problem solving, and place value, addition and subtraction.” More specifically, in regards to writing a multiplication problem she reported:

I was thinking about the multiplication problem type because I wanted to be developing their sense of number, their place value, and their base-ten understandings. So, I was looking at multiplication with groups of 10 because I thought that that would help do that.

Further evidence was found in the following excerpt as Ms. S discussed her rationale for the number choices she made in the multiplication problem she alluded to above:

- 1) Ms. S: I chose the numbers because I was, like for example, 14 groups with 10 in
- 2) each bag the same idea? I’m wondering if they’ll remember, they’ll see that same
- 3) relationship. Twenty-four groups with 10 in each bag [*indicated (24, 10)*], um, I
- 4) have a lot of kids that can tell me really easily what 7 tens is, what 9 tens is, but
- 5) I’m wondering if they can think about what 24 tens is, and that it’s 240.

Her rationale for the numbers choices seemed to suggest that the goal of developing an understanding of place value influenced the choices. This was demonstrated by the apparent objective to move her students beyond their understanding of the tens place to an understanding of the hundreds place in terms of the number of 10s in a three-digit decade number (lines 1- 2, and lines 4 - 5). Note, that the objective in this episode was a well-defined target whereas the goal of “understanding place value” is much more expansive.

Ms. T expressed that, “So, you know, for example, this year I have a lot of kids who are struggling very much with the concept of tens. I keep giving them numbers with tens, and hoping they’ll see tens.” And in a subsequent interview she reported:

That’s how I feel this year. I feel like, it’s interesting, and it’s interesting with the change of standards. Uh, my main thing, that I’m hopeful for them is that they have place value understanding enough to work with numbers mentally.

A specific example of a number choice that Ms. T made to support the claim that developing an understanding of place value is a goal for her students was found in the excerpt from the SSR interview after a lesson on multiplication:

- 1) Ms. T: Um, and I thought, well, this will be, it’s just, you know, it’s like I try to
- 2) just pick things that I thought they would see. I’ve been playing with the elevens
- 3) a lot [*indicated the number pair (99, 11)*], and they’re still not getting them. So,
- 4) like, I try to sort of throw things in that I thought that they would be able to see
- 5) with place value, and so forth too.

Thus, it seemed that Ms. T felt that the number choice that entailed multiplying 99 by 11 might support her students’ thinking of place value (lines 3 - 6) because she had been focusing on working with 11s in previous lessons (lines 2 - 3), and perhaps the fact that 99 was close to 100 might offer her students some insight.

Gain sophistication in strategies

As described in the case study reports, one of the ways the participants measured learning was through their respective students’ growing sophistication in the strategies they would employ during problem solving. Thus, it was not surprising that they seemed to hold the learning goal for their students to move towards more sophisticated strategies.

For example, during a SSR interview after a lesson on subtraction conducted in late spring, Ms. F offered:

But at this point in the year, I wanted to see if they could do some adjusting, and I think it was successful with many of the kids [*indicated the number pair (15, 6)*]. Not all of them, so if they were having trouble, I said, “So what if this was a 5?” and if they still couldn’t get it, then we had to back up to a tool.

In this excerpt, Ms. F reported that during the lesson she used a line of questioning designed to extend some students thinking towards using a strategy (round and adjust) that was more sophisticated than a direct modeling strategy, or using a tool as assistance.

Ms. T explained during the final clinical interview conducted in mid-January the attributes of what she considered to be mid-level thinking students in terms of strategy sophistication:

Well, they’re a kid who might start out, um, needing to draw things out to make sense of a problem, but then has enough like number sense to be able to translate like a multiplication problem into skip counting instead of still having to feel needing to draw every single like dot, or count every single bean...Um, they’re a kid who can start grouping and start breaking things apart for me at this time in the year, into tens and ones, and so forth, to work with them a little bit, even though it might not be super comfortable and easy for them...And they can translate it sort of from the base ten blocks into the counting, into the drawing, into to like, you know, skip counting, into, maybe, multiplication facts at this point?

Here, Ms. T described the learning trajectory that a student might have followed when solving a multiplication problem beginning with direct modeling, moving to a counting strategy, and then using derived, or known facts. These last two examples were specific instances in which Ms. F and Ms. T seemed to support their belief that students’ demonstrated learning through an increase in the sophistication of their strategies.

Which, in turn, seemed to support the argument that they both held the learning goal of interest.

It was presented in the previous section that Ms. S had been emphasizing place value concepts during the time of this study. In addition, she also reported that she had been emphasizing problem solving, and strategies. The following is an excerpt from the clinical interview conducted during the first round of data collection:

- 1) R: Um, what math concepts have you been emphasizing lately?
- 2) Ms. S: Well we've been working really hard on place value. Um, we've been
- 3) working, um, just basic kind of problem solving ideas? So, um, I don't want to
- 4) say 'emphasizing' but I've been trying really hard to support my learners that are
- 5) direct modeling with 10s, to support them in that idea. Um, and then also have, I
- 6) kinda have two groups of kids. I have kids that were direct modeling by 1s at the
- 7) the beginning of the year and now are starting to direct model by 10s so I'm
- 8) trying really hard to support them in that. And then I have the other end of the
- 9) spectrum, students that are transitioning from direct modeling with 10s to sort of
- 10) invented algorithms. So those are kinda the two places that I, we've been doing a
- 11) lot of, a lot of work.

In this excerpt, Ms. S inferred that she wanted to see her students advance along the typical CGI learning trajectory presented in Chapter 2. Ms. S reported that she had students that were direct modeling by 1s, some direct modeling by 10s, and some that were transitioning to invented algorithms (lines 6 - 10). As for counting strategies, in a later SSR interview, she filled in the gap in the learning trajectory when she reported, "I want them [students] to be more efficient if they're counting up in their counting. So that

the idea of getting to a friendly number and then counting by something, you know, by a 10 as to by 1s.”

Although it seemed that the participants wanted all of their respective students to reach a level of thinking whereby each student’s problem solving repertoire would include mental strategies and invented algorithms, each teacher did seem to take a pragmatic approach in regard to specific individual students. For example, Ms. F described her thoughts regarding the strategies of two students during a lesson on multiplication. The first student, Chance, as reported by Ms. F, had entered first grade without having had math in kindergarten beyond number recognition. For Chance, Ms. F said, “I wanted him to count by 5s. I wanted to see him know that he didn’t need to direct model every single thing and count by 1s.” Whereas with, Sally, who was one of her higher level thinkers, Ms. F said, “I want to see her start breaking apart numbers.” Since the respective lesson had occurred late in the year, it appeared that Ms. F held different learning goals for Chance and Sally in regards to their sophistication in strategies at the end of the school year. Similarly, Ms. S seemed to hold a pragmatic learning goal for one of her students, Henry, to whom she would give easier numbers than the rest of the class. As Ms. S said, “I have one student I always, uh, differentiate for, before we start.”

Finally, Ms. T also seemed to hold a reasonable learning goal for one of her students, Carter. As she said, “Well, Carter, struggles with the confidence of even like getting started; he always wants to guess at what to do.” To which she added regarding a year-end goal that she held for him:

I want him to be confident. I mean, that's my number one goal with him is to look at a problem...and that he actually can, you know, have a strategy for approaching it."

Thus, Ms. T reported that she wanted Carter to acquire strategies to solve the various problem types, and to gain confidence in his problem solving abilities.

This section discussed the four influencing factors, and evidence was presented to support inclusion of the categories in the framework illustrated in Figure 5.1. The next section examines the seven design-objectives that seemed to serve as important benchmarks towards reaching the two learning goals, and the means through which the teachers meant to meet the objectives. Essentially, the following section focuses on the arrow found in Figure 5.1 that runs from the category Design-Objectives to that of Learning Goals. The arrow represented the idea that the seven identified design-objectives served as benchmarks towards the two learning goals that emerged as particularly relevant for the participants in this portion of the study. Arguments will also be presented in the next section to support the claim that the goal/objective/means triad was an integral to developing a problem for a mathematics lesson. Figure 5.2 illustrates the triad by focusing more closely on the arrow between the categories in Figure 5.1.

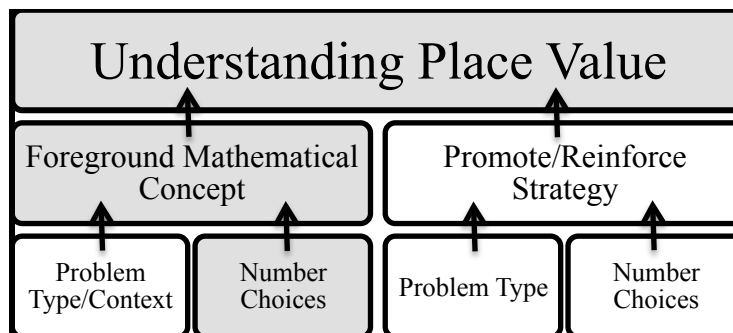


Figure 5.2: A learning goal/design-objective/means triad.

The three shaded elements in Figure 5.2 illustrate one of the goal/objective/means triads that are discussed in detail in the next section.

Design-Objectives in the Development of a Task

Table 5.1 provided definitions of the seven design-objectives that emerged in the analysis process. This section provides the evidence to support the inclusion of each as an important consideration in the task design process, specifically, that the design-objectives served as benchmarks towards reaching the two identified learning goals. To reiterate, the seven design-objectives were: (a) assess student thinking, (b) promote/reinforce strategy, (c) provide mathematical access/rigor, (d) problem makes sense, (e) foreground mathematical concept, (f) meet district standards, and (7) promote fact fluency.

Assess students' thinking

As described earlier in this chapter, for each of the case study teachers, assessment of their students' thinking was a continuous process. Therefore, when developing, or selecting a problem, each tended to hold the objective to assess their students' mathematical thinking. At times they may have looked to assess their students' thinking in more general terms, and sometimes they seemed to look to assess their students' thinking concerning specific mathematical ideas. In either case, the problem type and number choices acted as the means for doing so.

Regarding the objective to assess students' thinking in general, at times, Ms. F and Ms. T made comments that supported their intent to "just see" what their students would do with the problem. For example, Ms. F, when selecting a two-step problem involving subtraction and partitive division offered, "I just want to see what they do with

a two-step problem.” In a similar fashion, for a partitive division problem, Ms. T said, “So, I was just sort of throwing out a division problem to see what they’d do with it.” Here, it appeared that Ms. T wanted to assess her students’ understanding of division, but the scope of her assessment seemed less focused than she wanted to assess her students’ understanding of an idea in a more refined manner such as assessing her students’ thinking in regards to regrouping ones into tens. In both of these examples, the problem type supported the objective of assessing students’ thinking in general.

Next I present examples from each teacher that demonstrate they held the objective to assess students’ mathematical thinking regarding specific concepts, or ideas. Included in each example is evidence that the problem and the number choices supplied the means for assessing students’ thinking of particular mathematical concepts. The first example is from the SSR recall interview with Ms. F when she was asked to talk about the number choices for the multiplication problem:

Lana has ____ bags of cookies. She has ____ cookies in each bag. How many cookies does Lana have altogether?
(8, 5) (7, 10) (6, 4) (19, 2) (20, 5) (7, 12) (3, 25)

- 1) R: Yeah, you mentioned in passing, you said these were like your
- 2) favorite numbers.
- 3) Ms. F: Yes.
- 4) R: Can you talk about that?
- 5) Ms. F: This tells me a lot about the children, and you can go a lot of places with it.
- 6) Um, can they count by 5s accurately [*pointed to (8, 5)*] I wanted to see if people
- 7) can count by 10s [*pointed to (7, 10)*] without using the tens-frame, or see if they

- 8) can do it without their fingers at this point in the year... here [*pointed to (19, 2)*]
 9) I wanted, I do want to see if kids can count by 2s at this point of the year.

Ms. F appeared to have the objective to assess her students' ability to skip count, specifically by 2s, 5s, and 10s (lines 6, 7, & 9). Multiplication problems afford opportunities for students to use skip counting as part of their strategy, and the number choices that Ms. F made supported her students skip counting by the desired numbers. Note that the 2, the 5, and the 10 were the second number in each of the respective pairs. This was important because a student that used direct modeling as a strategy, as many of Ms. F's students might have done, would tend to follow the action of the problem. That is, for the number pair (8, 5), a direct modeler might have 'built' 8 bags of cookies (e.g., drawn the bags, or used fingers as the bags) with 5 cookies in each bag, and then counted the total number of cookies. Hence, there was the likelihood that such a student might have tried to skip count by 5s as part of the strategy. If the number pair had been reversed and presented as (5, 8), a skip counting strategy would have been unlikely since skip counting by 8s is less natural for young children--they are not usually taught to rote count by 8s as they are with 5s and 10s. The same argument can be made for the number pairs (7, 10) and (19, 2).

The next example is from the SSR interview with Ms. T concerning the problem involving adding fractions:

I baked a pie and cut it into ____ slices. Abby ate ____ of the pie, Nancy ate ____ of the pie, and Mary ate ____ of the pie. How much pie did they eat in all?

(5, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{2}{5}$) (8, $\frac{2}{8}$, $\frac{2}{8}$, $\frac{1}{4}$) (16, $\frac{3}{16}$, $\frac{2}{16}$, $\frac{1}{4}$)
 (6, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$) (12, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{12}$)

- 1) R: Why this problem today?

- 2) Ms. T: Cause I feel that all we ever do with fractions is, like, fair sharing, so I felt
- 3) like let's add some fractions. Play around a bit.
- 4) R: Um, what were you expecting today?
- 5) Ms. T:...I just wanted to see what they know about fractions, and what they could
- 6) do with fractions, and if they could figure out the equivalent fractions. That kind
- 7) of stuff.

The particular wording of the problem was such that Ms. T might have determined if her students could make the connection between the number of slices of the pie and the respective fractional amounts of fifths, eighths, sixteenths, sixths, and twelfths. As she said later, "Because I wanted them to figure out that is one fifth. One slice, one fifth of the pie. Cause I wanted that to be a piece of it, that was purposeful." Thus, the problem type, and the selection of the number of pieces in which to cut the pie supplied the means through which Ms. T could assess her students' ability to partition by the respective fractions. In addition, she voiced the objective of assessing whether her students could figure out equivalent fractions (line 6), and the last four number choices all included sets of equivalent fractions to support this objective.

The final example is from a clinical interview with Ms. S prior to a lesson that included the Part-Part-Part-Whole, Whole-Unknown problem:

Mrs. Freed is collecting rocks. She collects ____ bumpy rocks, ____ shiny rocks, and ____ smooth rocks. How many rocks does she have all together?

(23, 17, 27) (47, 38, 22) (126, 144, 214) (234, 179, 141)

- 1) R: So, why these problems today?
- 2) Ms. S: Well, we're doing a Part-Part-Whole, Whole-Unknown with

- 3) three addends, and a multiplication problem with some groups of ten.
- 4) And I chose the Part-Part-Whole with the three addends because we've
- 5) been, um, I have some kids that have been, are thinking about invented
- 6) algorithms and kind of doing direct modeling, and also direct modeling
- 7) with tens, but I wanted to do three addends to make it a little bit more
- 8) difficult? And the, um, numbers I chose, the initial numbers, the
- 9) smaller numbers [*pointed to (23, 17, 27)*] I'm looking to see if
- 10) anybody notices, or they're looking at the ones, that there's, can they
- 11) make a 10? With the 7 and the 3? Or, the idea of the 7 and 7 making a
- 12) 14. To see if they notice any facts that they know. Here [*pointed to*
- 13) *(47, 38, 22)*] I was looking to see if they, again, I'm wondering what
- 14) they'll, we've been talking a lot, playing a lot of games about doubles?
- 15) So to see if they notice anything about the 7 and the 8 in terms of
- 16) pulling out a doubles fact, either 7 and 7, or 8 and 8. Here again, 8
- 17) and 2, the 8 and the 2 could make 10. So I'm kind of wondering if
- 18) they'll notice any of those relationships. And the tens here are a little
- 19) bit more difficult because it's gonna go over a hundred.

The problem type with the inclusion of three addends provided Ms. S the means to assess her students thinking concerning adding three numbers, which she felt was more difficult for them (lines 7-8). As for the number choices, she articulated that they allowed her to gauge her students' ability to use facts that they have been emphasizing such as tens-friends (e.g., $7 + 3 = 10$, or $8 + 2 = 10$; lines 10-12), and doubles (e.g., $7 + 7 = 14$, or $8 + 8 = 16$; lines 15-17). In addition, when explaining her rationale for including

the number set, (47, 38, 22), she mentioned that this set made the problem more difficult because the total was greater than one hundred (lines 18-19). It seemed as if that particular number set allowed Ms. S to assess her students' thinking concerning a facet of place value, specifically, if they could regroup the 10 tens to make 100.

Promote/reinforce strategy

As presented in the previous section on Learning Goals, the participants viewed a child's increasing sophistication in strategies as one indicator of the child's mathematical learning. In an effort to cultivate their students learning a variety of strategies, they seemed to hold an objective to promote or reinforce different strategies during their lessons. To meet this objective, each indicated that the problem type and the number choices could combine to support the promoting of, or the reinforcement of, one or more specific strategies during a lesson. As Ms. T explained, "I like them to try different strategies."

For Ms. F and Ms. T, the strategy of 'round and adjust' was one in particular that emerged more than once during the interviews. To them, a round and adjust strategy was invoked when a child rounded one of the numbers in a given number set to a benchmark number that was easier to perform a specific operation on and finished the problem by adjusting for the change inherent in the rounding process. The following two examples illustrate whereby Ms. F and Ms. T used nuanced number choices to promote the strategy.

When discussing a Separate-Result-Unknown problem she had developed and implemented, Ms. F pointed at the number pair (15, 6), and said, "But at this point in the year, I wanted to see if they could do some adjusting, and I think it was successful with

many of the kids.” For a child following the action of a SRU problem, the operation is to subtract 6 from 15. A child using a round and adjust strategy might round the 6 to 5, removing a 1 from the subtrahend, so the difference becomes 10, which is greater by a factor of 1 because of the decreased subtrahend. To finish, the child would have to adjust the difference of 10 by removing 1 to reach a final answer of 9. In essence, subtracting 1 from the 6 to make the subtrahend 5 increased the distance between the minuend, 15, and new subtrahend, 5, to 10. This distance is one unit greater than the original distance of 9 between the minuend, 15, and the original subtrahend, 6, (think of measuring 10 feet on a board before cutting it, and you want a 9 foot piece, you have to cut off the extra foot).

She also offered the number pair (152, 49) in the problem because, as she explained:

That one? [*pointed at the 49*] I want to see if they can go to a 50, round up to a 50. And rounding is really important later too, in third grade. It's really a difficult skill for a lot of kids. So if they're already automatically starting to do it, I hope they continue to use that skill, that skill later.

In this passage, Ms. F talked about the importance of rounding, which she viewed as an advanced mathematical skill, and that the skill would become part of her students' repertoire going forward. This seemed to indicate that she held the objective to promote and reinforce the round and adjust strategy when writing certain problems, and she included number sets that supported students in employing the strategy.

Ms. T talked about the same strategy but within the context of a multiplication problem. In the problem of interest, a number of reindeer were given a number of carrots, and the students were to find the total number of carrots distributed. One of the number sets she put in the problem was (99, 7), which in the context of the problem was 99 reindeer each given 7 carrots:

- 1) R: So, do you write, or include numbers to promote a strategy?
- 2) Ms. T: Yes.
- 3) R: Can you give me an example?
- 4) Ms. T: Well, like I said like if I put in the, uh, last problem I did, uh, it was 7,
- 5) 99 and 7. Ninety-nine reindeer, and 7 carrots, or whatever. Which was actually
- 6) inverted but I wanted the number. I didn't want to give 99 carrots to a reindeer.
- 7) ...but I put in, uh, 99 and 7 specifically cause I wanted to see kids round. And be
- 8) able to know that then they had to subtract because there was an extra reindeer.
- 9) And they said, a hundred times 7 it was like, well, there's a hundred reindeer you
- 10) have to take away the carrots that that one reindeer would have gotten.

In this example, Ms. T indicated that she wanted to promote a round and adjust strategy similar to that described by Ms. F (lines 8 - 12), and she put in the number pair, (99, 7), as a way to encourage students to use the strategy.

Another strategy that Ms. F. and Ms. T seemed to promote and reinforce with their students was one that employed a feature of the distributive property when solving a multiplication problem. As Ms. F said, "I like numbers that can be taken apart easily too," which mirrored Ms. T's earlier comment, "I have to throw in some numbers that kids can break apart." For both, in a multiplication problem, the numbers 12 and 15 were examples of numbers that could be easily broken apart, namely into $10 + 2$, and $10 + 5$, respectively as part of a solution path. For example, in the Bags of Cookies problem, Ms. F gave her students (7, 12) as one of the number sets because she "likes to see them take that 10 out first," and she also added, "And I've, I've used 8 and 15 a lot too because 5s are easy to count by." So, Ms. F liked to see her students solve the multiplication, 7×12 ,

by breaking apart the 12 to form $7 \times (10 + 2)$, and then using the distributive property to first find 7×10 , and then 7×2 , and finally adding $70 + 14$ to arrive at 84.

An example that Ms. S also held the objective to promote or reinforce a strategy when writing a problem was found in her description as to why she presented her students with the Join-Change-Unknown problem:

Camille is reading a new version of the Gingerbread Man. She has read ____ pages. The book has ____ pages. How many more pages does she need to read?

(9, 40) (8, 52) (16, 67) (34, 148)

As Ms. S explained, the problem type (JCU) supported students using a counting up strategy (lines 1 - 2 & lines 4-6):

- 1) Ms. S: Due to the nature that the unknown, is the, you know, missing in the
- 2) middle...Most kids look at it that way, as a counting up problem, where
- 3) they're, you're counting up, you don't know. So that's why I chose the first
- 4) problem. Um, and a lot of my children in terms of the counting up are getting
- 5) really efficient at counting up by something other than just by 1s? Most of
- 9) them are counting up to something and then counting up by larger increments?

Later, in the same interview, she detailed why she chose specific number pairs as well to support students that might use the same strategy:

- 10) Ms. S: Yeah, so, they've been thinking a lot about, um, counting up? And so, if
- 11) you were going to count up from 9 to 40? I chose 9 because I thought it was
- 12) really close to 10, and so it would be easy to think, okay, I'm going to count to
- 13) 10, and then I can count forward easily, either by 10s or even a larger jump, or
- 14) chunk, or amount to 40. And 40 is a nice, um, even decade number. Uh, the same

- 15) here with 8 and, uh, 52. We've been talking a lot about number combinations, or
 16) trying to reinforce the number combinations to 10 so here I chose 8, and it's a
 17) number combination to 10, and then I was thinking they would count forward to
 18) 52. And I was hoping they would use 10s? Or they would get to 10 and say 'oh,
 19) it's 40 to 50.'

Thus, through the problem type and specific number choices, Ms. S supplied the means to promote or reinforce her kids using a counting up strategy. As she explained, the fact that 9 was close to 10 (lines 10 - 13) might promote her students to count up to 10, and then continue counting up until they reached 40, a number that also served a specific purpose due to it being a decade number (line 13). The rationale was similar for the number pair, (8, 52). She also went on to explain why she chose the last two number pairs in the problem:

- 20) Ms. S: ...and then I was, these were just...it's not as easy of a number to look at
 21) and see, oh, it's really close to 10 here with the 16? Um, and I have a couple
 22) of kids that are, um, thinking about this, beginning to think about this as a
 23) Separate-Change-Unknown? A subtraction problem?...So, I tried to be careful, I
 24) didn't want to add any element of necessarily having to regroup?
 25) *[indicated (34, 148)]*...So, I was trying to think of numbers too where they would
 26) not have to do that.

In this excerpt, Ms. S spoke to having the objective to reinforce a subtraction strategy for some of her students (lines 20 - 23). Moreover, (lines 23 - 26), she seemed to provide further support for those students that might try to use subtraction, as she was careful to insure that regrouping would not be an issue for them with the number pair,

(34, 148). It is subtleties of this sort that reiterate the prominence these teachers placed on the structure of the numbers that they chose.

Provide access/rigor

Each of the teacher participants spoke to the wide range of abilities, understandings, and needs of their current students, and they stressed the importance of affording access to a task for all students, as well as offering challenges for all students. Thus, an emergent theme was the objective to provide mathematical access and rigor when developing a task, and the inclusion of a multiple number choice structure offered the means to accomplish the objective. First, I present evidence that each teacher held the objective to provide access to, and to challenge, students when designing a task. I follow with a specific example that illustrates how the problem type, as well as a multiple number choice structure was used to support meeting the objective.

Consider the following statements made by the participants. When speaking of her experiences with first grade students, Ms. F presented:

You wanna make some numbers that are accessible to all the kids, at first, therefore the 3 and the 8. Something like that, to start with, so they have some good understanding. And then you'd wanna put numbers in that would challenge kids as well.

It seemed clear that when developing a task, Ms. F wanted to insure that all of her students could find an entry point to the problem; thus, she chose numbers that she was confident all of her students would understand within the context of the problem. Yet, she also selected numbers that she thought would prove challenging. Ms. T offered a similar sentiment regarding third grade students:

Um, there's always a range. So I don't think that it's specific to this year's group. I always have the direct modelers and the builders, and I

always have the kids who are super high level, that challenge me to like ‘Oh my gosh.’ So I always try and put in a huge range... of, like, numbers. So that I’m addressing all of their issues.

And Ms. S referenced her use of a multiple number choice structure to address the same issues:

I put a lot of thought into these numbers before we start. To see, to try to make sure that the numbers, that the four number sets, or the five number sets that I choose can meet a wide range of needs and instructional goals.

Next, I present a specific example of Ms. S describing her choice of numbers when she designed a Join-Change-Unknown problem. The example illustrates the utility that both the problem type, and multiple number sets can play in regulating the mathematical rigor of a task. The following are excerpts are from the SSR interview after a lesson based on the Gingerbread cookie problem presented above:

- 1) Ms. S: I’m trying to choose more complicated problem types? So we’ve done
- 2) some Join-Result-Unknowns and a few Separate-Result-Unknowns... We’ve done
- 3) some Join-Change-Unknowns and some Separate-Change-Unknowns, but not a
- 4) ton.... So, in the last, um, I’d say maybe month to six weeks we’ve been doing
- 5) more Join- Change-Unknowns. And some more Separate-Change-Unknowns
- 6) because it’s a little bit more difficult. Due to the nature that the unknown, is the,
- 7) you know, missing in the middle.

In this excerpt, Ms. S voiced the intent to present her students with a more complicated problem type (line 1), which was met by selecting a Join-Change-Unknown problem. She described this type as being more challenging because the unknown is the second addend in the problem (lines 5 - 7). That is, the naked number sentence for the

first number pair would be, $9 + \underline{\quad} = 40$. In this case, the problem type was a way to increase the mathematical rigor of the task as her students had been mainly solving problem types where the unknown was the sum. Now, consider her explanation for selecting the respective number pairs:

- 8) Ms. S: Yeah, so, they've been thinking a lot about, um, counting up? And so if
 9) you were going to count up from 9 to 40, I chose 9 because I thought it was really
 10) close to 10, and so it would be easy to think, okay, I'm going to count to 10, and
 11) then I can count forward easily, either by 10s or even a larger jump, or chunk, or
 12) amount to 40? And 40 is a nice, um, even decade number.

Ms. S indicated (lines 9 - 10) that the first number pair, (9, 40), would be easy for her students to think about in terms of the strategy (counting up to 10, and then incrementing up to 40) that she anticipated many of them would employ. She reported a manner through which she added rigor to the problem:

- 13) Um, and then I was, these were just, it's not as, it's not as easy of a number to
 14) look at it and see, it's really close to 10, here with the 16? [*indicated (16, 67)*]
 15) Um, and I have a couple kids that are thinking about this, beginning to think about
 16) this as a Separate-Result-Unknown? A subtraction problem? So I tried to be
 17) careful, I didn't want to add any element of necessarily having to regroup?
 18) [*indicated (34, 148)*]...I was trying to think of numbers too where they would not
 19) have to do that? If they looked at it as a subtraction, they would not have to.

Ms. S indicated that the number pair, (16, 67) might provide more of a challenge to her students (line 14) because the 16 was greater than, and not close to 10; therefore, it presented more of a challenge to a student using a counting up from strategy. The pair,

(34, 148), was selected for students that were beginning to view a Join-Change-Unknown problem type as a subtraction problem (lines 15 - 16), which research shows is a more advanced type of thinking because the students are reversing the action of the problem (Carpenter et al., 1999). It was interesting to note the level of detail that Ms. S considered in selecting the pair, (34, 148), as she did not want to present a student that was going to use subtraction the additional difficulty of having to regroup across any of the place values.

Problem makes sense

Another factor considered by the participant teachers when developing a task was the objective that the task would *make sense* to the students. Because the idea of making sense of a task can take on different meanings, and a certain degree of inference was necessary during data analysis, it is important to explain what I took to be the meaning of *making sense* of a task. I followed the lead of Ms. T, who, when referring to the Common Core State Standards, said of her students, “I want them to read and make sense of the problem.” Hence, it seemed appropriate to appropriate the idea of making sense of a task from the first CCSS Standard for Mathematical Practice: *Make sense of problems and persevere in solving them*, which states, in part:

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. (CCSSM, 2010, pg. 9)

The above excerpt lent itself to the notion that making sense of a problem entailed looking for meaning, or relevance of the problem, and gaining access to the problem by

analyzing the known and unknown quantities and other parameters. Ms. S offered a similar line of thinking when she said:

I want to have a class of kids that learn that math is not just about finding an answer. That math is about thinking about, and visualizing, conceptualizing a problem. And that math should make sense.

Whereas, Ms. F never used a phrase akin to “making sense” during interviews, she did allude to the importance of matching the task to her students’ current level of mathematical understandings. For example, when asked about her objectives for a particular lesson and problem, she said:

...when I looked through my collection of problems I look at the numbers that I have chosen, and I see the, where I think the kids are struggling, and see if I have, um, problems with situations and numbers that work for my kids.

Thus, it appeared that to Ms. F, a problem should align with her models of her students’ abilities, and by presenting them a problem “...with situations and numbers that work...” she seemed to be implicitly providing them with a platform for making sense of the problem.

To support students in making sense of a problem, each of the teachers used similar means through which to do so. For example, each tended to relate a problem to the classroom by situating it within a context that was familiar to the students. They related to students’ personal lives by using their names, names of family members, or names of other commonly known persons, places, or things within a problem. In addition, they incorporated into the context of problems recent classroom events, or vocabulary that related to other disciplines the class was studying. The following three

problems illustrate some of the ways that the participants used context as a means for making sense of a task. This was a Separate-Result-Unknown problem written by Ms. F:

There are _____ macaws in a tree. _____ fly away.
How many are left?
(7, 2) (25, 16) (102, 73) (32, 20) (152, 49) (1,927, 972)

Her rationale for the context was, “We’ve just been to Sea World and they have macaws in it. We’re also studying habitats so this goes with the rainforest we’ve been talking about.” Thus, she related the problem to a recent classroom field trip, as well as a science topic the class had been discussing.

Regarding the Part-Part-Part-Whole, Whole-Unknown problem presented above that referenced collecting rocks, Ms. S reported:

It’s what we’ve been studying in science. So, we’ve been studying rocks in science. Yesterday in science we, um, were describing rocks? Using these, some of these words--bumpy, shiny, smooth. We were talking about texture and luster. And so that was basically what we’ve been studying in science. So, and Mrs. Freed is my job share partner.

Again, this was an example that related the context to a recent topic in another discipline. She also included the name of the students’ other teacher, which would be familiar to the class. The final example was the Toys for Tots partitive division problem written by Ms. T. She wrote the problem during the winter holiday, and her reason for the context was:

...And I thought, well, we’re doing this collection for Toys for Tots right now, and they’re, they’ve been bringing in presents and stuff so it’s topical, it’s what were working on. So, it’s a holiday thing I’ll bring it in.

Furthermore, when talking about this last problem, Ms. T explained that she wanted the numbers to make sense to the students relative to the context of the problem. That is, she was concerned that as the numbers got larger, that the students would not be able to relate to the number of presents that each family would receive. As she described:

Uh, it was interesting because I thought, well, these numbers aren't really gonna make a lot of sense. Like it was kinda hard to think of numbers that would make sense in the context of how many each family would get?...I mean, you know, like this morning when I was thinking about it, and then I thought, yeah, but then the numbers are gonna make no sense whatsoever. Like, why would you give a family like, you know, whatever, three hundred and twenty seven gifts? Like that's crazy.

Whereas, Ms. T was explicit in detailing that she wanted the numbers to make contextual sense to the students, Ms. F and Ms. S did not use such precise language. However, an argument can be made that Ms. F and Ms. S also considered number sets to be an important part of supporting their students in making sense of a problem. Part of the basis for the argument is found in the prior section on Provide Access/Rigor. In that section, evidence was presented that each of the participants included easy number sets in every problem to provide access to the task for all of their students. Since “looking for entry points” to the solution of a problem is part of the CCSS description (and my interpretation) for making sense of a problem, it followed that Ms. F and Ms. S also considered the numbers they included in a problem to be part of supporting their students in making sense of a problem.

That said, providing easy numbers to students appeared to provide more than just access to a problem; easy numbers also seemed to help make the problem realistic and imaginable to the students as aligned with Gravemeijer and Dorrman's (1999) conceptualization of Realistic Mathematics Education (RME). In RME a problem is defined to be a context problem “if the problem situation is experientially real to the student” (Gravemeijer & Dorrnam, 1999, pg. 111). For example, in the Toys for Tots problem presented above, one can imagine that a third grader might have a difficult time

contextualizing the number set (4,590; 90), that is, 90 families sharing 4,590 presents, thus being unable to perceive the problem as being realistic. By reading the problem to her students using the number set (30, 6), Ms. T seemed to be providing an “experientially real” context to the students in the sense that they would be able to imagine 6 families sharing 30 presents equally. To summarize, an emergent objective was to have the task make sense to the students, and the supporting means included relating the context to the students and the classroom, and providing numbers that were realistic, and imaginable to the students.

Foreground mathematical concept

The problem type and specific number choices offered a way to foreground a mathematical concept, or topic during a lesson. This will be demonstrated below by an example from each participant, and this is followed by evidence from each regarding foregrounding a common idea involving money. First, is an example from an interview with Ms. F that showed how she supported the principle of the commutative property of multiplication by choosing the number pair (6, 4) in a multiplication problem:

- 1) Ms. F: Um..this number I chose to see if they could flip it? [*pointed to (6, 4)*]
- 2) As one of them because it's a small, um, product.
- 3) R: The commutative property?
- 4) Ms. F: Yes. It's small enough for them that they could do it just by counting by
- 5) 1s. And I could see if some kids could count by 4s. So, my objective with this is
- 6) to see if kids can take apart numbers and count by 5s, 10s, 20s, and talk about
- 7) the commutative property.

So, the number choice of (6, 4), because the pair formed a small product (line 2), was a means for Ms. F to promote the commutative property of multiplication. In the next example, Ms. S talked about how a Join-Change-Unknown problem could foreground the reciprocal relationship between the operations of addition and subtraction:

- 1) Ms. S: Um, well there used to be a second grade standard about fact families.
- 2) But we don't, we don't have that anymore. I mean we don't have that, I don't
- 3) think that that standard is necessarily in the Common Core. But, um, you know,
- 4) we're talking about it all the time in terms of the reciprocal nature of addition and
- 5) subtraction. Um, it, usually we're talking about it in terms of, um, you know, a
- 6) lot of times with a known fact like 14 minus 8 is 6. Well, how did you why?
- 7) I knew it because 6 plus 8 is 14. That's usually how the conversation comes out.

Research suggests that students will follow the action of a JCU problem and make it an addition problem, but as their mathematical sophistication grows, they might reverse the action and turn it into a subtraction problem (e.g., Carpenter et al, 1995). Thus, for Ms. S, a JCU problem with accessible numbers seemed to be a means to begin a conversation about the reciprocal relationship between addition and subtraction (lines 3-7).

In the next example, Ms. T used the following problem to foreground, among other things, the idea of equivalent fractions:

I baked a pie and cut it into ____ slices. Allison ate ____ of the pie, Natalie ate ____ of the pie, and Maddie ate ____ of the pie. How much pie did they eat in all?

(5, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{2}{5}$) (8, $\frac{2}{8}$, $\frac{2}{8}$, $\frac{1}{4}$) (16, $\frac{3}{16}$, 2, 16, $\frac{1}{4}$)
 (6, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$) (12, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{12}$)

In an excerpt from the SSR interview conducted a couple of hours after the lesson, Ms. T was asked if the lesson was mainly an assessment day:

- 1) Ms. T: It wasn't intended to be that way, but everyday is an assessment day, and I
- 2) felt that this problem when I come back to fractions, I'm finding again and again
- 3) that, oh yeah, they don't really know anything yet so, but they do know some, and
- 4) I think that every time I do it, they know a little bit more. And, uh, yeah, so, I
- 5) think that, yes, that that was a big piece of it. But I also do think that they had the
- 6) opportunity to learn a few new things within it, like equivalent fractions, and
- 7) maybe the relationship with dividing into pieces, and pieces. How many pieces
- 8) go into how many pieces?

In line 7, Ms. T suggested that the concept of equivalent fractions was one that might be supported by the problem, and the second number set, (8, $\frac{2}{8}$, $\frac{2}{8}$, $\frac{1}{4}$), introduced the concept to her students.

The final example in this section illustrates how each participant foregrounded the idea of money for their students when designing a task. Ms. F, and Ms. S utilized a multiplication problem with specific number choices to do so, while Ms. S met the objective through the context of a problem. First, Ms. F said:

And then here with the 25s [*pointed to (3, 25)*] we're still working with money, a lot. And so I want to know if kids know quarters yet. There's a lot of kids that don't. So I, it's small enough, the 3 was small enough, to just put the three 25s there to see if they could look at it.

Ms. T addressed foregrounding money in a multiplication problem in a similar fashion:

So I was sort of playing around with these, just easy numbers. With these [*pointed to (100,25)*] I always bring in quarters to see who's gonna get it, and I just did the hundred in quarters and didn't do any other sort of money things cause they're still struggling. And you saw, like, I had kids who are still saying 5 quarters equal a dollar.

Ms. S, on the other hand inserted the concept of money into the context of a Part-Part-Whole, Whole-Unknown problem:

Nicole has been earning extra money at her house. She earns ____ dollars in January, ____ dollars in February, and ____ dollars in March. How much money has Nicole earned?
(13, 27, 25) (59, 46, 41) (243, 165, 327)

Thus, the problem type, multiplication for Ms. F and Ms. T, and specific number choices afforded foregrounding or promoting the idea of money for students. In addition, the idea of money could also be inserted into the context of a problem as demonstrated by Ms. S.

Meet district standards

In both of the school districts where the study took place, teachers were expected to align their mathematics lessons with a set of grade level standards that included learning goals related to understanding place value. Each participant demonstrated that they held the design-objective to meet district math standards for their respective grade levels. However, the emphasis that was placed on this objective varied among the teachers, with Ms. F and Ms. T placing less emphasis on meeting standards than did Ms. S. In this section, I compare and contrast the differences that emerged among the participants relative to the role that meeting district math standards played in the development of a task.

In the district where Ms. F and Ms. T teach, the transition to adopting the State's version of the Common Core State Standards for Mathematics was still an ongoing

process. As Ms. F described the process, “We’re transitioning into it,” a statement that was echoed by Ms. T, and both brought up the fact that their district did not yet have a set curriculum in place. Ms. T remarked, “But I don’t feel that we’ve been given anything that’s actually different to use? As curriculum. So, it’s kinda odd to me that the whole district doesn’t have sort of a curriculum, a standards base guide of pacing guide, or whatever.” However, neither appeared overly concerned with how they were going to meet district standards as both conveyed confidence that CGI was already aligned with the CCSSM.

For example, when asked how Ms. F thought CGI aligned with, or supported the CCSSM, she answered:

Perfectly. I don’t think it’s going to be hard to go, to do anything mathematically at all using CGI. I think it’s, it’s the one thing that’s gonna stay. We’re, next year we have no, uh, consumables. So, in second grade they’re super worried about not having any math worksheets... So they’re gonna have to depend on some of us to help them. They, it’s, CGI’s already written into the Common Core.

And in a subsequent interview, she added:

And we haven’t memorized what the standards are gonna be for first grade, but I do know this—CGI fits in perfectly with math, and you can probably do all of your mathematics, with the Common Core, just by using CGI.

These last two passages seemed to support the claim that Ms. F was convinced that she could meet most of the first grade standards through the use of CGI problems in her math lessons. Therefore, to Ms. F, it appeared that the objective to meet district standards was accomplished through developing, or selecting from her library, problems that she viewed as suitable for her students.

Ms. T held a similar view towards meeting the district standards. She described the links between CGI and the CCSSM as follows:

As in every standard I use CGI for, and every problem I do in CGI covers about twelve standards on the Common Core... And I invited in the superintendent to sort of say like CGI hits the Common Core standards, and I want you to come see what we're doing in our classroom...I wrote a problem, and I printed out like the standards and I highlighted all the standards that were hit by that problem, and it was something like 85 percent of all the standards were hit in one problem. Okay? For Common Core math. Period. For third grade. I mean it's like, I mean you're not gonna get geometry, or whatever, in there necessarily, but it had, you know like, if you're looking under, whatever it was, multiplication, it had like everything.

So, Ms. T also seemed to consider that she met the CCSSM in a manner similar to Ms. F; that is, when developing a task, the means for Ms. T to reach the design-objective Meet District Standards were the types of problems and the number sets she selected. I am not claiming that either Ms. F, or Ms. T believed that they could meet all of the district standards by grounding their mathematics lessons in CGI. Rather, it seemed that both considered the problem types with particular number choices as the means through which they might meet the objective.

In contrast, Ms. S was much more explicit in her describing her practice in terms of meeting her District's standards. As described in the earlier case study report, for Ms. S, assessments of students' learning included a formal District trimester assessment that was linked to a set of District standards. Thus, when writing a problem for a lesson, she paid particular attention to the Proficiency Level Descriptors and the objective that her students would reach being secure in meeting the standards. As she said, when referring to the District standards:

And so I, I use those to sort of backwards plan. So, I say, okay, if this is what secure looks like, at the end of Trimester 2, what do we need to do to get most, if not all kids, there, or past there? Um, so the performance level descriptors which is a little bit different from the exam, help me to backwards plan? The exam, um, sometimes it gives me things that I think, oh, you know what we need to reteach that.

The idea of backwards planning from the District standards was a theme that emerged over the course of the interviews with Ms. S. The following transcript excerpts illustrate her considerations in regards to the District standards and District assessments when planning a task:

- 1) Ms S: This is from the District, so for numbers and operations in base 10 this is
- 2) the report card line item—Uses place value understanding in the properties of
- 3) operation.
- 4) R: It goes right back to the Common Core.
- 5) Ms. S: Right. And so this is what ‘secure’, this is a description of what ‘secure’
- 6) would look like in Trimester 1. Right? And so then I have to think it the, so I
- 7) look at this, and I use this concept to help me kind of. *[indicated third column*
- 8) *‘Secure in Trimester 1’]* So like in, so for right now in February’s the end of
- 9) Trimester 2. So I’ve looked at this one, and I get so that, okay, what do we
- 10) need to do to be able to meet this by February? So this is secure in Trimester 2,
- 11) and then this is secure in Trimester 3, So it’s a descriptor of the behaviors or
- 12) understandings that you would see for a child to be secure in this report card line
- 13) item.

Because the interview took place in February, Ms. S was designing problems to meet the Proficiency Level Descriptors for trimester 2 (Lines 8-10). Later in the interview she was more explicit in describing meeting specific District standards:

- 14) Ms. S: But we've been trying to imbed some of the tenets, or main
- 15) understandings of CGI in it. So this is "Numbers and Operations in Base Ten" so
- 16) it's essentially place value understandings? But so you can see 'Secure in
- 17) Trimester 1' they would use place value and properties of operation to add and
- 18) subtract within 100, to solve word problems, and then it lists the kinds of word
- 19) problems. And as the trimesters go on...the number size grows.

In lines 15-19, Ms. S referred to the standard of using place value and properties of operation to add and subtract within one hundred in regards to specific word problems. For this particular District standard the problem types were: (a) Join-Result-Unknown, (b) Separate-Result-Unknown, and (c) Part-Part-Whole, both Addends Unknown. In addition, in line 19, Ms. S referred to the increasing size of the numbers that the students should be secure in operating with in Trimester 2, and Trimester 3 respectively. Therefore, the means through which Ms. S met the District Standards were the problem types, and the number choices she made.

A final example illustrates the manner in which Ms. S used a specific problem type to meet a different District standard. The standard Domain was Operations and Algebraic Thinking, with the Report Card Line Item: *Represents and solves word problems involving addition and subtractions within 100*. To be secure in Trimester 3 a student "can write an equation that follows their strategy and one that follows the word problem." Associated with this objective was an assessment item on the Trimester 3

exam, which Ms. S referred to when talking about the Separate-Change-Unknown (SCU) problem she had written for the lesson that day:

So it's something like this [*wrote* $43 - \underline{\quad} = 19$] And so I thought, I'm trying to kind of access this thinking? And link it with a story problem. You know, cause they're gonna see this.

Here, Ms. S demonstrated that the SCU problem was her means to link to the Trimester 3 assessment item since the structure of a SCU problem is such that a student writing the equation $43 - \underline{\quad} = 19$ would necessarily follow the referents within the problem.

In summary, it seemed that district standards could serve as explicit, well-defined design-objectives, as was the case with Ms. S, or district standards might be more backgrounded in the task development process, as was the case with Ms. F and Ms. T. In either case, the problem type and associated number choices seemed to serve as the means through which to realize the design-objective of Meet District Standards.

Promote fact fluency

The final design-objective held by the participants was for their students to gain 'fact fluency', which because of the disparity in district expectations was viewed from different perspectives. In the case of Ms. F, she seemed motivated by her colleagues teaching second grade because in second grade the teachers want them to "have their number facts memorized." To Ms. F, fact fluency for first graders consisted of knowledge of "addition and subtraction facts," which she extended to include multiplication when describing what she might focus on during the last three weeks of the school year. As she said, "Because I think, I that's a real important thing to understand when they're going into multiplication. That's really important for them to understand, as is addition. They [second grade teachers] want them to know the, their fact families

well.” Ms. F seemed to support her students’ learning their multiplication and addition facts through problem solving, and the use of number sets that fostered fact fluency such as multiplying by 2s, 5s, and 10s, or including tens-friends pairs in addition problems.

Ms. T seemed to hold the perspective that fact fluency supported students in their problem solving strategies, and, in turn, she seemed to select numbers to promote the learning of number facts. For example, she included the number set, (99, 11), in a division problem because she had “been playing with the 11s a lot and they’re still not getting them.” In addition, from the first SSR interview regarding a lesson on multiplication she reported:

- 1) Ms. T: And I’m finding also that they don’t know, um, like any multiplication
- 2) facts. Like a lot of them don’t have any of that...So they’re all like sort of skip
- 3) counting up, even by 5s. And 2s. Like they don’t have any of those facts. So I’m
- 4) gonna get started on that, which I think is extremely helpful...even though I’m
- 5) not a, like, big proponent of you should memorize this and that, they do have to
- 6) have those facts so that they can work with them at a certain point. And, um, it
- 7) does make it harder for them to have a variety of strategies if they can’t just look
- 8) at it and go, well, 10 times 5 is 50... And, um, they should know their addition
- 9) facts too, and their tens-friends but they don’t, a lot of them this year.

In lines 1 - 3, Ms. T described that her students didn’t seem to know a lot of the facts that she would normally associate with a typical third grade class. In lines 4 - 8, she noted that knowing certain facts would support the students in their problem solving strategies. Finally, in lines 8 - 9, she offered a similar sentiment towards addition facts. Furthermore, in the case study report on Ms. T it was reported that because it was in the

second half of the school year, she was going to give more attention to students learning particular facts such as the multiplication facts up to, and including, 12.

In the previous section, Meet District Standards, I presented evidence that for Ms. S part of the task design process was to meet specific objectives within the standards of her district, and within these standards were objectives for fact fluency. Thus, for Ms. S, promoting fact fluency might also fall under the category of Meet District Standards. However, I present it here as a contrasting point of view to that of Ms. F and Ms. T, and also to foreshadow the penultimate section in this chapter that explores the relationships between the factors considered by the participants in the development of a task.

Ms. S also tendered that she wanted her students to gain fact fluency, but as stated above, it seemed that to her gaining fact fluency might be better described as an objective in meeting her district standards, and the problem type and numbers served as the means towards meeting the objective. As she described:

And also just in terms of wanting to develop some fluency with facts, so that's why I've been, um, when I do like a Join-Result-Unknown, or a Part-Part-Whole, Whole-Unknown, I'm trying to choose numbers that help to develop fact fluency as well.

And in a later interview when describing the standards that she may have met during a lesson that focused on a Join-Change-Unknown problem:

...and so I'm trying to pull out all of those pieces of fact fluency because there's a second grade, I look at all of those as sort of building fact fluency, um, a second grade standard about fact fluency to 20.

In the above quote, Ms. S was referring to the district standard and the Report Card Item: "Adds and subtracts within 20, using mental strategies," under which a student being secure in Trimester 3, "...fluently adds and subtracts numbers up to 20 using mental

strategies.” This particular district standard was linked to the CCSSM standard 2.OA.2 wherein second grade students: “Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers” (CCSSM, 2010, pg. 19).

Whereas it seemed that Ms. S held an objective of meeting her District standards for fact fluency, Ms. F and Ms. T did not refer to any specific standards when discussing fact fluency. Admittedly, in the case of Ms. S, there appeared to be an apparent blurring, or overlap, of the two design-objectives, Meet District Standards and Promote Fact Fluency. This overlap served as another example of a relationship between two factors that might be considered by the participants when developing a problem, and this topic is explored in greater depth in the next section in this chapter.

Relationships Among Factors

This section illustrates the complex nature of the practice of developing a problem for a mathematics lesson. It became apparent that the 11 categories of emergent factors were not discreet elements of the practice; rather, there was interplay, or overlap between and among the factors. For example, it has already been argued that due to the innate structure of the goal/objectives/means framework the seven design-objectives seemed to be directly related to the two identified learning goals in that the design-objectives served as benchmarks towards reaching the two learning goals. Moreover, at times, a statement made by a participant was interpreted as evidence in support of two, or more of the emergent factors in what might be described as patterns of interlacing strands of the problem development process. For example, in the section on the influencing factor, Teaching Practices, evidence was provided that the two teaching practices--anticipating

students' thinking and anticipating a response to students' thinking--were coupled. In addition, in the previous section there appeared to be some overlap between the design-objectives Promote Fact Fluency and Meet District Standards.

To further the argument that the design-objectives served as benchmarks for both of the learning goals, understanding place value, and gaining sophistication in strategies, I offer as evidence excerpts from a clinical interview with Ms. F that was conducted immediately prior to a lesson with a focus on the Separate-Result-Unknown problem:

There are ____ macaws in a tree. ____ fly away. How many are left?
(7, 2) (25, 16) (102, 73) (32, 20) (152, 49) (1,927; 972)

To begin, I offer that by definition the design-objective, Promote/Reinforce Strategy, supported the learning goal for students to gain sophistication in strategies. Therefore the bulk of the argument supports the claim that promoting a strategy served as a benchmark towards the learning goal of understanding place value. In the first excerpt Ms. F was explaining the number choice (102, 73):

- 1) Ms. F: It helps a lot to get to that 'ten-ness' when you're estimating your answer.
- 2) They'll need that later on....same with this one? [*indicated (102, 73)*] It's very
- 3) close to a hundred? So, later when they're doing harder division problems and
- 4) things like that, if they have a good, solid place value knowledge, they tend to do
- 5) better in the upper grades.

In line 1, Ms. F referred to the idea of ten-ness as part of her conception of place value understanding. In lines 2 - 3, she explained that because 102 is close to 100, it might help students in estimating an answer (line 1). Later, she explained the number choice (152, 49):

- 6) R: And how about the one fifty two and 49?
- 7) Ms. F: That one? I want them to see if they can go to a 50, round up to a 50,
- 8) [*pointed at 49*] and rounding is really important later too in third grade. It's really
- 9) a difficult skill for a lot of kids...So, if I ask if this was a 50? What would it be?
- 10) And then to adjust their thinking in the middle of solving that problem.

In this excerpt, Ms. F alluded to the 'round and adjust strategy,' whereby a student might round the 49 to the benchmark number 50, take the 50 away from 152, and to the difference, 102, add back the 1 that was put aside in the rounding process, thus producing a final answer of 103. It seemed that by promoting the round and adjust strategy, Ms. F was supporting her students in developing better place value understanding, in particular developing a better conceptualization of ten-ness, and the use of benchmark numbers like 50 (line 7) and 100 (lines 2 - 3). Therefore, a student that became proficient using the round and adjust strategy in such a manner would have demonstrated to Ms. F both an increased understanding of place value, as well as growth in sophistication of strategies, which underscored the claim that the design-objective, Promote/Reinforce Strategy, served as an objective towards reaching both of the learning goals.

In a similar interview with Ms. S, it seemed that the design-objectives of Meet District Standards (and, in her case, Promote Fact Fluency) also supported reaching the learning goal of place value understanding. She was discussing the Part-Part-Part-Whole, Whole-Unknown problem:

Mrs. Freed is collecting rocks. She collects ____ bumpy rocks, ____ shiny rocks, and ____ smooth rocks. How many rocks does she have all together?

(23, 17, 27) (47, 38, 22) (126, 144, 214) (234, 179, 141)

- 1) Ms. S: And the, um, numbers I chose, the initial numbers, the
- 2) smaller numbers [*pointed to (23, 17, 27)*] I'm looking to see if
- 3) anybody notices, or they're looking at the ones, that there's, can they
- 4) make a 10? With the 7 and the 3? Or, the idea of the 7 and 7 making a
- 5) 14. To see if they notice any facts that they know. Here [*pointed to*
- 6) *(47, 38, 22)*] I was looking to see if they, again, I'm wondering what
- 7) they'll, we've been talking a lot, playing a lot of games about doubles?
- 8) So to see if they notice anything about the 7 and the 8 in terms of
- 9) pulling out a doubles fact, either 7 and 7, or 8 and 8. Here again, 8
- 10) and 2, the 8 and the 2 could make 10. So I'm kind of wondering if
- 11) they'll notice any of those relationships. And the tens here are a little
- 12) bit more difficult because it's gonna go over a hundred.

In this excerpt Ms. S seemed to be supporting her students learning place value by providing them opportunities to use known facts to create bundles of tens. For instance, in lines 2 - 5, she offered that one reason she included the number set, (23, 17, 27), was because the set included a tens-friend (7 and 3) in the ones-digit. This reasoning was repeated in the lines 9 - 10 for the number set, (47, 38, 22), which offered an additional place value challenge as their sum required regrouping tens into a hundred (lines 11 - 12).

I have already presented an argument that for Ms. S there seemed to be an overlap between the design-objectives Meet District Standards and Promote Fact Fluency. There also appeared to be relationships between other design-objectives. For instance, in the immediate example the interview excerpt was also used as evidence that Ms. S held the

design-objective of Assess Students' Thinking. This seemed to suggest that the problem type and number choices afforded the means through which Ms. S could assess her students' thinking related to number facts (e.g., lines 2 - 5) as well as to promote her students in gaining fact fluency. The idea that assessing students' thinking might be coupled to another design-objective seemed to support Ms. T statement that "...but everyday is an assessment day," and that different objectives could be met through like means.

Another example was taken from the SSR interview with Ms. T following a lesson with a multiplication problem. The two design-objectives that seemed to overlap, or were coupled, were Promote/Reinforce Strategy and Provide Mathematical Access/Rigor. The following excerpt revisited Ms. T when she was discussing the process of writing a problem (much of the transcript was presented earlier to support the importance the participants placed on writing their own problems), and it picked up when she began to talk about selecting number choices:

- 1) Ms. S: ...and then once I pick, I start putting numbers in, and I try and think
- 2) about the numbers. I mean, it's mostly about the numbers for me. And so, I
- 3) always try and start with numbers that are approachable that, like, even my direct
- 4) modelers can draw out and do and make sense of the problem. And then I start
- 5) thinking about, it depends on what kind of math it is, like, if it's multiplication,
- 6) well, I gotta throw in some numbers that are round-able. I have to throw in some
- 7) numbers that kids can break apart. I have to throw in some numbers, um, you
- 8) know, like, with money. Or whatever, things that I think that they can approach
- 9) with mental math, without algorithms, and then by the time to the end I know I

- 10) gotta put some in that are super challenging for those kids who are higher, and
- 11) who can do all of those mentally really quickly. And I also will think about, like,
- 12) well, do I want to do some double-digit by double-digit multiplication because
- 13) that's something that, you know, will put off a new thing. ...While I'm working
- 14) on this too, and my goal really usually is to be able to pull a variety of strategies. I
- 15) mean I'm thinking about what strategies the kids will use, but I'm thinking about
- 16) the numbers I want to put in.

In lines 2 - 4, Ms. T appeared to consider the strategy of direct modeling as it related to students gaining mathematical access to the problem. In lines 4 - 10, she talked about numbers that promoted the strategy of round and adjust ("...numbers that are round-able") and the strategy of decomposing numbers to use the distributive property when multiplying ("...numbers that kids can break apart"), and seemed to tie these particular strategies to making the problem more rigorous and challenging for some students. The idea that more complex strategies were coupled with providing challenges for students also seemed to support Ms. T's conception that learning is demonstrated by an increase in the sophistication of the strategies employed by a student over time. Thus, it was not surprising to find a link between the two design-objectives of interest.

Although I presented an argument to support my contention of a relationship between the design-objectives, Promote/Reinforce Strategy and Provide Mathematical Access/Rigor, there also appeared to be other embedded design-objectives present in the excerpt. For instance, in lines 2 - 4, Ms. T seemed to suggest that making sense of the problem, another design-objective, was linked to providing mathematical access to the problem. Additionally, in lines 7 - 8, she mentioned putting in numbers to support

working with the idea of money, and in lines 12 - 13, the concept of double-digit by double-digit multiplication was considered, both of which seemed linked to foregrounding a mathematical concept, as well as increasing mathematical rigor, and promoting different strategies. Figure 5.3 illustrates what seemed to be the embedded nature of the complex relationships between the factors in this example.

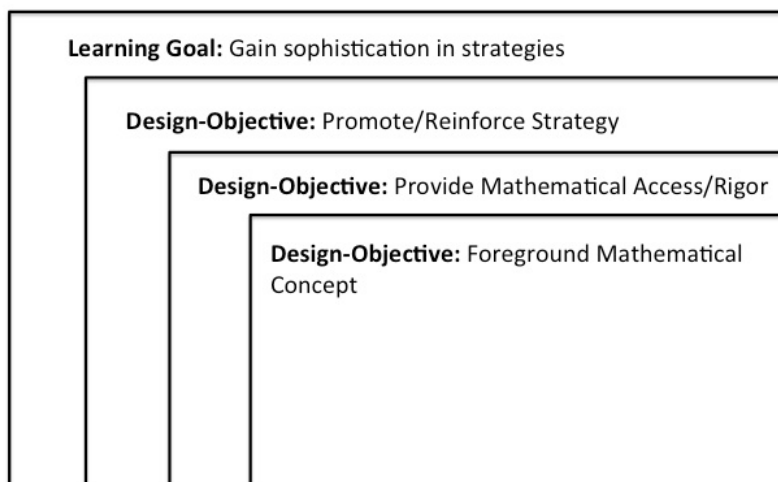


Figure 5.3: Embedded nature of design phase factors

The final example demonstrates that the teaching practice of reflecting on a lesson served a key function in the formative assessment of a lesson. To introduce adding fractions to her students, Ms. T developed the problem:

I baked a pie and cut it into ____ slices. Abby ate ____ of the pie, Nancy ate ____ of the pie, and Millie ate ____ of the pie. How much pie did they eat in all?
 (5, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{2}{5}$) (8, $\frac{2}{8}$, $\frac{2}{8}$, $\frac{1}{4}$) (16, $\frac{3}{16}$, 2, 16, $\frac{1}{4}$)
 (6, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$) (12, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{12}$)

Later, Ms. T essentially redesigned the problem during the follow up SSR interview because of concerns that arose during the lesson. First, she spoke to issues that her students seemed to have partitioning a circle into five equal parts. As she said:

...just starting with a circle cut into fifths was not a very smart thing...so it immediately starts out with those unequal pieces, which it's like, okay, is that because they don't understand, or is that because it's really hard to cut a circle into 5 even pieces?

Research suggests that children may find it difficult to partition circles into odd numbers of equal portions, and that some use a horizontal partitioning strategy that does not result in equal parts (e.g., Charles & Nason, 2000). The context of the problem suggested that Ms. T's students use a circle to represent a pie so when it was read to them with the first number set with 5 slices, this seemed to make access to the problem difficult for some students. Subsequently, Ms. T offered that with the second number set in the problem she introduced the concept of equivalent fractions too soon in the number choice sequence:

- 1) Ms. T: But I would like to not start with fifths. I would like to start with
- 2) something that's much easier to cut a pie into. I would like to do several that are
- 3) just adding fractions that all have like denominators. Before I go into playing
- 4) around with equivalent fractions? That might have been a nice transition. So to
- 5) have just the one, and then to go that? That was like a little bit rough. But I felt
- 6) that once they could do the drawing of this, every kid who had it was successful
- 7) with it.

In lines 2-4, Ms. T explained that she introduced equivalent fractions too early in the number choice sequence, and that the next time she selected this problem for a lesson she would add more number sets that all had common denominators before introducing a set with equivalent fractions. She also offered that it seemed that once the students could represent the pie cut into the appropriate number of equal pieces that they found success

with the problem. This seemed to underscore the fact that Ms. T had realized that the students immediately encountered a conceptual hurdle working with the first number set, (5, $1/5$, $2/5$, $2/5$), and that she would change the first set to offer more access to the problem (lines 1-2). Consequently, at the end of the interview Ms. T demonstrated that her reflection on the lesson was a major influence for her to redesign the problem:

- 8) Ms. T: So can I use your pen to just put my notes for numbers?
- 9) R: Yes, you can. Absolutely. Since I have this one running, tell me what you're
- 10) doing there.
- 11) Ms. T: I'm just writing down what I said before.
- 12) R: Such as...
- 13) Ms. T: Don't start with fifths. Add a few more like denominators.
- 14) R: So would you say that what happened today influences what you're doing
- 15) right there?
- 16) Ms. T: Yeah... Well, yeah, clearly. I mean, when I come back to it, often I'll be
- 17) like...that wasn't, I needed a few transitional numbers, or whatever. You know?
- 18) So, I'll often put that in for myself. And because, like now, I just keep them in a
- 19) folder and reuse sometimes, it's like I'm not thinking of where my kids are when
- 20) I'm writing them? And that's a problem.

Ms. T's reflection on the lesson served as a final example that relationships exist among the identified factors in this chapter. Because the initial number set proved difficult for students to find an entry point to the problem, Ms. T decided that next time she would start with fractions that students could partition more easily. Thus, the practice of reflecting on a lesson seemed to be linked to offering access to a problem. The

practice also appeared to be linked to foregrounding a mathematical concept as Ms. T decided to present more number sets with common denominators before introducing the idea of equivalent fractions (a link to providing access also seemed to exist in this example). And in lines 16 - 20, it seemed as if Ms. T was also reconsidering the manner in which she utilized her problem library, and that she should always base her number choices on the understandings that her students hold at that moment in time. Therefore, the practice of reflecting on a lesson seemed to be instrumental in the assessment of the lesson which lead to a restructuring of the number sets for a future lesson. That is, by reflecting on the lesson, Ms. T found that she was not reaching one of her design-objectives, providing mathematical access to the problem; hence, she changed the means attached to reaching that objective, namely the first number set, to something that she felt would work better the next time she taught that lesson.

Given the complex nature of teaching, it was not unexpected that the findings suggested that relationships existed between, and among, design-objectives, as well as associations with the identified influencing factors. However, it was beyond the scope of this study to investigate the intricate mosaic of relationships among the many factors, as the intent was to establish a basis for the practice of developing mathematical tasks from the perspective of expert teachers--not to produce an exhaustive narrative of the practice. Accordingly, the subject was not specifically addressed during the data collection process. For now, it will have to suffice that the existence of such relationships seem highly probable, and worthy of further investigation.

Summary

There was a lot presented in this chapter. It began with a discussion of a set of four factors that seemed to influence the teachers' decisions when developing a problem, and one of these influencing factors was directly linked to set of seven design-objectives that the teachers seemed to consider in the process. Figure 5.1 illustrated the hierarchy of the 11 identified factors and it is reproduced below to reiterate the link between the influencing factor Learning Goals and the design-objectives. What Figure 5.1 does not illustrate is the web of relationships that seemed to exist among the many factors. In Figure 5.1, the arrow running from the category Design-Objectives to that of Learning Goals represented the idea that the seven identified design-objectives served as benchmarks towards the two learning goals that emerged as particularly relevant for the participants in this portion of the study.

Arguments were also presented in this chapter to support the claim that the goal/objective/means triad existed as an integral part of developing a problem for a mathematics lesson. Figure 5.2 illustrated the triad by focusing more closely on the arrow between the categories in Figure 5.1, and linking the learning goal of understanding place value to two of the design objectives that were argued served as benchmarks towards reaching the goal. Figure 5.4 combines both dynamics into a single illustration of by a scenario where two design-objectives support reaching the learning goal, and the design-objectives are each in turn supported by to a respective pair of means. As in Figure 5.2, the shaded entries represent the particular goal/objective/means triad of Understanding Place Value/Foreground Mathematical Concept/Number Choices.

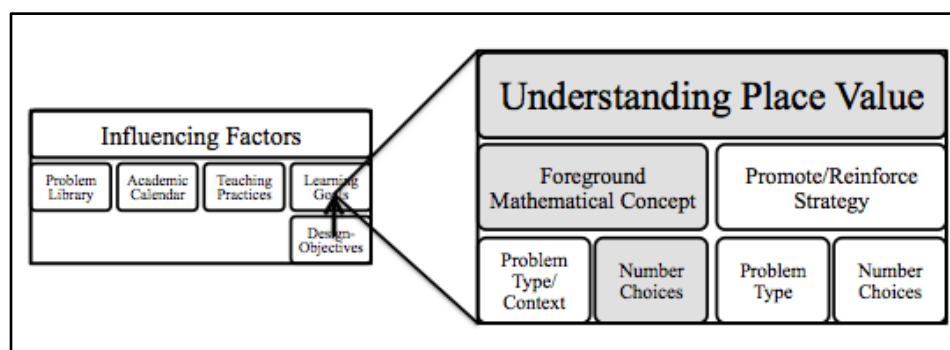


Figure 5.4: The dynamic structure of the design-objectives framework

Furthermore, Figure 5.4 somewhat illustrates the relationships that can exist between all of the factors in the emergent framework. The other three influencing factors each play a role in the design phase of a mathematics lesson with their impact on a teacher's learning goals, and their influence filters down to the design-objectives, or, what the teacher wants to accomplish with the lesson, and they thereby affect the means and the development of a task for the lesson. For example, in the previous section, it was demonstrated that the influencing factor of the teaching practice of reflecting on a lesson could serve as part of a formative assessment loop of the lesson by leading to a redesign of the spotlighted problem for use in a future lesson. This is a more substantive example of the filtering down metaphor of an influencing factor. To summarize, taken together, the factors that emerged as integral elements of developing a mathematical task in the design phase of a lesson and the web of relationships among the factors provide the foundation for the next chapter.

In Chapter 6, the focus turns to the enactment phase of a lesson, and it was found that the same structure of factors seemed to exist for modifying a problem for a student. To foreshadow the findings presented in Chapter 6, a subset of five of the seven design-objectives seemed to serve as the *mod-objectives* behind offering a student new numbers

with which to work a problem. The mod-objectives served as benchmarks towards the same two learning goals as did the design-objectives. In addition, the means for reaching the mod-objectives were found to be the structure of the new numbers, or a change of the context of the problem for a student. Thus the findings in Chapter 6 are similar to those presented in this chapter.

CHAPTER 6: FACTORS CONSIDERED WHEN MODIFYING A TASK DURING A MATHEMATICS LESSON

In this chapter, I present my findings from my analysis of the interview data, which addressed the second research question:

RQ2) What factors do elementary teachers who have an extensive understanding of children's mathematical thinking consider when adapting, or modifying mathematical tasks in the enactment phase of mathematics lessons?

The results were similar to the findings presented in Chapter 5 in the sense that the participant teachers seemed to hold specific objectives in mind when modifying, or adapting, a task for a student, and they employed particular means through which they intended to meet the objectives. I offer evidence to support the claim that there were five categories of objectives that the participants seemed to consider when modifying a problem during a teacher-student interaction. Table 6.1 lists and defines these *mod-objectives*, the categories of objectives identified during the enactment phase of a lesson.

Mod-Objectives and Means

Table 6.1 also demonstrates that the five mod-objectives form a subset of the seven design-objectives presented in Chapter 5. This finding seemed reasonable because the students were working on problems that the teachers had developed with the design-objectives in mind. Thus, it was not surprising to find that a modification to a problem seemed to fall within the same goal/objective/means triad found within the design of the problem. Moreover, it seemed also reasonable that the two categories of design-objectives, Meet District Standards and Problem Makes Sense, did not manifest during the enactment phase of a lesson. I am not claiming that the two objectives never manifested during a lesson, just that there was not sufficient evidence for including them

as mod-objectives in the working framework. For example, in the last section of this chapter that discusses the manner in which changing the context of a problem could serve as a means towards reaching a mod-objective I present an excerpt of a classroom episode between Ms. T and a student, Sandy, who was having difficulty connecting her answer to the referents found in the problem. Ms. T made the decisions to change the numbers and the context of the problem for Sandy, and an argument could be made that Ms. T held a mod-objective to help Sandy make sense of the problem. However, this episode was a singular instance from the corpus of data so there was insufficient evidence for including the category in the model.

Table 6.1: Definitions of identified mod-objectives

Mod-Objective	Definition
1) Assess Students' Thinking	The intent to assess students' mathematical thinking and understandings.
2) Promote/Reinforce Strategy	The intent to support the development of a particular strategy(s)
3) Provide Mathematical Access/Rigor	The intent to match the problem to a variety of students' mathematical understandings.
4) Foreground Mathematical Concept	The intent to focus on a specific mathematical concept, or topic
5) Promote Fact Fluency	The intent to support the development of fact fluency within the basic arithmetic operations.

In addition to the intersection of the sets of objectives that seemed to be instrumental to both the development of a task prior to a lesson and modifications of the task during a lesson, were the four influencing factors identified in Chapter 5. These influencing factors also seemed to overlap both processes. To reiterate, the influencing

factors were: (a) problem library, (b) academic calendar, (c) teaching practices, and (d) learning goals. The overlap between the influencing factors was expected because any modification of a task could occur only *after* a task had been developed, and later introduced to students in a lesson to be made available for modification. That is, a teacher could only act on, and modify an existing problem.³ Thus the four influencing factors identified in Chapter 5 seemed to play roles in both the design phase and the enactment phase of a lesson.

However, discussion is warranted on a particular teaching practice (an influencing factor) that emerged only during the enactment phase--the practice of eliciting students' mathematical thinking. Each participant engaged in this particular practice during all observed teacher-student interactions that resulted in the modification of a problem. That is, every modification of a problem occurred after a participant had questioned the student concerning the student's thinking when solving, or attempting to solve the problem. Therefore, it appeared that the decision to change the numbers in a problem, or the context of the problem, for a student was influenced by the student's explanation of their written work during the teacher-student interaction.

Complimentary to the mod-objectives were a set of identified means that emerged from analysis that seemed to support the teachers in meeting the mod-objectives. As discussed in Chapter 5, when the participants developed a task the predominant means towards meeting a design-objective were the problem type, and the number structures. In the enactment phase of a lesson, the number structures also seemed to play a predominant

³ While it was possible that a teacher might develop, or offer a student a different task during a lesson, this practice was not observed during the timeline of this study.

role when adapting a task for a student; however, three additional categories of means also emerged from the analysis. Table 6.2 presents, and defines, the four identified *mod-means*: (a) build on immediate, or prior work; (b) number structures; (c) known facts; and (d) change, or remove context.

Table 6.2: Definitions of identified mod-means

Mod-Means	Definition
1) Build on immediate, or prior work	The modification was linked to the student's immediate work, or a past strategy
2) Number Structure	The new numbers were intended to support or extend the student's thinking
3) Known facts	Commonly known number facts were presented to the student
4) Change or Remove Context	The context of the problem was changed to support or extend students' thinking

In Chapter 5, I presented my findings organized by each category of design-objective. In this chapter, I organize the discussion by teacher, as the definitions and underlying concepts for each category of objective have already been reported, and evidence was presented to support the conjecture that relationships exist among the objectives. Therefore, it seemed more economical to report on the five mod-objectives delineated by teacher because multiple objectives seemed to emerge within a single modification of a task for a student. I present evidence from the teachers' explanations of their thinking immediately after enacting a task modification and evidence from SSR interviews to support the claim that the identified objectives seemed to provide the foundations for the decisions made by the teachers to modify a problem for a student.

Furthermore, each modification occurred after a student had solved, or had attempted to solve the problem assigned that lesson; therefore, I also categorize each modification as a supporting, or extending teaching move as delineated by Jacobs & Ambrose (2008). To give a sense of the frequency for which the participants changed the numbers, or the context of task, during the three rounds of data collection the researcher investigated instances wherein Ms. F offered 27 such modifications to various students; Ms. S presented 9 modifications of interest; and, Ms. T modified tasks 9 times for students. Additional evidence is provided through copies of artifacts of students' work, and excerpts from transcripts of the teacher-student classroom interactions of interest.

To reiterate part of the methodology used to access the rationale behind each teacher's decision to modify a problem, they were asked to explain their thinking immediately following some of the modifications. However, they were not asked to explain their rationale for every decision to modify a problem because the researcher could either make a strong inference for the decision, or the researcher did not want to interrupt the flow of the lesson. In either case, the reasons for the modifications were investigated during each respective follow-up SSR interview. In those instances in which the lesson was interrupted by the researcher to question the teacher, the teacher had finished interacting with the student and was moving to work with another student. Thus, her interaction with the researcher occurred between working with students whom continued to work uninterrupted.

Ms. F: Task modifications

The first episode of interest occurred during the lesson that focused on the multiplication problem:

Lily has ____ bags of cookies. She has ____ cookies in each bag. How many cookies does Lily have altogether?

(8, 5) (7, 10) (6, 4) (19, 2) (20, 5) (7, 12) (3, 25)

In this episode, the student, Carl, had just worked with the number pair, (19, 2), and he had found two conflicting answers: 38 when solving 2 bags with 19 cookies each, but 34 when working with 19 bags of cookies with 2 cookies in each. His work (Figure 6.1), the transcript of the classroom interaction, including the interruption by the researcher, and an excerpt from the subsequent SSR interview is presented below.

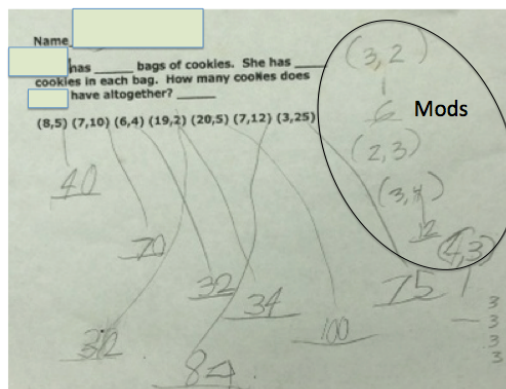


Figure 6.1: Carl's work with Ms. F's modifications highlighted

Note that in this transcript of a classroom interaction, and those to follow in this chapter, that the interruption of the researcher is separated from the teacher-student by a line, and the researcher's question is presented in bold font. Also, as was the case in Chapter 5 gestures are indicated by italicized print inside a pair of italicized brackets.

- | | |
|-----|---|
| 1) | Ms. F: So, what do you think? We're looking at this one. |
| 2) | Carl: Which one? |
| 3) | Ms. F: <i>[pointed to (19, 2)]</i> Cause this, this is two, 19s, <i>[pointed to answer of 38]</i> |
| 4) | That's correct. Nineteen, 2s, you got 34. Do you think that's correct for |
| 5) | nineteen, 2s? |
| 6) | Carl: Yeah. |
| 7) | Ms. F: Okay, let me ask another question here. Lets see if I can find you one |
| 8) | here. Three, we have 3 bags, 2 in each, what is the answer to that? |
| 9) | R: What were you thinking there? |
| 10) | Ms. F: Me? Well, I want to see if he can understand that it is the commutative |
| 11) | property. If we can get there. So, I went to simpler numbers. |

In line 8, Ms. F decided to modify the problem for Carl by changing the numbers from (19, 2) to (3, 2). The evidence suggested that for the change of numbers to (3, 2), Ms. F held four of the five mod-objectives. She wanted assess the students' thinking (line 10) by foregrounding mathematical concept of the commutative property of multiplication (lines 10 -11). The new numbers, (3, 2) were selected to provide mathematical access to the idea of the commutative property, and to promote the strategy that the community property might afford. The means that she employed to support the objectives were to connect to the student's immediate work done with the number pair, (19, 2), and to offer the student easier numbers with which to work, (3, 2). The claims are supported further in an excerpt from the follow-up SSR interview. It is important to note that later during the lesson, Ms. F also modified the task for Carl by changing the numbers to (2, 3), and later, (3, 4) then (4, 3) as highlighted above in Figure 6.1.

- 12) Ms. F: Now we're going to the commutative property.
- 13) R: Mm hmm, and why did you select those?
- 14) Ms. F: Because they were easy for him to do. And I wanted to do it quickly,
- 15) because he, he said that he was sure that two, 19s was thirty eight, and nineteen,

- 16) 2s was thirty four. But he was counting by 2s and it, that's the reason I put my,
 17) *[gestured putting out 9 fingers]* I modeled for the kids after they got ten, 2s I put
 18) my fingers out so they wouldn't get mixed up. They had a marker...I wasn't
 19) watching Carl so I didn't know if he had a marker or not for his 19. But I know
 20) he can count by 2s. So I wanted to see, um, if we visited the 3 and the 4, and
 21) turned them around, if he could go back to the 2 and the 19, or 19 and 2 and say
 22) which, which of those answers was correct. Was it the 34, or the 38? That's
 23) where I was going with him.

Arguments for each of the aforementioned categories of mod-objectives follow.

Assess students' thinking

When asked for her reasoning behind the modification (line 9), Ms. F offered that she wanted to see if the student could understand the commutative property of multiplication (lines 10 - 11). In addition, in lines 20 - 23, Ms. F explained her rationale for going on to change the numbers to (3, 4) and (4, 3) with the objective to assess his thinking in regards to the principle of the commutative property for multiplication, and to see if he could make the connection back to his work with the number pair, (19, 2).

Foreground mathematical concept

In line 10, Ms. F indicated that she was foregrounding the commutative property for Carl. The same argument applied to lines 20 -22 where Ms. F reported, "if we visited the 3 and the 4, and turned them around, if he could go back to the 2 and the 19, or 19 and 2 and say which, which of those answers was correct." The idea that turning around the 3 and the 4 seemed synonymous with the commutative property, and by offering Carl

another set of easier numbers to multiply, Ms. F seemed to view this as a way for Carl to see the commutative property in action.

Provide mathematical access

Ms. F seemed to be motivated for the modification with Carl finding two different answers when he commuted the product of 2 and 19 (lines 1 - 6). In lines 11 and 14 she explained that she changed the numbers to 3 and 2, “because they were easy for him to do.” Thus, Ms. F offered Carl numbers that were easier for him to multiply to provide access to the concept that she wanted to foreground--the commutative property.

Promote/Reinforce strategy

In lines 16 - 20, Ms. F talked about students using a strategy of skip counting by 2s when finding the product of 19 and 2, and that Carl was employing the strategy (line 16). Thus, she appeared to reinforce this strategy by changing the numbers to (3, 2); a number set that within the context of the problem was similar in structure to that of the pair, (19, 2). Therefore, it was reasonable to infer that Carl might have skip counted by 2s to find the product of 3 and 2; however, it could be that with such a friendly number pair that it could have been a known fact for him. The last two sub-sections also illustrated the complexity of the relationships among mod-objectives and means as the new numbers supported two different objectives. This was similar to the ideas introduced near the end of Chapter 5.

Promoting fact fluency

To finish the argument that Ms. F, at times, seemed to hold one or all of the mod-objectives I present an example that included the objective of Promoting Fact Fluency. In this classroom interaction, Ms. F was interacting with a girl, Taylor, who was working

the same multiplication problem presented above with the number pair, (7, 10). I present the transcript of the classroom interaction between Ms. F and Taylor, which is followed by an excerpt from the corresponding SSR interview. During this particular classroom episode the researcher did not interrupt to question Ms. F about her reasoning as it seemed clear to the researcher that she was questioning the student concerning the idea of “ten-ness” and 10s facts of multiplication.

- 1) Taylor: Is this 70?
- 2) Ms. F: What did you do?
- 3) Taylor: I just used my hands. Like, 10, 20, 30, 40, 50, 60, 70 [*counted on fingers*]
- 4) Ms. F: Yep...Tyler, can I ask a quick question? What are two, 10s?
- 5) Taylor: Two 10s? Twenty.
- 6) Ms. F: What are three, 10s?
- 7) Taylor: *paused* Thirty.
- 8) Ms. F: Okay, so you're still counting by 10s. You don't know it automatically yet.
- 9) If I said, what are six, 10s, what would you say?
- 10) Taylor: *quickly* Sixty.
- 10) Ms. F: Yeah, you trusted yourself. What are five, 10s?
- 11) Taylor: Fifty.

Excerpt from the SSR interview:

- 13) R: That was interesting to me because it was a very subtle way of changing the
- 14) problem a little bit. What were you thinking there?
- 15) Ms. F: Well, I, at this point I would love it to be automatic, that kids know that,
- 16) six, 10s is gonna be 60, eight, 10s is gonna be 80. And a lot of kids have it, but
- 17) Taylor, she still doesn't trust herself. She's a good math thinker, but she, she
- 18) reverts back to, um, less sophisticated strategies because she really doesn't trust
- 19) herself as a math student. And today she did. For instance, she will, when she's,
- 20) um, subtracting she will make a bunch of tally marks for the big number, and then
- 21) actually cross them out, that was a strategy that she used for a long time....And

- 22) today I noticed that she wasn't doing tally marks, she was, she made the bags, but
- 23) she could count by 5s and get the answer *pointed at work for (8, 5)* (Figure 6.2).
- 24) So she was, she was starting to trust herself there. Which I thought was just great.

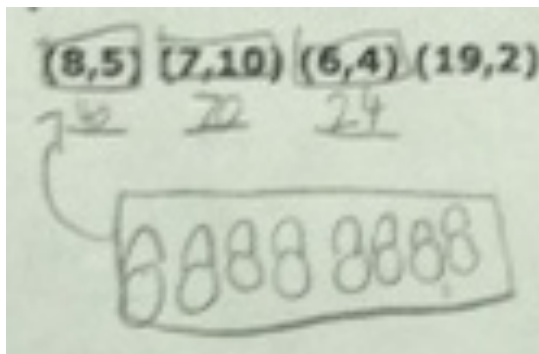


Figure 6.2: Taylor's direct modeling strategy for 8 bags of 5 cookies

In lines 2 - 3, Ms. F seemed to determine that Taylor was using a strategy of direct modeling by 10s. She then proceeded to offer Taylor a quick series of modifications of the original number pair, (7, 10), by changing them to (2, 10), (3, 10), (6, 10), and (5, 10) respectively (lines 4 - 10). For her reasoning behind the modifications, Ms. F offered that, "Well, I, at this point I would love it to be automatic, that kids know that, six, 10s is gonna be 60, eight, 10s is gonna be 80." (lines 15 - 16). Her reasoning seemed to support that she held the objective to promote fluency with multiplying 10 by single digits with Taylor.

Furthermore, in lines 17 - 23, Ms. F seemed to view that Taylor was on the cusp of gaining sophistication with her strategies, that is, Taylor seemed to be moving from direct modeling with 1s (lines 20 -22) to skip counting strategies, in the example, skip counting by 5s (lines 21 - 24). Thus, it appeared that not only was Ms. F promoting fact fluency with the series of modifications, it seemed that she was also promoting a more

advance strategy for solving multiplication problems for Taylor. The means that Ms. F used was to connect a series of similar numbers to Taylor's thinking about groups of tens. A similar argument could be made for Ms. F holding the objective of assessing Taylor's thinking regarding groups of tens during the classroom interaction. The next section presents similar arguments for the case of Ms. S.

Ms. S: Task modifications

The first example from the classroom of Ms. S was taken from a lesson that centered on the Part-Part-Part-Whole, Whole Unknown problem:

Mrs. Freed is collecting rocks. She collects ____ bumpy rocks, ____ shiny rocks, and ____ smooth rocks. How many rocks does she have all together?
(23, 17, 27) (47, 38, 22) (126, 144, 214) (234, 179, 141)

The teacher-student interaction of interest occurred immediately after the student, Jay, had explained his thinking about his strategy using the number set, (234, 179, 141) after which Ms. S decided to change the numbers to (375, 279, 632) for him. The student's work is presented in Figure 6.3, and is followed by the transcript of the classroom interaction between Ms. S and the researcher who had asked her to explain her reasoning for the modification, and then an excerpt from the SSR interview conducted later that day. For the modification, Ms. S seemed to have held the mod-objectives of (a) assess students' thinking, (b) promote/reinforce strategy, (c) provide mathematical rigor, and (d) foreground a mathematical concept.

Handwritten work showing the solution to $234 + 179 + 141 = 554$ using an incrementing strategy:

$$234 + 179 + 141 = 554$$

$$200 + 100 + 100 \rightarrow 400 + 70$$

$$\rightarrow 470 + 30 \rightarrow 500 + 40 = 540$$

$$\rightarrow + 9 \rightarrow 549 + 4 = 553 + 1$$

$$\rightarrow 554$$

Figure 6.3: Jay's work solving $234 + 179 + 141$

As mentioned the following transcript of the classroom intervention includes an interruption by the researcher:

- | | |
|-----|---|
| 1) | Jay: This should be 1. |
| 2) | Ms. S: Yeah. That's a lot of thinking here. And then you still added the 1 |
| 3) | (J: Mm hmm) and you got? |
| 3) | J: Four. |
| 4) | Ms. S: Okay. So, four hundred and fifty-nine plus 4, gotcha. I see just what you did. |
| 5) | So do me a favor, Jay. Will you do the same problem, but let's pretend there's |
| 6) | three and seventy-five bumpy, two hundred and seventy nine shiny, and six hundred |
| 7) | and thirty-two smooth. (J: Okay) Okay? You don't have as much room there. |
| 8) | R: What were you thinking there? |
| 9) | Ms. S: Um, Jay is really good at incrementing up? He increments up by the 100s and |
| 10) | by the 10s, and so I've been trying, we've been working with doing that |
| 11) | progressively, larger, more complicated numbers to see if he can do, um, essentially |
| 12) | larger chunks, or more efficient chunks of incrementing. |

An excerpt from the follow-up SSR interview:

- 13) R: Can you expound on that for me?
- 14) Ms. S: Um, so when he first started kinda incrementing he would increment, like,
- 15) let's just say it was 47 plus 23. He'd say 47 plus 10 is 57. Plus 10 is 67. Plus,
- 16) plus the 3. So he would break the 20 into 10 and 10. And now he would say, he

- 17) would be more likely to say ‘47 plus 20, I know, is 67.’ So, he’s using a, he’s
- 18) decomposing the 20 into like a bigger chunk of number? So, it’s a little bit more
- 19) efficient? And so I’ve been trying to work with bigger numbers, more
- 20) complicated numbers, to see if he can use more efficient, um, decomposing
- 21) numbers in a more efficient way so it’s less steps? Does that make sense?

The next sections are arguments to support the claim that Ms. S held the aforementioned mod-objectives when deciding to modify the problem for Jay.

Assess students’ thinking

In lines 9 - 10, Ms. S reported that Jay was proficient in employing an incrementing up strategy that included the decomposition of numbers into ‘chunks’ like 10s and 100s. In lines 11 - 12, she noted that she wanted to see if Jay could increment using larger, more efficient chunks, and she reiterated the objective of assessing his thinking in these regards in the SSR interview (lines 20 - 21). Her means for doing so appeared to include a set of numbers, (375, 279, 632), that were designed to link to the immediate work of the student and, and mirroring the structure of the previous number set, by offering Jay a set of three, 3-digit numbers to add. In addition, by providing Jay with larger, more complicated numbers, it seemed that Ms. S wanted to assess his thinking when he worked with more challenging numbers.

Promote/Reinforce strategy

Evidence that supported the claim that Ms. S held to objective to reinforce the incrementing strategy that Jay had just used was found in lines 10 and 11 of the classroom transcript where Ms. S said, “we’ve been working with doing that [incrementing strategy, with] progressively, larger, more complicated numbers.” This

statement appeared to indicate the objective to reinforce Jay's strategy by pushing him to increment using larger chunks of numbers, which the structure of the new numbers, (375, 279, 632), might support. In the problem that Jay had just worked the sum of the numbers was 554, and the sum of the new numbers was 1,286. Therefore, the modification offered him the opportunity to work with larger chunks of numbers. For example, he seemed to start with the hundreds place with his strategy (e.g., Figure 6.3), which with the new number set would have entailed him adding 300 and 200 to make 500. Next, the third number, 632, might have lead him to decompose the 600 from into 500 and 100, which could have resulted in Jay incrementing up first by the 500, a larger 'chunk', (thus reaching a sub-total of 1,000), before adding the remaining 100 and moving to working with the tens place. Figure 6.4 illustrates that Jay did indeed increment by a larger chunk; he incremented up by 600, and then moved to the tens place.

$$375 + 279 + 632$$

$$300 + 200 + 600 \rightarrow 1100 + 70 \rightarrow 1170$$

Figure 6.4: Jay's work with the new number set, $375 + 279 + 632$

Provide mathematical rigor

Evidence that Ms. S held this objective when she decided to modify the task for Jay has been couched within the arguments presented above. For instance, in lines 10 - 11, Ms. S. offered that she had been working with Jay with his incrementing strategy combined with "progressively, larger, more complicated numbers." A comparison between the original number set, (234, 179, 141), and the new set, (375, 279, 632), showed that each of the new numbers was larger than its respective counterpart in the

original set. The idea that the new numbers were more complicated was illustrated in the previous section by the fact that the new numbers summed to a value greater than 1000, which involved regrouping, whereas adding the original set did not entail regrouping. Thus, with the new number set, Ms. S seemed to meet the objective to make the problem more challenging for Jay, and as was already argued using this same evidence, Ms. S also seemed to be reinforcing Jay's strategy with the modification. The evidence common for both arguments was another example of the relationships that might exist among the mod-objectives (and other factors) when a task was modified by one of the participants. Another such relationship is also illustrated in the argument for the next mod-objective.

Foreground mathematical concept

In the preceding section, it was argued that Ms. S had the objective to increase the mathematical rigor of the problem for the student, Jay, by presenting him with "larger more complicated numbers," and it was argued that the new numbers were more complicated because the sum of the three involved the of regrouping. Therefore, Ms. S seemed to have held the objective to foreground the concept of regrouping hundreds into thousands, which is an important step towards understanding place value for students. This conjecture was further supported later on during the interview when after Ms. S had reviewed Jay's work (see Figure 6. 3), she remarked:

I'm impressed that he was able to know that this was eleven hundred. Right off the bat. Um, we haven't done a ton of work going over, I've done work going over a thousand sometimes? With some students? But not a ton with Jay, going over a thousand.

So, it appeared that Ms. S had been working with Jay on regrouping across 1000, which supports the claim that she wanted to foreground the idea for him with the change in numbers.

Promote fact fluency

In another interaction with Jay during the same lesson, but focused on a different problem, Ms. S seemed to hold the objective of extending, or promoting, Jay's fact fluency, specifically facts associated with multiplying by 10. The problem of interest was:

Mrs. Freed is organizing her rock collection. She makes _____ bags of rocks. In each bag she puts _____ rocks. How many rocks does she have all together?

(7, 10) (4, 15) (4, 18) (14, 10) (24, 10)

In the classroom interaction, Jay had just solved the problem with the number set (24, 10), but he had used a direct modeling strategy, whereby he had written the numerals 1 to 24 in order (to represent the bags of rocks) and then assigned each 'bag' the number 10 for the amount of rocks in each bag. The classroom transcript picks up with Ms. S confirming that this was his strategy, and then she modified the task by changing the number of bags of rocks in the problem to 10 bags.

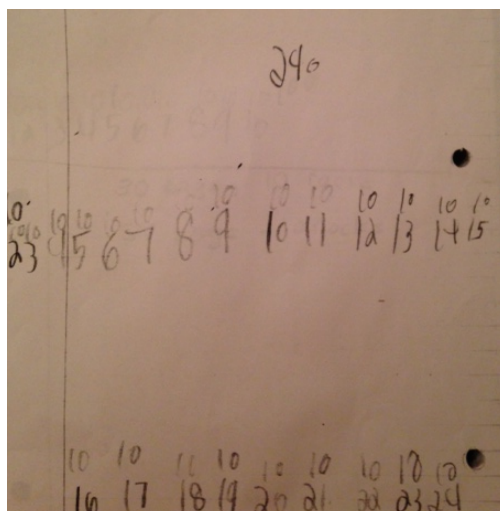


Figure 6.5: Jay's direct modeling strategy to solve 24×10

Classroom transcript. The researcher did not interrupt during the classroom episode.

- | | |
|-----|---|
| 1) | Ms. S: Are you drawing the bags? |
| 2) | Jay: Mm hmm. |
| 3) | Ms. S: That's a lot of work, huh? Jackson, I have a question for you. What you're |
| 4) | doing is very, very smart, but I have a question to ask you first. Um, let's pretend it |
| 5) | was 24, I'm sorry, 10 bags, and each bag had 10 rocks. How much would she have |
| 6) | then? |
| 7) | Jay: Two hundred and forty. |
| 8) | Ms. S: You think it's gonna be two hundred and forty? For the 24 bags. When you |
| 9) | touched each one, what were you counting by? |
| 10) | Jay: Ten. |
| 11) | Ms. S: You were counting by 10s, fantastic. Write down two hundred and forty so |
| 12) | you don't forget. Now, I'm wondering, if it was 10 bags. Pretend it's not 24 |
| 13) | for a second. If it was 10 bags and each one had 10 rocks, how much would that be? |
| 14) | I'm wondering if you can do that in your head? |

An excerpt from the subsequent SSR interview indicated that she seemed to hold the aforementioned objective:

- 16) R: What were you thinking there?
- 17) Ms. S: I'm surprised that he's writing 24 and then writing 10 at each. I know he
- 18) has, he has a pretty good sense of 10. And, um, so I was thinking that I thought
- 19) that he would know, what'd I change it to? Ten and 10? Right? I thought he

- 20) would know that 10 and 10 would be a...10 bags, 10 in each would be a 100?
- 21) And I was wondering, if thinking about that would help him to know something
- 22) about 24. And if he had said, 'that was a 100,' I was gonna say next, did I say
- 23) next, what would 20 be? I can't remember.

So, by offering Jay 10 bags of rocks (lines 5, 12 - 13), Ms. S seemed to hold the objective to work with Jay on his fluency with multiplying two-digit numbers by 10, which is further supported by her comment in lines 22 - 23 where said that she had already anticipated her next move of changing the number of bags to 20. Moreover, consider line 15, wherein Ms. S said to Jay, "I'm wondering if you can do that in your head?" This seemed to indicate that she did not want to reinforce the direct modeling strategy that Jay had employed when solving the problem initially. Rather, Ms. S was prepared for a series of quick modifications with friendly numbers (easy to multiply) to work towards Jay learning multiplication facts with 10s. This is demonstrated in the next excerpt from the classroom video:

- | | |
|-----|--|
| 24) | Ms. S: So let's pretend for a second that it's, um, 30 bags, and each one has 10 rocks |
| 25) | inside. (J: Okay) And you told me 20 bags is 200 (J: Mm hmm), 10 bags is 100, what |
| 26) | would 30 bags be? |
| 27) | Jay: Three hundred. |
| 28) | Ms. S: How'd you know lickety-split? |
| 29) | Jay: Then, if twenty's, 200, and 100, ten is 100, then just one more 100 is 300. |
| 30) | Ms. S: Will be 300? Very, very interesting. What if it was 57 bags, and each one has |
| 31) | 10 rocks? |

And her rationale from the SSR interview indicated that she seemed to hold that aforementioned objective:

- 32) R: What were you thinking in that interaction?

- 33) Ms. S: I was wondering if, okay, so now he knows ten, 10s is...a hundred, 20,
 34) 10s is 200, and he's kinda playing around with that idea that 30, 10s is 300. Fifty-
 35) seven, I was wondering if, bags, with 10 rocks in each bag, if he would be able to
 36) kind of extrapolate on that and know that it was five hundred and seventy.

Here, Ms. S offered that her objective was to see if Jay could build on what he had just done (lines 33 - 34), and make a connection to multiplying 57 by 10 (lines 35 - 36). Thus, it appeared that Ms. S used a series of modifications with each new number of bags selected to build on Jay's thinking with the previous number of bags in an attempt to move Jay towards fact fluency with multiplying by 10. Finally, it is worth pointing out that Ms. S also seemed to offer Jay the modification of 57 bags of rocks with the objective to also assess his thinking at that point (line 35: "I was wondering..."), which is another example of the linkages that seemed to exist between mod-objectives.

The previous five sub-sections demonstrated that five mod-objectives were foundational to the decisions that Ms. S made to modify problems for students. The next section presents arguments that Ms. T also considered the mod-objectives when deciding to modify problems for her students.

Ms. T: Task modifications

The first example from the classroom of Ms. T was taken from a lesson that centered on the partitive division problem:

The students at FCC collected ____ toys for the Toys for Tots program.
 If there are ____ families with children in the hospital, who need help
 with toys this season, how many toys can each family receive?
 (30, 6) (27, 9) (100, 25) (1000, 100) (4690, 2345) (555, 111)
 (120, 15) (99, 11) (1,296; 3) (4,590; 90) (8,400; 200) (126, 19)

During the classroom interaction Ms. T was working with Abby who had just solved the problem using a fair-sharing direct modeling strategy for the number sets, (30, 6), and (27, 9) whereby she drew circles to represent the number of families and then distributed the presents in an equal manner. Her work is shown in 6.6. Abby was using the same strategy for the number pair, (100, 25), when Ms. T interrupted her to question her about the multiplication fact of 25 times 4 equals 100 (that is the hand writing of Ms. T in the bottom portion of Figure 6.6). The in class transcript picked up after this exchange and it ended with Ms. T deciding to change the numbers to 40 presents and 5 families. The researcher interrupted to ask Ms. T to explain her thinking in the episode.

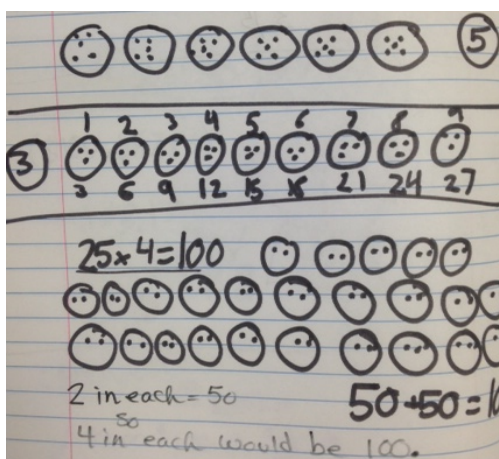


Figure 6.6: Abby's direct modeling strategy for the first 3 number pairs

Classroom transcript:

- 1) Ms. T: Do it with dots. See what I mean when you're looking at these numbers?
 2) They were getting kinda big? So what if I give you one, see if you can use a
 3) multiplication fact? Like you did here. To solve it. Okay, so what if I had, um, 40
 4) families, and no 40 presents (Abby: Yeah) and 5 families.
 5) Abby: And you can just count it by 5s. And see..
 6) Ms. T: Okay. And that would be your multiplication fact. Right? Okay. Try it.
-
- 7) **R: What were you thinking there?**
 8) Ms. T: I'm just thinking..that we can switch from drawing to dots to skip counting. If
 9) she can make sense of it because she used the multiplication here [*gestured circling*
 10) *student's work*] as opposed to this because it was a known fact. Where these ones
 11) weren't? And can she use some facts that she knows to flip it around.

An excerpt from the follow-up SSR interview:

- 12) R: Could you expound on that for me?
- 13) Ms. T: Well, when she came to this, okay, so here she had to direct model it.
- 14) [*pointed to student's work with (30, 6)*] That's what I mean by direct modeling
- 15) She drew out her 6 families, she passed out all the dots for the first one. She did
- 16) the same thing for the second problem [(27, 9)]. For the third problem [100, 25],
- 17) this is the one actually where she said she knew that fact. That was not this
- 18) problem, and she did get the right answer. She did 25 times 4 equals a hundred.
- 19) But she had the dots written I think also. The 25 circles with 2 dots in each...So,
- 20) I looked at that and said, okay, well, if she started by knowing this fact and have
- 21) this answer, then that means that she knows that this is a multiplication problem,
- 22) err, it's a division problem but that she can solve it through multiplication, like,
- 23) 25 times something equals a hundred...So, my thought was, well, that's a much
- 24) more efficient, maybe sophisticated, definitely more efficient strategy than like
- 25) drawing, and the numbers were starting to get really big, that was the jumping off

- 26) point for the big numbers cause then we went to a thousand...But since she was
 27) starting to use multiplication, I wanted to keep her going with that, and it was
 28) interesting that she said, ‘well, I could skip count.’ But skip counting is sort of, is
 29) a way to get to a multiplication problem. And maybe in her head she is skip
 30) counting. So what she’s doing [is] multiplication. So, did that answer the
 31) question? I wanted to push her forward with this strategy because I thought she
 32) was gonna need it as she moved forward.

The following sub-sections offer evidence to support the claim that all five of the mod-objectives were foundational to the decision made by Ms. T to modify the numbers for Abby. Instances of relationships between and among the mod-objectives are also provided.

Assess student’s thinking

In lines 8 - 9, Ms. T voiced that she wanted to see if Abby could “make sense” of a shift from a direct modeling to the strategy of using a multiplication fact (e.g., $5 \times 8 = 40$) to solve a division problem. That is, during the classroom interaction, Ms. T seemed to be motivated by the objective to assess the student’s thinking concerning the use of the new strategy, to use the operation of multiplication to solve a division problem. As Ms. T explained, if Abby could do so it would be a more sophisticated and efficient strategy for her (lines 23 -25) than the less efficient direct modeling strategy that Abby had used for the first two number sets. I argue that to meet this objective, that Ms. T linked to the work that the student had just done (lines 19 - 22), and that Ms. T presented the student with a set of friendly numbers because Ms. T felt that the student might possess the known fact of the multiplication involved (line 11).

Promote/Reinforce strategy

The mod-objectives to promote a strategy and to assess a student's thinking seemed to be inextricably linked in the decision that Ms. T made to present Abby with the new number pair, (40, 5). Not only did Ms. T want to see if Abby could make sense of a new strategy, she seemed to be building on her perception that Abby was beginning to understand that "it's a division problem but that she can solve it through multiplication," (line 22). In addition, Ms. T had offered, "I wanted to push her forward with this strategy because I thought she was gonna need it as she moved forward," (lines 32 -33), which seemed to also indicate that Ms. T held the objective to promote a new strategy for Abby. As was the case for the objective of assessing Abby's thinking, the means through which Ms. T proposed to promote the new strategy was to connect to Abby's immediate work, and to change to friendly numbers that, for Abby, might be linked to a known multiplication fact.

Provide mathematical access/rigor and foreground mathematical concept

I grouped these two categories together because the discussion of one contains references to the other. For example, inherent within the strategy that Ms. T seemed to promote for Abby was the principle of the inverse relationship between division and multiplication (lines 21 - 22), as well as the commutative property of multiplication. Ms. T seemed to provide access to these ideas by presenting Abby with the number pair, (40, 5), a number pair linked to the multiplication fact, 8 times 5 is 40, and its commutative partner, 5 times 8 is 40. It was interesting that upon closer investigation of the strategy that Abby employed when solving the problem with this set that she did not

use the same fair-sharing strategy that she had used for the first two number sets (see Figure 6.6)

Abby's new strategy appeared to be to skip count by 5s, which was the number of families receiving presents, and not to distribute the presents equally among the families as she did when solving the problem with the number sets (30, 6) and (27, 9). Thus, the commutative property became foregrounded in the sense that her final answer of 8 was found by counting the number of 5s she had used on the way to reaching 40. This strategy ran counter to her initial strategy that entailed counting the number of dots (i.e., presents) in each circle (i.e., family) as she did when solving the problem with the first two number pairs (Figure 6.6).

Promote fact fluency

During the same lesson, Ms. T employed another extending move when she went on to change the numbers for Abby after she had solved the problem for the pair, (40, 5). Figure 6.7 presents Abby's work for solving the problem with the new number set, (48, 6). Ms. T explained her reasoning for the modification in the following excerpt from the subsequent SSR interview.

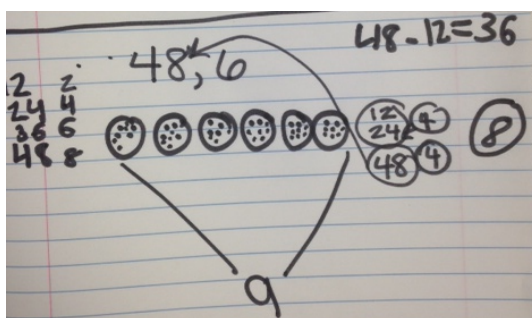


Figure 6.7: Abby's work with the pair (48, 6)

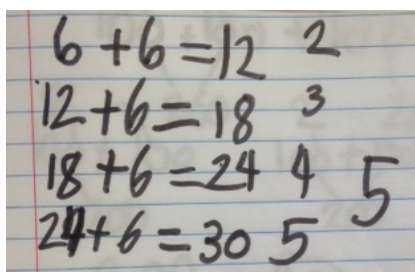
- 1) I was trying to give her one that she would know the multiplication fact for, but
- 2) she, I think that's why, now that I'm watching it back, you asked me why, or we
- 3) looked at why I did this instead of doing something she could skip count by?
- 4) R: Oh, this one? [*pointed to (48, 6)*] Okay.
- 5) Ms. T: I was trying to, well, I don't know if she would have known that fact
- 6) either, but I wanted the numbers to be small enough to be accessible if they
- 7) needed to be.

By modifying the task for Abby with a change of numbers to (48, 6), it seemed that Ms. T wanted to promote multiplication fact fluency for Abby by building on the work that Abby had just done with the number pair, (40, 5). Ms. T was “trying to give her one that she would know the multiplication fact for,” (line 1). The structure of the pair, (48, 6), was very similar to the structure of (40, 5), as 48 was the next multiple of 8 after 40. In lines 5 - 7, Ms. T remarked that she did not know if Abby knew the fact that 8 times 6 is 48, so in a sense Ms. T also seemed to be assessing Abby's thinking in terms of number facts. It was noted that Ms. T seemed to keep the objective of making her number choices, and, hence, the mathematics involved, accessible for Abby (lines 6 - 7). Ultimately, this objective seemed to be of importance since Abby reverted back the direct modeling strategy that she had employed with the problem initially (e.g., Figure 6.6). The last sub-section of findings in this chapter illustrates the manner in which a change of context can be used as a means towards an objective

Change or remove context as a means

In Table 6.2, it was presented that to change or remove the context of a task was a means for the participants to meet one of the five mod-objectives. This topic has yet to

be discussed. In this section, I present evidence from an interaction between Ms. T and a student, Sandy, to support the conjecture. In the episode, Ms. T was questioning Sandy about her strategy solving the same partitive division problem as Abby. Figure 6.8 shows that Sandy was using an incrementing up by 6s strategy to solve the problem with the first number pair, (30, 6); she built upon each successive sum by adding 6 to each subtotal until she reached 30.



$$\begin{array}{rcl}
 6 + 6 & = & 12 \quad 2 \\
 12 + 6 & = & 18 \quad 3 \\
 18 + 6 & = & 24 \quad 4 \quad 5 \\
 24 + 6 & = & 30 \quad 5
 \end{array}$$

Figure 6.8: Sandy's strategy for the numbers (30, 6)

In the following transcript from the classroom interaction, Ms. T was pressing Sandy to explain the meaning of the answer, 5, relative to the referent (number of presents given to each family) in the problem, and Sandy was having difficulty doing so. The change of context occurred in lines 20 - 21, whereby Ms. T changed the context from distributing toys to *families* to distributing toys to *kids*, a teaching move that seemed to support Sandy's thinking. Furthermore, the change of context followed a modification of the task whereby Ms. T changed the number of toys distributed from 30 to 6 toys for Sandy to share equally between 6 families (lines 14 - 15). Ms. T explained her rationale for both modifications in the follow-up SSR interview, from which excerpts are presented after the in-class episode. The researcher did not interrupt the flow of the interaction. First, the transcript of the classroom interaction between Ms. T and Sandy is presented:

- 1) Ms. T: So when you say like, well, 6 plus 6 equals 12, and that's 2. (S: Yeah) That's
- 2) two, 6s. Well what does that have to do with the problem? What does that have to do
- 3) with the gifts and the families?
- 4) Sandy: It has to do with, um, how many, so, I, um, it's, it's basically splitting two 6s
- 5) up. And both of them, so each kid would get 2.
- 6) Ms. T: But we're talking about families. Each family would get 2?
- 7) Sandy: Yeah.
- 8) Ms. T: And how does that work?
- 9) Sandy: Um..
- 10) Ms. T: So, like if there's 6 families, okay? If I had 6 presents..
- 11) Sandy: It's...
- 12) Ms. T: What if I had 6 families and 6 presents.
- 13) Sandy: You would make it up..
- 14) Ms. T: I would make what up? Let's change the problem. We collected 6 toys, there
- 15) are 6 families. How many toys can each family get?
- 16) Sandy: Um, each family would get..so,...
- 17) Ms. T: If there's 6 toys and 6 families, how many toys can each family get?
- 18) Sandy: Each family would get..um...
- 19) Ms. T: If I had 6 toys and 6 families, how many toys can I give to each family? Six
- 20) toys, 6 families. What if I take families out? I have 6 toys and 6 kids. How many
- 21) toys can I give to each kid?
- 22) Sandy: You would give each kid..5.
- 23) Ms. T: If I had 6 toys and 6 kids, I can give each kid 5 toys?
- 24) Sandy: One.
- 25) Ms. T: One toy, right?
- 26) Sandy: Yeah.

It appeared that Sandy was having difficulty connecting her strategy, namely her running total of 6s (see Figure 5.8), to the quantities given in the problem. Each 6 that Sandy kept in her running total referred to 1 present given to each family, so that her final answer of 5 was the number of presents that each of the 6 families would have received from 30 presents being distributed equally among the families. However, Sandy seemed to be having a difficult time making the connection. Ms. T explained her rationale for the two modifications she made for Sandy in excerpts from the SSR interview:

- 27) She's skip counting because she knows that she can get the answer by skip
- 28) counting. But does she know how that represents back into the problem? So, does
- 29) she know that like, well, every time I give out 6 presents each family gets 1

- 30) present? So now I'm giving out another 6 presents each family gets another
 31) present. Now I'm giving out another 6 presents each family gets a third present,
 32) or whatever. So, um, I just was curious if she knew why she was doing what she
 33) was doing, and why it worked...

First, Ms. T seemed to hold the objectives to assess Sandy's thinking concerning the answer she gave and "does she know how that represents back into the problem?" (line 27 - 28). Second, Ms. T also seemed to hold the objective for Sandy to provide access to the problem by relating her numerical answer, 5, to the quantity that it represented in the problem, namely, the number of presents each family would have received. In lines 14 - 15 of the classroom transcript, it appeared that Ms. T tried to help Sandy in making the connection by changing the number of families to 6, which in terms of Sandy's strategy would have represented her final answer, and Ms. T was trying to help her understand that "...every time I give out 6 presents each family gets 1 present..." This did not appear to have been successful (lines 16 - 20), whereupon Ms. T changed the context from 6 families to 6 kids to whom the presents would be distributed.

Ms. T explained her reasoning for changing the context in the following excerpt from the SSR interview:

- 34) ...I don't know but I felt that it was harder. And I didn't think about it when I
 35) wrote the problem. But when I was doing it I felt that it was an obstruction that if
 36) it was just one on one, like, you could go 'here's one for you, here's one for you.'
 37) But when we're talking about whole families it's, like, a little bit harder. I don't
 38) know why. I didn't anticipate it. But when I got there I felt that that may have
 39) been making it harder.

In line 34, Ms. T explained that she found that the idea of sharing presents among families seemed to be a conceptual hurdle for some students once they began working the problem. She did not anticipate the issue when she wrote the problem (line 35), but when some students encountered the obstacle during the lesson, she seemed to realize that “when we’re talking about whole families it’s, like, a little bit harder.” Thus, Ms. T changed the context as the means to overcome the obstacle, and to provide access to the problem for Sandy by relating her numeric answer to the corresponding quantity in the problem. The above episode was an example of modifying the context within the problem to serve as the means towards meeting various objectives during the enactment of a lesson, and it is the last argument to be made in this section. A summary of the chapter follows.

Summary

In this chapter, I presented evidence to support the claim that the participants held one, or more of the five identified mod-objectives when coming to the decision to modify a task for a student. In addition, it emerged that the mod-objectives formed a subset of the design-objectives discussed in Chapter 5. I further argued that the in-class dialogue between a teacher and student influenced a decision of this type when the teacher elicited the student’s thinking concerning her work on a problem. Thus, the modification of a problem seemed to be in response to the information the teacher had garnered during the dialogue, and to what the teacher had noticed during the interaction. Moreover, while each of the modifications could be classified as a supporting move, or an extending move (Jacobs & Ambrose, 2008), and each modification could be linked to the student’s

immediate work, the choice of numbers and inherent number structures seemed to emerge as the primary means employed by the teachers to reach the mod-objectives.

Because the five mod-objectives included in the theoretical framework were a subset of the seven-design objectives, essentially, the three participant teachers seemed to be redesigning the problem on the fly whenever they changed the numbers or the context for a student. As an exercise, considers all of the examples of one of the teachers changing the numbers for her students presented in this chapter. In what can be described as extraordinary cognizance, in a matter of seconds, the teacher appeared to redesign and tailor the problem according to what she noticed during the classroom interaction, aligned this new information with her dynamic model of the respective student's current mathematical understandings, and made a decision intended to support or extend the student's thinking.

One can only surmise as to the degree of mathematical knowledge for teaching that these special teachers invoked to make such astute decisions during the hectic environment of a mathematics lesson. Figure 6.9 presents a small moment in a lesson when a teacher comes to the decision to change the numbers for a student. It also merges the Learning Goal/Mod-Objective/Means framework of this investigation with the Jacobs and Ambrose (2008) extending/supporting children's mathematical thinking framework, and the Jacobs and colleagues (2010) framework of the professional noticing of children's mathematical thinking.

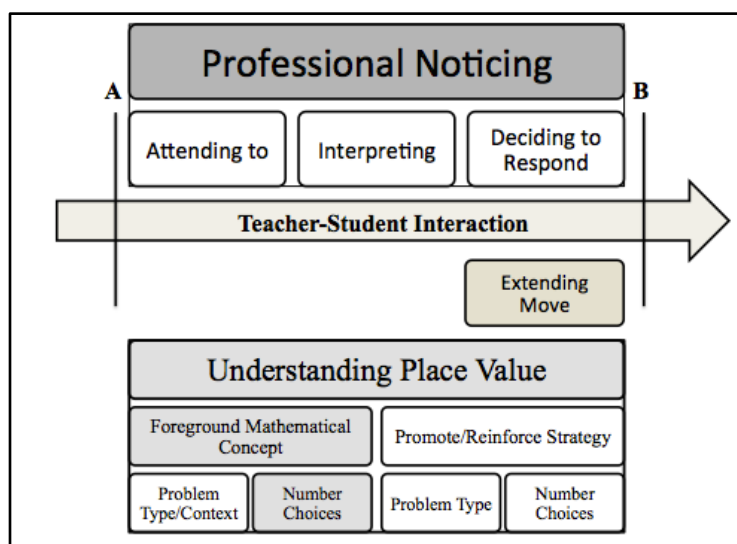


Figure 6.9: Overlay of the aforementioned frameworks on a teacher-student interaction

In Figure 6.9, the arrow represents a portion of the timeline of a teacher-student interaction during a mathematics lesson. At point A, the child presents the teacher with a correct answer to the problem, and the teacher asks the child to explain her thinking. Between points A and B, the teacher invokes her considerable skills on many different levels. First, because the teacher had decided to ask the child a question, the professional noticing frameworks essentially reboots so the teacher is now attending to and interpreting new information such as the child's written work and the child's verbal explanation of that work. I contend that at the same time, the teacher is evaluating her decision to respond to the child based upon the teacher's long-term learning goal for the child to understand place value, and she is incorporating into the decision to respond the mod-objective to foreground a mathematical concept (e.g. regrouping) by changing the numbers for the student. As the teacher nears a decision, she decides upon an extending move because the child presented a correct answer. One of the extending moves in the Jacobs and Ambrose framework is to ask the child to solve the same problem with more

challenging numbers. Hence, the mod-objective to provide mathematical rigor might enter the picture along with the associated means of selecting numbers whose structure provides an appropriate level of rigor. See, easy. And for the teachers in this study the timeline in the figure was measured in minutes if not seconds.

The culminating chapter of this dissertation discusses implications of the results of the investigation as they relate to teacher education and professional development. Chapter 7 contains also a discussion of the methodology employed to access the teachers' in-the-moment thinking and it's potential within the field of mathematics education research. Constraints, challenges and limitations of the study are also examined.

CHAPTER 7: CONCLUSIONS

In this chapter, I present a summary of the findings of this dissertation study. The findings are organized around the two research questions this investigation sought to answer and are presented in order. Then I discuss the significance of the study, methodological considerations, limitations of the study, and future research directions.

Summary of Findings Research Question 1

One goal of this investigation was to answer the research question: What factors do elementary teachers who have an extensive understanding of children's mathematical thinking consider when developing mathematical tasks in the design phase of mathematics lessons? In answering this question, four categories of influencing factors were identified, and associated with one, Learning Goals, were seven categories of design-objectives that served as measurable benchmarks towards reaching the learning goals. In addition, it was found that the problem type, problem context, and the number choices within the problem were the means employed by the participants to attain the design-objectives. Collectively, these formed the Learning Goal/Design-Objective/Means framework, and it was through this framework that the reasoning of the teachers when developing, or selecting, a problem during the design phase of a lesson was described.

All of the teachers in this study were identified as holding the learning goal for their students to understand place value. When developing a task to support meeting this goal, each teachers would first have her students work towards a more realizable objective such as regrouping ones into tens within the context of a problem that her

students would in all likelihood use an addition strategy to solve. In addition, the structure of the number choices became the main means through which the teachers aspired to reach the objective. For example, Ms. S composed a Join-Change-Unknown problem because she knew that her students would probably employ an incrementing up strategy. Thus she included sets of numbers whose sum would involve regrouping ones across the tens place; yet at the same time she made sure the problem did not involve regrouping across the hundreds place because she did not want to have that particular conversation with her students that day.

In conjunction with identifying the individual factors in the Learning Goal/Design-Objective/Means framework was the finding that all of the factors seemed to be linked within an intricate web of relationships. For example, the academic calendar influenced the problem types that Ms. T selected early in the year, while later in the year the timing of a lesson influenced her focusing on fluency with multiplication facts, especially those related to multiplying by 10. Thus, during the design phase of a lesson any, or all of the factors could be in play when writing a problem for a lesson.

Summary of Findings Research Question 2

To answer the second research question, I directed the focus of the study on the enactment phase of a mathematics lesson. The second research question was: What factors do elementary teachers who have an extensive understanding of children's mathematical thinking consider when modifying or adapting mathematical tasks during the enactment phase of mathematics lessons? In answering this question, five of the seven design-objectives were identified as being particularly relevant when the teachers decided to change the numbers for students during a lesson. This subset of design-

objectives was relabeled as mod-objectives to delineate the design phase from the enactment phase of a lesson. Four categories of means that supported reaching the five mod-objectives were also identified: (a) linking to the immediate work of the student; (b) incorporating known facts; (c) changing the context of the problem; and (d) most importantly, offering the student numbers with a structure that fit the mod-objective. For example, Ms. F wanted to foreground the commutative property of multiplication for a student so she changed the numbers from the product of 19 and 2 to the product of 3 and 2 because the second pair supported the student accessing the commutative property.

In answering the second research question, two additional theoretical frameworks were incorporated from the literature to serve as lenses through which to make inferences about the participants' reasoning when modifying a problem for a student. The supporting and extending children's mathematical thinking framework presented by Jacobs and Ambrose (2008) provided a clean demarcation to the decision reached by a teacher to modify a problem. If the child had correctly solved the problem, the decision was in all likelihood to extend the child's thinking, and conversely, if the child had not solved the problem, the decision was more likely made to support the child's thinking.

In a similar fashion, the professional noticing of children's mathematical thinking framework (Jacobs, Lamb, & Phillip, 2010) provided a platform for inferring what the teachers had attended to, how they had interpreted the information they received, and how these two skill sets worked in conjunction with the teachers coming to a decision to modify a problem for a student. The professional noticing framework, when integrated with the Learning Goal/Mod-Objective/Means framework, offered a refined perspective towards the process through which a teacher arrived at a decision to modify a problem.

For instance, in addition to what a teacher was noticing within the realm of children's mathematical thinking when interacting with a student, she was also considering the learning goals she held for the student, and her decision to modify the problem included specific objectives and the means to reach those objectives. As was the case with the answer for Research Question 1, the answer for the second research question of this dissertation unearthed the complexity of the relationships between the emergent factors of this study as well as with the two aforementioned frameworks.

Significance

The findings reported in this study illustrated the complexity of two teaching practices: (a) developing mathematical tasks when designing a lesson, and (b) modifying mathematical tasks when enacting a lesson. Perhaps the most compelling argument made in this dissertation was that the two practices are inextricably entwined. The practice of modifying a task is directly linked to the development of the task upon which the modification is enacted. By engaging teachers in discussions centered on products of their practices followed by conversations about decisions they made during their practice, a single framework emerged that encompassed the scope of this study. Succinctly, by interviewing teachers before, during and after a lesson lead to the development of a framework that filled a void in the task literature. The Learning Goals/Objectives/Means framework offers a basis for decomposing each of the aforementioned teaching practices into a set of distinct well-defined components that are salient to the fields of teacher education and teacher professional development.

The practice of developing a problem for a mathematics lesson has mainly been investigated from the perspective of expert researchers and educators. For example, Land

and colleagues (2014) reported on the utility of the number choices within problems similar to those found in the CGI literature, but their discussion was from their expert points of view. In their investigations, they did not ask the teachers to offer their perspective towards choosing number sets to meet the needs of their respective students (personal communication with T. J. Land, February 13, 2015). In contrast, I reported from the perspective of exemplary teachers the number choices they made when developing problems to support their students' mathematical thinking. The emergent Learning Goals/Objectives/Means framework affords a level of nuance that can be leveraged with the work of other researchers such as Land and colleagues to contribute to the decomposition of the practice of developing problems.

As an example of how the Learning Goals/Objectives/Means framework might be utilized in a course for prospective teachers, consider one of the number pairs that Ms. F frequently uses with multiplication problems, (7, 12). As discussed in Chapter 5, Ms. F might choose this set to support various design-objectives such as foregrounding the distributive property in conjunction with promoting the 'break apart' strategy when multiplying the two numbers. As an activity, prospective teachers could develop multiplication problems through the lens of the Learning Goals/Objectives/Means. First they would consider the learning goals they want to work towards then they would decide how they might best support kids' thinking by incorporating specific design-objectives and means into the process. In this manner, by approximating the practice of an expert like Ms. F, prospective teachers might begin to view problems in the same manner. For example, prospective teachers may begin to see the utility in number choices like (7, 12)

and the affordances offered within the number structures in the same manner as say, Ms. F.

Another significant result of this study lies in a relationship between the Learning Goals/Objectives/Means framework with the teaching practice of reflecting on a lesson. Essentially, the process of reflecting on a lesson acts as a formative assessment of the lesson, including the efficacy of the problem presented, which might lead to a redesign of the lesson. This was demonstrated through Ms. T reflecting on a lesson on adding fractions and how her choice of number sets did not seem to offer access to the problem, or to the concept of equivalent fractions. Thus, she did not meet two of the design-objectives that she held for that particular problem and lesson. During her reflection, Ms. T redesigned the problem for a future lesson. Prospective teachers might emulate the practice of developing a problem, and in an exemplar classroom environment, present it to children, and then reflect upon how well their problem met their objectives in supporting the children's mathematical thinking. Moreover, to complete their formative assessments the prospective students would redesign their tasks based on what they had learned while 'enacting' their lesson.

This investigation also offered methodological insights for the field of mathematics education. For instance, the decision to interject myself into the lessons by asking the teachers to explain their reasons for changing the numbers for students can be construed as an existence proof of the technique. In other words, it seems there exists a setting in which interrupting a teacher is effective without creating too much anxiety for the teacher or the students. By asking the teachers in this study a simple question, "What were you thinking there?" immediately after they changed the numbers for a student

seemed to offer access to their reasoning processes that other methodological techniques might not. Moreover, conflating quick in-the-moment interviews with the semi-structured stimulated interview methodology of this study seemed to provide a sense of triangulation for the data collected. The in-the-moment questions essentially provided snapshots of the teachers' classroom reasoning that could be further explored in the subsequent SSR interview.

Methodological considerations

As discussed in the previous section, the methodologies employed in this investigation proved crucial in generating data concerning the reasons supporting the decisions the teachers made when both developing and when modifying problems. First, by focusing SSR interviews on the teachers' reasons for their problem and number choices it began to emerge that the teachers' held specific objectives. The time lag between an SSR interview and a subsequent clinical interview provided an opportunity to conduct a preliminary analysis of the data, which informed the protocol for the follow up interview. Thus, a teacher's reasoning could be explored at a level of detail that was not possible with a single interview. Finally, a teacher's answers to the questions in the SSR interview could be clarified and explored in greater depth in the follow up clinical interview.

Asking a teacher to explain her reasoning during the enactment of a lesson was linked to the subsequent SSR interview. A teacher's quick response to the in-the-moment questions provided data that could be further mined during the SSR interview. Moreover, the video clips of the teacher-student interactions coupled with the researcher's interjections offered what I would argue were unique stimuli for the SSR interviews. The

in-class questions seemed to serve as points of reference for the teachers for further discussion during the SSR interviews.

It was fortuitous that on three separate occasions a teacher was available to discuss her reasoning for developing a problem prior to the lesson. These short interviews offered access to their thinking before they enacted the lesson; hence, the data were anticipatory instead of retrospective. Thus, in hindsight, adding this component to the first step of the research design (Step I) would have offered a larger corpus of data from which stronger inferences might have been made as to the teachers' intentions for their lessons. The next section further discusses other constraints, challenges and limitations of the study in addition to thoughts concerning future research.

Limitations and Future Research

This dissertation study had two goals, both of which required access to the decisions that teachers made during the day-to-day work of their practice. The focus of the study was narrow in the sense that the research questions centered on two specific teaching practices while ignoring the plethora of other things associated with practice. And yet, the research design of the study could have been improved to better capture the breadth of the factors involved. For example, in the previous section I noted that during the occasional times when I conducted a brief pre-instructional interview the data provided some insight to what the teachers might anticipate going into the lesson. I infer that the data collection could have been enriched by consistently incorporating these types of interviews.

In addition, the study was constrained by the number of rounds of data collection per teacher. Although the data corpus was extensive enough to provide significant

results, the study was limited by the number of problems that served as focal points of query as well as by the number of problem types. For example, both Ms. F and Ms. T tended to avoid using some of the more challenging types of problems, such as Separate-Change-Unknown problems. Including more teachers might have lead to a wider variety of problem types used during instruction. Moreover, collecting data across a wider span of the academic year might have also enriched the study by offering more insight into the roles played in the design phase by the influencing factors, Academic Year and Learning Goals, and the evolution of the number choices.

Furthermore, more rounds of data collection would offer more evidence to support factors that seemed salient but could not be substantiated. Thus, future research would include investigating these ‘factors’ that in all probability were part of the framework but that were excluded for lack of evidence. The first example was presented in the episode when Ms. T changed the context of a division problem for a student to help her understand the meaning of her answer relative to the referents of the problem. It seemed that Ms. T held a mod-objective for the student to ‘make sense of the problem’; which is a design-objective, but the episode was singular. A strong argument could not be made to include the category as a mod-objective.

In addition, only two learning goals were included in the framework, while the teachers likely held additional learning goals. A longitudinal study would have collected additional data from which further claims regarding other learning goals may have emerged. For example, there was evidence to *infer* that the learning goal, Understand Place Value, was part of a broader learning goal of Develop Number Sense. However, number sense, as defined by researchers such as Sowder (1992), includes several

components that did not emerge in the analysis, but that seemed to exist in the consciousness of the participants. For example, the teachers seemed to hold goals for their students to develop a sense of relative magnitudes of numbers, relational thinking regarding numbers, and moving between reasoning quantitatively and abstractly about numbers. However, including developing a high level of number sense as a learning goal in the framework would have been the result of a high degree of inference by the researcher.

A final area of future research aligned with this study would be to frame a research question around the influence of the affective domain in the two teaching practices of interest. Often, Ms. F and Ms. T would voice a goal for their students to gain confidence, or to be successful in their problem solving skills. However, Ms. S did not attend to the affective domain in the same manners. This is not to say that Ms. S did not hold the same objectives for her students; rather, she did not explicitly allude to the attitudes and emotions of her students. Thus, while the affective domain appeared to influence some of the decisions of Ms. F and Ms. T, analysis across the three cases could neither support including the affective domain as an influencing factor nor as an objective with the framework. Thus, further research in this area is warranted.

Appendix A

Pre-Interview Protocol: Background and Beliefs Towards Teaching and Learning

Background questions:

- 1) How long have you been teaching?
 - At what schools? Which grades?
 - How long have you emphasized using CGI in your classroom?
 - Can you describe how you went about shifting your instructional practices from what you used previously to using CGI?
 - Can you describe the process?
 - Was it easily accomplished?
 - Can you describe any tensions that may have existed between yourself and say administration when you decided to change the classroom instruction?
 - Describe your participation in any professional development programs that focused on the principles of CGI?
 - Do you still participate in PD?
 - If so, what type, how often?
 - Do you communicate with other teachers that employ CGI?

Beliefs and Attitudes questions:

- 1) Describe your personal goals as a teacher, specifically for mathematics.
 - Be prepared to follow up if the teacher refers to:
 - CGI
 - Students' prior knowledge and capabilities, learning trajectory etc.
 - Problem solving versus traditional lecture, or mixture of both
 - Goals for class as a whole versus individuals
 - How do you think that students learn math?
 - Possible follow-up topics:
 - Constructivist or similar viewpoint (students build their own knowledge)
 - What role do you play in their learning?
 - Sociocultural viewpoint and ZPD, but don't use these terms
 - If it hasn't come up: Can you describe how your teaching practices support students learning and understanding math?
 - Possible follow-up topics:
 - Emphasis on problem solving and multiple strategies
 - Students justifying their strategies
 - Introduction of topics and curriculum alignment

- Topics specific to CGI

Appendix B

Semi-Structured Stimulated Recall Interview Protocol

First, the rationale for the problem and the number choices is discussed:

- 1) Why did you pick this problem type for today's/yesterday's class?
 - Could probe for how the problems fit within the curriculum
 - What learning goals did you incorporate into the problem?
 - With this problem, what were you looking for in terms of students' strategies?
- 2) How did you decide on these particular number choices?
 - Clarify and probe depending on the response
 - Clarify potential factors as previously identified
 - Be aware of new potential factors

Selecting phenomena of interest.

I will use the videotape of that day's lesson to:

- Identify events of interest for the stimulated recall interview
 - Instances of the teacher changing numbers for a student.
 - Instances of the teacher changing the context of the problem.
 - Instances of teacher-student interactions of interest
 - A student thinking in an unexpected manner
 - Particular difficulties a student may be having
- Identify examples of student work to serve as recall support for the SSR
- Formulate questions for the interview--clarifying and probing

Stimulated Recall Interview Instructions (read to teacher)

Now we are going to watch *x-number* clips from the second group. I am interested in what you were thinking at the time of your interactions with the students. I can see how you were interacting with the students, but I don't know what you were thinking at that moment. What I'd like you to do is tell me what was going through your mind at the time you were watching or interacting with the students.

Because the video captures the entire table, you will be able to see things that you didn't see during the lesson and I ask that you ignore these and focus on what you were seeing and thinking at the time. I will start the video and I will pause the video at certain times to ask you questions. Also, if you want to tell me anything that comes to your mind you just push pause. Adapted from (Gass & Mackey, 2012)

When either the teacher or I stop the video during the SR portion of the interview, these instances form the foundation for my inferences concerning the teacher's professional noticing during the session. I anticipate exploring these inferences in greater detail by asking:

- 1) Where was your attention at that moment?
 - a) What were you looking at?
 - b) What were you listening for?
- 2) How did you interpret what was going on?

- 3)
 - a) What did you interpret in regards to the student's thinking/strategy at that moment?
- 4) On what did you base your decision to:
 - a) change the numbers?
 - b) modify the problem?

Appendix C

Clinical Interview Protocol

Foundational Questions for Each Round of Data Collection

Revisit the problem from the lesson of interest:

- 1) Why did you pick this problem type for today's/yesterday's class?
 - Could probe for how the problems fit within the curriculum
 - What learning goals did you incorporate into the problem?
 - With this problem, what were you looking for in terms of students' strategies?
- 2) How did you decide on these particular number choices?
 - Clarify and probe depending on the response
 - clarify potential factors as previously identified
 - be aware of new potential factors
 - Before, you mentioned that you thought the number choices might have been too difficult/easy. Can you tell me more about why you thought that?
- 3) What do you look for while students are working?
 - What do you look for in a student's strategy?
 - Try to have specific instances of student work to refer the teacher to
- 4) How do you infer or interpret what the student is thinking?
 - Have specific episodes to refer to: Yesterday when *Name* was working with the problem, I saw her..... What do you think she was thinking?
 - How did her work yesterday fit with what you think she understands?
 - Can you give an example of how you assist a student that doesn't understand the problem?
- 5) What goes into your decision to make a teaching move or action?
 - Be prepared to probe for distal associations: Could you talk about any relationships between working with today's class and past students?
 - I noticed that one teaching move was to change the numbers for students. What do you base this decision on?
 - Again, have a specific episode in mind
- 6) Were you surprised with what any of the students did? If so, why?
- 7) Can you give an example of how you provided assistance to move a student towards a learning goal.
 - How about extending a student's thinking?
 - Could present an example of an episode for stimulate recall

- 8) What did you take away from today's math center?
- What did you learn that might help you when the students again work this type of problem?

Additional Protocol for Final Clinical Interview

1) So, when I was filming a lesson, at times I interrupted you right after you made a decision, like, to change the numbers, or change the task. Can you speak to that process from your point of view?

- So do you think it affected the lesson at all?
- Do you think it had any affect on the students? With their thinking, their dispositions?
- Were you ever anticipating me jumping in?
- How did the process make you feel?

2) As a result of your years of experience teaching and years of professional development, you have quite a wealth of knowledge and skills in regards to teaching math. Can you talk about how all of this knowledge plays into writing a math problem for a lesson?

- a) Let me ask a clarifying question. You said kids a lot. Are you talking about kids in *general* or the kids in this room?

3) You also seem to have an understanding or interpretation of what each individual student *knows*- their mathematical understandings and their thinking-How does this knowledge of your current students, play into writing a problem?

4) Could you talk about your thoughts as to the relationships between these two knowledge bases? How they interact or mesh.

- a) All of your experience, your knowledge that you've gained over the years and your knowledge that you've developed of your kids this year.. How do they interact
- b) You talk about this wide range of abilities in general. What does someone look like, or what are the understandings that a child has that's in the middle?
-

5) How about the same questions for when you decide to change the numbers for a student during a lesson. What role does your experience and knowledge gained over the years play into changing the numbers for a student?

6) So, what role does your knowledge of your students in this class play with your decision to change numbers?

7) I want to try something. We're talking about the knowledge base of your experience, and the knowledge base of your kids. And your decision to change a number. Can you draw a diagram that reflects how you see all of that?

8) Reaction to diagram: So, suppose this is young Mike here and you see Mike doing this *making tally marks* Are your questions, or decisions specific for Mike? Or, someone that's like Mike?

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