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Eisenstein Cocycles for Powers of the Multiplicative Group

A dissertation submitted in partial satisfaction
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by

Timothy J. Smits

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ABSTRACT OF THE DISSERTATION

Eisenstein Cocycles for Powers of the Multiplicative Group

by

Timothy J. Smits

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2024

Professor Romyar Thomas Sharifi, Chair

This dissertation generalizes work of Sharifi and Venkatesh to construct a canonical choice of cocycle Θ via a "lifting" process that represents a cohomology class

$$[\Theta] \in H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), K_n^M(\mathbb{Q}(\mathbb{G}_m^n)) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{(n+1)!}])$$

for an integer $n \geq 2$. We show that $[\Theta]$ is Eisenstein with respect to the action of certain Hecke operators.

The dissertation of Timothy J. Smits is approved.

Burt Totaro

Chandrashekhar Khare

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University of California, Los Angeles

2024

To my father, Anthony, who would have been proud.

TABLE OF CONTENTS

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VITA

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CHAPTER 1

Introduction

There is a philosophy that is frequently floated around by Sharifi, which is that for a global field F and integer $n \geq 2$, there should be a relation between the geometry of GL_n/F (near a boundary component) and the arithmetic of GL_{n-1}/F (which is made more precise in [12]). When $n = 2$ and $F = \mathbb{Q}$, this philosophy says there should be a relation between the geometry of *modular curves near cusps* and the arithmetic of *cyclotomic fields*. One such connection has long been known, which was one of the major ideas that went into the proof of the Iwasawa main conjecture of Mazur and Wiles [19].

For $N \geq 1$, let $X_1(N)$ denote the usual closed modular curve over $\mathbb{Q}$. Sharifi's connection between arithmetic and geometry is encoded by a map

$$\Pi_N : H_1(X_1(N), \mathbb{Z}) \to K_2(\mathbb{Z}[\zeta_N]) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}].$$

This map was independently constructed by Busuioc [7] and Sharifi [24], and arises from the restriction of the map

$$\Pi_N^\circ : H_1(X_1(N), C_1^\circ(N), \mathbb{Z}) \to K_2(\mathbb{Z}[\tfrac{1}{N}, \zeta_N]) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}]$$

defined on the larger homology group relative to the "non-infinite" cusps of $X_1(N)$. This homology group is easier to understand: it has a well-known set of generators known as *Manin symbols*, which are special classes of geodesics in the extended upper half plane $\mathbb{H}^*$ between cusps. Explicitly, Π_N° has a remarkably simple description:

$$\Pi_N^\circ([u : v]) = \{1 - \zeta_N^u, 1 - \zeta_N^v\},$$

mapping Manin symbols to Steinberg symbols of cyclotomic units.

In [24], Sharifi conjectured that Π_N is *Eisenstein*. This means that for primes $\ell \nmid N$ and $x \in H_1(X_1(N), C_1^\circ(N), \mathbb{Z})$,

$$\Pi_N(T_\ell x) = (\ell + \sigma_\ell)\Pi_N(x),$$

where T_ℓ is the ℓth Hecke operator and $\sigma_\ell \in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ is the automorphism induced by $\zeta_N \mapsto \zeta_N^\ell$. For primes $\ell \mid N$, he additionally conjectured that

$$\Pi_N(U_\ell^* x) = \Pi_N(x),$$

where U_ℓ^* is ℓth adjoint Hecke operator.

Progress towards these conjectures was made by Fukaya and Kato in [11], where they prove both conjectures hold for the p-adic realizations $\Pi_N \otimes \mathrm{id}_{\mathbb{Z}_p}$ for primes $p \mid N$.

In [25], Sharifi and Venkatesh provide a new construction of the map Π_N and verify the Eisenstein condition for the Hecke operators T_ℓ. They do this by constructing a 1-cocycle

$$\Theta : \mathrm{GL}_2(\mathbb{Z}) \to K_2(\mathbb{Q}(\mathbb{G}_m^2))/\langle\{-z_1, -z_2\}\rangle,$$

where z_i denotes the ith coordinate of $\mathbb{G}_m^2$. By "specializing" their cocycle at an N-torison point of $\mathbb{G}_m^2$, they obtain a cocycle

$$\Theta_N : \tilde{\Gamma}_0(N) \to K_2(\mathbb{Q}(\zeta_N))/\langle\{-1, -\zeta_N\}\rangle,$$

where $\tilde{\Gamma}_0(N)$ is the congruence subgroup of $\mathrm{GL}_2(\mathbb{Z})$ of matrices with lower left entry divsible by N. The cocycle Θ is parabolic, and by restricting to $\Gamma_1(N)$ they are able to recover the map Π_N.

Now, the construction of the cocycle Θ is carried out as follows: there is a Gersten-type complex arising from the coniveau spectral sequence

$$K_\bullet : K_2(\mathbb{Q}(\mathbb{G}_m^2)) \to \bigoplus_{x \in Y_1} K_1(\mathbb{Q}(x)) \to \bigoplus_{x \in Y_2} K_0(\mathbb{Q}(x)),$$

where Y_p denotes the set of codimension p points of $\mathbb{G}_m^2$. The first map sends a Steinberg symbol to its tame symbol, while the second map sends an element of $K_1(\mathbb{Q}(x)) = \mathbb{Q}(x)^\times$ to its divisor. This complex carries a natural action of the monoid $\mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ via pullback (with $\gamma \in \mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ inducing an endomorphism of $\mathbb{G}_m^2$ via right multiplication).

The kernel of the first map is the motivic cohomlogy group $H^2(\mathbb{G}_m^2, 2)$, and the complex

$$0 \to K_2(\mathbb{Q}(\mathbb{G}_m^2))/H^2(\mathbb{G}_m^2, 2) \to \bigoplus_{x \in Y_1} K_1(\mathbb{Q}(x)) \to \bigoplus_{x \in Y_2} K_0(\mathbb{Q}(x)) \to 0$$

is acyclic. The cocycle Θ is constructed by choosing "lifts" of the $\mathrm{GL}_2(\mathbb{Z})$-fixed element $e \in \bigoplus_{x \in Y_2} K_0(\mathbb{Q}(x))$ corresponding to the generator $1 \in K_0(\mathbb{Q}(x)) \cong \mathbb{Z}$ for x the identity of $\mathbb{G}_m^2$.

By working with so called "trace-fixed parts" of these groups, the authors produce a cocycle class $[\Theta] \in H^1(\mathrm{GL}_2(\mathbb{Z}), K_2(\mathbb{Q}(\mathbb{G}_m^2))/\langle\{-z_1, -z_2\}\rangle)$. They verify that this class satisfies the Eisenstein condition for the operators T_ℓ, from which they are able to deduce corresponding property for Π_N after specializing.

There are several other Eisenstein cocycles that have been constructed in the literature. In particular, we bring attention to [27], where the author essentially gives an alternate construction of the cocycle constructed in this thesis, and provides a realization map to a construction of Lim and Park [17].

1.1 Overview

The main purpose of this thesis is to generalize the cocycle construction of [25] to $\mathbb{G}_m^n$ for $n \geq 2$. The outline of the thesis is as follows. Chapter 2 provides many of the algebraic preliminaries necessary to carry out the constructions in later chapters. We review the definition of motivic cohomology, its relation with Milnor K-theory, and the key properties needed to carry out the cohomology computations performed in Chapter 3. In Section 2.2, we review the coniveau spectral sequence. In particular, the coniveau spectral sequence is used to produce a Gersten-type complex $K_\bullet$ with a natural action of the monoid $M_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$ via pullback.

Chapter 3 consists of two parts. One is dedicated to developing the theory of *trace modules* over a ring R, which are modules over the ring $R[\mathbb{Z}_{\geq 1}]$ for the semigroup of positive integers under multiplication. This is done in Section 3.2. Our attention is mostly restricted to a subcategory of so called *nice trace modules* over $\mathbb{Z}$, those which have a particular notion of a generalized eigenspace decomposition with respect to the linear maps induced via the semigroup action. Our main result about nice trace modules is Proposition 3.2.16, which says that after inverting finitely many primes, there are *canonical* choices of projection operators to generalized eigenspaces. The existence of such operators is crucial in showing that the category of nice trace modules is abelian.

The second part of Chapter 3 is the computation of motivic cohomology groups of certain families of schemes $(Z_p)_{p\in\mathbb{Z}}$ arising from Gysin type sequences called *toric flags* (which are built from kernels of characters $v : \mathbb{G}_m^n \to \mathbb{G}_m$, hence the terminology). This is done in order to fix ambiguity inherent to the cocycle class that is constructed in Chapter 4. Section 3.1 deals with computations pertaining to the motivic cohomology groups $H^i(\mathbb{G}_m^n, j)$, showing that they are nice trace modules, where the semigroup action is via pushfoward $[m]_*$ of the multiplication by m map. The main result of Chapter 3 is the existence of a special exact sequence of motivic cohomology groups, after projecting to so called *generalized trace-fixed parts*:

Theorem 3.3.8. *Let $\mathbb{Z}' = \mathbb{Z}[\frac{1}{(n+1)!}]$, and let $Z = (Z_p)_{p\in\mathbb{Z}}$ be a toric flag. There is an exact sequence of trace modules over $\mathbb{Z}'$*

$$0 \to (H^n(\mathbb{G}_m^n, n)\otimes_{\mathbb{Z}}\mathbb{Z}')^0 \to (H^{n-1}(\mathbb{G}_m^n - Z_1, n-1)\otimes_{\mathbb{Z}}\mathbb{Z}')^0 \to \ldots \to (H^0(Z_n, 0)\otimes_{\mathbb{Z}}\mathbb{Z}')^0 \to 0,$$

where notationally A^0 denotes the generalized trace-fixed part of a trace module A. The exact sequence above is extracted from a right half-plane spectral sequence attached to a toric flag that we call the *toric flag spectral sequence*:

$$E_1^{p,q}(Z) : H^{q-p}(Z_p - Z_{p+1}, n-p) \implies H^{p+q}(\mathbb{G}_m^n, n).$$

The vanishing of (almost all) motivic cohomology groups of interest after projecting to generalized trace fixed parts (which requires working over $\mathbb{Z}'$) is proved in Section 3.3, and a homological algebra argument in conjunction with analyzing the spectral sequence proves Theorem 3.3.8.

Chapter 4 is devoted to generalizing the cocycle construction of [25], and heavily relies on Theorem 3.3.8. The cocycles we construct arise from a general algebraic technique: given an acyclic complex of $\mathbb{Z}[G]$-modules

$$0 \to C_n \to \ldots \to C_0 \to 0,$$

there is a map

$$\delta : C_0^G \to H^{n-1}(G, C_n)$$

defined by compositions of connecting morphisms, which we call the *lifting morphism.* Given a map of complexes $C_\bullet \to D_\bullet$, by pushing forward the lifting morphism, one obtains a map

$$\delta_{C_\bullet,D_\bullet} : C_0^G \to H^{n-1}(G, D_n).$$

These details are worked out in Section 4.1.

In light of the above, for any $c \in C_0^G$, explicit choices of "lifts" of c inside $C_\bullet$ give a representative for the cocycle class $\delta_{C_\bullet,D_\bullet}(c)$.

For our applications, we work with a special complex $K_\bullet$ called the *Gersten complex for* $\mathbb{G}_m^n$, which is a homological complex that arises from the coniveau spectral sequence. The complex $K_\bullet$ is not easy to work with directly, so to construct a cocycle we use a special "parameterizing complex" $C_\bullet^{line}$ and construct a map of complexes of $\mathbb{Z}'[\mathrm{GL}_n(\mathbb{Z})]$-modules

$$C_\bullet^{line} \otimes_\mathbb{Z} \mathbb{Z}' \to K_\bullet \otimes_\mathbb{Z} \mathbb{Z}'.$$

This then results in a cocycle class

$$[\Theta] \in H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), K_n \otimes_Z \mathbb{Z}')$$

by pushing forward the lifting morphism applied to the element $1 \in (C_0^{line})^{\mathrm{GL}_n(\mathbb{Z})} \cong \mathbb{Z}$.

For technical reasons, the morphism of complexes

$$C_\bullet^{line} \otimes_\mathbb{Z} \mathbb{Z}' \to K_\bullet \otimes_\mathbb{Z} \mathbb{Z}'$$

must be constructed by first factoring through a special acyclic complex $C_\bullet^{lim}$ called the *limit complex for* $\mathbb{G}_m^n$. This complex arises from a direct limit over all toric flags of the complexes obtained by Theorem 3.3.8. The details of the construction of the map

$$C_\bullet^{line} \otimes_\mathbb{Z} \mathbb{Z}' \to C_\bullet^{lim}$$

is the content of Theorem 4.4.7 and is the main result of Chapter 4. Explicit lifts of $1 \in (C_0^{line})^{\mathrm{GL}_n(\mathbb{Z})}$ are described in Theorem 4.4.12, which then gives an explicit choice of representative for the cocyle class $[\Theta]$.

Section 4.2 defines the complex C^{line}, along with an alternate parameterizing complex, C^{ray}, which is more closely related to the construction in [25]. Integral cocycle constructions for $n = 2, 3$ are carried out in Section 4.3 using the complex C^{ray}, and we show that for $n = 2$ this construction precisely recovers that of [25].

In Section 4.5, we examine actions of Hecke operators on the cocycle class $[\Theta]$. The main result here is Theorem 4.5.2, where we prove that $[\Theta]$ is Eisenstein.

CHAPTER 2

Algebraic Background

The purpose of this expository chapter is twofold:

- Review facts about motivic cohomology and Milnor K-theory that are used throughout the thesis.
- Review the coniveau spectral sequence, and use it to construct a Gersten-type complex in motivic cohomology that will be important in Chapters 3 and 4.

2.1 Motivic cohomology and Milnor K-theory

Motivic cohomology arose from trying to construct a cohomology theory for schemes that maintains many of the key properties that singular cohomology has for topological spaces. Historically, there are several different ways to define motivic cohomology. One way is due to Bloch, using higher Chow groups. These arise from a scheme-theoretic analogue of simplicial homology groups in algebraic topology. A second way, due to Voevodsky, defines motivic cohomology as hypercohomology of so called "motivic complexes" on the Zariski site. For sufficiently nice schemes, these two theories agree with each other, and so there need not be a distinction.

Milnor K-theory, on the other hand, was a purely "ad hoc" construction of Milnor [21] that was an attempt to understand what algebraic K-theory should look like for fields. While it turned out that Milnor's definition was not quite the right generalization of higher K-groups that had been hoped for, remarkably Milnor's definition led to a rather interesting theory in its own right.

There is a classical result of Bloch that relates (rational) algebraic K-theory and higher Chow groups:

$$K_n(X) \otimes \mathbb{Q} \cong \bigoplus_{i=0}^{\infty} \mathrm{Ch}^i(X,n) \otimes \mathbb{Q}.$$

In particular, when $X = \mathrm{Spec}(F)$, surprisingly the higher Chow groups $\mathrm{Ch}^n(F,n)$ are isomorphic to Milnor K-groups. This is worked out in a paper of Totaro [26].

Due to its deep connections with algebraic K-theory, Milnor K-theory, and étale cohomology, motivic cohomology is a very promising cohomology theory. Unfortunately, many of its properties are conjectural, and most known results about motivic cohomology are very hard theorems.

In this section, we will give a general overview of the properties of motivic cohomology and Milnor K-theory that will be used throughout the rest of this thesis. There are many good references on motivic cohomology and its connections to the various other theories listed. In particular, we refer the reader to either [20] or [10] for a good treatment of the subject.

2.1.1 Definitions and key properties

Our starting point will be Bloch's *higher Chow groups*. Our main point of reference will be [5].

Let X be a quasi-projective scheme of finite type over a perfect field F. For each integer $k \geq 0$, define

$$\Delta^k = \mathrm{Spec}(F[t_0, \ldots, t_k]/(t_0 + \ldots + t_k - 1)),$$

which is the algebraic analogue of the standard k-simplex. We define a *face* of Δ^k to be a closed subscheme corresponding to $t_{i_1} = \ldots = t_{i_\ell} = 0$ for $0 \leq i_1 < \ldots < i_\ell \leq k$ and some $\ell \leq k$. Note that a face is therefore isomorphic to $\Delta^{k-\ell}$. Let $Z^k(X)$ denote the group of codimension k cycles on X, and let $z^k(X,i) \subset Z^k(X \times \Delta^i)$ denote the subgroup of cycles that have proper intersection with $X \times \Phi$ for each face Φ of Δ^i.

For each i, there is an embedding $\partial_{k,i} : \Delta^{k-1} \to \{t_i = 0\} \subset \Delta^k$ called the *face map*, which extends in the obvious way to give face maps on $X \times \Delta^k$. Via pullback of cycles, these give rise to morphisms $\partial_{k,i}^* : z^k(X,i) \to z^k(X,i-1)$. One may then define a boundary operator $d_k = \sum_{i=0}^{k}(-1)^i \partial_{k,i}^*$ via the usual alternating sum of face maps, which makes $(z^k(X,\cdot), d_k)$ a chain complex.

Definition 2.1.1. Let i, k be non-negative integers. The higher Chow group $\mathrm{Ch}^k(X,i)$ is defined as

$$\mathrm{Ch}^k(X,i) = H_i(z^k(X,\cdot)).$$

Definition 2.1.2. Let X be a quasi-projective scheme of finite type over a perfect field F. For integers i, j with $j \geq 0$, we define the *motivic cohomology group* $H^i(X,j)$ of degree i and weight j by

$$H^i(X,j) = \mathrm{Ch}^j(X, 2j-i)$$

When $j < 0$ we set $H^i(X,j) = 0$.

When X is smooth, then there is a natural isomorphism

$$H^i(X,j) \cong H^i(X,\mathbb{Z}(j)),$$

where $H^i(X,\mathbb{Z}(j))$ is the degree i and weight j motivic cohomology group constructed by Voevodsky (which is the usual definition of motivic cohomology when X is not smooth). Therefore, when applicable, we will use the notation on the left instead. The equivalence of the two constructions can be found in [20].

Below, we will list many of the standard properties of motivic cohomology groups that will be relevant. Many of these are taken (essentially verbatim) from Section 2.1 of [25], where references are provided, but all are found in [20].

- If $F \subset L$ is a finite separable extension, and X (viewed as a scheme over L) is smooth, then $H^i(X,j)$ does not depend on whether or not X is viewed as a scheme over F or over L.
- If $X = \bigsqcup_{m=1}^{n} X_m$ is a union of disjoint schemes, then $H^i(X,j) \cong \bigoplus_{m=1}^{n} H^i(X_m, j)$.
- $H^i(X,j) \cong H^i(X \times_F \mathbb{A}^1, j)$ via pullback by the projection morphism $X \times_F \mathbb{A}^1 \to X$.
- $H^0(X,0) \cong \mathbb{Z}$ if X is connected and $H^i(X,0) = 0$ for $i \neq 0$.

- If X is smooth, then $H^1(X,1) \cong \mathcal{O}(X)^\times$, the group of global units on X. Further, $H^2(X,1) \cong \mathrm{Pic}(X)$, and $H^i(X,1) = 0$ for $i \notin \{1,2\}$.

- If X is smooth, then $H^i(X,j) = 0$ for $i > j + \dim(X)$.

- For $f : X \to Y$, there is a pullback map $f^* : H^i(Y,j) \to H^i(X,j)$. If f is proper of relative dimension d, there is a pushforward map $f_* : H^i(X,j) \to H^{i-2d}(Y, j-d)$.

- If $f : X \to Y$ is a finite locally free morphism of quasi-projective schemes of finite type (and therefore proper of relative dimension 0), then $f_* f^*$ is multiplication by the degree of f.

When X is equidimensional, for any closed subscheme $\rho : Z \to X$ of pure codimension c and its complement $\iota : U \to X$, there is an exact Gysin sequence

$$\ldots \to H^i(X,j) \xrightarrow{\iota^*} H^i(U,j) \xrightarrow{\partial} H^{i-2c+1}(Z, j-c) \xrightarrow{\rho_*} H^{i+1}(X,j) \to \ldots$$

Motivic cohomology has cup products

$$\cup : H^i(X,j) \times H^{i'}(X,j') \to H^{i+i'}(X, j+j'),$$

and in particular, when $X = \mathrm{Spec}(F)$ is the spectrum of a field, there is an isomorphism of graded rings

$$\bigoplus_{i=0}^{\infty} H^i(F,i) \cong \bigoplus_{i=0}^{\infty} K_i^M(F),$$

where $K_i^M(F)$ denotes the i-th *Milnor K-group* of the field F. The construction of these groups are as follows:

Definition 2.1.3. Let F be a field and let $T(F^\times) = \mathbb{Z} \oplus F^\times \oplus (F^\times \otimes F^\times) \oplus \cdots$ denote the tensor algebra of $F^\times$. The graded ring $K_*^M(F)$ is defined by

$$K_*^M(F) = T(F^\times)/I,$$

where I is the homogeneous ideal generated by $x \otimes (1-x)$ for all $x \neq 0, 1$. For $n \geq 0$, the n-th Milnor K-group is defined as the subgroup of elements of degree n.

For $x_1, \ldots, x_n \in F^\times$ we write $\{x_1, \ldots, x_n\}$ to denote the image of $x_1 \otimes \ldots \otimes x_n$ in $K_n^M(F)$ and call such an element a *Steinberg symbol*.

Explicitly, $K_0^M(F) = \mathbb{Z}$ and $K_1^M(F) = F^\times$, and the Steinberg symbols $\{x_1, \ldots, x_n\}$ generate the group $K_n^M(F)$. These symbols are bilinear and are zero when $x_i + x_j = 1$ for some i, j.

The isomorphism of graded rings is then induced by the identification of both sides with $\mathbb{Z}$ and $F^\times$ in degrees 0 and 1, and in degree n by the $\mathbb{Z}$-bilinear extension of $x_1 \cup \ldots \cup x_n \mapsto \{x_1, \ldots, x_n\}$ for $x_i \in F^\times$.

Remark 2.1.4. For $0 \leq n \leq 2$, there is an isomorphism of groups

$$K_n^M(F) \xrightarrow{\sim} K_n(F),$$

where $K_n(F)$ is nth *algebraic* K-group of Quillen. For $n = 2$, this is a theorem of Matsumoto. When $n \geq 3$, the relation between these groups is not that well understood. When F is a number field, the following was shown by Shapiro [23]:

- The map $K_3^M(F) \to K_3(F)$ is injective.
- The map $K_4^M(F) \to K_4(F)$ is not injective.
- The map $K_n^M(F) \to K_n(F)$ is zero for $n \geq 5$.

As we will be interested in constructing cocycles valued in Milnor K-theory, it will be helpful to review some basic properties. The proposition below is rather elementary, but these sorts of manipulations on symbols in Milnor K-theory will become common later, so we include the proof for instructive purposes.

Proposition 2.1.5. *Let $x, y \in F^\times$. Then $\{x, -x\} = 0$ and $\{x, y\} = -\{y, x\}$.*

Proof. From the identity $-x = \frac{1-x}{1-x^{-1}}$ and the Steinberg relations, one has $\{x, -x\} = \{x, 1-x\} + \{x, \frac{1}{1-x^{-1}}\} = -\{x, 1-x^{-1}\} = \{x^{-1}, 1-x^{-1}\} = 0$. Then, note that $0 = \{xy, -xy\} = \{x, y\} + \{y, x\}$. ☐

Corollary 2.1.6. *The ring $K_*^M(F)$ is graded-commutative, i.e. for $m, n \geq 0$ and $x \in K_n^M(F), y \in K_m^M(F)$, one has*

$$\{x, y\} = (-1)^{mn}\{y, x\}$$

In particular, for $x_1, \ldots, x_n \in F^\times$ and $\pi \in S_n$ one has

$$\{x_1, \ldots, x_n\} = sign(\pi)\{x_{\pi(1)}, \ldots, x_{\pi(n)}\}.$$

Proof. This easily follows from induction and the previous proposition, and can be found in [21]. □

Now, let $v : F^\times \to \mathbb{Z}$ be a discrete valuation on F with valuation ring $\mathcal{O}_v$ and residue field $k(v)$. Let π be a uniformizer, i.e. an element $\pi \in F^\times$ such that $v(\pi) = 1$. Any $x \in F^\times$ may be written as $u\pi^k$ for some integer $k \geq 0$ and unit $u \in \mathcal{O}_v^\times$, so it immediately follows by the previous propositions that $K_n^M(F)$ is generated by symbols of the form $\{\pi, u_2, \ldots, u_n\}$ for $u_i \in \mathcal{O}_v^\times$.

Proposition 2.1.7. *Let v be a discrete valuation on $F^\times$. For each $n \geq 1$, there exists a unique surjective homomorphism*

$$\partial_v : K_n^M(F) \to K_{n-1}^M(k(v))$$

that satisfies

$$\partial_v(\{\pi, u_2, \ldots, u_n\}) = \{\overline{u}_2, \ldots, \overline{u}_n\}$$

for all uniformizers π and collections of units $u_2, \ldots, u_n \in \mathcal{O}_v^\times$, where $\overline{u}_i$ denotes the image of i in $k(v)$. The kernel of ∂_v contains all symbols of the form $\{u_1, \ldots, u_n\}$.

Proof. This is Lemma 2.1 in [21]. □

The map ∂_v in the above proposition is called a *higher tame symbol*, or a *residue map*. In particular, for $n = 1$ the map $\partial_v : K_1^M(F) \to K_0^M(k(v))$ is simply the valuation map $v : F^\times \to \mathbb{Z}$, while for $n = 2$ the map $\partial_v : K_2^M(F) \to K_1^M(k(v))$ is given by the formula

$$\partial_v(\{x, y\}) = (-1)^{v(x)v(y)}\overline{y^{v(x)}x^{-v(y)}}$$

This can be seen by first fixing a choice of uniformizer π and then writing $x = \pi^{v(x)}u_1$ and $y = \pi^{v(y)}u_2$ for $u_1, u_2 \in \mathcal{O}_v^\times$ and using the definition of ∂_v and properties of Steinberg symbols. Indeed, this agrees with the classical definition of the tame symbol, hence the terminology.

Returning to motivic cohomology, we will often need to compare pullbacks and push-forwards. The following two results are therefore essential:

Proposition 2.1.8 (Base change). *Suppose that*

$$\begin{tikzcd} X' \arrow[r, "\pi_X"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "\pi_Y"] & Y \end{tikzcd}$$

is a Cartesian diagram of smooth quasi-projective schemes of finite type over F, with π_Y flat and f proper. Then π_X is flat, f' is proper, and

$$(f')_* \pi_X^* = \pi_Y^* f_*$$

as morphisms

$$H^i(X, j) \to H^{i+\dim Y - \dim X}(Y', j).$$

Proposition 2.1.9 (Projection formula). *Let $f : X \to Y$ be a proper morphism of smooth quasi-projective schemes of finite type over F, and let $\alpha \in H^i(X, j)$ and $\beta \in H^{i'}(Y, j')$. Then*

$$f_*(\alpha \cup f^*(\beta)) = f_*(\alpha) \cup \beta \in H^{i+i'+\dim Y - \dim X}(Y, j + j').$$

These are 2.1.1 and 2.1.2 in [25], and we refer the interested reader there for a proof.

Conjecturally, negative degree motivic cohomology groups should vanish. A reference for this is [10, II.4, Conjecture 5].

Conjecture 2.1.10 (Beilinson-Soulé vanishing conjecture). *Let X be a smooth scheme over a field F. Then for $i < 0$,*

$$H^i(X, j) = 0.$$

There are not many cases where this conjecture is known. One important case for us will be the following:

Theorem 2.1.11 (Borel). *Let $X = \mathrm{Spec}(F)$ for a number field F. Then the Beilinson-Soulé vanishing conjecture holds.*

Proof. This follows from work of Borel [6] and Beilinson [4]. $\square$

2.1.2 Coniveau spectral sequences

In this section, we review the coniveau spectral sequence for motivic cohomology. This spectral sequence will be of central importance, as it provides complexes that compute motivic cohomology groups in many situations of interest.

Proposition 2.1.12. *Let Y be a finite type smooth connected variety over a field F and $n = \dim(Y)$. There is a right half-plane spectral sequence*

$$E_1^{p,q} = \bigoplus_{x \in Y_p} H^{q-p}(F(x), n-p) \implies H^{p+q}(Y, n),$$

where Y_p denotes the set of points of codimension p of Y.

The spectral sequence in the above proposition is called the *coniveau spectral sequence* for Y. A reference for the construction is [8].

Roughly, the coniveau spectral sequence is obtained by taking a direct limit of spectral sequences attached to Gysin sequences. Somewhat more precisely, let $Z = (Z_p)_{p \in Z}$ be a (descending) flag of closed F-subschemes of Y with the following properties:

- Z_p is of pure codimension p for $1 \leq p \leq n$.
- $Z_p - Z_{p+1}$ is smooth for all $p \in \mathbb{Z}$.
- $Z_p = Y$ for $p \leq 0$.
- $Z_p = \emptyset$ for $p > n$.

We shall call such a flag Z a *good flag* for Y. Good flags for Y admit Gysin sequences

$$\ldots \to H^i(Z_p, n-p) \to H^i(Z_p - Z_{p+1}, n-p-1) \xrightarrow{\partial} H^{i-1}(Z_{p+1}, n-p) \to \ldots$$

for $0 \leq p \leq n-1$.

Set $D^{p,q} = H^{q-p}(Z_p, n-p)$ and $E^{p,q} = H^q(Z_p - Z_{p+1}, n-p)$. Then there is an exact couple (D, E, f, g, h) obtained from the maps $D^{p+1,q} \to D^{p,q}$, $E^{p,q-p} \to D^{p+1,q}$, and $D^{p,q} \to E^{p,q-p}$ given by the corresponding maps between terms in the Gysin sequence. Therefore, attached to this flag is a convergent right half-plane spectral sequence

$$E_1^{p,q}(Z) = H^{q-p}(Z_p - Z_{p+1}, n-p) \implies H^{p+q}(Y, n).$$

For two good flags Z, Z' of Y, we can define an ordering by declaring that $Z \leq Z'$ if Z'_p is a closed subscheme of Z_p for all $p \in \mathbb{Z}$. This ordering then gives rise to a morphism of spectral sequences $E_1^{p,q}(Z) \to E_1^{p,q}(Z')$ via the composition $j^* \iota_*$ of pushforward and pullback along

$$Z'_p - Z'_{p+1} \xrightarrow{\iota} Z_p - Z'_{p+1} \xleftarrow{j} Z_p - Z_{p+1}.$$

The coniveau spectral sequence arises as the spectral sequence obtained by taking a direct limit over all possible good flags Z.

The $q = n$ row in the E_1-page of the coniveau spectral sequence is a complex of the form

$$K_\bullet = K_\bullet(Y): \quad K_n^M(F(Y)) \to \bigoplus_{x \in Y_1} K_{n-1}^M(F(x)) \to \ldots \to \bigoplus_{x \in Y_n} K_0^M(F(x)),$$

which we refer to as the *Gersten complex* for Y [1]

Now, set $\Delta = M_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$. Then there is a natural monoid action of Δ on $\mathbb{G}_m^n$ via right multiplication. That is to say, for any $\gamma = (a_{ij}) \in \Delta$ and $(z_1, \ldots, z_n) \in \mathbb{G}_m^n$, we have

$$(z_1, \ldots, z_n) \cdot \gamma = (z_1^{a_{11}} z_2^{a_{21}} \cdots z_n^{a_{n1}}, \ldots, z_1^{a_{1n}} z_2^{a_{2n}} \cdots z_n^{a_{nn}}).$$

Since this is a right action on $\mathbb{G}_m^n$, it induces a *left* action on the motivic cohomology group $H^i(\mathbb{G}_m^n, j)$ via pullback, and the residue maps in Gysin sequences are equivariant with respect to this action. There is also an action of Δ on the Gersten complex. Explicitly, for $x \in Y_d$ and $\gamma \in \Delta$, then pullback induces a map $\gamma^* : K_d^M(\mathbb{Q}(x)) \to K_d^M(\mathbb{Q}(x \cdot \gamma^{-1}))$. The pullback action on the Gersten complex is then the sum of these maps.

[1] Our choice of terminology is because this complex arises as the complex of total sections in Kato's variant of the Gersten complex, first constructed in [13].

CHAPTER 3

Trace Modules

The purpose of this chapter is to develop the theory of *trace modules*. For us, these will be certain kinds of motivic cohomology groups that admit actions of trace operators $[p]_*$ for primes p. These trace maps are induced via pushforwards by the multiplication by p map on the underlying scheme. We start with a review of [25], where the motivic cohomology groups $H^i(\mathbb{G}_m^n, j)$ are computed. There, the authors developed the idea that trace operators make these groups easier to understand. The main result of this chapter is Theorem 3.3.8, which states that for so-called *toric flags*, we can, after inverting suitably many primes, extract an exact sequence of *generalized trace-fixed parts* of motivic cohomology groups. This complex will play a crucial role in Chapter 4.

3.1 Motivic cohomology of $\mathbb{G}_m^n$

Of crucial importance to us is the computation of the motivic cohomology groups $H^i(\mathbb{G}_m^n, j)$. This is worked out in [25, Section 3.1], which we record here for convenience. For a field F, we use the coordinates $\mathbb{G}_m^n = \mathrm{Spec}(F[z_1^{\pm 1}, \ldots, z_n^{\pm 1}])$.

Proposition 3.1.1. *Let Y be an equidimensional quasi-projective scheme of finite type over F. There is a natural isomorphism*

$$H^i(Y, j) \oplus H^{i-1}(Y, j-1) \cong H^i(\mathbb{G}_m \times Y, j),$$

where the map on the first summand is pullback under projection to the second factor and the map on the second summand is left exterior product with $-z_1$, considered as a class in $H^1(\mathbb{G}_m, 1)$.

A recursive argument employing Proposition 3.1.1 yields the following:

Proposition 3.1.2. *For a field F and $n \geq 1$, there is a natural isomorphism*

$$H^i(\mathbb{G}_m^n, j) = \begin{cases} \bigoplus_{k=0}^{\min(j,n)} H^{i-k}(F, j-k)^{\binom{n}{k}} & i \leq j \\ 0 & i > j \end{cases}.$$

Our main tool for trying to understand the motivic cohomology groups $H^i(\mathbb{G}_m^n, j)$ will be *trace operators.* For a positive integer m, the multiplication by m map

$$[m] : \mathbb{G}_m^n \to \mathbb{G}_m^n$$

defined on coordinate rings by $z_i \mapsto z_i^m$ induces via pushforward the *trace map*

$$[m]_* : H^i(\mathbb{G}_m^n, j) \to H^i(\mathbb{G}_m^n, j).$$

Definition 3.1.3. The *trace-fixed part* $H^i(\mathbb{G}_m^n, j)^{(0)}$ of the group $H^i(\mathbb{G}_m^n, j)$ is defined by

$$H^i(\mathbb{G}_m^n, j)^{(0)} = \{x \in H^i(\mathbb{G}_m^n, j) : ([p]_* - 1)x = 0 \text{ for all primes } p \neq \mathrm{char}(F)\}.$$

Trace maps allow us to gain more "control" over the motivic cohomology groups, in the sense that these trace operators kill off almost all of the cohomology classes in the decomposition of Proposition 3.1.2, giving a rather simple description of the trace fixed part. This is worked out below for characteristic 0 fields, following the arguments given in [25].

Proposition 3.1.4. *The element $-z \in H^1(\mathbb{G}_m, 1)$ generates $H^1(\mathbb{G}_m, 1)^{(0)}$.*

Proof. The group $H^1(\mathbb{G}_m, 1)$ consists of the global units on $\mathbb{G}_m$, and therefore each element is uniquely of the form $\eta(-z)^k$ for some $\eta \in F^\times$ and $k \in \mathbb{Z}$. Now, suppose that $[m]_*$ fixes such a cohomology class for m coprime to $\mathrm{char}(F)$. From the computation

$$[m]_*(-z) = \prod_{i=0}^{m-1}(-\zeta_m^i z^{1/m}) = -z,$$

one sees that $-z$ is trace fixed. Therefore, such a trace-fixed class must satisfy $[m]_*\eta = \eta^m$. If $\mathrm{char}(F) \neq 2$, then $\eta^2 = \eta$, so $\eta = 1$. Otherwise, $\eta^3 = \eta$ yields the same conclusion. □

Proposition 3.1.5. *Suppose that F is a field of characteristic 0. Then $H^i(\mathbb{G}_m^n, n)^{(0)} = 0$ except when $i = n$, in which case $H^n(\mathbb{G}_m^n, n)^{(0)} \cong \mathbb{Z}$, generated by the class of $(-z_1) \cup (-z_2) \cup \ldots \cup (-z_n)$.*

Proof. The case of $n = 2$ was done in [25]. We use the same idea to prove the general result. If $i > n$ this follows trivially, so assume $i \leq n$. The motivic cohomology group $H^i(\mathbb{G}_m^n, n)$ breaks up as direct sum

$$H^i(\mathbb{G}_m^n, n) \cong \bigoplus_{j=0}^{n} H^{i-j}(F, n-j)^{\binom{n}{j}}.$$

Explicitly, this isomorphism is described as follows. Pullback via the map $s : \mathbb{G}_m^n \to \mathrm{Spec}(F)$ produces maps $s^* : H^{i-j}(F, n-j) \to H^{i-j}(\mathbb{G}_m^n, n-j)$. For $j > 0$, on each of the $\binom{n}{j}$ summands on the right-hand side, the map to $H^i(\mathbb{G}_m^n, n)$ is defined by $\eta \mapsto s^*(\eta) \cup \big((-z_{k_1}) \cup \ldots \cup (-z_{k_j})\big)$ where $1 \leq k_1 < \ldots < k_j \leq n$, taken over all $\binom{n}{j}$ such tuples $(k_1, \ldots, k_j)$. For $j = 0$, the map simply sends η to $s^*(\eta) \cup 1$. Since any class η pulled back from $\mathrm{Spec}(F)$ satisfies $[m]^*\eta = \eta$, we see from the projection formula that for any $\alpha \in H^j(\mathbb{G}_m^n, j)$ and such $\eta \in H^{i-j}(\mathbb{G}_m^n, n-j)$,

$$[m]_*(\alpha \cup \eta) = [m]_*(\alpha \cup [m]^*\eta) = ([m]_*\alpha) \cup \eta.$$

That is to say, taking trace-fixed parts preserves summands, so it suffices to consider each one individually.

Now, on each summand corresponding to j in the sum, we have that $[m]_*\alpha = m^{n-j}\alpha$ for any α of the form $(-z_{k_1}) \cup \ldots \cup (-z_{k_j})$. We can see this as follows. The class $-z_i \in H^1(\mathbb{G}_m^n, 1)$ may be written as $-z_i = \pi_i^*(-z)$, where $-z \in H^1(\mathbb{G}_m, 1)$ and $\pi_i : \mathbb{G}_m^n \to \mathbb{G}_m$ is the projection onto the ith coordinate. Therefore,

$$[m]_*((-z_{k_1}) \cup \ldots \cup (-z_{k_j})) = [m]_*(\pi_{k_1}^*(-z) \cup \ldots \cup \pi_{k_j}^*(-z)).$$

We write $[m]_{i*}$ to denote the map $[(1, \ldots, m, \ldots, 1)]_*$, the pushforward of the map that is multiplication by m in the i-th coordinate. Now, note that for $i_1 \neq i_2$ that $\pi_{i_1} \circ [m]_{i_2} = \pi_{i_1}$, and therefore $[m]_{i_2}^* \pi_{i_1}^*(-z) = \pi_{i_1}^*(-z)$. In particular, we see that

$$[m]_{k_1}^* \pi_{k_i}^*(-z) = \pi_{k_i}^*(-z)$$

for $i \neq 1$. Therefore,

$$[m]_*(\pi_{k_1}^*(-z) \cup \ldots \cup \pi_{k_j}^*(-z)) = (\prod_{i \neq k_1} [m]_{i*}) \circ [m]_{k_1 *}\Big(\pi_{k_1}^*(-z) \cup [m]_{k_1}^*\big(\pi_{k_2}(-z) \cup \ldots \cup \pi_{k_j}^*(-z)\big)\Big).$$

By the projection formula, we must have

$$[m]_{k_1*}\Big(\pi_{k_1}^*(-z)\cup[m]_{k_1}^*\big(\pi_{k_2}(-z)\cup\ldots\cup\pi_{k_j}^*(-z)\big)\Big) = [m]_{k_1*}\pi_{k_1}^*(-z)\cup\pi_{k_2}(-z)\cup\ldots\cup\pi_{k_j}^*(-z),$$

and

$$[m]_{k_1*}\pi_{k_1}^*(-z)\cup\pi_{k_2}(-z)\cup\ldots\cup\pi_{k_j}^*(-z) = \pi_{k_1}^*(-z)\cup\ldots\cup\pi_{k_j}^*(-z)$$

because $\pi_{k_1}^*(-z)$ is trace fixed. Using that the cup product is alternating and a similar argument, we see that the value of $[m]_*$ on $\pi_{k_1}^*(-z)\cup\ldots\cup\pi_{k_j}^*(-z)$ depends only the value of $[m]_{i*}$ for $i \neq k_1, \ldots, k_j$. The computation of $[m]_*\Big(\pi_{k_1}^*(-z)\cup\ldots\cup\pi_{k_j}^*(-z)\Big)$ can be handled similarly: for $i \neq k_1, \ldots, k_j$, we note that $[m]_i^*\pi_{k_\ell}^*(-z) = \pi_{k_\ell}^*(-z)$ for $\ell = 1, \ldots, j$ and therefore

$$[m]_i^*(\pi_{k_1}^*(-z)\cup\ldots\cup\pi_{k_j}^*(-z)) = \pi_{k_1}^*(-z)\cup\ldots\cup\pi_{k_j}^*(-z).$$

Since $[m]_{i*}[m]_i^*(\pi_{k_1}^*(-z)\cup\ldots\cup\pi_{k_j}^*(-z)) = m(\pi_{k_1}^*(-z)\cup\ldots\cup\pi_{k_j}^*(-z))$ because $[m]_i$ is a degree 1 map, it follows that

$$[m]_{i*}(\pi_{k_1}^*(-z)\cup\ldots\cup\pi_{k_j}^*(-z)) = m(\pi_{k_1}^*(-z)\cup\ldots\cup\pi_{k_j}^*(-z)).$$

As $[m]_* = \prod_{i=1}^n [m]_{i*}$, we then find that $[m]_*(\pi_{k_1}^*(-z)\cup\ldots\cup\pi_{k_j}^*(-z)) = m^{n-j}(\pi_{k_1}^*(-z)\cup\ldots\cup\pi_{k_j}^*(-z))$ as desired.

Returning back to our prior task, saying that $\alpha\cup\eta$ is trace fixed is the same as saying $(p^{n-j}-1)(\alpha\cup\eta) = 0$ for all primes p.

First, suppose $n > j$. Let $a \geq 1$, and consider the ideal $I = \big(p^a - 1 : p \text{ prime}\big)$ as an ideal of $\mathbb{Z}$. Let $I = (d)$ for some $d \in \mathbb{Z}$ with $d \geq 1$. For any prime q dividing d, we must have $q^a - 1 \equiv -1 \equiv 0 \bmod q$, which is clearly impossible if $d > 1$. Therefore, $I = (1)$ and so we may find a $\mathbb{Z}$-linear combination of elements of I that sum to 1. This proves that $\alpha\cup\eta = 0$, so the corresponding trace fixed part is 0.

The remaining case is $n = j$. Here, if $i = n$ then this follows because $(-z_1)\cup(-z_2)\ldots\cup(-z_n)$ is trace fixed. Otherwise, $i < n$ in which case we have a weight 0 motivic cohomology group with negative degree, which is 0 from Section 2.1.1. $\square$

3.2 Trace modules

The purpose of this section is to develop the theory of *trace modules.*

Definition 3.2.1. An abelian group A is called a *trace module* if for each prime $p \in \mathbb{Z}$ there are $\mathbb{Z}$-linear endomorphisms

$$[p]_* : A \to A$$

such that for primes $p \neq q$,

$$[p]_* \circ [q]_* = [q]_* \circ [p]_*.$$

We call such endomorphisms *trace maps.* A morphism of trace modules is a $[p]_*$-equivariant map of $\mathbb{Z}$-modules for all primes p.

Remark 3.2.2. Equivalently, a trace module is a $\mathbb{Z}[\mathbb{Z}_{\geq 1}]$-module for the semigroup $\mathbb{Z}_{\geq 1}$ under multiplication, and a morphism of trace modules is just a $\mathbb{Z}[\mathbb{Z}_{\geq 1}]$-linear map. For any commutative ring R, it therefore makes sense to speak more generally of *trace modules over* R as $R[\mathbb{Z}_{\geq 1}]$-modules. In particular, if A is a trace module, then the extension of scalars $A \otimes_{\mathbb{Z}} R$ is an $R[\mathbb{Z}_{\geq 1}]$-module. By a slight abuse of notation, we will also refer to the module $A \otimes_{\mathbb{Z}} R$ as A, but refer to A as a trace module over R, to avoid ambiguity.

The motivating example for this definition is, of course, the following:

Example 3.2.3. The abelian groups $H^i(\mathbb{G}_m^n, j)$ for integers i, j are trace modules with trace maps $[p]_*$ coming from pushforward by the pth power maps.

Definition 3.2.4. Let A be a trace module. For an integer $r \geq 0$, the rth *eigenspace* of A is defined by

$$A^{(r)} = \{a \in A : ([p]_* - p^r)a = 0 \text{ for almost all primes } p\}.$$

The rth *generalized eigenspace* of A is defined by

$$A^r = \{a \in A : \exists N \geq 1 \text{ with } ([p]_* - p^r)^N a = 0 \text{ for almost all primes } p\}.$$

Remark 3.2.5. In the above definitions, the set of bad primes may vary from element to element. Similarly, the exponent in a generalized eigenspace may vary from element to element. For a finite set of primes S and integer $N \geq 1$, we use the notations $A_S^{(r)}$ and $A_{S,N}^r$ to denote eigenspaces/generalized eigenspaces with uniform bad set of primes S, and, in the second case, universal exponent N.

Our inspiration for the notion of a trace module is [25, Proposition 6.1.2]. Our goal will be to show that after inverting a suitable number of primes, we can construct projection operators onto eigenspaces of trace modules, so that trace modules admit eigenspace decompositions, which makes working with them much easier. To do so, we first need to make precise what we mean by this:

Definition 3.2.6. Let A be a trace module. We say that A has an *eigenspace decomposition* if there exists an integer $n \geq 0$ and a finite set of primes S such that

$$A = \bigoplus_{r=0}^{n} A_S^{(r)}.$$

We also make the following definition:

Definition 3.2.7. Let $N \geq 1$, and A be a trace module. We say that A is *nice for* N if there exists $n \geq 0$ and a finite set of primes S such that

$$A = \bigoplus_{r=0}^{n} A_{S,N}^r.$$

In particular, we note that if A is a trace module that is nice for 1, then A admits an actual eigenspace decomposition.

A key example for us that motivates our definition of eigenspaces "away from S" is the following:

Proposition 3.2.8. *Suppose that $F = \mathbb{Q}$, and let $n \geq 1$. Let $T = \mu_{m_1} \times \cdots \times \mu_{m_k}$ be a product of roots of unity with $m_1 \cdots m_k = N$. Let S be the set of primes dividing N. Then $H^i(T \times \mathbb{G}_m^n, n)_S^{(0)} = 0$ unless $i = n$.*

Proof. The argument is similar to that of Proposition 3.1.5. By applying Proposition 3.1.1 recursively, one sees that there is a natural isomorphism

$$H^i(T \times \mathbb{G}_m^n, n) = \begin{cases} \bigoplus_{j=0}^{n} H^{i-j}(T, n-j)^{\binom{n}{j}} & i \leq n \\ 0 & i > n \end{cases}.$$

When $i > n$, this follows from $T \times \mathbb{G}_m^n = \bigsqcup_i \operatorname{Spec}(E_i) \times_{\mathbb{Q}} \mathbb{G}_m^n$ for appropriate cyclotomic fields E_i, and the fact that $H^i(\operatorname{Spec}(E_i) \times_{\mathbb{Q}} \mathbb{G}_m^n, j) = H^i(\mathbb{G}_m^n, j) = 0$ (the latter $\mathbb{G}_m^n$ viewed as a scheme over E_i) because motivic cohomology does not depend on the choice of base scheme. Therefore, we assume that $i \leq n$. Pullback via the projection map $\pi_1 : T \times \mathbb{G}_m^n \to T$ produces maps on cohomology $\pi_1^* : H^{i-j}(T, n-j) \to H^{i-j}(T \times \mathbb{G}_m^n, n-j)$, and the map on each summand of the right-hand side of the direct sum decomposition is given by $\eta \mapsto \pi_1^*(\eta) \cup \pi_2^*((-z_{k_1}) \cup \ldots \cup (-z_{k_j}))$ where $1 \leq k_1 < \ldots < k_j \leq n$, taken over all $\binom{n}{j}$ such tuples $(k_1, \ldots, k_j)$, and $\pi_2 : T \times \mathbb{G}_m^n \to \mathbb{G}_m^n$ is the other projection map (here, we think of $(-z_{k_1}) \cup \ldots \cup (-z_{k_j}) \in H^j(\mathbb{G}_m^n, j)$).

It then follows that for any $\eta \in H^j(T, j)$ and $\alpha \in H^{i-j}(\mathbb{G}_m^n, n-j)$ that

$$[m]_*(\pi_2^*(\alpha) \cup \pi_1^*(\eta)) = ([m]_*\pi_2^*(\alpha)) \cup ([m]_*\pi_1^*(\eta)).$$

Now, the claim is for $(m, N) = 1$, that $[m]_*\pi_1^*(\eta) = \pi_1^*(\eta)$. Proposition 3.2.8 then follows from the factorization of the map $[m]_*$ as in the proof of Proposition 3.1.5. By base change, this is the same as proving that $[m]_* : H^j(T, j) \to H^j(T, j)$ is the identity map. Now, the group scheme T breaks up as a disjoint union of irreducible components, and therefore it's sufficient to prove the claim for such a component C. An irreducible component C of T looks like $C = \operatorname{Spec}(\mathbb{Q}(\zeta_{d_1})) \times \cdots \times \operatorname{Spec}(\mathbb{Q}(\zeta_{d_k}))$ for $d_i \mid m_i$. Using the definition of $H^j(C, j)$ as a higher Chow group and looking at the level of cycles, it follows that because $[m]$ preserves C and irreducible cycles of $C \times \Delta^j$ of codimension j are of the form $C \times \{\mathrm{pt}\}$, that $[m]_*$ fixes elements of $H^j(T, j)$ as desired. □

The first key property of trace modules is that after inverting enough primes, one can construct explicit projection operators onto their eigenspaces after extending scalars. These projections will be written in terms of the following operator, and will be frequently used throughout the rest of the section.

Definition 3.2.9. Fix an integer $n \geq 0$. Let $r \geq 0$, ℓ prime, and A a trace module. We define $\phi_r(\ell) : A \to A$ to be the operator

$$\phi_r(\ell) = \prod_{\substack{s=0 \\ s \neq r}}^{n} ([\ell]_* - \ell^s).$$

We are now ready to begin developing our theory.

Proposition 3.2.10. *Let A be a trace module. Suppose that there exists an integer $n \geq 0$ and a finite set of primes S such that $A = \sum_{r=0}^{n} A_S^{(r)}$. Then as a trace module over $\mathbb{Z}' = \mathbb{Z}[\frac{1}{(n+1)!}]$, there exist finitely many primes $\ell_1, \dots, \ell_k \notin S$ and operators*

$$\phi_r = \sum_{t=1}^{k} c_{t,r} \phi_r(\ell_t)$$

for $0 \leq r \leq n$ such that

$$\phi_r : A \twoheadrightarrow A_S^{(r)}$$

are orthogonal projections. In other words, A has an eigenspace decomposition.

Proof. For any $a \in A$, we can write $a = \sum_{r=0}^{n} a_r$ where $a_r \in A_S^{(r)}$. For any $\ell \notin S$, we then have by definition that

$$\phi_r(\ell) a_r = c_r(\ell) a_r,$$

where

$$c_r(\ell) = \prod_{\substack{s=0 \\ s \neq r}}^{n} (\ell^r - \ell^s),$$

and $\phi_r(\ell) a_i = 0$ for $i \neq r$. Now, consider the ideal $I_r = \big(c_r(\ell) : \ell \notin S\big)$. As this is an ideal of $\mathbb{Z}$, then we may write $I_r = (d)$ for some $d \geq 1$. Now, let q be a prime dividing d. If $q > n+1$, then $c_r(\ell) \equiv 0 \bmod q$ for all $\ell \notin S$. By choosing ℓ to be a primitive root mod q, we find that this is impossible. Therefore, $q \leq n+1$, and so $I_r = (1)$ as an ideal of $\mathbb{Z}'$. This means there exist $c_{1,r}, \dots, c_{k,r} \in \mathbb{Z}'$ and primes $\ell_1, \dots, \ell_k$ such that $\sum_{i=1}^{k} c_{i,r} c_r(\ell_i) = 1$. We then define $\phi_r = \sum_{i=1}^{k} c_{i,r} \phi_r(\ell_i)$, so that $\phi_r a = a_r$ for all $a \in A$, which proves that $\phi_r : A \to A_S^{(r)}$ is a projection. The proposition statement follows by taking the set of primes to be the union of the finite sets of primes $\{\ell_k\}$ after performing this for each r once we have shown that ϕ_r are orthogonal and sum to 1.

To see this, it's clear that $\sum_{r=0}^{n} \phi_r = 1$, so we just need to see that these operators are orthogonal. Suppose that $r \neq r'$. If $a \in A_S^{(r)} \cap A_S^{(r')}$ we must have $([p]_* - p^r)a = 0$ and $([p]_* - p^{r'})a = 0$ for all $p \notin S$, i.e. $(p^r - p^{r'})a = 0$ for all $p \notin S$. As above, the only primes dividing $p^r - p^{r'}$ for all $p \notin S$ are at most $n+1$, and therefore $a = 0$. $\square$

Corollary 3.2.11. *Suppose that the projections ϕ_r of Proposition* 3.2.10 *can be constructed using all but some finite set S of primes. Then for any finite set of primes S' with $S \subset S'$, ϕ_r can be constructed using primes not in S'.*

Proof. This follows immediately because any trace module A that satisfies the conditions of Proposition 3.2.10 for S also satisfies the conditions for S'. $\square$

Remark 3.2.12. The operators ϕ_r constructed in Proposition 3.2.10 depend *only* on the set of bad primes S for A. In particular, for any trace modules A and A' over $\mathbb{Z}'$ with the same set of bad primes S, we may choose operators ϕ_r that project A and A' simultaneously onto their S-eigenspaces. Corollary 3.2.11 further says that this is possible even if A and A' have differing sets of bad primes. Indeed, if A has set of bad primes S and A' has set of bad primes S', then by constructing projections using primes that avoid $S \cup S'$, we can construct operators ϕ_r that simultaneously project A and A' onto their S and S'-eigenspace respectively.

For us, the following special case of Proposition 3.2.10 will be of crucial importance:

Corollary 3.2.13. *Let $F = \mathbb{Q}$, and let $T = \mu_{m_1} \times \cdots \times \mu_{m_k}$ for some $m_1, \ldots, m_k \geq 0$. Then for integers $i \leq n$, as trace modules over $\mathbb{Z}'$:*

- *The motivic cohomology groups $H^i(\mathbb{G}_m^n, n)$ admit eigenspace decompositions with $S = \emptyset$.*
- *The motivic cohomology groups $H^i(T \times \mathbb{G}_m^n, n)$ admit eigenspace decompositions with S the set of primes dividing $m_1 \cdots m_k = N$.*

Proof. The first statement follows from the second, taking $T = 1$ and $S = \emptyset$.

Let $\alpha \in H^i(T \times \mathbb{G}_m^n, n)$. The proof of Proposition 3.2.8 shows that we may write

$$\alpha = \sum_{j=0}^{n} \alpha_j$$

where $\alpha_j \in H^{i-j}(T, n-j)^{\binom{n}{j}}$ and $[p]_* \alpha_j = p^{n-j} \alpha_j$ for $p \notin S$. By abuse of notation, we relabel these elements so that $[p]_* \alpha_j = p^j \alpha_j$, i.e. $\alpha_j \in H^i(T \times \mathbb{G}_m^n, n)_S^{(j)}$. We are then in the setting of Proposition 3.2.10. $\square$

Next, we would like to understand generalized eigenspaces of trace modules. Our first result shows that for trace modules with an eigenspace decomposition, there is no difference between these concepts over $\mathbb{Z}'$.

Proposition 3.2.14. *Suppose that A has an eigenspace decomposition $A = \bigoplus_{r=0}^n A_S^{(r)}$ with set of bad primes S. Then over $\mathbb{Z}'$, we have*

$$A^r = A^r_{S,1} = A_S^{(r)} = A^{(r)}$$

for $0 \le r \le n$. In particular, A is nice for $N = 1$ with set of bad primes S.

Proof. The equality of $A^r_{S,1}$ and $A_S^{(r)}$ is trivial (and therefore the statement about niceness), and because A has an eigenspace decomposition, the equality of $A_S^{(r)}$ and $A^{(r)}$ is also evident. Therefore, the only statement of content is that $A^r = A_S^{(r)}$, which is what we prove. One containment is clear, so suppose that $a \in A^r$. By definition, there exists an $N = N_{a,r}$ depending on a and r and a set of bad primes S_a such that $([p]_* - p^r)^{N_{a,r}} a = 0$ for all $p \notin S_a$. From the eigenspace decomposition, we write $a = \sum_{i=0}^n a_i$ with $a_i \in A_S^{(i)}$, which then says $([p]_* - p^r)^{N_{a,r}} a_i = 0$ for $p \notin S_a$ and all i because $a_i = \phi_i(a)$ and $([p]_* - p^r)^{N_{a,r}} \phi_i(a) = \phi_i(([p]_* - p^r)^{N_{a,r}} a) = 0$. If we restrict to $p \notin S_a \cup S$, then this means that $(p^i - p^r)^{N_{a,r}} a_i = 0$ for $i \neq r$. For each i, consider the ideal $\big(p^i - p^r : p \notin S_a \cup S\big)$ of $\mathbb{Z}$. By the same argument used in Proposition 3.2.10, over $\mathbb{Z}'$ we can find finitely many primes $\ell_1, \ldots, \ell_k$ with $\sum_j c_j(\ell_j^i - \ell_j^r) = 1$, i.e., $\big(\ell_1^i - \ell_1^r, \ldots, \ell_k^i - \ell_k^r\big) = 1$ as an ideal of $\mathbb{Z}'$. It then follows that $\big((\ell_1^i - \ell_1^r)^{N_{a,r}}, \ldots, (\ell_k^i - \ell_k^r)^{N_{a,r}}\big) = 1$ as an ideal of $\mathbb{Z}'$, which implies that $a_i = 0$. This means that $a = a_r$, so that $a \in A_S^{(r)}$ as desired. This proves that $A_S^{(r)} = A^r$. $\square$

As before, the following special case will be useful to us later:

Corollary 3.2.15. *Let i, j be integers, and let F be a field of characteristic 0. Then as trace modules over $\mathbb{Z}'$, $H^i(\mathbb{G}_m^n, j)^{(r)} = H^i(\mathbb{G}_m^n, j)^r$ for $0 \le r \le n$.*

Next, we would like to understand the behavior of the operators ϕ_r on the generalized eigenspaces of nice trace modules. What we will see is that after inverting enough primes, these nicely described operators "almost" define projection operators to generalized eigenspaces.

Proposition 3.2.16. *Suppose that A is nice for N with set of bad primes S. Then as a trace module over $\mathbb{Z}'$, there exist finitely many primes $\ell_1, \dots, \ell_k \notin S$ and operators*

$$\phi_r = \sum_{t=1}^{k} c_{t,r}\phi_r(\ell_t)$$

for $0 \le r \le n$ such that $\phi_r^N : A \to A_{S,N}^r$ is a surjection. Furthermore, there exist $U_r \in \mathbb{Z}'[x_1, \dots, x_k]$ where $x_t = [\ell_t]_* - \ell_t^r$ such that $U_r\phi_r^N : A \to A_{S,N}^r$ is a projection.

Proof. We prove this by induction on N. If A is nice for $N = 1$, then $A_{S,1}^r = A_S^{(r)}$ and so A has an actual eigenspace decomposition. Therefore, this follows from Proposition 3.2.10.

Inductively, we suppose we know that the proposition holds true for any trace module that is nice for $N-1$. Let A be a trace module that's nice for N with set of bad primes S, and write $A = \bigoplus_{r=0}^n A_{S,N}^r$ as in the statement. Let $B = \bigoplus_{r=0}^n A_S^{(r)}$, which is then nice for 1 by definition. Therefore over $\mathbb{Z}'$ there are finitely many primes $\ell_1, \dots, \ell_k \notin S$ and operators $\phi_r = \sum_{t=1}^k c_{t,r}\phi_r(\ell_t)$ such that $\phi_r : B \to A_S^{(r)}$ define projections. Now, we have an exact sequence of trace modules over $\mathbb{Z}'$

$$0 \to B \to A \to \bigoplus_{r=0}^{n} A_{S,N}^r / A_S^{(r)} \to 0.$$

We set $C = \bigoplus_{r=0}^n A_{S,N}^r / A_S^{(r)}$. Observe that C is nice for $N-1$, with set of bad primes S: indeed, for $a \in A_{S,N}^r$ and $p \notin S$, by definition we find $x = ([p]_* - p^r)^{N-1}a$ satisfies $([p]_* - p^r)x = 0$, so $x \in A_S^{(r)}$, and therefore $([p]_* - p^r)^{N-1}a \equiv 0 \bmod A_S^{(r)}$. Therefore, by the induction hypothesis, $\phi_r^{N-1} : C \to A_{S,N}^r / A_S^{(r)}$ is a surjection for all $0 \le r \le n$.

We therefore have an exact sequence

$$0 \to B \to A \to C \to 0$$

of trace modules over $\mathbb{Z}'$ as well as operators ϕ_r for $0 \le r \le n$, such that $\phi_r : B \to A_S^{(r)}$ and $\phi_r^{N-1} : C \to A_{S,N}^r / A_S^{(r)}$ are surjections. Our first claim is that $\phi_r^N : A \to A_{S,N}^r$ is a surjection.

To that end, choose $a \in A$ and suppose $a \mapsto c \in C$. Then $\phi_r^{N-1}(a) \mapsto \phi_r^{N-1}(c) \in C_{S,N-1}^r$, and therefore for any $p \notin S$ we have $([p]_* - p^r)^{N-1}\phi_r^{N-1}(a) \mapsto 0$. By exactness, we may choose $b \in B$ such that $b \mapsto ([p]_* - p^r)^{N-1}\phi_r^{N-1}(a)$. Applying ϕ_r, this means $\phi_r(b) \mapsto ([p]_* - p^r)^{N-1}\phi_r^N(a)$, and therefore $0 = ([p]_* - p^r)\phi_r(b) \mapsto ([p]_* - p^r)^N\phi_r^N(a)$. This therefore means $([p]_* - p^r)^N\phi_r^N(a) = 0$ for all $p \notin S$, so that we have a well-defined map $\phi_r^N : A \to A_{S,N}^r$. It remains to be seen that this map is surjective.

We do this as follows. By construction, we may write $\phi_r = \sum_{t=1}^k c_{t,r}\phi_r(\ell_t)$. We view $A_{S,N}^r$ as a $\mathbb{Z}'[x_1, \ldots, x_k]$-module by letting x_t act via $[\ell_t]_* - \ell_t^r$. Using the definition of $\phi_r(\ell_t)$, we may write

$$\phi_r(\ell_t) = \prod_{\substack{s=0 \\ s\neq r}}^{n}(x_t + \ell_t^r - \ell_t^s).$$

Further, using the definition of $c_r(\ell_t)$ from Proposition 3.2.10, we find that

$$\phi_r(\ell_t) = c_r(\ell_t) + p_t(x_t),$$

where $p_t(x_t)$ is divisible by x_t. Therefore, $\phi_r = 1 + \sum_{t=1}^k c_{t,r}p_t(x_t) = 1 + p(x_1, \ldots, x_k)$ viewed as an element of $\mathbb{Z}'[x_1, \ldots, x_k]$, where $p(x_1, \ldots, x_k)$ has no constant term.

Viewed this way, we note that the argument in the earlier paragraph shows that $x_i^N a = 0$ for $a \in A_{S,N}^r$, so that we may view $A_{S,N}^r$ as a $\mathbb{Z}'[x_1, \ldots, x_k]/I$-module for $I = \left(x_i^N : i \in \{1, \ldots, k\}\right)$. In this ring, $p(x_1, \ldots, x_k)$ is nilpotent, and therefore the restriction $\phi_r^N : A_{S,N}^r \to A_{S,N}^r$ is invertible. Let $U_r \in \mathbb{Z}'[x_1, \ldots, x_k]$ be a lift of the inverse of $\phi_r^N|_{A_{S,N}^r}$. We've then shown that $\phi_r^N : A \to A_{S,N}^r$ is surjective, as desired. It only remains to show that $U_r\phi_r^N : A \to A_{S,N}^r$ form orthogonal projections, and then we're done. To do so, we show that the collection $\{U_r\phi_r^N\}$ is a family of orthogonal idempotents that sum to 1.

We first start by by showing that the operators ϕ_r^N are orthogonal, from which it follows that $U_r\phi_r^N$ are also orthogonal. We tack this on as part of the induction, i.e., we show that the operators ϕ_r^N that we've constructed are not just surjections, but also orthogonal as well. Indeed, in the $N = 1$ case, the operators ϕ_r are already orthogonal, because if $a \in A_{S,1}^r \cap A_{S,1}^{r'}$ for $r \neq r'$, this means $(p^r - p^{r'})a = 0$ for all $p \notin S$. Therefore, the same ideal argument of Proposition 3.2.10 shows that over $\mathbb{Z}'$, $a = 0$. Now inductively, suppose that the operators ϕ_r^{N-1} are orthogonal for any trace module that's nice for $N-1$.

Using the short exact sequence

$$0 \to B \to A \to C \to 0$$

above, suppose that $a \mapsto c$. Then for any $r \neq r'$, we have $\phi_r^{N-1}\phi_{r'}^{N-1}(a) \mapsto \phi_r^{N-1}\phi_{r'}^{N-1}(c) = 0$, and therefore there is $b \in B$ with $b \mapsto \phi_r^{N-1}\phi_{r'}^{N-1}(a)$. Applying $\phi_r\phi_{r'}$, this means $0 = \phi_r\phi_{r'}(b) \mapsto \phi_r^N\phi_{r'}^N(a)$, so by injectivity this means that $\phi_r^N\phi_{r'}^N(a) = 0$ as desired.

Therefore, the operators ϕ_r^N on A are orthogonal, and therefore $U_r\phi_r^N$ are orthogonal, too. These operators are clearly idempotent, so it remains to see they sum to 1. However, this follows more or less immediately. By definition, for any $a \in A$ we may write $a = \sum_{r=0}^n a_r$ with $a_r \in A_{S,N}^r$, and the orthogonality makes it clear that $U_r\phi_r^N(a) = a_r$. $\square$

Remark 3.2.17. With $\phi_r = 1 + p(x_1, \ldots, x_k) \in \mathbb{Z}'[x_1, \ldots, x_k]$ as above, suppose that f is such that $p(x_1, \ldots, x_k)^f$ annihilates $A_{S,N}^r$. Then via the usual formula for the inverse of a geometric series, we can explicitly choose

$$U_r = (1 - p(x_1, \ldots, x_k) + p(x_1, \ldots, x_k)^2 + \ldots + (-1)^{f-1}p(x_1, \ldots, x_k)^{f-1})^N$$

as an inverse for ϕ_r^N on $A_{S,N}^r$.

In particular, much like how it is possible to construct simultaneous projections for trace modules A and A' over $\mathbb{Z}'$, we can similarly construct simultaneous projections for nice trace modules A, A' over $\mathbb{Z}'$. Indeed, if A is nice for N with set of bad primes S and A' is nice for M with set of bad primes S', then running the argument with the common set of bad primes $S \cup S'$ produces operators ϕ_r such that $\phi_r^N : A \to A_{S,N}^r$ and $\phi_r^M : A' \to A'_{S',M}$ are well-defined. As a polynomial operator, choose a suitably large power f such that $p(x_1, \ldots, x_k)^f$ annihilates both $A_{S,N}^r$ and $A'^r_{S',M}$. One may then pick a common $U_r \in \mathbb{Z}'[x_1, \ldots, x_k]$ such that $(U_r\phi_r)^N$ and $(U_r\phi_r)^M$ are the produced projections $A \to A_{S,N}^r$ and $A' \to A'^r_{S',M}$ respectively. Therefore, the common operator $(U_r\phi_r)^{N+M}$ simultaneously projects A and A' onto their generalized eigenspaces.

In order to further investigate nice trace modules, it's helpful to introduce some categorical language.

Definition 3.2.18. For a commutative ring R, we let $\mathcal{C}_{NTM}(R)$ denote the category of nice trace modules over R. This is the full subcategory of $R[\mathbb{Z}_{\geq 1}]$-modules whose objects are trace modules over R that are nice for some $N \geq 1$.

Similarly to modules with an actual eigenspace decomposition, we have the following:

Lemma 3.2.19. *Let $A \in \mathcal{C}_{NTM}(\mathbb{Z}')$ that is nice for N with set of bad primes S. Then $A^r = A^r_{S,N}$ for all $0 \leq r \leq n$.*

Proof. The containment $A^r_{S,N} \subset A^r$ is clear, so we show the other containment. Fix $r \geq 0$ and $a \in A^r$. By definition, there is a finite set of primes S_a and an integer N_a such that $([p]_* - p^r)^{N_a} a = 0$ for all $p \notin S_a$. Write $a = \sum_{i=0}^n a_i$ with $a_i \in A^i_{S,N}$. By avoiding primes in $S \cup S_a$, construct ϕ_r so that $\phi_r^N : A \to A^r_{S,N}$ is a surjection.

Let $\ell_1, \dots, \ell_k$ be the primes used to construct ϕ_r. Setting $x_i = [\ell_i]_* - \ell_i^r$, then as an element of $\mathbb{Z}'[x_1, \dots, x_k]$, we have that ϕ_r acts by $1+p(x_1, \dots, x_k)$ for some $p(x_1, \dots, x_k) \in \mathbb{Z}'[x_1, \dots, x_k]$ with no constant term. Now, $([p]_* - p^r)^{N_a} a = 0$ for all $p \notin S_a \cup S$, and therefore this holds true for each a_i as well. Suppose that $i \neq r$. This means $x_j^{N_a} a_i = 0$ for all j, so $p(x_1, \dots, x_k)^{N_a} a_i = 0$. Therefore, ϕ_r acts invertibly on a_i with inverse some $U_r \in \mathbb{Z}'[x_1, \dots, x_k]$. By Remark 3.2.17 this operator U_r may be chosen so that $U_r \phi_r = 1$ on a_i and $(U_r \phi_r)^N$ is the projection $A \to A^r_{S,N}$. In particular,we have $(U_r \phi_r)^N (a_i) = 0$, which then means $a_i = 0$. Therefore, $a = a_r$, which yields the desired containment. $\square$

Now, we investigate how the property of niceness behaves with respect to short exact sequences.

Lemma 3.2.20. *Suppose that*

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of trace modules over $\mathbb{Z}'$. If A is nice for N and C is nice for M, then B is nice for $N + M$.

Proof. Let the set of bad primes for A be S and the set of bad primes for C be S'. Let ϕ_r be the operators constructed via Proposition 3.2.16 with common set of bad primes $S \cup S'$. The same argument of that proposition shows that $\phi_r^{N+M} : B \to B^r_{S \cup S', N+M}$ are well-defined orthogonal maps, that may be multiplied by a polynomial operator to become isomorphisms when restricted to $B^r_{S \cup S', N+M}$. Remark 3.2.17 says that we may choose simultaneous orthogonal projections π_r to the generalized eigenspaces of A, B, C.

It remains to be seen that π_r sum to 1 on B, which would show that B is nice for $N+M$ with set of bad primes $S \cup S'$. To see this, suppose that $b \in B$ and $b \mapsto c \in C$. Then $b - \sum_{r=0}^n \pi_r(b) \mapsto c - \sum_{r=0}^n \pi_r(c) = 0$, and therefore there is $a \in A$ such that $a \mapsto b - \sum_{r=0}^n \pi_r(b)$. Applying π_r and using that these are orthogonal idempotents, this means $\pi_r(a) \mapsto 0$ for all r, and so by injectivity this means $\pi_r(a) = 0$ for all r. As $a = \sum_{r=0}^n \pi_r(a)$, this means $b - \sum_{r=0}^n \pi_r(b) = 0$ as desired. $\square$

Lemma 3.2.21. *Let $B \in \mathcal{C}_{NTM}(\mathbb{Z}')$ with $A \subset B$ an inclusion of trace modules over $\mathbb{Z}'$. Then $A, B/A \in \mathcal{C}_{NTM}(\mathbb{Z}')$.*

Proof. Suppose that $B = \bigoplus_{r=0}^n B_{S,N}^r$ is nice for N with set of bad primes S, and $0 \to A \to B$ an inclusion of trace modules over $\mathbb{Z}'$. Let $\pi_r := U_r \phi_r^N : B \to B_{S,N}^r$ be the orthogonal projection operators of Proposition 3.2.16. As A is a $\mathbb{Z}'[x_1, \dots, x_k]$-submodule of B, it then follows that the restriction of π_r to A is valued in $B_{S,N}^r \cap A$. It's clear by definition that $A_{S,N}^r = B_{S,N}^r \cap A$, so it immediately follows that $A = \bigoplus_{r=0}^n A_{S,N}^r$, so A is nice for N with set of bad primes S.

Now, the claim is that B/A is nice for N with set of bad primes S. To see this, the induced map $\overline{\pi}_r$ on the quotient defined by $\overline{\pi}_r(b \mod A) = \pi_r(b) \mod A$ is well-defined, and certainly $([p]_* - p^r)^N \pi_r(b) \in A$, so we have a map $\overline{\pi}_r : B/A \to (B/A)_{S,N}^r$. This means that $\sum_{i=0}^n \operatorname{im}(\overline{\pi}_i) \subset \sum_{i=0}^n (B/A)_{S,N}^i$. However, the maps $\overline{\pi}_r$ certainly form a set of complete orthogonal projections for B/A, because they satisfy this property for B. In particular, this means $B/A = \bigoplus_{i=0}^n \operatorname{im}(\overline{\pi}_i) \subset \bigoplus_{i=0}^n (B/A)_{S,N}^i$, which then forces $B/A = \bigoplus_{r=0}^n (B/A)_{S,N}^r$ as desired. $\square$

Proposition 3.2.22. *$\mathcal{C}_{NTM}(\mathbb{Z}')$ is abelian.*

Proof. It suffices to prove that $\mathcal{C}_{NTM}(\mathbb{Z}')$ contains sums, kernels, and cokernels. Let $A, B \in \mathcal{C}_{NTM}(\mathbb{Z}')$ and $f : A \to B$ a morphism. By Lemma 3.2.21, $\ker(f) \in \mathcal{C}_{NTM}(\mathbb{Z}')$. We have $A \oplus B \in \mathcal{C}_{NTM}$ by applying Lemma 3.2.20 to the short exact sequence

$$0 \to A \to A \oplus B \to B \to 0.$$

Finally, $\operatorname{im}(f) \in \mathcal{C}_{NTM}(\mathbb{Z}')$ because $\operatorname{im}(f) \cong A/\ker(f)$ is nice by Lemma 3.2.21, and therefore $\operatorname{coker}(f) = B/\operatorname{im}(f) \in \mathcal{C}_{NTM}(\mathbb{Z}')$ as well. $\square$

Proposition 3.2.23. *Let $r \geq 0$. The functor $\mathcal{F}_r : \mathcal{C}_{NTM}(\mathbb{Z}') \to \mathcal{C}_{NTM}(\mathbb{Z}')$ defined by $A \mapsto A^r$ is exact.*

Proof. Consider a short exact sequence

$$0 \to A \to B \to C \to 0$$

with $A, B, C \in \mathcal{C}_{NTM}(\mathbb{Z}')$. Let the sets of bad primes for A, B, C be S_A, S_B, S_C respectively. By constructing projection operators avoiding primes in $S = S_A \cup S_B \cup S_C$, we obtain simultaneous projections π_r that project A, B, C onto their generalized eigenspaces, so without loss of generality, we may assume that A, B, C are all nice for the same common set of primes S. Now, the left exactness of this functor is clear, so we just prove right exactness. To that end, suppose that $c \in C^r$. Then there exists $b \in B$ such that $b \mapsto c$. Applying π_r, this says $\pi_r(b) \mapsto \pi_r(c) = c$, which is what we wanted. $\square$

3.3 Motivic cohomology of toric flags

The theory of trace modules developed in the previous section is somewhat general, but at the end of the day, we ultimately are only interested in applying our theory to very specific kinds of trace modules. For us, these will be motivic cohomology groups that arise from Gysin-type sequences over $\mathbb{Q}$.

Definition 3.3.1. A *good flag* for $\mathbb{G}_m^n$ is a decreasing system $(Z_p)_{p \in \mathbb{Z}}$ of closed $\mathbb{Q}$-subschemes such that:

- $Z_p = \mathbb{G}_m^n$ for $p \leq 0$.
- $Z_p = \emptyset$ for $p > n$.
- Z_p is of pure codimension p for $1 \leq p \leq n$.
- $Z_p - Z_{p+1}$ is smooth.

From the construction of the coniveau spectral sequence in Section 2.1.2, attached to this good flag is a right half-plane spectral sequence

$$E_1^{p,q} : H^{q-p}(Z_p - Z_{p+1}, n-p) \implies H^{p+q}(\mathbb{G}_m^n, n),$$

which we will refer to as the *toric flag spectral sequence.*

Definition 3.3.2. Let $I = \{v_1, \ldots, v_N\} \subset \mathbb{Z}^n$ be a finite set of primitive vectors with $N \geq n$ such that $v_i \neq \pm v_j$ for all $i \neq j$, and let S_{v_i} be the kernel of the character $v_i : \mathbb{G}_m^n \to \mathbb{G}_m$. A *toric flag* is a good flag of the form

$$Z_p = \bigcup_{(i_1,\ldots,i_p)\in\{1,\ldots,N\}^p} S_{v_{i_1}} \cap \ldots \cap S_{v_{i_p}}$$

for $1 \leq p \leq n$, where the union is taken over p-tuples $(i_1, \ldots, i_p) \in \{1, \ldots, N\}^p$ such that no coordinates are equal.

For a toric flag $(Z_p)_{p\in\mathbb{Z}}$, it makes sense to speak of trace maps on the cohomology groups $H^i(Z_p - Z_{p+1}, j)$. These motivic cohomology groups have many nice properties, of which we will frequently make use. For example, trace maps coming from different primes commute (and so these are trace modules), and residue maps in Gysin sequences attached to toric flags are morphisms of trace modules. The arguments are essentially identical to those found in [14, Lemmas 2.1.3 and 2.1.4].

Using the theory of trace modules, we will prove the following theorem:

Theorem 3.3.3. *Set $\mathbb{Z}' = \mathbb{Z}[\frac{1}{(n+1)!}]$ and suppose that $Z = (Z_p)_{p\in\mathbb{Z}}$ is a toric flag. There is an exact sequence of trace modules over $\mathbb{Z}'$*

$$0 \to (H^n(\mathbb{G}_m^n, n) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 \to (H^n(\mathbb{G}_m^n - Z_1, n) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 \to \ldots \to (H^0(Z_n, 0) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 \to 0.$$

We will break this up as a series of lemmas proven throughout the rest of this section. The general outline is as follows:

(i) For a toric flag $(Z_p)_{p\in\mathbb{Z}}$, we prove that $H^i(Z_p - Z_{p+1}, n-p) = 0$ for $i > n-p$ or $i < 0$ so that almost all the terms on the E_1-page of the toric flag spectral sequence vanish.

(ii) We prove that all the terms on the E_1-page of the toric flag spectral sequence belong to the category $\mathcal{C}_{NTM}(\mathbb{Z}')$ of nice trace modules over $\mathbb{Z}'$, after extending scalars to $\mathbb{Z}'$.

(iii) We construct a new spectral sequence

$$E_1^{p,q} : (H^{q-p}(Z_p - Z_{p+1}, n-p) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 \implies (H^{p+q}(\mathbb{G}_n^n, n) \otimes_{\mathbb{Z}} \mathbb{Z}')^0$$

that computes the generalized trace-fixed parts of these cohomology groups over $\mathbb{Z}'$.

(iv) Finally, we show that $(H^k(Z_p - Z_{p+1}, n-p) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 = 0$ for all $0 \leq p \leq n$ and $0 \leq k < n$. This shows that the E_1-page of the above spectral sequence vanishes outside of the $q = n$ row, which combined with $(H^k(\mathbb{G}_m^n, n) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 = 0$ for $k \neq n$ and the convergence gives us what we want.

We start with step (i). First, we prove a technical lemma to make the proof easier to digest.

Lemma 3.3.4. *Let $M_1, \ldots, M_j \geq 0$ be integers, and let $S = \mu_{M_1} \times \ldots \times \mu_{M_j}$. Then $H^k(\mathbb{G}_m^n - S \times \mathbb{G}_m^{n-j}, n) = 0$ for $k > n$ or $k < 0$.*

Proof. We have a Gysin sequence

$$\ldots \to H^k(\mathbb{G}_m^n, n) \to H^k(\mathbb{G}_m^n - S \times \mathbb{G}_m^{n-j}, n) \to H^{k-1}(S \times \mathbb{G}_m^{n-j}, n-1) \to \ldots$$

By the base change property of motivic cohomology, viewing $S \times \mathbb{G}_m^{n-j}$ as a scheme over $\mathbb{Q}(\mu_{M_1 \cdots M_j})$ we see that $H^{k-1}(S \times \mathbb{G}_m^{n-j}, n-1)$ breaks up as a direct sum of copies of $H^{k-1}(\mathbb{G}_m^{n-j}, n-1)$. When $k > n$ this follows immediately from Proposition 3.1.5, and when $k < 0$ it follows from the same proposition upon noting that each term in the direct sum decomposition is zero by Borel's theorem. $\square$

Lemma 3.3.5. *Let $(Z_p)_{p \in \mathbb{Z}}$ be a toric flag. Then for $0 \leq p \leq n$,*

$$H^k(Z_p - Z_{p+1}, n-p) = 0$$

if $k > n - p$ or $k < 0$.

Proof. Suppose that $|I| = N$ and that our flag is built out of the tori $S_1, \ldots, S_N$. We first start with the proof for $p = 0$. We wish to show $H^k(\mathbb{G}_m^n - Z_1, n) = 0$ for $k > n$ or

$k < 0$. The main technique in the proof will be computing motivic cohomology groups by removing "one term at a time".

The previous lemma shows that $H^k(\mathbb{G}_m^n - S_1, n) = 0$ for $k > n$ or $k < 0$. Suppose we know for some integer $1 \leq M \leq N$ that $H^k(\mathbb{G}_m^n - \bigcup_{i=1}^{M-1} S_i, n) = 0$ for $k > n$ or $k < 0$. We will show that $H^k(\mathbb{G}_m^n - \bigcup_{i=1}^{M} S_i, n) = 0$, from which what we want follows from induction on M. To do so, we form another Gysin sequence

$$\ldots \to H^k(\mathbb{G}_m^n - \bigcup_{i=1}^{M-1} S_i, n) \to H^k(\mathbb{G}_m^n - \bigcup_{i=1}^{M} S_i, n) \to H^{k-1}(S_M - \bigcup_{i=1}^{M-1} (S_M \cap S_i), n-1) \to \ldots,$$

and therefore we're done once we show $H^{k-1}(S_M - \bigcup_{i=1}^{M-1}(S_M \cap S_i), n-1) = 0$ when $k > n$ or $k < 0$.

For the sake of readability, we introduce the notation $T_{M,a} = \bigcap_{i=1}^{a} S_{M-a+1}$, so that $T_{M,1} = S_M$. Now, the previous lemma shows that $H^{k-1}(S_M - S_M \cap S_1, n-1) = 0$ for $k > n$ or $k < 0$ because $S_M \cap S_1 \cong \mu_{M_1} \times \mu_{M_2} \times \mathbb{G}_m^{n-2}$ for some integers M_1, M_2, where $S_M \cong \mu_{M_1} \times \mathbb{G}_m^{n-1}$. Therefore, the same inductive reasoning removing "one term at a time" as in the prior paragraph shows that the vanishing of $H^{k-1}(T_{M,1} - \bigcup_{i=1}^{M-1}(T_{M,1} \cap S_i), n-1)$ follows from the vanishing of $H^{k-2}(T_{M,2} - \bigcup_{i=1}^{M-2} T_{M,2} \cap S_i, n-2)$. By continuing the chain of inductive implications, we see that the vanishing of $H^{k-a}(T_{M,a} - \bigcup_{i=1}^{M-a}(T_{M,a} \cap S_i), n-a)$ follows from the vanishing of $H^{k-a-1}(T_{M,a+1} - \bigcup_{i=1}^{M-a-1}(T_{M,a+1} \cap S_i), n-a-1)$. In particular, for $a = M-1$, we note that vanishing of $H^{k-M+1}(T_{M,M-1} - (T_{M,M-1} \cap S_1), n-M+1) = H^{k-M+1}(T_{M,M-1} - T_{M,M}, n-M+1)$ depends on the vanishing of $H^{k-M}(T_{M,M}, n-M)$. We're now happy, because $T_{M,M} = \bigcap_{i=1}^{M} S_i$ is a zero dimensional smooth scheme (namely, a finite product of roots of unity), and therefore $H^{k-M}(T_{M,M}, n-M) = 0$ for $k > n$ by properties of motivic cohomology. When $k < 0$, this follows by Theorem 2.1.11 and the base change property of motivic cohomology (viewing $T_{M,M}$ as a scheme over an appropriate choice of cyclotomic field).

To summarize, we've finished the proof in the case $p = 0$. The case of $p = n$ is immediate, because $H^k(Z_n, 0) = 0$ for $k \neq 0$ by the properties of motivic cohomology discussed in Section 2.1.1.

Therefore, we just need handle $1 \leq p \leq n-1$, which we do as follows. For a subset $J \subset \{1, \ldots, N\}$ we write $S_J := \bigcap_{j \in J} S_j$. There is a decomposition of motivic cohomology groups

$$H^k(Z_p - Z_{p+1}, n-p) = \bigoplus_{|J|=p} H^k(S_J - \bigcup_{i \notin J} S_{J \cup \{i\}}, n-p),$$

where the direct sum is taken over all such subsets of size p. Without loss of generality, we let $J = \{1, \dots, p\}$, so we wish to show that $H^k(T_{1,p} - \bigcup_{i=p+1}^N (T_{1,p} \cap S_i), n-p) = 0$ for $k > n-p$ or $k < 0$.

Now, $T_{1,p} \cong T \times \mathbb{G}_m^{n-p}$ for some T a finite product of roots of unity, and so working over an appropriate choice of cyclotomic field F, it follows by the base change property that $H^k(T_{1,p} - \bigcup_{i=p+1}^N (T_{1,p} \cap S_i), n-p)$ decomposes into a sum of cohomology groups $H^k(\mathbb{G}_m^{n-p} - \bigcup_{i=p+1}^N S_i', n-p)$, with $\mathbb{G}_m^{n-p}$ and S_i' viewed as schemes over $\mathrm{Spec}(F)$, for some codimension 1 tori S_i'. The vanishing of these cohomology groups then follows from the same argument as in the $p = 0$ case. □

This concludes step (i). Now, we move to step (ii).

Lemma 3.3.6. *Let $(Z_p)_{p \in \mathbb{Z}}$ be a toric flag. For $0 \le p \le n$ and $0 \le k \le n-p$, the motivic cohomology groups $H^k(Z_p - Z_{p+1}, n-p) \otimes_{\mathbb{Z}} \mathbb{Z}'$ are nice trace modules over $\mathbb{Z}'$.*

Proof. Suppose that $|I| = N$ and that our toric flag is built out of $S_1, \dots, S_N$. The technique will be the same as the one used in the previous lemma. In fact, the proof is almost identical, replacing "$= 0$" with "is nice". By abuse of notation, we will write $H^i(X, j)$ to mean $H^i(X, j) \otimes_{\mathbb{Z}} \mathbb{Z}'$, viewing them as trace modules over $\mathbb{Z}'$.

As before, we use the notation $T_{M,a} = \bigcap_{i=1}^a S_{M-a+i}$. Starting with $p = 0$, the standard Gysin sequence

$$\dots \to H^k(\mathbb{G}_m^n, n) \to H^k(\mathbb{G}_m^n - S_1, n) \to H^{k-1}(S_1, n-1) \to \dots$$

shows that $H^k(\mathbb{G}_m^n - S_1, n)$ is nice after applying Lemma 3.2.20 (after breaking up the long exact sequence into short exact sequences) because both $H^k(\mathbb{G}_m^n, n)$ and $H^{n-1}(S_1, n-1) \cong H^{k-1}(\mathbb{G}_m^{n-1}, n-1)$ are nice.

Now, if we know $H^k(\mathbb{G}_m^n - \bigcup_{i=1}^{M-1} S_i, n)$ is nice for some $1 \le M \le N$, the computation in the argument of the previous lemma shows that the niceness of $H^k(\mathbb{G}_m^n - \bigcup_{i=1}^M S_i, n)$ can be deduced from that of $H^{k-M}(T_{M,M}, n-M)$. Now, $T_{M,M} = S_1 \cap \dots \cap S_M \cong$

$\mu_{N_1}\times\ldots\times\mu_{N_M}\times\mathbb{G}_m^{n-M}$ for some integers $N_1,\ldots,N_M$. It follows that $H^{k-M}(T_{M,M},n-M)$ is nice by Corollary 3.2.13.

This handles the case $p=0$. The direct sum decomposition of $H^k(Z_p-Z_{p+1},n-p)$ shows that $H^k(Z_p-Z_{p+1},n-p)$ are nice for $0\le k\le n-p$ using the argument of the $p=0$ case (after base changing to an appropriate choice of cyclotomic field).

The remaining case is $p=n$, and we wish to show that $H^0(Z_n,0)$ is nice. Write $Z_n=\bigcup_{|J|=n}S_J$. There is an isomorphism $H^0(Z_n,0)\cong H^0(Z_n-\bigcup_{i=n+1}^N S_{J\cup\{i\}},0)$ by examining the corresponding Gysin sequence. Breaking this up as a direct sum, it suffices to show that $H^0(T_{1,n}-\bigcup_{i=n+1}^N T_{1,n}\cap S_i,0)$ is nice. Fitting into another Gysin sequence, this reduces to to showing that $H^0(\bigcup_{i=n+1}^N T_{1,n}\cap S_i,0)$ is nice, because $H^0(T_{1,n},0)$ is nice by Proposition 3.2.8. Repeating this reduces down to showing that $H^0(T_{1,N},0)$ is nice, which follows from Proposition 3.2.8. $\square$

This completes step (ii). Now, we move to step (iii).

Lemma 3.3.7. *Let $(Z_p)_{p\in\mathbb{Z}}$ be a toric flag. Then there is a spectral sequence*

$$E_1^{p,q}:(H^{q-p}(Z_p-Z_{p+1},n-p)\otimes_{\mathbb{Z}}\mathbb{Z}')^0\implies(H^{p+q}(\mathbb{G}_m^n,n)\otimes_{\mathbb{Z}}\mathbb{Z}')^0$$

Proof. Let $C=(D,E,f,g,h)$ be the exact couple that produces the toric spectral sequence starting from the first page, and denote this spectral sequence E'. By Lemma 3.3.6, after extending scalars to $\mathbb{Z}'$, the finitely many non-zero terms of E_1' all belong to the category $\mathcal{C}_{NTM}(\mathbb{Z}')$. Applying the exact functor $-\otimes_{\mathbb{Z}}\mathbb{Z}'$ to C produces a spectral sequence $E_1'':H^{q-p}(Z_p-Z_{p+1},n-p)\otimes_{\mathbb{Z}}\mathbb{Z}'\implies H^{p+q}(\mathbb{G}_m^n,n-p)\otimes_{\mathbb{Z}}\mathbb{Z}'$. Applying the exact functor $\mathcal{F}_0$ to $C\otimes_{\mathbb{Z}}\mathbb{Z}'$ then produces the desired spectral sequence. $\square$

We're now ready to prove the main result of this chapter.

Theorem 3.3.8. *Set $\mathbb{Z}'=\mathbb{Z}[\frac{1}{(n+1)!}]$. Suppose that $Z=(Z_p)_{p\in\mathbb{Z}}$ is a toric flag. There is an exact sequence of trace modules over $\mathbb{Z}'$*

$$0\to(H^n(\mathbb{G}_m^n,n)\otimes_{\mathbb{Z}}\mathbb{Z}')^0\to(H^n(\mathbb{G}_m^n-Z_1,n)\otimes_{\mathbb{Z}}\mathbb{Z}')^0\to\ldots\to(H^0(Z_n,0)\otimes_{\mathbb{Z}}\mathbb{Z}')^0\to 0.$$

Proof. Let

$$E_1^{p,q}:(H^{q-p}(Z_p-Z_{p+1},n-p)\otimes_{\mathbb{Z}}\mathbb{Z}')^0\implies(H^{p+q}(\mathbb{G}_m^n,n)\otimes_{\mathbb{Z}}\mathbb{Z}')^0$$

be the spectral sequence of generalized trace-fixed parts of Lemma 3.3.7.

By Lemma 3.3.5, the only possible non-zero terms on the E_1-page are $E_1^{p,q}$ for $0 \le p \le n$ and $0 \le q \le n$. The claim is that $(H^k(Z_p - Z_{p+1}, n-p) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 = 0$ for $0 \le k < n-p$, which rules out everything except the $q = n$ row. Once we have this, the exactness of the sequence in Theorem 3.3.8 is then immediate from the convergence of the spectral sequence and Proposition 3.1.5.

To prove the claim, we use the same argument as Lemmas 3.3.5 and 3.3.6. Suppose that $|I| = N$ and our toric flag is built from $S_1, \dots, S_N$. Consider the exact sequence

$$\dots \to (H^k(\mathbb{G}_m^n, n) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 \to (H^k(\mathbb{G}_m^n - S_1, n) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 \to (H^{k-1}(S_1, n-1) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 \to \dots$$

obtained by tensoring the Gysin sequence with $\mathbb{Z}'$ and taking generalized trace-fixed parts. As $H^k(\mathbb{G}_m^n, n)^0 = 0$ for $0 \le k < n$ by Proposition 3.1.5 and Corollary 3.2.15, and $S_1 \cong \mathbb{G}_m^{n-1}$, it follows that $(H^k(\mathbb{G}_m^n - S_1, n) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 = 0$ for $0 \le k < n$. Inductively, if we know for some $1 \le M \le N$ that $(H^k(\mathbb{G}_m^n - \bigcup_{i=1}^{M-1} S_i, n) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 = 0$ for $0 \le k < n$, the same Gysin sequence chase of Lemma 3.3.5 reduces down to showing that $(H^{k-M}(T_{M,M}, n-M) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 = 0$ for $0 \le k < n$, which follows from Proposition 3.2.8. This proves the $p = 0$ case. The claim holds for $1 \le p \le n-1$ by decomposing $(H^k(Z_p - Z_{p+1}, n-p) \otimes_{\mathbb{Z}} \mathbb{Z}')^0$ into its direct sum decomposition and using the argument for the $p = 0$ case. The last case is $p = n$, for which the claim is immediate, so we're done. $\square$

CHAPTER 4

Explicit Cocycle Constructions for $\mathbb{G}_m^n$

Our goal in this chapter is to use Theorem 3.3.8 to construct a "canonical" cocycle Θ that represents a class

$$[\Theta]_{line} \in H^{n-1}(\mathrm{GL}_n(\mathbb{Z}), K_n^M(\mathbb{Q}(\mathbb{G}_m^n)) \otimes_{\mathbb{Z}} \mathbb{Z}'),$$

where $\mathbb{Z}' = \mathbb{Z}[\frac{1}{(n+1)!}]$. This construction is carried out in Theorem 4.4.7. We then prove in Theorem 4.5.2 that the class $[\Theta]_{line}$ is "Eisenstein", in a certain sense.

The cocycle Θ is a higher dimensional analogue of the cocycle constructed in Proposition 3.3.1 of [25], although its construction requires inverting finitely many primes. In Section 4.3., we give a construction of a canonical cocycle representing a class $[\Theta]_{ray}$ for $n = 2, 3$ that does not require inverting any primes. This cocycle is valued in a *quotient* of $K_n^M(\mathbb{Q}(\mathbb{G}_m^n))$ (which is easily described for $n = 2$, and the author expects to be describable for $n = 3$).

Sections 4.1 and 4.2 develop the theory needed to carry out our constructions.

4.1 Cocycles via lifting

Our method of constructing cocycles comes from the following elementary idea in group cohomology.

Let G be a group and let

$$0 \to C_n \xrightarrow{d_n} C_{n-1} \to \ldots \to C_0 \to 0$$

be an exact sequence of $\mathbb{Z}[G]$-modules. The short exact sequence

$$0 \to \ker(d_1) \to C_1 \xrightarrow{d_1} C_0 \to 0$$

induces a long exact sequence on group cohomology, and in particular, gives rise to the connecting homomorphism

$$\delta_0 : C_0^G \to H^1(G, \ker(d_1)).$$

As $\ker(d_1) = \operatorname{im}(d_2)$, there is another short exact sequence

$$0 \to \ker(d_2) \to C_2 \to \ker(d_1) \to 0$$

and therefore another connecting morphism

$$\delta_1 : H^1(G, \ker(d_1)) \to H^2(G, \ker(d_2)).$$

Continuing this process, at the last stage, we have the connecting morphism

$$\delta_{n-2} : H^{n-2}(G, \ker(d_{n-2})) \to H^{n-1}(G, C_n),$$

which follows upon noting that $\ker(d_{n-1}) = \operatorname{im}(d_n) = C_n$ by exactness of the complex. The composition

$$\delta = \delta_{n-2} \circ \ldots \circ \delta_0$$

then defines a map

$$\delta : C_0^G \to H^{n-1}(G, C_n).$$

Definition 4.1.1. The *lifting morphism* of an exact sequence of G-modules

$$0 \to C_n \xrightarrow{d_n} C_{n-1} \to \ldots \to C_0 \to 0$$

is defined as the map

$$\delta : C_0^G \to H^{n-1}(G, C_n)$$

described above.

Remark 4.1.2. The connecting morphisms δ_n can be explicitly described via a "lifting" process, hence the terminology. For example, for $c \in C_0^G$, choose any lift $\hat{c} \in C_1$, which is possible via exactness. Then

$$\delta_0(c) = [\gamma \mapsto (\gamma - 1)\hat{c}] \in H^1(G, \ker(d_1)).$$

Now, our main method of constructing cocycles comes from the following simple observation. Suppose we have a morphism of complexes of $\mathbb{Z}[G]$-modules

$$\begin{tikzcd} 0 \arrow[r] & C_n \arrow[r, "d_n^C"] \arrow[d, "f_n"] & \dots \arrow[r, "d_1^C"] & C_0 \arrow[r] \arrow[d, "f_0"] & 0 \\ 0 \arrow[r] & K_n \arrow[r, "d_n^K"] & \dots \arrow[r, "d_1^K"] & K_0 \arrow[r] & 0 \end{tikzcd}$$

where the top complex is acyclic outside of homological degree n, and the bottom complex is acyclic. Then by pushing forward lifts in $C_\bullet$, we can construct a map

$$\delta_{C_\bullet, K_\bullet} : C_0^G \to H^{n-1}(G, K_n).$$

Explicitly, we start by composing the connecting morphisms $\delta_{n-2}^C \circ \dots \circ \delta_0^C$ to form the map

$$\delta' : C_0^G \to H^{n-1}(G, \ker(d_{n-1}^C)).$$

Since $f_\bullet$ is a morphism of complexes, it induces a morphism on group cohomology, and so we consider the induced map

$$f_* : H^{n-1}(G, \ker(d_{n-1}^C)) \to H^{n-1}(G, \ker(d_{n-1}^K)).$$

As the bottom complex is exact, we have $\ker(d_{n-1}^K) = \mathrm{im}(d_n^K) = K_n$, and so the composition of δ' with f_* produces a map

$$\delta_{C_\bullet, K_\bullet} : C_0^G \to H^{n-1}(G, K_n).$$

4.2 Ray and Line complexes

We begin by describing auxiliary complexes which will be crucial in our construction. The main purpose of these complexes will be to "parameterize" certain symbols in Milnor K-theory. Using the lifting morphism, we can construct cocycles valued in these complexes, which we can then push forward to obtain cocyles valued in Milnor K-theory after constructing appropriate maps of complexes.

Let $T = \mathbb{G}_m^n$, and let $X = X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ be the co-character group of T. Set $X_\mathbb{R} = X \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^n$ and fix an orientation on $X_\mathbb{R}$. We view the sphere S^{n-1} as the quotient of $X_{\mathbb{R}} - \{0\}$ by positive scaling:

$$S^{n-1} = (X_{\mathbb{R}} - \{0\})/\mathbb{R}_+$$

Therefore, points $x \in S^{n-1}$ are identified with rays $\mathbb{R}_+ x \subset X_{\mathbb{R}}$ for $x \in X_{\mathbb{R}}$ non-zero. A ray is called *rational* if it is the image of an element of X, i.e. passes through a point of X. We denote the set of rational points of S^{n-1} by $S^{n-1}_{\mathbb{Q}}$. Any rational point in $S^{n-1}_{\mathbb{Q}}$ passes through a unique primitive vector $v \in \mathbb{Z}^n$, and we will often identify such a primitive vector with its corresponding ray.

We define a chain complex as follows:

- $C_0 = \mathbb{Z}$.
- For $1 \leq k \leq n$, we define C_k to be the abelian group with generators the k-tuples $[v_1, \dots, v_k]$ of rays $v_i \in S^{n-1}_{\mathbb{Q}}$ such that $\{v_1, \dots, v_k\}$ extends to a $\mathbb{Z}$-basis of $\mathbb{Z}^n$, with the relations $[v_1, \dots, v_k] = (-1)^{|\sigma|}[v_{\sigma(1)}, \dots, v_{\sigma(k)}]$ for any permutation $\sigma \in S_k$, where $|\sigma|$ denotes the sign of σ.
- The boundary map $d_1 : C_1 \to C_0$ is defined as the augmentation map, and for $k \geq 2$ the boundary map is defined via the usual alternating sum:

$$d_k([v_1, \dots, v_k]) = \sum_{i=1}^{k} (-1)^{i-1} [v_1, \dots, \hat{v}_i, \dots, v_k].$$

We call the resulting complex $(C_\bullet, d_\bullet)$ the *ray complex* for $S^{n-1}_{\mathbb{Q}}$ and denote it by C^{ray}. Taking $\Gamma = \mathrm{GL}_n(\mathbb{Z})$, there is a natural equivariant action of Γ on the ray complex defined by

$$\gamma \cdot [v_0, \dots, v_k] = [v_0\gamma^{-1}, \dots, v_k\gamma^{-1}].$$

Here, we think of vectors v_i as *row* vectors so that the matrix multiplication is happening on the right. On C_0^{ray}, we let Γ act trivially.

We have the following aycliclity result for this complex:

Proposition 4.2.1. *The complex*

$$0 \to C_n^{ray} \to C_{n-1}^{ray} \to \dots \to C_1^{ray} \to C_0^{ray} \to 0$$

is acyclic outside of degree n.

Proof. The complex $C_\bullet^{ray}$ is the augmented complex of the "partial basis complex" of $\mathbb{Z}^n$. By [18], the claim then follows. $\square$

Now, we define another complex, which we call the *line complex*. This is defined as follows:

- $C_0 = \mathbb{Z}$ and for $1 \leq k \leq n$, we define C_k to be the abelian group with generators the k-tuples $[v_1, \ldots, v_k]$ of distinct $v_i \in S^{n-1}(\mathbb{Q})$ such that all v_i generate different lines in $\mathbb{Z}^n$ (i.e., no two v_i are antipodal).
- The boundary map $d_1 : C_1 \to C_0$ is defined as the augmentation map, and for $k \geq 1$, d_k is defined via the usual alternating sum

$$d_k([v_1, \ldots, v_k]) = \sum_{i=1}^{k} (-1)^{i-1} [v_1, \ldots, \hat{v}_i, \ldots, v_k].$$

We identify all tuples in this complex under the antipodal map, i.e., the rays v and $-v$ are identified, so all tuples may be written as $[v_1, \ldots, v_k]$ such that v_i are distinct primitive vectors in the upper half space of $\mathbb{R}^n$. We call the resulting complex the *line complex* C^{line}. As with the ray complex, there is a natural Γ-action on C^{line} induced by matrix multiplication on the right, and we have the following acyclicity result:

Proposition 4.2.2. *The complex*

$$0 \to C_n^{line} \to C_{n-1}^{line} \to \ldots \to C_0^{line} \to 0$$

is acyclic outside degree n.

Proof. This is obvious for degree 0, so let $1 \leq i \leq n-1$. Let $s = \sum_{j=1}^{k} c_j \alpha_j$ be an arbitrary element of C_i^{line}, for tuples α_j and $c_j \in \mathbb{Z}$. Let x be any vector that does not appear in any of the components of α_j. We define $\theta_x : C_i^{line} \to C_{i+1}^{line}$ by $\theta_x([v_1, \ldots, v_i]) = [x, v_1, \ldots, v_i]$ if x does not appear among the v_i, and 0 otherwise. It then follows that $d_{i+1} \circ \theta_x(s) + \theta_x \circ d_i(s) = s$. In particular, if $s \in \ker(d_i)$ we find that $d_{i+1}(\theta_x(s)) = s$ as desired. $\square$

4.3 $\mathbb{G}_m^n$-cocycle for $n = 2, 3$

We start by illustrating our strategy in the cases $n = 2, 3$. In Section 4.4, however, we will need to take a more complicated approach, which requires using the theory of trace modules.

As usual, we let $K_\bullet$ denote the Gersten complex for $\mathbb{G}_m^n$ over $\mathbb{Q}$. This complex is the $q = n$ row of the E_1-page of the coniveau spectral sequence and is of the form

$$0 \to K_n^M(\mathbb{Q}(\mathbb{G}_m^n)) \to \bigoplus_{x \in Y_1} K_{n-1}^M(\mathbb{Q}(x)) \to \ldots \to \bigoplus_{x \in Y_n} K_0(\mathbb{Q}(x)) \to 0,$$

where Y_p denotes the set of codimension p points of $\mathbb{G}_m^n$, and we let $\overline{K}_\bullet$ denote the *reduced* Gersten complex, i.e.

$$\overline{K}_i = K_i$$

for $0 \leq i \leq n-1$ and

$$\overline{K}_n = K_n / \ker(K_n \to K_{n-1}).$$

Proposition 4.3.1. *Let $n = 2, 3$. Then $\overline{K}_\bullet$ is acyclic.*

Proof. The case of $n = 2$ is worked out in [25]. For $n = 3$, we examine the coniveau spectral sequence for $\mathbb{G}_m^3$. For $0 \leq p \leq 3$ and $0 \leq q \leq 3$, this looks like

$$\begin{array}{ccccccc}
K_3^M(\mathbb{Q}(\mathbb{G}_m^3)) & \longrightarrow & \bigoplus_{x \in Y_1} K_2(\mathbb{Q}(x)) & \longrightarrow & \bigoplus_{x \in Y_2} K_1(\mathbb{Q}(x)) & \longrightarrow & \bigoplus_{x \in Y_3} K_0(\mathbb{Q}(x)) \\
H^2(\mathbb{Q}(\mathbb{G}_m^3), 3) & \longrightarrow & \bigoplus_{x \in Y_1} H^1(\mathbb{Q}(x), 2) & \longrightarrow & 0 & & 0 \\
H^1(\mathbb{Q}(\mathbb{G}_m^3), 3) & \longrightarrow & \bigoplus_{x \in Y_1} H^0(\mathbb{Q}(x), 2) & \longrightarrow & 0 & & 0 \\
H^0(\mathbb{Q}(\mathbb{G}_m^3), 3) & \longrightarrow & \bigoplus_{x \in Y_1} H^{-1}(\mathbb{Q}(x), 2) & \longrightarrow & 0 & & 0
\end{array}$$

Note that the spectral sequence degenerates at the second page. From the top row we recover the Gersten complex, and its homology computes $H^i(\mathbb{G}_m^3, 3)$ for $4 \leq i \leq 6$ from the second term onward. Since these motivic cohomology groups vanish, the complex is acyclic outside the left, which therefore means $\overline{K}_\bullet$ is acyclic. $\square$

Before we start constructing cocycles, we will need some notation to parameterize certain elements inside the Gersten complex that we will frequently use throughout the rest of this chapter.

- We have $K_0 = \bigoplus_x K_0(\mathbb{Q}(x))$ where the sum is taken over all codimension n cycles of $\mathbb{G}_m^n$, and $K_0(\mathbb{Q}(x)) \cong \mathbb{Z}$. We pick out the special element $e \in K_0$ corresponding to $1 \in \mathbb{Z}$ supported at the identity of $\mathbb{G}_m^n$.
- For the other groups K_i for $1 \leq i \leq n-1$, we will use subscript notation to denote symbols which are supported at single codimension i-points of a certain type. E.g. for $x \in Y_i$, we write $\{x_1, \ldots, x_i\}|_x \in K_i$ to mean the symbol $\{x_1, \ldots, x_i\} \in K_i^M(\mathbb{Q}(x))$.

We are primarily concerned with indexing symbols that live on specific tori. For a primitive (row) vector $v \in \mathbb{Z}^n$, we identify $v = (a_1, \ldots, a_n)$ with the associated character

$$v : \mathbb{G}_m^n \to \mathbb{G}_m,$$

defined on coordinates by

$$(z_1, \ldots, z_n) \mapsto z_1^{a_1} \cdots z_n^{a_n}.$$

We set

$$S_v = \ker(v),$$

so that $S_v \in Y_1$. In particular, for $i \leq j$ we will also frequently use the notation

$$T_{i,j} = S_{e_i} \cap \ldots \cap S_{e_j},$$

noting that $T_{i,j} \in Y_{j-i+1}$.

Below, we give a concrete example of the sort of notation that will become common.

Example 4.3.2. The functions $1 - z_1$ and $1 - z_2$ are both invertible functions on $T_{3,n} - ((T_{3,n} \cap S_{e_1}) \cup (T_{3,n} \cap S_{e_2}))$ and so may be thought of as elements of $K_1(\mathbb{Q}(T_{3,n}))$. The symbol $\{1 - z_1, 1 - z_2\}|_{T_{3,n}}$ is then identified with the image of the cup product $(1 - z_1) \cup (1 - z_2) \in H^2(T_{3,n} - ((T_{3,n} \cap S_{e_1}) \cup (T_{3,n} \cap S_{e_2})), 2)$ under the morphism to $K_2(\mathbb{Q}(T_{3,n}))$.

A collection of primitive vectors $v_1, \dots, v_k$ naturally defines a morphism

$$A : \mathbb{G}_m^k \to \mathbb{G}_m^n$$

via the product of cocharacters

$$(z_1, \dots, z_k) \mapsto v_1(z_1) \cdots v_k(z_k).$$

We have an embedding

$$i : \mathbb{G}_m^k \to \mathbb{G}_m^n$$

given by

$$(z_1, \dots, z_k) \mapsto (z_1, \dots, z_k, 1, \dots, 1)$$

with image $i(\mathbb{G}_m^k) = T_{k+1,n}$.

Now, suppose we have a collection of linearly independent primitive vectors $v_1, \dots, v_k$. Then the morphism A is finite and therefore proper. Let γ be any extension of the matrix $\begin{pmatrix} -v_1- \\ \vdots \\ -v_k- \end{pmatrix} \in M_{k\times n}(\mathbb{Z})$ associated to A to a matrix in $\mathrm{GL}_n(\mathbb{Q}) \cap M_n(\mathbb{Z})$. Now, γ defines a morphism $\gamma : \mathbb{G}_m^n \to \mathbb{G}_m^n$, and we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{G}_m^k & \xrightarrow{A} & \mathbb{G}_m^n \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{G}_m^n & \xrightarrow{\gamma} & \mathbb{G}_m^n \end{array}$$

which then induces a commutative diagram of pushforward maps on motivic cohomology

$$\begin{array}{ccc} H^i(\mathbb{G}_m^k, j) & \xrightarrow{A_*} & H^i(\mathbb{G}_m^n, j) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \mathrm{id}_*} \\ H^i(\mathbb{G}_m^n, j) & \xrightarrow{\gamma_*} & H^i(\mathbb{G}_m^n, j) \end{array}$$

so that the pushforward A_* is identified with the pushforward γ_*. In the special case where the matrix associated to A can be extended to a matrix $\gamma \in \mathrm{SL}_n(\mathbb{Z})$, the pushforward is easy to describe: it follows from base change that

$$\gamma_* = (\gamma^{-1})^*$$

which is simply the pullback action of γ^{-1} on $H^i(\mathbb{G}_m^n, j)$.

Taking $i = j = r$ for $0 \leq r \leq k$ and applying the natural map $H^r(\mathbb{G}_m^n, r) \to K_r^M(\mathbb{Q}(\mathbb{G}_m^n))$[1], we obtain another commutative diagram:

$$\begin{tikzcd}
H^r(\mathbb{G}_m^k, r) \arrow[r, "A_*"] \arrow[d, "i_*"] & H^r(\mathbb{G}_m^n, r) \arrow[d, "\mathrm{id}_*"] \\
H^r(\mathbb{G}_m^n, r) \arrow[r, "\gamma_*"] \arrow[d] & H^r(\mathbb{G}_m^n, r) \arrow[d] \\
K_r^M(\mathbb{Q}(\mathbb{G}_m^n)) \arrow[r, "\gamma_*"] & K_r^M(\mathbb{Q}(\mathbb{G}_m^n))
\end{tikzcd}$$

The point of all this is that starting with the vectors $e_1, \ldots, e_k \in \mathbb{Z}^k$, the complement $\mathbb{G}_m^k - (S_{e_1} \cup \ldots \cup S_{e_k})$ has image $T_{k+1,n} - ((S_{e_1} \cup \ldots \cup S_{e_k}) \cap T_{k+1,n})$ under the embedding i. Choosing a collection of linearly independent primitive vectors $v_1, \ldots, v_k$ allows us to compare the pushforward map A_* with a pushforward map $\gamma_* : K_r^M(\mathbb{Q}(T_{k+1,n})) \to K_r^M(\mathbb{Q}(T_{k+1,n}))$, and for $\gamma \in \mathrm{SL}_n(\mathbb{Z})$ this map can be explicitly described.

Example 4.3.3. We work out the explicit description of this pushforward map when $k = r = 1$. Here, we start with a primitive vector $v \in \mathbb{Z}^n$. Choose any matrix $\gamma \in \mathrm{SL}_n(\mathbb{Z})$ with first row equal to v. Viewing $\mathcal{O}(T_{2,n} - \{1\})^\times \hookrightarrow K_1(\mathbb{Q}(T_{2,n}))$, the value of the map v_* on the invertible function $1 - z_1$ on the torus $T_{2,n} - \{1\}$ is given by $v_*(1 - z_1) = (\gamma^{-1})^*(1 - z_1)$ on the torus $T_{2,n}\gamma^{-1} - \{1\} = S_{\gamma^{-1}e_1} \cap \ldots \cap S_{\gamma^{-1}e_n} - \{1\}$ (counter to our usual convention, the vector $\gamma^{-1}e_i$ means the i-th column of γ^{-1}).

For a concrete example, suppose $n = 2$ and $v = (a, c)$. Choose any (b, d) such that $ab + cd = 1$. Taking $\gamma = \left(\begin{smallmatrix} a & c \\ -d & b \end{smallmatrix}\right)$, then explicitly one sees that

$$v_*(1 - z_1|_{S_{(0,1)}}) = 1 - z_1^b z_2^d|_{S_{(-c,a)}}.$$

To construct representative choices of cocycles for $\mathbb{G}_m^n$ in the cases of $n = 2, 3$, our parameterizing complex of choice will be the ray complex. However, we note that there is always map of complexes from the ray complex to the Gersten complex, regardless of n.

[1]By which we mean the map $H^r(\mathbb{G}_m^r, r) \to H^r(\mathbb{Q}(\mathbb{G}_m^r), r) \cong K_r^M(\mathbb{Q}(\mathbb{G}_m^n))$ arising from the coniveau spectral sequence.

Proposition 4.3.4. *There is a morphism of complexes $\mathbb{Z}[\Gamma]$-modules*

$$\begin{tikzcd}
0 \arrow[r] & C_n^{ray} \arrow[r] \arrow[d, "f_n"] & C_{n-1}^{ray} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \arrow[r] & C_1^{ray} \arrow[r] \arrow[d, "f_1"] & C_0^{ray} \arrow[r] \arrow[d, "f_0"] & 0 \\
0 \arrow[r] & K_n \arrow[r] & K_{n-1} \arrow[r] & \dots \arrow[r] & K_1 \arrow[r] & K_0 \arrow[r] & 0
\end{tikzcd}$$

with maps defined as follows:

- $f_0(1) = e$
- $f_k([l_1, \dots, l_k]) = \begin{pmatrix} -l_1- \\ \vdots \\ -l_k- \end{pmatrix}_* (\{1 - z_1, \dots, 1 - z_k\}|_{T_{k+1,n}})$

Proof. We start by checking commutativity of the diagram. We wish to check that

$$\partial f_k([l_1, \dots, l_k]) = \sum_{i=1}^{k} (-1)^{i-1} f_{k-1}([l_1, \dots, \hat{l}_i, \dots, l_k]).$$

Note that the residue map commutes with pushforwards, and the boundary of the symbol $\{1-z_1, \dots, 1-z_k\}|_{T_{k+1,n}}$ is given by $\sum_{i=1}^{k} (-1)^{i-1} \{1-z_1, \dots, \widehat{1-z_i}, \dots, 1-z_k\}|_{S_{e_i} \cap T_{k+1,n}}$. Let $A = \begin{pmatrix} -l_1- \\ \vdots \\ -l_k- \end{pmatrix}$ be the matrix associated to the vectors $l_1, \dots, l_k$, and let A_i denote the matrix obtained by deleting the i-th row of A. Then the morphisms induced by the matrices A and A_i are the same on the torus $S_{e_i} \cap T_{k+1,n}$, which means they induce the same pushforward map on motivic cohomology, and therefore on Milnor K-theory. For the rightmost square, we explicitly work out the computation to see the commutativity. Explicitly, $f_1([l]) = (\gamma^{-1})^*(1 - z_1|_{T_{2,n}})$ for any $\mathrm{SL}_n(\mathbb{Z})$-extension of the vector l. From this description and Example 4.3.2, it is then apparent that $\partial f_1([l]) = e$, which is what we wanted.

It remains to check that these morphisms are actually Γ-equivariant. This is obvious for f_0. For $k \geq 2$, pick $\gamma \in \Gamma$. By definition, we find that $f_k(\gamma \cdot [l_1, \dots, l_k]) = (A\gamma^{-1})_*(\{1 - z_1, \dots, 1 - z_k\}|_{T_{k+1,n}})$. Now, pushing forward is an *anti-action* because of our convention of right matrix multiplication, and therefore we have $(A\gamma^{-1})_* = (\gamma^{-1})_* A_* = \gamma^* A_*$ as desired. For f_1, the pushforward $(l\gamma^{-1})_*$ is computed by choosing any $\mathrm{SL}_n(\mathbb{Z})$-extension of the vector $l\gamma^{-1}$. Certainly if A is such an extension for l, then $A\gamma^{-1}$ is such an extension for $l\gamma^{-1}$, and therefore $(l\gamma^{-1})_* = \gamma^* l_*$ via the same reasoning as before. □

Remark 4.3.5. Since elements in the ray complex are tuples of vectors that extend to $\mathbb{Z}$-bases of $\mathbb{Z}^n$, we very well could have defined the maps via *pullbacks* instead of *pushforwards.* Indeed, such a construction would be the natural generalization of the construction given in Section 3.2 of [25]. However, our construction in the next section will be done using pushforwards, so for the purposes of exposition, we keep the convention here the same.

Corollary 4.3.6. *Set $v_0 = (-1, 0)$. The map*

$$\Theta : \mathrm{GL}_2(\mathbb{Z}) \to \overline{K}_2$$

defined by

$$\gamma \mapsto f_2(\alpha_\gamma)$$

for any element $\alpha_\gamma \in C_2^{ray}$ lifting $(\gamma - 1) \cdot [v_0]$ is a 1-cocycle representing the class of the cocycle $[\Theta] = \delta_{C_\bullet^{ray}, \overline{K}_\bullet}(1) \in H^1(\mathrm{GL}_2(\mathbb{Z}), \overline{K}_2)$

Proof. Starting with the element $1 \in C_0^{ray}$, we choose the lift $[v_0] \in C_1$. Then for any $\gamma \in \mathrm{GL}_2(\mathbb{Z})$, the element $(\gamma - 1) \cdot [v_0]$ has trivial boundary, and therefore there exists $\alpha_\gamma \in C_2^{ray}$ such that $d_2 \alpha_\gamma = (\gamma - 1) \cdot [v_0]$. Firstly, note that the choice of α_γ is irrelevant for the value of $f_2(\alpha_\gamma)$: indeed, if α'_γ is any other choice of lift of $(\gamma - 1) \cdot [v_0]$ in C_2^{ray}, it follows that $\partial_2(f_2(\alpha_\gamma - \alpha'_\gamma)) = f_1(0) = 0$, and so by exactness this means $f_2(\alpha_\gamma) = f_2(\alpha'_\gamma)$. It then follows that $\gamma \mapsto f_2(\alpha_\gamma)$ a 1-cocycle, as the cocycle condition is easily verified because equality in $\overline{K}_2$ is determined by boundary in K_1.

That the cocycle we have constructed represents the class of $\delta_{C_\bullet^{ray}, \overline{K}_\bullet}(1)$ follows from the explicit description of the connecting map $C_0^{ray} \to H^1(G, \ker(d_1))$ as well as the fact that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_2^{ray} & \longrightarrow & C_1^{ray} & \longrightarrow & C_0^{ray} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{f}_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \\ 0 & \longrightarrow & \overline{K}_2 & \longrightarrow & K_1 & \longrightarrow & K_0 & \longrightarrow & 0 \end{array}$$

is a map of complexes. □

Remark 4.3.7. Explicitly, we have $f_1([v_0]) = 1 - z_1^{-1}|_{S_{(0,1)}}$ and therefore the cocycle constructed above recovers [25, Proposition 3.3.1].

Now, we move on to the cocycle construction for $n = 3$. The main difference with the $n = 2$ case is that explicitly picking lifts becomes more complicated, due to the extra group C_2^{ray} in the ray complex.

For vectors $v_1, v_2 \in \mathbb{Z}^3$ that are primitive and linearly independent, the vector $v_1 + v_2$ is a non-zero vector in $\mathbb{Z}^3$, and therefore $[v_1 + v_2] \in C_1^{ray}$ is well defined.

The following is rather elementary, but important:

Proposition 4.3.8. *Let $v_1, v_2 \in \mathbb{Z}^3$ be primitive, linearly independent vectors. Let e_i denote the first standard basis vector such that $\{v_1, v_2, e_i\}$ is $\mathbb{Z}$-linearly independent. Let*

$$A = \begin{pmatrix} -v_1- \\ -v_2- \\ -e_i- \end{pmatrix}.$$

If $|\det(A)| > 1$, then $0 < |\det(A_j)| \leq |\det(A)|$, where for $j = 1, 2$, the matrix A_j is obtained by replacing the j-th row of A with the primitive generator of the ray $[v_1 + v_2] \in C_1^{ray}$.

Proof. The primitive generator of $v_1 + v_2$ is obtained by simply dividing $v_1 + v_2$ by the gcd of the entries. It's then immediate from basic linear algebra that $\det(A_j)$ is reduced by a factor of this gcd. $\square$

Definition 4.3.9. Let $v_1, v_2 \in \mathbb{Z}^3$ be primitive, linearly independent vectors. The *unimodularization* $[v_1, v_2]_* \in C_2^{ray}$ is defined recursively as follows:

- If $[v_1, v_2] \in C_2^{ray}$, then $[v_1, v_2]_* = [v_1, v_2]$. Otherwise, we replace the (not defined) symbol $[v_1, v_2]$ by the sum $[v_1, v_1 + v_2] + [v_1 + v_2, v_2]$. These symbols may also not be defined, but are "closer" to being well defined by Proposition 4.3.8.

- Repeat the above on each of the symbols $[v_1, v_1 + v_2]$ and $[v_1 + v_2, v_2]$.

- Eventually this process must produce a sequence of vectors $v_1 = u_1, \ldots, u_k = v_2$ such that $[u_i, u_{i+1}] \in C_2^{ray}$ and

$$d_2\Big(\sum_{i=1}^{k-1} [u_i, u_{i+1}]\Big) = [v_2] - [v_1].$$

We then define

$$[v_1, v_2]_* = \sum_{i=1}^{k-1} [u_i, u_{i+1}].$$

Remark 4.3.10. The "unimodularization" of the (potentially undefined) symbol $[v_1, v_2]$ is the analogue of the connecting sequences used in Section 3.3 of [25]. It is similar, in spirit, to the *modular symbol algorithm* of Ash and Rudolph [2].

Using unimodularizations, we can construct a canonical choice of cocycle.

Corollary 4.3.11. *The map*

$$\Theta : \mathrm{GL}_3(\mathbb{Z})^2 \to \overline{K}_3$$

defined by

$$(\gamma, \gamma') \mapsto f_3(\alpha_{\gamma,\gamma'})$$

for any choice of lift $\alpha_{\gamma,\gamma'} \in C_3^{ray}$ of $\eta_\gamma \in C_2^{ray}$ defined in equation (4.1)*, is a* 2*-cocycle that represents the class of $[\Theta] = \delta_{C_\bullet^{ray}, \overline{K}_\bullet}(1) \in H^2(\mathrm{GL}_3(\mathbb{Z}), \overline{K}_3)$.*

Proof. We perform the construction as follows. Starting with $1 \in C_0^{ray}$, we lift it up to $[v_0] \in C_1^{ray}$ for $v_0 = -e_1$. The element $(\gamma - 1) \cdot [v_0]$ therefore has trivial boundary, and so there exists a lift $\eta_\gamma \in C_2^{ray}$ with $d_2\eta_\gamma = (\gamma - 1) \cdot [v_0]$. Explicitly, we will define the lift as follows:

$$\eta_\gamma = \begin{cases} [v_0, \gamma \cdot v_0]_* & \{v_0, \gamma \cdot v_0\} \text{ linearly independent} \\ \gamma \cdot ([e_1, e_2] + [e_2, v_0]) & \gamma \cdot v_0 = -v_0 \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Indeed, if $\{v_0, \gamma \cdot v_0\}$ is a linearly independent set, then by construction $[v_0, \gamma \cdot v_0]_*$ has boundary $(\gamma - 1)[v_0]$. If $\gamma \cdot v_0 = v_0$, then certainly $(\gamma - 1) \cdot [v_0] = 0$, so 0 is a valid choice of lift. Finally, if $\gamma \cdot v_0 = -v_0$, then note that $d_2\eta_\gamma = \gamma \cdot ([v_0] - [e_1]) = (\gamma - 1) \cdot [v_0]$. One may then similarly verify in all cases that $\gamma \cdot \eta_{\gamma'} + \eta_\gamma - \eta_{\gamma\gamma'}$ has trivial boundary. Therefore, there exists $\alpha_{\gamma,\gamma'} \in C_3^{ray}$ such that $d_3(\alpha_{\gamma,\gamma'}) = \gamma \cdot \eta_{\gamma'} + \eta_\gamma - \eta_{\gamma\gamma'}$.

As before, the choice of lift $\alpha_{\gamma,\gamma'}$ does not matter, because elements of $\overline{K}_3$ are determined by their boundaries. To see that the map $(\gamma, \gamma') \mapsto f_3(\alpha_{\gamma,\gamma'})$ is a 2-cocycle, one must check that for $\gamma, \gamma', \gamma'' \in \Gamma$, that

$$\gamma^* f_3(\alpha_{\gamma',\gamma''}) - f_3(\alpha_{\gamma\gamma',\gamma''}) + f_3(\alpha_{\gamma,\gamma'\gamma''}) - f_3(\alpha_{\gamma,\gamma'}) = 0$$

in $\overline{K}_3$. This is done by verifying they have the same residue in K_2, which by commutativity is the same as checking that

$$f_2(d_3(\gamma \cdot \alpha_{\gamma',\gamma''} - \alpha_{\gamma\gamma',\gamma''} + \alpha_{\gamma,\gamma'\gamma''} - \alpha_{\gamma,\gamma'})) = 0.$$

This follows because the boundary of the inside term is 0.

By unwinding the definitions of the connecting maps on group cohomology, one finds that this cocycle is a representative of $\delta_{C_\bullet^{ray},\overline{K}_\bullet}(1)$ □

Remark 4.3.12. As the images of all symbols under the map of complexes are *trace-fixed*, we know that the cocycle Θ constructed for $n = 2$ lifts to a cocycle valued in $K_2^{(0)}/(\mathbb{Z} \cdot \{-z_1, -z_2\})$ because $\ker(K_2 \to K_1)^{(0)} = H^2(\mathbb{G}_m^2, 2)^{(0)}$, as worked out in Section 4.1 of [25]. The author expects that an analysis of the coniveau spectral sequence for $n = 3$ can show that $\ker(K_3 \to K_2)^{(0)} = H^3(\mathbb{G}_m^3, 3)^{(0)}$, which would produce a cocycle valued in $K_3^{(0)}/(\mathbb{Z} \cdot \{-z_1, -z_2, -z_3\})$ in this case. However, this is not quite as obvious as when $n = 2$, and we do not work it out.

4.4 $\mathbb{G}_m^n$ cocycle: general case

When $n > 3$, there are several problems that make the ideas of the previous section hard to generalize:

- It is no longer easy to work with the coniveau spectral sequence, because of the conjectural vanishing of negative degree motivic cohomology groups that appear as terms on the E_1-page. In particular, it is no longer obvious that the reduced Gersten complex is acyclic. Even if it were, the group $H_n(K_\bullet)$, which would control the indeterminacy of the cocycle, is not easily described.
- It is no longer obvious how to pick canonical choices of lifts inside the ray complex, and therefore even if $\overline{K}_\bullet$ were known to be exact, it's no longer obvious how to pick a canonical representative for $[\Theta]_{ray}$.

These problems are fixable, but they come at the cost of working over $\mathbb{Z}' = \mathbb{Z}[\frac{1}{(n+1)!}]$. Our approach is then as follows:

- We construct an intermediary acyclic complex $C_\bullet^{lim}$, which is built using the exact sequences obtained from Theorem 3.3.8.

- By modifying the maps in Proposition 4.3.4, we obtain a morphism of complexes

$$C_\bullet^{line} \otimes_{\mathbb{Z}} \mathbb{Z}' \xrightarrow{f_\bullet} C_\bullet^{lim} \xrightarrow{g_\bullet} \overline{K}_\bullet \otimes_{\mathbb{Z}} \mathbb{Z}'$$

  which maps to *fully symmetrized* symbols in Milnor K-theory.

By pushing forward, we obtain a cocycle class

$$g_*(\delta_{C_\bullet^{line} \otimes_{\mathbb{Z}} \mathbb{Z}', C_\bullet^{lim}}(1)) := [\Theta]_{line} \in H^{n-1}(\Gamma, \overline{K}_n \otimes_{\mathbb{Z}} \mathbb{Z}').$$

Factoring through the limit complex $C_\bullet^{lim}$ makes it easier for us to prove certain relations among symbols in Milnor K-theory hold, which is ultimately what becomes necessary to produce a morphism of complexes, whereas working with $C_\bullet^{line}$ instead of $C_\bullet^{ray}$ over $\mathbb{Z}'$ makes it significantly easier to pick lifts, and therefore pick out a (canonical) representative cocycle.

Before we begin, it is worth pointing out that the obstruction to picking canonical lifts in $C_\bullet^{ray}$ over $\mathbb{Z}$ is proving that certain relations hold at the level of Milnor K-theory. The author expects that canonical lifts can be chosen by picking "unimodularizations" of symbols in the ray complex, analogous to the $n = 3$ case, but it is not easy to prove that a map of complexes $C_\bullet^{ray} \to \overline{K}_\bullet$ is obtained this way. If one were able to prove such relations, then the modified approach we've taken here is unnecessary.

First, we start by defining the limit complex.

Definition 4.4.1. Let $Z = (Z_p)_{p \in \mathbb{Z}}$ be a toric flag for $\mathbb{G}_m^n$ built out of the set of tori $I_Z = \{S_1, \ldots, S_N\}$. We call the exact sequence of trace modules over $\mathbb{Z}' = \mathbb{Z}[\frac{1}{(n+1)!}]$

$$0 \to (H^n(\mathbb{G}_m^n, n) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 \to (H^n(\mathbb{G}_m^n - Z_1, n) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 \to \ldots \to (H^0(Z_n, 0) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 \to 0$$

extracted via Theorem 3.3.8 the *toric complex* associated to the flag Z, which we denote by K_Z.

Given toric flags Z and Z', we can define a partial ordering by $Z \leq Z'$ if Z'_p is a closed subscheme of Z_p for all $p \in \mathbb{Z}$. As in the construction of the coniveau spectral sequence, this induces a map on the toric flag spectral sequences attached to Z and Z', and therefore a map of complexes $K_{Z'} \to K_Z$ after tensoring with $\mathbb{Z}'$ and taking generalized trace fixed parts. This allows us to speak of the direct limit complex $\varinjlim_Z K_Z$, which is acyclic and supported in degrees $[0, n+1]$. As our parameterizing complexes are only supported in degrees $[0, n]$, it will be convenient to modify this complex so that degrees match up.

Definition 4.4.2. The *limit complex* $C_\bullet^{lim}$ is defined by

$$C_i^{lim} = (\varinjlim_Z K_Z)_i$$

for $0 \leq i \leq n-1$, and

$$C_n^{lim} = (\varinjlim_Z K_Z)_n / (H^n(\mathbb{G}_m^n, n) \otimes_{\mathbb{Z}} \mathbb{Z}')^0.$$

As mentioned, our goal is to produce a map of complexes

$$C_\bullet^{line} \otimes_{\mathbb{Z}} \mathbb{Z}' \xrightarrow{f_\bullet} C_\bullet^{lim} \xrightarrow{g_\bullet} \overline{K}_\bullet \otimes_{\mathbb{Z}} \mathbb{Z}'.$$

We then will take

$$g_*(\delta_{C_\bullet^{line} \otimes_{\mathbb{Z}} \mathbb{Z}', C_\bullet^{lim}}(1)) := [\Theta]_{line} \in H^{n-1}(\Gamma, \overline{K}_n \otimes_{\mathbb{Z}} \mathbb{Z}').$$

From the construction of the coniveau spectral sequence and the definition of the reduced Gersten complex $\overline{K}_\bullet$, we already have a map of complexes of $\mathbb{Z}'[\Gamma]$-modules

$$C_\bullet^{lim} \xrightarrow{g_\bullet} \overline{K}_\bullet \otimes_{\mathbb{Z}} \mathbb{Z}'$$

which sends cup products to Steinberg symbols. Therefore, we need only to construct the map $f_\bullet$.

In order to do so, it will be necessary to describe the image of the pushforward maps induced via tuples of cocharacters inside the complex $C_\bullet^{lim}$.

Recall that a lattice $\mathcal{L} \subset \mathbb{Z}^n$ is called *saturated* if the quotient $\mathbb{Z}^n/\mathcal{L}$ is torsion free. For a lattice $\mathcal{L}$, we let $\mathcal{L}_{sat} = \{x \in \mathbb{Z}^n : \exists k \neq 0 \in \mathbb{Z} \text{ such that } kx \in \mathcal{L}\}$ denote the saturation of $\mathcal{L}$.

Proposition 4.4.3. *Let $\mathcal{L} \subset \mathbb{Z}^n$ be a saturated lattice of rank r. Then any basis of $\mathcal{L}$ extends to a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

Proof. Let $\{v_1, \ldots, v_r\}$ be a basis of $\mathcal{L}$ and $I = (\det(v_1, \ldots, v_r, x_{r+1}, \ldots, x_n) : x_i \in \mathbb{Z}^n) = (d)$ for some $d \geq 1$. If $d > 1$, choose a prime p with $p \mid d$. Working over $\mathbb{F}_p$, we must have $\overline{v}_1, \ldots, \overline{v}_r$ are linearly dependent, and so we can find $c_i \in \mathbb{Z}$ not all divisible by p and $y \in \mathbb{Z}^n$ such that $c_1 v_1 + \ldots + c_r v_r = py$. As $\mathcal{L}$ is saturated this means $y \in \mathcal{L}$, and so this forces $p \mid c_i$ for all i, which is a contradiction. Therefore, $d = 1$, which is what we wanted. $\square$

Proposition 4.4.4. *Let $v_1, \ldots, v_k \in \mathbb{Z}^n$ be linearly independent primitive vectors. Consider the map*

$$A : \mathbb{G}_m^k \to \mathbb{G}_m^n$$

defined by

$$(z_1, \ldots, z_k) \mapsto v_1(z_1) \cdots v_k(z_k)$$

corresponding to the product of co-characters. Then $X_*(\mathrm{im}(A)) = Span_{\mathbb{Z}}\{v_1, \ldots, v_k\}_{sat}$, the saturation of the lattice spanned by $v_1, \ldots, v_k$ inside $\mathbb{Z}^n$, where $X_*(-)$ is the functor that sends a torus to its co-character group.

Proof. By [9, Proposition 5.1], there is a short exact sequence

$$0 \to X^*(\mathbb{G}_m^n / \mathrm{im}(A)) \to X^*(\mathbb{G}_m^n) \to X^*(\mathrm{im}(A)) \to 0,$$

where $X^*(-) = \mathrm{Hom}(-, \mathbb{G}_m)$ is the functor that sends a torus to its character group. As $X_*(-) = \mathrm{Hom}_{\mathbb{Z}}(X^*(-), \mathbb{Z})$, we obtain a short exact sequence

$$0 \to X_*(\mathrm{im}(A)) \to X_*(\mathbb{G}_m^n) \to X_*(\mathbb{G}_m^n / \mathrm{im}(A)) \to 0.$$

The group $X^*(\mathbb{G}_m^n / \mathrm{im}(A))$ is free because $X^*(\mathbb{G}_m^n) = \mathbb{Z}^n$, so certainly its dual $X_*(\mathbb{G}_m^n / \mathrm{im}(A))$ is free as well. Thus, $X_*(\mathrm{im}(A))$ is saturated as a sub-lattice of $X_*(\mathbb{G}_m^n) = \mathbb{Z}^n$. On the other hand, consider the map

$$X_*(A) : X_*(\mathbb{G}_m^k) \to X_*(\mathbb{G}_m^n).$$

Then $\mathrm{im}(X_*(A)) = \mathrm{Span}_{\mathbb{Z}}\{v_1, \ldots, v_k\}$. Now, $\mathrm{im}(X_*(A)) \subset X_*(\mathrm{im}(A))$, and these two lattices have the same rank. Since $X_*(\mathrm{im}(A))$ is saturated, this forces it to be the saturation of $\mathrm{im}(X_*(A))$ as desired. $\square$

Remark 4.4.5. For any lattice $\mathcal{L} \subset \mathbb{Z}^n$, we note that the orthogonal complement $\mathcal{L}^\perp$ is saturated. Since tori are uniquely determined by their co-character lattices, the above proposition gives an explicit description of $\mathrm{im}(A)$. In particular, if we view $\mathbb{G}_m^k$ as a subgroup of $\mathbb{G}_m^n$ under the embedding $i : \mathbb{G}_m^k \to T_{k+1,n}$, then $\mathrm{im}(A)$ is identified with $S_{w_{k+1}} \cap \ldots \cap S_{w_n}$, where $\{w_{k+1}, \ldots, w_n\}$ is a basis of $(\mathrm{Span}_{\mathbb{Z}}\{v_1, \ldots, v_k\}_{sat})^\perp$.

We're now ready to begin constructing our map of complexes. Before we begin, we will need some more notation.

Definition 4.4.6. For an integer $1 \leq k \leq n$, we define the *standard symmetrized cup product for* $\mathbb{G}_m^n$, $\langle e_1, \ldots, e_k \rangle$, by

$$\langle e_1, \ldots, e_k \rangle := (1-z_1)(1-z_1^{-1}) \cup \ldots \cup (1-z_k)(1-z_k^{-1}) \in H^k\Big(T_{k+1,n} - \big(\bigcup_{i=1}^{k} S_{e_i} \cap T_{k+1,n}\big), k\Big).$$

For linearly independent vectors $v_1, \ldots, v_k \in \mathbb{Z}^n$, we consider the corresponding (finite) map $A = \begin{pmatrix} -v_1- \\ \vdots \\ -v_k- \end{pmatrix}$ with domain $T_{k+1,n}$. We let $v_1', \ldots, v_k'$ be a basis of $\mathrm{Span}_{\mathbb{Z}}\{v_1, \ldots, v_k\}_{sat}$. Then $\{v_1', \ldots, v_k'\}$ extends to a $\mathbb{Z}$-basis $\{v_1', \ldots, v_k', y_{k+1}, \ldots, y_n\}$ of $\mathbb{Z}^n$, so that the map A is the same as the map induced by the matrix action of $\gamma = \begin{pmatrix} -v_1'- \\ \vdots \\ -y_n- \end{pmatrix} \in \mathrm{SL}_n(\mathbb{Z})$. If γ^{-1} has columns $w_1, \ldots, w_n$, it then follows that $\{w_{k+1}, \ldots, w_n\}$ is a basis of $(\mathrm{Span}_{\mathbb{Z}}\{v_1, \ldots, v_k\}_{sat})^\perp$, and so by Remark 4.4.5, we have $\mathrm{im}(A) = S_{w_{k+1}} \cap \ldots \cap S_{w_n}$. It then follows that

$$A_*(\langle e_1, \ldots, e_k \rangle) \in H^k\Big(\bigcap_{i=k+1}^{n} S_{w_i} - \bigcup_{j=1}^{k} \big(S_{w_j} \cap \bigcap_{i=k+1}^{n} S_{w_i}\big), k\Big),$$

which appears as one of the terms in the decomposition of $H^k(Z_{n-k} - Z_{n-k+1}, k)$ for the toric flag built out of the tori $S_{w_1}, \ldots, S_{w_n}$. Therefore, $A_*(\langle e_1, \ldots, e_k \rangle) \in C_k^{lim}$, so we can use the same ideas as the previous section to obtain the map of complexes we seek.

The majority of the remainder of the chapter is devoted to proving this:

Theorem 4.4.7. *Set* $\mathbb{Z}' = \mathbb{Z}[\frac{1}{(n+1)!}]$. *There is a morphism of complexes of* $\mathbb{Z}'[\Gamma]$*-modules*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C_n^{line} \otimes_{\mathbb{Z}} \mathbb{Z}' & \longrightarrow & C_{n-1}^{line} \otimes_{\mathbb{Z}} \mathbb{Z}' & \longrightarrow & \ldots & \longrightarrow & C_0^{line} \otimes_{\mathbb{Z}} \mathbb{Z}' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f_n} & & \big\downarrow{\scriptstyle f_{n-1}} & & & & \big\downarrow{\scriptstyle f_0} & & \\ 0 & \longrightarrow & C_n^{lim} & \longrightarrow & C_{n-1}^{lim} & \longrightarrow & \ldots & \longrightarrow & C_0^{lim} & \longrightarrow & 0 \end{array}$$

where the vertical maps are defined as follows:

- $f_0(1) = e \in H^0(\{1\}, 0)$
- $f_k([v_1, \ldots, v_k]) = \begin{cases} \frac{1}{2^k} A_*(\langle e_1, \ldots, e_k \rangle) & \{v_1, \ldots, v_k\} \textit{ linearly independent} \\ 0 & \textit{otherwise} \end{cases}$

where $A = \begin{pmatrix} -v_1- \\ \vdots \\ -v_k- \end{pmatrix}$.

The difficulty of the theorem comes entirely from showing that the maps may be taken to be 0 on tuples of linearly dependent vectors in the line complex. We also note that the equivariance of the maps f_i follows formally by the same argument given in Proposition 4.3.4.

We will break up the proof into several lemmas throughout the rest of the section, and our strategy will be to inductively construct the diagram.

We start by proving the following two results, which will essentially become our base case.

Lemma 4.4.8. *Let $n = 2$, and let f_i be defined as in Theorem 4.4.7. There is a morphism of complexes of $\mathbb{Z}'[\Gamma]$-modules*

$$\begin{tikzcd} 0 \arrow[r] & C_2^{line} \otimes_{\mathbb{Z}} \mathbb{Z}' \arrow[r] \arrow[d, "f_2"] & C_1^{line} \otimes_{\mathbb{Z}} \mathbb{Z}' \arrow[r] \arrow[d, "f_1"] & C_0^{line} \otimes_{\mathbb{Z}} \mathbb{Z}' \arrow[r] \arrow[d, "f_0"] & 0 \\ 0 \arrow[r] & C_2^{lim} \arrow[r] & C_1^{lim} \arrow[r] & C_0^{lim} \arrow[r] & 0 \end{tikzcd}$$

Proof. We start by showing commutativity of the right square. For $[l] \in C_1^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$, we have $f_0 d_1[l] = f_0(1) = e$ by definition. On the other hand, we have $f_1[l] = \frac{1}{2}(l)_* \langle e_1 \rangle$. Explicitly, we choose any extension $\{l, y_2, \ldots, y_n\}$ to a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Then $\frac{1}{2}(l)_* \langle e_1 \rangle = \frac{1}{2}(\gamma^{-1})^*((1 - z_1)(1 - z_1^{-1}))$, and so it follows that $\partial_1 f_1[l] = \frac{1}{2}(e + e) = e$ as desired.

Now, we show the left square commutes. Start with $[l_1, l_2] \in C_2^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$ with $\{l_1, l_2\}$ linearly independent, so that $d_2([l_1, l_2]) = [l_2] - [l_1]$. Then $f_1(d_2([l_1, l_2])) = f_1([l_2] - [l_1]) = \frac{1}{2}((l_2)_* - (l_1)_*)(\langle e_1 \rangle) \in H^1(S_{l_2} - \{1\}, 1) \oplus H^1(S_{l_1} - \{1\}, 1)$. On the other hand, $f_2([l_1, l_2]) = \frac{1}{4} A_*(\langle e_1, e_2 \rangle) \in H^2(\mathbb{G}_m^2 - (S_{l_1} \cup S_{l_2}), 2)/(H^2(\mathbb{G}_m^2, 2) \otimes_{\mathbb{Z}} \mathbb{Z}')^0$ for $A = \begin{pmatrix} -l_1- \\ -l_2- \end{pmatrix}$. As in Proposition 4.3.4, the point is that the boundary maps in motivic cohomology commute with pushforwards, and it's evident that $\partial \langle e_1, e_2 \rangle = 2(\langle e_2 \rangle - \langle e_1 \rangle)$, so commutativity is verified. The only remaining case is when $\{l_1, l_2\}$ are not linearly independent.

We're now done, because by definition of the line complex, it is not possible to have a symbol of the form $[l_1, l_2]$ with l_1, l_2 linearly dependent in the $n = 2$ case (the upper half space condition would force $l_1 = l_2$, but such a symbol doesn't exist). $\square$

The point is that having the commutativity of the above diagram helps us prove certain relations hold inside of the group $(\varinjlim_Z K_Z)_2$. Before moving on, it will make our lives a little easier to allow for a slight abuse of notation: for a set of vectors $\{v_1, \ldots, v_k\}$, we will write $[v_1, \ldots, v_k] \in C_k^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$ to mean 0 if v_i and v_j are antipodal for some i, j (as such a symbol does not actually exist in the line complex).

Lemma 4.4.9. *Let $\{v_1, v_2, v_3\}$ be a set of primitive vectors in $\mathbb{Z}^2$. Write $\partial[v_1, v_2, v_3]$ to denote $[v_2, v_3] - [v_1, v_3] + [v_1, v_2] \in C_2^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$. Then $f_2(\partial[v_1, v_2, v_3]) = 0$ in $(\varinjlim_Z K_Z)_2$.*

Proof. Our first observation is that by exactness of

$$0 \to C_2^{lim} \to C_1^{lim},$$

the value of f_2 in C_2^{lim} is completely determined by its residue in C_1^{lim}.

We start with the case that $\{v_1, v_2, v_3\}$ spans a rank 1 submodule. In this case, all vectors live on the same line and so by our definition, $\partial[v_1, v_2, v_3] = 0$. Clearly, $f_2(0) = 0$.

Now, we suppose that $\{v_1, v_2, v_3\}$ span a rank 2 submodule, and consider $\partial[v_1, v_2, v_3] = [v_2, v_3] - [v_1, v_3] + [v_1, v_2]$. It's quite clear that if one of these symbols is 0, that $\partial[v_1, v_2, v_3] = 0$. Therefore, we may assume that all three symbols are non-zero in $C_2^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$. As $d_2(\partial[v_1, v_2, v_3]) = 0$, it follows from Lemma 4.4.8 that $f_2(\partial[v_1, v_2, v_3]) = 0$ in $C_2^{lim} \otimes_{\mathbb{Z}} \mathbb{Z}'$. Therefore in $(\varinjlim_Z K_Z)_2$, it follows that $f_2(\partial[v_1, v_2, v_3]) = n((-z_1) \cup (-z_2))$ for some $n \in \mathbb{Z}'$, because $(H^2(\mathbb{G}_m^2, 2) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 = \mathbb{Z}'\langle(-z_1) \cup (-z_2)\rangle$ by Proposition 3.1.5.

Now, by definition we may write

$$f_2(\partial[v_1, v_2, v_3]) = \frac{1}{4} \sum_{i=1}^{3} (A_i)_* \langle e_1, e_2 \rangle,$$

where A_i denotes the matrix $A = \begin{pmatrix} -v_1- \\ -v_2- \\ -v_3- \end{pmatrix}$ with ith row deleted, and so in particular, everthing may be thought of as happening inside $H^2(\mathbb{G}_m^2 - (S_{v_1} \cup S_{v_2} \cup S_{v_3}), 2) \otimes_{\mathbb{Z}} \mathbb{Z}'$ (which is $H^2(Z_0 - Z_1, 2) \otimes_{\mathbb{Z}} \mathbb{Z}'$ for the toric flag built from $\{S_{v_1}, S_{v_2}, S_{v_3}\}$).

The symmetrized cup product $\langle e_1, e_2\rangle$ is invariant under the pullback action by $\pm I_2$ and $\pm\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, so it then follows that $4f_2(\partial[v_1, v_2, v_3]) = 0$ in $(\varinjlim_Z K_Z)_2$ because

$$(-z_1) \cup (-z_2) + (-z_1^{-1}) \cup (-z_2) + (-z_1) \cup (-z_2^{-1}) + (-z_1^{-1}) \cup (-z_2^{-1}) = 0.$$

Therefore, $f_2(\partial[v_1, v_2, v_3]) = 0$ in $(\varinjlim_Z K_Z)_2$ as desired. $\square$

Lemma 4.4.10. *Let $r \geq 2$, and let $\{u_1, \ldots, u_{r+1}\}$ be a set of primitive vectors in $\mathbb{Z}^r$ such that $\mathrm{Span}_{\mathbb{Z}}\{u_1, \ldots, u_{r+1}\}$ has rank $\leq r$. Let*

$$\partial[u_1, \ldots, u_{r+1}] := \sum_{i=1}^{r+1} (-1)^{i-1}[u_1, \ldots, \hat{v}_i, \ldots, u_{r+1}] \in C_r^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'.$$

If

$$f_r(\partial[u_1, \ldots, u_{r+1}]) = 0$$

in $(\varinjlim_Z K_Z)_r$ for $\mathbb{G}_m^r$ for all such sets of vectors, then

$$\begin{tikzcd} C_{r+1}^{line} \otimes_{\mathbb{Z}} \mathbb{Z}' \arrow[r] \arrow[d, "f_{r+1}"] & C_r^{line} \otimes_{\mathbb{Z}} \mathbb{Z}' \arrow[d, "f_r"] \\ C_{r+1}^{lim} \arrow[r] & C_r^{lim} \end{tikzcd}$$

commutes in dimension $r + 1$.

Proof. Pick $[v_1, \ldots, v_{r+1}] \in C_{r+1}^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$. If $\mathrm{Span}_{\mathbb{Z}}\{v_1, \ldots, v_{r+1}\}$ has rank $r + 1$, then the commutativity of this diagram follows formally, because boundary maps in motivic cohomology commute with pushforwards. If $\mathrm{Span}_{\mathbb{Z}}\{v_1, \ldots, v_{r+1}\}$ has rank $\leq r - 1$, then so does each subset that defines a symbol in the boundary expression $d_{r+1}([v_1, \ldots, v_{r+1}])$. By definition of the maps f_{r+1} and f_r, both $[v_1, \ldots, v_{r+1}]$ and $d_{r+1}([v_1, \ldots, v_{r+1}])$ map to 0, so certainly the diagram commutes. Therefore, the only interesting case is when $\mathrm{Span}_{\mathbb{Z}}\{v_1, \ldots, v_{r+1}\}$ has rank exactly r.

Now, we examine $f_r(d_{r+1}([v_1, \ldots, v_{r+1}]))$, and we wish to show that this is 0 in C_r^{lim}. The saturation of the lattice spanned by $\{v_1, \ldots, v_{r+1}\}$ and any rank r subset must be the same, so let $\mathcal{L} = \mathrm{Span}_{\mathbb{Z}}\{v_1, \ldots, v_{r+1}\}_{sat}$, and pick a basis $\{v_1', \ldots, v_r'\}$ of $\mathcal{L}$. We extend $\{v_1', \ldots, v_r'\}$ to a (positively oriented) $\mathbb{Z}$-basis $\{v_1', \ldots, v_r', y_{r+1}\}$ of $\mathbb{Z}^{r+1}$. It follows from the discussion after Definition 4.4.6 that

$$f_r(d_{r+1}([v_1,\ldots,v_{r+1}])) \in H^r\Big(S_{w_{r+1}} - \bigcup_{i=1}^{r}(S_{w_i}\cap S_{w_{r+1}}), r\Big),$$

where $w_1,\ldots,w_{r+1}$ are the columns of γ^{-1}, and $\gamma = \begin{pmatrix} -v_1'- \\ \vdots \\ -y_{r+1}- \end{pmatrix} \in \mathrm{SL}_{r+1}(\mathbb{Z})$.

Now, by definition we may write

$$\gamma^* f_r(d_{r+1}([v_1,\ldots,v_{r+1}])) = \frac{1}{2^r}\gamma^*\Big(\sum_{i=1}^{r+1}(-1)^{i-1}(A_i)_*\langle e_1,\ldots,e_r\rangle\Big),$$

where A_i is the matrix obtained by deleting the i-th row from $A = \begin{pmatrix} -v_1- \\ \vdots \\ -v_{r+1}- \end{pmatrix}$, and $(A_i)_*$ is interpreted to be 0 if $\mathrm{rank}(A_i) < r$. The point then is that because $\gamma^* = (\gamma^{-1})_*$ and the map f_r is Γ-equivariant, we find that

$$\gamma^* f_r(d_{r+1}([v_1,\ldots,v_{r+1}])) = \frac{1}{2^r}\sum_{i=1}^{r+1}(-1)^{i-1}(A_i\gamma^{-1})_*\langle e_1,\ldots,e_r\rangle.$$

The key observation here is that the last column of each matrix $A_i\gamma^{-1}$ is 0, by definition of γ. Let B be the matrix $A\gamma^{-1}$ with the last column (of all 0's) removed, and write $B = \begin{pmatrix} -u_1- \\ \vdots \\ -u_{r+1}- \end{pmatrix}$ for $u_i \in \mathbb{Z}^r$. Then algebraically, this means the expressions $(A_i\gamma^{-1})_*\langle e_1,\ldots,e_r\rangle \in H^r(S_{e_{r+1}} - \bigcup_{i=1}^r(S_{e_i}\cap S_{e_{r+1}}), r)$ and $(B_i)_*\langle e_1,\ldots,e_r\rangle \in (\varinjlim_Z K_Z)_r$ agree. (Note: the latter cup product $\langle e_1,\ldots,e_r\rangle$ is the standard symmetrized cup product in $\mathbb{G}_m^r$, so that $(B_i)_*\langle e_1,\ldots,e_r\rangle$ is valued in the direct limit complex for $\mathbb{G}_m^r$.)

However, we can certainly write $\frac{1}{2^r}\sum_{i=1}^{r+1}(-1)^{i-1}(B_i)_*\langle e_1,\ldots,e_r\rangle$ as $f_r(\partial[u_1,\ldots,u_{r+1}])$ (using the map f_r in dimension r), and by our assumption, this equals 0. Therefore, $\gamma^* f_r(d_{r+1}([v_1,\ldots,v_{r+1}])) = 0$, which certainly means $f_r(d_{r+1}([v_1,\ldots,v_{r+1}])) = 0$ in C_r^{lim} (for $\mathbb{G}_m^{r+1}$). This proves the commutativity of the desired square. $\square$

Lemma 4.4.11. *Suppose that*

$$\begin{tikzcd} C_{r+1}^{line}\otimes_{\mathbb{Z}}\mathbb{Z}' \arrow[r] \arrow[d, "f_{r+1}"] & C_r^{line}\otimes_{\mathbb{Z}}\mathbb{Z}' \arrow[d, "f_r"] \\ C_{r+1}^{lim} \arrow[r] & C_r^{lim} \end{tikzcd}$$

commutes, using the corresponding complexes for $\mathbb{G}_m^{r+1}$*. Then for any set* $\{v_1, \ldots, v_{r+2}\}$ *of primitive vectors in* $\mathbb{Z}^{r+1}$ *that span a submodule of rank* $\leq r+1$*,*

$$f_{r+1}(\partial[v_1, \ldots, v_{r+2}]) = 0$$

in $(\varinjlim_Z K_Z)_{r+1}$.

Proof. Since $d_{r+1}(\partial[v_1, \ldots, v_{r+2}]) = 0$ in $C_r^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$, from the commutativity it follows that $f_{r+1}(\partial[v_1, \ldots, v_{r+2}]) = 0$ in C_{r+1}^{lim}, which means that $f_{r+1}(\partial[v_1, \ldots, v_{r+2}]) \in (H^{r+1}(\mathbb{G}_m^{r+1}, r+1) \otimes_{\mathbb{Z}} \mathbb{Z}')^{(0)}$.

Therefore, there is $n \in \mathbb{Z}'$ such that $f_{r+1}(\partial[v_1, \ldots, v_{r+2}]) = n((-z_1) \cup (-z_2) \cup \ldots \cup (-z_{r+1})) \in H^{r+1}(\mathbb{G}_m^{r+1}, r+1) \otimes_{\mathbb{Z}} \mathbb{Z}'$. Let $\gamma_i = \mathrm{Diag}(1, \ldots, -1, \ldots, 1)$ be the diagonal matrix with -1 in the (i,i)th entry. As symbols in $C_{r+1} \otimes_{\mathbb{Z}} \mathbb{Z}'$ are invariant under the matrix actions of $\pm I_{r+1}$ and $\pm\gamma_i$, and the map f_{r+1} is Γ-equivariant, it follows that

$$2^{r+1} f_{r+1}(\partial[v_1, \ldots, v_{r+2}]) = n\Big(\sum_{\epsilon_i \in \{\pm 1\}} (-z_1^{\epsilon_1}) \cup \ldots \cup (-z_{r+1}^{\epsilon_{r+1}}) \Big) = 0$$

as desired. □

With these lemmas proven, We are now ready to prove Theorem 4.4.7.

Proof of Theorem 4.4.7*.* We begin as follows. Lemma 4.4.8 shows the theorem holds for $\mathbb{G}_m^2$, and Lemma 4.4.9 tells us the "right relations" hold true in dimension 2. Lemma 4.4.10 kicks in, so in diagram of the theorem for $\mathbb{G}_m^3$, we find that the leftmost square commutes. It then follows from Lemma 4.4.11 that the "right relations" hold inside the group $(\varinjlim_Z K_Z)_3$ that lives in the direct limit complex for $\mathbb{G}_m^3$. Inductively applying Lemmas 4.4.10 and 4.4.11, we see that the leftmost square *always* commutes, and the "right relations" *always* hold in $(\varinjlim_Z K_Z)_r$ inside the direct limit complex for $\mathbb{G}_m^r$ for all $r \geq 2$.

Now, we need to show that Theorem 4.4.7 holds true in $\mathbb{G}_m^r$ for $r \geq 2$. We've already shown the left most square always commutes, so we need to show all squares to the right commute as well, which we will do recursively.

We take a look at

$$\begin{tikzcd}
C_k^{line} \otimes_{\mathbb{Z}} \mathbb{Z}' \arrow[r] \arrow[d, "f_k"] & C_{k-1}^{line} \otimes_{\mathbb{Z}} \mathbb{Z}' \arrow[d, "f_{k-1}"] \\
C_k^{lim} \arrow[r] & C_{k-1}^{lim}
\end{tikzcd}$$

where $3 \leq k \leq r-1$ (noting that the squares for $k = 1, 2$ basically commute for free).

If we start with a tuple $[v_1, \ldots, v_k] \in C_k^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$, by the argument of Lemma 4.4.10 the only interesting case to check is when the tuple has rank exactly $k-1$. As usual, we write $\partial[v_1, \ldots, v_k] = \sum_{i=1}^{k} (-1)^i [v_1, \ldots, \hat{v}_i, \ldots, v_k] \in C_{k-1}^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$. We set $\mathcal{L} = \mathrm{Span}_{\mathbb{Z}}\{v_1, \ldots, v_k\}_{sat}$ and choose a basis $\{v_1', \ldots, v_{k-1}'\}$ of $\mathcal{L}$. Extending to a $\mathbb{Z}$-basis $\{v_1', \ldots, v_{k-1}', y_k, \ldots, y_r\}$ of $\mathbb{Z}^r$, we let $w_1, \ldots, w_r$ denote the columns of γ^{-1}, where $\gamma = \begin{pmatrix} -v_1'- \\ \vdots \\ -y_r- \end{pmatrix}$. Therefore, $f_{k-1}(\partial[v_1, \ldots, v_k]) \in H^{k-1}(S_{w_k} \cap \ldots \cap S_{w_r} - \bigcup_{i=1}^{k-1}(S_{w_i} \cap S_{w_k} \cap \ldots \cap S_{w_r}), k-1)$.

It then follows that

$$\gamma^* f_{k-1}(\partial[v_1, \ldots, v_k]) \in H^{k-1}\Big(T_{k,r} - \bigcup_{i=1}^{k-1}(S_{e_i} \cap T_{k,r}), k-1\Big)$$

and

$$\gamma^* f_{k-1}(\partial[v_1, \ldots, v_k]) = \frac{1}{2^{k-1}} \sum_{i=1}^{k} (-1)^i (A_i \gamma^{-1})_* \langle e_1, \ldots, e_{k-1} \rangle$$

where $A = \begin{pmatrix} -v_1- \\ \vdots \\ -v_k- \end{pmatrix}$.

As in Lemma 4.4.10, the point is that by definition of γ, the last $r-k+1$ columns of $A_i\gamma^{-1}$ are all 0. Therefore, algebraically, the expression $\gamma^* f_{k-1}(\partial[v_1, \ldots, v_k])$ agrees with that of $\frac{1}{2^{k-1}} \sum_{i=1}^{k} (-1)^i (B_i)_* \langle e_1, \ldots, e_{k-1} \rangle$ where B is the matrix $A\gamma^{-1}$ with the last $r-k+1$ columns (of 0's) removed (and again, this latter standard symmetrized cup product is the one for $\mathbb{G}_m^{k-1}$). Writing $B = \begin{pmatrix} -u_1- \\ \vdots \\ -u_k- \end{pmatrix}$, this expression is $f_{k-1}(\partial[u_1, \ldots, u_k])$ (using the map f_{k-1} for $C_{k-1}^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$ for $\mathbb{G}_m^{k-1}$), and therefore is 0 by what we said at the beginning of the proof. Therefore, the square commutes, and since r was arbitrary, we're done. □

Theorem 4.4.12. *The map*

$$\Theta : \mathrm{GL}_n(\mathbb{Z})^{n-1} \to \overline{K}_n \otimes_{\mathbb{Z}} \mathbb{Z}'$$

defined by

$$(\gamma_1, \ldots, \gamma_{n-1}) \mapsto (g_n \circ f_n)([e_1, \gamma_1 \cdot e_1, \ldots, (\gamma_1 \cdots \gamma_{n-1}) \cdot e_1]_{ext})$$

is an $(n-1)$*-cocycle representing the class of*

$$g_*(\delta_{C_\bullet^{line} \otimes_{\mathbb{Z}} \mathbb{Z}', C_\bullet^{lim}}(1)) = [\Theta]_{line} \in H^{n-1}(\Gamma, \overline{K}_n \otimes_{\mathbb{Z}} \mathbb{Z}'),$$

where the extended symbol $[e_1, \gamma_1 \cdot e_1, \ldots, (\gamma_1 \cdots \gamma_{n-1}) \cdot e_1]_{ext} \in C_n^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$ *is taken to mean* 0 *if the set* $\{e_1, \gamma_1 \cdot e_1, \ldots, (\gamma_1 \cdots \gamma_{n-1}) \cdot e_1\}$ *contains an antipodal pair of vectors, and is* $[e_1, \gamma_1 \cdot e_1, \ldots, (\gamma_1 \cdots \gamma_{n-1}) \cdot e_1]$ *otherwise.*

Proof. The element $1 \in C_0^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$ may be lifted to $[e_1] \in C_1^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$. Consider $(\gamma_1 - 1) \cdot [e_1]$, which has trivial boundary, and therefore lifts to some element of $C_2^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$. If $e_1, \gamma_1 \cdot e_1$ are not antipodal, then $[e_1, \gamma_1 \cdot e_1] \in C_2^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$ certainly works. Otherwise, they are antipodal, in which case we have $(\gamma_1 - 1) \cdot [e_1] = 0$, so 0 is a lift. Therefore, $[e_1, \gamma_1 \cdot e_1]_{ext}$ is a lift of $(\gamma_1 - 1) \cdot [e_1]$.

Now, for any $k \geq 1$ consider the expression

$$\sum_{i=1}^{k+1} (-1)^i [e_1, \gamma_1 \cdot e_1, \ldots, \widehat{(\gamma_1, \ldots, \gamma_i) \cdot e_1}, \ldots, (\gamma_1 \cdots \gamma_{k+1}) \cdot e_1]_{ext}$$

If $\{e_1, \ldots, (\gamma_1 \cdots \gamma_{k+1}) \cdot e_1\}$ does not contain an antipodal pair, then the expression above is just the boundary of $[e_1, \ldots, (\gamma_1 \cdots \gamma_{k+1}) \cdot e_1] \in C_{k+1}^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$.

Otherwise, some pair is antipodal. Without loss of generality, we assume this pair is $e_1, \gamma \cdot e_1$. It's then rather clear that

$$\sum_{i=1}^{k+1} (-1)^i [e_1, \gamma_1 \cdot e_1, \ldots, \widehat{(\gamma_1, \ldots, \gamma_i) \cdot e_1}, \ldots, (\gamma_1 \cdots \gamma_{k+1}) \cdot e_1]_{ext} = 0,$$

so 0 is a lift. Either way, $[e_1, \gamma_1 \cdot e_1, \ldots, (\gamma_1 \cdots \gamma_{k+1}) \cdot e_1]_{ext}$ is a lift.

By unwinding the definition of the composition of connecting maps, it then follows that

$$(\gamma_1, \ldots, \gamma_{n-1}) \mapsto [e_1, \gamma_1 \cdot e_1, \ldots, (\gamma_1 \cdots \gamma_{n-1}) \cdot e_1]_{ext}$$

is an $(n-1)$ cocycle representing the class of $(\delta_{n-2} \circ \ldots \circ \delta_0)(1) \in H^{n-1}(\Gamma, \ker(d_n))$. Applying f_n yields a representative of $\delta_{C_\bullet^{line} \otimes_{\mathbb{Z}} \mathbb{Z}', C_\bullet^{lim}}(1) \in H^{n-1}(\Gamma, C_n^{lim})$, and applying g_n proves the theorem. $\square$

Remark 4.4.13. The cocycle Θ may be lifted to a cocycle valued in $K_n \otimes_{\mathbb{Z}} \mathbb{Z}'$, and not just $\overline{K}_n \otimes_{\mathbb{Z}} \mathbb{Z}'$. This is because for any $[v_1, \dots, v_n] \in C_n^{line} \otimes_{\mathbb{Z}} \mathbb{Z}'$, the value of $f_n([v_1, \dots, v_n]) \in C_n^{lim}$ is fixed by the actions of $\pm I_n$ and $\pm\gamma_i$ for $\gamma_i = \mathrm{Diag}(1, \dots, -1, \dots, 1)$, where the -1 is in the (i,i)th coordinate, because of the definition of the symmetrized cup product $\langle e_1, \dots, e_n \rangle$. As $(H^n(\mathbb{G}_m^n, n) \otimes_{\mathbb{Z}} \mathbb{Z}')^0 = \mathbb{Z}'\big((-z_1) \cup \dots \cup (-z_n)\big)$, summing up the pullback actions over all these matrices on $\big((-z_1) \cup \dots \cup (-z_n)\big)$, we obtain

$$\sum_{\epsilon_i \in \{\pm 1\}} (-z_1^{\epsilon_1}) \cup \dots \cup (-z_n^{\epsilon_n}) = 0,$$

so that $(H^n(\mathbb{G}_m^n, n) \otimes_{\mathbb{Z}} \mathbb{Z}')^0$ is killed by the sum of pullbacks. Since the sum of pullbacks on $f_n([v_1, \dots, v_n])$ is the same as $2^n f_n([v_1, \dots, v_n])$, we may view $2^n f_n([v_1, \dots, v_n]) \in (\varinjlim_Z K_Z)_n$, and not just C_n^{lim}, by choosing a section of the quotient map $(\varinjlim_Z K_Z)_n \to C_n^{lim}$, and the resulting cochain is a cocycle valued in this group. Dividing by 2^n, we have $(g_n \circ f_n)([v_1, \dots, v_n]) \in K_n \otimes_{\mathbb{Z}} \mathbb{Z}'$, so Θ may be thought of as being valued in $K_n \otimes_{\mathbb{Z}} \mathbb{Z}'$.

4.5 Hecke operators

We would now like to make sense of actions of Hecke operators on the cocycle class $[\Theta]_{line} \in H^{n-1}(\Gamma, K_n \otimes_{\mathbb{Z}} \mathbb{Z}')$. All of the theory here is explicitly worked out in [22], albeit with right actions on cohomology groups instead of left actions. We write only the essentials and refer the reader there for a more complete treatment.

Let G be a group and let $\Gamma \leq G$ be a subgroup. The *commensurator* $\tilde{\Gamma}$ of Γ is defined by

$$\tilde{\Gamma} = \{g \in G : g^{-1}\Gamma g \sim \Gamma\},$$

where the notation $g^{-1}\Gamma g \sim \Gamma$ means the two groups are commensurable, i.e. $[G : \Gamma \cap g^{-1}\Gamma g] < \infty$. More generally, we say that subgroups $\Gamma, \Gamma' \leq G$ are commensurable in G if

$$[\Gamma : \Gamma \cap \Gamma'] < \infty, \quad [\Gamma' : \Gamma \cap \Gamma'] < \infty.$$

Let Δ denote a subsemigroup of G, and let $\mathcal{C}(\Delta)$ denote the set of subgroups of G such that

$$\Gamma \subset \Delta \subset \tilde{\Gamma}$$

for all $\Gamma \in \mathcal{C}(\Delta)$.

For $\Gamma \in \mathcal{C}(\Delta)$, we let $\mathcal{H}(\Gamma, \Delta)$ denote the free $\mathbb{Z}$-module generated by double cosets $\Gamma\alpha\Gamma$ with $\alpha \in \Delta$. There is a multiplicative structure on $\mathcal{H}(\Gamma, \Delta)$ that turns it into a ring with multiplicative identity [16]. We call the ring $\mathcal{H}(\Gamma, \Delta)$ the *Hecke algebra* for Γ.

Given a double coset $\Gamma g\Gamma \in \mathcal{H}(\Gamma, \Delta)$, we may decompose the double coset as a finite union

$$\Gamma g \Gamma = \bigsqcup_{t=1}^{v} g_i \Gamma$$

as a union of left cosets. For a $\mathbb{Z}[\Delta]$-module M, there is an action of this double coset on (inhomogeneous) cochains as follows, which is worked out in [15]. Given

$$f : \Gamma^k \to M \in C^k(\Gamma, M),$$

we define the *Hecke operator* of $\Gamma g\Gamma$ on f by

$$(T(g)f)(\gamma) = \sum_{t=1}^{v} g_{\sigma(t)} f(\mu_t),$$

where for $\gamma = (\gamma_1, \dots, \gamma_k) \in \Gamma^k$, the elements $\sigma \in S_v$ and $\mu_t \in \Gamma^k$ for $1 \leq t \leq v$ are defined as follows: recursively setting $h_t^{(k)} = g_t$ and

$$\gamma_w h_t^{(w)} = h_t^{(w-1)} \mu_{t,w}$$

with $\mu_{t,w} \in \Gamma$ and $h_r^{(w)} \in \{g_1, \dots, g_v\}$ for $1 \leq w \leq k$, we take $\mu_t = (\mu_{t,1}, \dots, \mu_{t,k})$ and let $\sigma \in S_v$ be the unique permutation such that $g_{\sigma(t)} = h_t^{(0)}$ for each $1 \leq i \leq k$.

In general, the cochain $T(g)f$ depends on the choice of representatives in the decomposition of the coset $\Gamma g \Gamma$. However, [22] shows that such a choice of representatives does not matter for producing a well-defined action on group cohomology:

Proposition 4.5.1. *The operation defined above on cochains induces a left action of Hecke operators on the cohomology groups $H^k(\Gamma, M)$ independent of the choice of representatives of the double coset $\Gamma g \Gamma$. Furthermore, $T(g)$ provides a morphism of δ-functors.*

Proof. This is worked out in [15]. □

As a consequence of the above, it follows that if

$$0 \to L \to M \to N \to 0$$

is a short exact sequence of Γ-modules, then we must have

$$(\Gamma g \Gamma) \circ \delta_k = \delta_k \circ (\Gamma g \Gamma)$$

as morphisms

$$H^k(\Gamma, N) \to H^{k+1}(\Gamma, L)$$

via the description of the connecting morphism δ_k given earlier.

For our purposes, we take $G = \mathrm{GL}_n(\mathbb{Q})$, $\Delta = M_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$, and $\Gamma = \mathrm{GL}_n(\mathbb{Z})$. Now, fix a prime ℓ. For $i = 1, \dots, n-1$, we set

$$g^{(i)} = \mathrm{Diag}(\ell, \ell, \dots, \ell, 1, \dots, 1),$$

the diagonal matrix consisting of i ℓ's on the diagonal. We consider the Hecke operators $T_\ell^{(i)} := T(g^{(i)})$ for the double coset decomposition

$$\Gamma g^{(i)}\Gamma = \bigsqcup_{j=0}^{v_i} g_j^{(i)}\Gamma.$$

Explicit choices of representatives $g_j^{(i)}$ for the double coset decomposition are known, and can be found, for example, in [1, Lemma 2.18]. Following their convention, a complete set of representatives for $T_\ell^{(i)}$ consists of all matrices $A \in M_n(\mathbb{Z})$ such that:

- A is upper triangular and $\det(A) = \ell^i$.
- All entries on the diagonal are either 1 or ℓ.
- If a diagonal entry is ℓ, then all entries in that column are 0.
- Non-zero entries can only appear in rows that contain an ℓ, and the possible non-zero values are $1, 2, \ldots, \ell - 1$.

The number of representatives v_i is given by $\binom{n}{i}_\ell$, the number of i-dimensional subspaces of $(\mathbb{Z}/\ell\mathbb{Z})^n$.

Via the pullback action of Δ, we may also define a corresponding endomorphism of the Gersten complex $K_\bullet \otimes_\mathbb{Z} \mathbb{Z}'$ by

$$T_\ell^{(i)} = \sum_{j=0}^{v_i} (g_j^{(i)})^* : K_\bullet \otimes_\mathbb{Z} \mathbb{Z}' \to K_\bullet \otimes_\mathbb{Z} \mathbb{Z}'.$$

This endomorphism certainly depends on the choice of coset representatives used and therefore is primarily used a computational aid. However, as mentioned earlier, such choices do not matter at the level of cohomology. We are now ready to prove the following theorem:

Theorem 4.5.2. *Let ℓ be a prime. The cocycle $[\Theta]_{line} \in H^{n-1}(\Gamma, K_n \otimes_\mathbb{Z} \mathbb{Z}')$ satisfies*

$$T_\ell^{(i)}[\Theta]_{line} = \left(\binom{n-1}{i-i}_\ell [\ell]^* + \binom{n}{i}_\ell - \binom{n-1}{i-1}_\ell \right) [\Theta]_{line},$$

where $[\ell] = Diag(\ell, \ldots, \ell)$, and $\binom{n}{k}_\ell$ denotes the ℓ-binomial coefficient. That is to say, $[\Theta]_{line} \in H^{n-1}(\Gamma, K_n \otimes_\mathbb{Z} \mathbb{Z}')$ is Eisenstein.

Proof. By definition, the cocycle class $[\Theta]_{line}$ is constructed via pushing a forward composition of connecting morphisms under morphisms of complexes of $\mathbb{Z}'[\Gamma]$-modules. It then follows that the class $[\Theta]_{line}$ could have also been obtained by starting with $e \in H^0(\Gamma, K_0 \otimes_{\mathbb{Z}} \mathbb{Z}')$ and performing the lifting process by taking lifts to be the images of the lifts up top under the morphism of complexes $C_\bullet^{line} \otimes_{\mathbb{Z}} \mathbb{Z}' \to K_\bullet \otimes_{\mathbb{Z}} \mathbb{Z}'$. Therefore, by the earlier remark about connecting maps, it suffices to compute the action of $T_\ell^{(i)}$ on $e \in K_0 \otimes_{\mathbb{Z}} \mathbb{Z}'$.

This can be done by reducing to a linear algebra problem. For each $0 \leq j \leq c_i^n$, the pullback $(g_j^{(i)})^* e$ is given by $\ker(g_j^{(i)})$, which is a cyclic subgroup of μ_ℓ^n. Now, the claim is that different integers j, j' correspond to distinct kernels. To see this, choose two coset representatives $g_j^{(i)}$ and $g_{j'}^{(i)}$. If these matrices have diagonal ℓ's in differing locations, then it's rather clear from their definition that the kernels must be different. Therefore, we may assume that they have diagonal ℓ's in exactly the same places, and have the same "shape". From the restriction on the non-zero entries, it's also rather clear the kernels are distinct if an entry differs somewhere.

Therefore, we want to compute the formal sum $\sum_{j=0}^{v_i} \ker(g_j^{(i)})$ inside μ_ℓ^n. This is now a counting problem: we need to know how many times each non-identity element of μ_ℓ^n shows up in this sum. First, note that each kernel is a rank i subgroup of μ_ℓ^n. The number of rank i subgroups of μ_ℓ^n that do *not* contain $x \neq e$ is given by $\ell^i \binom{n-1}{i}_\ell$, by simply counting the number of possible ways to choose a basis for a rank i subgroup of μ_ℓ^n that do not contain any multiple of x. This means there are $\binom{n}{i}_\ell - \ell^i \binom{n-1}{i}_\ell = \binom{n-1}{i-1}_\ell$ such subgroups that *do* contain x. Adding $\binom{n-1}{i-1}_\ell$ identity elements, we get $\binom{n-1}{i-1}_\ell$ copies of μ_ℓ^n. There are then $\binom{n}{i}_\ell - \binom{n-1}{i-1}_\ell$ leftover copies of the identity. We're done after noticing that $[\ell]^* e = \mu_\ell^n$. □

We would like to end with the following remark. In Section 4.2 of [25], the authors are able to specialize their cocycle at the N-torsion point

$$s : \mathrm{Spec}(\mathbb{Q}(\mu_N)) \to \mathbb{G}_m^2$$

with value $(1, \zeta_N) \in \mathbb{G}_m(\mathbb{Q}(\mu_N))^2$ to obtain a cocycle

$$\Theta_N : \tilde{\Gamma}_0(N) \to K_2(\mathbb{Q}(\mu_N))/\langle\{-1, -\zeta_N\}\rangle,$$

where

$$\tilde{\Gamma}_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) : N \mid c \right\}.$$

Now, Bass and Tate [3] proved the following:

Proposition 4.5.3. *Let F be a number field and let r_1 denote the number of real embeddings of F. Then for $n \geq 3$, there is an isomorphism*

$$K_n^M(F) \cong (\mathbb{Z}/2\mathbb{Z})^{r_1}.$$

In particular, this shows for $n \geq 3$ and $N \geq 3$ that $K_n^M(\mathbb{Q}(\mu_N)) = 0$, and therefore the same idea would not prove to be fruitful.

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