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A power spectrum for the geomagnetic dipole moment

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ABSTRACT

We use stochastic models for fluctuations in the dipole moment to derive a geomagnetic power spectrum. The theoretical spectrum is represented as a function of frequency f in the general form Af^{-n} with smooth transitions between changes in the exponent n . A flat spectrum ($n = 0$) at low frequencies changes to $n = 2$ at intermediate frequencies. We attribute the transition frequency to the decay time of dipole fluctuations. Accounting for correlated noise in the stochastic model introduces another transition from $n = 2$ to 4, where the transition frequency is set by the correlation time of the noise. Numerical geodynamo models suggest that the correlation time is less than the convective overturn time and may be related to the lifetime of helical eddies. Applying these results to paleomagnetic estimates of the power spectrum yields a correlation time of 100 to 200 yrs. Evidence for a transition between $n = 0$ and $n = 2$ in paleomagnetic power spectra is interpreted as a constraint on the electrical conductivity of the core. Additional correlation times in the noise model are expected to produce a transition to $n = 6$, possibly associated with diffusion of the dipole field through a magnetic boundary layer at the top of the core.

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1. Introduction

Fluctuations in the geomagnetic field are observed over a broad range of time scales (Courillot and LeMouél, 1988). Long-term trends in the mean reversal rate (Johnson et al., 1995) are observed with periods of 10^8 yrs or more. Such low-frequency phenomena are commonly attributed to slow convective processes in the Earth's mantle (e.g. Biggin et al., 2012). Dynamo processes are more naturally associated with higher frequency variability. A representative fluid velocity can sweep magnetic fields across the core in 10^2 yrs (Holme and Olsen, 2006), whereas the dipole decay time is roughly 10^4 yrs (Gubbins and Roberts, 1987). Waves with periods of 10^1 to 10^3 yrs can also contribute to the variability in the geomagnetic field (Gillet et al., 2010; Buffett, 2014; Canet et al., 2014).

A power spectrum for the geomagnetic field quantifies the variability and offers insights into the underlying processes (Constable and Johnson, 2005). An important source of information is obtained from measurements of relative paleointensity in marine sediments (Valet, 2003). Records are stacked and calibrated using independent estimates of absolute paleointensity to construct models for the virtual axial dipole moment (VADM) over the past

two million years (Valet et al., 2005; Ziegler et al., 2011). Higher resolution records from a combination of archeomagnetic and lake sediment data have also been used to construct a low-degree spherical harmonic expansion of the geomagnetic field over the past 10 kyr (Korte et al., 2011; Korte and Constable, 2011). Even higher resolution records are available from historical observations (Jackson et al., 2000). Taken together these models provide a broadband estimate of fluctuations in the axial dipole moment.

A quantitative interpretation of the observed power spectrum requires physical models for the underlying processes. Numerical geodynamo models offer insights (Sakuraba and Hamano, 2007; Olson et al., 2012), but concerns about unrealistic model parameters invariably complicate a direct comparison with observations. We propose an alternative approach that uses stochastic models for the axial dipole moment to construct a theoretical power spectrum. The parameters of a stochastic model can be recovered from a realization of the process, so it is possible to use paleomagnetic estimates of the VADM to guide the construction of a power spectrum. The same procedures can also be applied to the output of geodynamo models. By comparing the theoretical power spectrum from the stochastic model with that computed directly from the geodynamo model, we build confidence in the approach and attach physical significance to features in the resulting power spectrum. Extending these ideas to paleomagnetic estimates of the VADM provides a physical basis for interpreting the observed power spectrum.

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2. Stochastic description of dipole fluctuations

Stochastic models are often applied to problems where only a small subset of variables are observable. The approach is well suited to study of the geomagnetic dipole moment because most of the convective flow that sustains the dipole field is not detected at the surface. A stochastic model separates the evolution of the dipole field into two parts, based on the inherent time scales of the processes involved. One part describes the slow adjustment of the dipole field toward a time-averaged state, while the second part represents the effect of short-period convective fluctuations in the core. These short-period fluctuations are treated as a random process in the time evolution. We denote the axial dipole moment by $x(t)$ and describe its time evolution using a standard Langevin model (Van Kampen, 2007)

$$\frac{dx}{dt} = v(x) + \sqrt{D(x)}\Gamma(t) \quad (1)$$

where the drift term, $v(x)$, describes the slow evolution and the noise term, $D(x)$, defines the amplitude of random fluctuations. The time dependence of the random process, $\Gamma(t)$, is assumed to be Gaussian with a vanishing time average

$$\langle \Gamma(t) \rangle = 0. \quad (2)$$

We also assume that the correlation time of the noise source is short compared with the sampling of $x(t)$, although this assumption is relaxed in later sections. In the limit of short correlation times the autocovariance function of $\Gamma(t)$ is approximated by a Dirac delta function,

$$\langle \Gamma(t_1)\Gamma(t_2) \rangle = 2\delta(t_1 - t_2) \quad (3)$$

where the factor of two is a common convention (e.g. Risken, 1989).

Estimates for $v(x)$ and $D(x)$ can be extracted from a realization of the stochastic process. The drift term is given by

$$v(x) = \frac{\langle x(t + \Delta t) - x(t) \rangle}{\Delta t} \quad (4)$$

and the noise term is approximated by

$$D(x) = \frac{\langle [x(t + \Delta t) - x(t)]^2 \rangle}{2\Delta t} \quad (5)$$

where time averages are assigned according to the position of $x(t)$ in discrete bins that span the range of dipole moments (see Buffett et al., 2014 for details). The time increment, Δt , is chosen to be long enough to ensure that the noise source, $\Gamma(t)$, is adequately approximated by an uncorrelated random process. Fig. 1 shows an example using the output of a numerical geodynamo model. The drift term can be represented by

$$v(x) = -\gamma(x - \langle x \rangle) \quad (6)$$

where $\langle x \rangle$ denotes the time-averaged moment and γ is a constant that characterizes the time scale for slow adjustments of the dipole (Buffett et al., 2014). A similar representation for $v(x)$ was recovered from paleomagnetic estimates (e.g. Buffett et al., 2013). Indeed very similar values for the constant, $\gamma \approx 34 \text{ Myr}^{-1}$, were reported for the SINT-2000 model of Valet et al. (2005) and the PADM2M model of Ziegler et al. (2011). By comparison, the noise term in Fig. 1 reveals a weak dependence on x . It suffices for our purposes to adopt the approximation $D(x) = D_{eq}$, where D_{eq} denotes the value of the noise term at $x = \langle x \rangle$. We explore the validity of this approximation below.

Fig. 1. Drift and noise terms as a function of dipole moment x from the geodynamo model *Calypso* (see text). The drift term is well approximated by a linear function of x , whereas the noise term is nearly constant.

2.1. Power spectrum for the dipole moment

A power spectrum is computed for deviations $\epsilon(t) = x(t) - \langle x \rangle$ around the time average. The equation for $\epsilon(t)$ becomes

$$\frac{d\epsilon}{dt} = -\gamma\epsilon + \sqrt{D_{eq}}\Gamma(t) \quad (7)$$

and solutions are obtained using Fourier transforms. We define the Fourier transform of $\epsilon(t)$ in terms of frequency, f , as

$$\epsilon(f) = \int_{-\infty}^{\infty} \epsilon(t) e^{-2\pi i f t} dt \quad (8)$$

An analogous expression defines $\Gamma(f)$. Taking the Fourier transform of (7) yields

$$\epsilon(f) = \frac{\sqrt{D_{eq}}}{(\gamma + i2\pi f)} \Gamma(f) \quad (9)$$

Consequently, the power spectrum of the dipole moment becomes

$$S_{\epsilon}(f) \equiv \epsilon(f)\epsilon(f)^* = \frac{D_{eq}}{(\gamma^2 + 4\pi^2 f^2)} S_{\Gamma}(f) \quad (10)$$

where the power spectrum of the noise

$$S_{\Gamma}(f) \equiv \Gamma(f)\Gamma(f)^* = 2 \quad (11)$$

is obtained by taking the Fourier transform of the autocovariance function for $\Gamma(t)$ in (3). Thus the predicted spectrum in (10) is nearly constant at low frequency and decreases at higher frequency as f^{-2} . A qualitatively similar power spectrum was proposed by Barton (1982) on the basis of paleomagnetic directions. More quantitative agreement is found with the study of Olson et al. (2012), which analyzed the dipole moment from a suite of numerical geodynamo models. The typical spectrum in their study was flat at low frequency and decreased as $f^{-1.8 \pm 0.1}$ at higher frequencies. Similar results were obtained in the study of Davies and Constable (2014), which recovered a frequency exponent of $n = 2.1 \pm 0.2$.

On the basis of the stochastic model, the transition in the power spectrum is approximated by $f \approx \gamma/2\pi$, where γ^{-1} represents the time scale for slow adjustments in the dipole moment.

Integrating the spectrum over frequency yields the total variance in the dipole moment

$$\sigma^2 = \int_{-\infty}^{\infty} S_{\epsilon}(f) df = \frac{D_{eq}}{\gamma} \quad (12)$$

This expression illustrates that $S_{\epsilon}(f)$ is defined for both positive and negative frequencies. Since the spectrum is symmetric about $f = 0$, we multiply $S_{\epsilon}(f)$ by a factor of two when making comparisons to numerical calculations of one-sided spectra (i.e. $0 < f < \infty$).

2.2. Extensions for correlated noise

The assumption of uncorrelated noise in the stochastic model is reasonable when the dipole moment is sampled at large time steps, corresponding to low frequencies. However, a correlation in the noise affects the power spectrum at high frequencies. We explore the influence of correlated noise by introducing a finite correlation time, τ , into the autocovariance function for $\Gamma(t)$. We start by replacing the white noise assumption in Eq. (3) with

$$\langle \Gamma(t_1) \Gamma(t_2) \rangle = \frac{1}{\tau} e^{-|t_1 - t_2|/\tau} \quad (13)$$

Taking the Fourier transform of the autocovariance function gives

$$S_{\Gamma}(f) = \frac{2\alpha^2}{(\alpha^2 + 4\pi^2 f^2)} \quad (14)$$

where $\alpha = \tau^{-1}$. Using (14) in (10) gives

$$S_{\epsilon}^c(f) \equiv \epsilon(f)\epsilon(f)^* = \frac{2D_{eq}\alpha^2}{(\gamma^2 + 4\pi^2 f^2)(\alpha^2 + 4\pi^2 f^2)} \quad (15)$$

where the superscript c signifies the inclusion of correlated noise. When τ is small (or α is large) the spectrum at low frequency is unaltered by the addition of correlated noise. We obtain a nearly flat spectrum with an f^0 dependence at low frequency and an f^{-2} dependence at intermediate frequencies. At higher frequencies ($f > \alpha/2\pi$) the spectrum exhibits an f^{-4} dependence. A similar high frequency behavior was reported by Olson et al. (2012), where the exponent was $n = 4 \pm 0.2$. The present study suggests that the transition to high-frequency behavior is set by the correlation time of the noise in the stochastic model. It is interesting to note that a more complicated noise model with two distinct correlation times would give rise to another transition in the spectrum. A steep f^{-6} dependence would be expected at very high frequencies (e.g. Davies and Constable, 2014).

3. Comparison with geodynamo simulations

Geodynamo simulations can be used to test the validity of stochastic models. One example was reported in the study of Buffett et al. (2014), which constructed a stochastic model using the axial dipole moment from the geodynamo model *Calypso* (Matsui et al., 2014). The numerical solution was specified by a set of control parameters, which included the Ekman number, $E = 5 \times 10^{-5}$, the Prandtl number, $Pr = 1$ and the magnetic Prandtl number, $Pm = 0.5$. The vigor of convection was specified by a modified Rayleigh number, $Ra = 1400$, which was roughly $17.5\times$ the critical value for the onset of convection without a magnetic field. A time-dependent, dipolar field was produced outside the core, although the polarity of the field did not reverse over the simulation. A somewhat larger modified Rayleigh number ($Ra = 1600$) succeeded in producing a reversing solution, although we exclude this solution from the present analysis because the magnetic field

did not have a strong dipole. However, we include a second numerical solution at lower Rayleigh number ($Ra = 600$) to explore the consequence of changing the vigor of convection.

Time in the numerical simulation is scaled by the viscous diffusion time, L^2/ν , where L is the thickness of the fluid shell and ν is the kinematic viscosity. We convert time in our simulation to physical units by adopting a realistic value for the magnetic diffusivity η . This choice gives a reasonable estimate for the dipole decay time. Setting $\eta = 1.6 \text{ m}^2 \text{ s}^{-1}$ and $L = 2.259 \times 10^6 \text{ m}$ gives $\nu = Pm \eta = 0.8 \text{ m}^2 \text{ s}^{-1}$ and a viscous diffusion time of 203 kyr. The corresponding dipole decay time $\tau_d = 24.3 \text{ kyr}$. Estimates for the convective overturn time, $\tau_c = L/V_{rms}$, are inferred from the time-averaged root-mean-square velocity. We obtain $\tau_c = 1.1 \text{ kyr}$ from the solution at $Ra = 1400$ and $\tau_c = 2.4 \text{ kyr}$ from the solution at $Ra = 600$. Finally, we quote the magnetic Reynolds number

$$Rm = \frac{V_{rms}L}{\eta} \quad (16)$$

for these two calculations. We get $Rm = 90$ and 42 for the higher and lower values of Ra , respectively. Higher values for Rm are often achieved in numerical calculations when $Pm > 1$, but this choice underestimates the influence of magnetic diffusion relative to viscous and thermal diffusion.

Estimates for the drift and noise terms from the numerical simulation at $Ra = 1400$ are shown in Fig. 1. The drift term is compatible with the linear representation in (6), where the slow time scale is set by $\gamma = 106 \text{ Myr}^{-1}$. The noise term, $D(x)$, decreases weakly with x , suggesting that the amplitude of convective fluctuations become smaller when the dipole field is strong. A nominal value for $D(x)$ is evaluated at $x = \langle x \rangle$, giving $D_{eq} = 33 \times 10^{44} \text{ A}^2 \text{ m}^4 \text{ Myr}^{-1}$. The predicted variance from (12) agrees well with the variance of the input time-series. (Specifically, we obtain $\sigma = \sqrt{D_{eq}/\gamma} = 0.56 \times 10^{22} \text{ Am}^2$ from the stochastic model versus $\sigma = 0.59 \times 10^{22} \text{ Am}^2$ from the input time-series.) Much of the disagreement between the predicted and actual variance of the time series is probably due to treating $D(x)$ as a constant, although the consequences for the power spectrum are small (see below). Analysis of the numerical solution at $Ra = 600$ yields qualitatively similar results, but we obtain different numerical values: $\gamma = 45 \text{ Myr}^{-1}$ and $D_{eq} = 4.7 \times 10^{44} \text{ A}^2 \text{ m}^4 \text{ Myr}^{-1}$. Thus the slow time scale γ^{-1} is longer and the amplitude of the noise term is smaller when the vigor of convection is weaker.

The theoretical power spectrum, $S_{\epsilon}(f)$, is set by the values for γ and D_{eq} in the stochastic model. When we allow for correlation in noise source, the predicted power spectrum is modified at high frequencies according to $S_{\epsilon}^c(f)$ in (15). Predictions for $S_{\epsilon}(f)$ and $S_{\epsilon}^c(f)$ are tested by evaluating the power spectrum directly from the geodynamo model (see Fig. 2). A multi-taper method (e.g. Percival and Walden, 1993) is used to compute the power spectrum, based on an implementation in Matlab (function pmtm). The time-averaged dipole moment is subtracted from the time series to define the deviation, $\epsilon(t)$, and the frequency resolution of the spectrum is controlled by the number of points in the fast-Fourier transform (fft). For this specific calculation we set the number of points in the fft equal to the length of the time series, but in some subsequent calculations we restrict the frequency resolution by limiting the number of points in the fft. Because the low-frequency part of the theoretical spectrum is specified by the values for γ and D_{eq} , the good agreement with the computed spectrum in Fig. 2 means that the stochastic model successfully captures the variability of the geodynamo model at a sampling interval greater than Δt .

Deviations between $S_{\epsilon}(f)$ and the computed power spectrum at high frequency indicate the presence of correlated noise. Setting a correlation time of $\tau = 333 \text{ yrs}$ ($\alpha = 3000 \text{ Myr}^{-1}$) in $S_{\epsilon}^c(f)$ gives

Fig. 2. Power spectrum for the axial dipole moment from *Calypso* model (green), compared with theoretical spectra $S_\epsilon(f)$ and $S_\epsilon^C(f)$. A correlation time of $\tau = 333$ yrs is used in $S_\epsilon^C(f)$ to fit the high-frequency part of the geodynamo model. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

a much better fit at high frequency without altering the agreement at low frequency. The effective correlation time for the noise term is roughly 1/3 of the convective overturn time. A somewhat longer correlation time ($\tau \approx 700$ yrs) is needed to reproduce the power spectrum from the numerical solution at $Rm = 600$. In this case the overturn time is roughly 2400 yrs, so the relative correlation time in both calculations is consistent with 1/3 of the overturn time.

4. Interpretation of the transition frequencies

Stochastic models with correlated noise can account for the general form of power spectra in geodynamo models (Olson et al., 2012; Davies and Constable, 2014). The frequency transition from f^0 to f^{-2} dependence is predicted to occur at $f \approx \gamma/2\pi$. This first transition frequency can be understood in terms of the autocovariance function for $\epsilon(t)$, which is defined by

$$C_\epsilon(\delta t) = \langle \epsilon(t)\epsilon(t + \delta t) \rangle \quad (17)$$

where δt represents an arbitrary time lag or lead. The long-term (large t) autocovariance function for the stochastic model is obtained from $S_\epsilon(f)$ by taking the inverse Fourier transform to yield

$$C_\epsilon(\delta t) = \sigma^2 \exp(-\gamma|\delta t|) \quad (18)$$

where σ^2 is the variance from (12). We see that γ^{-1} defines the correlation time for fluctuations $x(t) - \langle x \rangle$. If $x(t)$ is sampled at time intervals large compared with γ^{-1} then the resulting time series can be approximated as uncorrelated random noise about the time-averaged dipole moment. The resulting power spectrum at sufficiently low frequency should correspond to white noise with no dependence on f . Thus the transition from a flat spectrum is set by the longest correlation time for fluctuations in $x(t)$.

Buffett et al. (2014) attributed this correlation time to the dipole decay time. Quantitative estimates for γ from the *Calypso* models are found to depend on the vigor of convective. For example, the geodynamo solution at $Ra = 1400$ yields $\gamma = 106 \text{ Myr}^{-1}$, which implies a correlation time of 9.4 kyr. Detailed analysis of this geodynamo model shows that the spatial structure of a typical dipole fluctuation can be described by the first few decay modes.

Quantitative agreement with γ^{-1} requires the average dipole fluctuation to be comprised of the first and second decay models with roughly comparable amplitudes. A longer correlation time, $\gamma^{-1} = 22 \text{ kyr}$, is obtained when the Rayleigh number decreases to $Ra = 600$. This result suggests that a typical dipole fluctuation has a simpler spatial structure. Recall that the first decay mode has a decay time of 24.3 kyr for our choice of η , so the fluctuations at $Ra = 600$ must be comprised primarily of the first decay mode.

The preceding discussion suggests that the longest correlation time in the geodynamo model is set by the dipole decay time. More vigorous convection might make the dipole fluctuations spatially complex, which shortens the associated decay time, but we do not expect the correlation time to exceed the dipole decay time for the first (slowest) mode. This expectation is supported by several power spectra from the study of Davies and Constable (2014). However a few spectra from both Davies and Constable (2014) and Olson et al. (2012) have transition frequencies that imply a correlation time longer than the dipole decay time. This suggests that processes other than dipole decay can also play a role.

To understand this result we compare the autocorrelation function

$$R_\epsilon(\delta t) = \frac{C_\epsilon(\delta t)}{\sigma^2} \quad (19)$$

from one of the *Calypso* models with one of the numerical models from the study of Olson et al. (2012). We consider case c from Olson et al. (2012), which was associated with a magnetic Reynolds number of $Rm = 375$. Other relevant parameters for this solution include the Ekman number $E = 3 \times 10^{-4}$ and the magnetic Prandtl number $Pm = 3$. We compare this solution with the *Calypso* model at $Ra = 1400$, corresponding to $Rm = 90$. Both solutions are scaled to physical quantities in exactly the same way. We adopt a realistic value for η and determine the viscosity and viscous time scale using the relevant value for Pm . The *Calypso* model has $Pm = 0.5$, which enhances magnetic diffusion and accounts for the lower value of Rm .

Fig. 3a shows the computed and theoretical autocorrelation function for the *Calypso* model. The theoretical autocorrelation function has a wider central peak than the direct calculation, although the absolute values of the correlations at $\delta t = \pm 20 \text{ kyr}$ are comparable. There is some complexity in the computed autocorrelation, which is absent from the theoretical prediction, but the two results give reasonable agreement for a representative correlation time. (Consider the value of δt when $|R_\epsilon|$ drops below 0.2.) By comparison, the autocorrelation for case c in Olson et al. (2012) has a much larger correlation time, corresponding to a lower frequency for the transition between f^0 and f^{-2} dependences (see Fig. 2 of Olson et al., 2012).

The origin of the long correlation time in case c appears to be connected with the presence of magnetic reversals. This numerical solution undergoes a series of polarity reversals with an average rate of 2 Myr^{-1} . The duration of the transition state is variable, but a typical transition persists for several hundred thousand years before a stable amplitude is re-established. Such long transition states are expected to affect the autocorrelation of the dipole moment. Constable et al. (1998) developed an idealized model in which the dipole moment was fixed at a constant amplitude A during stable polarities, but dropped to zero during transitions. The occurrence of reversals was specified by a Poisson process with rate λ and a transition duration of δ . An example of the predicted autocorrelation function is shown in Fig. 3b using $\lambda = 2 \text{ Myr}^{-1}$ and $\delta = 0.15 \text{ Myr}$. While the idealized model and the geodynamo simulation (case c) exhibit somewhat different correlation times, the comparison does illustrate that persistent transition states can cause long correlation times (e.g. longer than the dipole decay time). On the other hand, transition states in the

Fig. 3. Autocorrelation function for the dipole moment from (a) geodynamo model *Calypso* and (b) case c in Olson et al. (2012). Case c is a reversing solution that exhibits a long correlation time. Broadly similar correlations are predicted by the model of Constable et al. (1998) for a representative transition duration of $\delta = 0.15$ Myr. Such persistence correlation at large δt causes a transition from f^0 to f^{-2} dependence at low f . Shorter transitions in the recent paleomagnetic field would imply much shorter correlation times and higher values for the corner frequency in the power spectrum.

recent paleomagnetic field are thought to be much shorter. Reversals in the field direction occur over timescales less than 10 kyr (Clement, 2004), whereas the duration of weak intensities may persist for 20 to 25 kyr (Ziegler et al., 2011). The dipole decay time is comparable or longer, particularly with recent upward revisions of the electrical conductivity in the core (Pozzo et al., 2012; de Koker et al., 2012). Thus the first transition from f^0 to f^{-2} dependence in the paleomagnetic power spectrum is more likely due to dipole decay.

The second transition between f^{-2} and f^{-4} dependence is associated with the correlation time of the noise term. Power spectra from the two Calypso models are best explained by a correlation time of one-third the overturn time, whereas several other high-resolution models from the study of Olson et al. (2012) give a consistent transition frequency that corresponds to roughly two-thirds the overturn time. Both results imply with a short-lived process. The noise term in the stochastic models characterizes fluctuations in dipole generation about the time-averaged generation

Fig. 4. Power spectra for PADM2M (Ziegler et al., 2011), CALS10k.1b (Korte et al., 2011) and CALS3k.4 (Korte and Constable, 2011), compared with theoretical spectra $S_\epsilon(f)$ and $S_\epsilon^\tau(f)$. Theoretical spectrum $S_\epsilon(f)$ accounts for the low-frequency part of PADM2M. Allowing for correlated noise in $S_\epsilon^\tau(f)$ with $\tau = 100$ yrs improves the fit to CALS10k.1b, but underestimates the power in CALS3k.4.

(Buffett et al., 2014). These fluctuations are thought to be associated helical flow in the core. Variations in the intensity of helical eddies can cause fluctuations in dipole generation, so it would be reasonable to expect the correlation time of these fluctuations to be related to the lifetime of the convective eddies. The inference drawn from the geodynamo models is that the lifetime of these helical structures is a fraction of their rise time through the outer core.

5. Comparison with paleomagnetic observations

Paleomagnetic estimates of the dipole moment can also be used to construct a stochastic model. As an example we consider the PADM2M model of Ziegler et al. (2011). The recovered drift term, $v(x)$, is approximated by (6) in the vicinity of the time average (x) with $\gamma = 34 \text{ Myr}^{-1}$. Similarly the noise term can be approximated by a constant value $D_{eq} = 69 \times 10^{44} \text{ A}^2 \text{ m}^4 \text{ Myr}^{-1}$ (Buffett et al., 2013). In Fig. 4 we compare the predicted power spectrum $S_\epsilon(f)$ with the computed power spectrum for PADM2M. A relatively flat spectrum at low frequency changes to f^{-2} dependence beyond $f = \gamma/2\pi$, consistent with the behavior in numerical geodynamo models. However, the power spectrum for PADM2M falls below the theoretical spectrum above $f = 300$ cycles/Myr. One possible explanation is due to the acquisition of magnetization in sediments, which averages the relative paleointensity over short time intervals (Roberts and Winkholfer, 2004). Alternatively, the regularization used in estimating PADM2M penalizes high-frequency fluctuations. Finally, we may be seeing the influence of correlated noise, which is not included in the construction of the stochastic model.

A steep slope for the power spectrum of PADM2M at high f argues against the influence of correlated noise. Correlated noise causes an f^{-4} dependence, whereas the actual spectrum has a much steeper slope. In addition the power in PADM2M at high f falls well below the power in two other high-resolution paleomagnetic models. Models CALS3k.4 (Korte and Constable, 2011) and CALS10k.1b (Korte et al., 2011) resolve the paleomagnetic field into a low-degree spherical harmonic expansion over the past 3 kyr and 10 kyr, respectively. The longer 10-kyr model has less spatial and temporal resolution than the shorter 3-kyr model, which accounts for differences in power at overlapping frequencies. On the other hand, both of these models have power well above the trend

Fig. 5. Autocorrelation function for PADM2M model (Ziegler et al., 2011) and a 10-Myr realization of the stochastic model. A total of eleven polarity reversals occur during this particular realization. Good agreement between the 10-Myr realization and the theoretical autocorrelation function in (18) supports the theoretical model.

of PADM2M, so it seems likely that PADM2M underestimates the geomagnetic power spectrum at frequencies above 300 cycles/Myr, possibly due to a combination of time averaging and regularization.

Reconciling the two high-resolution paleomagnetic models presents another challenge. The power spectrum for CALS10k.1b can be approximated by $f^{-4.1}$ in the lower frequency range. This dependence is consistent with the high-frequency behavior of $S_{\epsilon}^{\epsilon}(f)$, suggesting that a transition to the f^{-4} dependence has already occurred by $f \approx 10^3$ cycles/Myr (see Fig. 4). Adopting a correlation time of $\tau = 200$ yrs gives a transition frequency of $f \approx 1/2\pi\tau \approx 800$ cycles/Myr, which could plausibly account for the slope of CALS10k.1b. Alternatively, we might attempt to fit a single power spectrum to both PADM2M and CALS10k.1b. Introducing a correlation time of $\tau = 100$ yrs fits the mid-range frequencies of CALS10k.1b, but underestimates the power in CALS3k.4. The slope of CALS3k.4 is compatible with an f^{-6} dependence, suggesting the presence of a second and shorter correlation time in the noise model. A second correlation time of $\tau \approx 50$ yrs would allow the transition to an f^{-6} dependence at $f \approx 3000$ cycles/Myr. This choice would yield an f^{-6} dependence in the frequency range covered by CALS3k.4, but it would not account for the offset in the amplitude of the various power spectra.

A separate question surrounds the approximations used to represent the drift and noise terms in the stochastic model. Estimates for $v(x)$ and $D(x)$ have been recovered from PADM2M at discrete values of x around the time average $\langle x \rangle$. While $v(x)$ can be approximated by (6) near $x = \langle x \rangle$, we find deviations from (6) over the full range of values for x . Similarly, $D(x)$ can be approximated as a constant near $x = \langle x \rangle$, but variations in $D(x)$ were recovered from PADM2M as a function of x . Allowing for a more complete description of $v(x)$ and $D(x)$ in the calculation of the power spectrum is not straightforward, but a simpler alternative is possible. The full description of $v(x)$ and $D(x)$ can be used to obtain realizations of the stochastic process by numerically integrating (1). Each realization can be used to evaluate a power spectrum or an autocorrelation function. Fig. 5 shows the autocorrelation function for one 10-Myr realization. This particular realization includes eleven geomagnetic reversals, but other realizations give similar results. The autocorrelation function for this realization is compared with the approximate autocorrelation function from (18), as well as the

actual autocorrelation function of the PADM2M model. Good agreement between the realization and the theoretical function in (18) lends support to the approximations used for $v(x)$ and $D(x)$. This agreement is not entirely surprising because the realization spends most of the time near the time average. Consequently, the properties of $v(x)$ and $D(x)$ near the time average should weigh heavily in the calculation of the autocorrelation function or the power spectrum. On the other hand, a higher correlation in the simulation at large δt may represent the influence of magnetic reversals (Constable et al., 1998).

The autocorrelation function for PADM2M in Fig. 5 exhibits complexity that is reminiscent of the geodynamo models. While our estimate for γ gives a reasonable value for the nominal correlation time, the details of the autocorrelation function are not fully reproduced by the simple model. Extensions to this model would be needed to account for the oscillatory structure in the autocorrelation function of the PADM2M model.

6. Concluding remarks

Stochastic models are found to give a concise and quantitative description of the variability in both geodynamo simulations and paleomagnetic observations. The transition between f^0 and f^{-2} dependence appears to be set by the decay time of dipole fluctuations. The longest decay time is associated with the slowest decay mode, so the corner frequency in the PADM2M spectrum places a lower bound on the value of electrical conductivity in the core. Dipole fluctuations involving only the slowest decay mode would require an electrical conductivity of $0.6 \times 10^6 \text{ Sm}^{-1}$. Fluctuations with more complex spatial structure require higher electrical conductivities, consistent with recent theoretical calculations (Pozzo et al., 2012; de Koker et al., 2012).

The noise term in the stochastic model characterizes random fluctuations in dipole generation. A correlation in the noise term introduces a second transition in the power spectrum, where the frequency dependence changes from f^{-2} to f^{-4} . The nominal transition frequency is $f = (2\pi\tau)^{-1}$, where τ is the correlation time. Two geodynamo models with $Pm = 0.5$ and low Rm suggest that the correlation time is roughly 1/3 the overturn time. Several other geodynamo models from the study of Olson et al. (2012) suggest a value closer to 2/3 the overturn time. Given a correlation time of $\tau = 200$ yrs from the paleomagnetic models, we infer a convective overturn time of 300 to 600 yrs. Broadly similar estimates are obtained using velocities inferred from geomagnetic secular variation (Gubbins, 2007).

The physical significance of the correlation time is not established by the stochastic models, but the consequences for the power spectrum are evident in both geodynamo models and the low-frequency part of the CALS10k.1b model. A plausible interpretation of the correlation time is based on the lifetime of helical eddies in the core (e.g. Parker, 1969). However, other sources of correlation may also contribute. For example, we expect the generation of magnetic field in the interior of the core to be conveyed to the surface by diffusion through a magnetic boundary layer and to a lesser extent through the mantle (Holme and de Viron, 2013). The time scale for diffusion is liable to be much shorter than $\tau = 200$ yrs, which would imply a second, independent correlation time and possibly another transition in the power spectrum to an f^{-6} dependence. Evidence for a steeper slope in the power spectrum at very high frequency is found in both numerical models (Olson et al., 2012; Davies and Constable, 2014) and observations (Barton, 1982).

Our derivation of a geomagnetic power spectrum from stochastic models yields a simple dependence of the form f^{-n} , where $n = 0, 2, 4, \dots$ is assigned even integer values. Each transition is associated with a correlation time, which smoothly alters the slope

of the spectrum from one regime to another. The first transition between $n = 0$ and 2 is associated with the decay time of dipole fluctuations. Taking the inverse Fourier transform of $S_\epsilon(f)$ defines the autocovariance function in (18), which clearly identifies γ^{-1} as the primary correlation time for the dipole moment. Accounting for correlated noise adds another smooth transition between $n = 2$ and 4. Additional processes with shorter correlation times may give rise to further transitions, as noted above.

We see little evidence for fractional exponents, like $n = 5/3$ or $3/2$, as might be expected from models of hydrodynamic or magnetohydrodynamic turbulence (e.g. Consolini et al., 2002). Taylor's hypothesis is routinely invoked to establish a linear relationship between spatial and temporal variations, but the basis of this assumption for the dipole moment is not obvious. Only part of the convective flow contributes to dipole generation, and this selectiveness is not captured by Taylor's hypothesis. On the other hand, available observations permit large uncertainties on the slope of the power spectrum. Better observations are needed to constrain key parts of the spectrum and better theoretical models are required to link these parts together. Stochastic models and related methods provide a useful tool for accomplishing these goals.

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