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INVARIANCE AS A BASIS FOR NECESSITY AND LAWS

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Many philosophers are baffled by necessity. Some are at a loss to explain the necessity of logical laws. They regard the necessity of logic, along with the nature of logicality, as too basic to be given a theoretical explanation. Others are at a loss to account for natural necessity. Humeans, in particular, are deeply disturbed by the thought of necessary laws of nature. They view the idea of such laws as an arcane idea of secret, inexplicable, mystical powers governing the world, a leftover from the idea of an omnipotent ruler governing the world.

In this paper I offer a down-to-earth yet systematic explanation of necessity and laws in terms of *invariance.* Much has been said about invariance, necessity, and laws in the philosophical literature, and some about the connection between the three. The present account adds another dimension to this literature.

The type of invariance used to explain necessity and laws in this paper is based on the so-called Tarski-Sher thesis (Tarski 1966/86, Sher 1991) – a meta-logical thesis that characterizes logicality in terms of *invariance*. The invariance used in this thesis is, essentially, invariance of *properties* under *1-1 and onto replacements of individuals*. In this paper I extend this use of invariance in several directions. First, I extend its use from logical properties to properties in general. In so doing, I show how prevalent this type of invariance is in all areas of knowledge and how natural it is to understand the general idea of property in terms of invariance. This, in turn, enables me to show that the fruitfulness of invariance in understanding logicality isn't an accident, but part of a broader phenomenon. Second, I establish a systematic correlation between invariance and *necessity* – or more precisely, between *degrees of invariance* and *degrees of necessity* – in a variety of fields, from logic and mathematics to the natural sciences. Since the idea of invariance is both common-sensical and quantitative, an explanation of *necessity* in terms of invariance is likely to reduce its mystery.

Another way in which the present account puts the necessity of natural laws and logical-mathematical laws on a par is by grounding both in the *world* (rather than grounding the latter in our mind or language alone.). The difference between logical necessity and physical necessity is explained by the different facets of the world they're grounded in: formal facets (associated with maximally-invariant properties) in the case of logic, highly-invariant (but less than maximally-invariant) natural facets in the case of physics. In this way the account supports *realism* both in physics and in logic while also differentiating between them.

The present work exhibits both commonalities and differences with other works on invariance, necessity, and laws, in particular Sher's (1991, 1996, 2016) and Lange's (2000, 2005, 2009).[1]

Like Sher (*op.cit.*), it uses invariance to explain logical necessity and to ground it in formal facets of the world. Like Lange (*op.cit.*), it shows how natural laws can be both necessary and contingent. And like both, it (i) involves both actual and counterfactual elements, (ii) affirms multiple necessities, and (iii) uses invariance to explain why and in what sense some laws (e.g., logical laws) have a stronger necessity than other laws (e.g., natural laws). But the present account differs from the earlier accounts in significant ways as well. It differs from Sher (*op.cit.*) in (i) placing invariance in a universal setting, (ii) extending the invariance test to natural properties, and (iii) using this extension to explain natural necessity and laws. And it differs from Lange (*op.cit.*) both in the type of invariance it uses and in its explanatory scope: (i) The present account employs a different type of invariance from the one employed by Lange, namely, invariance of *properties,* while Lange employs invariance of *facts*. (ii) The present account introduces a continuous progression from the accidental to the necessary, while Lange draws a sharp division between them. (iii) The present account distinguishes between logical necessity and other types of non-physical necessity, such as conceptual and metaphysical necessity, whereas Lange's account doesn't.[2] (iv) The present account deals with laws in general, including

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[1] There is a connection between Lange (2005) and Sher (1996).

[2] Lange doesn't object to this distinction, but it's an open question whether his tools are sufficient to draw it.

single laws (one law at a time), while Lange's account deals only with collections, or "strata", of laws, and more especially physical laws. These differences are partly related to differences in our goals.

The present work is related to that of other philosophers as well. By applying invariance to *properties* I align myself with philosophers of science who place properties at the center of their account of laws, such as Lewis (1983) and Armstrong (1983). Unlike Lewis and Armstrong, however, I don't distinguish between "natural" ("non-artificial", "carving nature at the joints") and "unnatural" properties or between "universals" and other properties. My account shows that all highly-invariant properties of natural (physical, biological, ...) objects can support laws of nature *as far as the necessity of such laws is concerned.* But which of these properties in fact serves as a basis for laws of nature is determined on a variety of grounds, including, but not limited to, necessity.

This paper, however, doesn't focus on comparisons with other accounts. My goal is to explain a certain conception of the relation between invariance and necessity in the clearest way possible, within the limits of a journal article.

I will begin with a short introduction to the general idea of invariance.

I.

In its simplest form, invariance is a binary relation: *X is invariant under Y*. Intuitively, to say that X is invariant under Y is to say that X doesn't "notice", doesn't "pay attention" to, is "blind" to, is "immune" to, or isn't "affected" by, changes in Y. It "abstracts" from such changes. A few examples of invariance from different fields and on different levels are:

- Logical truths are invariant under changes in Tarskian models. They are not affected by such changes. You can replace one Tarskian model by another, and the logical truths won't "notice". They hold in all.

- The property of being a Euclidean triangle is invariant under transformations of space that preserve ratios of distances between points, regardless of the distances themselves.

Euclidean triangles are not affected by such transformations: the image of a Euclidean triangle under such a transformation is a Euclidean triangle. This holds for all Euclidean properties: they "abstract" from mere differences in distance. Geometrical properties of other types – say, topological properties – abstract from more differences. They abstract not just from the size of geometrical objects but also from many aspects of their shape. For example, they're oblivious to the differences between triangles and rectangles.

- The laws of special relativity are invariant under changes of inertial reference frames. They're the same in all such frames. They're indifferent to, hence not affected by, replacement of one inertial frame by another.

- The laws of universal grammar are invariant under variations in natural language. They hold in all natural languages. They don't distinguish between one natural language and another. They don't even distinguish between actual and linguistically possible (but not actual) natural languages.

Invariance, as these examples suggest, plays a central role in many theories and fields of knowledge. Its fruitfulness, explanatory power, and objectivity are widely recognized. Here are a few citations:

> [T]here is a structure in the laws of nature which we call the laws of *invariance*. ... [L]aws of nature could not exist without principles of *invariance*. [Wigner 1967: 29. My emphases]

> [E]xplanatory relations must be *invariant* relations, where a relation is *invariant* if it remains stable or unchanged as we change various other things. ... [L]aws describe *invariant* relationships. [Woodward 1997: S26-7. My emphases]

> Questions about objectiveness depend upon the range of transformations under which something is *invariant*. [Nozick 2001: 10. My emphases]

Symmetries, which are so central to physics, are also *invariances*:

> Symmetries are transformations (technically one-to-one functions which map onto their codomain) that leave all relevant structure intact – the result is always exactly like the original, in all *relevant* respects. ... [Such] transformations leave each individual the same in all relevant respects. ... [They] leave... the [relevant properties] of the individuals *invariant*. [van Fraassen 1989: 243-4]

Holton (1973: 380) reports that "for the first two years Einstein, in his letters, preferred to call [Special Relativity] not 'relativity theory' but exactly the opposite: *Invariantentheorie*". And in a (1921) letter, Einstein said that the name "invariance theory" captures the *method* of special relativity.[3]

Kronecker exalted invariance in mathematics: "when the concept of invariants ... is tied ... to the general concept of *equivalence*, … [it] reaches the most general realm of thought". (Cited from Mancosu 2016:15)

A well-known invariantist project is Klein's Erlangen program. (The geometrical example above is based on it.) This project classifies the different geometrical fields as more or less general by comparing the class of transformations under which they, or their notions, are invariant. Narrower geometries (such as Euclidean geometry) are invariant under fewer transformations than more general geometries (such as topology).

Generalizing Klein's project, Tarski proposed a demarcation of logical notions in terms of invariance:

> [C]onsider the class of *all* one-one transformations of the space, or universe of discourse, or 'world', onto itself. What will be the science which deals with the notions *invariant* under this widest class of transformations? Here we will have ... notions … of a very general character. I suggest that they are the logical notions, that we call a notion 'logical' if it is *invariant* under all possible one-one transformations of the world onto itself. [Tarski 1966/86: 149. Last two emphases are mine]

In contemporary terminology, Tarski says that a notion is logical iff (if and only if) it's invariant under all permutations. The so-called Tarski-Sher thesis says that a notion, constant, or property is logical iff it's invariant under all *isomorphisms*, or under all *bijections on domains of individuals*.

The type of invariance I discuss in this paper generalizes the Tarski-Sher invariance. Let me now turn to this invariance.

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[3] "Der Name Invarianz-Theorie würde die Forschungs*methode* der [Relativitäts-] Theorie bezeichen" (pp. 294).

II.

My starting point is a simple and common-sensical picture of the world, one that is independent of any specific philosophical or mathematical theory. According to this picture, there are *objects* of various levels in the world: *individuals* (level 0) and properties (levels 1,2,3,...). Individuals are definite and distinct. Individuals have *properties* (including *relations*), these properties themselves have *properties*, and so on, and it's determined for each property and each object in its range whether it holds of this object. This picture, which underlies much of our thinking in philosophy, mathematics, and science, is convenient for introducing invariance.[4]

Having this picture in mind, we may proceed to a simple yet significant observation about properties:

(Selectivity)   *Properties* in general are *selective* in character. They "pay attention to", "are attuned to", "notice", "discern" *some* differences between individuals, but *not* others. Accordingly, they may distinguish between *some* individuals but *not all*.

For example, the 1st-level property *is-a-human* distinguishes between Alfred Tarski and Mt. Everest, but not between Tarski and Meryl Streep or between Everest and Grand Canyon.[5] You can replace Tarski by Streep or Everest by Grand Canyon and the property *is-a-human* won't notice. But if you replace Tarski by Everest it will. This can be expressed in terms of "invariance": *is-a-human* is *invariant* under a replacement of Tarski by Streep and of Everest by Grand Canyon, but it *isn't invariant* under a replacement of Tarski or Streep by Everest or Grand Canyon.

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[4] (i) In principle, though, the present idea of invariance can be formulated by reference to other conceptions of the world.

  (ii) For the sake of simplicity, I limit my attention to non-vague and non-paradox-generating properties and I assume bivalence.

  (iii) I don't elaborate the notions of individual and property beyond what is needed for the present paper.

[5] Note: Throughout the paper I use words in italics separated by dashes (as in "*is-a-human*") for 1st-level properties (relations).

Although the language of the Selectivity observation is metaphorical, it's so just for the sake of stimulating our common-sense intuitions. There is nothing metaphorical about its *content* as captured by invariance. Given a (non-empty) domain of individuals, D, a 1st-level property P – say, *is-a-human* – divides it into two sub-domains. One contains all the individuals in D that have the property P, the other contains all the individuals in D that don't have P. P is *invariant* under all replacements of individuals that have P by individuals that have P and of individuals that don't have P by individuals that don't have P. It's *not invariant* under any replacements of individuals that have P by individuals that don't have P and vice versa.

Many properties have distinct extension and intension.[6] The selectivity of properties that have a distinct extension and intension is determined by their *intension*. For the purpose of this paper it is sufficient to assume an intuitive distinction between intension and extension, according to which the extension of a property is the set of actual objects falling under it and its intension has to do with its satisfaction conditions or the meaning of the concept corresponding to it (if there is one). Two properties can have the same extension without having the same intension, but not the other way around. A paradigm example is *has-a-kidney* and *has-a-heart*. Their extensions are the same, but their intensions differ.

Since "invariance", as we will presently see, is an *extensional* notion in the sense of being expressed in objectual terms (i.e., in terms of individuals and properties), to accurately express the selectivity of properties in terms of invariance, we have to *"extensionalize"* their intension. We do this by extending the extension of properties to *counterfactual* individuals. A counterfactual individual is a possible but not actual individual (where at this initial stage the notion of possibility is a merely intuitive notion, left unspecified). To indicate the difference in intension between *has-a-kidney* and *has-a-heart* we say that their actual-counterfactual

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[6] It's more common to talk about the extension and intension of concepts (rather than properties), but because the present discussion focuses on properties rather than on concepts, it is more convenient to attribute extension and intension to properties. The underlying idea is that we can treat the concept "has a heart" as determining the property *has-a-heart*, and talk about the extension and intension of this property in much the same way that we usually talk about the extension/intension of the corresponding concept, saying, e.g., that the properties *has-a-heart* and *has-a-kidney* have the same extension but not the same intension. (We will shortly see how this works.)

extensions are the classes of all actual-counterfactual (actual and/or counterfactual) individuals that have them. This enables us to indicate that although they have the same actual extension they do not have the same actual-counterfactual extension. From now on I use this expanded notion of extension instead of "intension". I distinguish between actual, counterfactual, and actual-counterfactual extensions. Unless otherwise indicated (either explicitly or implicitly from the context) the intended use of "extension" is "actual-counterfactual extension".

It's important to indicate that our focus is on properties, and counterfactual individuals are introduced only in order to talk about the invariance of properties. By introducing counterfactual individuals, we're able to take account of differences between properties that are not reflected in their actual extension yet are real and important for understanding the aspects of necessity and laws that interest us here. This enables us to measure the selectivity of a given property in terms of invariance, by checking what replacements of *actual-counterfactual* individuals it's invariant under.

Introducing something like counterfactual individuals is indeed useful for understanding properties in general. Take the property *has-a-mass* (some mass, one-or-another). The selectivity of this physical property depends on its satisfaction conditions, and these do not distinguish between actual and counterfactual physical individuals. To get the selectivity/invariance of this property right, we have to realize that if Earth had a second moon, *has-a-mass* wouldn't have distinguished between it and its actual moon. Similarly, *is-a-human* is invariant not just under a replacement of Tarski by any actual human but also under a replacement of Tarski by any counterfactual human.

The same holds for higher-level properties. Consider the 2nd-level property IS-A-GEOLOGICAL-PROPERTY.[7] The selectivity of this property is reflected by the fact that it doesn't distinguish between any actual-counterfactual individuals that have 1st-level geological properties, it doesn't distinguish between any actual-counterfactual individuals that don't have

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[7] Note: Throughout the paper I use words in small caps separated by dashes (as in "IS-A-GEOLOGICAL-PROPERTY") for 2nd-level properties. In contrast, "Everest" names an individual.

1st-level geological properties, but it does distinguish between actual-counterfactual individuals that have and those that don't have 1st-level geological properties. Thus, IS-A-GEOLOGICAL-PROPERTY doesn't distinguish between any mountains, between any canyons, between any mountain and canyon, between any humans, between any numbers, between any human and number, but it does distinguish between mountains and humans, between canyons and humans, and so on. In terms of invariance, this 2nd-level property is invariant under some replacements of actual-counterfactual individuals but not under others.

Although the notion of actual-counterfactual individual we use to identify the selectivity (invariance) of properties is completely intuitive and pre-theoretical, it's hard for philosophers steeped in the philosophical literature on counterfactuals to retreat to a pre-theoretical conception of counterfactuality. To accommodate readers who would like to know more about my use of "actual" and "counterfactual" let me add a few further clarifications:[8]

Counterfactual Individuals. First, let me note that since our discussion of invariance is generalist in character, not only does the paper not need specialized accounts of its background notions – "individual", "property", "actual", "counterfactual", etc. – but it is open to diverse precisifications of these notions. That is to say, most of what we say here is not dependent on any particular precisification of these notions. In fact, a detailed precisification of the background notions will stand in the way. The danger of precisifying beyond what is absolutely necessary to understand the general ideas expressed here is that a reader's prior objections to a particular precisification of some background notion might interfere with her openness to the discussion of invariance and necessity pursued here, which is independent of any particular precisification of this notion. Where needed, I further clarify certain notions as we go along. But for the most part, I rely primarily on readers' pre-theoretical understanding of the background notions. This said, here are a few clarifications:

(i)   To explain the general idea of invariance in terms of actual-counterfactual

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individuals, there's no need for a full-scale modal apparatus, and I don't use one here.

(ii)   There are two types of individuals: actual and counterfactual. When we abstract from the status of an individual as actual or counterfactual, we view it as an actual-counterfactual individual, or as an individual simpliciter.[9] Later on in the paper I will distinguish several types of possibility, with ramifications for counterfactuality. But at this (early) stage I assume a single, common-sensical, pre-theoretical, unspecified type of possibility. As an example of a counterfactual individual we can think of a second moon of Earth.

(iii)   All individuals have properties (stand in relations). An actual individual has the properties it actually has. A counterfactual individual is a non-actual individual, hence the cluster of properties it has isn't identical to the cluster of properties of any actual individual.

(iv)   There's no cross-actual-counterfactual identity: no actual individual is identical to any counterfactual individual. Informally, one may treat counterfactual individuals which are property-wise similar to a given actual individual in certain ways (to be specified outside invariance theory) as its "counterparts", but no specific theory of counterparts is assumed here.

(v)   At some point we introduce domains of individuals. Individuals are not domain relative.

(vi)   Domains are non-empty collections of actual-counterfactual individuals.

(vii)   Given a unary 1st-level property P, we may talk about its extension simpliciter, where by this we mean its extensions in the class of all actual-counterfactual individuals. We may also talk about its extension in domains D. The extension of P in domain D is the class of all actual-counterfactual individuals in D that have this property. These notions of extension are expanded to relational properties (an n-place property/relation P) in the usual way, namely, by talking about n-tuples of individuals having (or standing in) it.

(viii) We also expand these notions to higher-level properties. For example, the extension of a unary 2nd-level property is the class of all 1st-level properties that have it. Later on we will explain how to discuss the extension of higher-level properties in a context involving domains.

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[9] As for the status of mathematical individuals, in my discussion of mathematics in Section IV I will delineate two options relevant to their status. At this point we may assume that either all mathematical individuals are actual or all mathematical individuals are counterfactual.

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Given the limitation of size on a journal paper, I leave further clarifications to future work.[10]

To set the ground for the ensuing discussion of invariance, necessity, and laws let me offer a methodological remark.

Methodology. The methodology I use is *holistic.* "Holism" means different things to different people. My approach is holistic in a sense that is most naturally represented by the *Neurath boat* metaphor (on a *realist* interpretation). Knowledge is knowledge of the world, or some facet of the world, and this means that we always aim at truth (in a broad correspondence sense). But there's no Archimedean standpoint. To cognitively reach the world, we start from where we stand at the moment, use available tools, employ our critical and creative faculties, and begin, or continue, theorizing. Theorizing typically involves back-and-forth movement. Starting from elements we have, we develop new elements and then use these elements (together with others we've discovered or developed along the way) to turn back, re-examine the elements we started with, replace, revise, or keep these elements in place, and go on. Incorporating knowledge from various fields (both philosophical and others) is *par de course.* Partial circularity is also *par de course.* This, I believe, is the way progress in theorizing is usually achieved by humans.

We are now ready to define the notion of invariance associated with the selectivity of properties. I will call this notion "*property invariance*". To define "property invariance" without

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[10] Still I will consider a puzzle pointed out by an anonymous reviewer:

Let P_1, P_2, P_3 be the properties *is-Trump* (x=t), *is-Sanders* (x=s), and *is-the-only-president-of-the-US-in-2019.* Let a be an actual individual that has properties P_1 and P_3. Let b be a counterfactual individual that has properties P_2 and P_3. Given that a and b have property P_3, they are identical. But given that a is an actual individual and b is a counterfactual individual, they are not identical according to our account.

This puzzle may be solved in different ways by different precisifications of the relevant notions. One way to solve it is to introduce an apparatus of possible worlds and index properties to worlds. There's an actual world, w_1, in which there's an (actual) individual, a, which has the properties P_1w_1 (P_1 indexed to w_1) and P_3w_1. There's also another, counterfactual, world, w_2, with a (counterfactual) individual, b, which has the properties P_2w_2 and P_3w_2. Since $P_3w_1 \neq P_3w_2$, a and b need not be identical. I'd like to thank the anonymous reviewer for suggesting this way of solving the puzzle.

In the remainder of the paper, however, I will not assume that properties are indexed to possible worlds. I will assume that the puzzle can be resolved in one way or another and that there are ways to incorporate adequate solutions into our account.

commitment to a particular mathematical background theory, I will use neutral terminology (not specific to a particular theory) and avoid terms of art as much as possible.

Let D be a *domain* – a non-empty class of actual-counterfactual individuals. As noted above, each 1st-level property P divides D into two sub-classes: the class of individuals (n-tuple of individuals) in D that have P, and its complement, the class of individuals (n-tuple of individuals) in D that don't have P.

Let **r** be a *replacement function*, where a replacement function is a *1-1* function from a domain D_1 *onto* a domain D_2 (possibly $D_1=D_2$). (Since the **r**'s are 1-1 and onto, they apply only to pairs of equinumerous domains.) I will say that each **r** is *indexed* to a pair of (equinumerous) domains, $<D_1,D_2>$. Sometimes, I will speak of **r** as indexed to a single domain, D, in which case the second domain will be the (exact) range of **r**.

Property-invariance or, for short, *invariance*, is then the relation **P** *is invariant under* **r**, where **P** is a property of any level and **r** is a replacement function (on individuals). To include higher-level properties in the definition of invariance, I introduce the notion of "property P restricted to D" – "P_D". If **P** is a 2nd-level property, P_D is a 1st-level property whose actual-counterfactual extension is restricted to the domain D. Below, I will identify P_D with its extension (i.e., the extension of P in D).

To make the definition of "**P** is invariant under **r**" – "INV(**P**,**r**)" – clear, I will focus on three simple cases:

Definition of "P is invariant under r" – "INV(P,r)":

Case 1: **P** is a unary 1st-level property, **r** is a replacement function indexed to $<D_1,D_2>$.

  INV(**P**,**r**) $=_{Df} (\forall x)(\forall y)[[x \in D_1 \ \& \ y \in D_2 \ \& \ y=r(x)] \rightarrow [\mathbf{P}(x) \leftrightarrow \mathbf{P}(y)]]$

  More concisely: if **r** is indexed to D (see above), then

  INV(**P**,**r**) $=_{Df} (\forall x)[x \in D \rightarrow (\mathbf{P}(x) \leftrightarrow \mathbf{P}(r(x)))]$

Case 2: **P** is an n-place 1st-level property, **r** is as above (i.e., indexed to D).

  INV(**P**,**r**) $=_{Df} (\forall x_1)...(\forall x_n)[<x_1,...,x_n> \in D^n \rightarrow (\mathbf{P}(x_1,...,x_n) \leftrightarrow \mathbf{P}(r(x_1),...,r(x_n)))]$

Case 3: **P** is a unary 2nd-level property of unary 1st-level properties, **r** is as above.

$$\text{INV}(\mathbf{P},\mathbf{r}) =_{Df} (\forall P_D)[\mathbf{P}(P_D) \leftrightarrow \mathbf{P}(\mathbf{r}^*(P_D))], \text{ where } \mathbf{r}^*(P_D) \text{ is the image of } P_D \text{ under } \mathbf{r}.[11]$$

The full definition of INV($\mathbf{P}$,$\mathbf{r}$) for all cases is a natural extension of the above definitions.

Examples

Consider:
$\mathbf{P_1}$ = *is-human*,
$\mathbf{P_2}$ = *is-identical-to*,
$\mathbf{P_3}$ = IS-A-(UNARY-)GEOLOGICAL-PROPERTY.
And let:
$\mathbf{r_1}$, $\mathbf{r_2}$ be indexed to $<D_1,D_2>$, where:
D_1 = {Tarski, Everest, Shamu (the whale)},
D_2 = {Streep, Grand Canyon, Titanic},
$\mathbf{r_1}$(Tarski)=Streep, $\mathbf{r_1}$(Everest)=Titanic, $\mathbf{r_1}$(Shamu)=Grand Canyon,
$\mathbf{r_2}$(Tarski)=Titanic, $\mathbf{r_2}$(Everest)=Grand Canyon, $\mathbf{r_2}$(Shamu)=Streep .

It's easy to see that:
(i) $\mathbf{P_1}$ is invariant under $\mathbf{r_1}$ but not under $\mathbf{r_2}$.[12]
(ii) $\mathbf{P_2}$ is invariant under both $\mathbf{r_1}$ and $\mathbf{r_2}$.[13]
(iii) $\mathbf{P_3}$ is invariant under $\mathbf{r_2}$ but not under $\mathbf{r_1}$.[14]

In a similar way we can determine whether any scientific and logical/mathematical property is invariant under a given $\mathbf{r}$. For example, the property *is-subject-to-gravity* is invariant under $\mathbf{r}$'s that replace individuals that are subject to gravity by individuals that are subject to gravity and individuals that are not subject to gravity by individuals that are not subject to

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[11] (i) Note that while $\mathbf{r}$ is a replacement of individuals of D, $\mathbf{r}^*$ is a replacement, *induced* by $\mathbf{r}$, of 1st-level properties restricted to D (P_D's). The point is that since the P_D's are, as indicated above, identified with their extensions (in D), every replacement $\mathbf{r}$ of the individuals in D *induces* a replacement $\mathbf{r}^*$ of the P_D's. For example: If $\mathbf{r}$ is a 1-1 function from D=D_1={a,b} onto D_2={c,d} and P_D={a}, then $\mathbf{r}^*(P_D)$= {$\mathbf{r}$(a)}.

(ii) The fact that $\mathbf{r}$ – a function on individuals – induces a function on 1st-level properties – $\mathbf{r}^*$ – means that we can determine the invariance of 2nd-level properties by focusing on individuals.

[12] $\mathbf{r_1}$ assigns humans to humans and non-humans to non-humans; $\mathbf{r_2}$ does not.

[13] a=b iff $\mathbf{r_i}$(a)=$\mathbf{r_i}$(b) for i=1,2. (Here I use "=" for "*is-identical-to*"). For questions concerning the necessity of identity statements see Section III, including fn. 20, below.

[14] D_1 has 3 individuals, so there are (up to extensional equivalence) 8 distinct 1st-level properties restricted to D_1 (P_{D1}'s). Since geological individuals have geological properties and non-geological individuals don't, the only 1st-level geological properties on D_1 and D_2 are, extensionally, the P_{D1}'s {Everest} and {Grand Canyon}. (For simplicity, I leave Ø out.) The image of {Everest} under $\mathbf{r_2}$ is {Grand Canyon}. I.e., the image of each 1st-level geological property restricted to D_1 under $\mathbf{r_2}$ is a 1st-level geological property restricted to D_2 (and vice versa), and the image of each non-geological 1st-level property restricted to D_1 under $\mathbf{r_2}$ is a non-geological 1st-level property restricted to D_2 (and vice versa). This is not the case with $\mathbf{r_1}$.

gravity. I.e., it's invariant under any replacement of physical individuals by physical individuals and under any replacement of non-physical individuals by non-physical individuals. But it is not invariant under any replacement of physical by non-physical individuals and vice versa.

This notion of property invariance is a generalization of the notion of invariance used in the Tarski-Sher thesis. The idea is that the latter notion, which has until now been used only in connection with logic, mathematics, and linguistics, can be fruitfully extended to explain the necessity of laws in general.[15] In the next section I will state four theses that establish the connection between property-invariance and necessity (necessary laws).

III.

Four Theses on Invariance, Necessity, and Laws:

Thesis 1.   Every property is invariant under some 1-1 and onto replacement(s) of individuals.

Thesis 2.   Some properties have a higher degree of invariance than others; some, but not all, properties have maximal invariance.

Thesis 3.   The higher the degree of invariance of a given property, the greater the degree of necessity of the laws/principles governing/describing it.

Thesis 4.   The higher the degree of invariance of a given field of knowledge, the greater the degree of necessity of, or available to, its laws/principles.

Explanation:

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[15] In the literature on logic there are a couple of alternative proposals for what I call here "property-invariance", due to Feferman (1999) and Bonnay (2008). The main difference between the alternative proposals and the Tarski-Sher proposal, expressed in the present terminology, is that the latter sets stronger requirements on **r**. (**r** has to be both 1-1 and onto. Feferman's proposal, for example, doesn't require **r** to be 1-1.) The reason I prefer to generalize the Tarski-Sher notion rather than the Feferman or Bonnay notion is my belief that the former is more fruitful and philosophically significant than the latter. Explaining this in detail here will divert us from our main subject, so let me just refer the reader to criticisms of Feferman's proposal in van Benthem (2002: 431), Sher (2008: 324-38), and Bonnay (2008: 42-4). I should add that (i) some of the criticisms of Feferman's proposal apply to Bonnay as well (for example, *ad hocness* and lack of theoretical philosophical justification), (ii) Feferman withdrew from his proposal, adopting (Feferman 2015) an altogether different approach to logicality. Of course, nothing I say here should discourage other philosophers from trying to generalize any of the alternative proposals to non-logical properties and compare the different generalizations.

Thesis 1. Every property is invariant under some 1-1 and onto replacement(s) of individuals. This is trivial, since every property is invariant under the *identity* replacement of individuals, i.e., replacement of each individual by itself. Even properties as particular as *is-Everest* or *is-Tarski* are invariant under this replacement. But many properties, including those in the earlier examples, are invariant under non-trivial replacements of individuals as well. *Has-a-mass*, for example, is invariant not only under replacement of Earth by Earth, but also under replacement of Earth by Mars (and many other actual-counterfactual physical individuals).

Thesis 2. Some properties have a higher degree of invariance than others; some, but not all, properties have maximal invariance. We've seen that all properties are invariant under *some* 1-1 and onto replacement(s) **r** of individuals. But are any properties invariant under *all* 1-1 and onto replacements of individuals? Yes. The 1st-level property *is-identical-to* is. (In one of the examples in Section II we saw that it's invariant under both r_1 and r_2, but in fact, it's invariant under *any* **r**.) The 2nd-level property IS-A-NON-EMPTY-PROPERTY (the existential-quantifier property)[16] is also invariant under any **r**.[17] In this sense both *identity* and NON-EMPTINESS are *maximally-invariant*. I define:

(Max-INV)   **P** is *maximally-invariant* iff $(\forall \mathbf{r})$INV(**P**,**r**).

I.e., **P** is *maximally-invariant* iff **P** is invariant under every replacement function **r**.

Note: Although sometimes maximally-invariant properties **P** are universal (**P** holds of all individuals if it is a 1st-level property, of all 1st-level properties if it is a 2nd-level property), this isn't generally the case. For example, the property *is-not-identical-to* is maximally-invariant but it doesn't hold of any individual in any domain, and the property IS-A-NON-EMPTY-PROPERTY doesn't hold of any empty 1st-level property in any domain. Similarly, the property EXACTLY-

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[16] "$(\exists x)Px$" says: "P is non-empty".

[17] For any 1-1 and onto **r** on D_1, P_{D1} is NON-EMPTY iff $\mathbf{r}^*(P_{D1})$ is NON-EMPTY. Example (see fn. 11 above): Let **r** be a 1-1 function from $D=D_1=\{a,b\}$ onto $D_2=\{c,d\}$ such that $\mathbf{r}(a)=c$ and $\mathbf{r}(b)=d$. Then $\mathbf{r}^*$ is the function on P_{D1}'s induced by r. Let $P_{D1}=\{a\}$. Then $\mathbf{r}^*(P_{D1})=\{\mathbf{r}(a)\}=\{c\}$. It is easy to see that P_{D1} is NON-EMPTY iff $\mathbf{r}^*(P_{D1})$ is not empty. It is also easy to prove that the same holds for any D_1, D_2, 1-1 function **r** from D_1 onto D_2, and P_{D1}.

TWO[18] (which holds of a 1st-level property P in a given domain D iff P holds of exactly two individuals in D) is maximally-invariant, yet is not universal in any D.

Not all properties are maximally-invariant. As we've seen above, *is-a-human*, *is-subject-to-gravity*, IS-A-GEOLOGICAL-PROPERTY, etc. are *not* maximally-invariant.

If P_1 is maximally-invariant and P_2 isn't maximally-invariant, we'll say that P_1 has a *higher degree of invariance* than P_2. Here, having a higher degree of invariance means being invariant under more **r**'s, where "more" is understood in terms of proper inclusion. (If the class of all **r**'s under which P_1 is invariant properly includes the class of all **r**'s under which P_2 is invariant, then P_1 has a higher degree of invariance than P_2.)[19] Both *is-identical-to* and IS-A-NON-EMPTY-PROPERTY have a higher degree of invariance than *is-a-human*, *is-subject-to-gravity*, and IS-A-GEOLOGICAL-PROPERTY. These examples of maximally- and non-maximally-invariant properties are sufficient to establish the present thesis.

Thesis 3. The higher the degree of invariance of a given property, the greater the degree of necessity of the laws/principles governing/describing it. To understand this Thesis, let's start with maximally-invariant properties, say, *is-identical-to* and IS-NON-EMPTY, and let's consider some principles that govern/describe them, for example:

(Id)   Every individual *is-identical-to* itself

and

(NE)   If P IS-NON-EMPTY and every individual that has P has Q, then Q IS-NON-EMPTY.

Let's start with (Id). Given that *is-identical-to* is invariant under *any* 1-1 and onto replacement of *any* actual-counterfactual individuals, i.e., doesn't distinguish between any actual-counterfactual individuals, the principles governing/describing it cannot distinguish between any actual-counterfactual individuals either. (If they did, they wouldn't accurately describe identity.) I.e., (Id) is a *maximally-necessary* principle. Since it holds of some actual-

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[18] Definable in standard 1st-order logic.

[19] In Sections IV and V below I will give precise definitions of "higher degree of invariance".

counterfactual individuals (e.g., Tarski), it holds of all. As such (Id) has the kind of *necessity* that is required for *laws*, indeed, for laws of the *strongest* modal force. As far as its necessity is concerned, (Id) is thus an *admissible candidate* for a *maximally-necessary law*.[20] This demonstrates that to the extent that necessity is concerned, laws *are* possible.

The same holds for (NE). Given that IS-NON-EMPTY is invariant under *any* 1-1 and onto replacement of *any* actual-counterfactual individuals, hence doesn't distinguish between non-empty properties that hold of individuals of different kinds, the principles governing/describing it, such as (NE), can't distinguish between such properties either. (NE) holds of non-empty properties of any individuals, including properties that are non-empty only in domains of extremely counterfactual individuals, such as the property *is-both-all-red-and-yellow-at-the-same-time*. (If (NE) didn't hold of non-empty properties of that kind, it wouldn't accurately describe IS-NON-EMPTY.)

In Section V we will see how this result extends to laws governing/describing properties that are highly, but not maximally, invariant, such as the physical property *is-subject-to-gravity*.

For those philosophers of logic (e.g., nominalists) who are worried about the mysteriousness of necessity in logic, it's important to note that there is *nothing mysterious* about the way the present account explains the possibility of maximally-necessary principles/laws such as (Id) or (NE). The maximal necessity of these principles/laws is due to the *non-mysterious* fact that identity and non-emptiness are maximally-invariant, i.e., *invariant under any* **r**.

Thesis 4. The higher the degree of invariance of a given field of knowledge, the greater the necessity of, or the necessity available to, its laws/principles. The correlation between higher degrees of invariance and higher degrees of necessity (greater modal force) can be extended to fields of knowledge. We can characterize the degree of invariance of a given field as the degree of invariance of its most highly-invariant properties, and the degree of necessity of, or available

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[20] Note that the fact that (Id) is maximally-necessary doesn't mean that every identity statement is maximally-necessary (or maximally-impossible). While "Hesperus is Hesperus" is maximally-necessary in our sense, "Hesperus is Phosphorus" is not. (in this paper I leave it open whether it is necessary in some weaker sense.)

to, its laws/principles as the degree of necessity of its most highly-necessary laws/principles.

(Here, I mean by "properties of a field X" properties that are distinctive of it. For example, *has-a-mass* and *is-subject-to-gravity* are physical properties, but although logical and mathematical properties apply to physics, they are not distinctly physical, so they are not physical properties.) In Section V I will further discuss *properties* of, and *necessity* in, scientific fields.

Using "DI" for "degree of invariance", "DN" for "degree of necessity", "$\succ$" for the relation "higher than" between DI's, and "$>$" for the relation "higher than" between DN's, it follows from Theses 3 and 4 that:

$$DI(logic) \succ DI(physics),$$

hence:

$$Highest\ Available\ DN(logical\ laws/principles) >$$
$$Highest\ Available\ DN(physical\ laws/principles),$$

or in short:

$$DN(logic) > DN(physics).$$

In the next section I will discuss the application of these theses to logic and mathematics.

IV.

The application of invariance to logic and mathematics yields rich results. Technically, these results employ the above definitions of "**P** is invariant under **r**" (INV(**P**,**r**)) and "**P** is maximally-invariant" (Max-INV), as well as the following definition of "higher degree of invariance" ($\succ$):

($\succ$) The degree of invariance of $\mathbf{P}_1$ is *higher than* the degree of invariance of $\mathbf{P}_2$ iff the class of all **r**'s under which $\mathbf{P}_1$ is invariant *properly includes* the class of all **r**'s under which $\mathbf{P}_2$ is invariant.

In symbols:

$DI(\mathbf{P}_1) \succ DI(\mathbf{P}_2)$ iff $\{\mathbf{r}: INV(\mathbf{P}_1,\mathbf{r})\} \supset \{\mathbf{r}: INV(\mathbf{P}_2,\mathbf{r})\}$.

Clearly, $\succ$ is anti-reflexive, anti-symmetric, and transitive. I.e., it is a strong partial ordering.[21]

In the existent literature, the rich results of the applications of invariance to logic are commonly formulated in terms of "invariance under isomorphisms/bijections", but this is just another way to express "invariance under 1-1 and onto replacements of individuals" as it's defined here. These results are closely related to the Tarski-Sher thesis. The most relevant works for the present discussion are Tarski (1966/86) and Sher (1991, 2016).

While Tarski (*op. cit*) doesn't view himself as concerned with the philosophical question "What is logic?", Sher (*op. cit*) does. The starting point of Sher (2016) is epistemic: given humans' epistemic aspirations on the one hand and their cognitive limitations on the other, they greatly benefit from a powerful method of inference applicable to all or most fields. In particular, they benefit from the development of a type of inference, or *consequence*, that transmits truth from premises to conclusion with an especially strong modal force. The question is how, theoretically, do we build such a system. The Tarski-Sher thesis approaches this question from the perspective of admissible choices of logical constants. Using the present terminology, the question is "Which choice of logical constants yields a logical system that sanctions all and only logical consequences/principles/laws that are maximally-necessary?". The demarcation of admissible logical constants is given in terms of (property-) invariance.[22] Here I limit myself to a brief discussion of a few theoretical results of the above theses for logic and mathematics.

Before turning to these results, however, let me address a thorny methodological point. It's natural to formulate the results of the above theses for logic and mathematics using

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[21] Although $\succ$ is transitive, in its present formulation it's only vacuously transitive, since the antecedent of "(x$\succ$y & y$\succ$z) → x$\succ$z" is empty. (Only maximally-invariant properties have a higher degree of invariance than other properties.) This partial ordering is sufficient for our applications of invariance to logic and mathematics (where, as we'll see below, only differences in degree of invariance between maximally-invariant and non-maximally-invariant properties play a role). In Section V we'll refine the definition of $\succ$ in a way that renders its transitivity non-vacuous (so its non-trivial applications are extended to pairs of non-maximally-invariant properties).

[22] Note: Adherents of this demarcation focus on theoretical considerations, such as logic's role in knowledge, and have no prior stand on the relation between logic and mathematics. Critics usually focus on intuitive considerations, natural language, and/or prior commitments concerning the relation between logic and mathematics. This creates a disconnect between adherents and critics. For discussions see, e.g., McGee (1996), Feferman (1999), Sagi (2015), Griffiths & Paseau (2016).

mathematical terms-of-art, but this is likely to create the *false* impression that the results themselves involve commitment to a particular background mathematical theory. To avoid this, it's desirable to conduct the discussion in two steps. In the first step we limit ourselves to general philosophical and everyday terminology, in order to emphasize the mathematical neutrality of the discussion. The second step is a step of precisification, using the resources of a specific mathematical theory (yet still without commitment to either its unique or perfect adequacy for this task). In describing property-invariance above in terms of "replacement of individuals", I have used only general terminology. But limitations of space prevent me from a strict separation of the general and specific discussion in what follows. Readers, however, should keep in mind the intended division into two levels of explanation.

There are various orders in which one could present the ramifications of invariance for logic and mathematics. Given the concerns of this paper, I will start with the relation between *invariance* and *formality*.

It's natural to think of formality (in the objectual sense, which is relevant here, as opposed to the syntactic sense) as strong structurality. For example, it's natural to think of logic and mathematics as formal in the sense of being highly-structural. Strong structurality, in turn, is naturally characterized as invariance under all isomorphisms. But invariance under all isomorphisms is (as we've noted above) equivalent to maximal property-invariance, namely, invariance of properties under all 1-1 and onto replacements of individuals. Accordingly, *maximal-invariance* can be viewed as a *criterion of formality*. Note that this use of "formality" implies that there are no formal individuals, only formal properties.

Given this conception of formality we have:

> *Result 1: A property is formal iff it is maximally-invariant; a law/principle is formal iff it governs/describes formal properties. It follows from Thesis 3 that formal laws/principles are maximally-necessary. Now, it has been shown (Tarski-Sher Thesis) that all the logical properties of standard logic (those denoted by its logical constants) are formal. Hence the logical principles/laws (those governing/describing these properties), whatever they are, are highly-necessary.*

Proceeding to the next result, we note that by identifying maximal-invariance with formality, we are led to an additional fruitful idea, the idea of *formal necessity/possibility*. The point is that the category of formally-possible situations is extremely broad. It encompasses not just situations involving physically-possible individuals, but also situations involving physically-impossible individuals, such as individuals that are both all-red and yellow (at the same time). Formally necessary laws/principles hold in all such situations. Accordingly, formal necessity is extremely strong. Laws/principles governing formal properties are formally necessary.

We can use the "formality" terminology to set important conditions on an adequate semantic or model-theoretic definition of logical consequence, such as the Tarskian definition,

(T)   A sentence S is a logical consequence of a set of sentences Γ iff there is no model in which all the sentences of Γ are true and S is false.

If we view a genuine logical consequence as one that transmits truth from sentences (premises) to a sentence (conclusion) with an especially strong modal force and based on the logical structure (distribution and identity of logical constants) of the sentences involved, then we have:

> *Result 2 (Adequacy of Definition of Logical Consequence): To be adequate (i.e., to identify only formally-necessary consequences as logical), a Tarskian definition of logical consequence has to require that (i) logical constants denote formal properties, and (ii) the totality of models represents all formally-possible situations.[23]*

This result has further ramifications for the scope of logic. If we think of logical consequence as dependent on the logical structure of the sentences involved and of logical structure as dependent on the division of constants into logical and non-logical constants, then the scope of logic is significantly dependent on this division. Result 2 shows, theoretically, that any formal constant (one that denotes a formal property) can, in principle, be considered logical as far as the requirement of transmission-of-truth-with-an-especially-strong-modal-force is concerned. Of course, if we set additional requirements on an adequate logical system, some

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[23] This result explains why Tarskian models are *not* "representational" in Etchemendy's (1990) sense, which does not distinguish between formal and general-metaphysical possibility/necessity. (See Sher 1996.)

formal constants might be ruled out. But as far as the modal requirement is concerned, all formal constants are in principle admissible.

As an example, consider the 2nd-level property (quantifier) MOST[24]. This property, it's easy to see, is maximally-invariant, just like the existential- and universal-quantifier properties, IS-NON-EMPTY and IS-UNIVERSAL[25]. Accordingly, the consequence

(1)   (Most x)Px; therefore (∃x)Px,

transmits truth from premise to conclusion with the same modal force as

(2)   (∀x)Px; therefore (∃x)Px.[26]

Put otherwise, the logical principle expressed by (1) has the same modal force as that expressed by (2).

To establish the full scope of formal properties we turn to another result (Tarski 1966/68, Sher 1991, McGee 1996):

> *Result 3 (Scope of Formal Properties). All higher-level mathematical properties and several 1st-level mathematical properties (e.g., equality – that is, identity – and non-equality) are formal (maximally-invariant).*

This result has several significant ramifications.

First, it enables us to delineate in a precise way a maximalist conception of the scope of logic: as far as the maximal necessity of logic is concerned, *any* higher-level and several 1st-level mathematical properties are admissible denotations of logical constants.

Second, Result 3 yields a highly significant result for mathematics:

> *Result 4. All higher-level mathematical laws/principles are maximally-necessary.*

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[24] P_D has the property MOST iff the cardinality of P_D is greater than the cardinality of $\overline{P}_D$ – the property of not having the property P, restricted to D.

[25] P_D has the property UNIVERSAL iff $\overline{P}_D$ is empty.

[26] Assuming, for simplicity, that universes of models are not empty.

That is, the connection between maximal invariance and maximal modal force (Thesis 3), together with Result 3, establish the maximal necessity of a large portion of mathematics (mathematical principles/laws).

What about 1st-level/order mathematics? Results due to Tarski (1966/86)[27] show that 1st-level mathematical properties have a weaker degree of invariance than logical properties and higher-level mathematical properties. Two approaches to 1st-level mathematics that are compatible with Result 4 are: (a) 1st-level mathematics is different from higher-level mathematics and has a weaker degree of necessity than the latter (and logic). (b) 1st-level mathematics *represents* higher-level mathematics: 1st-level arithmetic laws *represent* higher-level laws of finite cardinalities and 1st-level set-theoretical laws *represent* higher-level formal laws in general. The maximal necessity of *higher-level* mathematical laws is then extended to *1st-level* mathematical laws in virtue of their *representational* role. Tarski (1966/86) is indifferent between the first approach and approaches that are more similar to the second approach. Sher (2016: Chapters 8, 10) favors the second approach for reasons that have to do with its ability to explain (i) mathematical truth in general, (ii) the strong necessity of mathematical laws, and (iii) the similarity between logic and mathematics in their applicability to science. On the second approach, the continuum hypothesis, for example, is either necessarily true or necessarily false in the strongest sense of necessity, but the current axioms of set theory are not sufficient to decide whether it is (necessarily) true or (necessarily) false.[28]

I will conclude with a result concerning *logical and mathematical realism.* This result is based on the observation that property-invariance is *worldly* or *objectual*, rather than linguistic or conventional. It deals with properties of objects rather than with words or conventions. Objects have properties of many kinds, including maximally-invariant – formal – properties. Accordingly, the invariantist necessity of laws/principles governing/describing these properties –

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[27] And derivable as well from Lindström (1966) and (partly) from Mostowski (1957).

[28] Unlike logicism, however, this approach does not *assimilate* or *reduce* mathematics to logic altogether, since it assigns different roles – a division of labor – to logic and to pure mathematics (including higher-level mathematics) in the pursuit of knowledge (see below and *op.cit.*).

logical and mathematical laws/principles – is also objectual or worldly rather than linguistic or conventional. In this sense, the proper attitude toward mathematical and logical laws/principles is realist. We may be unsure what the laws/principles of formal properties are, yet sure, based on invariance results, that whatever they are, they have a very strong degree of necessity.

> *Result 5 (Logical-Mathematical Realism). Formal laws hold of objects in the world (individuals, properties of individuals, and so on), hence theories stating formal laws are true or false in the objectual sense, i.e., true or false about the world, broadly understood. Accordingly, the appropriate stance toward such theories is realist (rather than conventionalist, or purely pragmatist, or linguistic).*

Let me clarify that this realism is Aristotelian rather than Platonistic in character. The underlying picture is that of *one* world (rather than two, physical and Platonic), with objects that have *both* physical and formal properties. Talk of counterfactual individuals is a device used to identify the selectivity of different properties (or rather their non-selectivity, which is the basis for their invariance) and doesn't involve commitment to Platonic worlds. Accordingly, the laws/principles governing/describing different properties – including logical and mathematical properties, which are formal in character – are real in a (broad) Aristotelian sense.

Logic and mathematics, on this view, are closely interconnected yet not identical. Mathematics studies objectual laws/principles governing/describing formal properties, and logic studies laws/principles of reasoning (inference, consequence) grounded in formal laws/principles. The two fields develop in a back-and-forth process (of the kind described in my discussion of methodology above).

This Aristotelian realism has interesting ramifications for the relation between mathematics and physics as well. Among other things, it leads to a straightforward answer to Wigner's (1960) question about the applicability of mathematics to physics. *Formal* properties, being maximally-invariant, don't distinguish between individuals of any kind. Therefore, the laws governing them – both logical and mathematical laws – are applicable to individuals (and properties) of any kind, hence to physical individuals (and properties). There is nothing surprising or mysterious about their applicability to physics.

This concludes our discussion of invariance as a basis for logical and mathematical necessity. Let's turn to natural necessity (the necessity of laws of nature).

V.

So far we've shown that formal, i.e., logical and higher-level mathematical, properties are maximally-invariant and that the connection between invariance and necessity implies that, and explains why, formal principles/laws have a maximal degree of necessity. We've also shown that natural (physical, biological, ...) properties are not maximally-invariant, hence natural principles/laws have a lower degree of necessity than logical and mathematical principles/laws.

Can we use the simple ordering relation, $\succ$, as it now stands, to compare the degrees of invariance of natural properties? No. Let me first show that this, indeed, is the case, then analyze why it's the case, and finally delineate a refinement of $\succ$ that can perform such comparisons.

Consider two distinct non-maximally-invariant properties (distinct in the sense of not being invariant under exactly the same **r**'s), say, *is-subject-to-gravity* and *is-a-star*. Based on our informal considerations above, the degree of invariance of *is-subject-to-gravity* is higher than the degree of invariance of *is-a-star* (because all replacements of stars by stars are also replacements of individuals that are subject to gravity by individuals that are subject to gravity but some replacements of individuals subject to gravity by individuals subject to gravity are replacements of stars by non-stars). The $\succ$-test, as it now stands, misses this result: according to the present version of this test neither property has a higher degree of invariance than the other. To see this, it's sufficient to present two **r**'s, r_1 and r_2, such that one of these properties is invariant under r_1 but not r_2 and the other is invariant under r_2 but not r_1.[29] Here are such **r**'s:

r_1 and r_2 are indexed to $<D_1,D_2>$, where
$D_1 = \{a_1,b_1,c_1\}$, $D_2 = \{a_2,b_2,c_2\}$, and where
the a's are individuals that are subject to gravity but are not stars,
the b's are both stars and subject to gravity,
and the c's are neither (say the c's are mathematical individuals).
$r_1(a_1)=b_2$, $r_1(b_1)=a_2$, $r_1(c_1)=c_2$,

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[29] I'd like to thank a person who was present at my inaugural talk at the Wittgenstein symposium 2018 for bringing this point to my attention. Although I wrote down this person's name at the time, it was unfortunately lost.

$\mathbf{r_2}(a_1)=c_2$, $\mathbf{r_2}(b_1)=b_2$, $\mathbf{r_2}(c_1)=a_2$.

This result is generalizable. The present version of $\succ$ suffices to show that (i) all maximally-invariant (logical) properties have a higher degree of invariance than all non-maximally-invariant (e.g., natural) properties and (ii) all maximally-invariant (logical) properties have the same degree of invariance. But for the most part it renders distinct non-maximally-invariant properties (including natural properties) invariance-wise incommensurate.

Why does the $\succ$-test work properly when at least one of the properties is maximally-invariant but not when neither is maximally-invariant? Why does it work properly in comparing the degrees of invariance of *identity* and *gravity* but not of *gravity* and *star*?

The reason is that in all cases the $\succ$-test, as it now stands, takes into account $\mathbf{r}$'s indexed to domains with *any* actual-counterfactual individuals, but only when (at least) one of the properties is maximally-invariant are all these $\mathbf{r}$'s relevant and should be taken into account. We may say that maximally-invariant properties are global; others are local, and to discern differences in degree of invariance between the latter, $\succ$ has to "localize" the $\mathbf{r}$'s taken into account by weeding out replacements of "irrelevant" actual-counterfactual individuals. For example, to discern that DI(*is-subject-to-gravity*) is higher than DI(*is-a-star*), $\succ$ should take into account only replacements of physically viable (actual-counterfactual) individuals, excluding other individuals such as mathematical individuals and individuals that are both all-red and yellow. It's the fact that we took into account mathematical individuals that skewed our result in the example above. Had we limited ourselves to replacements of physically viable actual-counterfactual individuals, we would have obtained the right result.

This suggests that to render the $\succ$-test applicable to pairs of non-maximally-invariant properties we need to refine it by adjusting the replacements taken into account. The decision of which replacements to take into account is done outside invariance theory, holistically, based on our interests and our current knowledge/understanding of the properties in question.

Technically, there are a few ways to proceed. Without going into details, here is a brief outline of one of these. We proceed in two steps:

A. For each pair of properties whose degree of invariance we wish to compare we identify a range of *relevant* **r**'s reflecting the selectivity of the properties involved. For example, it's reasonable to limit the **r**'s to domains of physically-viable actual-counterfactual individuals when applying $\succ$ to IS-A-PHYSICAL-PROPERTY or *is-subject-to-gravity*. It's reasonable to limit the relevant **r**'s to domains of physically-viable disks of uranium and gold when comparing the degrees of invariance of properties of such disks (see example below). And so on. Based on our earlier results, we stipulate that all individuals simpliciter are relevant to formal (logical, higher-level mathematical) properties.

B. We apply the $\succ$-test using relevant **r**'s.

Technically, we may add the notion "replacement function relevant to a pair of properties $\langle P_1,P_2\rangle$", or "$r_{P1,P2}$":

($r_{P1,P2}$)   Given a pair of properties $\langle P_1,P_2\rangle$, $r_{P1,P2}$ is a replacement function indexed to a pair of domains, $\langle D_1,D_2\rangle$, such that D_1,D_2 contain only individuals that are relevant to this pair of properties.

And refine the definition of "$\succ$" as follows:

($\succ$*) $DI(P_1) \succ^* DI(P_2)$ iff $\{r_{P1,P2}: INV(P_1,r_{P1,P2})\} \supset \{r_{P1,P2}: INV(P_2,r_{P1,P2})\}$.

$\succ$*, like $\succ$, is a strong partial-ordering.[30]

Assuming reasonable limits on the **r**'s used in the examples below, it follows from the definition of $\succ$* that

$$DI(\textit{is-identical-to}) \succ^* DI(\textit{is-subject-to-gravity}) \succ^* DI(\textit{is-a-star}),$$

$$DI(\text{IS-A-LOGICAL-PROPERTY}) \succ^* DI(\text{IS-A-PHYSICAL-PROPERTY}) \succ^* DI(\text{IS-A-BIOLOGICAL-PROPERTY}).[31]$$

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[30] But unlike $\succ$, it is non-vacuously transitive (see examples below).

[31] An anonymous reviewer of this paper suggested that $\succ$* might be replaceable by a simpler notion. This is an open question. Two possibilities based on the reviewer's comments are:

[1]: $P_1 \prec^* P_2$ only if P_1 is a proper subset of P_2 in all domains consisting of relevant individuals.
[2]: $P_1 \prec^* P_2$ only if P_1 is a proper subset of P_2 in the union of all domains consisting of relevant

We may also introduce the notions "physically maximally-invariant property" and "biologically maximally-invariant property". A property is physically/biologically *maximally-invariant* iff it is invariant under all **r**'s relevant to *is-a-physical/biological-individual*. Clearly,

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individuals (or in the class of all relevant individuals).

First, let me note that we need an "iff" to replace $\prec^*$ by another notion.

Second, let me note that the above "only if" do not hold in all cases. To show this I present four cases. Both [1] and [2] don't hold in the first three cases; [1] also doesn't hold in the fourth case. In each case, I present two properties, **P₁** and **P₂**, choose (prima-facie) reasonable relevant individuals, and identify a specific domain D of such individuals, where $\mathbf{P_1} \prec^* \mathbf{P_2}$ yet **P₁** is not a proper subset of **P₂** in D / the class of all relevant individuals. I use "*m*", "*s*", "*f*", "*d*" (sometimes with subscripts) to indicate a mathematical individual, a star, a frog, and a dog.

Case 1: **P₁**=IS-A-(1ST-LEVEL-UNARY)PHYSICAL-PROPERTY, **P₂**=IS-A-(1ST-LEVEL-UNARY-)LOGICAL-PROPERTY. Relevant individuals: any actual-counterfactual individual. D=$\{s,m\}$. In this case we have two 2nd-level properties, **P₁** and **P₂**. $\mathbf{P_1} \prec^* \mathbf{P_2}$. Explanation: There are exactly two logical 1st-level unary properties (properties satisfying **P₂**) – the property that does not hold of any actual-counterfactual individual (we can refer to it as "$x \neq x$") and the property that holds of every actual-counterfactual individual ("$x=x$"). Every **r** indexed to any $<D_1,D_2>$ takes $(x \neq x)_{D1}$ to $(x \neq x)_{D2}$ and $(x=x)_{D1}$ to $(x=x)_{D2}$ and it takes every P_{D1} that is different from both $(x \neq x)_{D1}$ and $(x=x)_{D1}$ to a P'_{D2} that is different from both $(x \neq x)_{D2}$ and $(x=x)_{D2}$. So **P₂** is maximally-invariant. But **P₁** is not maximally invariant, because for some **r**'s, the induced **r***'s take 1st-level unary physical properties to 1st-level unary mathematical, hence non-physical, properties. Now, for [1], let's go to D above. There, **P₁**=$\{\{s\}\}$, **P₂**=$\{\varnothing,\{s,m\}\}$; therefore, **P₁** is not a proper subset of **P₂** in D. For [2], the intersection of logical properties and physical properties in the class of all actual-counterfactual individuals is empty, hence, **P₁** is not a proper subset of **P₂** in this class.

Case 2: **P₁**=*is-a-star*, **P₂**=*is-not-self-identical* ($x \neq x$). Relevant individuals: any actual-counterfactual individual. D=$\{s,f\}$. In this case we have two 1st-level properties, one (**P₁**) non-logical, the other (**P₂**) logical. As such, $\mathbf{P_1} \prec^* \mathbf{P_2}$. But since **P₂** is empty in D / the class of all actual-counterfactual individuals and **P₁** is not, **P₁** is not a proper subset of **P₂** in D / the class of all actual-counterfactual individuals.

Case 3: **P₁**=*is-a-frog*, **P₂**=*is-immortal*. Relevant individuals: any actual-counterfactual biological individual (assuming no actual-counterfactual biological individual is immortal). D=$\{d,f\}$. This case is similar to Case 2, but **P₂** is not a logical property.

Case 4: **P₁**=*is-a-star*, **P₂**=*is-subject-to-gravity*. Relevant individuals: any actual-counterfactual physical individual. D=$\{s_1,s_2\}$. Here we have two non-empty 1st-level properties. Given that only physical individuals are relevant, **P₂** is invariant under all **r**'s indexed to domains of relevant individuals, but **P₁** is not. **P₁**, however, is universal in the particular D we are considering. Hence it is not a proper subset of **P₂** in that D.

While the above possibilities won't do as they stand, in principle, as I noted above, there could be a systematic connection between $\succ^*$ and some simpler notion that could replace it. Due to limitations of space and for the sake of accessibility, I leave further discussion of this possibility for another paper. I should emphasize, though, that what this paper aims to show is that it's possible to explain significant aspects of necessity and laws based on the simple observation that properties are selective in character, that such an explanation can be systematized (in one way or another) using invariance, and that this reduces the mystery that is often attributed to necessity and laws. Whether it's possible to explain this in other terms, not involving invariance, is left open in this paper.

I'd like to thank the reviewer for a comment that led to this clarification.

is-a-physical-individual is physically maximally-invariant, but (given our current state of knowledge) *is-subject-to-gravity* is also physically maximally-invariant. *Is-a-star*, however, is not physically maximally-invariant. *Is-a-biological-individual* is biologically maximally-invariant, but neither *is-a-mammal* nor IS-A-HEREDITARY-PROPERTY are.

It's useful to introduce the notion "close to being physically/biologically maximally-invariant" as well. I won't undertake the task of defining this notion here, but an example of a physical property which (depending on how we demarcate it) is either physically maximally-invariant or close to being physically maximally-invariant is *has-a-mass*.

We are now ready to apply Thesis 3, which connects invariance and necessity, to the natural sciences. The greater the invariance of a given property, the more replacements ("more" in the sense of inclusion) of actual-counterfactual individuals it doesn't "notice". And if a property doesn't notice replacements of certain individuals, the laws/regularities governing/describing it cannot notice them either. (If they do, they are not laws of *this* property.) If P doesn't distinguish between *a* and *b*, its laws (if any) cannot distinguish between them either. If they hold of one, they hold of both. In particular, principles/laws governing/describing physically maximally-invariant properties don't distinguish between a great many actual-counterfactual individuals, and so do their laws/regularities. They hold in the entire class of physically actual-counterfactual individuals. This means that the actual-counterfactual scope of the laws/regularities governing physically maximally-invariant properties is very large. It's so large as to render these laws/regularities physically-*highly*-necessary. And the laws/regularities governing/describing properties that are close to being physically maximally-invariant render the laws/regularities of many of these *physically-fairly-highly-necessary*.[32] An example of a principle/law that, if true, is highly or at least fairly highly necessary is Newton's law of gravity. This law,

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[32] Assuming all the predicates appearing in a canonical formulation of these principles/laws denote properties that are at least close to being physically maximally-invariant.

$$F = G \frac{m_1 m_2}{r^2}, \quad [33]$$

connects the gravitational force between two bodies to their masses and the distance between the centers of their masses. In our terminology, Newton's law connects three properties that are at least close to being physically-maximal, and as such has at least a fairly high degree of necessity. Indeed, given that the properties in question have a higher degree of necessity than almost all physical properties, including both most physical properties "proper" and all natural properties outside physics proper, the degree of necessity of Newton's law of gravity is indeed very high – as high as we'd require the necessity of laws of nature to be.

Applying Thesis 4 to the natural sciences, we have:

DI(Physics)≻*DI(Biology),

Hence (using ">*" for the relation "higher than" between degrees of necessity correlated with ≻*):

Highest Available DN(physical laws/principles) >*
Highest Available DN(biological laws/principles),

or in short:

DN(physics) >* DN(biology).

Philosophical Notes:

A. Invariance and Metaphysics

Two questions concerning invariance and metaphysics naturally arise: (i) Is invariance theory a metaphysical theory? (ii) What is the relation between metaphysical invariance/ necessity and other types of invariance/necessity, e.g., logical and physical invariance/necessity?

(i) Invariance theory, as it is conceived here, does not belong to a single branch of philosophy. It belongs to a cluster of philosophical fields (sub-fields) which includes, in addition

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[33] Where: F – gravitational force between two bodies; G – gravitational constant; m_1, m_2 – masses of the two bodies; r – distance between centers-of-mass of these bodies.

to metaphysics, also epistemology, philosophy of logic, philosophy of science, and possibly others.

(ii) From the point of view of invariance, metaphysics is a highly heterogeneous field of knowledge, since it deals with heterogeneous topics, from the topic of *objects simpliciter,* associated with the maximally-invariant property *is-an-object(-simpliciter)*, to such topics as *causality* and *free will*, which, in spite of the fact that they're associated with properties of far lower degrees of invariance, stand on a par with the former topic within metaphysics. It's thus difficult to assign a degree of invariance/necessity to metaphysics as a whole. It's tempting to place metaphysical properties/principles in the interval between formal properties/principles and physical properties/principles. But in fact this is a complex issue that requires a separate inquiry.

B. Invariance and Science

1. Invariance, Necessity, Natural Laws, and Humeanism

(a) Humeans believe that the idea of laws of nature is mysterious or otherworldly. We can conjure up laws in our minds, but in nature, there is just a mosaic of particular physical objects, each with its own properties. The connection between invariance and necessity delineated in this paper shows, however, that if the natural mosaic itself is not mysterious, the idea of necessary principles governing/describing the behavior of some properties in this mosaic is not mysterious either. Given that (i) every property, including properties acceptable to Humeans, is invariant under some 1-1 and onto replacements of individuals, that (ii) some properties, including ones acceptable to Humeans, have a fairly high degree of invariance, that (iii) high degree of invariance is connected to, and explains, the necessity of principles/laws governing/describing highly-invariant properties, and that (iv) none of (i)-(iii) is mysterious, it's neither surprising nor mysterious that there are necessity-wise admissible candidates for necessary laws. The claim that, due to their necessity, to accept natural laws is to commit oneself to something mysterious or otherworldly, is unfounded. To the extent that individuals in the world have properties and that some significant properties have a high degree of invariance, there is an *infrastructure* in *nature itself* for necessary laws.

What do I say to a Humean who accepts the claim that some physical properties have a high degree of invariance (though not maximal invariance) yet refuses to accept the consequence that natural laws associated with such properties can in principle be significantly necessary, on the ground that they do not hold of all formally (or logically) possible objects?

My response has two parts: (i) The Humean of the last paragraph seems to recognize only maximal – logical – necessity as "real" necessity. In so doing, he discards Thesis 3 of this paper, which connects degrees of invariance to degrees of necessity on all levels, not just the maximal level, without offering any criticism, let alone refutation, of this thesis. (ii) The Humean seems to assume a rigid binary bifurcation which leaves room only for two modal states: logical-necessity and contingency. I am wary of rigid binary bifurcations of this kind, which include, in addition to the necessary-contingent bifurcation, also the analytic-synthetic and the apriori-aposteriori bifurcations. These bifurcations lump together everything that does not belong to one extreme case (the purely analytic, purely apriori, logically necessary), losing sight of theoretically significant differences between elements that are thus lumped together. In the case of necessity and contingency, our Humean loses sight of the considerable difference in modal force between such extremely modally weak facts as the fact that I am wearing a black sweater today and modally far stronger facts, such as the fact that I, along with all material bodies, am subject to gravity. In contrast, the invariantist offers a non-mysterious way to recognize, make sense of, and introduce structure into, such differences. (iii) In saying that laws of nature are not necessary since they don't hold of all logically-possible objects, the humean doesn't realize how deeply different formally-but-not-physically-possible objects are from objects that are reasonably viewed as possible in scientific contexts. Think, for example, of an object that is both entirely red and yellow, of a person who is both dead and has a beating heart, a functioning brain, and so on, of a particle which is both sub-atomic and six feet long, of a body which is both a moon of Earth and a black hole, and so on. Logic doesn't distinguish between such physically/biologically impossible objects and physically/biologically possible objects. But the natural sciences do and should. The logical possibility of such objects isn't relevant to science. Natural necessity isn't

applicability to such objects. Natural necessity is applicability to non-actual objects that are physically or biologically possible, such as a second moon of Earth or a species of humanoids with DNA instructions for six rather than five fingers in each hand. Physical necessity is, and ought to be recognized as, different from logical necessity, and one should not reject it on the ground that it doesn't satisfy the standards of logical necessity.

(b) While the connection between invariance and the necessity of laws captures something basic and significant about laws, it's important to recognize its limits.

First, this connection shows that the world is "ready", in certain significant ways, or has an appropriate "infrastructure" in place, for necessary physical laws, but whether there are, in fact, such laws is not determined by this connection. The fact that there's an infrastructure for necessary physical laws suggests that, and explains why, searching for laws of nature is reasonable. But it neither guarantees that scientists will find (or have found) such laws nor determines what these laws are or with which highly-invariant properties they're associated.

Second, necessity may not be limited to principles governing highly-invariant properties. If, and to the extent that, there are other types of necessity, the present account doesn't explain them. For example, if, and to extent that, some laws obtain their status by (mere) stipulation, their necessity is not explained by our account. If, and to the extent that, there are singular laws – laws describing singular features of nature – their necessity may not be explained by the present account. Here the main question is whether the singular feature described by a putative law is an isolated feature or is related to features of highly-invariant properties. This question arises in the case of the principle/law stating that nothing moves faster than light, for example.

What about necessary principles/laws of the kind "Given the way the world actually is, X must be the case in the world"? Here the necessity is due to the connection between the way the world is and X. If the connection is formal – logical or mathematical – then its necessity *is* explained by our account. If it's based on a natural principle/law that governs/describes certain highly-invariant properties, its necessity is also explained by our account. But if it's based on something altogether different, it wouldn't be explained by our account.

Another limit of our account is that it doesn't distinguish between necessary principles and bona-fide laws. In this respect our account follows the common practice in logic, where all logical principles/truths/consequences are on a par. For various reasons, some scientists and philosophers of science do distinguish between the two. In determining which principles should be presented as laws, they appeal to many considerations, only one of which is modal force. The additional considerations may be pragmatic, methodological, or based on scientific inquiry. They may focus on nature itself ("laws of nature") or on what our theories should present as laws ("laws of theories of nature"). A few examples of considerations other than modal force are projectability considerations (Goodman 1955/65), naturalness considerations (Lewis 1983), and invariance considerations that relate to other types of invariance than the one discussed here (Lange 2000, Woodward 2018). But at least some of the worries underlying these considerations, such as the worries about "artificial" properties, are irrelevant to the invariantist, who is concerned only with *necessity-wise admissible candidates* for laws.

(c) The connection between necessity and property-invariance implies, as we've seen above, that *necessity and possibility (of the kinds we focus on here) come in degrees*. The higher the degree of invariance of a given property is, the higher the degree of necessity of laws governing/describing it. The degree of necessity of logical laws is higher than that available to physical laws, and the degree of necessity available to physical laws is higher than that available to biological laws/principles, such as evolutionary principles. This explains why survival of the fittest, for example, may be evolutionarily necessary, but is neither logically nor physically necessary.

(d) The introduction of types or "spaces" of possibility enables the invariantist to account for the resolution of issues requiring a differentiation between necessity and accidental generality in a simple and systematic manner. Consider the widely discussed difference between the non-existence on Earth of spheres of gold one mile in diameter and the non-existence on Earth of spheres of uranium one mile in diameter. The former is a matter of the limited amount of gold on Earth, the latter of the inner structure of uranium (regardless of its location). But if there is just

one type of possibility, logical possibility, there is no difference between the two. Logically, the non-existence-on-Earth of both is accidental. But as soon as we distinguish between physical and logical possibility, the explanation is simple. Whereas in the space of logical possibilities neither *is-a-uranium-sphere* nor *is-a-gold-sphere* is invariant under all 1-1 replacements of small-diameter spheres by large-diameter spheres, this is not the case in the space of physically-possible uranium and gold spheres. Here, no such replacement preserves the property *is-a-uranium-sphere*, but some such replacements preserve *is-a-gold-sphere*.

Humeans have always recognized differences between outwardly similar phenomena, such as the absence of large gold spheres on Earth and the absence of large uranium spheres on Earth. But our account goes a step further by systematically explaining differences of this kind in terms of invariance, and through it, physical possibility.

2. *Abstraction.* The connection between invariance and laws relieves another worry about laws of nature, namely, their *abstract* character. The worry is that what the laws abstract from is just what is real. All there is in nature are particular things, and the abstractness of laws violates this particularity. To abstract is to falsify. What is abstraction? Abstraction is often characterized in a way that is similar to our characterization of selectivity and invariance. To abstract is to "leav[e] things out", to "ignor[e] things" (Godfrey-Smith 2009: 47). Those who associate abstraction with falsehood think that the only source of abstraction is the human mind. There is no abstraction in nature. It's we who abstract from natural objects when we think of nature in terms of laws. And in so doing we distort nature.

Contrary to this view, invariance theory shows that it isn't we (or not just or primarily we) who are the source of abstraction. Nature itself is its source. Natural properties are selective *in nature*. They have a *non-trivial degree of invariance*. And to have a non-trivial degree of invariance – to be selective – is to *abstract* from some features of natural objects, i.e., to abstract from the particularity of natural objects. But if natural properties are abstract in nature, then abstraction is, in principle, *true* to nature. Abstraction resides in the world itself. What we do is, primarily, bring it into view. Natural properties such as *is-a-star*, *is-an-electron*, *emits-light*, *is-*

subject-to-gravity, *is-subject-to-evolution,* IS-A-HEREDITARY-PROPERTY, IS-AN-ASTRONOMICAL-PROPERTY, differ from each other in what they pay attention to and what they overlook or abstract from. Some abstract from more features of objects than others (have a higher degree of invariance than others). This is a factual, objective, matter; not a human fabrication. If the truth of the matter is that a particular object has a property with a high degree of invariance, then *it is neglecting this fact, rather than bringing it to light, that leads us to stray from truth*. Since abstraction is built into nature, the possibility of abstract laws is built into it as well.[34]

3. *Scientific Realism.* Having a high degree of invariance is, as we've noted, an *objectual* or *worldly* feature of properties. As a result, the necessity of principles/laws governing/ describing highly-invariant properties is also primarily worldly (rather than epistemic, conceptual, mental, or linguistic). Worldly necessity is sometimes called "*metaphysical* necessity", but "metaphysical" here is ambiguous. It can be understood as referring to worldly necessity as opposed to non-worldly (e.g., linguistic) necessity. But it can also be understood as referring to a type of necessity that is different from other types of necessity, say logical, physical, and/or biological necessity. (See discussion of metaphysics above.) For that reason I prefer the expression "worldly necessity".

The worldly character of natural necessity is one of the building blocks of *scientific realism*. Psillos (2014) says that *the reality of laws is based on the reality of patterns and regularities*. The idea is that "there is a network of natural patterns in nature" (*ibid.:* 9), these natural patterns are the basis for natural regularities, and these worldly regularities are, in turn, the basis for natural laws. Invariance theory adds another layer to Psillos's conception of scientific realism. The high degree of invariance of some natural properties is the basis for the necessity of principles, and eventually laws, describing regularities in the patterns they exhibit.[35]

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[34] (i) It's worthwhile noting that this has nothing to do with the character of natural laws as deterministic or probabilistic. If properties abstract from certain differences between individuals, then their laws must abstract from them too, whether they're deterministic or probabilistic.

 (ii) For the connection between invariance and abstraction in mathematics see Mancosu (2016).

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