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PROPERTIES AND SOLUTIONS OF THE EIGENSYSTEM OF NONPROPORTIONALLY DAMPED LINEAR DYNAMIC SYSTEMS

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Abstract

This report reviews currently available methods for solving dynamic response of nonproportionally damped linear structural systems. A general condition in which the system is not limited to be lightly damped is considered to accomodate the possibility of various applications. Mode superposition is used to evaluate the response of the system subjected to initial conditions and external loadings. For this purpose, the subspace iteration technique is extended to extract the least dominant set of eigensystems. The iteration vectors are modified in such a way that only real arithmetic is employed when dealing with complex vectors. Hence, both storage space and computation time are reduced tremendously for the extraction of the damped modes.

Contents

1. INTRODUCTION

In the analysis of dynamic response of structures, the equation of motion of a damped linear system can be expressed as

$$\mathbf{M}\,\ddot{\mathbf{q}}\,(t) + \mathbf{C}\,\dot{\mathbf{q}}\,(t) + \mathbf{K}\,\mathbf{q}\,(t) = \mathbf{f}\,(t) \tag{1.1}$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are the mass, damping, stiffness matrices respectively and $\ddot{\mathbf{q}}\,(t)$, $\dot{\mathbf{q}}\,(t)$, $\mathbf{q}\,(t)$ are the acceleration, velocity, displacement vectors respectively.

A common method used to solve this problem is to find the normal modes of vibration of the corresponding undamped system; that is, to solve the following eigenproblem (e.g., see [C2])

$$\mathbf{K}\,\boldsymbol{\psi} = \omega^2\,\mathbf{M}\,\boldsymbol{\psi} \tag{1.2}$$

where ω and $\boldsymbol{\psi}$ are the modal frequency and shape of the undamped system. From these modal quantities, a set of uncoupled equations of motion can be obtained which are in terms of the normal-coordinates of the system. Each of these equations corresponds to each mode of vibration and can be solved individually. The effect of damping is taken into account by adding into these equations a term which represents a reasonable amount of damping ratio. The solution of the problem can then be obtained by superposition of various modal contributions. This leads to the so-called *mode-superposition method.*

This solution is exact only when the damping matrix $\mathbf{C}$ is *proportional* ; that is, $\mathbf{C}$ is of such a form that it can be diagonalizd by the same transformation that uncouples the undamped systems [C1]. When $\mathbf{C}$ is not proportional, the above method often is still used and the solution thus found is usually used as an approximation to the exact one. This approximation may be close when damping is very small. For large damping or concentrated damping, it may not be appropriate to retain the approximate method. Therefore, the purpose of this report is to find the procedure to solve Eq.(1.1) considering a general damping matrix $\mathbf{C}$.

2. PROPERTIES OF EIGEN-SOLUTIONS

Equation (1.1) possesses a homogeneous solution of the type

$$\mathbf{q}\,(t) = e^{\lambda t}\,\mathbf{w} \tag{2.1}$$

where λ and $\mathbf{w}$ are the eigenvalue and eigenvector of the system and both are to be determined. Introducing this solution into Eq.(1.1), we obtain the characteristic equation

$$(\,\lambda^2\,\mathbf{M} + \lambda\,\mathbf{C} + \mathbf{K}\,)\,\mathbf{w} = 0 \tag{2.2}$$

This is a quadratic eigenproblem and hence is computationally more complicated than the linear eigenproblem which arises from undamped systems. The nontrivial solutions exist if, and only if, the determinant of the coefficient matrix of Eq.(2.2) is zero; that is,

$$\det (\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K}) = 0 \tag{2.3}$$

Since the left side of Eq.(2.3) is a real polynomial of degree $2n$ in λ, the equation possesses $2n$ roots, although they are not necessarily all distinct. These roots may either be real or be complex conjugate pairs. For a real pair of eigenvalue, the associated pair eigenvectors are also real. For a complex conjugate pair of eigenvalues, the associated eigenvectors are also a complex conjugate pair. For the purpose of discussion, we assume there are *nc* complex conjugate pairs of eigenpairs and *nr* real pairs of eigenpairs. That is, the eigenvalues are

$$\alpha_1 , \bar{\alpha}_1 , \ldots , \alpha_{nc} , \bar{\alpha}_{nc} , \beta_1 , \hat{\beta}_1 , \ldots , \beta_{nr} , \hat{\beta}_{nr} \tag{2.4}$$

with the following convention

$$\alpha_j = \alpha_{Rj} + \alpha_{Ij} \, i^* \qquad \bar{\alpha}_j = \alpha_{Rj} - \alpha_{Ij} \, i^* \tag{2.5a}$$

$$\beta_j \geq \hat{\beta}_j \tag{2.5b}$$

and the corresponding eigenvectors are

$$\boldsymbol{\phi}_1 , \bar{\boldsymbol{\phi}}_1 , \ldots , \boldsymbol{\phi}_{nc} , \bar{\boldsymbol{\phi}}_{nc} , \boldsymbol{\psi}_1 , \hat{\boldsymbol{\psi}}_1 , \ldots , \boldsymbol{\psi}_{nr} , \hat{\boldsymbol{\psi}}_{nr} \tag{2.6}$$

with

$$\boldsymbol{\phi}_j = \boldsymbol{\phi}_{Rj} + \boldsymbol{\phi}_{Ij} \, i^* \qquad \bar{\boldsymbol{\phi}}_j = \boldsymbol{\phi}_{Rj} - \boldsymbol{\phi}_{Ij} \, i^* \tag{2.7}$$

where α_{Rj}, α_{Ij}, β_j and $\hat{\beta}_j$ are real-valued scalars; $\boldsymbol{\phi}_{Rj}$, $\boldsymbol{\phi}_{Ij}$, $\boldsymbol{\psi}_j$ and $\hat{\boldsymbol{\psi}}_j$ are real-valued vectors. Of course, i^* is the imaginary unit defined by $\sqrt{-1}$. Note that the eigenpairs mentioned above have been arranged in the following order,

$$| \alpha_1 | = | \bar{\alpha}_1 | < | \alpha_2 | = | \bar{\alpha}_2 | < , \ldots , < | \alpha_{nc} | = | \bar{\alpha}_{nc} | \tag{2.8a}$$

$$| \beta_1 | < | \beta_2 | < , \ldots , < | \beta_{nr} | \tag{2.8b}$$

Remark 2.1 For a *stable* system, the real parts of all the eigenvalues have to be negative; that is, all α_{Rj} , β_j , $\hat{\beta}_j$ are negative. □

Proportionally damped case

For the special case of a proportionally damped system, all the eigenvectors are real. That is, all the $\boldsymbol{\phi}_{Ij}$ are zero and consequently all the $\bar{\boldsymbol{\phi}}_j$ are equal to $\boldsymbol{\phi}_j$. Moreover, all the $\hat{\boldsymbol{\psi}}_j$ coincide with $\boldsymbol{\psi}_j$. As a result, there are only n independent eigenvectors, i.e., $\mathbf{w}_1$, $\mathbf{w}_2$, ..., $\mathbf{w}_n$ and these eigenvectors are the same as those obtained by solving Eq.(1.2). That is, the eigenvectors of the proportionally damped system coincide with the eigenvectors of the corresponding undamped system, which is obtained simply by neglecting the damping matrix; i.e., letting $\mathbf{C} = 0$. Since both $\mathbf{M}$ and $\mathbf{K}$ are symmetric, these eigenvectors are orthogonal with respect to $\mathbf{M}$ and $\mathbf{K}$ and also with respect to $\mathbf{C}$ due to the fact of proportionality. That is, we have for i and j from 1 to n

$$\mathbf{w}_i^T \mathbf{M} \mathbf{w}_j = \delta_{ij} \, m_j \tag{2.9a}$$

$$\mathbf{w}_i^T \mathbf{C} \mathbf{w}_j = \delta_{ij} c_j \tag{2.9b}$$

$$\mathbf{w}_i^T \mathbf{K} \mathbf{w}_j = \delta_{ij} k_j \tag{2.9c}$$

where m_j , c_j and k_j are called the generalized mass, damping and stiffness of the j^{th} mode in consideration. If we substitute $\mathbf{w}_j$ for $\mathbf{w}$ in Eq.(2.2); i.e., we force the system to vibrate in that particular mode; then we can pre-multiply the equation by $\mathbf{w}_j^T$ to obtain the j^{th} modal characteristic equation

$$m_j\lambda^2 + c_j\lambda + k_j = 0 \tag{2.10}$$

The solution of Eq.(2.10) are

$$\lambda_j = -\frac{c_j}{2m_j} \pm \sqrt{(\frac{c_j}{2m_j})^2 - \frac{k_j}{m_j}} \tag{2.11}$$

if we let $2\sqrt{k_j m_j}$ be the critical damping corresponding to this mode, then we have the damping ratio expressed as

$$\xi_j = \frac{c_j}{2\sqrt{k_j m_j}} \tag{2.12}$$

and the undamped frequency and damped frequency expressed as

$$\omega_j = \sqrt{\frac{k_j}{m_j}} \qquad \omega_{Dj} = \omega_j\sqrt{1 - \xi_j^2} \tag{2.13}$$

Depending upon the numerical value of ξ_j, we can rewrite Eq.(2.11) as either one of

$$\lambda_j = -\xi_j\omega_j \pm \omega_{Dj}\, i^* \qquad \xi_j<1 \tag{2.14a}$$

$$\lambda_j = -\xi_j\omega_j \pm \omega_j\sqrt{\xi_j^2 - 1} \qquad \xi_j>1 \tag{2.14b}$$

The underdamped solution expressed by Eq.(2.14a) corresponds to α_j and $\bar{\alpha}_j$ while the overdamped solution expressed by Eq.(2.14b) corresponds to β_j and $\hat{\beta}_j$ according to aforementioned notation. As we can see that the *primary* root β_j is the one with the plus sign in Eq. (2.14b) while the *secondary* root $\hat{\beta}_j$ is the one with the minus sign according to the convention. [L1] Notice that the undamped frequencies of this proportionally damped system is the same as the frequencies of the corresponding undamped system, which is obtained by neglecting the damping matrix. This is because the generalized mass and stiffness of the proportionally damped system is the same as those of the corresponding undamped system. It is this property which enables us to use the simplified method, where we analyze the undamped system first to obtain the generalized mass and stiffness and then add an arbitrary amount of generalized damping or the equivalent amount of damping ratio into the solution.

Nonproportionally damped case

For the general case of a nonproportionally damped system, there exit $2n$ eigenvectors. Each eigenvector of the proportionally damped system becomes two eigenvectors when the damping becomes nonproportional. Among these $2n$ eigenvectors, the nc complex conjugate

pairs of them are associated with the underdamped modes and the nr real pairs of them are associated with the overdamped modes. We propose to construct modal properties for the nonproportionally damped system in a similar manner as for the proportionally damped system. That is, for the system vibrating in the j^{th} mode, we can pre-multiply the corresponding Eq.(2.2) by the transpose of its *paired* eigenvector to obtain a equation of the form of Eq.(2.10). For underdamped modes, the equation would be either of the two

$$\alpha^2 \bar{\boldsymbol{\phi}}^T \mathbf{M} \boldsymbol{\phi} + \alpha \bar{\boldsymbol{\phi}}^T \mathbf{C} \boldsymbol{\phi} + \bar{\boldsymbol{\phi}}^T \mathbf{K} \boldsymbol{\phi} = 0 \tag{2.15a}$$

$$\bar{\alpha}^2 \boldsymbol{\phi}^T \mathbf{M} \bar{\boldsymbol{\phi}} + \bar{\alpha} \boldsymbol{\phi}^T \mathbf{C} \bar{\boldsymbol{\phi}} + \boldsymbol{\phi}^T \mathbf{K} \bar{\boldsymbol{\phi}} = 0 \tag{2.15b}$$

Here $\bar{\boldsymbol{\phi}}^T \mathbf{M} \boldsymbol{\phi}$ and $\boldsymbol{\phi}^T \mathbf{M} \bar{\boldsymbol{\phi}}$ are the same real-valued scalar and may be written as m for brevity. We define c and k in a similar manner. Now, we can easily see that α and $\bar{\alpha}$ are simply the solutions to the characteristic equation

$$m\lambda^2 + c\lambda + k = 0 \tag{2.16}$$

For obvious reason, we can define m, c and k as the *generalized mass, generalized damping* and *generalized stiffness* for the underdamped modes of the system. Moreover, we can define the critical damping, damping ratio, undamped frequency and damped frequency by these generalized quantities as before for the proportionally damped system. But it should be pointed out that the generalized mass and stiffness of the damped system are different from those of the corresponding undamped system in general. Consequently, the undamped frequencies of the damped system are different from the frequencies of the corresponding undamped system. Hence, one must be specific when he mentions the term "undamped frequencies" of the nonproportionally damped system. One way to avoid this ambiguity is to call the frequencies of the undamped system as undamped frequencies while to call the undamped frequencies of the damped system as *pseudo* undamped frequencies.

For the overdamped modes, the equation corresponding to the motion of the j^{th} mode is

$$\beta^2 \hat{\boldsymbol{\psi}}^T \mathbf{M} \boldsymbol{\psi} + \beta \hat{\boldsymbol{\psi}}^T \mathbf{C} \boldsymbol{\psi} + \hat{\boldsymbol{\psi}}^T \mathbf{K} \boldsymbol{\psi} = 0 \tag{2.17a}$$

$$\hat{\beta}^2 \boldsymbol{\psi}^T \mathbf{M} \hat{\boldsymbol{\psi}} + \hat{\beta} \boldsymbol{\psi}^T \mathbf{C} \hat{\boldsymbol{\psi}} + \boldsymbol{\psi}^T \mathbf{K} \hat{\boldsymbol{\psi}} = 0 \tag{2.17b}$$

Where $\hat{\boldsymbol{\psi}}^T \mathbf{M} \boldsymbol{\psi}$ and $\boldsymbol{\psi}^T \mathbf{M} \hat{\boldsymbol{\psi}}$ are the same scalar and can be written as m for brevity. We define c and k in a similar way. Again, we can easily see that β and $\hat{\beta}$ are simply the solutions to the characteristic equation

$$m\lambda^2 + c\lambda + k = 0 \tag{2.18}$$

For this reason, we can define m, c and k as the *generalized mass, damping* and *stiffness* respectively for the overdamped modes of the system.

Remark 2.2 Note that if one forms the following equation

$$\lambda^2 \boldsymbol{\psi}^T \mathbf{M} \boldsymbol{\psi} + \lambda \boldsymbol{\psi}^T \mathbf{C} \boldsymbol{\psi} + \boldsymbol{\psi}^T \mathbf{K} \boldsymbol{\psi} = 0 \tag{2.19}$$

then only one root of the equation is β while the other is not an eigenvalue of the system. The

similar conclusion is reached if $\boldsymbol{\psi}$ is replaced by $\hat{\boldsymbol{\psi}}$. □

Reduced equation of motion

From the above discussion, it is clear that the generalized quantities which characterize the system can still be constructed from the **M** , **C** , **K** matrices and their eigenvectors. But these eigenvectors are not orthogonal with respect to any of **M** , **C** or **K** for a nonproportionally damped system. The fact that an n degree-of-freedom system can only have exactly n independent vectors to span the whole space should overthrow the possibility of these $2n$ eigenvectors being orthogonal to one another with respect to **M** , **C** or **K**. Accordingly, Eq.(1.1) cannot be uncoupled by its eigenvectors. This suggests that another form of the equation of motion for the system be used instead of Eq.(1.1). An effective way is to substitute for Eq.(1.1) an equivalent first order system [F3], which is obtained by the introduction of auxiliary variables. If $\dot{\mathbf{q}}\,(t)$ is adopted as the auxiliary vector variable, then combination of Eq.(1.1) with the matrix identity

$$\mathbf{M}\,\dot{\mathbf{q}}\,(t) - \mathbf{M}\,\dot{\mathbf{q}}\,(t) = 0 \tag{2.20}$$

gives a linear system represented by

$$\mathbf{A}\,\dot{\mathbf{Q}}\,(t) - \mathbf{B}\,\mathbf{Q}\,(t) = \mathbf{F}\,(t) \tag{2.21}$$

where **A** and **B** are super-matrices represented by

$$\mathbf{A} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \tag{2.22}$$

and $\mathbf{Q}\,(t)$ and $\mathbf{F}\,(t)$ are super-vectors represented by

$$\mathbf{Q}\,(t) = \begin{Bmatrix} \mathbf{q}\,(t) \\ \dot{\mathbf{q}}\,(t) \end{Bmatrix} \qquad \mathbf{F}\,(t) = \begin{Bmatrix} \mathbf{f}\,(t) \\ \mathbf{0} \end{Bmatrix} \tag{2.23}$$

Here a system of n second-order equations is reduced to a system of $2n$ first-order equations. Eq.(2.21) is thus refered to as the *reduced* form of Eq.(1.1). A similar reduced form can be obtained by using **K** instead of **M** in Eq.(2.20). This leads to the same equation as Eq.(2.21) but with

$$\mathbf{A} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{C} \end{bmatrix} \tag{2.24}$$

and

$$\mathbf{Q}\,(t) = \begin{Bmatrix} \mathbf{q}\,(t) \\ \dot{\mathbf{q}}\,(t) \end{Bmatrix} \qquad \mathbf{F}\,(t) = \begin{Bmatrix} \mathbf{0} \\ -\mathbf{f}\,(t) \end{Bmatrix} \tag{2.25}$$

Note that if **M** , **C** and **K** are symmetric; then **A** and **B** are also symmetric although neither is positive-definite.

3. ORTHOGONALITY PROPERTIES

To solve the eigenproblem associated with the nonproportionally damped linear system, it is more convenient to use the reduced form of the equation of motion, given by Eq.(2.21). The resulting eigenproblem is a linear system represented by

$$\lambda \mathbf{A}\,\mathbf{z} = \mathbf{B}\,\mathbf{z} \tag{3.1}$$

where **A** and **B** are defined by Eq.(2.22) or (2.24) and **z** is a super-vector represented by

$$\mathbf{z} = \begin{Bmatrix} \mathbf{w} \\ \lambda \mathbf{w} \end{Bmatrix} \tag{3.2}$$

Eq.(3.1) is a generalized eigenproblem and is in the same form as Eq.(1.2). Therefore, techniques similar to those used in the treatment of Eq.(1.2) may also be employed. Since **K** is nonsingular after all rigid-body modes have been eliminated from the system; **B** will be invertible in general. We can obtain the following standard eigenproblem

$$\mathbf{D}\,\mathbf{z} = \frac{1}{\lambda}\,\mathbf{z} \tag{3.3}$$

where $\mathbf{D} = \mathbf{B}^{-1}\mathbf{A}$ is called the *damped dynamic matrix* and plays a similar role as the dynamic matrix $\mathbf{K}^{-1}\mathbf{M}$ does during the analysis. Eq.(3.1) and Eq.(3.3) are equivalent, both can be used to find the eigensolutions to the original system. The eigenvalues resulting from the solution of Eq.(3.1) or (3.3) are the same as those given in Eq.(2.4) while the eigenvectors now correspond to Eq.(2.6) but in the form of super-vectors

$$\boldsymbol{\Phi}_1 , \overline{\boldsymbol{\Phi}}_1 , \ldots , \boldsymbol{\Phi}_{nc} , \overline{\boldsymbol{\Phi}}_{nc} , \boldsymbol{\Psi}_1 , \hat{\boldsymbol{\Psi}}_1 , \ldots , \boldsymbol{\Psi}_{nr} , \hat{\boldsymbol{\Psi}}_{nr} \tag{3.4}$$

with

$$\boldsymbol{\Phi}_j = \boldsymbol{\Phi}_{Rj} + \boldsymbol{\Phi}_{Ij}\, i^* \qquad \overline{\boldsymbol{\Phi}}_j = \boldsymbol{\Phi}_{Rj} - \boldsymbol{\Phi}_{Ij}\, i^* \tag{3.5}$$

where

$$\boldsymbol{\Phi}_{Rj} = \begin{Bmatrix} \boldsymbol{\phi}_{Rj} \\ \alpha_{Rj}\boldsymbol{\phi}_{Rj} - \alpha_{Ij}\boldsymbol{\phi}_{Ij} \end{Bmatrix} \qquad \boldsymbol{\Phi}_{Ij} = \begin{Bmatrix} \boldsymbol{\phi}_{Ij} \\ \alpha_{Rj}\boldsymbol{\phi}_{Ij} + \alpha_{Ij}\boldsymbol{\phi}_{Rj} \end{Bmatrix} \tag{3.6}$$

and

$$\boldsymbol{\Psi}_j = \begin{Bmatrix} \boldsymbol{\psi}_j \\ \beta_j \boldsymbol{\psi}_j \end{Bmatrix} \qquad \hat{\boldsymbol{\Psi}}_j = \begin{Bmatrix} \hat{\boldsymbol{\psi}}_j \\ \hat{\beta}_j \hat{\boldsymbol{\psi}}_j \end{Bmatrix} \tag{3.7}$$

The fact that both **A** and **B** are symmetric gives the property [F1] that the eigenvectors are orthogonal with respect to both **A** and **B** . That is, for any two eigenvectors which correspond to different eigenvalues, we have

$$\mathbf{z}_j^T\,\mathbf{A}\,\mathbf{z}_k = 0 \qquad \mathbf{z}_j^T\,\mathbf{B}\,\mathbf{z}_k = 0 \tag{3.8}$$

For underdamped modes it is useful to derive the orthogonality relationships in terms of the real and imaginary parts of the eigenvectors. Introducing various different eigenvector into the

first of Eq.(3.8) , we obtain the following possible conditions

$$(\Phi_{Rj} - \Phi_{Ij}\, i^*)^T \mathbf{A} (\Phi_{Rj} + \Phi_{Ij}\, i^*) = 0 \tag{3.9}$$

$$(\Phi_{Rj} + \Phi_{Ij}\, i^*)^T \mathbf{A} (\Phi_{Rk} + \Phi_{Ik}\, i^*) = 0 \tag{3.10}$$

$$(\Phi_{Rj} - \Phi_{Ij}\, i^*)^T \mathbf{A} (\Phi_{Rk} + \Phi_{Ik}\, i^*) = 0 \tag{3.11}$$

$$\Psi_k^T \mathbf{A} (\Phi_{Rj} + \Phi_{Ij}\, i^*) = 0 \tag{3.12}$$

Expanding Eq.(3.9) , we obtain

$$\Phi_{Rj}^T \mathbf{A}\, \Phi_{Rj} + \Phi_{Ij}^T \mathbf{A}\, \Phi_{Ij} = 0 \tag{3.13}$$

Solving Eq.(3.10) and (3.11) simultaneously and noting that both real and imaginary parts must vanish separately, we obtain

$$\Phi_{Rj}^T \mathbf{A}\, \Phi_{Rk} = 0 \qquad \Phi_{Ij}^T \mathbf{A}\, \Phi_{Ik} = 0 \tag{3.14}$$

$$\Phi_{Rj}^T \mathbf{A}\, \Phi_{Ik} = 0 \qquad \Phi_{Ij}^T \mathbf{A}\, \Phi_{Rk} = 0 \tag{3.15}$$

Similarly, we obtain from Eq.(3.12)

$$\Psi_j^T \mathbf{A}\, \Phi_{Rk} = 0 \qquad \Psi_j^T \mathbf{A}\, \Phi_{Ik} = 0 \tag{3.16}$$

Eqs.(3.14), (3.15) and (3.16) express the orthogonality property of the eigenvectors in terms of their real and imaginary parts. These relationships are developed with **A** as the weighting matrix. It is clear that the same relationships hold if we replace **A** by **B** as the weighting matrix.

Normalization of eigenvectors

The eigenvectors can be normalized such that

$$\mathbf{z}_j^T \mathbf{A}\, \mathbf{z}_j = 1 \qquad \mathbf{z}_j^T \mathbf{B}\, \mathbf{z}_j = \lambda_j \tag{3.17}$$

For overdamped modes, it simply is

$$\Psi_j^T \mathbf{A}\, \Psi_j = 1 \qquad \Psi_j^T \mathbf{B}\, \Psi_j = \beta_j \tag{3.18a}$$

Remark 3.1. Since neither **A** or **B** is positive-definite, it is not always possible to obtain Eq.(3.18a) by using real vectors. As a result, we may want to use the following alternative to avoid pure imaginary vectors.

$$\Psi_j^T \mathbf{A}\, \Psi_j = -1 \qquad \Psi_j^T \mathbf{B}\, \Psi_j = -\beta_j \tag{3.18b}$$

Remark 3.2. Although it is not explicitly expressed in the above formulae, it is clear that Eqs.(3.15) and (3.17) hold equally if we can substitute any primary real eigenvector Ψ_j by its corresponding secondary eigenvector $\hat{\Psi}_j$. □

For underdamped modes, we can expand Eq.(3.17) to have

$$\Phi_{Rj}^T \mathbf{A}\, \Phi_{Rj} - \Phi_{Ij}^T \mathbf{A}\, \Phi_{Ij} = 1 \tag{3.19}$$

$$\boldsymbol{\Phi}_{Rj}^T \mathbf{A} \boldsymbol{\Phi}_{Ij} + \boldsymbol{\Phi}_{Ij}^T \mathbf{A} \boldsymbol{\Phi}_{Rj} = 0 \tag{3.20}$$

and

$$\boldsymbol{\Phi}_{Rj}^T \mathbf{B} \boldsymbol{\Phi}_{Rj} - \boldsymbol{\Phi}_{Ij}^T \mathbf{B} \boldsymbol{\Phi}_{Ij} = \alpha_{Rj} \tag{3.21}$$

$$\boldsymbol{\Phi}_{Rj}^T \mathbf{B} \boldsymbol{\Phi}_{Ij} + \boldsymbol{\Phi}_{Ij}^T \mathbf{B} \boldsymbol{\Phi}_{Rj} = \alpha_{Ij} \tag{3.22}$$

Eq.(3.19), together with Eq.(3.13), gives

$$\boldsymbol{\Phi}_{Rj}^T \mathbf{A} \boldsymbol{\Phi}_{Rj} = \frac{1}{2} \qquad \boldsymbol{\Phi}_{Ij}^T \mathbf{A} \boldsymbol{\Phi}_{Ij} = -\frac{1}{2} \tag{3.23}$$

Eq.(3.20) itself gives

$$\boldsymbol{\Phi}_{Rj}^T \mathbf{A} \boldsymbol{\Phi}_{Ij} = 0 \qquad \boldsymbol{\Phi}_{Ij}^T \mathbf{A} \boldsymbol{\Phi}_{Rj} = 0 \tag{3.24}$$

By similar manipulation, we can obtain

$$\boldsymbol{\Phi}_{Rj}^T \mathbf{B} \boldsymbol{\Phi}_{Rj} = \frac{1}{2}\alpha_{Rj} \qquad \boldsymbol{\Phi}_{Ij}^T \mathbf{B} \boldsymbol{\Phi}_{Ij} = -\frac{1}{2}\alpha_{Rj} \tag{3.25}$$

$$\boldsymbol{\Phi}_{Rj}^T \mathbf{B} \boldsymbol{\Phi}_{Ij} = \frac{1}{2}\alpha_{Ij} \qquad \boldsymbol{\Phi}_{Ij}^T \mathbf{B} \boldsymbol{\Phi}_{Rj} = \frac{1}{2}\alpha_{Ij} \tag{3.26}$$

These formulae and the aforementioned orthogonality properties will be useful in the process of solving the eigenpairs of the system. They will also be useful in the solution of Eq.(1.1) to determine the response of the system.

4. POWER METHOD

The power method may be used to extract the dominant eigenpair of Eq.(3.3). The way it works is the same as the way it works in solving Eq.(1.2) except that there is a difficult point. The difficulty is that there will be two dominant eigenpairs instead of one. As shown before, the underdamped eigensolutions of Eq.(3.3) generally exist in complex conjugate pairs. Therefore, a pair of eigenvalues with the same modulus are dominant. The usual iteration procedure will converge not to either of them directly but rather to their linear combination. In other words, we perform the iteration by the following formula

$$\mathbf{v}^{(k+1)} = \mathbf{D}\, \mathbf{v}^{(k)} \tag{4.1}$$

where $\mathbf{v}$ is an approximate to the dominant eigenvector and superscript k is the iteration number. When k is large enough, all but the components of two dominant eigenvectors will be eliminated and $\mathbf{v}$ will become

$$\mathbf{v}^{(k)} \approx \alpha_1 \lambda_1^k \mathbf{z}_1 + \bar{\alpha}_1 \bar{\lambda}_1^k \bar{\mathbf{z}}_1 \tag{4.2}$$

where λ_1 and $\mathbf{z}_1$ is the dominant eigenpair and $\bar{\lambda}_1$ and $\bar{\mathbf{z}}_1$ is its conjugate; α_1 is a constant and $\bar{\alpha}_1$ is its conjugate.

There are several methods available to determine the complex conjugate eigenpairs. (e.g., see [H1], [M1]) An easier way than that proposed in [H1] and [M1] is to keep three successive approximates, say $\mathbf{v}^{(k)}$, $\mathbf{v}^{(k+1)}$ and $\mathbf{v}^{(k+2)}$, during iteration process. If λ_1 and $\bar{\lambda}_1$ are the roots of the quadratic equation

$$\lambda^2 + b\,\lambda + c = 0 \tag{4.3}$$

then it follows that [F2]

$$\mathbf{v}^{(k+2)} + b\,\mathbf{v}^{(k+1)} + c\,\mathbf{v}^{(k)} = 0 \tag{4.4}$$

as can be verified by substituting Eq.(4.2) into Eq.(4.4). The constants b and c can be found from any two equations of the set Eq.(4.4), but a more suitable way [J1] would be to find least square solution of Eq.(4.4). After b and c are found, λ_1 and $\bar{\lambda}_1$ can be obtained by solving Eq.(4.3) and the corresponding eigenvectors can be calculated from

$$\mathbf{z}_1 \approx \mathbf{v}^{(k+2)} - \bar{\lambda}_1\,\mathbf{v}^{(k+1)} \qquad \bar{\mathbf{z}}_1 \approx \mathbf{v}^{(k+2)} - \lambda_1\,\mathbf{v}^{(k+1)} \tag{4.5}$$

The power method may only be used to find the dominant eigenpair. To find other eigenpairs, such techniques as shifting or sweeping can be used. This renders the solutions inefficient or less accurate compared to the solution for the dominant eigenpairs. Therefore, a more suitable method needs to be pursued.

5. SUBSPACE ITERATION METHOD

The subspace iteration method has been recognized as an efficient method to extract the least dominant set of eigenpairs of the undamped eigenproblem represented by Eq.(1.2); therefore, it is desirable to extend this method for solving the damped eigenproblem represented by Eq.(3.1). From a theoretical point of view, solving Eq.(3.1) is conceptually the same as solving Eq.(1.2), although the former is computationally more complicated than the latter because the solutions of Eq.(3.1) are complex in general while the solutions of Eq.(1.2) are real. The subspace iteration algorithm [B1] , which is uesd to solve Eq.(1.2), is shown in Box 5.1 for convenience. In the following, we will show that the same steps as those in Box 5.1 can be followed, with some modification in the iteration vectors, to solve the damped eigenproblem; i.e., Eq.(3.1).

As in the algorithm listed in Box 5.1 , we form a set of vectors **U** which approximate the required eigenvectors. Assuming we have performed many cycles of iterate in the following way and almost eliminated the undesirable higher mode components in **U**, we start the next cycle to further improve **U**. The first step is to solve for a new set of vectors **V** from the following formula

$$\mathbf{B}\,\mathbf{V} = \mathbf{A}\,\mathbf{U} \tag{5.1}$$

Box 5.1 *Algorithm for Subspace Iteration Method*

For k = 1,2,3,....

1. Iterate from E_k to E_{k+1} :

$$\mathbf{K}\,\mathbf{V}_{k+1} = \mathbf{M}\,\mathbf{U}_k$$

2. Find the projections of the operators **K** and **M** onto E_{k+1} :

$$\mathbf{K}_{k+1} = \mathbf{V}_{k+1}^{\mathrm{T}}\,\mathbf{K}\,\mathbf{V}_{k+1} \qquad \mathbf{M}_{k+1} = \mathbf{V}_{k+1}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{V}_{k+1}$$

3. Solve the eigensystem of the projected operators :

$$\mathbf{K}_{k+1}\,\mathbf{X}_{k+1} = \mathbf{M}_{k+1}\,\mathbf{X}_{k+1}\,\mathbf{\Lambda}_{k+1}$$

4. Find an improved approximation to the eigenvectors :

$$\mathbf{U}_{k+1} = \mathbf{V}_{k+1}\,\mathbf{X}_{k+1}$$

Then, provided that the eigenvectors in $\mathbf{U}_1$ are not orthogonal to one of the required eigenvectors, we have

$$\mathbf{\Lambda}_{k+1} \to \mathbf{\Lambda} \quad \text{and} \quad \mathbf{U}_{k+1} \to \mathbf{\Phi} \quad as\ k \to \infty$$

where **U** contains the approximate eigenvectors from the previous iterate and can be arranged in the following form

$$\mathbf{U} = [\ \mathbf{a}_1+\mathbf{b}_1\mathbf{i}^*\ ,\ \mathbf{a}_1-\mathbf{b}_1\mathbf{i}^*\ ,\ \mathbf{a}_2+\mathbf{b}_2\mathbf{i}^*\ ,\ \mathbf{a}_2-\mathbf{b}_2\mathbf{i}^*\ ,\ .....\ ,\ \mathbf{d}_1\ ,\ \mathbf{d}_2\ ,\ .....\ ] \tag{5.2}$$

As **A** and **B** are both real, **V** will have the same form as **U** ; say,

$$\mathbf{V} = [\ \mathbf{r}_1+\mathbf{s}_1\mathbf{i}^*\ ,\ \mathbf{r}_1-\mathbf{s}_1\mathbf{i}^*\ ,\ \mathbf{r}_2+\mathbf{s}_2\mathbf{i}^*\ ,\ \mathbf{r}_2-\mathbf{s}_2\mathbf{i}^*\ ,\ .....\ ,\ \mathbf{t}_1\ ,\ \mathbf{t}_2\ ,\ .....\ ] \tag{5.3}$$

The second step is to form projection matrices **G** and **H** by

$$\mathbf{G} = \mathbf{V}^{\mathrm{T}}\,\mathbf{B}\,\mathbf{V} \qquad \text{and} \qquad \mathbf{H} = \mathbf{V}^{\mathrm{T}}\,\mathbf{A}\,\mathbf{V} \tag{5.4}$$

The matrices **G** and **H** have special structure due to the way we arrange the matrices **U** and **V**. To identify this structure, we partition **G** into the following 4 submatrices

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{cc} & \mathbf{G}_{rc} \\ \mathbf{G}_{cr} & \mathbf{G}_{rr} \end{bmatrix} \tag{5.5}$$

where $\mathbf{G}_{cc}$ is constructed entirely from complex vectors, $\mathbf{G}_{cr}$ is constructed from complex vectors by real vectors, $\mathbf{G}_{rc}$ is the transpose of $\mathbf{G}_{cr}$, and $\mathbf{G}_{rr}$ is constructed entirely from real vectors. If we further partition $\mathbf{G}_{cc}$ into 2 by 2 submatrices which correspond to a complex conjugate pair of vectors by a complex conjugate pair of vectors, then all submatrices have the following structure

$$\begin{bmatrix} (\mathbf{r}_i^T\mathbf{B}\mathbf{r}_j - \mathbf{s}_i^T\mathbf{B}\mathbf{s}_j) + (\mathbf{r}_i^T\mathbf{B}\mathbf{s}_j + \mathbf{s}_i^T\mathbf{B}\mathbf{r}_j)\mathbf{i}^* & (\mathbf{r}_i^T\mathbf{B}\mathbf{r}_j + \mathbf{s}_i^T\mathbf{B}\mathbf{s}_j) - (\mathbf{r}_i^T\mathbf{B}\mathbf{s}_j - \mathbf{s}_i^T\mathbf{B}\mathbf{r}_j)\mathbf{i}^* \\ (\mathbf{r}_i^T\mathbf{B}\mathbf{r}_j + \mathbf{s}_i^T\mathbf{B}\mathbf{s}_j) + (\mathbf{r}_i^T\mathbf{B}\mathbf{s}_j - \mathbf{s}_i^T\mathbf{B}\mathbf{r}_j)\mathbf{i}^* & (\mathbf{r}_i^T\mathbf{B}\mathbf{r}_j - \mathbf{s}_i^T\mathbf{B}\mathbf{s}_j) - (\mathbf{r}_i^T\mathbf{B}\mathbf{s}_j + \mathbf{s}_i^T\mathbf{B}\mathbf{r}_j)\mathbf{i}^* \end{bmatrix} \tag{5.6}$$

$\mathbf{G}_{cr}$ can also be partitioned into 2 by 1 submatrices which correspond to a complex conjugate pair by a real vector, they look like

$$\begin{bmatrix} (\mathbf{r}_i^T\mathbf{B}\mathbf{t}_j) + (\mathbf{s}_i^T\mathbf{B}\mathbf{t}_j)\mathbf{i}^* \\ (\mathbf{r}_i^T\mathbf{B}\mathbf{t}_j) - (\mathbf{s}_i^T\mathbf{B}\mathbf{t}_j)\mathbf{i}^* \end{bmatrix} \tag{5.7}$$

$\mathbf{G}_{rr}$ has elements of the following usual form

$$\mathbf{t}_i^T\mathbf{B}\mathbf{t}_j \tag{5.8}$$

Likewise, $\mathbf{H}$ has exactly the same form as $\mathbf{G}$ except that $\mathbf{B}$ is replaced by $\mathbf{A}$ in the foregoing formulae. The third step is to solve the projected eigensystem

$$\mathbf{G}\,\mathbf{X} = \mathbf{H}\,\mathbf{X}\,\mathbf{\Lambda} \tag{5.9}$$

where $\mathbf{\Lambda}$ and $\mathbf{X}$ are eigenvalues and eigenvectors of the projected system. The last step is to improve the approximation to the eigenvectors of the original system by

$$\mathbf{U} = \mathbf{V}\,\mathbf{X} \tag{5.10}$$

This procedure is continued until a specified convergence criterion is fulfilled; then, we have $\mathbf{\Lambda}$ and $\mathbf{U}$ as the required eigenpairs.

As can be seen above, the major difficulty of using subspace iteration method is that both the iteration vector and the projected matrices contain complex numbers. This requires not only twice as much storage space as for real numbers but also an eigen-solver which can handle complex matrices. To overcome this difficulty, we make some modifications to avoid the use of complex vectors. Note that all the information of a complex conjugate pair of vectors is contained in the real and imaginary parts of the complex vectors. During the iteration process, We can use the real part and imaginary part of a complex vector to take the positions of a pair of complex conjugate vectors and we record the real vectors without any modification. That is, we use

$$\mathbf{U}^* = [\,\mathbf{a}_1\,,\,\mathbf{b}_1\,,\,\mathbf{a}_2\,,\,\mathbf{b}_2\,,\,\ldots\ldots\,,\,\mathbf{d}_1\,,\,\mathbf{d}_2\,,\,\ldots\ldots\,] \tag{5.11}$$

instead of $\mathbf{U}$ in the iteration process. As a result of this substitution, $\mathbf{V}^*$ becomes

$$\mathbf{V}^* = [\,\mathbf{r}_1\,,\,\mathbf{s}_1\,,\,\mathbf{r}_2\,,\,\mathbf{s}_2\,,\,\ldots\ldots\,,\,\mathbf{t}_1\,,\,\mathbf{t}_2\,,\,\ldots\ldots\,] \tag{5.12}$$

and the modified projection matrices $\mathbf{G}^*$ and $\mathbf{H}^*$ can be obtained by Eq.(5.4) with $\mathbf{V}$ replaced by $\mathbf{V}^*$. We partition the matrix $\mathbf{G}^*$ as before. The 2 by 2 submatrices of $\mathbf{G}_{cc}^*$ now become

$$\mathbf{G}_{cc}^* = \begin{bmatrix} \mathbf{r}_i^T\mathbf{B}\mathbf{r}_j & \mathbf{r}_i^T\mathbf{B}\mathbf{s}_j \\ \mathbf{s}_i^T\mathbf{B}\mathbf{r}_j & \mathbf{s}_i^T\mathbf{B}\mathbf{s}_j \end{bmatrix} \tag{5.13}$$

The 2 by 1 submatrices of $\mathbf{G}^*_{cr}$ now become

$$\mathbf{G}^*_{cr} = \begin{bmatrix} \mathbf{r}_i^T \mathbf{B} \mathbf{t}_j \\ \mathbf{s}_i^T \mathbf{B} \mathbf{t}_j \end{bmatrix} \tag{5.14}$$

The submatrix $\mathbf{G}^*_{rr}$ remains unchanged and thus is equal to $\mathbf{G}_{rr}$. The matrix $\mathbf{H}^*$ is identical to $\mathbf{G}^*$ with $\mathbf{B}$ replaced by $\mathbf{A}$. As can be seen, three quarters of the computational effort and half of the storage space can be saved through forming $\mathbf{G}^*_{cc}$ instead of $\mathbf{G}_{cc}$ and half of the computational effort and the storage space can be saved through forming $\mathbf{G}^*_{rc}$ and $\mathbf{G}^*_{cr}$ instead of $\mathbf{G}_{rc}$ and $\mathbf{G}_{cr}$.

we justify next that these modified matrices, which are distinguished by superscript *, can take the place of the original ones without changing the final solution. For this purpose, we notice that any complex conjugate pair of numbers can be *factorized* into their components and their units in the following way

$$[\ \alpha+\beta i \quad \alpha-\beta i\ ] = [\ \alpha \quad \beta\ ]\ \mathbf{L} \tag{5.15}$$

where $\mathbf{L}$ contains the real and imaginary unit and can be found as

$$\mathbf{L} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \tag{5.16}$$

Using this type of factorization, we can obtain relationship between the original iteration vectors and the modified iteration vectors as

$$\mathbf{U} = \mathbf{U}^*\ \mathbf{N} \qquad \text{and} \qquad \mathbf{V} = \mathbf{V}^*\ \mathbf{N} \tag{5.17}$$

where $\mathbf{N}$ is made up of $\mathbf{L}$ matrix in the first *nc* 2 by 2 diagonal blocks, *nr* unities along the rest of the diagonal and zero everywhere else. The relationship between the original projected matrices and the modified projected matrices is, accordingly, found as

$$\mathbf{G} = \mathbf{N}^T\ \mathbf{G}^*\ \mathbf{N} \qquad \text{and} \qquad \mathbf{H} = \mathbf{N}^T\ \mathbf{H}^*\ \mathbf{N} \tag{5.18}$$

This indicates that $\mathbf{G}^*$ and $\mathbf{H}^*$ are obtained by *equivalent transformation* from $\mathbf{G}$ and $\mathbf{H}$ respectively and, furthermore, they have the same transformation matrix $\mathbf{N}$. The fact that two *pencils* ($\mathbf{G}^*$, $\mathbf{H}^*$) and ($\mathbf{G}$, $\mathbf{H}$) are equivalent ensures [P1] that

$$\mathbf{\Lambda}^* = \mathbf{\Lambda} \qquad \text{and} \qquad \mathbf{X}^* = \mathbf{N}\ \mathbf{X} \tag{5.19}$$

This can be checked by the following procedure. Premultiplying Eq.(5.9) by $\mathbf{H}^{-1}$ and then introducing Eq.(5.18) into it, we have

$$\mathbf{N}^{-1}\ \mathbf{H}^{*-1}\ \mathbf{G}^*\ \mathbf{N}\ \mathbf{X} = \mathbf{X}\ \mathbf{\Lambda} \tag{5.20}$$

This confirms the validity of Eq.(5.19). Substituting $\mathbf{N}\ \mathbf{X}$ for $\mathbf{X}^*$ into counterpart of Eq.(5.10), we obtain $\mathbf{U}^* = \mathbf{V}^*\ \mathbf{N}\ \mathbf{X} = \mathbf{V}\ \mathbf{X}$, which is equal to $\mathbf{U}$. This concludes that the same eigensolution will be obtained by using these modified matrices. That is, the transformed iteration vectors represented by Eq.(5.11) have been corrected by the subspace iteration algorithm itself after one complete cycle of iteration and no inverse transformation is required to obtain the

final solution.

The main features of the above method can be summarized as follows. The same algorithm as that described in Box 5.1 is used except that the iteration vectors are modified to minimize computation effort and storage requirement. All matrices involved are real-valued and hence only real arithmetic is required during iteration process. Note that $\mathbf{G}^*$ and $\mathbf{H}^*$ are both real. The solution of the projected eigensystem can be easily accomplished by some available software packages, e.g., the RGG routines in EISPACK. [G1], [S1]

6. SOLUTION OF TRANSIENT PROBLEM :

After the eigenpairs, λ_j and $\mathbf{z}_j$, are found, the solution of Eq.(1.1) can simply be obtained by modal analysis. The general procedures, which are analogous to those used in the analysis of undamped systems, are described next. To take advantage of orthogonality properties developed before, we work with the reduced form of the equation of motion. Since the $2n$ eigenvectors constitute a complete basis, the super-vector $\mathbf{Q}(t)$ can be expressed as the sum of modal components [F1] ; that is,

$$\mathbf{Q}(t) = \sum_{j=1}^{2n} \mathbf{z}_j \, p_j(t) = \mathbf{Z}\,\mathbf{p}(t) \tag{6.1}$$

where $\mathbf{p}(t)$ is a vector whose elements $p_i(t)$ are generally complex-valued and

$$\mathbf{Z} = \left[\mathbf{z}_1 \, \mathbf{z}_2 \, ... \, \mathbf{z}_{2n} \right] \tag{6.2}$$

is the modal matrix which serves to transform from the normal coordinates $\mathbf{p}(t)$ to geometric coordinates $\mathbf{Q}(t)$. Introducing Eq.(6.1) into Eq.(2.16), pre-multiplying both sides by $\mathbf{z}_j^T$ and considering the orthonormality conditions represented by Eq.(3.8) and (3.17), we have

$$\dot{p}_j(t) - \lambda_j p_j(t) = \mathbf{w}_j^T \mathbf{f}(t) \qquad j = 1,2,3,....,2n \tag{6.3}$$

These are uncoupled equations and thus can be solved individually. For any one of them, the complete solution may be written as the sum of the homogeneous solution and the particular solution. The homogeneous solution is due to the initial conditions while the particular solution is due to the externally applied load. They are described separately in the following.

Free Vibration

The homogeneous solution to the Eq.(6.3) is represented by

$$p_j(t) = p_j(0) e^{\lambda_j t} \tag{6.4}$$

where the participation factor is determined from the initial conditions of the system by

$$p_j(0) = \mathbf{z}_j^T \mathbf{A}\,\mathbf{Q}(0) \tag{6.5}$$

which can be expanded as

$$p_j(0) = \lambda_j \, \mathbf{w}_j^T \mathbf{M} \, \mathbf{q}\,(0) + \mathbf{w}_j^T \mathbf{C} \, \mathbf{q}\,(0) + \mathbf{w}_j^T \mathbf{M} \, \dot{\mathbf{q}}\,(0) \tag{6.6}$$

As can be seen from Eq.(6.6), the participation factor can either be complex-valued or real-valued depending on the corresponding mode being complex or real. Combining the j^{th} complex conjugate pair of solution, we obtain the response due to this underdamped mode as

$$\mathbf{Q}\,(t) = p_j(0)\, e^{\alpha_j t}\, \boldsymbol{\Phi}_j + \bar{p}_j(0)\, e^{\bar{\alpha}_j t}\, \bar{\boldsymbol{\Phi}}_j \tag{6.7}$$

To understand the physical meanings of this modal solution, we can rearrange the equation to obtain the entirely real-valued expression. For this purpose, $p_j(0)$ can be expressed in terms of its modulus and phase angle as

$$p_j(0) = \frac{1}{2}\, a_j \, e^{i \theta_j} \tag{6.8}$$

and make use of Eq.(2.14a) and (3.5) and of the identity between exponential and trigonometric functions to rewite [B2] Eq.(6.7) as

$$\mathbf{Q}\,(t) = a_j \, e^{-\xi_j \omega_j t} [\, \boldsymbol{\Phi}_{Rj} \cos(\tilde{\omega}_j t + \theta_j) - \boldsymbol{\Phi}_{Ij} \sin(\tilde{\omega}_j t + \theta_j) \,] \tag{6.9}$$

The above equation indicates [V1] that the j^{th} underdamped modal solution consists of the sum of two exponentially decaying harmonic motions with a circular frequency, $\tilde{\omega}_j$, and a damping ratio, ξ_j . The component motions lag one another by one-quarter the period $\tilde{T}_j = 2\pi \,/\, \tilde{\omega}_j$, and they are in different spatial configurations. As a result, each point of the system undergoes a simple harmonic motion, but the configuration of the system does not remain constant but changes continously, repeating itself at intervals $\tilde{T}_j$. The quantity $\tilde{T}_j$ is thus known as the j^{th} damped natural period of the system.

Combining the j^{th} real pair of solution, we obtain the response due to this overdamped mode as

$$\mathbf{Q}\,(t) = p_j(0)\, e^{\beta_j t}\, \boldsymbol{\Psi}_j + \hat{p}_j(0)\, e^{\hat{\beta}_j t}\, \hat{\boldsymbol{\Psi}}_j \tag{6.10}$$

This indicates that the j^{th} overdamped modal solution consists of the superposition of two exponentially decaying functions, which are in different spatial configurations. Hence, the resulting motion is non-oscillatory. The configuration of the system is not constant, either.

The response of the system to an arbitrary initial conditions is thus given by the superposition of the various modal solutions

$$\mathbf{Q}\,(t) = \sum_{j=1}^{nc} a_j \, e^{-\xi_j \omega_j t} [\, \boldsymbol{\Phi}_{Rj} \cos(\tilde{\omega}_j t + \theta_j) - \boldsymbol{\Phi}_{Ij} \sin(\tilde{\omega}_j t + \theta_j) \,]$$

$$+ \sum_{j=1}^{nr} [\, p_j(0) e^{\beta_j} \, \boldsymbol{\Psi}_j + \hat{p}_j(0) e^{\hat{\beta}_j t} \, \hat{\boldsymbol{\Psi}}_j \,] \tag{6.11}$$

Forced Vibration

The particular solution to the Eq.(6.2) for an arbitrary loading $\mathbf{f}\,(t)$ may be written in terms of a convolution integral

$$p_j(t) = \mathbf{w}_j^T \int_0^t e^{\lambda_j(t-\tau)} \mathbf{f}(\tau)\, d\tau \tag{6.12}$$

For complex modes, it may be desirable to express the solution in terms of the real-valued quantities which are physically interpretable. To this end, we recall that the *impulse response function* for a damped SDOF system of unit mass is

$$h_j(t) = \frac{1}{\tilde{\omega}_j} e^{-\xi_j \omega_j t} \sin\tilde{\omega}_j t \tag{6.13}$$

and its derivative with respect to time is

$$\dot{h}_j(t) = e^{-\xi_j \omega_j t} \cos\tilde{\omega}_j t - \xi_j \omega_j h_j(t) \tag{6.14}$$

In terms of these well-defined functions, we have the following expression

$$e^{\lambda_j t} = [\, \dot{h}_j(t) + \xi_j \omega_j h_j(t) \,] + \tilde{\omega}_j h_j(t)\, \mathrm{i}^* \tag{6.15}$$

Accordingly, the real and imaginary part of $p_j(t)$ is represented by,

$$p_{Rj}(t) = \boldsymbol{\phi}_{Rj}^T \int_0^t [\, \dot{h}_j(t-\tau) + \xi_j \omega_j h_j(t-\tau) \,] \mathbf{f}(\tau)\, d\tau - \boldsymbol{\phi}_{Ij}^T \int_0^t \tilde{\omega}_j\, h_j(t-\tau) \mathbf{f}(\tau)\, d\tau \tag{6.16a}$$

$$p_{Ij}(t) = \boldsymbol{\phi}_{Ij}^T \int_0^t [\, \dot{h}_j(t-\tau) + \xi_j \omega_j h_j(t-\tau) \,] \mathbf{f}(\tau)\, d\tau + \boldsymbol{\phi}_{Rj}^T \int_0^t \tilde{\omega}_j\, h_j(t-\tau) \mathbf{f}(\tau)\, d\tau \tag{6.16b}$$

and this j^{th} modal response is represented by

$$\mathbf{Q}(t) = 2\, [\, \boldsymbol{\Phi}_{Rj}\, p_{Rj}(t) - \boldsymbol{\Phi}_{Ij}\, p_{Ij}(t) \,] \tag{6.17}$$

Eq.(6.17) indicates that the j^{th} underdamped modal solution consists of two vibratory motions with spatial configurations $\boldsymbol{\Phi}_{Rj}$, $\boldsymbol{\Phi}_{Ij}$ and temporal variations p_{Rj} , p_{Ij} respectively.

The j^{th} overdamped modal solution is simply

$$\mathbf{Q}(t) = [\, \boldsymbol{\psi}_j^T \int_0^t e^{\beta_j(t-\tau)} \mathbf{f}(\tau)\, d\tau \,]\, \boldsymbol{\Psi}_j + [\, \hat{\boldsymbol{\psi}}_j^T \int_0^t e^{\hat{\beta}_j(t-\tau)} \mathbf{f}(\tau)\, d\tau \,]\, \hat{\boldsymbol{\Psi}}_j \tag{6.18}$$

This simply indicates that the j^{th} overdamped modal solution consists of two exponentially decaying motions with spatial configurations $\boldsymbol{\Psi}_j$, $\hat{\boldsymbol{\Psi}}_j$ and decaying factors β_j , $\hat{\beta}_j$ respectively.

The response of the system due to the applied loadings can thus be represented by the superposition of the tributary modal solutions. That is,

$$\mathbf{Q}(t) = 2 \sum_{j=1}^{nc} [\, \boldsymbol{\Phi}_{Rj}\, p_{Rj}(t) - \boldsymbol{\Phi}_{Ij}\, p_{Ij}(t) \,] + \sum_{j=1}^{nr} [\, \{\, \boldsymbol{\psi}_j^T \int_0^t e^{\beta_j(t-\tau)} \mathbf{f}(\tau)\, d\tau \,\}\, \boldsymbol{\Psi}_j + \{\, \hat{\boldsymbol{\psi}}_j^T \int_0^t e^{\hat{\beta}_j(t-\tau)} \mathbf{f}(\tau)\, d\tau \,\}\, \hat{\boldsymbol{\Psi}}_j \,] \tag{6.19}$$

The general response of the system to both initial conditions and externally applied loadings can be obtained simply by combining the solutions represented by Eq.(6,11) and (6.19).

7. Examples

Two examples are presented in this section to illustrate the eigensolutions of the nonproportionally damped system and the capablitity of the extended subspace iteration method to find these solutions. For this purpose, we concentrate on the eigensolutions of the system at present. The eigensolutions consist of eigenvalues and eigenvectors. The eigenvalues contain the information of modal damping and frequencies of the system as has been shown in Eq.(2.14). The eigenvectors represent the vibration mode shapes of the system. These damped modes are different from normal modes, which are obtained by analyzing undamped system. As shown previously, the damped modes are composed of linear combination of two shapes, each of which undergoes a different temporal variation. For underdamped modes, all points in the system undergo motion at the same frequency but at different phase angle. In other words, not all parts of the system necessarily pass through their equilibrium position simultaneously as they do for normal modes. As a result, there is no stationary *node* or fixed shape in these underdamped modes. If undamped modes can be thought os as a *standing wave* pattern, then damped modes can be thought of as a *traveling wave* through a structure. The representation of these moving damped modes thus requires continuous description over the span of the period of this mode. For the present discussion, we simply plot in Appendix B the two tributary shapes, which is just the eigenvectors. But it should be understood that the actual mode shape, which is the linear combination of the two shapes, is no longer stationary.

Example 1 : Find the eigensolutions of a cantilever beam with a concentrated translational viscous-damper attached at tip. The geometry and physical properties of the beam are shown in Fig. 1. Consider c = 0 , c = 5 and c = 5000 three cases.

The complete eigensolutions obtained by RGG routines are shown in Appendix B. The CPU time required for these computations are shown in Table 1 and the CPU time required for the computations of the partial eigensolutions by the extended subspace iteration method is shown in Table 2.

Table 1 *CPU time required for complete eigensolutions*

damping	total CPU time
c = 0	9.0
c = 5	10.2
c = 5000	11.0

For the c = 0 case, all eigenvalues are pure imaginary and all eigenvectors are real. For the c = 5 and c = 5000 cases, both eigenvalues and eigenvectors are complex. Note the first mode in c = 0 case becomes the overdamped mode in the damped cases. Hence the first underdamped mode in the damped cases evolve from the second mode of the undamped case. The location of the damper may explain why the first mode of undamped case becomes overdamped modes of damped cases.

Table 2 *CPU time required for partial eigensolutions*

damping	no. of vectors & guard vectors	no. of iterations	CPU time for RGG	total CPU time
c = 5	3,1	6	3.1	6.7
	3,2	4	3.6	6.9
	3,3	5	8.0	12.7
c = 5000	3,1	10	5.4	10.5
	3,2	5	4.9	8.8
	3,3	6	9.9	14.9

Example 2 : Find the eigensolutions of the system shown in Fig. 2, where two beams are connected by a rotational viscous-damper. Consider c = 0 , c = 5 and c = 5000 three cases.

The complete eigensolutions obtained by RGG routines are shown in Appendix B. The CPU time required for these computations is shown in Table 3 and the CPU time required for the computations of the partial eigensolutions by the extended subspace iteration method are shown in Table 4.

Table 3 *CPU time required for complete eigensolutions*

damping	total CPU time
c = 0	34.0
c = 5	55.6
c = 5000	55.1

Contrary to example 1, the overdamped mode of damped cases evolve from the highest mode of the undamped case. The solution of undamped case appears to be those of two independent same beams. That is, there are ten modes associated with the vibration of one beam with the other beam motionless and the other ten modes are the other way around. But the solution of the damped cases clearly shows that there actually are ten symmetric and ten anti-symmetric modes. When damping is supplied into the system, all the symmetric modes are changed into complex modes while all the anti-symmetric modes remain unaffected and real. The fact that damping takes effect only in symmetric modes is physiclly reasonable because there is no relative rotational displacement, thus no relative rotational velocity, around damper when the system vibrates in anti-symmetric modes.

Table 4 *CPU time required for partial eigensolutions*

damping	no. of vectors & guard vectors	no. of iterations	CPU time for RGG	total CPU time
c = 5	5,1	9	17.1	29.8
	5,2	10	28.1	45.1
	5,3	9	38.1	55.8
	5,4	5	28.9	41.6
	5,5	5	37.6	51.7
c = 5000	5,1	20	34.4	61.0
	5,2	12	30.5	51.1
	5,3	9	33.0	51.8
	5,4	5	25.6	37.8
	5,5	5	34.8	48.9

From these examples, we find out an important feature pertaining to the solution of quadratic eigenproblem. This feature has been mentioned in secton 2, where we have shown that eigenvalues of the system appear in pairs. For complex modes, α_j and $\bar{\alpha}_j$ appear simultaneously. For real modes, β_j and $\hat{\beta}_j$ appear simultaneously. For the heavily damped case such as c = 5000 case in the above examples, the modulus of the primary eigenvalue of the first overdamped mode β_1 is very small while the modulus of the secondary eigenvalue $\hat{\beta}_1$ is very large. But they show up simultaneously in the subspace iteration process. As a result, the primary one converges faster during the iteration process while the secondary one converges slower or even will not converge at all if it is so large that it exceeds all the eigenvalues associated with those eigenvectors which span the subspace in consideration. This fact explains

why it takes more iterations to get solution for c = 5000 cases than for c = 5 cases in Table 2 and 4.

Remark 7.1 It is known that the computer time required for subspace iteration method to find the solution depends completely on how close the starting space is to the subspace spanned by the required eigenvectors. The trial vectors which are used to obtain the solutions in Table 2 and 4 are constructed by evenly distributing the n unities on nv vectors with zeroes at other locations. This arbitrarily chosen starting vectors is generally not a good one. Other choices, based on some knowledge or the characterisic of the system being solved, are more desirable. □

8. Closure

The subspace iteration method has been extended to extract the least dominant set of complex eigensolutions. These eigensolutions may be used to uncouple the equation of motion of a damped dynamic system. Each of the differential equations may be solved independently and then modal solutions may be superposed to give the response of the original system. Only symmetric damping and stiffness matrices, which corresponds to a damped nongyroscopic system, are considered in this report. In the conservative gyroscopic system the damping matrix, which sometimes is refered to as *Coriolis* matrix, is skew-symmetric. More generally, the damping matrix may be asymmetric if it includes both gyroscopic forces and viscous forces. Therefore, further study is required to accomodate the gyroscopic system or the general system.

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Appendix A: Algorithm for computing eigensolutions of damped systems

The general aspects for obtaining the eigenpairs of damped systems have been described in section 5. The computation algorithm is presented here. Since it requires additional effort to assemble and store matrices **A** and **B**, we work directly with matrices **M**, **C** and **K** during the iteration process. Moreover, the lower part on matrices **A** and **B** comes from an identity equation, the lower part of the new iteration matrix is just the upper part of the old iteration matrix. This implies that it is more efficient to perform the projected eigenproblem calculation every two inverse iteration instead of every one inverse iteration. For this reason, we partition the iteration matrice into upper and lower parts and work with these submatrices, which correspond directly to matrices **M**, **C** and **K**. The complete procedure is detailed in the following :

(1) Select trial initial vectors **U**

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_u \\ \mathbf{U}_l \end{bmatrix} \tag{A.1}$$

(2) Form **A U** (and store it in **W**)

$$\mathbf{W}_u = \mathbf{C}\,\mathbf{U}_u + \mathbf{M}\,\mathbf{U}_l \tag{A.2}$$

$$\mathbf{W}_l = \mathbf{M}\,\mathbf{U}_u \tag{A.3}$$

(3) Form **V**

$$\mathbf{V}_u = -\,\mathbf{K}^{-1}\,\mathbf{W}_u \tag{A.4}$$

$$\mathbf{V}_l = \mathbf{U}_u \tag{A.5}$$

(4) Form **A V** (and store it in $\underline{\mathbf{U}}$ ‡)

$$\underline{\mathbf{U}_u} = \mathbf{C}\,\mathbf{V}_u + \mathbf{M}\,\mathbf{V}_l \tag{A.6}$$

$$\underline{\mathbf{U}_l} = \mathbf{M}\,\mathbf{V}_u \tag{A.7}$$

(5) Scale $\mathbf{v}_j$, where j ranges from 1 to $2n$, such that

$$| \mathbf{v}_j^{\mathrm{T}}\,\mathbf{B}\,\mathbf{v}_j | = 1.0 \tag{A.8}$$

(6) Form **G**

$$\mathbf{G} = \mathbf{V}_u^{\mathrm{T}}\,\mathbf{W}_u + \mathbf{V}_l^{\mathrm{T}}\,\mathbf{W}_l \tag{A.9}$$

(7) Form **H**

$$\mathbf{H} = \mathbf{V}_u^{\mathrm{T}}\,\underline{\mathbf{U}_u} + \mathbf{V}_l^{\mathrm{T}}\,\underline{\mathbf{U}_l} \tag{A.10}$$

---

‡ The underline used here means that the matrix is temporarily employed to store some information.

(8) Solve $\mathbf{H}\,\mathbf{X} = \mathbf{G}\,\mathbf{X}\,\mathbf{\Lambda}^{-1}$ by the RGG routine from EISPACK.

(9) Compare eigenvalues $\mathbf{\Lambda}$ of the current and the previous iteration. If convergence is achieved , go to (12); otherwise continue.

(10) Form $\mathbf{A}\,\mathbf{U}$ (and store it in $\mathbf{W}$) and $\mathbf{U}$ for the next cycle of approximation.

$$\mathbf{W}_u = \underline{\mathbf{U}_u}\,\mathbf{X} \tag{A.11}$$

$$\mathbf{U}_u = -\,\mathbf{K}^{-1}\,\mathbf{W}_u \tag{A.12}$$

$$\mathbf{U}_l = \mathbf{V}_u\,\mathbf{X} \tag{A.13}$$

$$\mathbf{W}_u = \mathbf{C}\,\mathbf{U}_u + \mathbf{M}\,\mathbf{U}_l \tag{A.14}$$

$$\mathbf{W}_l = \mathbf{M}\,\mathbf{U}_u \tag{A.15}$$

(11) Go to (3)

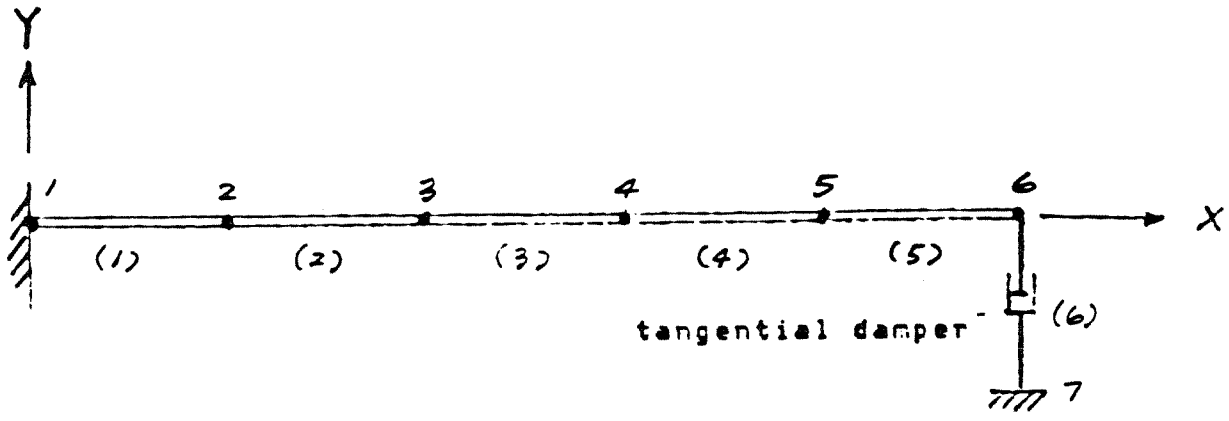
(12) Form eigenvectors $\mathbf{U}$

$$\mathbf{U}_u = \mathbf{V}_u\,\mathbf{X} \tag{A.16}$$

$$\mathbf{U}_l = \mathbf{V}_l\,\mathbf{X} \tag{A.17}$$

(13) Exit

In step (8) above, we can equally solve $\mathbf{G}\,\mathbf{X} = \mathbf{H}\,\mathbf{X}\,\mathbf{\Lambda}$ but $\mathbf{G}$ appears to be more diagonally dominant. Hence, $\mathbf{H}\,\mathbf{X} = \mathbf{G}\,\mathbf{X}\,\mathbf{\Lambda}^{-1}$ may be easier to be solved by the RGG routine.



nodal coordinates

node	1 coord	2 coord
1	0.	0.
2	1.0000	0.
3	2.0000	0.
4	3.0000	0.
5	4.0000	0.
6	5.0000	0.
7	5.0000	-1.0000

beam element material properties

modulus	0.1000e+03
area	0.1000e+01
inertia	0.1000e+01
density	0.1000e+01

edge nodal b.c.

node	1 b.c.	2 b.c.	3 b.c.
1	1	1	1
2	-1	0	0
3	-1	0	0
4	-1	0	0
5	-1	0	0
6	1	0	0
7	1	1	1

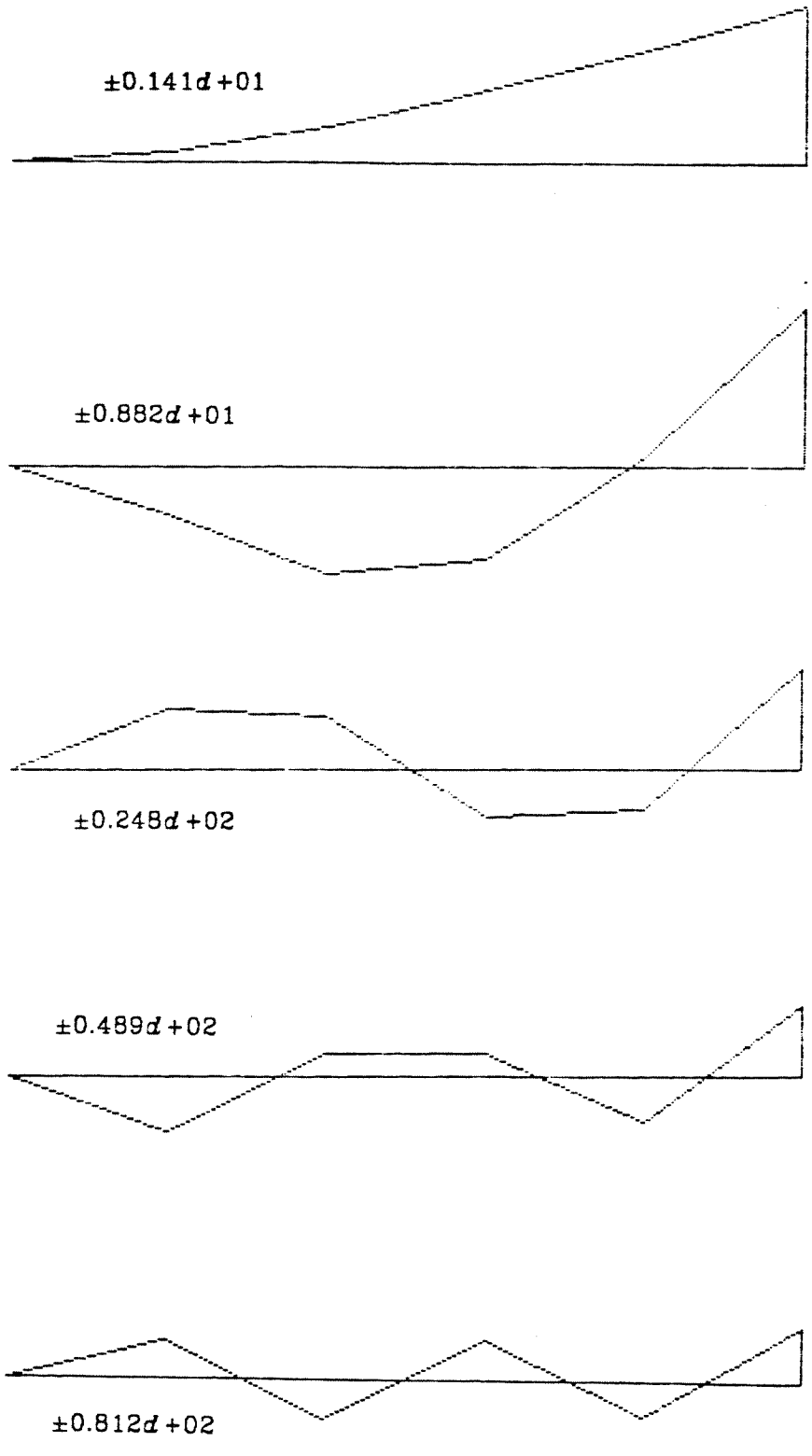


Figure 1 *Modal Solution for c = 0*

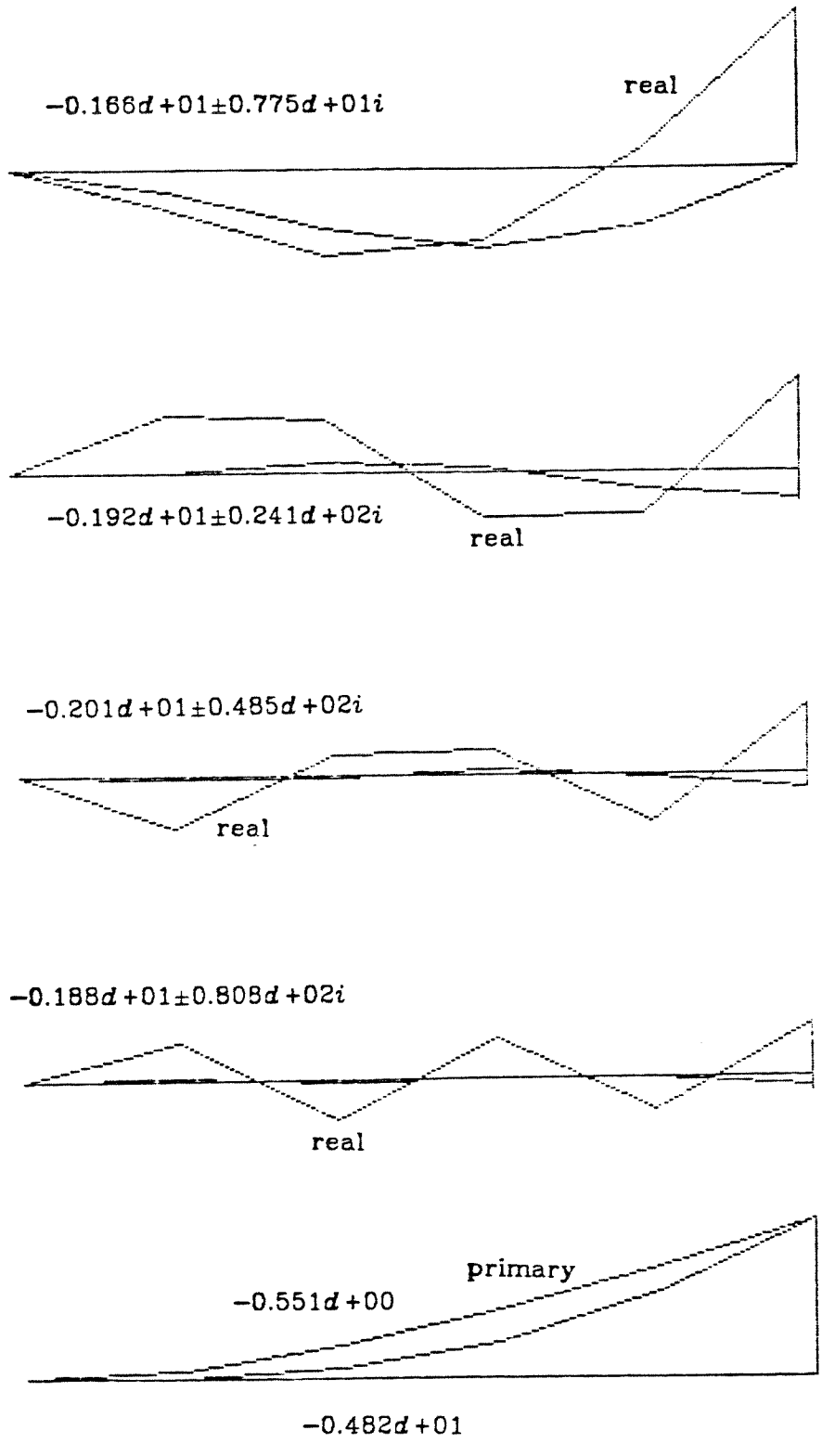


Figure 2 *Modal Solution for c = 5*

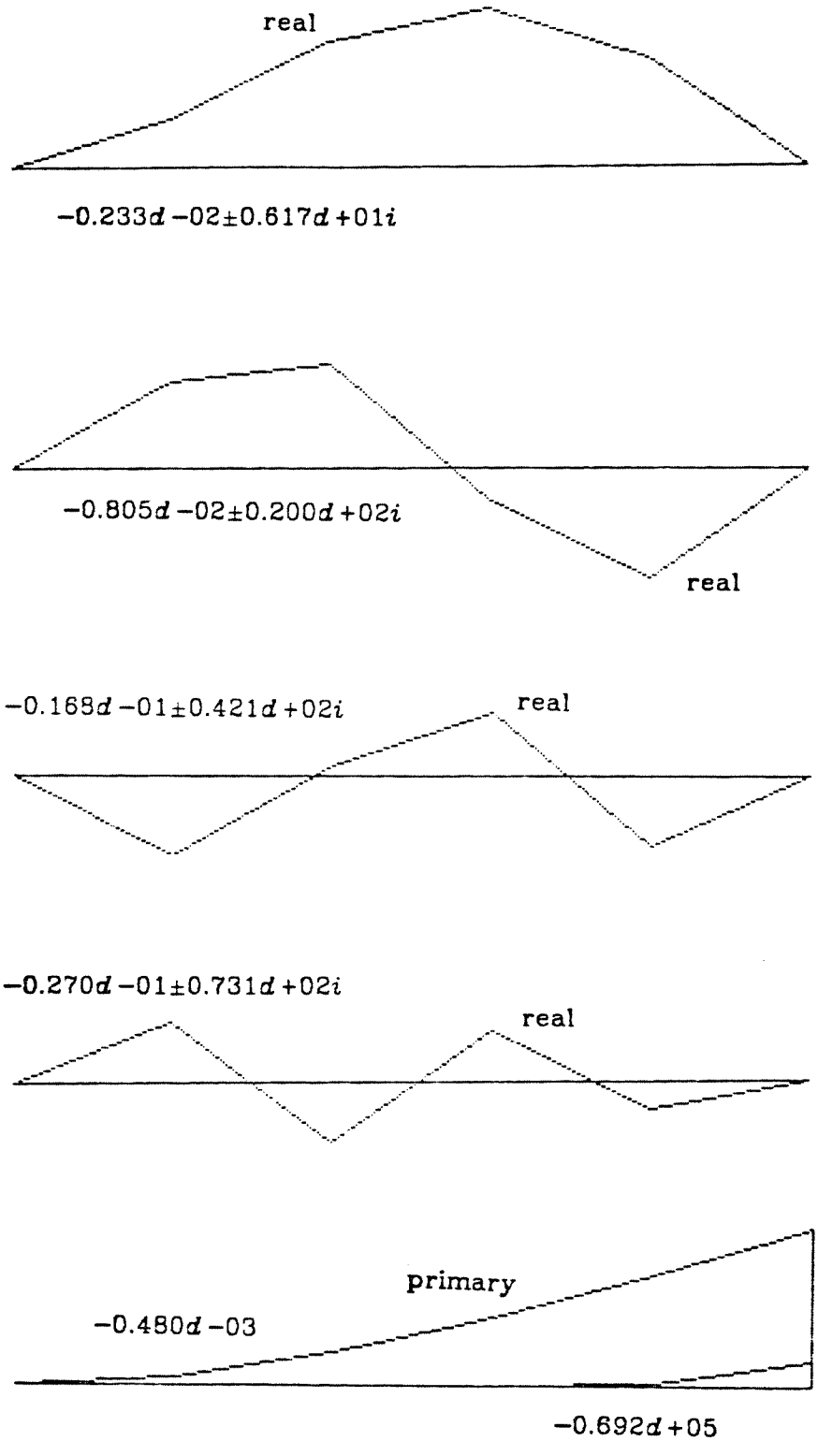
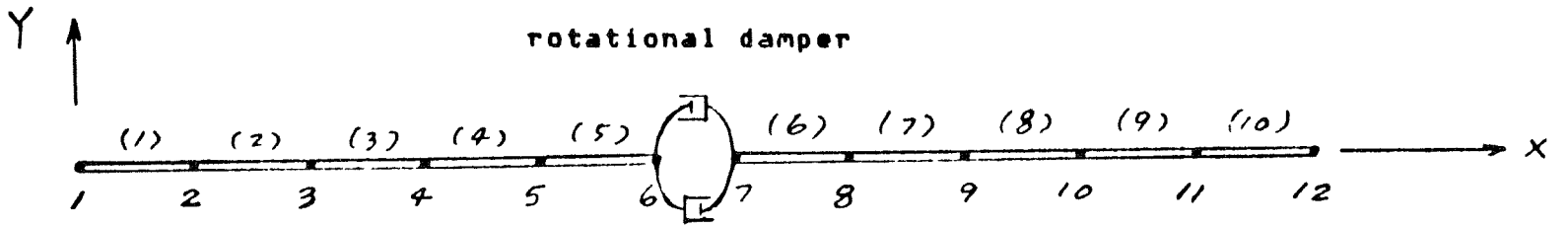


Figure 3 *Modal Solution for c = 5000*



nodal coordinates

node	1 coord	2 coord
1	0.	0.
2	2.0000	0.
3	4.0000	0.
4	6.0000	0.
5	8.0000	0.
6	10.0000	0.
7	10.1000	0.
8	12.1000	0.
9	14.1000	0.
10	16.1000	0.
11	18.1000	0.
12	20.1000	0.

n o d a l b. c.

node	1 b.c.	2 b.c.	3 b.c.
1	-1	1	0
2	-1	0	0
3	-1	0	0
4	-1	0	0
5	-1	0	0
6	-1	1	0
7	-1	1	0
8	-1	0	0
9	-1	0	0
10	-1	0	0
11	-1	0	0
12	1	1	0

beam element material properties

modulus	0.1000e+03
area	0.1000e+01
inertia	0.1000e+01
density	0.1000e+01

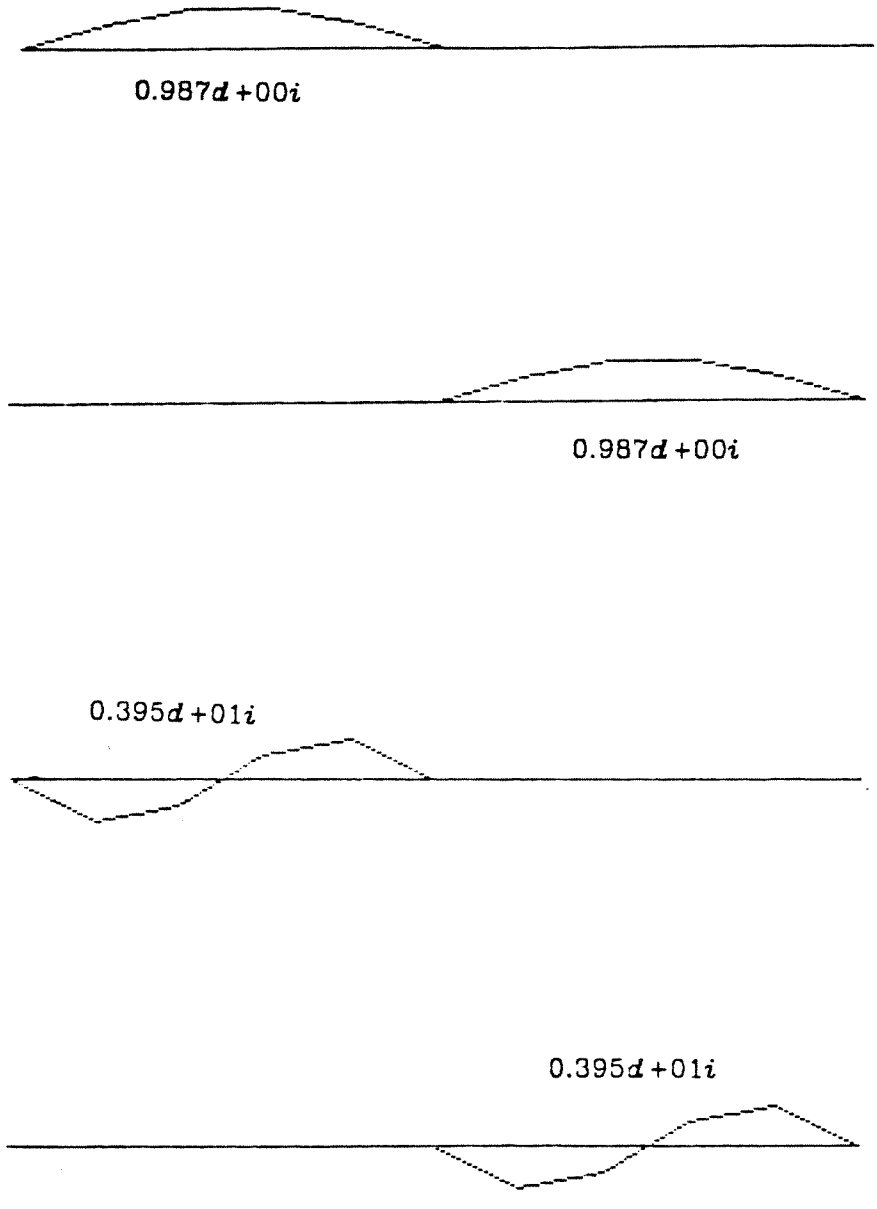
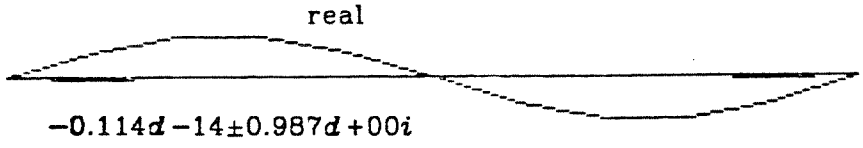
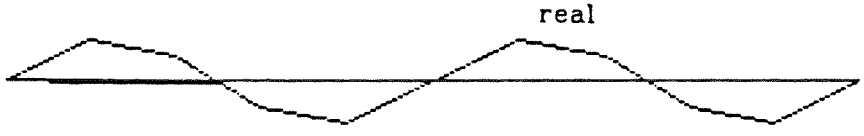


Figure 4 *Modal Solution for c = 0*

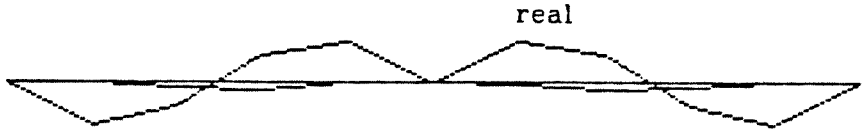


$-0.114d-14\pm0.987d+00i$

$-0.997d-01\pm0.996d+00i$



$-0.139d-14\pm0.395d+01i$



$-0.391d+00\pm0.406d+01i$

$-0.215d+02$

$-0.187d+03$

Figure 5 *Modal Solution for c = 5*

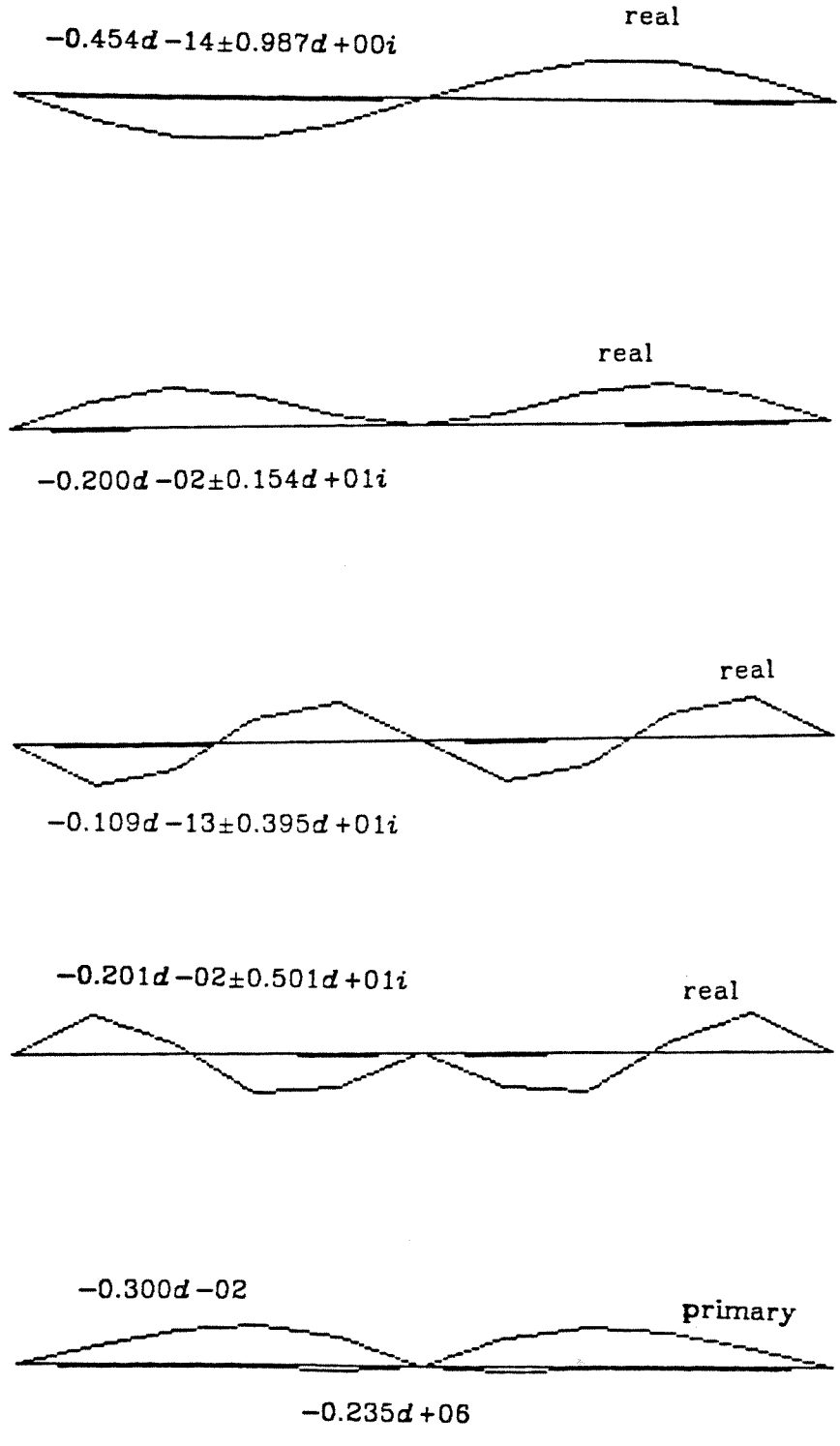


Figure 6 *Modal Solution for c = 5000*