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ABSTRACT

The question of the existence of the Freund-Karplus natural boundary in the two-body scattering amplitude is examined. We consider a possibility not previously discussed of a cancellation of singularities, but it is found to be insufficient to prevent the natural boundary from being present.

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I. INTRODUCTION

We wish to consider whether the singularities and sheet structure of the two-body scattering amplitude $A(s, z)$ implied by unitarity necessarily act together to produce a natural boundary, as was first suggested by Freund and Karplus.¹ In particular we shall show that the dense accumulation of singularities is not averted either by sheet structure or cancellations of singularities. Our claim is that the feature of the scattering amplitude which is discussed here does not prevent the occurrence of a natural boundary. We wish to emphasize, however, that no opinion is expressed as to what might happen if other mechanisms come into play. For simplicity we shall restrict our attention to a theory containing only a single spinless meson with $G = +1$. We are not concerned here with the dynamical question of whether such a system can successfully bootstrap itself.

As usual, we assume that the only singularities of $A(s, z)$ in s on the physical sheet are a pole for $s = m^2$ and branch points at the many-particle thresholds $s = n^2 m^2$, $n = 2, 3, \dots$. We have, of course, the same fixed singularities in the crossed-channel energy variable t and u . If we take $m = 1$ and define a new variable

$$z_0 = 1 + \frac{2}{s - 4} = \cos \theta_0, \quad (1)$$

the position of the pole in the t channel is $z = z_0$.

Assume the amplitude $A(s, z)$ on the physical sheet is given; then the elastic unitarity equation

$$A^{(2)}(s, z) = A(s, z) - \frac{i}{4} \sqrt{\frac{s-4}{s}} \int A(s, z) A^{(2)}(s, z^*) d\Omega \quad (2)$$

may be regarded as an integral equation for the amplitude $A^{(2)}(s, z)$ on

the first unphysical sheet in the s plane, reached by going through the cut associated with the square-root singularity at $s = 4$. Since $A(s, z)$ has a pole for $z = z_0$, $A^{(2)}(s, z)$ must also have a pole at $z = z_0$. Then by the method of successive iterations, which will be discussed shortly, one deduces the existence of Landau surfaces whose positions are given by $z = \cos n\theta_0$.¹⁻³ If one knows that these Landau surfaces were singular on all sheets of $A^{(2)}(s, z)$ in the s variable one could immediately deduce the existence of a natural boundary for $-1 \leq z_0 \leq 1$, since then for all z the singularities would densely accumulate at this interval on all sheets.

II. SINGULARITY STRUCTURE IN THE MANDELSTAM REGION

In analyzing the singularity structure implied by Eq. (2), one first "pinches" the pole at $z' = z_0$ with the pole at $z'' = z_0$ to deduce the existence of a new singularity in $A^{(2)}(s, z)$. Next one pinches this new singularity with the pole in $A(s, z)$ to produce still another singularity, and so forth. Let us define a sequence of functions containing these singularities, dropping unimportant factors:

$$D_1(s, z) = \int \frac{1}{z'' - z_0} \frac{1}{z' - z_0} d\Omega, \quad (3a)$$

$$D_n(s, z) = \int D_{n-1}(s, z'') \frac{1}{z' - z_0} d\Omega, \quad (3b)$$

It is well known from the work of Mandelstam⁴ that, if one considers $D_{n-1}(s, z)$ and analytically continues it, starting from the physical region where it is known to be real, by increasing z and keeping s fixed, then the first singularity encountered is located at $z = \cos n\theta_0$.

In other words continuation along a path λ_0 described by increasing z as above, circling $z = \cos n\theta_0$ with a small loop, and decreasing z back to its original value gives a nonvanishing discontinuity. This result can most easily be proved by induction, using Eqs. (3a) and (3b). Therefore in the region $\{z > 1, z_0 > 1\}$, which we shall call the Mandelstam region, there is no possibility of cancellation of singularities on sheets reached directly from the physical region (i.e., by paths such as λ_0), and hence these singularities are all present in $A^{(2)}(s, z)$. From this fact Freund and Karplus immediately deduced the existence of the natural boundary. What they did not discuss is that the function $D_n(s, z)$ also contains Landau surfaces for $z = \cos(n-2)\theta_0$, $z = \cos(n-4)\theta_0, \dots$ and that these surfaces are singular on sheets reached by first looping $z = \cos n\theta_0$, say. It therefore appears to be necessary to check whether in the vicinity of the proposed natural boundary the Landau surfaces get tangled up, revealing these additional singular surfaces in the separate iterations and thereby allowing a possibility of systematic cancellation of singularities in the complete expression $A^{(2)}(s, z)$.

In our discussion so far we have made no mention of the singularities arising from pinches involving the threshold singularities. Although the resulting singularities also accumulate densely at the interval $-1 \leq z_0 \leq 1$, they do not have the same locations and therefore could not reasonably be expected simply to cancel the singularities we have already found. The more difficult problem is that they introduce additional sheet structure which does in fact have the feature that the singularities at $z = \cos n\theta_0$ may be absent on certain sheets. This point still requires

further investigation. It is obvious, of course, that the inclusion of singularities arising from pinches with u-channel singularities would not affect our conclusions.

III. DESCRIPTION OF PATHS

For a given value of z , the Landau surfaces in the z_0 plane will all lie inside an ellipse passing through z with foci at $z_0 = \pm 1$. In fact, putting $z = \cos \theta$, the positions of the Landau surfaces in the z_0 plane are given by¹

$$z_0 = \cos \left(\frac{\theta}{n} + \frac{2\pi k}{n} \right), \quad \text{for } k = 0, 1, \dots, n-1 \quad (4)$$

and $n = 1, 2, \dots$

To be specific we shall consider the discontinuity of $A^{(2)}(s, z)$ arising from continuation along the following path λ : Start at a physical point $P(-1 < z < 1, z_0 > 1)$ and increase $\text{Im } z$ to ϵ (small and positive, for convenience). Now fix z there and increase $\text{Im } z_0$ so that $\text{Im } z_0 > \epsilon$; next decrease $\text{Re } z_0$ until $\text{Re } z_0$ coincides with a value determined by $z = \cos m\theta_0$, for any given m . Then decrease $\text{Im } z_0$ to almost reach the point satisfying $z = \cos m\theta_0$, circle it with a small loop (i.e., not enclosing any other Landau surfaces) and then retrace the path back to P . If all but a finite number of paths of this type give rise to a nonvanishing discontinuity, then it follows that $A^{(2)}(s, z)$ contains the natural boundary, even when considered as a function of two complex variables.

A few of the surfaces $z = \cos n\theta_0$ in the $\text{Re } z - \text{Re } z_0$ plane are shown in Fig. 2 for $\text{Im } z = 0$. When $\text{Im } z = \epsilon$ they are changed quite drastically; for one thing they no longer intersect (one could, for

example, imagine $\text{Im } z_0$ to be out of the plane of the paper). As discussed in Section II, the path λ_0 gives a nonzero discontinuity. The question therefore becomes whether the path λ can be slipped back (homotopy) to coincide with λ_0 without getting "hooked up" on curves of higher n ; in the former case it will give the same discontinuity.

It is very difficult to demonstrate the homotopy of λ and λ_0 discussed above in terms of the variables z and z_0 . However, with some care the question becomes tractable if we use the variables θ and θ_0 instead. In these variables the relevant equations are

$$\begin{aligned} \text{for } n &= 0, 1, \dots, \\ \theta &= n\theta_0 + 2\pi k \\ \text{and } k &= 0, \pm 1, \pm 2, \dots \end{aligned} \tag{5}$$

There is, of course, a lot of redundancy due to the multiple-valuedness of the arc cos function. One could, for example, consider only those surfaces which pass through the origin. The important point is that $\text{Im } \theta_0 = \frac{1}{n} \text{Im } \theta$ for the n th set of surfaces. However, the detailed demonstration that λ and λ_0 are homotopic still requires picturing the situation in four dimensions and is relegated to an appendix in which the homotopy is explicitly displayed.

IV. CONCLUSION

To summarize the situation, each successive set of singularities is on a smaller and smaller ellipse, for a given z . We have proved that a Landau surface of given n will be singular when approached starting from the physical region, say, if singularities of higher n have not already been looped around. It therefore follows that analytic continuation

of $A^{(2)}(s,z)$ is impossible along any path (even if it depends on both variables) which crosses $-\infty < s < 3$. Actually, as we mentioned earlier, the proof is rigorous only if we neglect the singularities produced by pinches involving threshold branch points.

We remark that recently Omnes and Karplus (unpublished work) have given a satisfactory proof of the same result using quite different methods.

APPENDIX

We wish to show here that the paths λ and λ_0 , defined in Sections II and III respectively, are homotopic. We first describe λ and λ_0 in the variables θ and θ_0 and then pictorially display the homotopy. In order to draw in a plane curves in the space of two complex variables we use the technique⁵ of representing a point by a vector, with the understanding that the point to which it is attached gives the real coordinates while its magnitude and direction give the two imaginary coordinates.

In Fig. 3 we display the path λ_0 in the complex θ plane, since λ_0 is fixed for this curve. To be explicit we draw the case $n = 2$. In Fig. 4, we draw points along the path λ_0 (numbered successively) in the full four-dimensional space, and similarly in Fig. 5 for the path λ . The claim is that if one stares at Figs. 4 and 5 long enough λ_0 and λ are seen to be homotopic, but to make it somewhat easier to visualize we draw a case in Fig. 6 which is roughly halfway between.

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FIGURE CAPTIONS

1. The path λ in the z_0 plane.
2. The paths λ and λ_0 in the $\text{Re } z - \text{Re } z_0$ plane.
3. The path λ_0 in the θ plane.
4. The path λ_0 in the four-dimensional $\theta - \theta_0$ space.
5. The path λ in the four-dimensional $\theta - \theta_0$ space.
6. A path homotopic to both λ and λ_0 in the four-dimensional $\theta - \theta_0$ space.

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Fig. 1

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Fig. 2

Fig. 3

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Fig. 4

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Fig. 5

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Fig. 6

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