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Note on Linear Operators.

By *Edward O. Thorp*, Los Angeles, Cal.

Introduction. This note answers two questions. 1.) If X and Y are Banach spaces, with Y inseparable, does it ever happen that all the bounded operators from X into Y are compact? 2.) If X and Y are Banach spaces, consider the following subset of the space of bounded operators from X into Y with the uniform topology: the operators (compact operators) with both a) non-dense range in Y , and b) which send some non-zero vector into zero. Does this subset of operators (compact operators) ever have an interior point?

Notation. X and Y denote Banach spaces. $[X, Y]$ is the set of bounded operators from X into Y . $[X, Y]_c$ is the subspace of compact operators from X into Y . $l^p(S)$, $\infty > p \geq 1$, is the set of scalarvalued functions on S , an arbitrary set, having at most a countable number of non-zero coordinates and such that the norm of the function x (where we denote $x(s)$, s in S , by x_s) is given by

$$\|x\| = \left(\sum_{s \in S} |x_s|^p \right)^{\frac{1}{p}} < \infty.$$

We define $l^\infty(S)$, $c_0(S)$ and $c(S)$ analogously [4]. If S is countable we omit the (S) . With a few exceptions, explained when they occur, the notation is taken from [8].

Some known cases in which all bounded operators are compact.

(a) $[l^p, l^q]$, $\infty > p > q \geq 1$; $[c_0, l^q]$, $[c, l^q]$, $\infty > q \geq 1$. For $[l^p, l^q]$, $\infty > p > q > 1$ see [3, p. 700] and [7, thm. 1]. The other cases are contained in (b), below.

(b) $[X, l^1]$ where X is either quasi-reflexive [2, thm. 5.1], or weakly complete [6, Cor. 4.11], or has a separable conjugate space [6, Cor. 4.11]. [Note: 4.11 states "X a separable conjugate space . . ." This is evidently false as the example $X = l^1$ (the separable conjugate of c_0) shows. The proof of 4.11 indicates the corrected form.]

(c) Either X or Y is finite dimensional.

(d) If all the operators in $[Y', X']$ are compact, then all the operators in $[X, Y]$ are compact [8].

All the Y in the above list are separable. Also the range of a compact operator is separable [1]. These facts motivated question 1.

The spaces $l^q(T)$, $1 \leq q < \infty$, are not separable when T is uncountable, so the following theorem answers question 1 by showing that all the bounded operators may be compact even if Y is not separable.

Theorem 1. *All the bounded operators are compact in the following cases:*

(a) $[l^p(S), l^q(T)]$, $\infty > p > q \geq 1$; $[c_0(S), l^q(T)]$, $[c(S), l^q(T)]$, $\infty > q \geq 1$.

(b) $[X, l^1(T)]$ where X is either quasi-reflexive or weakly complete or has a separable conjugate space.

Proof. Let A be a bounded linear operator. For case (a), let $X(S)$ be the domain and $Y(T)$ the range. If (x_n) is a bounded sequence in $X(S)$, (x_n) has only a countable number of non-zero coordinates among the totality of its members, so the closed span of (x_n) , written $\overline{sp}(x_n)$, is linearly isometric to $X(J)$. Similarly, $\overline{sp}(Ax_n)$ is isometric to $Y(L)$ where L is either finite or J . Since A is continuous and linear, $A(\overline{sp}(x_n)) \subset \overline{sp}(Ax_n) = \overline{sp} A(x_n)$. Therefore we can consider the restriction of A as an operator $\tilde{A}$ on $\overline{sp}(x_n)$ into $\overline{sp}(Ax_n)$. But the list of known results, listed in section 2, shows $\tilde{A}$ is compact hence $(\tilde{A}x_n) = (Ax_n)$ has a convergent subsequence, proving A is compact.

The preceding argument applies verbatim to case (b) because each of the properties involved is inherited by closed subspaces. For quasi-reflexivity, note that a norm-closed subspace of a weakly complete space is weakly closed, hence weakly complete [5; ch. 6 thm. 22]. If X' is separable, use the Hahn-Banach extension theorem.

Theorem 2. *If $X = l^p(S)$, $Y = l^q(T)$, $\infty > p > q > 1$, with S and T uncountable, then every bounded linear operator A from X into Y has a non-dense range, a non-zero kernel, and is compact.*

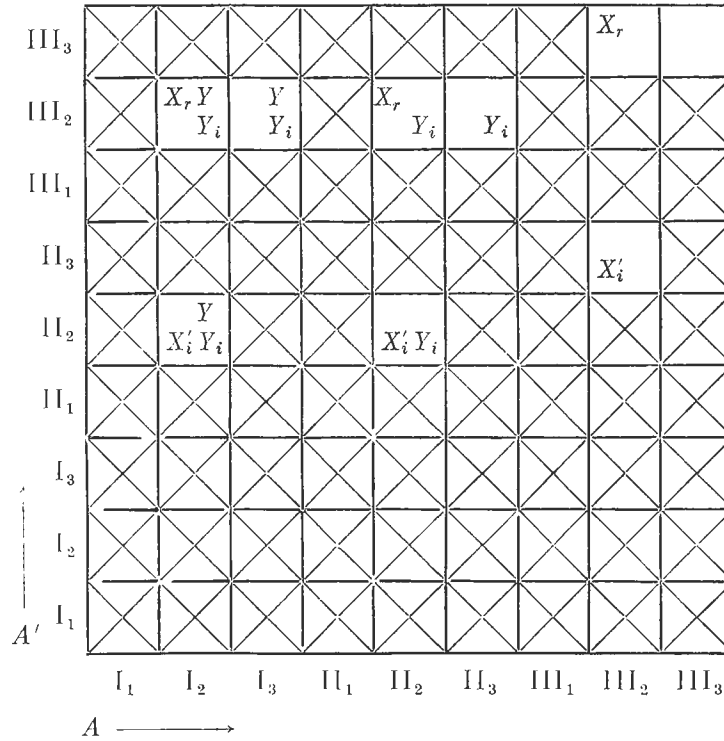
Proof. We will need the following lemmas.

Lemma 1. If A is compact, the range of A is separable [1].

Lemma 2. If A is compact and A has a bounded inverse, the dimensions of X and of the range of A are finite and equal.

The proof is immediate.

State Diagram for Compact Operators.



Y : impossible if Y is complete.

X_r : impossible if X is reflexive.

Y_i : impossible if Y is inseparable.

X'_i : impossible if X' is inseparable.

Now refer to the "state diagram" of [8] (or [9]). Lemma 1 shows that states I and II are impossible for A when Y is not separable. Similarly, if X' is not separable, states I and II are impossible for A' . Lemma 2 says that if X and Y are both infinite dimensional (the assumption throughout this paper), state I is impossible for either A or A' .

The result actually is the precise state diagram for compact operators. To prove this requires a list of examples to show that each remaining state is possible except as restricted. Since we only need to know which states are not possible for compact operators, we do not give these examples here.

Since Theorem 1 shows that all the bounded operators are compact, the state diagram for compact operators applies. But X' and Y are inseparable and X is reflexive so the only state possible for $A \in [X, Y] = [X, Y]_c$ is (III_3, III_3) . Thus question 2 is answered.

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