Recursion theory and countable Borel equivalence relations

by

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Abstract

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We investigate the question of what equivalence relations from recursion theory are universal countable Borel equivalence relations. While this problem is interesting in its own right, it has also been a particularly rich source of connections between recursion theory, countable Borel equivalence relations, and Borel combinatorics. Tools developed by this investigation have proved very applicable to other problems in these fields.

In Chapter 2, we prove a model universality theorem, and introduce several themes of the thesis. A corollary of this first theorem is that polynomial time Turing equivalence is a universal countable Borel equivalence relation.

Slaman and Steel have shown that arithmetic equivalence is a universal countable Borel equivalence relation. In Chapter 3, we combine this fact with the existence of a cone measure for arithmetic equivalence to prove several structural results about universal countable Borel equivalence relations in general. We show that universality for Borel reductions coincides with universality for Borel embeddings, and a universal countable Borel equivalence relation is always universal on some nullset with respect to any Borel probability measure. We also settle questions of Thomas [51] and Jackson, Kechris, and Louveau [20] by showing that a smooth disjoint union of non-universal countable Borel equivalence relations is non-universal. This result can be significantly strengthened by assuming a conjecture of Martin which states that every Turing invariant function is equivalent to a uniformly Turing invariant function on a Turing cone.

In Chapter 4, we investigate uniformity of homomorphisms among equivalence relations from recursion theory. We pose several open questions in this context, and investigate the implications of the uniformity that they imply. We introduce the concept of a Borel metric on a countable Borel equivalence relation, and show that this concept is closely connected to a weakening of the notion of a uniform homomorphism. Using this language of metrics and the machinery of Slaman and Steel for proving the universality of arithmetic equivalence, we construct an example of a homomorphism between equivalence relations coarser than Turing equivalence which is not uniform on any pointed perfect set. This is the first example of

a nonuniform homomorphism in this sort of recursion-theoretic context, and it places some limits on how abstract a proof of Martin's conjecture could be.

In Chapter 5, we turn to the question of whether recursive isomorphism is a universal countable Borel equivalence relation. Improving prior results of Dougherty and Kechris [12] and Andretta, Camerlo, and Hjorth [2], we show that recursive isomorphism on 3^ω is a universal countable Borel equivalence relation. We isolate a question of Borel combinatorics for which a positive answer would imply that recursive isomorphism on 2^ω is universal. We show that this question is equivalent to the problem of whether ω many 2-regular Borel graphs on the same space can be simultaneously Borel 3-colored so that there are no monochromatic points. We then show that this question has an affirmative answer if and only if many-one equivalence on 2^ω is a uniformly universal countable Borel equivalence relation. Thus, we have an exact combinatorial calibration of the difficulty of this universality problem.

In Chapter 6, we consider the question of whether there exist disjoint Borel complete sections for every pair of aperiodic countable Borel equivalence relations. We show that this question is very robust, and has many equivalent formulations. A positive answer to this question would positively answer the combinatorial question of the previous paragraph, while a negative answer would settle several open questions of Borel combinatorics. We also show that this question is true in both the measure and category context, in all its equivalent forms. One application of this fact is that every Borel bipartite 3-regular graph has measurable and Baire measurable edge colorings with 4 colors. This is a descriptive analogue of a special case of Vizing's theorem on edge colorings from classical combinatorics. Finally, we see that recursive isomorphism on 2^ω is measure universal. Thus, purely measure-theoretic tools cannot be used to prove that it is not universal.

To my parents, with gratitude.

Contents

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Chapter 1

Background

1.1 Borel equivalence relations

Our basic references for descriptive set theory and effective descriptive set theory are the books of Kechris [23] and Sacks [42].

The field of descriptive set theory gives a general setting for studying the difficulty of classification problems in mathematics. A *Borel equivalence relation* E on a Polish space X is an equivalence relation on X that is Borel as a subset of $X \times X$. If F is also a Borel equivalence relation on the Polish space Y, then we say that E is *Borel reducible* to F, noted $E \leq_B F$, if there is a Borel function $f : X \to Y$ so that for all $x, y \in X$, we have $xEy \iff f(x)Ff(y)$. Such a function induces an injection from X/E to Y/F. We can view Borel reducibility as comparing the difficulty of classifying E and F by invariants, where if $E \leq_B F$, then any complete set of invariants for F can be used as a set of complete invariants for E. The class of Borel equivalence relations under $\leq_B$ has a rich structure that been a major topic of research in descriptive set theory in the past couple decades. The field has had remarkable success both in calibrating the difficulty of classification problems of interest to working mathematicians, and also in understanding the structure of the space of all classification problems.

A Borel equivalence relation E is said to be *smooth* if E is Borel reducible to $\Delta(2^\omega)$, the equality relation on 2^ω. Rephrasing, smooth Borel equivalence relations are those that we can classify using real numbers as invariants. By a theorem of Silver [45], every Borel equivalence relation E either has either countably many equivalence classes, or $\Delta(2^\omega) \leq_B E$. An early structural result in the field of Borel equivalence relations is the dichotomy theorem of Harrington, Kechris, and Louveau, which shows that there is a canonical obstruction to a Borel equivalence relation being smooth. Let E_0 be the equivalence relation of equality mod finite on 2^ω. Simple ergodicity arguments show that E_0 is not smooth.

Theorem 1.1.1 (Harrington, Kechris, and Louveau [15]). *If E is a Borel equivalence relation, then either E is smooth, or $E_0 \leq_B E$.*

Beyond E_0, the structure of the Borel equivalence relations under $\leq_B$ becomes much more complex. For example, by a result of Louveau and Veličković [31], the partial order of inclusion mod finite on $\mathcal{P}(\omega)$ embeds into the partial order of $\leq_B$ on the Borel equivalence relations.

A Borel equivalence relation is said to be *countable* if all of its equivalence classes are countable. The class of countable Borel equivalence relations is still quite complicated. For instance, by a theorem of Adams and Kechris [1], there are continuum many $\leq_B$-incomparable countable Borel equivalence relations. However, the countable Borel equivalence relations also have several nice structural properties that are not shared by the class of Borel equivalence relations in general.

Suppose that E is a countable Borel equivalence relation on the Polish space X. Then by Lusin-Novikov uniformization (18.10, 18.15 in [23]), there must be a countable set $\{\varphi_i\}_{i\in\omega}$ of partial Borel functions $\varphi_i : X \to X$ such that xEy if and only if there exists an i and j such that $\varphi_i(x) = y$ and $\varphi_j(y) = x$. Conversely, a *generating family of partial Borel functions* on X is any collection $\{\varphi_i\}_{i\in\omega}$ of partial Borel functions on X that is closed under composition and includes the identity function. We let $E^X_{\{\varphi_i\}}$ be associated equivalence relation where $xE^X_{\{\varphi_i\}}y$ if there exists an i and j such that $\varphi_i(x) = y$ and $\varphi_j(y) = x$. In this case we say $xE^X_{\{\varphi_i\}}y$ via (i, j).

A countable Borel equivalence relation E is said to be *universal* if for all countable Borel equivalence relations F, we have $F \leq_B E$. Universal countable Borel equivalence relations exist [10]–this follows from the representations of the previous paragraph–and they arise naturally in many areas of mathematics. Isomorphism of finitely generated groups [55], conformal equivalence of Riemann surfaces [17], and isomorphism of locally finite connected graphs [25] are all universal countable Borel equivalence relations[1].

Recursion theory is a source of many interesting countable Borel equivalence relations, and a central topic of this thesis is the question of their position in the hierarchy of countable Borel equivalence relations. While this question is interesting in its own right, it has been a rich source of connections between these fields and has stimulated results in both areas.

For example, motivated by recursion-theoretic concerns, Slaman and Steel defined the notion of hyperfiniteness [47], which has since become a fundamental part of the study of countable Borel equivalence relations. A countable Borel equivalence relation E is said to be *hyperfinite* if $E = \bigcup_{i\in\omega} F_i$ where $F_0 \subseteq F_1 \subseteq \ldots$ is an increasing sequence of Borel equivalence relations each having only finite equivalence classes. Slaman and Steel showed that Turing equivalence is not hyperfinite and they proved that a Borel equivalence relation is hyperfinite if and only if it is induced by a Borel $\mathbb{Z}$ action. They obtained these results independently from the work that was beginning on the field of Borel equivalence relations at the time. Their last result is due independently to Weiss [57]. In the same paper, Slaman and Steel posed some further structurability questions about Turing equivalence. These were answered

[1]While the these examples are not strictly countable Borel equivalence relations, they are *essentially countable* as we can isolate countably many representatives of each equivalence class in a Borel way.

by Kechris [22] via methods associated with the concept of amenability. Amenability has since played a large role in the study of Borel equivalence relations.

A major theme of recursion theory is that its degree structures are often as rich and complicated as possible. This viewpoint evolved though the 1970s and 1980s from earlier, more naïve intuitions. For instance, in the early history of r.e. degrees, Schoenfield conjectured [43] that the r.e. degrees were an ω-saturated uppersemilattice. This would imply that the theory of the r.e. degrees admits quantifier elimination, and would hence be decidable. Schoenfield's conjecture was quickly refuted based on the existence of a minimal pair of r.e. degrees, but it took several years before it was realized how completely wrong the conjecture was. The theory of the r.e. degrees is in fact as undecidable as possible; it can interpret true arithmetic [40].

The natural manifestation of this phenomenon of richness and complexity in the context of countable Borel equivalence relations would be for many recursion-theoretic equivalence relations to be universal. Along these lines, Kechris [25] has conjectured that Turing equivalence is a universal countable Borel equivalence relation. This conjecture remains open, but a few other equivalence relations from recursion theory are known to be universal. For example, Slaman and Steel have shown that arithmetic equivalence is a universal countable Borel equivalence relation [33].

If E and F are Borel equivalence relations on the Polish spaces X and Y, then a *homomorphism* from E to F is function $f : X \to Y$ such that xEy implies $f(x)Ff(y)$. Likewise, a *cohomomorphism* from E to F is a function $f : X \to Y$ such that $f(x)Ff(y)$ implies xEy. Hence, a Borel reduction from E to F is a Borel function that is simultaneously a homomorphism and a cohomomorphism from E to F. A *weak Borel reduction* from E to F is a Borel homomorphism from E to F that is countable-to-one. Say that a countable Borel equivalence relation E is *weakly universal* if for every countable Borel equivalence relation F, there is a weak Borel reduction from F to E. We can equivalently characterize the weakly universal countable Borel equivalence relations as follows. (The equivalence of (1) and (3) is due to Kechris and Miller. We establish the equivalence of (2) with (1) and (3) in Section 1.3):

Theorem 1.1.2 (Kechris-Miller [51])**.** *Suppose E is a countable Borel equivalence relation. The following are equivalent.*

1. *E is weakly universal.*
2. *If F is any countable Borel equivalence relation, then there exists an injective Borel homomorphism from F to E.*
3. *There exists a universal countable Borel equivalence relation F such that $F \subseteq E$.*

Weakly universal countable Borel equivalence relations are known to be quite complicated in the hierarchy of countable Borel equivalence relations. Perhaps the strongest known theorem in this direction is due to Thomas:

Theorem 1.1.3 (Thomas [51]). *If E is a weakly universal countable Borel equivalence relation, then E is not Borel reducible to any countable Borel equivalence relation arising as the orbit equivalence relation of a free Borel action of a countable group.*

From the perspective of recursion theory, weakly universal countable Borel equivalence relations are those which permit us to do coding. It is easy to see that almost all equivalence relations from recursion theory are weakly universal. By a result of Feldman and Moore [14], every countable Borel equivalence relation on a Polish space X arises as the orbit equivalence relation of a Borel action of $\mathbb{F}_\omega$ on X. Let $\{w_i\}_{i\in\omega}$ be a recursive listing of all the words of $\mathbb{F}_\omega$, and let $f : X \to 2^\omega$ be a Borel injection. Then define the function $\hat{f} : X \to 2^\omega$ by

$$\hat{f}(x) = \bigoplus_{i\in\omega} f(w_i \cdot x)$$

so that $\hat{f}(x)$ is the recursive join of the $f(w_i \cdot x)$. It is clear that for any w_i, we can obtain $\hat{f}(w_i \cdot x)$ from $\hat{f}(x)$ by recursively permuting the columns of $\hat{f}(x)$. Hence, any equivalence relation on 2^ω closed under such permutations is weakly universal.

It is an open question of Hjorth whether every weakly universal countable Borel equivalence relation is universal.

Question 1.1.4 (Hjorth [2], [20]). *If E is a weakly universal countable Borel equivalence relation, must E be universal?*

In general, there is no relationship between containment and Borel reducibility amongst countable Borel equivalence relations. Thomas [53] has shown that there exist countable Borel equivalence relations $E \subseteq F$ such that $E \not\leq_B F$. Thus, if Hjorth's Question 1.1.4 has an affirmative answer, then the proof must use weak universality in an essential way.

There is a rather simple-minded approach that one might try to show that Hjorth's question has a positive answer. We have seen above that if E is weakly universal, then there is a scheme for coding any countable Borel equivalence relation F into E using an injective homomorphism. For example, the functions $\hat{f}$ as defined above are such a scheme for equivalence relations from recursion theory. One might then attempt to show that a sufficiently generic instance of such an injective homomorphism (e.g. forming $\hat{f}$ from a sufficiently generic f) must yield a Borel reduction.

This approach is doomed to failure. For instance, there cannot be any such function $\hat{f}$ witnessing that Turing equivalence is a universal countable Borel equivalence relation; it is known that if Turing equivalence is universal, this cannot be witnessed in a uniform manner. The situation appears quite complex and closely connected to deep issues of uniformity in recursion-theoretic constructions.

The most well known such connection is to a longstanding conjecture of Martin which would strongly refute Kechris' conjecture that Turing equivalence is universal, and therefore settle Hjorth's question in the negative. Martin's conjecture and Kechris' conjecture lie at opposite ends of a spectrum of possible conceptions of the Turing degrees. In contrast to the vast and complex landscape of degree invariant constructions which must exist if Turing equivalence is universal, Martin's conjecture implies that there is a simple and complete classification of all such functions.

1.2 Martin's conjecture

Martin's conjecture on Turing invariant functions is one of the oldest and deepest open problems on the global structure of the Turing degrees. Inspired by Sacks' question on the existence of a degree-invariant solution to Post's problem [41] and by the wellfoundedness of the Wage hierarchy, Martin made a sweeping conjecture that says in essence, the only nontrivial definable Turing invariant functions are the Turing jump and its iterates through the transfinite.

Let $\leq_T$ be Turing reducibility on the Cantor space 2^ω, and let $\equiv_T$ be Turing equivalence. Given $x \in 2^\omega$, let x' be the Turing jump of x. The Turing degree of a real $x \in 2^\omega$ is the $\equiv_T$ equivalence class of x. A *Turing invariant function* is a homomorphism from $\equiv_T$ to $\equiv_T$, that is, a function $f : 2^\omega \to 2^\omega$ such that for all reals $x, y \in 2^\omega$, if $x \equiv_T y$, then $f(x) \equiv_T f(y)$. The Turing invariant functions are those which induce functions on the Turing degrees.

Martin's conjecture is set in the context of ZF + DC + AD, where AD is the axiom of determinacy. We assume ZF+DC+AD for the rest of this section. The results we will discuss all localize so that their consequences for Borel functions follow from Borel determinacy.

To state Martin's conjecture, we need to recall the notion of Martin measure. A *Turing cone* is a set of the form $\{x : x \geq_T y\}$. The real y is said to be the *base* of the cone $\{x : x \geq_T y\}$. A *Turing invariant set* is a set $A \subseteq 2^\omega$ that is closed under Turing equivalence: for all $x, y \in 2^\omega$, if $x \in A$ and $x \equiv_T y$, then $y \in A$.

Theorem 1.2.1 (Martin [34] [35]). *Assume* ZF + AD. *Then every Turing invariant set either contains a Turing cone, or is disjoint from a Turing cone.*

Note that the intersection of countably many cones contains a cone; the intersection of the cones with bases $\{y_i\}_{i\in\omega}$ contains the cone whose base is the join of the y_i. Hence, under AD, the function

$$\mu(A) = \begin{cases} 1 & \text{if } A \text{ contains a Turing cone} \\ 0 & \text{if the complement of } A \text{ contains a Turing cone} \end{cases}$$

is a measure on the σ-algebra of Turing invariant sets. This measure is called *Martin measure*. Martin measure is a fundamental tool in recursion theory that has had a profound effect on our understanding of the Turing degrees. In part, this is because of the ubiquity of relativization; the Turing degrees retain much of their complexity and structural properties on Turing cones. For the rest of this section, by a.e. we will mean almost everywhere with respect to Martin measure. Since we will care only about the behavior of functions a.e., we will occasionally deal with functions which are only defined a.e.

For Turing invariant $f, g : 2^\omega \to 2^\omega$, let $f \leq_m g$ if and only if $f(x) \leq_T g(x)$ a.e. Similarly, $f \equiv_m g$ if and only if $f(x) \equiv_T g(x)$ a.e. Say that f is increasing a.e. if $f(x) \geq_T x$ a.e. Finally, say that f is constant a.e. if there exists a $y \in 2^\omega$ such that $f(x) \equiv_T y$ a.e. (i.e. the induced function on Turing degrees is constant a.e.).

We are now ready to state Martin's conjecture on Turing invariant functions.

Conjecture 1.2.2 (Martin [27, p. 281])**.** *Assume* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$*. Then*

I. If $f : 2^\omega \to 2^\omega$ is Turing invariant and f is not increasing a.e., then f is constant a.e.

II. $\leq_m$ prewellorders the set of Turing invariant functions which are increasing a.e. If f has $\leq_m$-rank α, then f' has $\leq_m$-rank $\alpha + 1$, where $f'(x) = f(x)'$ for all x.

While Martin's conjecture remains open, significant progress has been made towards establishing its truth. We will begin by discussing some partial results for uniform functions.

Definition 1.2.3. Suppose $E^X_{\{\varphi_i\}}$ and $E^Y_{\{\psi_i\}}$ are countable Borel equivalence relations on the Polish spaces X and Y, induced by the generating families $\{\varphi_i\}_{i\in\omega}$ and $\{\psi_i\}_{i\in\omega}$. Say that a homomorphism $f : X \to Y$ is *uniform (with respect to $\{\varphi_i\}_{i\in\omega}$ and $\{\psi_i\}_{i\in\omega}$)* if there exists a function $u : \omega^2 \to \omega^2$ such that for all $x, y \in X$, if $x E^X_{\{\varphi_i\}} y$ via (i, j), then $f(x) E^Y_{\{\psi_i\}} f(y)$ via $u(i, j)$.

The Turing reductions are a canonical generating family for Turing equivalence, and for the remainder of this thesis, we equip Turing equivalence with this generating family unless we say otherwise. Say that a Turing invariant function f is *uniformly Turing invariant* if it is uniform with respect to the generating family of Turing reductions.

The first progress on Martin's conjecture was made by Steel [48] and was continued by Slaman and Steel [47]. They proved that Martin's conjecture is true when restricted to the class of uniformly Turing invariant functions.

Theorem 1.2.4 (Slaman and Steel [47])**.** *Part I of Martin's conjecture holds for all uniformly Turing invariant functions.*

Theorem 1.2.5 (Steel [48])**.** *Part II of Martin's conjecture holds for all uniformly Turing invariant functions.*

Theorems 1.2.4 and 1.2.5 also imply that Martin's conjecture is true when restricted to the larger class of functions f so that $f \equiv_m g$ for some uniformly Turing invariant g. Steel has conjectured that this is true of all Turing invariant functions.

Conjecture 1.2.6 (Steel [48])**.** *Assume* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$*. If $f : 2^\omega \to 2^\omega$ is Turing invariant, then there exists a uniformly Turing invariant g such that $f \equiv_m g$.*

Assuming Conjecture 1.2.6, Steel [48] has computed the $\leq_m$-rank of many familiar such jump operators. Steel also proves that Conjecture 1.2.6 implies that if $f(x) \in L[x]$ a.e., then f has a natural normal form in terms of master codes in Jensen's J.

The original intent of Martin's conjecture was to be a precise way of stating that the only definable non-constant Turing invariant functions are the Turing jump and its transfinite iterates such as $x \mapsto x^{(\alpha)}$ for $\alpha < \omega_1$, $x \mapsto \mathcal{O}^x$, and $x \mapsto x^\sharp$. Becker has shown that Conjecture 1.2.6 precisely captures this idea. In [3], Becker defines the notion of a reasonable pointclass, and shows that for any such reasonable pointclass Γ, there is a universal $\Gamma(x)$ set for every x, where $\Gamma(x)$ is the relativization of Γ to x. (Such a universal set is not unique, but the universal $\Gamma(x)$ subset of ω will be unique up to Turing equivalence.) For instance, if we consider the pointclass of Π^1_1 sets, the universal $\Pi^1_1(x)$ subset of ω is $\mathcal{O}^x$. Becker has shown that the strictly increasing uniformly Turing invariant functions are precisely the functions which map x to the universal $\Gamma(x)$ subset of ω for some reasonable pointclass Γ.

Theorem 1.2.7 (Becker [3]). *Assume* ZF+DC+AD. *Let $f : 2^\omega \to 2^\omega$ be a Turing invariant function so that $f(x) >_T x$ a.e. Then f is uniformly Turing invariant if and only if there is a reasonable pointclass Γ, and a Turing invariant g so that $g(x)$ is the universal $\Gamma(x)$ subset of ω, and $f \equiv_m g$.*

Suppose f is Turing invariant. Then say that f is *order preserving* if $x \geq_T y$ implies that $f(x) \geq_T f(y)$. Say that f is *uniformly order preserving* if there exists a function $u : \omega \to \omega$ so that $x \geq_T y$ via i implies $f(x) \geq_T y$ via $u(i)$. It is clear that if f is uniformly order preserving then f is uniformly Turing invariant. A corollary of Becker's work is that for any Turing invariant f, there exists a uniformly Turing invariant g so that $g \equiv_m f$ if and only if there exists a uniformly order preserving h so that $h \equiv_m f$.

Two more cases of Martin's conjecture are known. They are especially interesting because they do not require uniformity assumptions.

Theorem 1.2.8 (Slaman and Steel [47]). *If f is a Borel order preserving Turing invariant function that is increasing a.e., then there exists an $\alpha < \omega_1$ so that $f(x) \equiv_T x^{(\alpha)}$ a.e.*

The proof of this theorem uses a generalization of the Posner-Robinson theorem for iterates of the Turing jump up through ω_1. To generalize this theorem beyond the Borel functions, it would be enough to generalize the Posner-Robinson theorem further through the hierarchy of jump operators. For instance, Woodin [58] has proved a generalization of the Posner-Robinson theorem for the hyperjump. This can be used to show that if f is increasing and order preserving a.e., and not Borel, then $f(x) \geq_T \mathcal{O}^x$ a.e.

The last known case of Martin's conjecture is for all recursive functions.

Theorem 1.2.9 (Slaman and Steel [47]). *Assume* ZF + DC + AD. *Suppose $f(x) \leq_T x$ a.e. Then either $f(x) \equiv_T x$ a.e., or f is constant a.e.*

The proof of this theorem uses both game arguments and a significant amount of recursion theory. Generalizing this theorem past the $\mathbf{\Delta}^0_1$ functions appears to be difficult, and the proof does not give much of an idea of how to do this.

The special case of a degree invariant solution to Post's problem has also received considerable attention. Lachlan [29] has shown that there are no uniform solutions to Post's problem. This result predated Theorem 1.2.5, which generalized it. Downey and Shore [13] later put further restrictions on any possible solution. By using Theorem 1.2.5, they showed that any degree invariant solution to Post's problem must be low_2 or high_2. On the positive side, Slaman and Steel (unpublished) have proved that there is a degree invariant solution to Post's problem restricted to the domain of Σ^0_3 sets. Finally, Lewis has constructed a degree invariant solution to Post's problem on a set of sufficiently generic degrees [30].

Martin's conjecture has also inspired a couple theorems for functions from 2^ω to $\mathcal{P}(2^\omega)$. Steel [48] has proved the existence of a natural prewellorder on inner model operators using the uniform case of Martin's conjecture. Slaman [46] has proved an analogue of Martin's conjecture for all Borel functions from 2^ω to $\mathcal{P}(2^\omega)$ satisfying certain natural closure conditions. This proof uses a technique that is reminiscent of Theorem 1.2.8, and relies on a sharpening of the generalized Posner-Robinson theorem due to Shore and Slaman [44].

The metamathematics of Martin's conjecture has been the source of some interesting results. Chong and Yu [8] have constructed uniformly Turing invariant Π^1_1 counterexamples to Martin's conjecture when the hypothesis of $\mathrm{ZF}+\mathrm{DC}+\mathrm{AD}$ is replaced with $\mathrm{ZFC}+\mathrm{V}=\mathrm{L}$. They raised the question of the consistency strength of Theorem 1.2.5. Chong, Wang, and Yu [7] have proved that the restriction of Theorem 1.2.5 to $\boldsymbol{\Pi}^1_{2n+1}$ functions is equivalent to $\boldsymbol{\Sigma}^1_{2n+2}$ determinacy for all $n \geq 0$.

For what follows, we will generally be interested only in the restriction of Martin's conjecture to Borel functions. Thus, in the rest of the thesis, when we assume Martin's conjecture we will really mean its consequences for Borel functions. The following characterization of Martin's conjecture for Borel functions is an easy consequence of Theorems 1.2.4 and 1.2.5.

Theorem 1.2.10 (Slaman and Steel)**.** *The following are equivalent:*

1. *Martin's conjecture restricted to Borel functions.*
2. *If f is Borel and Turing invariant, then either f is constant a.e., or there is an ordinal $\alpha < \omega_1$ so that $f(x) \equiv_T x^{(\alpha)}$ a.e.*
3. *If f is Borel and Turing invariant, then there exists a uniformly Turing invariant g such that $f \equiv_m g$.*

Assuming Martin's conjecture, there is a particular fact about Turing invariant functions that we will use several times. Given any subset A of 2^ω, the $\equiv_T$-*saturation* of A is defined to be the smallest Turing invariant set containing it. Let $f : 2^\omega \to 2^\omega$ be a countable-to-one function that is Turing-invariant. Then the $\equiv_T$-saturation of $\mathrm{ran}(f)$ must contain a Turing cone. This is because f cannot be constant a.e. and so it must be that $f(x) \geq_T x$ a.e. Hence, the complement of the $\equiv_T$-saturation of $\mathrm{ran}(f)$ cannot contain a Turing cone.

1.3 A note on definitions of uniformity

Our definition of a uniformly Turing invariant function f is slightly different than some of the papers we reference. For example, the definition used in [47] merely requires f to be uniform (in our sense) when restricted to some pointed perfect set. We briefly explain the connection between these definitions.

Recall that a perfect subset of a Polish space X is a closed subset of X with no isolated points. Every perfect subset of 2^ω can be realized as the paths $[T]$ through some infinite perfect tree T in $2^{<\omega}$. A *pointed perfect tree* T is a tree T so that for all $x \in [T]$, $x \geq_T T$, and T is *uniformly pointed* if there is a single Turing reduction φ_i which witnesses $x \geq_T T$ for all $x \in [T]$. A *pointed perfect set* is the paths $[T]$ through some pointed perfect tree T. Likewise, a *uniformly pointed perfect set* is the paths through some uniformly pointed perfect tree. Pointed perfect sets arise naturally in determinacy arguments, and have many nice properties. Given a perfect subset $[T]$ of 2^ω, it is clear that 2^ω is homeomorphic to $[T]$ via a canonical homeomorphism that preserves the ordering on 2^ω. Let $f : 2^\omega \to [T]$ be this canonical homeomorphism. This homeomorphism f will preserve the Turing degrees above T if T is pointed: for all $x \geq_T T$, we have $x \equiv_T f(x)$, since both x and $f(x)$ can compute a representation of T and hence also a representation of f.

We sketch a proof of the following useful lemma, which is a slight variation of Martin's cone theorem in [35]. While Martin measure cannot be extended to the σ-algebra of all subsets of 2^ω under AD, pointed perfect sets often provide a suitable replacement when we would like to measure a set which is not Turing invariant.

Lemma 1.3.1 (Martin [35])**.** *Assume* $\mathrm{ZF} + \mathrm{AD}$*. Suppose $B \subseteq 2^\omega$ is a set such that for all $y \in 2^\omega$, there exists an $x \in B$ such that $x \geq_T y$. Then B contains a uniformly pointed perfect set.*

Proof. Consider the game where I plays $i \in \omega$ followed by $x \in 2^\omega$, and II plays $y \in 2^\omega$, where the players alternate playing bits of these reals as usual. Let II lose unless $y \geq_T x$, and if the game is not decided by this condition, then I wins if and only if $x \geq_T y$ via the Turing reduction φ_i.

Given any strategy s for II coded as a real in 2^ω, I can win against this strategy by playing some $x \geq_T s$ and an index for a Turing reduction which computes s from x and then uses it to compute II's play according to s.

Thus, I has a winning strategy s in this game. Our desired uniformly pointed perfect set is I's winning plays against II playing all reals of the form $s \oplus z$. $\square$

We will often use the following Corollary of the above:

Corollary 1.3.2 (Folklore)**.** *Assume* $\mathrm{ZF} + \mathrm{AD}$*. Then every function $c : 2^\omega \to \omega$ is constant on a uniformly pointed perfect set.*

The following theorem is implicit in Slaman and Steel's paper [47].

Theorem 1.3.3 (Slaman and Steel [47]). *Assume* ZF + AD*. If f is Turing invariant, then $f \equiv_m g$ for some uniformly Turing invariant g if and only if f is uniform when restricted to some pointed perfect set.*

Proof. We begin with the forward direction. Suppose there is a uniformly Turing invariant g such that $f(x) \equiv_T g(x)$ on a Turing cone of x. Then let $c : 2^\omega \to \omega^2$ be the function mapping x to the lexicographically least pair (i, j) such that $f(x) \equiv_T g(x)$ via (i, j). Then by Corollary 1.3.2, c is constant on a pointed perfect set P. Hence f is uniformly Turing invariant on P.

Conversely, suppose that f is uniform on some pointed perfect set P. Then by Lemma 1.3.1, P contains a uniformly pointed perfect subset $P^* = [T]$. Let h be the canonical homeomorphism from 2^ω to P^*. Then let g be the restriction of $f \circ h$ to the Turing cone $\{x : x \geq_T T\}$. □

Thus, the difference between our definition and that of [47] is harmless. We have not changed what it means for a function on the Turing degrees to be represented by a uniformly invariant function on a Turing cone. We use our definition because of its simplicity, and because it generalizes more readily to equivalence relations other than Turing equivalence.

We can use Corollary 1.3.2 to provide a quick proof of Theorem 1.1.2. First recall the statement of the Theorem, where the equivalence of (1) and (3) is due to Kechris-Miller:

Theorem 1.1.2 (Kechris-Miller [51]). *Suppose E is a countable Borel equivalence relation. The following are equivalent.*

1. *E is weakly universal.*

2. *If F is any countable Borel equivalence relation, then there exists an injective Borel homomorphism from F to E.*

3. *There exists a universal countable Borel equivalence relation F such that $F \subseteq E$.*

Proof. (1) $\implies$ (2). Suppose that E is weakly universal. Then there is a countable-to-one Borel homomorphism f from $\equiv_T$ to E. Using uniformization, we can find Borel $\{A_i\}_{i\in\omega}$ such that f is injective on each A_i and $\bigcup_{i\in\omega} A_i = 2^\omega$. By Lemma 1.3.1, one of the A_i must contain a pointed perfect set. To finish, note that Turing equivalence restricted to any pointed perfect set is weakly universal, and this can always be witnessed with an injective homomorphism.

(2) $\implies$ (3) and (3) $\implies$ (1) are obvious. □

It is interesting to point out that Corollary 1.3.2 is a reversal of quantifiers away from proving Steel's Conjecture 1.2.6. There exist finitely many Turing reductions $\{\psi_0, \ldots, \psi_n\}$ such that every Turing reduction can be written as a composition of the $\{\psi_0, \ldots, \psi_n\}$. For instance, we could begin by letting $\psi_0(x) = \varphi_i(x)$, where φ_i is the ith Turing reduction, and i is the least number such that $x(i) = 1$. If P is a uniformly pointed perfect set, then Turing equivalence restricted to P is also finitely generated in this way. Indeed, if i is a uniform index for a pointed perfect set, let $\{\psi_{i,0}, \ldots, \psi_{i,n}\}$ be the associated finite set of Turing reductions which depend only on i. To verify that a Turing invariant function is uniformly Turing invariant on a pointed perfect set, it suffices to demonstrate that it is uniform with respect to these finitely many Turing reductions.

Let $c(x, i)$ be the function mapping x to the tuple of ω^n witnessing how $f(x)$ computes $f(\psi_{i,0}(x)), \ldots, f(\psi_{i,n}(x))$. We would like for $c(x, i)$ to be constant on all x in a uniformly pointed perfect set P with uniform index i. However, this sort of fixed point does not exist in general. For example, let $c(x, i) = 0$ if x is the leftmost path in a uniformly pointed perfect set with index i, and 1 otherwise.

1.4 Martin's conjecture and countable Borel equivalence relations

Martin's conjecture has turned out to have many interesting implications for the theory of countable Borel equivalence relations. Indeed, while Martin's conjecture says something very beautiful and fundamental about Turing reducibility and the hierarchy of definability, it is not so surprising that Martin's conjecture has been more applicable in this area, rather than in recursion theory. Martin's conjecture gives a complete classification of all homomorphisms from Turing equivalence to itself, and homomorphisms are a basic object of study in the area of countable Borel equivalence relations.

The most obvious consequence of Martins' conjecture for the theory of countable Borel equivalence relations is that $\equiv_T$ is not universal. Essentially, if there is a reduction from $\equiv_T \sqcup \equiv_T$ to $\equiv_T$, then the range of the reduction on one of the copies of $\equiv_T$ must be disjoint from a cone. See [11] for a more thorough discussion of this topic. Recent work of Thomas has showed that Martin's conjecture has many more implications than merely the non-universality of $\equiv_T$.

Let E be a countable Borel equivalence relation on a Polish space X, and μ be a probability measure on X. Given a countable Borel equivalence relation F, we say that E is *F-ergodic* with respect to μ if every Borel homomorphism from E to F maps a set of measure 1 into a single F-class. E is said to be simply *ergodic* with respect to μ if it is $\Delta(Y)$-ergodic with respect to μ for every Polish space Y, where $\Delta(Y)$ is the equivalence relation of equality on Y. Define a subset A of X to be *E-invariant* if for all $x, y \in X$, if $x \in A$, and xEy, then $y \in A$. Equivalently, E is ergodic with respect to μ if and only if every Borel E-invariant set has measure 0 or 1.

For the above definitions of ergodicity to make sense, μ can be a measure on merely the σ-algebra of E-invariant Borel subsets of X, as Martin measure is for $\equiv_T$. For example, $\equiv_T$ is ergodic with respect to Martin measure.

Strong ergodicity results for $\equiv_T$ and Martin measure would be very interesting. For example, Thomas [50] has shown that if $\equiv_T$ is E_0-ergodic with respect to Martin measure, then $\equiv_T$ is not Borel bounded. Borel boundedness is closely connected to the long open increasing union problem for hyperfinite equivalence relations [4]. It is currently open whether there are any Borel equivalence relations that are not Borel bounded.

Thomas has shown that Martin's conjecture implies that $\equiv_T$ is E_0-ergodic with respect to Martin measure, and in fact, Martin's conjecture implies the strongest ergodicity for $\equiv_T$ that is possible with respect to Martin measure. Clearly, if E is weakly universal, then $\equiv_T$ is not E-ergodic with respect to Martin measure, since there is a countable-to-one Borel homomorphism from $\equiv_T$ to E. Assuming Martin's conjecture, Thomas has proved the remarkable fact that the converse is true:

Theorem 1.4.1 (Thomas [50]). *Assume Martin's conjecture is true. Let E be any countable Borel equivalence relation. Then exactly one of the following hold:*

1. *E is weakly universal.*

2. *$\equiv_T$ is E-ergodic, with respect to Martin measure.*

The proof of this theorem uses the fact that Martin's conjecture implies that the saturation of the range of a countable-to-one Turing invariant function must contain a Turing cone. We will discuss Theorem 1.4.1 more in Chapter 3.

Martin's conjecture appears to be closely connected to the structure of the weakly universal countable Borel equivalence relations, and has a number of implications for them. Thomas [50] has shown that assuming Martin's conjecture, there are continuum many pairwise $\leq_B$-incomparable weakly universal countable Borel equivalence relations. These equivalence relations are products of the form $\equiv_T \times E_\alpha$ where $\{E_\alpha : \alpha \in 2^\omega\}$ is a family of non weakly universal Borel equivalence relations on 2^ω so that if $\alpha \neq \beta$, then E_α is E_β-ergodic with respect to Lebesgue measure. The proof of this result uses Popa's cocycle superrigidity theorem to establish the existence of such a family of E_α, and then applies Theorem 1.4.1 to prove the $\leq_B$-incomparability of the product equivalence relations.

Thomas has also used Martin's conjecture to investigate weak universality in another context. Say that a countable group G is *weakly action universal* if there is a Borel action of G on a Polish space X so that the induced orbit equivalence relation E^X_G is weakly universal. Assuming Martin's conjecture, Thomas has shown that G is weakly action universal if and only if the conjugacy relation on the subgroups of G is weakly universal [54].

One appeal of Martin's conjecture is that it provides a dimension of analysis that is completely orthogonal to measure theory. This is particularly interesting because all other known techniques for analyzing non-hyperfinite countable Borel equivalence relations are measure-theoretic. Assuming Martin's conjecture, Thomas [50] has proved that the complexity of a weakly universal countable Borel equivalence relation always concentrates on a nullset. This is strong evidence that techniques that are not purely measure-theoretic are needed to unravel the structure of the weakly universal countable Borel equivalence relations.

Theorem 1.4.2 (Thomas [50]). *Assume Martin's conjecture. If E is a weakly universal countable Borel equivalence relation on a Polish space X, and μ is a Borel probability measure on X, then there is a Borel set $B \subseteq X$ with $\mu(B) = 1$ such that $E \upharpoonright B$ is not weakly universal.*

Thomas has applied this theorem to show that assuming Martin's conjecture, there does not exist a strongly universal countable Borel equivalence relation. We will discuss this result more in Chapter 3.

Finally, some of the partial results on Martin's conjecture have also found applications in the field of countable Borel equivalence relations. Thomas [49] has used Theorem 1.2.9 to show the nonexistence of continuous Borel reductions between several equivalence relations. These results are significant because in practice, most Borel reductions are continuous.

It is worth noting that uniformity in a rather different context than Martin's conjecture has already had a tremendous impact on field of countable Borel equivalence relations. Suppose G and H are countable groups equipped with Borel actions on the Polish spaces X and Y respectively. Let E_G^X and E_H^Y be the orbit equivalence relations arising from these actions. Now suppose the action of H on Y is free, so that if $xE_H^Y y$, then there is a unique witness $h \in H$ such that $h \cdot x = y$. In this setting, if f is a Borel homomorphism from E_G^X to E_H^Y, then the *cocycle associated to* f is the function $\alpha : G \times X \to H$ witnessing that f is a homomorphism; $\alpha(g,x)$ is the unique $h \in H$ such that $\alpha(g,x) \cdot f(x) = f(g \cdot x)$. Note that this α must satisfy the cocycle relation

$$\alpha(g_1 g_2, x) = \alpha(g_1, g_2 \cdot x)\alpha(g_2, x).$$

A *strict cocycle* for a given H and a Borel action of G on X is defined to be any Borel function $\alpha : G \times X \to H$ satisfying the cocycle relation. Cocycles are generally studied in the setting where X is equipped with a Borel probability measure μ. There, a *cocycle* is defined to be a Borel function satisfying the cocycle relation μ-a.e.

Two cocycles α and β are said to be cohomologous if there is a Borel map $x \mapsto h_x$ such that $\alpha(g,x) = h_{g\cdot x}\beta(g,x)h_x^{-1}$. If α is the cocycle associated to a Borel homomorphism f from E_G^X to E_H^Y, then α and β are cohomologous if and only if β is the cocycle associated to a Borel homomorphism g from E_G^X to E_H^Y such that $f(x) E_H^Y g(x)$ μ-a.e.

If the value of a cocycle $\alpha(g,x)$ is independent of x, we see that α is a group homomorphism from G to H. Hence, in this context, the analogous kind of uniformity to that posited by Martin's conjecture is for every cocycle arising from a Borel homomorphism between two equivalence relations to be cohomologous to a group homomorphism. This is the phenomenon of cocycle superrigidity as introduced by Zimmer [60]. It is certainly not the case that every cocycle is cohomologous to a group homomorphism (see for instance, Chapter III of [24]). However, superrigidity theorems have been proved for certain classes of groups, and these results have had a large number of applications in the field of countable Borel equivalence relations. See for example Thomas' paper [51] applying recent superrigidity results of Popa.

We remark that the machinery of cocycles does not appear to be suited to studying Martin's conjecture. Recall that Thomas' Theorem 1.1.3 above implies that Turing equivalence cannot be generated by the free Borel action of a countable group. Suppose that f is a Turing invariant function. In an attempt to skirt this problem, consider the map $\alpha_f :\equiv_T \to \mathcal{P}(\omega^2)$ which maps Turing equivalent x and y to the set of (i,j) such that $f(x) \equiv_T f(y)$ via (i,j).

We might hope that α_f, which encodes how the Turing invariance of f is witnessed, might behave similarly to a cocycle. However, from α_f, it is trivial to recover f, and so there is no real difference in working with f and α_f. Of course, the loss of information inherent in working with cocycles instead of homomorphisms is their main utility.

In Chapter 4, we will take a rather different approach to this problem. If we are prepared to lose finitely much information as to how $f(x) \equiv_T f(y)$ is witnessed, we will see that metrics on equivalence relations can play a similar role to cocycles for analyzing a weaker form of uniformity which we call being uniform-mod-finite.

Chapter 2

Overture

In this chapter we will prove a simple universality theorem in the context of computational complexity theory. A corollary of our result is that poly-time Turing equivalence is a universal countable Borel equivalence relation. We will then discuss several leitmotifs of the thesis.

2.1 A simple universality theorem

We begin by defining some notation and conventions. Given any set S, we will often exploit the bijection (via characteristic functions) between $\mathcal{P}(S)$ and 2^S, and move freely between these two representations. The Cantor space, noted 2^ω, is the Polish space of functions from ω to 2. We use the notation 2^n for the set of finite binary strings of length n which approximate elements of 2^ω, and $2^{\leq n}$ for finite binary strings of length $\leq n$. The set of all finite strings is noted $2^{<\omega}$, and we will use the letters r, s and t to note its elements.

Define $\mathcal{P}(2^{<\omega})$ to be the Polish space of subsets of $2^{<\omega}$, which are known as *languages*. If $x, y \in \mathcal{P}(2^{<\omega})$, then their recursive join is $x \oplus y = \{0^\frown r : r \in x\} \cup \{1^\frown r : r \in y\}$. The recursive join of finitely many elements of $\mathcal{P}(2^{<\omega})$ is defined analogously.

Suppose $f : \omega \to \omega$ is a function. If $x, y \in \mathcal{P}(2^{<\omega})$ and x is Turing reducible to y via a Turing reduction φ that runs in $f(n)$ time on strings of length n, then we say x is *f-time Turing reducible* to y. We use the standard convention that oracle strings of length n may be queried only after time n. Of course, the relation of f-time Turing reducibility is not necessarily transitive. However, we can obtain a transitive relation by considering time bounded Turing reducibility with respect to a class of functions with suitable closure properties.

Definition 2.1.1. Suppose $g : \omega \to \omega$ is a strictly increasing function where $g(n) \geq n^2$, and $g(n)$ is Turing computable in $O(g^k(n))$ time for some k. For $x, y \in \mathcal{P}(2^{<\omega})$, let $x \leq_{O(g^k)} y$ if there exists some k such that x is f-time Turing reducible to y for some $f(n) \in O(g^k(n))$. The resulting reducibility is transitive, and we let $\equiv_{O(g^k)}$ be the associated equivalence relation.

For example, if we take the function $g(n) = n^2$, then the associated reducibility $\leq_{O(g^k)}$ is poly-time Turing reducibility.

Theorem 2.1.2. *Let g be a function as in Definition 2.1.1. Then the associated equivalence relation $\equiv_{O(g^k)}$ is a universal countable Borel equivalence relation.*

Proof. By [10], there is a universal countable Borel equivalence relation E_∞, generated by a continuous action of $\mathbb{F}_2 = \langle a, b\rangle$ on 2^ω. Indeed, $\mathbb{F}_2$ may act via recursive permutations on the elements of 2^ω. We will show that $\equiv_{O(g^k)}$ is universal by constructing a continuous embedding $\hat{f}$ of E_∞ into $\equiv_{O(g^k)}$.

The key to our proof is that given $w \in \mathbb{F}_2$, we can code $\hat{f}(w \cdot x)$ into $\hat{f}(x)$ rather sparsely so that if $|w| = k$, then strings of length n in $\hat{f}(w \cdot x)$ are coded by strings of length greater than $g^k(n)$ in $\hat{f}(x)$. From here our basic idea is as follows: given $x, y \in 2^\omega$ such that $x \not{E}_\infty y$ and a Turing reduction that runs in g^k time, we wait till we have finite initial segments of x and y witnessing that $w \cdot x \neq y$ for any w of length $\leq k$. Then if n is large enough, we can change the value of strings of length n in $\hat{f}(x)$ without changing $\hat{f}(y)$ restricted to strings of length $\leq g^k(n)$, and so it is easy to diagonalize. The remaining technical wrinkle of the proof is that we must be able to simultaneously do a lot of this sort of diagonalization.

We now give a precise definition of the coding we will use. Let $c : 2^{<\omega} \to 2^{<\omega}$ be the function where $c(r)$ is r concatenated with $g(r)$ zeroes. It is clear that if $x \in \mathcal{P}(2^{<\omega})$, then x can compute $c(x) = \{c(r) : r \in x\}$ in $O(g(n))$ time.

Given $f : 2^\omega \to \mathcal{P}(2^{<\omega})$, let $\hat{f} : 2^\omega \to \mathcal{P}(2^{<\omega})$ be defined by

$$\hat{f}(x) = f(x) \oplus c\left(\hat{f}(a \cdot x) \oplus \hat{f}(a^{-1} \cdot x) \oplus \hat{f}(b \cdot x) \oplus \hat{f}(b^{-1} \cdot x)\right).$$

While this definition of $\hat{f}$ is self-referential, it is not circular, as one can see by repeatedly expanding the definition of $\hat{f}$ on the right hand side. It is clear that given any f, the associated $\hat{f}$ is a homomorphism from E_∞ to $\equiv_{O(g^k)}$. We claim that if f is a sufficiently generic continuous function, then $\hat{f}$ will also be a cohomomorphism from E_∞ to $\equiv_{O(g^k)}$.

We begin by defining a partial order for constructing a generic Lipschitz continuous function from 2^ω to $\mathcal{P}(2^{<\omega})$. Our partial order $\mathbb{P}$ will consist of functions $p : 2^n \to \mathcal{P}(2^{\leq n})$ such that if $m < n$ and $r_1, r_2 \in 2^n$ extend $r \in 2^m$, then $p(r_1)$ and $p(r_2)$ agree on all strings of length $\leq m$. Given such a p and $r \in 2^n$, we will often think of $p(r)$ as a function from $2^{\leq n}$ to 2. If p has domain 2^n, then we will say p has height n. If $p, p^* \in \mathbb{P}$ are such that the height of p is m and the height of p^* is n, then say that p^* extends p, noted $p^* \leq_{\mathbb{P}} p$ if for all $r^* \in 2^n$ extending $r \in 2^m$, $p^*(r^*)$ extends $p(r)$ as functions.

If $p \in \mathbb{P}$, then we can define $\hat{p}$ analogously to the definition of $\hat{f}$ above. In particular, for each $w \in \mathbb{F}_2$, we have some partial information about $p(w \cdot r)$ based on the longest finite initial segment of $w \cdot r$ that we know (recall that the action of F_2 on 2^ω is continuous). Hence, given a finite string r, $\hat{p}$ will map r to a partial function from finite strings to 2 that amalgamates this partial information.

Because of our coding scheme, if $r \in 2^{<\omega}$, the length of r is $|r| = n$, and $w \in \mathbb{F}_2$ is of length k, then whether $r \in \hat{f}(w \cdot x)$ is canonically coded into $\hat{f}(x)$ at some string of length greater than $g^k(n)$.

Let p_0 be the condition of height 1 where $p_0(r) = \emptyset$ for all r. Hence, if $f : 2^\omega \to \mathcal{P}(2^{<\omega})$ extends p_0, then every string in $f(x)$ must have length ≥ 2. For convenience, the generic function we construct will extend p_0. Now fix φ_e, which runs in g^k-time. Note that since $g(n) \geq n^2$, if f is $O(g^{k-1})$ for some k, then $f(n) \leq g(n)^k$ for sufficiently large n.

Suppose we are given $r, s \in 2^m$ such that $w \cdot r$ is incompatible with s for all $w \in \mathbb{F}_2$ where $|w| \leq k$, and a Turing reduction φ_e that runs in g^k time. Let $D_{r,s,k,e}$ be the set of p^* of height $\geq m$ such that if p^* has height i, then there exists an $n \geq m$ so that if $t \in 2^i$, then $\hat{p}(t)$ is defined on all strings of length $\leq g^k(n)$, and for all $r^*, s^* \in 2^i$ extending r and s, we have that φ_e is not a g^k-time reduction of $\hat{p}(s^*)$ to $\hat{p}(r^*)$ as witnessed by some string of length n. We claim that $D_{r,s,k,e}$ is dense below p_0. The theorem will follow from this fact.

Given any $p \leq_{\mathbb{P}} p_0$ where p has height j, we must construct an extension p^* of p so that $p^* \in D_{r,s,k,e}$. Fix an n and an i such that $i \gg n \gg j$. Define q of height i where for all r, $q(r)$ contains only the strings in $p(r \restriction i)$. Here we require $i \gg n$ so that for all $r \in 2^i$, $\hat{q}(r)$ is defined on all strings of length $\leq g^k(n)$. To determine how large n must be, we let n be variable for the next few paragraphs, while j and k are constant.

Given any $r^* \in 2^i$ we compute an upper bound on how many elements $\hat{q}(r^*)$ could have of length $\leq g^k(n)$. Clearly every $q(t)$ has less than 2^{j+1} elements. It will be enough to establish an upper bound on the number of words $w \in \mathbb{F}_2$ so that some element of $q(w \cdot r^*)$ is coded into $\hat{q}(r)$ via a string of length $\leq g^k(n)$.

Since q extends p_0, any element of any $q(w \cdot r^*)$ must be a string of length ≥ 2. Now if $w \in \mathbb{F}_2$ is such that some string of length ≥ 2 in $q(w \cdot r^*)$ is coded into $\hat{q}(r^*)$ below $g^k(n)$, then it must be that $g^{|w|}(2) \leq g^k(n)$. Since $g(n) \geq n^2$, we have that $g^{|w|}(2) \geq 2^{2^{|w|}}$. Hence, the length of w must be $\leq k + \log_2 \log_2(n)$. we see that there are most $O(\log_2(n))^2)$ such words w Since there are $\leq 4^{l+1}$ words of $\mathbb{F}_2$ of length $\leq l$.

Since each $q(w \cdot r^*)$ must have less than 2^{j+1} elements, $\hat{q}(r^*)$ contains $O((\log_2(n))^2)$ strings of length $\leq g^k(n)$. Let n be large enough so that this is less than n. Let S be this set of strings where $|S| < n$ such that every possibly nonzero element of $\hat{q}(r^*)$ is in S. Note that S does not depend on r^*.

We see now that amongst all of the r^* extending r, there are $\leq 2^{n-1}$ possibilities for what any $\hat{q}(r^*)$ could be: they are all elements of $\mathcal{P}(S)$. Let $u_0, u_1, \ldots$ be a listing of the elements of $\mathcal{P}(S)$. Recall that based on our definition of recursive join and $\hat{p}$, strings of length $n-1$ in $p(s^*)$ are coded into $\hat{p}(s^*)$ using strings of length n that begin with 0. Define p^* to be equal to q except on extensions s^* of s. There, if σ is lth string of length $n-1$, then put $\sigma \in p(s^*)$ if and only if σ is not accepted by φ_e run relative to u_l. □

Corollary 2.1.3. *Poly-time Turing equivalence is a universal countable Borel equivalence relation.*

Recall that by Martin's cone theorem, every Turing invariant Borel set either contains a Turing cone or is disjoint from a Turing cone. Martin's proof is remarkably robust, and generalizes to other natural reducibilities possessing the property that given a strategy s and a real x, the real obtained by playing the strategy s against x is reducible to $s \oplus x$. This property is not true of poly-time Turing reducibility since strategies grow exponentially in size with the number of moves made. However, the analogue of Martin's cone theorem is true for the reducibility arising from the function $g(n) = 2^n$.

We will see in Chapter 3 that having a cone measure on a universal countable Borel equivalence relation is powerful tool. We will use such a combination to prove several structural results about universal countable Borel equivalence relations, settling a couple open problems. In Chapter 3 we will use arithmetic equivalence and the arithmetic cone measure in this capacity, but many of our results could be replicated using $\equiv_{O(g^k)}$, where $g(n) = 2^n$. Our preference for arithmetic equivalence comes from fact that Turing equivalence is a subset of arithmetic equivalence. This allows us to prove additional results under the assumption of Martin's conjecture.

2.2 Uniform universality

We begin this section by pointing out a uniformity in our proof of Theorem 2.1.2 above. Given $w \in \mathbb{F}_2$, there is a pair of $O(g^k)$-time reductions φ_i and φ_j witnessing that $\hat{f}(x) \equiv_{O(g^k)} \hat{f}(w \cdot x)$ independently of x. Montalbán, Reimann, and Slaman have pointed out that all known universality proofs are uniform in this sense and have asked (in the context of equivalence relations generated by Borel actions of countable groups) whether every universal countable Borel equivalence relation must be uniformly universal. We will state a more general version of their question in the context of any generating family of partial Borel functions. While the Feldman-Moore theorem implies that every countable Borel equivalence relation can be generated by the Borel action of a countable group, we prefer to work in this more general context since many equivalence relations from recursion theory and not naturally generated by group actions.

Definition 2.2.1. Given a generating family $\{\varphi_i\}_{i\in\omega}$ of partial Borel functions on a Polish space X, say that the associated countable Borel equivalence relation $E^X_{\{\varphi_i\}}$ is a uniformly universal countable Borel equivalence relation if for every countable Borel equivalence relation $E^Y_{\{\psi_i\}}$, there exists a uniform Borel reduction from $E^Y_{\{\psi_i\}}$ to $E^X_{\{\varphi_i\}}$.

It is important that we specify a generating family when we consider whether a universal countable Borel equivalence relation is uniformly universal; a corollary of Theorem 3.1.6 is that every universal countable Borel equivalence relation is universal with respect to some generating family.

The arguments of Dougherty, Jackson, and Kechris in [10] show that the equivalence relation $E(\mathbb{F}_2, 2)$ generated by the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ is a uniformly universal countable Borel equivalence relation. Thus, to show that a countable Borel equivalence relation is uniformly universal, it is enough to uniformly reduce $E(\mathbb{F}_2, 2)$ as we have in Theorem 2.1.2 above.

It is an obvious corollary of Slaman and Steel's Theorem 1.2.4 that Turing equivalence is not uniformly universal as generated by the Turing reductions; there cannot be a uniform embedding of $\equiv_T \sqcup \equiv_T$ into $\equiv_T$.

Question 2.2.2 (Following Montalbán, Reimann, and Slaman). *Suppose that $E^X_{\{\varphi_i\}}$ is a universal countable Borel equivalence relation. Must $E^X_{\{\varphi_i\}}$ be uniformly universal?*

We will discuss this question more in Chapter 4. There we will pose an open question where an affirmative answer would imply both Martin's conjecture and an affirmative answer to Question 2.2.2. Uniform universality will also play an important role in Chapter 5, where we will consider the question of whether many-one equivalence is uniformly universal as generated by the many-one reductions. We will show that this question is equivalent to a problem of Borel combinatorics, and we will further investigate the associated combinatorial problem in Chapter 6.

We finish this section by discussing the problem of what equivalence relations from recursion theory are uniformly universal when generated in a natural way. Here, the central issue appears to be what reductions can be used for coding, and to what extent the remaining reductions can unravel this coding.

The equivalence relations $\equiv_{O(g^k)}$ of Theorem 2.1.2 are good examples of equivalence relations for which proving universality is easy. The $O(g^k)$-time Turing reductions are naturally organized into an ω-length hierarchy, and there is a system of coding that increases complexity in this hierarchy when it is iterated. Arithmetic equivalence is another equivalence relation with similar properties, and in Slaman and Steel's proof of the universality of arithmetic equivalence [33], the Σ_k reductions play an analogous role to the $O(g^k)$-time Turing reductions in our proof. In contrast, Turing equivalence has a tremendous amount of closure and is not uniformly universal. Given any uniform system of coding into Turing equivalence, there are Turing reductions which can completely unravel this coding.

Somewhere in the middle of this spectrum is the equivalence relation of recursive isomorphism. While we cannot prevent recursive isomorphisms from iterating whatever coding method we choose, they have no flexibility in how they use the information they gain from being able to iterate our coding. We will discuss recursive isomorphism more in Chapter 5.

Chapter 3

Universal countable Borel equivalence relations

In this chapter, we apply Slaman and Steel's result that arithmetic equivalence is universal to prove several structural theorems about universal countable Borel equivalence relations in general. For instance, we settle an open question of Jackson, Kechris, and Louveau [20] and show that if E is a universal countable Borel equivalence relation, then for every E-invariant Borel set B, either $E \upharpoonright B$ is universal, or $E \upharpoonright \overline{B}$ is universal.

The key to these proofs will be combining the cone measure for arithmetic equivalence with the relativized version of Slaman and Steel's theorem which shows that arithmetic equivalence restricted to any arithmetic cone is universal. In [51], Simon Thomas asked whether there exists a *strongly universal* countable Borel equivalence relation; a universal countable Borel equivalence relation E on a Polish space X equipped with an invariant ergodic Borel probability measure μ such that $E \upharpoonright B$ is universal whenever $\mu(B) = 1$. This question remains open, though Thomas' Theorem 1.4.2 shows that it has a negative answer assuming Martin's conjecture. However, if we allow ourselves to use the arithmetic cone measure instead of a Borel probability measure, then arithmetic equivalence is "strongly universal" with respect to its cone measure. Hence, we are able to obtain some of the same consequences that motivated Thomas' question.

In retrospect, cone measure from recursion theory are much more natural to use in this setting that Borel probability measures. While cone measures almost always preserve structural properties because of relativization, the relationship between an equivalence relation E and its restriction to sufficiently random reals is poorly understood. For example, say that a countable Borel equivalence relation E on a Polish space X is *measure hyperfinite* if given any Borel probability measure μ, there exists a set B with $\mu(B) = 1$ such that $E \upharpoonright B$ is hyperfinite. It is open whether every measure hyperfinite countable Borel equivalence relation is hyperfinite.

3.1 Consequences of the universality of arithmetic equivalence

If $x, y \in 2^\omega$, then x is said to be *arithmetically reducible* to y, noted $x \leq_A y$, if there is an n so that x has a Σ^0_n definition relative to y. Equivalently, $y \geq_A x$ if there is an n so that $y^{(n)} \geq_T x$, where $y^{(n)}$ is the nth iterate of the Turing jump relative to y. The associated countable Borel equivalence relation is called arithmetic equivalence and is noted $\equiv_A$.

A measure analogous to Martin measure exists for arithmetic equivalence. It is called the *arithmetic cone measure*. An arithmetic cone is a set of the form $\{x : x \geq_A y\}$. An arithmetically invariant set has measure 1 with respect to the arithmetic cone measure if it contains an arithmetic cone, otherwise it has measure 0. Martin's proof in [35] still works when Turing reducibility is replaced by arithmetic reducibility. Hence, this function is indeed a measure on the σ-algebra of arithmetically invariant sets.

Theorem 3.1.1 (Slaman and Steel [33])**.** *Arithmetic equivalence restricted to any arithmetic cone is a uniformly universal countable Borel equivalence relation as witnessed by a Borel embedding of $E(\mathbb{F}_2, 2)$ into it.*

In Chapter 4, we will use an extension of Slaman and Steel's technique for showing that arithmetic equivalence is universal to investigate questions of uniformity for arithmetically invariant functions. In doing so, we will reprove Slaman and Steel's theorem. For now, we combine the universality of arithmetic equivalence with the existence of the arithmetic cone measure to prove several theorems.

Jackson, Kechris, and Louveau [20] have asked the following question: suppose E is a universal countable Borel equivalence relation on X, and B is an E-invariant Borel subset of X. Is one of $E \restriction B$ or $E \restriction (X \setminus B)$ universal? The answer is yes, and we prove a stronger fact that was originally posed as a question by Thomas [51, question 3.20].

Theorem 3.1.2. *Suppose X and Y are Polish spaces, E is a universal countable Borel equivalence relation on X, and $f : X \to Y$ is any Borel homomorphism from E to $\Delta(Y)$, where $\Delta(Y)$ is the relation of equality on Y. Then there exists a $y \in Y$ so that the restriction of E to $f^{-1}(y)$ is a universal countable Borel equivalence relation.*

Proof. First, note that it is enough to prove this for arithmetic equivalence. Let E and f be as in the statement of the theorem, and let $g : 2^\omega \to X$ be a Borel reduction from $\equiv_A$ to E. Then arithmetic equivalence restricted to $(f \circ g)^{-1}(y)$ is universal if and only if E restricted to $f^{-1}(y)$ is universal.

Now let f be a homomorphism from $\equiv_A$ to $\Delta(Y)$. Since arithmetic equivalence is ergodic with respect to the arithmetic cone measure, there must be a $y \in Y$ so that $f^{-1}(y)$ contains an arithmetic cone. Arithmetic equivalence restricted to this set is thus universal. $\square$

The use of Borel determinacy in our proof raises an interesting metamathematical question: must any proof of this theorem use Borel determinacy? For instance, one could ask whether Theorem 3.1.2 implies Borel determinacy over some simple base theory. We ask a weaker question of whether Theorem 3.1.2 shares a metamathematical property of Borel determinacy:

Question 3.1.3. *Does a proof of Theorem 3.1.2 require the existence of ω_1 iterates of the powerset of ω?*

Let E be a countable Borel equivalence relation, and suppose f is a Borel homomorphism from $\equiv_A$ to E. Then f is also a Borel homomorphism from $\equiv_T$ to E. If $B \subseteq 2^\omega$ contains a Turing cone, then the $\equiv_A$-saturation of B must contain an arithmetic cone. It is therefore possible to use ergodicity results about Turing equivalence and Martin measure to obtain ergodicity results about arithmetic equivalence and the arithmetic cone measure. We will apply Thomas' Theorem 1.4.1 in this way to prove an analogous sort of ergodicity result for all universal countable Borel equivalence relations.

Theorem 3.1.4. *Assume Martin's conjecture is true. Suppose E is a universal countable Borel equivalence relation, and F is an arbitrary countable Borel equivalence relation. Then exactly one of the following holds:*

1. *F is weakly universal.*

2. *For every Borel homomorphism f of E into F, there exists a single F-class whose preimage B has the property that $E \restriction B$ is universal.*

Proof. As in the proof of Theorem 3.1.2, we only need to prove this when E is $\equiv_A$. Let f be a homomorphism from $\equiv_A$ to F. Then f is also a homomorphism from $\equiv_T$ to F, and hence by Theorem 1.4.1, either F is weakly universal, or there is a single F-class whose preimage B contains a Turing cone. In this latter case, since f is also a homomorphism from $\equiv_A$ to F, the preimage of this single F-class contains the $\equiv_A$-saturation of this Turing cone which is an arithmetic cone. Hence, since B contains an arithmetic cone, $\equiv_A \restriction B$ is universal. $\square$

In [50], Thomas proved a variant of this theorem where the assumption that E is universal is changed to say E is weakly universal, and option 2 is changed to say that $E \restriction B$ is weakly universal. Theorem 3.1.4 strengthens this fact, as one can see by applying Theorem 1.1.2.

In the proof of Theorem 3.1.4 we have used the ergodicity of $\equiv_T$ that follows from Martin's conjecture. However, we only need the weaker ergodicity which Martin's conjecture implies for arithmetic equivalence. We isolate this in the following conjecture. It may be that it is easier to prove ergodicity results for arithmetic equivalence and the arithmetic cone measure than it is for Turing equivalence and Martin measure.

Conjecture 3.1.5. *Let E be any countable Borel equivalence relation. Then exactly one of the following holds:*

1. *E is weakly universal.*

2. $\equiv_A$ is E-ergodic, with respect to the arithmetic cone measure.

A special case of the above conjecture is quite interesting. Thomas [50] has raised the question of whether $\equiv_T$ is E_0-ergodic with respect to Martin measure. It is weaker to ask whether arithmetic equivalence is E_0-ergodic with respect to the arithmetic cone measure, but this would have similarly nice consequences. For example, it would imply that $\equiv_A$ is not Borel bounded, and also that option 2 in Theorem 3.1.4 holds when F is E_0 without the assumption of Martin's conjecture.

If E and F are countable Borel equivalence relations, then we say that E is *Borel embeddable* in F and write $E \sqsubseteq_B F$ if there exists a Borel embedding of E into F. We can use Lemma 1.3.1 to derive the following fact about universality for embeddings.

Theorem 3.1.6. *Let E be a universal countable Borel equivalence relation. Then given any countable Borel equivalence relation F, it must be that $F \sqsubseteq_B E$. That is, not only is $F \leq_B E$ (since E is universal), we can always find an injective Borel reduction.*

Proof. First recall that in Theorem 3.1.1, the function witnessing the universality of $\equiv_A$ is a Borel embedding, and this remains true when the proof is relativized to any pointed perfect set.

Since E is countable universal, there is a Borel reduction f from $\equiv_A$ to E. Using uniformization, split 2^ω into countably many Borel pieces $\{B_i\}_{i\in\omega}$ so that f is injective on each B_i. One of these B_i must contain a pointed perfect set by Lemma 1.3.1. □

This theorem is an interesting counterpoint to the following theorem of Thomas. Recall that an equivalence relation is said to be *aperiodic* if all of its equivalence classes are infinite.

Theorem 3.1.7 (Thomas [52])**.** *There exist aperiodic countable Borel equivalence relations E and F such that $E \leq_B F$, and $F \leq_B E$, but it is not the case that $E \sqsubseteq_B F$.*

Lemma 1.3.1 also gives an easy proof of the following:

Theorem 3.1.8. *Suppose E is a universal countable Borel equivalence relation on a Polish space X, and let $\{B_i\}_{i\in\omega}$ be a partition of X into countably many (not necessarily E-invariant) Borel pieces. Then there exists some i such that $E \restriction B_i$ is a universal countable Borel equivalence relation.*

Proof. As in Theorem 3.1.2, we only need to prove this for $\equiv_A$. Let $\{B_i\}_{i\in\omega}$ be a partition of 2^ω into countably many Borel pieces. By Lemma 1.3.1 above, one of these pieces must contain a pointed perfect set, and the restriction of $\equiv_A$ to any pointed perfect set is countable universal. □

If the B_i in the above theorem are all E-invariant, this theorem follows from Theorem 3.1.2. From this, we could also conclude the general case since for any Borel B, $E \restriction B$ is universal if and only if $E \restriction [B]_E$ is universal, where $[B]_E$ is the E-saturation of B.

Theorem 3.1.8 associates two natural σ-ideals to every countable Borel equivalence relation.

Definition 3.1.9. Let E be a countable Borel equivalence relation on a Polish space X. Define the *non-universal ideal of* E to be the Borel subsets B of X on which $E \restriction B$ is not universal. Define the *non-weakly-universal ideal of* E to be the Borel subsets B of X on which $E \restriction B$ is not weakly universal.

We will discuss these σ-ideals more in what follows. They seem to be important for developing the theory of universal and weakly universal countable Borel equivalence relations.

We now proceed to prove a weak version of Theorem 1.4.2 without the assumption of Martin's conjecture.

Theorem 3.1.10. *Let E be a universal countable Borel equivalence relation on a Polish space X, and let μ be a Borel probability measure on X. Then there is a measure 0 subset B of X for which $E \restriction B$ is a universal countable Borel equivalence relation.*

First, recall the following theorem of Sacks (we give the relativized version of the theorem):

Theorem 3.1.11 (Sacks [41])**.** *If μ is a Borel probability measure on 2^ω, then for all $x \in 2^\omega$ such that there exists a representation $y \in 2^\omega$ of μ such that $x >_T y$, the cone $\{z : z \geq_T x\}$ has μ-measure 0.*

Hence, for any Borel probability measure, sufficiently complicated cones are always nullsets. The same theorem is also true when Turing reducibility is replaced with arithmetic reducibility. One way to see this is to first replace our measure μ with a $\equiv_A$-quasi-invariant measure ν that dominates μ. Then we can find a Turing cone with ν measure 0, and the $\equiv_A$-saturation of this cone will be the arithmetic cone with the same base.

Sacks' theorem implies that Martin measure and likewise the arithmetic cone measure cannot be extended to probability measures on all the Borel sets of 2^ω.

Proof of Theorem 3.1.10: Again, we need only prove this for arithmetic equivalence; given any other universal countable Borel equivalence relation E on the space X with measure μ, by Theorem 3.1.6 let f be a Borel embedding from $\equiv_A$ to E. Presuming the range of f has positive μ-measure, let ν be the measure on 2^ω defined by $\nu(A) = \frac{1}{\mu(f(2^\omega))}\mu(f(A))$. If $\equiv_A \restriction B$ is universal and $\nu(B) = 0$, then $E \restriction f(B)$ is also universal, and $\mu\left(f(B)\right) = 0$.

As we have shown above, given any Borel probability measure μ on $\equiv_A$, there is an arithmetic cone with measure 0, and $\equiv_A$ restricted to this cone is universal. □

The extra leverage that Thomas gets by assuming Martin's conjecture is that for all Borel sets B, $\equiv_T \restriction B$ is weakly universal if and only if B contains a pointed perfect set. Hence if B is $\equiv_T$-invariant, then $\equiv_T \restriction B$ is weakly universal if and only if $\equiv_T \restriction (2^\omega \setminus B)$ is not weakly universal.

This exact classification of the non-weakly-universal ideal for $\equiv_T$ that follows from Martin's conjecture seems very useful.

Question 3.1.12. *Are there "nice" characterizations of the non-weakly-universal ideals of naturally occurring weakly universal countable Borel equivalence relations?*

One could also ask the same question for universal countable Borel equivalence relations and the non-universal ideal. Theorem 3.1.10 seems to rule out characterizations that are based purely on measure theory. The fact that these ideals do not seem to be measure-theoretic is very interesting, since all known theorems in the field of countable Borel equivalence relations that distinguish between non-hyperfinite countable Borel equivalence relations are based on measure theory.

The following question seems to be relevant: define a countable Borel equivalence relation E to be *measure universal* if for every countable Borel equivalence relation F on a Polish space X equipped with a Borel probability measure μ, there exists a $B \subseteq X$ that is F-invariant with $\mu(B) = 1$ such that $F \restriction B$ is Borel reducible to E.

Question 3.1.13. *If E is a countable Borel equivalence relation that is measure universal, must E be universal?*

We will return to this question in Chapter 6, where we show that many-one equivalence and recursive isomorphism are measure universal. It remains open whether these equivalence relations are universal. The existence of a measure universal countable Borel equivalence relation that is not universal would strengthen Thomas' Theorem 1.4.2.

3.2 Ergodicity and ultrafilters

The theory of Borel equivalence relations is intimately connected with the study of the cardinality of quotients of the reals in models of determinacy. If E and F are Borel equivalence relations on the Polish spaces X and Y, then it is clear than whenever $E \leq_B F$, this implies that the cardinality of X/E is less than or equal to the cardinality of Y/F in any model of ZF. Less obviously, techniques for showing $E \not\leq_B F$ often generalize to show that the cardinality of X/E does not exceed the cardinality of Y/F in natural determinacy models. Moreover, some of the dichotomy theorems from Borel equivalence relations have been generalized to this setting as well. See for example [5], [18] and [19].

Woodin's theory AD^+ seems to be the natural setting for investigating the cardinality of quotients of the reals by equivalence relations with countable classes, as it implies enough uniformization to develop a reasonable theory [59]. We assume AD^+ for the rest of this section. Here, some of our results and questions above have interesting restatements. For example, Theorems 3.1.2 and 3.1.4 seems reminiscent of cofinality results. Recall that in ZFC, $\alpha < \mathrm{cf}(\kappa)$ if for all function $f : \kappa \to \alpha$, there exists a $\beta \in \alpha$ such that $f^{-1}(\beta)$ has cardinality κ. If choice fails, then given any set S, we can likewise investigate the class of sets R such that for every function $f : S \to R$, there exists a y such that $f^{-1}(y)$ has the same cardinality as S.

Under AD^+, strong ergodicity properties of cone measures from recursion theory can also be interestingly rephrased in terms of ultrafilters and Rudin-Keisler reducibility. Presently, very little is known about the structure of the ultrafilters on quotients of the reals in determinacy models, and many interesting questions are open.

We recall a few definitions. If U and V are ultrafilters on R and S, then U is said to be *Rudin-Keisler reducible* to V, noted $U \leq_{RK} V$, if there exists a function $f : S \to R$ such that for all $A \subseteq R$, we have $A \in U$ if and only if $f^{-1}(A) \in R$. The sets of Martin measure 1 correspond to an ultrafilter on the Turing degrees known as the *Martin ultrafilter*. Thus, the question of whether Turing equivalence is E_0-ergodic can be restated in following way:

Question 3.2.1. *Assume* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}^+$*. If* U *is an ultrafilter on* $2^\omega/E_0$*, and* U *is Rudin-Keisler reducible to the Martin ultrafilter, must* U *be principal?*

Zapletal has asked a related question. Let U_1 be the ultrafilter on $2^\omega/E_0$ arising from the sets of Lebesgue measure 1, and U_2 be the ultrafilter on $2^\omega/E_0$ arising from the comeager sets. It is easy to see that these two ultrafilters are incomparable with respect to the Rudin-Keisler ordering.

Question 3.2.2 (Zapletal)**.** *Assume* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}^+$*. If* V *is a nonprincipal ultrafilter on* $2^\omega/E_0$*, must* $V \geq_{RK} U_1$ *or* $V \geq_{RK} U_2$*?*

We finish with a broad question. By the analogue of Thomas' Theorem 1.4.1 under AD^+, if Martin's conjecture is true, then if E is a countable equivalence relation on a Polish space X, there exists a nonprincipal ultrafilter on X/E that is Rudin-Keisler reducible to Martin's ultrafilter if and only if E is weakly universal.

Question 3.2.3. *Assume* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}^+$*. Let* E *be a countable equivalence relation on a Polish space* X*. To what extent can we classify the ultrafilters on* X/E *up to Rudin-Keisler reducibility? To what extent is the structure of the ultrafilters on* X/E *related to the cardinality of* X/E*?*

Chapter 4

Uniformity and metrics

In the Borel setting, Martin's conjecture reduces purely to a question about uniformity, as shown in Theorem 1.2.10. If we embrace Martin's conjecture and ponder what larger principle it might embody, we are naturally led to the possibility that similar principles of uniformity might exist amongst a wider class of equivalence relations, even though the original form of Martin's conjecture appears to hinge on specific properties of Turing equivalence that do not generalize to many equivalence relations. This is an intriguing possibility that would lead to a compelling theory providing a systematic way to explain many phenomena. In this chapter, we shall adopt such a viewpoint and pose several questions about uniformity in broader contexts. Our questions will be phrased so that affirmative answers would be the most natural from the above perspective. However, even negative answers would be very interesting as they might provide starting points for constructing counterexamples to Martin's conjecture.

For example, we've seen in Chapter 3 that if every Borel homomorphism from arithmetic equivalence to Turing equivalence is equivalent to a uniform homomorphism on a cone, then we obtain many of the same consequence that Martin's conjecture would have. In particular, this would imply that Conjecture 3.1.5 is true, and hence that Theorem 3.1.4 is true without the assumption of Martin's conjecture. It would also imply that Turing equivalence is not a universal countable Borel equivalence relation. This is because it rules out an embedding of $\equiv_A$ into $\equiv_T$; for all α with $\omega \leq \alpha < \omega_1$, the map $x \mapsto x^{(\alpha)}$ is not a reduction of arithmetic equivalence restricted to any arithmetic cone into $\equiv_T$.

One might also wonder whether every Borel homomorphism from arithmetic equivalence to itself must be equivalent to a uniform such homomorphism on an arithmetic cone.

Question 4.0.1. *Equip $\equiv_A$ with the generating family of arithmetic reductions. If f is a Borel homomorphism from $\equiv_A$ to $\equiv_A$, must there exist a uniform Borel homomorphism g from $\equiv_A$ to $\equiv_A$ such that $f(x) \equiv_A g(x)$ on an arithmetic cone?*

Here, we are able to obtain more information than the case of Turing equivalence. We begin by introducing the notion of a metric on a Borel equivalence relation. Such metrics are closely connected to a weakening of the notion of a uniform homomorphism, which we call a uniform mod finite homomorphism. We show that there exist homomorphisms from $\equiv_A$ to $\equiv_A$ that are not uniform mod finite on any pointed perfect set if and only if there exists a Borel metric on $\equiv_A$ that is not bounded by a canonical metric d_A on any pointed perfect set. Using the same technology, we are also able to provide the first example of a homomorphism between "recursion-theoretic" equivalence relations that is not uniform on any pointed perfect set.

If $F_0 \subseteq F_1 \subseteq \dots$ is an increasing sequence of countable Borel equivalence relations, then there is a natural ultrametric on their union $\bigcup_{i \in \omega} F_i$. Our investigation of metrics on arithmetic equivalence and Turing equivalence leads us some natural questions regarding how these equivalence relations can represented as increasing unions. We finish with some more general questions on how the non-hyperfiniteness of a countable Borel equivalence relation may be witnessed.

This chapter is the most speculative of the thesis, and many of our results are relationships between open question.

In what follows, we assume that Turing equivalence and arithmetic equivalence are equipped with the generating families of Turing reductions and arithmetic reductions respectively, if we do not specify another generating family.

4.1 Are the Turing reductions the only way to generate Turing equivalence?

We begin this section by posing a precise version of the question given in its title.

Question 4.1.1. *Suppose that $\{\psi_i\}_{i\in\omega}$ is a generating family of partial Borel functions for Turing equivalence. Must there exist a pointed perfect set P and a function $u : \omega^2 \to \omega^2$ such that for all $x, y \in P$, and for all pairs of Turing reductions φ_i and φ_j, if $\varphi_i(x) = y$, $\varphi_i(y) = x$, and $u(i,j) = (k,l)$, then $\psi_k(x) = y$ and $\psi_l(y) = x$?*

We will see that a positive answer to this question implies a large amount of uniformity in many different contexts. Indeed, a positive answer to this question will imply a positive answer to every other question we pose in this chapter! We start by showing that it is really a restatement of the following strengthening of Steel's conjecture 1.2.6 (and hence also Martin's conjecture) for Borel homomorphisms from Turing equivalence to any countable Borel equivalence relation.

Theorem 4.1.2. *Equip $\equiv_T$ with the generating family of Turing reductions. Question 4.1.1 has an affirmative answer if and only if for every countable Borel equivalence relation $E^X_{\{\psi_i\}}$, every Borel homomorphism from $\equiv_T$ to $E^X_{\{\psi_i\}}$ is uniform on some pointed perfect set.*

Proof. For the forward direction, let $E^X_{\{\psi_i\}}$ and f be given. Now we construct another generating family for Turing equivalence. For $i, j \in \omega$ define the partial Borel function $\theta_{i,j} : 2^\omega \to 2^\omega$ by letting $\theta_{i,j}(x) = \varphi_i(x)$ if $\psi_j(f(x)) = f(\varphi_i(x))$, and letting $\theta_{i,j}(x)$ be undefined otherwise. It is clear that for every i, there is some j such that $\theta_{i,j}(x) = \varphi_i(x)$. Hence, if we add the identity function to this collection and then close off under composition, we obtain a generating family of Borel partial functions for Turing equivalence. Now applying the fact that Question 4.1.1 has a positive answer, we see that f must be uniform on some pointed perfect set.

For the reverse direction, suppose $\{\psi_i\}_{i\in\omega}$ is a generating family of partial Borel functions for Turing equivalence. Let f be the identity function, which is a homomorphism from Turing equivalence as generated by the Turing reductions to Turing equivalence as generated by $\{\psi_i\}_{i\in\omega}$. Then this homomorphism must be uniform on some pointed perfect set. $\square$

Recall that by Theorem 1.3.3, a Borel homomorphism from $\equiv_T$ to $\equiv_T$ is uniform on a pointed perfect set if and only if it is equivalent to a uniform Borel homomorphism from $\equiv_T$ to $\equiv_T$ on a Turing cone. The proof we gave is quite robust, and also applies to homomorphisms from $\equiv_T$ to any Borel $E^X_{\{\varphi_i\}}$. Even more generally, this fact remains true if we replace $\equiv_T$ with any natural equivalence relation coarser than $\equiv_T$, such as arithmetic equivalence. For the rest of this chapter, we will prefer the language of being uniform on a pointed perfect set rather than being equivalent to a uniform homomorphism on a cone.

We now show that a positive answer to Question 4.1.1 implies its analogue for arithmetic equivalence.

Theorem 4.1.3. *Suppose Question 4.1.1 has a positive answer. Suppose $\{\psi_i\}_{i\in\omega}$ is any generating family of partial Borel functions for arithmetic equivalence. Then there exists a pointed perfect set P and a function $u : \omega^2 \to \omega^2$ such that for all $x, y \in P$, given any pair of arithmetic reductions φ_i and φ_j, if $\varphi_i(x) = y$, $\varphi_i(y) = x$, and $u(i, j) = (k, l)$, then $\psi_k(x) = y$ and $\psi_l(y) = x$.*

Proof. Assuming Question 4.1.1 has a positive answer, let P be a uniformly pointed perfect set and u be a partial function from ω^2 to ω^2 so that if φ_i and φ_j are arithmetic reductions that are also Turing reductions, then $u(i, j)$ is defined, and for all $x, y \in P$, if $\varphi_i(x) = y$, $\varphi_i(y) = x$, and $u(i, j) = (k, l)$, then $\psi_k(x) = y$ and $\psi_l(y) = x$.

Let f be the canonical homeomorphism from 2^ω to P. Then a canonical representation of the Turing jump function inside P is $J_P : P \to P$ where $J_P(x) = f(x')$. Now using Corollary 1.3.2, find an n and a uniformly pointed perfect subset P^* of P such that for all $x \in P^*$, we have $\psi_n(x) = J_P(x)$. If $J_{P^*}(x)$ is the canonical representation of the Turing jump function inside P, then there is a pair of Turing reductions witnessing $J_{P^*}(x) \equiv_T J_P(x)$ for all x, and hence we can uniformly represent the Turing jump inside P^* using functions in $\{\psi_i\}_{i\in\omega}$. Thus, P^* is the pointed perfect set we desire. $\square$

Now by using an argument identical to that of Theorem 4.1.2, we obtain the following:

Corollary 4.1.4. *If Question 4.1.1 has a positive answer, then every Borel homomorphism from $\equiv_A$ to any countable Borel equivalence relation $E^X_{\{\psi_i\}}$ is uniform on a pointed perfect set.*

Corollary 4.1.5. *If Question 4.1.1 has a positive answer, then so does Question 2.2.2; every universal countable Borel equivalence relation $E^X_{\{\psi_i\}}$ must be uniformly universal.*

Proof. Let f be a Borel embedding of arithmetic equivalence into $E^X_{\{\psi_i\}}$ witnessing that $E^X_{\{\psi_i\}}$ is universal. Then f is uniform on some pointed perfect set. Since arithmetic equivalence restricted to any pointed perfect set is uniformly universal, we are done. □

4.2 The jump metric on arithmetic equivalence

Definition 4.2.1. Let E be an equivalence relation on a Polish space X. A *metric on E* is a non-negative function $d : E \to \mathbb{R}$ (that is, having domain $\{(x, y) \in X^2 : xEy\}$) that satisfies the usual axioms of a metric.

Alternatively, we could think of a metric on E as a $\mathbb{R} \cup \{\infty\}$-valued metric on X such that $d(x, y) < \infty$ if and only if xEy. Naturally, we will be interested in Borel metrics on Borel equivalence relations. The metrics we consider will all take integer values. We begin with a couple examples.

Example 4.2.2. *Let G be a countable group equipped with a set of generators and a Borel action on a Polish space X. Let E^X_G be the induced orbit equivalence relation. Then we can define a metric on E^X_G by mapping (x, y) to the least n such there exists a $g \in G$ with word norm $|g| = n$ such that $x = g \cdot y$.*

Example 4.2.3. *If $F_1 \subseteq F_2 \subseteq \ldots$ is an increasing sequence of countable Borel equivalence relations, then the function*

$$d(x,y) = \begin{cases} 0 & \textit{if } x = y \\ n & \textit{where } n \textit{ is least such that } xF_n y\textit{, otherwise.} \end{cases}$$

is a Borel metric on the union $E = \bigcup_i E_i$. Note that d is an ultrametric*; a metric where the triangle inequality can be replaced by the stronger statement that for all x, y, z, $d(x, z) \leq max(d(x, y), d(y, z))$.*

We now define a metric on arithmetic equivalence which will be important for our investigation.

Definition 4.2.4. The *jump metric* on $\equiv_A$ is defined on distinct x and y by $d_j(x, y) = n+1$ where n is the least number such that $x^{(n)} \geq_T y$ and $y^{(n)} \geq_T x$.

Metrics naturally arise from homomorphisms between equivalence relations as given by the following definition. In the next section, we will see that there is a close connection between uniformity and such metrics. Our main goal of this section is to establish Theorem 4.2.6 which will be useful in this context.

Definition 4.2.5. Suppose E and F are countable Borel equivalence relations on the Polish spaces X and Y. Suppose F is equipped with a Borel metric d, and $f : X \to Y$ is a Borel homomorphism from E to F. Then define d_f to be the Borel metric on E associated to f, where $d_f(x, y) = d(f(x), f(y))$.

Now consider the reverse situation, where we want to know what Borel metrics on E can arise from Borel homomorphisms from E to F. In the case where F is arithmetic equivalence equipped with the jump metric, we have the following theorem.

Theorem 4.2.6. *Equip $\equiv_A$ with the jump metric. Suppose E is a countable Borel equivalence relation on a Polish space X, and d is any Borel metric on E. Then there exists a Borel homomorphism f of E into $\equiv_A$ such that $d_f \geq d$, where d_f is the Borel metric on E associated to f.*

Our proof will be a variation of Slaman and Steel's proof that arithmetic equivalence is a universal countable Borel equivalence relation. Indeed, the homomorphisms we construct will all be embeddings. The crux of the proof is the following method of coding.

In what follows, we fix a recursive pairing function giving a bijection $\langle , \rangle : \omega^2 \to \omega$. If x is a real, then the nth column of x is defined by $x^{[n]}(i) = x(\langle n, i\rangle)$.

Definition 4.2.7. Given $y, z \in 2^\omega$, say that z *jump codes* y if for every n, $z^{[n]}$ has a limit, and $y(n) = \lim_m z^{[n]}(m)$. The *Skolem function* for this jump coding is the function from ω to ω that maps n to the least i such that $\forall j \geq i[z(\langle n, j\rangle) = z(\langle n, i\rangle)]$.

The name of this coding derives from the fact that if z jump codes y, then $z' \geq_T y$. Indeed, using z' as an oracle, we can compute both y and the Skolem function for this jump coding. Given n, find the least i so that $\forall j > i[z(\langle n, j\rangle) = z(\langle n, i\rangle)]$, using the oracle z'. Then the nth bit of y is $z(\langle n, i\rangle)$.

Definition 4.2.8. Let $x : \omega \to 2^{<\omega}$ be any function. For any real $y \in 2^\omega$, define $J(x, y) \in 2^\omega$ to be the real that jump codes y via x. Precisely, we mean that the nth column of $J(x, y)$ will be

$$(J(x,y))^{[n]} = \begin{cases} x(n)10000\ldots & \text{if y(n) = 0} \\ x(n)01111\ldots & \text{if y(n) = 1} \end{cases}$$

Hence, $J(x, y)$ jump codes y, and the Skolem function for this jump coding is $n \mapsto |x(n)| + 1$, where $|x(n)|$ is the length of the finite sequence $x(n)$.

If p is a partial function from ω to $2^{<\omega}$ and r is a partial function from ω to 2 with $\mathrm{dom}(p) \subseteq \mathrm{dom}(r)$, analogously define $J(p, r)$, a partial function from ω to 2, where the nth column of $J(p, r)$ is undefined if $n \notin \mathrm{dom}(p)$.

The idea of coding a real as a limit of columns has a long history in recursion theory. The proof we will present uses jump codings of "depth" ω. In this way, it is reminiscent of some constructions that have been used to investigate the structure $\langle \mathcal{D}, \leq_T, ' \rangle$ of the Turing degrees under $\leq_T$ and the jump operator. See the papers of Hinman and Slaman [16], and Montalbán [39].

Let $\mathbb{P}_{\omega,2^{<\omega}}$ be the partial order of finite partial functions from ω to $2^{<\omega}$ ordered under inclusion. Say that a function x from ω to $2^{<\omega}$ is *arithmetically generic* if it meets every arithmetically definable dense subset of $\mathbb{P}_{\omega,2^{<\omega}}$. Similarly, finitely many functions $x_1, \dots x_n$ from ω to $2^{<\omega}$ are *mutually arithmetically generic* if $(x_1, \dots, x_n)$ meets every arithmetically definable dense subset of $(\mathbb{P}_{\omega,2^{<\omega}})^n$.

We begin with a lemma whose proof is standard for the subject:

Lemma 4.2.9. *If x, z, and w are mutually arithmetically generic functions from ω to $2^{<\omega}$, then for all $n \in \omega$ and $y \in 2^\omega$,*

1. $\left(0^{(n)} \oplus J(x,y) \oplus z\right)' \equiv_T 0^{(n+1)} \oplus x \oplus y \oplus z$

2. $0^{(n)} \oplus J(x,y) \oplus z \not\geq_T w$

Proof. We prove part 1. Let $\mathbb{P}_{\omega,2}$ be the partial order of finite partial functions from ω to 2 ordered under inclusion.

Fix an e. Consider the set D of pairs $(p,q) \in \left(\mathbb{P}_{\omega,2^{<\omega}}\right)^2$ such that for every $r \in \mathbb{P}_{\omega,2}$ with $\mathrm{dom}(p) = \mathrm{dom}(r)$, either $\varphi_e(0^{(n)} \oplus J(p,r) \oplus q)$ halts, or for every extension of (p,q,r) to (p^*, q^*, r^*), we have that $\varphi_e(0^{(n)} \oplus J(p^*, r^*) \oplus q^*)$ does not halt. We claim that D is dense in $\left(\mathbb{P}_{\omega,2^{<\omega}}\right)^2$.

Suppose $(p,q) \in \left(\mathbb{P}_{\omega,2^{<\omega}}\right)^2$. We wish to show that (p,q) can be extended to meet D. Let $r_1, \dots, r_n$ be a list of all elements of $\mathbb{P}_{\omega,2}$ such that $\mathrm{dom}(p) = \mathrm{dom}(r_i)$. Let $s_0 = \emptyset$, and $q_0 = q$. We will define an increasing sequence $s_1 \subseteq \dots \subseteq s_n$ of elements of $\mathbb{P}_{\omega,2}$ and an increasing sequence $q_1 \subseteq \dots \subseteq q_n$ of elements of $\mathbb{P}_{\omega,2^{<\omega}}$.

Inductively, for $1 \leq i \leq n$, consider $0^{(n)} \oplus J(p, r_i) \sqcup s_{i-1} \oplus q_{i-1}$, a partial function from ω to 2. Either there no extension of this partial function that makes φ_e halt relative to it, or there is a finite such extension. If there is such an extension, let it be $0^{(n)} \oplus J(p, r_i) \sqcup s_i \oplus q_i$, where q_i extends q_{i-1}, where s_i extends s_{i-1}, and the domain of s_i is disjoint from $J(p, r_i)$. If there is no such extension, let $q_i = q_{i-1}$, and $s_i = s_{i-1}$.

Extend p to any $\hat{p}$ so that for every $\langle j,k \rangle \in \mathrm{dom}(s_n)$. $\hat{p}(j)(k) = s(\langle j,k \rangle)$. Note that this means that for any $r \in \mathbb{P}_{\omega,2}$ with $\mathrm{dom}(r) = \mathrm{dom}(\hat{p})$, $J(\hat{p}, r)$ will be an extension of $J(p,r) \sqcup s_n$. It is clear that $(\hat{p}, q_n)$ meets D.

If x and z are arithmetically generic, then for each e, $0^{(n+1)} \oplus x \oplus y \oplus z$ can compute a place where (x,z) meets D. Hence, from $0^{(n+1)} \oplus x \oplus y \oplus z$ we can compute the Σ^0_1 theory of $0^{(n)} \oplus J(x,y) \oplus z$, and thus $0^{(n+1)} \oplus x \oplus y \oplus z \geq_T \left(0^{(n)} \oplus J(x,y) \oplus z\right)'$. Obviously $\left(0^{(n)} \oplus J(x,y) \oplus z\right)' \geq_T 0^{(n+1)} \oplus x \oplus y \oplus z$.

We now proceed to part 2, whose proof is similar to part 1. Fix an e. The dense set that (x,z,w) must meet is the set of triples $(p,q,r) \in (\mathbb{P}_{\omega,2^{<\omega}})^3$ such that for every $s \in \mathbb{P}_{\omega,2}$ with $\mathrm{dom}(p) = \mathrm{dom}(s)$, there exists a k such that $\varphi_e(0^{(n)} \oplus J(p,s) \oplus q)(k)\downarrow \neq r(k)$, or for every extension of (p,q,s) to (p^*, q^*, s^*), we have that $\varphi_e(0^{(n)} \oplus J(p,s) \oplus q)(k)$ does not halt. We leave the rest of the proof to the reader. □

Note that for all $x_0, \ldots, x_n$ and $y_0, \ldots, y_n$, we have that

$$J(x_0, y_0) \oplus \ldots \oplus J(x_n, y_n) \equiv_T J(x_0 \oplus \ldots \oplus x_n, y_0 \oplus \ldots \oplus y_n)$$

and that if $x_0, \ldots, x_n$ and $z_0, \ldots, z_n$ are all mutually arithmetically generic, then $x_0 \oplus \ldots \oplus x_n$ and $z_0 \oplus \ldots \oplus x_n$ are mutually arithmetically generic. Therefore, we can conclude a more general fact:

Lemma 4.2.10. *If $x_0, \ldots, x_i$, $z_0, \ldots, z_j$, and w are mutually arithmetically generic functions from ω to $2^{<\omega}$, then for all $n \in \omega$ and $y_0, \ldots, y_i \in 2^\omega$*

1. $\left(0^{(n)} \oplus J(x_0, y_0) \oplus \ldots \oplus J(x_i, y_i) \oplus z_0 \oplus \ldots \oplus z_j\right)'$
$$\equiv_T 0^{(n+1)} \oplus x_0 \oplus \ldots \oplus x_i \oplus y_0 \oplus \ldots \oplus y_i \oplus z_0 \oplus \ldots z_j.$$
2. $0^{(n)} \oplus J(x_0, y_0) \oplus \ldots \oplus J(x_i, y_i) \oplus z_0 \oplus \ldots \oplus z_j \not\geq_T w$

We need one final definition before we proceed to the proof of Theorem 4.2.6.

Definition 4.2.11. If $n \geq 1$, then given $x : \omega \to 2^{<\omega}$ where $x = x_1 \oplus x_2 \oplus \ldots \oplus x_n$ and $y \in 2^\omega$, define the finitely iterated jump coding $J^n(x, y)$ as follows. $J^1(x, y) = J(x, y)$ and $J^n(x, y) = J(x_n, (J^{n-1}(x_1 \oplus x_2 \oplus \ldots \oplus x_{n-1}, y))$.

We now proceed to the proof of Theorem 4.2.6.

Proof of Theorem 4.2.6. We may as well assume that d takes values in ω by replacing d with $\lceil d \rceil$, which is also a metric. We also may assume that E is generated by a Borel action of some group G. Let $\{g_i\}_{i \in \omega}$ be a listing of all the elements of G.

Let $g : X \to (2^{<\omega})^\omega$ be a Borel function such that for every distinct $x_0, x_1, \ldots, x_n \in X$, we have that $g(x_0), g(x_1), \ldots, g(x_n)$ are all mutually arithmetically generic functions from ω to $2^{<\omega}$. Now we simultaneously define the functions $\{f_i\}_{i \in \omega}$ where $f_i : X \to 2^\omega$ by:

$$f_i(x) = J^{d(x, g_i \cdot x)+1}\left(g(x)^{[i]}, f_0(g_i \cdot x) \oplus f_{i+1}(x)\right).$$

where $g(x)^{[i]}$ is the ith column of $g(x)$. While the definitions of the f_i above are self-referential, they are not circular, as one can see by repeatedly expanding the definitions on the right hand side of the above equation.

It is clear that for all i, $f_0(x) \geq_A f_i(x) \geq_A f_0(g_i \cdot x)$. Hence, if we set $f = f_0$, then f is a homomorphism from E to $\equiv_A$. By inductively applying Lemma 4.2.10, we see that for all n, if $d(x, y) > n$, then $(f(x))^{(n)} \not\geq_T f(y)$. $\square$

4.3 Uniformity, metrics, and arithmetic equivalence

Suppose that $\{\varphi_i\}_{i\in\omega}$ is a generating family of partial Borel functions on a Polish space X, inducing the countable Borel equivalence relation $E^X_{\{\varphi_i\}}$. We will now proceed to define a natural metric on $E^X_{\{\varphi_i\}}$ associated to this generating family. For the next two sections, we will assume that φ_0 is the identity function in every generating family $\{\varphi_i\}_{i\in\omega}$. This avoids some trivialities in our definitions.

Consider the function $\tilde{d}_{\{\varphi_i\}} : E^X_{\{\varphi_i\}} \to \omega$ defined by $\tilde{d}_{\{\varphi_i\}}(x,y) = i + j$ where i is least such that $\varphi_i(x) = y$ and j is least such that $\varphi_j(y) = x$. In some sense, $\tilde{d}_{\{\varphi_i\}}$ captures the distance between x and y as measured by the generating family $\{\varphi_i\}_{i\in\omega}$. However, $\tilde{d}_{\{\varphi_i\}}$ is not necessarily a metric, as it may not satisfy the triangle inequality. If $\varphi_i(x) = y$ and $\varphi_j(y) = z$, then $(\varphi_j \circ \varphi_i)(x) = z$, but the index for $\varphi_j \circ \varphi_i$ may not be $\leq i + j$. Nevertheless, there is a simple way we can transform $\tilde{d}_{\{\varphi_i\}}$ into a metric.

Definition 4.3.1. Define $d_{\{\varphi_i\}}$, the canonical metric on $E^X_{\{\varphi_i\}}$ associated to the generating family $\{\varphi_i\}_{i\in\omega}$ by setting

$$d_{\{\varphi_i\}}(x,y) = \min_{x_0,\ldots,x_n} \sum_{0\leq i<n} \tilde{d}_{\{\varphi_i\}}(x_i, x_{i+1}),$$

where $\tilde{d}_{\{\varphi_i\}}$ is defined as above. That is, $d_{\{\varphi_i\}}(x,y)$ is the minimum distance from x to y along paths with edges weighted by $\tilde{d}_{\{\varphi_i\}}$.

Clearly $d_{\{\varphi_i\}} \leq \tilde{d}_{\{\varphi_i\}}$. Indeed, $d_{\{\varphi_i\}}$ is the maximal metric that is $\leq \tilde{d}_{\{\varphi_i\}}$. Further, there a function $\rho : \omega \to \omega$ such that $\tilde{d}_{\{\varphi_i\}}(x,y) \leq \rho(d_{\{\varphi_i\}}(x,y))$. This is because if $d_{\{\varphi_i\}}(x,y) = n$, then $x E^X_{\{\varphi_i\}} y$ via compositions of $\leq n$ of the functions in $\{\varphi_i\}_{i\in\omega}$ all of index $\leq n$.

We will often encounter this situation where for some pair of metrics d_0 and d_1, there exists some function $\rho : \omega \to \omega$ such that $d_0(x,y) \leq \rho(d_1(x,y))$. We define the following notation:

Definition 4.3.2. If d_0 and d_1 are metrics on the equivalence relation E, then we write $d_0 \preceq d_1$ if there exists some $\rho : \omega \to \omega$ such that $d_0(x,y) \leq \rho(d_1(x,y))$ for all $(x,y) \in E$. We say that $d_0 \preceq d_1$ on the set S if the restriction of d_0 to $E \restriction S$ is $\preceq$ the restriction of d_1 to $E \restriction S$. We write $d_0 \simeq d_1$ if $d_0 \preceq d_1$ and $d_1 \preceq d_0$.

For example, suppose $\{\varphi_i\}_{i\in\omega}$ is a generating family and $\{\psi_i\}_{i\in\omega}$ is a reordering of the same generating family. Then for the associated metrics $d_{\{\varphi_i\}}$ and $d_{\{\psi_i\}}$, we have that $d_{\{\varphi_i\}} \simeq d_{\{\psi_i\}}$.

Definition 4.3.3. Let d_T be the canonical metric on Turing equivalence arising from the generating family of Turing reductions. Let d_A be the canonical metric on Arithmetic equivalence arising from the generating family of Arithmetic reductions. These metrics are both well-defined up to $\simeq$.

In what follows, we will prefer to work with d_A rather than the jump metric on $\equiv_A$. Hence, we restate Theorem 4.2.6 using this canonical metric. Note that when we change from the jump metric to d_A, our conclusion changes to $d_f \succeq d$ instead of $d_f \geq d$.

Corollary 4.3.4. *Equip $\equiv_A$ with the canonical metric d_A. Suppose E is a countable Borel equivalence relation on a Polish space X, and d is any Borel metric on E. Then there exists a Borel homomorphism f of E into $\equiv_A$ such that $d_f \succeq d$, where d_f is the Borel metric on E associated to f.*

Proof. We claim that the same f as in Theorem 4.2.6 works to prove this theorem. It is clear that the jump metric $d_j \preceq d_A$. Now let $d_{f,j}$ be the metric arising from f and d_j and $d_{f,A}$ be the metric arising from f and d_A. That is, $d_{f,j}(x, y) = d_j(f(x), f(y))$, and $d_{f,A}(x, y) = d_A(f(x), f(y))$. Then we see that $d_{f,A} \succeq d_{f,j} \geq d$. □

There is a connection between metrics and a weakening of the notion of a uniform homomorphism. Recall that $\mathcal{P}_\omega(S)$ is the set of finite subsets of S.

Definition 4.3.5. If f is a homomorphism from $E^X_{\{\varphi_i\}}$ to $E^Y_{\{\psi_i\}}$, say that f is *uniform mod finite* if there is a function $u : \omega^2 \to \mathcal{P}_\omega(\omega^2)$ such that if $xE^X_{\{\varphi_i\}}y$ via (i, j), then $f(x)E^Y_{\{\psi_i\}}f(y)$ via one of the finitely many pairs in $u(i, j)$.

The connection then, between metrics and this weak form of uniformity is obvious.

Lemma 4.3.6. *If f is a homomorphism from $E^X_{\{\varphi_i\}}$ to $E^Y_{\{\psi_i\}}$, then f is uniform mod finite if and only if $d_f \preceq d_{\{\varphi_i\}}$, where d_f is the metric associated to f defined by $d_f(x, y) = d_{\{\psi_i\}}(f(x), f(y))$*

Now we proceed to use this lemma together with Theorem 4.2.6 to produce an example of a homomorphism between "recursion-theoretic" equivalence relations, which is not equivalent to any uniform homomorphism on any pointed perfect set.

Theorem 4.3.7. *Equip $\equiv_A$ with the generating family of arithmetic reductions. There exists a countable Borel equivalence relation $E_{\{\varphi_i\}}$ on 2^ω that is coarser than Turing equivalence and a Borel homomorphism f from $E_{\{\varphi_i\}}$ to $\equiv_A$ that is not uniform on any pointed perfect set.*

Proof. We first define the equivalence relation E by xEy if $x^{(\alpha)} \geq_T y$ and $y^{(\alpha)} \geq_T x$ for some $\alpha < \omega^\omega$. Define the Borel function $\theta : 2^\omega \to 2^\omega$ by setting $\theta(x) = x^{(\omega^n)}$ where $n \in \omega$ is the least even number such that $x(n) = 1$, and $\theta(x) = x$ if there is no such n. Our generating family $\{\varphi_i\}_{i\in\omega}$ will be the closure of the Turing reductions and θ under composition.

There is another way we could generate E. Let $\{\psi_i\}_{i\in\omega}$ be the generating family obtained by closing the Turing reductions and the functions $\{x \mapsto x^{\omega^{(n)}}\}_{n\in\omega}$ under composition.

Suppose P is a uniformly pointed perfect set, f is the canonical homeomorphism from $2^\omega \to P$, and z is a real representing the tree T such that $P = [T]$. Then consider the function $\theta_P : P \to P$, where $\theta_P(x) = f(\theta(f^{-1}(x) \oplus z))$. It is clear that for every n, there exists an $x \in P$ such that $\theta_P(x) \geq_T x^{\omega^{(n)}}$. In contrast, given any n, there must be an m such that if $d_{\{\psi_i\}}(x, y) \leq n$, then $y \leq x^{\omega^m}$. Hence, because $d_{\{\psi_i\}}(x, \theta_P(x))$ is unbounded on P, while $d_{\{\varphi_i\}}(x, \theta_P(x))$ is bounded, we see that $d_{\{\psi_i\}} \not\preceq d_{\{\varphi_i\}}$ on any pointed perfect set.

Now apply Corollary 4.3.4 to obtain a Borel homomorphism f from $E_{\{\varphi_i\}}$ to $\equiv_A$ such that the associated metric d_f on $E_{\{\varphi_i\}}$ is $\succeq d_{\{\psi_i\}}$. Hence, $d_f \not\preceq d_{\{\varphi_i\}}$ on any pointed perfect set, so f cannot be uniform mod finite on any pointed perfect set. □

The key to the above theorem is that our equivalence relation E is a nontrivial increasing union of equivalence relations. In particular, if we let xF_iy if $x^{(\alpha)} \geq_T y$ and $y^{(\alpha)} \geq_T x$ for some $\alpha < \omega^i$, then $E = \bigcup_{i \in \omega} F_i$. This increasing union is nontrivial in the sense that there does not exist a pointed perfect set P such that $E \restriction P = F_n \restriction P$ for some n. It is natural to ask whether such nontrivial increasing unions exist for Turing equivalence or arithmetic equivalence, and we will consider such questions shortly.

We now turn to the question of uniform mod finite homomorphisms on arithmetic equivalence. Here, Theorem 4.2.6 gives us some information. To begin, we ask the following weak version of Question 4.0.1.

Question 4.3.8. *Equip $\equiv_A$ with the generating family of arithmetic reductions. If f is a Borel homomorphism from $\equiv_A$ to $\equiv_A$, then must f be uniform mod finite on some pointed perfect set?*

Here, we can equivalently characterize this question in a number of ways.

Theorem 4.3.9. *The following are equivalent:*

1. *Every Borel homomorphism from $\equiv_A$ to $\equiv_A$ is uniform mod finite on a pointed perfect set.*
2. *If $E^X_{\{\varphi_i\}}$ is any countable Borel equivalence relation, then every Borel homomorphism from $\equiv_A$ to $E^X_{\{\varphi_i\}}$ is uniform mod finite on a pointed perfect set.*
3. *If d is any Borel metric on $\equiv_A$, then $d \preceq d_A$ on some pointed perfect set.*

Proof. (1) $\implies$ (2) and (3) $\implies$ (1) are obvious. $\neg(3) \implies \neg(2)$ follows from Corollary 4.3.4 and Lemma 4.3.6; a Borel metric d on $\equiv_A$ such that $d \not\preceq d_A$ on every pointed perfect set would allow us to construct a Borel homomorphism from $\equiv_A$ to $\equiv_A$ that is not uniform mod finite on any pointed perfect set. □

This theorem is evidence that the very strong uniformity principles we considered in Section 4.1 are not completely implausible, or at least not significantly more implausible than an affirmative answer to Question 4.0.1.

If Question 4.3.8 has an affirmative answer, this would have several corollaries.

Corollary 4.3.10. *If Question 4.3.8 has an affirmative answer, then every universal countable Borel equivalence relation $E^X_{\{\varphi_i\}}$ is uniformly universal mod finite, in the sense that given any $E^Y_{\{\psi_i\}}$, there is a uniform mod finite Borel reduction from $E^Y_{\{\psi_i\}}$ to $E^X_{\{\varphi_i\}}$.*

The proof of this fact is completely analogous to the proof of Corollary 4.1.5.

Corollary 4.3.11. *If Question 4.3.8 has an affirmative answer, then if $F_0 \subseteq F_1 \subseteq \dots$ is an increasing union of non-universal countable Borel equivalence relations, then $\bigcup_{i\in\omega} F_i$ is a non-universal countable Borel equivalence relation.*

Proof. For a contradiction, suppose that $\bigcup_{i\in\omega} F_i$ was universal. Equip each F_i with a generating family, and let $\bigcup_{i\in\omega} F_i$ be generated by the closure of the union of these generating families under composition. Now take a Borel embedding f of $\equiv_A$ into $\bigcup_{i\in\omega} F_i$. Then f must be uniform mod finite on a uniformly pointed perfect set P, since every pointed perfect set contains a uniformly pointed perfect set by Lemma 1.3.1.

Given any uniformly pointed perfect set P, there are finitely many injective arithmetic reductions $\{\psi_0, \psi_1, \dots, \psi_n\}$ whose inverses are also arithmetic reductions such that the closure of these ψ_i under composition generates arithmetic equivalence restricted to P. Now because f is uniform mod finite on P, we see that there must be some F_j so that for all $x \in P$, $f(x) F_j f(\psi_i(x))$ for all $i \leq n$. Hence, f is an embedding of arithmetic equivalence restricted to a pointed perfect set into F_j, and so F_j is universal. □

A very similar argument also shows the following.

Corollary 4.3.12. *If Question 4.3.8 has an affirmative answer, arithmetic equivalence is E_0-ergodic with respect to the arithmetic cone measure.*

4.4 Is d_T the maximal way to measure distance in the Turing degrees?

We now turn to similar questions for Turing equivalence. Lacking a version of Theorem 4.2.6 for Turing equivalence, the appropriate question seems to be the following.

Question 4.4.1. *If d is any Borel metric on $\equiv_T$, must there exist a pointed perfect set P such that $d \preceq d_T$ on P? Equivalently, if f is a Borel homomorphism from $\equiv_T$ to any countable Borel equivalence relation $E^X_{\{\varphi_i\}}$, must f be uniform mod finite on a pointed perfect set?*

It is probably worth pointed out that in Question 4.4.1, it is vital that d be a metric; there is a symmetric Borel function $c :\equiv_T \to \omega$ (which does not satisfy the triangle inequality) such that $c \not\preceq d_T$ on every pointed perfect set. To see this, define c as follows: if $d_T(x, y) = n$, then let $c(x, y) = m$ where m is the least number such that for all $k \leq n$, if the kth program halts relative to x or y in s steps, then $s \leq m$. That $c \not\preceq d_T$ on any pointed perfect set follows from the fact that every pointed perfect set contains x and y such that $d_T(x, y)$ is arbitrarily large.

A few of the results from the previous section have analogues for Turing equivalence via similar proofs. For example, If Question 4.4.1 has an affirmative answer, then Turing equivalence is E_0 ergodic with respect to Martin measure.

Question 4.4.1 is related to Question 4.3.8 in the following way:

Theorem 4.4.2. *If Question 4.4.1 has an affirmative answer, then so does Question 4.3.8.*

Proof. Suppose that d is a Borel metric on arithmetic equivalence. Let $d \restriction \equiv_T$ be the restriction of d to Turing equivalence. Then if $d \restriction \equiv_T \preceq d_T$ on some pointed perfect set P, we claim that $d \preceq d_A$ on some pointed perfect subset P^* of P. This follows via an argument like that used to establish Theorem 4.1.3. $\square$

We obtain an interesting weakening of Question 4.4.1 if we restrict our attention to ultrametrics. The resulting question can be equivalently phrased in terms of increasing unions as follows:

Question 4.4.3. *Suppose we write Turing equivalence as an increasing union of countable Borel equivalence relations $F_0 \subseteq F_1 \subseteq \ldots$. Must there exist some n and some pointed perfect set P such that $\equiv_T \restriction P = F_n \restriction P$?*

Slaman and Steel [47] were the first to consider these sorts of increasing unions. Their motivating concern was the construction of non-uniform Turing invariant functions. Suppose $F_0 \subseteq F_1 \subseteq \ldots$ are Borel equivalence relations such that $\bigcup_{i \in \omega} F_i$ is Turing equivalence. Then these F_i can be used in the construction of a Borel Turing-invariant function f in the following way. Construct f in ω many stages, where at stage i, we make commitments about how $f(x)$ is coded into $f(y)$ for all x and y such that $x F_i y$. If we want f to be non-uniform on every pointed perfect set, then we would like each F_i to be a simple enough cross section of Turing equivalence so that at every stage it is still possible to extend the approximation of f to diagonalize against being uniform on some pointed perfect set with a representation Turing reducible to x.

Slaman and Steel's investigation led to their proof that Turing equivalence is not hyperfinite. It is easy to see that for any hyperfinite countable Borel equivalence relation E, there is a countable-to-one homomorphism of E into $\equiv_T$ whose range is contained in the complement of a cone. Slaman and Steel then posed an open question asking whether Turing equivalence satisfied a strengthening of being non-hyperfinite. Their original question is posed in the context of AD. We state the Borel version of their question.

Question 4.4.4 (Slaman and Steel [47])**.** *Suppose we write Turing equivalence as an increasing union of countable Borel equivalence relations $F_0 \subseteq F_1 \subseteq \ldots$. Must there exist an x and an n such that $x \geq_T \langle y_i \rangle_{i \in \omega}$ for some sequence of y_i such that $x F_n y_i$ for all y_i?*

Of course, a positive answer to Question 4.4.3 implies a positive answer to Question 4.4.4. It is currently open whether a negative answer to Question 4.4.4 would allow the construction of a Borel Turing invariant function that is not uniform on any pointed perfect set.

We note that Slaman and Steel's question is equivalent to a much more general one. Recall that a countable Borel equivalence relation is *aperiodic* if all its equivalence classes are infinite.

Theorem 4.4.5. *Question 4.4.4 has a positive answer if and only if there exists some aperiodic countable Borel equivalence relation E and a Borel function S which maps each equivalence class C of E to $S(C)$, a countable set of infinite subsets of C such that S witnesses that E is not hyperfinite in the following strong way: if $F_0 \subseteq F_1 \subseteq \ldots$ is any increasing sequence of countable Borel equivalence relations such that $\bigcup_{i \in \omega} F_i = E$, then there exists some n, some equivalence class C of E, and some element R of $S(C)$ such that R is a subset of some equivalence class of F_n.*

Proof. The forward direction is obvious. For the reverse direction, we may as well assume that E is an equivalence relation on 2^ω. Let $z \in 2^\omega$ be such that both E and S have $\Delta^1_1(z)$ definitions. Suppose that $\alpha < \omega_1^z$ is such that for all x and y where xEy, we have that $(x \oplus z)^{(\alpha)} \equiv_T (y \oplus z)^{(\alpha)}$. Suppose further that for every equivalence class C of E, for every $x \in C$, and every $R \in S(C)$, we have that $x \geq_T \langle y_i \rangle_{i \in \omega}$ for some sequence of y_i such that $R = \{y_i : i \in \omega\}$. Then the function $f(x) = (x \oplus z)^{(\alpha)}$ is an injective homomorphism from E to $\equiv_T$. Given any increasing union of countable Borel equivalence relations $F_0 \subseteq F_1 \subseteq \ldots$ such that $\bigcup_{i \in \omega} F_i$ is Turing equivalence, these F_i can be pulled back along f to give an increasing sequence of countable Borel equivalence relations whose union is E. □

We now state an open question of intermediate strength between Question 4.4.3 and Question 4.4.4.

Question 4.4.6. *Does there exist an aperiodic countable Borel equivalence relation E and countably many Borel function $\{S_i\}_{i \in \omega}$ witnessing that E is not hyperfinite in the following way: the domain of each S_i is the equivalence classes of E, $S_i(C)$ is an infinite subset of C for all i and C, and for all increasing sequences $F_0 \subseteq F_1 \subseteq \ldots$ of countable Borel equivalence relations such that $E = \bigcup_{i \in \omega} F_i$, there exists an n, an i, and an equivalence class C of E such that $S_i(C)$ is a subset of some equivalence class of F_n.*

We finish this section by noting that a positive answer to Question 4.4.6 would require a proof of the existence of a non-hyperfinite countable Borel equivalence relation that does not use purely measure theoretic arguments.

Theorem 4.4.7. *Suppose E is a countable Borel equivalence relation on the Polish space X, and $\{S_i\}$ is a set of Borel functions as in Question 4.4.6. Then if μ is a Borel probability measure on X, then there exists an E-invariant set B of full measure and an increasing union of countable Borel equivalence relations $F_0 \subseteq F_1 \subseteq \ldots$ on B such that $\bigcup_{i \in \omega} F_i = E \restriction B$, and for all equivalence classes C of E, and for all i and n, $S_i(C)$ is not a subset of any equivalence class of F_n.*

Proof. Recall that a measure μ is said to be *E-quasi-invariant* if for all Borel sets A, if $\mu(A) = 0$, then $\mu([A]_E) = 0$. We may assume that μ is E-quasi-invariant, since every Borel probability measure is dominated by an E-quasi-invariant Borel probability measure. By Lemma 6.1.2, let $\{A_{i,j}\}_{(i,j)\in\omega^2}$ be a sequence of Borel sets such that $A_{i,j}$ meets every set in the range of S_i, and $\mu(A_{i,j}) \leq 2^{-j}$. Now let $A_k^* = \bigcup_{i\in\omega} A_{i,i+k}$. Hence, every A_k^* meets every set in the range of S_i for all i, and $\mu(A_k) \to 0$ as $k \to \infty$. Define B to be the complement of $[\bigcap_{k\in\omega} A_k^*]_E$. Finally, define F_i by setting xF_iy if xEy and $x \notin A_i^*$ and $y \notin A_i^*$. $\square$

Chapter 5

Recursive Isomorphism

We say that $x, y \in 2^\omega$ are *recursively isomorphic* if there exists a recursive bijection $\rho : \omega \to \omega$ such that $x(n) = y(\rho^{-1}(n))$. That is, we can obtain x by recursively permuting the bits of y. We let $\equiv_1$ denote the equivalence relation of recursive isomorphism on 2^ω. We can analogously define recursive isomorphism on ω^ω and on n^ω for $n \geq 2$. It is clear that the identity function witnesses that recursive isomorphism on n^ω is Borel reducible to recursive isomorphism on m^ω for $n \leq m$.

In this chapter, we will study the question of whether recursive isomorphism is a universal countable Borel equivalence relation. This question is interesting for a number of reasons. First, it is a theorem of folklore that $\equiv_T \leq_B \equiv_1$ via the map $x \mapsto x'$. Hence, if we are able to prove that $\equiv_1$ is not universal, then neither is $\equiv_T$.

Secondly, the universality of recursive isomorphism has arisen in the study of Hjorth's Question 1.1.4 of whether every weakly universal countable Borel equivalence relation is universal. For instance, an investigation of this question inspired the following theorem of Andretta, Camerlo, and Hjorth:

Theorem 5.0.1 (Andretta, Camerlo, and Hjorth, [2])**.** *Recursive isomorphism on 5^ω is a universal countable Borel equivalence relation.*

In fact, they proved a far more general theorem. Let S_∞ be the group of all permutations of ω. S_∞ acts on n^ω where for $\rho \in S_\infty$, we define $(\rho \cdot x)(n) = x(\rho^{-1}(n))$. Given any countable subgroup $G \subseteq S_\infty$, let $\equiv^n_G$ be the orbit equivalence relation arising from the restriction of this action to G. Andretta, Camerlo, and Hjorth showed that there is a group G_0 consisting solely of recursive permutations such that for any countable group G where $G_0 \subseteq G \subseteq S_\infty$, the associated equivalence relation $\equiv^5_G$ is a universal countable Borel equivalence relation.

Finally, work on the universality of recursive isomorphism has been a stepping stone to other universality proofs. For example, Theorem 5.0.1 was used in the same paper to prove the following theorem, improving an earlier result of Thomas and Veličković [55]. This theorem is also related to Hjorth's Question 1.1.4:

Theorem 5.0.2 (Andretta, Camerlo, and Hjorth, [2])**.** *If G is any countable group containing $\mathbb{F}_2$ as a subgroup, then the equivalence relation of conjugacy on subgroups of G is a universal countable Borel equivalence relation.*

Camerlo has also used Theorem 5.0.1 to show that the recursive isomorphism relation on countable trees, groups, Boolean algebras, fields and total orderings are all universal countable Borel equivalence relations [6].

Theorem 5.0.1 improved an earlier result of Dougherty and Kechris [11] [12] that recursive isomorphism on ω^ω is a universal countable Borel equivalence relation.

In this chapter, we will prove two main results. First, we identify an open question which takes the form of an abstract diagonalization problem. We show that a positive answer to this question implies the universality of recursive isomorphism on 2^ω. A corollary of our proof is that recursive isomorphism on 3^ω is universal, improving Theorem 5.0.1. Secondly, we provide a partial converse to this result by demonstrating that this question has an affirmative answer if and only if many-one equivalence on 2^ω is uniformly universal (with respect to the many-one reductions). In Chapter 6, we will see that our open question can be restated as a problem of Borel combinatorics. Hence, our result gives an exact combinatorial calibration of the difficulty of showing that many-one equivalence is uniformly universal.

5.1 Recursive isomorphism on 2^ω and 3^ω

We begin this section by stating the following open problem:

Question 5.1.1. *Suppose that X is a Polish space, and $\{g_i\}_{i\in\omega}$ is a countable collection of Borel injections $X \to X$. Must there exist a Borel $c : X \to 2^\omega$ such that for all $x \in 2^\omega$, if $g_i(x) \neq x$ for all i, then there exists an i such that $c(x)(i) \neq c(g_i(x))(i)$?*

This question will arise below as a diagonalization problem. If $c(x)(i) \neq c(g_i(x))(i)$, we will have diagonalized against x during stage i of some construction. If $g_i(x) = x$ for some i, then this corresponds to a situation where we don't need to diagonalize against x. We will further investigate Question 5.1.1 in Chapter 6, and demonstrate that it can be restated as a question about graph colorings.

Theorem 5.1.2. *If Question 5.1.1 has an affirmative answer, then recursive isomorphism on 2^ω is a universal countable Borel equivalence relation.*

Proof. Suppose that E is a countable Borel equivalence relation on a Polish space X. Assuming that Question 5.1.1 has an affirmative answer, we will construct a Borel embedding of E into recursive isomorphism on 2^ω.

By the Feldman-Moore theorem, we may assume that E is generated by a Borel action of $\mathbb{F}_\omega$, the free group on ω generators. Fix a recursive listing $\{w_i\}_{i\in\omega}$ of the words of $\mathbb{F}_\omega$

Given any function $f : X \to 2^\omega$, define

$$\hat{f}(x) = \bigoplus_{i\in\omega} f(w_i \cdot x).$$

It is clear that for any f, the associated $\hat{f}$ is a homomorphism from E to recursive isomorphism; we can obtain any $\hat{f}(w_i \cdot x)$ from $\hat{f}(x)$ by recursively permuting columns. We will construct a Borel f such that the associated $\hat{f}$ is also a cohomomorphism.

Our f will be constructed in three steps. At each step, there will be a set $S \subseteq \omega$ such that for all $x \in 2^\omega$, we will have determined $f(x)(n)$ if and only if $n \in S$. We will say that we have defined f on S.

Define the *height* of the number $\langle i, j\rangle$ to be j. Hence, the $\langle i, j\rangle$th bit of $\hat{f}(x)$ is the jth bit of $f(w_i \cdot x)$ so that all bits with height j in $\hat{f}(x)$ correspond to the jth bits of the $f(w_i \cdot x)$.

If ρ is a recursive bijection of ω, say that ρ changes height j if there is some $\langle i, j\rangle$ so that $\rho(\langle i, j\rangle) = \langle k, l\rangle$ where $j \neq l$. In this case, it is trivial to ensure that for all x, $\rho \cdot \hat{f}(x)$ is not in the range of $\hat{f}$. To do this, we can simply require that for all x, $f(x)(j) = 0$ and $f(x)(l) = 1$.

Using this idea, step 1 of our construction is to define f on some infinite-coinfinite $S_1 \subsetneq \omega$ such that for all recursive bijections ρ of ω, if ρ changes infinitely many heights, then $\rho \cdot \hat{f}(x)$ cannot be in the range of $\hat{f}$.

For step 2, let $h : X \to 2^\omega$ be a Borel reduction of $\Delta(X)$ to E_0, so that if $x \neq y$, then $h(x) \not\!\!E_0 h(y)$. Fix some infinite-coinfinite subset S_2 of $\omega \setminus S_1$. Now if m is the nth element of S_2, let $f(x)(m) = h(x)(n)$.

Finally, we define f on $S_3 = \omega \setminus (S_1 \cup S_2)$. To begin, partition S_3 into ω many infinite pieces $\{S_{3,i}\}_{i\in\omega}$ such that if ρ_i is the ith recursive bijection on ω that changes only finitely many heights, then ρ_i does not change the height of any $n \in S_{3,i}$.

Suppose that ρ_i is a recursive isomorphism such that ρ_i does not change heights $\geq n$. Let $R = \{\langle j, k\rangle : k \in S_2 \ \&\ k \geq n\}$. Given any x, we have already defined $\hat{f}(x)$ on R in step 2, and because ρ_i does not change heights above n, $\rho_i \cdot \hat{f}(x)$ will also be defined on R. Because we can recover $y \in X$ if we know all but finitely many bits of $h(y)$, there can be at most one y such that $\rho_i \cdot \hat{f}(x) \restriction R = \hat{f}(y) \restriction R$. Let $g_{\rho_i} : X \to X$ be the Borel function mapping x to this y if it exists. Note that g_{ρ_i} is injective; if $x_1, x_2 \in X$ and $x_1 \neq x_2$, then $h(x_1)$ and $h(x_2)$ must differ at infinitely many bits, so $\hat{f}(x_1) \restriction R \neq \hat{f}(x_2) \restriction R$.

Now given x and y as in the previous paragraph where $y = g_{\rho_i}(x)$, the $\langle 0, k\rangle$th bit of $\rho_i \cdot \hat{f}(x)$ corresponds to the kth bit of $f(w_j \cdot x)$ for the j such that $\rho(\langle j, k\rangle) = \langle 0, k\rangle$. Hence, if the kth bit of $\hat{f}(y)$ differs from the kth bit of $f(w_j \cdot x)$, then $\rho_i \cdot \hat{f}(x) \neq \hat{f}(y)$. In this case, we have successfully diagonalized, and $\rho_i \cdot \hat{f}(x)$ cannot be in the range of $\hat{f}$.

Thus, for each i, define the function $g_{\rho_i,k}$ for $k \in S_{3,i}$ by $g_{\rho_i,k}(y) = w_j \cdot g_{\rho_i}^{-1}(y)$, where j is such that $\rho(\langle j, k\rangle) = \langle 0, k\rangle$. For each i, we thus have a countable family $\{g_{\rho_i,k}\}_{k\in S_{3,i}}$ of injective partial Borel functions on X. It is clear that we can extend these $g_{\rho_i,k}$ to total injective Borel functions on $X \sqcup 2^\omega$.

Assuming that Question 5.1.1 has an affirmative answer, let c_i be a function such that for all y, there exists some k such that $g_{\rho_i,k}(y) = y$, or $c_k(y)(k) \neq c_k(g_{\rho_i,k}(y))(k)$. Then for $k \in S_{3,i}$, let $f(y)(k) = c_k(y)(k)$.

Given any $x, y \in X$ where $y = g_{\rho_i}(x)$, if there exists a k such that $g_{\rho_i,k}(y) = y$, then $y = w_j \cdot x$ for some j. If instead there exists some k such that $c_k(y)(k) \neq c_k(g_{\rho_i,k})(y))(k)$, then as we've argued above, $\rho_i \cdot \hat{f}(x)$ is not the range of $\hat{f}$. □

Corollary 5.1.3. *Recursive isomorphism on 3^ω is a universal countable Borel equivalence relation.*

Proof. Using an identical argument, we see that the analogous theorem is true if we replace 2^ω with 3^ω in both Question 5.1.1 and Theorem 5.1.2. However, the analogue of Question 5.1.1 for 3^ω is obviously true. Suppose $g : X \to X$ is a single injective Borel function. Then we can find a Borel $c : X \to 3$ such that if $x \neq f(x)$, then $c(x) \neq c(f(x))$. This is because g induces a graph where every vertex has degree ≤ 2, which can therefore be 3-colored; see [21]. □

Recall that given $x, y \in 2^\omega$, we say that x is *many-one reducible to* y, noted $x \leq_m y$, if there is a recursive function $r : \omega \to \omega$ such that $x = r^{-1}(y)$. The associated symmetrization of this reducibility is many-one equivalence, and is noted $\equiv_m$. Many-one equivalence is Borel reducible to recursive isomorphism via the function $x \mapsto \oplus_{i\in\omega} x$; the function mapping x to the recursive join of ω many copies of x. Indeed, for all reals x and y, we have that $\oplus_{i\in\omega} x$ and $\oplus_{i\in\omega} y$ are recursively isomorphic if and only if they are many-one equivalent. Hence, if many-one equivalence is a universal countable Borel equivalence relation, then so is recursive isomorphism.

Our proof in Theorem 5.1.2 also works for many-one equivalence.

Corollary 5.1.4. *If Question 5.1.1 has an affirmative answer, then many-one equivalence on 2^ω is a universal countable Borel equivalence relation.*

Proof. Let E be a universal countable Borel equivalence relation generated by an action of $\mathbb{F}_\omega$ such that for some $w \in \mathbb{F}_\omega$ not equal to the identity, $w \cdot x = x$ for all x. Let $\hat{f}$ be the Borel reduction witnessing that $E \leq_B \equiv_1$ from Theorem 5.1.2. Then each column of $\hat{f}$ is repeated ω many times, and $\hat{f}$ is also an embedding of E into many-one equivalence. □

5.2 A partial converse

In this section, we prove a partial converse to Corollary 5.1.4.

Theorem 5.2.1. *If many-one equivalence on 2^ω is a uniformly universal countable Borel equivalence relation, then Question 5.1.1 has a positive answer.*

Our proof of Theorem 5.2.1 will have the following rough outline. Given $\{g_i\}_{i\in\omega}$ as in Question 5.1.1, we will first construct an equivalence relation F such that $x \not F g_i(x)$ for all x and i, and $g_i(x) F g_j(x)$ for all x, i, and j. We will then use a reduction of F to many-one equivalence to construct the required c.

If E and F are equivalence relations on X, then we say that E and F are *independent* if there is no sequence $x_0, x_1, \ldots, x_n = x_0$ with $n \geq 2$ such that $x_0 E x_1 F x_2 E x_3 F x_4 \ldots$, and $x_i \neq x_{i+1}$. A key part of our proof will be the following theorem of Conley and Miller. Recall that if f is an automorphism of X, then it induces the equivalence relation E_f where $x E_f y$ if there exists an $n \in \mathbb{Z}$ such that $f^n(x) = y$.

Theorem 5.2.2 (Conley and Miller)**.** *Let E be a countable Borel equivalence relation on a Polish space X. Then there exists a Borel automorphism f of X such that the induced equivalence relation E_f is independent of E.*

A version of this result in the measure context appears in [37]. Their more general argument for proving Theorem 5.2.2 is an adaption of this proof to the Conley-Miller topology [36].

Definition 5.2.3 (The Conley-Miller topology)**.** Let X be a Polish space. The Conley-Miller topology is the topology on the space of Borel automorphisms of X that is generated by the basic open sets $O(\{P_i\}_{i\in\omega}, f)$ where $\{P_i\}_{i\in\omega}$ is a Borel partition of X into countably many pieces, f is a Borel automorphism of X, and g is in the open set $O(\{P_i\}_{i\in\omega}, f)$ if $f(P_i) \triangle g(P_i)$ is countable for every i.

The Conley-Miller topology has some similarities to the weak topology on $\mathrm{Aut}(X, \mu)$, the space of measure preserving automorphisms of a standard measure space (X, μ). It promises to be a very useful tool for generalizing arguments from the measuring setting to the pure Borel setting. While this topology is not Polish, it is Baire, and in fact, strong Choquet. Indeed, a generic automorphism suffices to prove Theorem 5.2.2.

Theorem 5.2.4 (Conley and Miller)**.** *The set of f satisfying the conclusion of Theorem 5.2.2 is comeager in the Conley-Miller topology.*

We will begin our proof of Theorem 5.2.1 with the following Lemma:

Lemma 5.2.5. *If $E^X_{\{\varphi_i\}}$ is a uniformly universal countable Borel equivalence relation, then for every countable Borel equivalence relation $E^Y_{\{\psi_i\}}$, there is a uniform reduction from $E^Y_{\{\psi_i\}}$ to $E^X_{\{\varphi_i\}}$ such that the uniformity function $u : \omega^2 \to \omega^2$ is recursive.*

The proof of this Lemma really comes down to two facts. First, the equivalence relation $E(\mathbb{F}_2, 2^\omega)$ is a uniformly universal countable Borel equivalence relation, and this can always be witnessed with a recursive uniformity function. Secondly, any uniform reduction of $E(\mathbb{F}_2, 2^\omega)$ to a uniformly universal $E^X_{\{\varphi_i\}}$ may also be witnessed by a recursive uniformity function, since $\mathbb{F}_2$ is a finitely generated recursive group.

Proof of Lemma 5.2.5. Let Y be a Polish space, and $E^Y_{\{\psi_i\}}$ be some equivalence relation on Y, generated by the family $\{\psi_i\}_{i\in\omega}$. Let $Y^* = Y \sqcup \{p\}$ be the Polish space of Y with a point adjoined. Let $\mathbb{F}_{\omega^2}$ be the free group on the ω^2 generators, $\{\gamma_{(i,j)}\}_{(i,j)\in\omega^2}$.

First, note that $E^Y_{\{\psi_i\}}$ embeds uniformly into the shift action of $\mathbb{F}_2$ on $(Y^*)^{\mathbb{F}_2}$ with a recursive uniformity function. To see this, recall that there is a recursive embedding $h : \mathbb{F}_{\omega^2} \to \mathbb{F}_2$ of $\mathbb{F}_{\omega^2}$ into $\mathbb{F}_2$. Let $\theta_{\gamma_{i,j}} : Y^* \to Y^*$ be

$$\theta_{\gamma_{(i,j)}}(y) = \begin{cases} \psi_i(y) & \text{if } y \in Y \text{ and } \psi_j(\psi_i(y)) = y \\ p & \text{otherwise} \end{cases}$$

Define $\theta_{\gamma^{-1}_{(i,j)}}$ to be $\theta_{\gamma_{(j,i)}}$. Finally, we can define θ_w for any reduced word $w \in F_{\omega^2}$ by composing the $\theta_{\gamma_{(i,j)}}$ and $\theta_{\gamma^{-1}_{(i,j)}}$ in the obvious way.

Now we define our reduction $f : Y \to (Y^*)^{\mathbb{F}_2}$ from $E^Y_{\{\psi_i\}}$ to $E(\mathbb{F}_2, Y^*)$:

$$f(y)(g) = \begin{cases} \theta_w(y) & \text{if } h(w) = g \\ p & \text{if } g \neq \operatorname{ran}(r) \end{cases}$$

This is clearly a uniform reduction. Further, the uniformity function is recursive. If $xE^Y_{\{\psi_i\}}y$ via (i,j), then $f(x)E(\mathbb{F}_2, Y^*)f(y)$ via the group element $h(\gamma_{(i,j)})$.

Finally, since $E^X_{\{\varphi_i\}}$ is uniformly universal, $E(\mathbb{F}_2, Y^*)$ reduces uniformly to $E^X_{\{\varphi_i\}}$. This reduction can also be witnessed by a recursive uniformity function; we only need to know how the uniformity of the reduction is witnessed for the two generators of $\mathbb{F}_2$. Thus, composing these two reductions gives a uniform reduction of $E^Y_{\{\psi_i\}}$ into $E^X_{\{\varphi_i\}}$, and the uniformity function is recursive: it is the composition of two recursive functions. □

Now we turn to the proof of the main theorem. Suppose that $F_0, F_1, \ldots$ are equivalence relations on X. Say that $F_0, F_1, \ldots$ are *mutually independent* if there does not exist a pair of sequences $x_0, x_1, \ldots, x_n = x_0$ and $k_0, k_1, \ldots k_{n-1}$ where $n \geq 2$ such that $k_i \neq k_{i+1}$, $x_i \neq x_{i+1}$, and $x_0 F_{k_0} x_1 F_{k_1} x_2 F_{k_2} x_3 \ldots$.

Proof of Theorem 5.2.1. Let g_i be a set of partial Borel functions on the space X. By Lemma 6.2.2, which we will prove in Chapter 6, we may assume that the g_i are total independent automorphisms, that $g_i(x) \neq x$ for all x, and that the induced equivalence relations E_{g_i} are all mutually independent. Assume that many-one equivalence is a uniformly universal countable Borel equivalence relation, as generated by the many-one reductions. We must construct a Borel c as demanded by Question 5.1.1.

Let E be the equivalence relation arising from the generating family obtained by closing $\{g_i\}_{i\in\omega}$ under composition. By Theorem 5.2.2, let f be an independent Borel automorphism from E. Now let F be the equivalence relation on X generated by all functions of the form $g_i \circ f^{-1}$. Note that E and F are independent because the E_{g_i} are all mutually independent. In particular, the only non-identity words in the group $\langle g_0 f^{-1}, g_1 f^{-1}, \ldots\rangle$ that do not contain

f or f^{-1} are of the form $g_{i_0} g_{i_1}^{-1} g_{i_2} g_{i_3}^{-1} \dots g_{i_{2n}} g_{i_{2n+1}}^{-1}$. Thus, x and $f(x)$ are not F-related for any x.

Now let h be a uniform reduction from F to $\equiv_m$ with a recursive uniformity function. We may assume that for all x, $h(x) = \oplus_{i \in \omega} y$ for some y, by replacing h with the function $x \mapsto \oplus_{i \in \omega} h(x)$ if necessary. Hence, for all x and y, $h(x)$ and $h(y)$ are many-one equivalent if and only if they are recursively isomorphic.

Because the uniformity of h is witnessed recursively, there is a uniformly recursive sequence of recursive bijections $\{r_i\}_{i \in \omega}$ witnessing that $h(f(x))$ and $h(g_i(x))$ are recursively isomorphic.

Now we use a back and forth argument to construct a pair of recursive bijections $\rho_1, \rho_2 : \omega \to \omega$ such that for all x, $h(f(x))(n) = h(g_{\rho_1(n)}(x))(\rho_2(n))$. At each stage of the construction, we will have defined ρ_1 and ρ_2 on the same finite set.

At even steps, we begin by picking the least n not in the domain of ρ_1 and ρ_2. Let m be the least number not in the range of ρ_1, and define $\rho_1(n) = m$. Now r_m witnesses that $h(f(x))$ and $h(g_m(x))$ are recursively isomorphic, so that $h(f(x))(n) = h(g_m(x))(r_m(n))$ for all x. Then if $r_m(n) = \langle i, j \rangle$, we can set $\rho_2(n) = \langle i^*, j \rangle$ were i^* is least such that $\langle i^*, j \rangle \notin \mathrm{ran}(\rho_2)$. Recall that every element in $\mathrm{ran}(h)$ is of the form $\oplus_{i \in \omega} y$ for some y, hence $h(g_m(x))(\langle i, j \rangle) = h(g_m(x))(\langle i^*, j \rangle)$ for all i^*.

At odd steps, we pick the least m and l that are not in the range of ρ_1 and ρ_2 respectively. Now again, r_m witnesses that $h(f(x))(n) = h(g_m(x))(r_m(n))$ for all x and n, so let $\langle i, j \rangle = r_m^{-1}(l)$. Now let i^* be least such that $\langle i^*, j \rangle$ is not in the domain of ρ_1 and ρ_2. Then define $\rho_1(\langle i^*, j \rangle) = m$, and $\rho_2(\langle i^*, j \rangle) = l$.

Let $c : X \to 2^\omega$ be $\rho_1 \rho_2^{-1} \cdot h$, so that $c(x)(n) = h(x)(\rho_2(\rho_1^{-1}(n)))$. Clearly c is still a reduction of E to many-one equivalence, and for all x, $c(x)$ and $h(x)$ are recursively isomorphic. We claim that c is the function we desire. For a contradiction, suppose there exists an x such that for all i, $c(x)(i) = c(g_i(x))(i)$. Then

$$c(x)(i) = c(g_i(x))(i) = h(g_i(x))(\rho_2(\rho_1^{-1}(i))) = h(f(x))(\rho_1^{-1}(i))$$

Hence, $c(x)$ is recursively isomorphic to $h(f(x))$ and thus $c(f(x))$, contradicting our assumption that h and thus also c are reductions from F to many-one equivalence; x and $f(x)$ are not F-related for any x. □

Question 5.2.6. *Are there combinatorial calibrations of the difficulty of showing that other equivalence relations from recursion theory are universal?*

Chapter 6

Two questions of Borel combinatorics

In this chapter, we further investigate Question 5.1.1 which arose in our investigation of the universality of recursive isomorphism. We provide an equivalent way of phrasing this question in terms of Borel combinatorics. We will then turn to the following question:

Question 6.0.1. *Suppose E and F are countable Borel equivalence relations on a Polish space X whose equivalence classes all have cardinality ≥ 3. Must there exist disjoint Borel sets A and B such that A is a complete section for E, and B is a complete section for F?*

An affirmative answer to this question would imply an affirmative answer to Question 5.1.1. Question 6.0.1 is remarkably robust, and we demonstrate that it can be restated in many equivalent ways. Roughly, it asserts the existence of a Borel way of mediating between two equivalent and conflicting requirements, and this is reflected in its many forms.

As part of our investigation, we show that Question 6.0.1 is true in both the context of measure and Baire category. A corollary of this result is that recursive isomorphism is measure universal.

6.1 Introduction to Borel combinatorics

Borel combinatorics is the study of classical combinatorial problems such as graph coloring, but where we only consider Borel objects, and Borel witnesses to their combinatorial properties. A *Borel graph* G on a Polish space X is an irreflexive symmetric Borel relation on X. We will occasionally think of such graphs as subsets of $[X]^2$. A graph is said have *degree at least* n if every vertex of the graph has degree $\geq n$. A graph is said to be *n-regular* if every vertex of the graph has degree n.

Graph coloring is a typical problem studied in the field of Borel combinatorics, where a *n-coloring* of G is a function $c : X \to n$ so that if $x, y \in X$ are neighbors, then $c(x) \neq c(y)$. For example, the following theorem is a Borel analogue of an easy classical fact:

Theorem 6.1.1 (Kechris, Solecki, and Todorcevic [21]). *Every Borel n-regular graph has a Borel $n + 1$-coloring.*

However, not every classical result has an analogue in the Borel setting. For instance, Kechris, Solecki, and Todorcevic have also shown that there are acyclic Borel graphs which have no Borel ω-coloring, though classically, such a graph must be 2-colorable [21].

Suppose X is a Polish space, and E is an equivalence relation. Then a *complete section* of E is a set $B \subseteq X$ that meets every equivalence class of E. Given a countable Borel equivalence relation E, a *vanishing sequence of Borel markers* is a decreasing sequence of Borel complete sections of E with empty intersection. We will often use the following Marker lemma of Slaman and Steel.

Lemma 6.1.2 (Marker Lemma, see [47] and Lemma 6.7 of [26])**.** *Suppose E is an aperiodic countable Borel equivalence relation. Then E admits a vanishing sequence of markers*

Let X be a Polish space. Then we let $[X]^{<\omega}$ note the Polish space of finite subsets of X. If E is a countable Borel equivalence relation, we let $[E]^{<\omega}$ be the Borel subset of $[X]^{<\omega}$ consisting of the $S \in [X]^{<\omega}$ such that S is a subset of some equivalence class of E.

The following Lemma which is very useful for organizing constructions in Borel combinatorics:

Lemma 6.1.3 (Lemma 7.3 of [26])**.** *Suppose E is a countable Borel equivalence relation and let G be the intersection graph on $[E]^{<\omega}$ where $(S,T) \in G$ iff $S \neq T$ and $S \cap T \neq \emptyset$. Then G has a Borel ω-coloring.*

We will often use this Lemma in the following way. Suppose that there is a Borel property of finite sets such that every equivalence class of E has at least one finite subset with this property. Then there is a Borel set $C \subseteq [E]^{<\omega}$ such that every $S \in C$ has this property, and all distinct R and S in C are disjoint.

Another useful consequence of Lemma 6.1.3 is the following:

Lemma 6.1.4 (Proposition 7.4 of [26])**.** *If n is a positive integer, and E is an aperiodic countable Borel equivalence relation, then E contains a finite Borel subequivalence relation $F \subseteq E$, all of whose classes are of cardinality n.*

Finally, we remark that in this chapter we will often need to break "ties" when there is some irrelevant choice in our constructions. In cases where we need to choose one of finitely many points, we will generally break ties by choosing the leftmost point. Here, we are presuming we have a Borel linear order on our Polish space X. One way of obtaining such an order is via a Borel homeomorphism with 2^ω. In cases where we need to choose one of countably many options, we can use uniformization.

6.2 An abstract diagonalization problem

In this section, we study the obstruction to the uniform universality of many-one equivalence that we exhibited in Chapter 5. First, recall the original form of the question:

Question 5.1.1. *Suppose that X is a Polish space, and $\{g_i\}_{i\in\omega}$ is a countable collection of Borel injections $X \to X$. Must there exist a Borel $c : X \to 2^\omega$ such that for all $x \in 2^\omega$, if $g_i(x) \neq x$ for all i, then there exists an i such that $c(x)(i) \neq c(g_i(x))(i)$?*

We begin our investigation of this question with a simple lemma. Suppose we are given a countable sequence $\{g_i\}_{i\in\omega}$ of Borel injections on X as in Question 5.1.1. Now suppose c is a partial function on X whose codomain is the set of partial functions from ω to 2. We will assume that if $x \in \mathrm{dom}(c)$, then $c(x)$ is nonempty. Say that a point $x \in X$ is *satisfied by c* if there exists an i such that $g_i(x) = x$, or there exists an i such that $c(x)(i) \neq c(g_i(x))(i)$. Say that such a c is *satisfied on its domain* if $x \in \mathrm{dom}(c)$ implies x is satisfied by c.

Lemma 6.2.1. *Suppose X is a Polish space, and $\{g_i\}_{i\in\omega}$ is a countable sequence of Borel injections $X \to X$. Suppose also that there is a partial Borel function c as above that is satisfied on its domain. Then we can extend c to a Borel c^* that is also satisfied on its domain, and such that* $\mathrm{dom}(c^*)$ *is saturated under the equivalence relation $E_{\{g_i\}}$ generated by the g_i.*

Proof. Given any single g_i, we can extend c in a Borel way to make its domain saturated under the equivalence E_{g_i} generated by the single g_i, and so that it remains satisfied on its domain. To see this, we may first assume that $c(x)(i)$ is defined for all $x \in \mathrm{dom}(c)$, by setting $c(x)(i) = 0$ if this is not the case. Now let b be the least extension of c such that for all $x \notin \mathrm{dom}(c)$ such that $g_i^k(y) = x$ for some k, we have $b(x) = (c(g_i^n(x)) + n) \bmod 2$, where n is either the least positive integer such that $g_i^n(x) \in \mathrm{dom}(c)$ if this exists, or else the greatest negative integer such that $g_i^n(x) \in \mathrm{dom}(c)$. This b is the extension we desired.

Our Lemma follows by iterating the above process. Let $\{i_j\}_{j\in\omega}$ be a sequence of elements of ω where each number appears infinitely often. Now let $c_0 = c$, and define c_{k+1} to be the extension of c_k as above that has domain $[\mathrm{dom}(c_k)]_{E_{g_{i_k}}}$ and is satisfied on this domain. Finally, let $c^* = \bigcup_{i\in\omega} c_i$. □

We now turn to proving the Lemma we used in Theorem 5.2.1.

Lemma 6.2.2. *Question 5.1.1 is equivalent to the special case where $g_i(x) \neq x$ for all x and i, every g_i is an automorphism of X, there are no finite E_{g_i} equivalence classes of even cardinality, and all the $E_{\{g_i\}}$ are mutually independent.*

Proof. This follows from Lemma 6.2.1. We can define a partial Borel c that is satisfied on its domain such that if x is a point witnessing the failure of the above properties, then $y \in \mathrm{dom}(c)$ for some y in the same $E_{\{g_i\}}$-equivalence class as x. Then we may use Lemma 6.2.1 to extend c so that it is only undefined on an $E_{\{g_i\}}$-invariant set that satisfies the hypothesis of the Lemma.

For the first three conditions, it is straightforward to construct such a c. Let c be the smallest partial function on X with the following properties. First, if $g_i(x) = x$, then $c(x)(i) = 0$. Second, if $x \notin \mathrm{ran}(g_i)$, then let $c(g_i^n(x))(i) = n \bmod 2$ for all $n \in \omega$. Finally, if x is the leftmost real in some E_{g_i} equivalence class of even cardinality, then let $c(g_i^n(x))(i) =$

n mod 2 for all n. Hence, we may assume that this first three conditions are satisfied, and we move to the last condition of the statement of the Lemma.

Suppose that x_i, j_i, and k_i for $i \leq n$ are sequences where $x_i \in X$, $j_i \in \mathbb{Z}$ and $k_i \in \omega$. Say that these sequences witness that failure of the mutual independence of the $E_{\{g_i\}}$ if $n > 1$, $x_i \neq x_{i+1}$ for all $i \leq n$, and $g_{j_i}^{k_i}(x_i) = x_{i+1}$ where $|k_i|$ is as small as possible so that this condition is satisfied. Now use Lemma 6.1.3 to fix a Borel collection of disjoint such finite sequences witnessing the failure of the mutual independence of the $E_{\{g_i\}}$ in every $E_{\{g_i\}}$-equivalence class containing such a sequence. Then for each of these sequences, define $c(g_{j_i}^l(x_i))(j_i) = l$ mod 2 for all $i \leq n$ and l between 0 and k_i inclusive. □

The following Lemma will be useful for the next two proofs, and follows trivially from the constructions used in [28]. A complete section of a graph G on X is a subset of X that meets all the connected components of G.

Lemma 6.2.3. *If G is a 2-regular Borel graph on a Polish space X, and B is a Borel complete section of G, then there is a Borel 3-coloring c of G such that if $c(x) = 2$, then $x \in B$.*

Indeed, if we wish to 3-color a 2-regular graph, then if we have any Borel discrete complete section A of G, then we can construct a Borel 3 coloring c by using one of the colors on the points in A, and then using the remaining two colors on $X \setminus A$. The above Lemma follows by taking a Borel discrete $A \subseteq B$.

Suppose we have a collection of graphs $\{G_i\}_{i\in\omega}$ on the same space X, and an associated collection of n-colorings $\{c_i\}_{i\in\omega}$ of the G_i. Then say a point $x \in X$ is *monochromatic* if there is some $k \in n$ such that $c_i(x) = k$ for all $i \in \omega$.

Theorem 6.2.4. *Question 5.1.1 is equivalent to the following. Suppose that $\{g_i\}_{i\in\omega}$ is a countable collection of Borel automorphisms of a Polish space X that induce the 2-regular Borel graphs $\{G_i\}_{i\in\omega}$. Must there exist a sequence $\{c_i\}_{i\in\omega}$ of Borel functions $c_i : X \to 3$ such that each c_i is a 3-coloring of G_i, and there are no monochromatic points?*

Proof. Suppose we have a countable collection $\{g_i\}_{i\in\omega}$ of Borel automorphisms of X as in Lemma 6.2.2. Associate to each g_i the Borel graph G_i induced by g_i, and let c_i be the collection of Borel 3-colorings of the G_i with no monochromatic points. Now we define the required function c.

$$c(x)(i) = \begin{cases} c_i(x) & \text{if } c_i(x) = 0 \text{ or } 1 \\ 1 - c_i(g_i^{-1}(x)) & \text{if } c_i(x) = 2 \end{cases}$$

Since for all x, there must exist some i where $c_i(x) \neq 2$, we see that for this i, $c(x)(i) \neq c(g_i(x))(i)$. Note that for this to work, we only needed to know that there were no monochromatic points of color 2.

Now suppose we such a countable collection of 2-regular Borel graphs on X induced by the Borel automorphisms $\{g_i\}_{i\in\omega}$. Let c be the function we obtain from Question 5.1.1.

Then it is clear that we can produce a Borel 3-coloring c_i of each G_i with the property that $c_i(x) = 2$ only if $c(x)(i) = c(g(x))(i)$. We can do this by first letting $c_i(x) = c(x)(i)$ for those x in the connected components of G_i where $x \mapsto c(x)(i)$ is already a coloring. Then we can apply Lemma 6.2.3 to define c_i on the remaining part of the graph, letting $B = \{x : c(x)(i) = c(g(x))(i)\}$.

If we 3-color the G_{g_i} with c_i in this way, then there are no points x such that $c_i(x) = 2$ for all i. So we have managed to show we can 3-color the G_i with no monochromatic points of color 2. To obtain the result we want, note that when we begin, we can partition our graphs G_i into three countably infinite sets. By coloring each piece of the partition in the above manner and then permuting the colors, we can obtain a coloring with no monochromatic points. □

A natural attempt to prove that Question 5.1.1 has a positive answer leads us to Question 6.0.1, which we started earlier. We finish this section by showing that a positive answer to Question 6.0.1 implies a positive answer to Question 5.1.1.

Lemma 6.2.5. *A positive answer to Question 6.0.1 implies a positive answer to Question 5.1.1.*

Proof. Let $\{g_i\}_{i\in\omega}$ be as in Lemma 6.2.2. If we take disjoint complete sections for E_{g_0} and E_{g_1} and then apply Lemma 6.2.3, we can obtain 3-colorings c_0 and c_1 of G_0 and G_1 such that there are no monochromatic points of color 2. By the proof of the above theorem, this is enough to conclude Question 5.1.1 has a positive answer. □

6.3 Disjoint complete sections

If G is a graph on X, an *antimatching* of G is a function $f : X \to X$ such that for all $x \in X$, there is an edge from x to $f(x)$, and $f(f(x)) \neq x$. A *partial antimatching* of G is a partial function $f : X \to X$ satisfying these conditions for all $x \in \mathrm{dom}(f)$.

An example of a Borel graph without a Borel antimatching is the graph on 2^ω defined in the following way. Let $\iota : 2^\omega \to 2^\omega$ be the involution which flips all the bits of x, and σ be the odometer on 2^ω defined by

$$\sigma(x) = \begin{cases} 0^n 1y & \text{if } x = 1^n 0y \\ 00\ldots & \text{if } x = 11\ldots \end{cases}$$

Let G be the 2-regular graph where there is an edge between x and y if $x = \sigma(y)$ or $x = \iota(\sigma(y))$. A Borel antimatching of this graph would give a Borel way of picking a single E_0 equivalence class from every pair of E_0 equivalence classes $[x]_{E_0}$ and $[y]_{E_0}$ such that $\iota(x) E_0 y$. Simple measure or category arguments show that this is impossible.

The following Lemma is useful when dealing with Borel antimatchings.

Lemma 6.3.1. *Suppose G is a locally countable graph, and f is a partial Borel antimatching of G such that if $x \in \operatorname{ran}(f)$, then $f(x) \in \operatorname{dom}(f)$, and every connected component contains some $x \in \operatorname{dom}(f)$. Then f can be extended to a total Borel antimatching f^*.*

Proof. Define f^* as follows. Let $f^*(x) = f(x)$ if $x \in \operatorname{dom}(f)$. Otherwise, let $f^*(x) = y$, where y is the neighbor of x such that the distance from y to an element of $\operatorname{dom}(f)$ is as small as possible (using uniformization to break ties). Then clearly $f^*(f^*(x)) \neq x$ since for any $x \notin \operatorname{dom}(f)$, we have that $f^*(x)$ is closer to some element of $\operatorname{dom}(f)$ than x. □

We now give several equivalent forms of Question 6.0.1. Recall that a Borel graph is said to be *Borel bipartite* if there are Borel sets witnessing its bipartiteness.

Theorem 6.3.2. *The following are equivalent.*

1. *Question 6.0.1 has a positive answer.*

2. *If E and F are independent aperiodic hyperfinite Borel equivalence relations on the same space, then there exists a Borel set B such that B is a complete section for E, and $\overline{B}$ is a complete section for F.*

3. *If G is a locally finite Borel graph having degree at least 3, then G has a Borel antimatching.*

4. *The following is true for some $n \geq 3$: every acyclic Borel bipartite n-regular graph has a Borel antimatching.*

5. *If E and F are countable Borel equivalence relations and E is aperiodic, then we can find a Borel set B such that B is a complete section for E, and $\overline{B}$ meets every infinite equivalence class of F.*

6. *If $F_0, F_1, \ldots, F_n$ are aperiodic countable Borel equivalence relations on a Polish space X, then there exists disjoint Borel sets $B_0, B_1, \ldots B_n \subseteq X$ such that B_i is a complete section for F_i.*

7. *If $F_0, F_1, \ldots, F_n$ are independent aperiodic hyperfinite Borel equivalence relations on X, then there exists a function $c : X \to n$ such that for all $i \leq n$ and all x, $c([x]_{F_i}) \neq \{i\}$.*

Proof. (1) $\implies$ (2) is obvious.

(2) $\implies$ (3). Let Y be a Polish space. We will begin by proving the special case where G is a 3-regular acyclic Borel graph on $\{0,1\} \times Y$ where $(0,x)G(1,y)$ iff $x = y$. For $i \in \{0,1\}$, let F_i be the equivalence relation on Y where xEy iff (i,x) and (i,y) are in the same connected component of $G \restriction \{i\} \times Y$. E and F are independent because G is acyclic, and E and F are hyperfinite because they have Borel graphings that are 2-regular. Let B be a Borel set such that B is a complete section for E and $\overline{B}$ is a complete section for F. We can use B to define a Borel antimatching. The rough idea is to direct elements of $\{0\} \times Y$ towards elements of B and direct elements of $\{1\} \times Y$ away from elements of B.

If $x \in B$, define $f((0,x)) = (1,x)$. Then let z be the point of B such that $(1,z)$ is closest to $(1,x)$ in $G \restriction \{1\} \times Y$, and define $f((1,x)) = (1,y)$ where $(1,y)$ is the neighbor of $(1,x)$ along the path from $(1,x)$ to $(1,z)$. Likewise, if $x \notin B$, define $f((1,x)) = (0,x)$, let z be the point of $\overline{B}$ such that $(0,z)$ is closest to $(0,x)$ in $G \restriction \{0\} \times Y$, and define $f((0,x)) = (0,y)$ where $(0,y)$ is the neighbor of $(0,x)$ along the path from $(0,x)$ to $(0,z)$.

Now let G be an arbitrary locally finite Borel graph having degree at least 3. We will engage in a rather naïve attempt to construct a Borel antimatching of G. If this construction fails, the obstruction will be a graph of the above type.

First, note that we may assume that G is acyclic. To see this, suppose that every connected component of G contained at least one cycle. In this case, it is easy to construct a Borel antimatching. First, use Lemma 6.1.3 to obtain a Borel set C of disjoint cycles that contains at least one cycle from each connected component of G. Now define a Borel antimatching f of G as follows. For each cycle $x_0, x_1, \ldots x_n = x_0$ in C, let $f(x_i) = x_{i+1}$ for $i < n$, and $f(x_n) = x_0$. Now use Lemma 6.3.1 to extend f to a total Borel antimatching f^*.

So assume that G is acyclic. Given $x \in X$, let $N(x)$ be the set of neighbors of x. Now let Y be the collection of finite subsets of X of the form $S(x) = \{x\} \cup N(x)$. Use Lemma 6.1.3 to ω-color the intersection graph on Y, and let A_i be the set of $x \in X$ such that $S(x)$ is assigned color i. Thus, each A_i is a discrete set in X, and $X = \bigcup_{i \in \omega} A_i$. We begin by defining a sequence $f_0 \subseteq f_1 \subseteq \ldots$ of partial Borel antimatchings. These f_i will have the following properties for all x: (i) if $x \in \mathrm{ran}(f_i)$ and $x \notin \mathrm{dom}(f_i)$, then there exists exactly two neighbors y_1 and y_2 of x that are not in $\mathrm{dom}(f_i)$. (ii) if $x \in \mathrm{dom}(f_i)$, then $y \in \mathrm{dom}(f_i)$ for all $y \in N(x)$ such that $y \neq f_i(x)$.

Condition (i) ensures we can always extend f to a total antimatching (though not necessarily a Borel one). Condition (ii) reflects the fact that if we have defined $f_i(x) = z$, then we may as well set $f_i(y) = x$ for all neighbors $y \neq z$ of x that are not yet in $\mathrm{dom}(f_i)$.

Let $f_0 = \emptyset$. Now we define $f_{i+1} \supseteq f_i$. Consider the elements of A_i. For every $x \in A_i$, if $x \notin \mathrm{dom}(f_i)$ and $x \notin \mathrm{ran}(f_i)$, then by condition (ii) above, none of the $y \in N(x)$ are in $\mathrm{dom}(f_i)$ either. In this case, we will define $f_{i+1}(y) = x$ for all $y \in N(x)$ except for the two lexicographically least $y \in N(x)$. To finish our definition of f_{i+1}, we extend in Borel way to satisfy condition (ii), as described above.

Let $f = \bigcup_{i \in \omega} f_i$. This f is a partial Borel antimatching of G. Let $A = X \setminus \mathrm{dom}(f)$ so that each $x \in A$ has exactly two neighbors from G that are in A. Thus, $G \restriction A$ is a 2-regular graph.

We would like to repeat this process to obtain a similar set B to A but so that $G \restriction B$ and $G \restriction A$ do not have any connected components that are equal. We can do this in the following way. Take a discrete Borel set of edges of $G \restriction A$ that contains at least one edge from each connected component of $G \restriction A$. Remove these edges from G to obtain the graph G'. Now perform the same process on G' as in the preceding paragraphs to obtain B, analogously to how we obtained A. (Of course, G' may have vertices of degree < 3. However, such vertices must have degree 2, and they must be a discrete set in G', so the above attempt to construct an antimatching of G' works without modification.)

Our situation now is that we have Borel sets A and B such that $G \restriction A$ and $G \restriction B$ are 2-regular graphs. These A and B correspond to places where we have failed to construct antimatchings. Hence, without loss of generality, we may assume that each connected component of G meets both A and B. By Lemma 6.1.3, let C be a Borel set of finite paths in G from elements of A to elements of B that contains at least one path from every connected component of G. We may assume that if $(x_0, \ldots, x_n)$ is a path in C, then x_0 is the only point of this path in A, and x_n is the only point of this path in B. (We allow paths consisting of a single point where A and B intersect). We may also assume that each pair of connected components of $G \restriction A$ and $G \restriction B$ are connected by at most one path in C.

Let $S \subseteq X$ consist of the connected components of $G \restriction A$ that meet only finitely many paths in C. $G \restriction A$ is smooth on S. Hence, we can apply Lemma 6.3.1 to obtain a Borel antimatching on $[S]_G$. An identical comment is true for B. Thus, without loss of generality, we can assume that for each connected component of $G \restriction A$ and $G \restriction B$, if there is a path in C that meets this connected component, then there are infinitely many.

Let Y be the collection of starting points of paths in C, and Z be the collection of ending points of paths in C. Note that Y and Z may have nonempty intersection. Define $W = \{0\} \times Y \cup \{1\} \times Z$. Consider the 3-regular Borel graph H on W, defined by the following three conditions. First, $(0,x)H(1,y)$ iff there is a path in C from x to y. Second, $(0,x)H(0,x)$ iff there is a path from x to y in $G \restriction A$ that does not contain any other element of Y. Third, $(1,x)H(1,y)$ iff there is a path from x to y in $G \restriction B$ that does not contain any other element of Z.

H is a graph of the type we discussed at the beginning of this proof, and hence we can find a Borel antimatching of H. Let A^* be the saturation of Y in $G \restriction A$, and B^* be the saturation of Z in $G \restriction B$. It is clear that we can turn the Borel antimatching of H into a partial Borel antimatching f of G whose domain is $A^* \cup B^* \cup \{x : \exists p \in C(x \in p)\}$, and such that if $x \in \operatorname{ran}(f)$, then $x \in \operatorname{dom}(f)$. We finish by applying Lemma 6.3.1.

(3) $\implies$ (1) Suppose first that all the equivalence classes of E and F are finite. Then let $Y \sqcup Z$ be the disjoint union of the equivalence classes of E and the equivalence classes of F. Clearly Y and Z are Polish spaces, being Borel subsets of $[X]^{<\omega}$. Now let G be intersection graph on $Y \sqcup Z$, where $(R,S) \in G$ iff $R \in Y$, $S \in Z$, and $R \cap S \neq \emptyset$. Extend G to a Borel graph G^* on $Y \sqcup Z \sqcup 2^\omega$ so that G^* has degree at least 3, $G^* \restriction Y \sqcup Z = G$, and such that if $x \in Y \sqcup Z$ is incident to an edge of G^* that is not in G, then the degree of x in G is ≤ 2. Note that a vertex in G of degree ≤ 2 corresponds to a pair of equivalence classes R and S of E and F such that $R \cap S$ has cardinality ≥ 2.

Now let f be a Borel antimatching of G^*. Let A_0 be the Borel set of $x \in X$ such that there exists an $R \in Y$ such that $f(R) \in Z$ and $R \cap f(R) = \{x\}$. Let A_1 be the Borel set of $x \in X$ such that there exists an $R \in Y$ and $S \in Z$ such that $R \cap S$ has cardinality ≥ 2, and x is the leftmost element of $R \cap S$. Let $A = A_0 \cup A_1$. Clearly A meets every equivalence class of E, and $\overline{A}$ meets every equivalence class of F.

For the general case, suppose that E and F are countable Borel equivalence relations, whose equivalence classes all have cardinality ≥ 3. By Lemma 6.1.4, we can find Borel equivalence relations $E^* \subseteq E$ and $F^* \subseteq F$ such that all the equivalence class of E and F are finite and have cardinality ≥ 3.

(3) $\implies$ (4) is obvious.

(4) $\implies$ (2). Suppose we have two aperiodic countable Borel equivalence relations E and F on a Polish space X. By Lemma 6.1.4 we can find E^* and F^*, finite Borel subequivalence relations of E and F whose equivalence classes all have cardinality n. The intersection graph of their equivalence classes is Borel bipartite and n-regular. If it has a Borel antimatching, as in our proof that (3) $\implies$ (1), we can use this antimatching to produce Borel disjoint complete sections for E and F.

(1) $\implies$ (5). Let E^* and F^* be extensions of E and F to aperiodic countable Borel equivalence relations on $X \sqcup 2^\omega$ such that $E = E^* \restriction X$ and $[x]_F = [x]_{F^*}$ whenever $[x]_F$ is infinite. Then let A be a Borel set such that A is a complete section for E^* and $\overline{A}$ is a complete section for F^*. Let $B = A \cap X$.

(5) $\implies$ (6). This is by induction. Let $F_0, F_1, \ldots F_{n+1}$ be independent aperiodic countable Borel equivalence relations on X, and suppose $B_0, B_1, \ldots B_n$ are Borel disjoint complete sections for $F_0, F_1, \ldots F_n$. We may assume $\bigcup_{i \leq n} B_i = X$. Let X_i be the set of $x \in B_i$ such that $[x]_{F_{n+1}}$ meets infinitely many points of B_i. Now for $i \leq n$, let C_i be a Borel set such that C_i is a complete section for $F_{n+1} \restriction X_i$ and $\overline{C_i}$ meets every equivalence class of $F_i \restriction X_i$ that is infinite. Then for $i \leq n$, set $B_i^* = \overline{C_i} \cup (B_i \setminus X_i)$, and $B_{n+1}^* = \bigcup_{i \leq n} C_i$. Then $B_0^*, B_1^*, \ldots B_{n+1}^*$ are disjoint complete sections for $F_0, F_1, \ldots, F_{n+1}$.

(6) $\implies$ (7). We may as well assume that $\bigcup_{i \leq n} B_i = X$. Then if i is such that $x \in B_i$, map x to $i + 1 \bmod n + 1$.

(7) $\implies$ (3). This is via an argument like the argument that (2) implies (3). We can reduce to the special case of a graph G on $(n-1) \times Y$ where $(i, x) G (j, x)$ for $i \neq j$, and each (i, x) has exactly two neighbors of the form (i, y). Let F_i be the equivalence relation on Y where $x F_i y$ if (i, x) and (i, y) are in the same connected component of $G \restriction \{i\} \times Y$. Let c be a Borel function such that $c([x]_{F_i}) \neq \{i\}$. Then we can construct a Borel antimatching f of G in the following. If $c(x) \neq i$, then map $f((i, x)) = (c(x), x)$. If $c(x) = i$, then map (i, x) towards the closet point (i, y) (as measured in $G \restriction \{i\} \times Y$) such that $c(y) \neq i$. □

These equivalent statements of Question 6.0.1 yield many interesting corollaries. Recall that a *perfect matching* of a graph G is a subset M of its edges such that every vertex is incident to exactly one edge in M.

Lemma 6.3.3. *Suppose G is a locally countable Borel bipartite graph on a Polish space X, with degree at least 2. Then if G has a Borel perfect matching, then G has a Borel antimatching.*

Proof. Let $M \subseteq G$ be the perfect matching of G. Let A and B be Borel sets witnessing that G is Borel bipartite, so that $X = A \sqcup B$. We can obtain a Borel antimatching f of G in the following way. For $x \in A$, define $f(x) = y$, where y is the unique element of B such that $(x, y) \in M$. For $x \in B$, use uniformization to define $f(x) = y$ for some y such that $(x, y) \notin M$. □

Corollary 6.3.4. *If Question 6.0.1 has a negative answer, then for every $n \geq 3$, there exists a Borel bipartite n-regular graph with no Borel perfect matching.*

Proof. Use (4) in Theorem 6.3.2, and Lemma 6.3.3. □

It is known that for every even n, there exists a Borel bipartite n-regular graph with no Borel perfect matching. The case for odd n remains stubbornly open [9].

We now note a connection between Question 6.0.1 and graph colorings.

Corollary 6.3.5. *If Question 6.0.1 has a negative answer, then for every $n \geq 2$, there exists an acyclic $2n$-regular Borel graph with no Borel n-coloring.*

Proof. Here we use (7) in Theorem 6.3.2. Since the F_i are hyperfinite, they are each induced by a single Borel automorphism f_i. Let G_i be the associated 2-regular graph where xG_iy iff $f_i(x) = y$ or $f_i(y) = x$. Let $G = \bigcup_{i \leq n} G_i$. Then G is a $2n$-regular acyclic graph. Clearly any Borel n-coloring c of G would have $c([x]_{F_i}) \neq \{i\}$ for all x. □

If Question 6.0.1 did turn out to have a negative answer, the above corollary would improve the best known lower bounds for the Borel chromatic number of such graphs. Conley and Kechris [9] have shown that for all n, there exist Borel acyclic $2n$-regular graphs having Borel chromatic number at least $\frac{n+\sqrt{2n-1}}{\sqrt{2n-1}}$. For large n, Lyons and Nazarov [32] have improved this lower bound to $n/\log(2n)$.

Another Corollary of Lemma 6.3.3 is as follows.

Corollary 6.3.6. *Question 6.0.1 in all its equivalent forms in Lemma 6.3.2 is true modulo an (appropriately invariant) meager set.*

Proof. By [38], every acyclic 3-regular Borel graph G has a Borel perfect matching modulo a G-invariant meager set. Hence by Lemma 6.3.3, every such graph also has a Borel antimatching modulo a G-invariant meager set. Our corollary follows, since the equivalence proofs we have given in Theorem 6.3.2 are all 'local'. □

For example, given any aperiodic countable Borel equivalence relations E and F on X, there exists a comeager Borel $E \vee F$-invariant set $Y \subseteq X$ and a Borel $B \subseteq Y$ such that B is a complete section for $E \restriction Y$, and $\overline{B}$ is a complete section for $F \restriction Y$, where $E \vee F$ is the join of E and F.

Using a rather different argument, we can see that Question 6.0.1 is also true in the measure context.

Lemma 6.3.7. *Let μ be a Borel probability measure on a Polish space X. Then if E and F are aperiodic countable Borel equivalence relations on X, then there exist disjoint Borel sets A and B such that A meets μ-a.e. equivalence class of E and B meets μ-a.e. equivalence class of F.*

Proof. First note that if C is a Borel complete section for E and $C_0 \subseteq C_1 \subseteq C_2 \ldots$ are Borel sets such that $\bigcup_{i \in \omega} C_i = C$, then for every ϵ there exists some i such that $[C_i]_E > 1 - \epsilon$. One way to see this is to associate in a Borel way to each $x \in C$ a countable set $D_x \subseteq [x]_E$ such that $x \neq y$ implies $D_x \cap D_y = \emptyset$, and so that $\bigcup_{x \in C} D_x = X$. Such a map exists by uniformization. Then we can construct a Borel probability measure on C where $\nu(B) = \bigcup_{x \in B} D_x$ for $B \subseteq C$.

By the Marker Lemma, we can find complete sections for E and F of arbitrarily small measure. We will now define sequences $\{A_i\}_{i \in \omega}$ and $\{B_i\}_{i \in \omega}$ of Borel complete sections for E and F respectively that converge wrt μ. Let A_0 be any Borel complete section for E. Now inductively, let B_i be some Borel complete section for F of the form $B_i = \overline{A_i} \cup B_i^*$ for some Borel B_i^* such that $\mu([A_i - B_i^*]_E) > 1 - 2^{-i}$. Likewise, let $A_{i+1} = \overline{B_i} \cup A_{i+1}^*$ for some Borel A_{i+1}^* such that $\mu([B_i - A_{i+1}^*]_F) > 1 - 2^{-i}$. We can find such A_{i+1}^* and B_i^* by the previous paragraph, and we may assume they are decreasing.

Let $A = \bigcup_{i \in \omega} \overline{B_i}$ and $B = \overline{A}$. Then A meets μ-a.e. equivalence class of E and B meets μ-a.e. equivalence class of F. $\square$

Corollary 6.3.8. *Question 6.0.1 in all its equivalent forms in Lemma 6.3.2 is true modulo a null set with respect to any Borel probability measure.*

Corollary 6.3.9. *Recursive isomorphism on 2^ω is measure universal.*

Proof. By Corollary 6.3.8, the combinatorial problem of Theorem 5.1.2 can be solved modulo an invariant null set with respect to any Borel probability measure. $\square$

Hence, if recursive isomorphism on 2^ω is not a universal countable Borel equivalence relation, we cannot use purely measure theoretic tools to prove this fact.

We will now establish a few more equivalent forms of Question 6.0.1. Recall that an *edge coloring* of a graph G is an assignment of colors to the edges of the graph G such that if two edges are incident, they are assigned different colors. It is a theorem of Vizing [56] that every n-regular graph has an edge coloring with $n+1$ colors. We show that Question 6.0.1 has an affirmative answer if and only if every 3-regular Borel bipartite graph has a Borel edge coloring with 4 colors.

Suppose G is a graph on X. A *directing* of G is a set $D \subseteq G$ that contains exactly one of (x, y) and (y, x) for every pair of neighbors $x, y \in X$. A *partial directing* of G is a subset of G that contains at most one of (x, y) and (y, x) for every pair of neighbors $x, y \in X$. Given a partial directing D of a graph G, say that a point $x \in X$ is a source if it is incident to at least one edge of D, and for all $y \in N(x)$, $(y, x) \notin D$. Similarly, say that a point $x \in X$ is a sink if it is incident to at least one edge of D, and for all $y \in N(x)$, $(x, y) \notin D$.

If f is an antimatching of a graph G, then if we extend the set $\{(x, f(x)) : x \in X\}$ to a directing D of G, then this directing will have no sinks.

Lemma 6.3.10. *Suppose that G is a graph, and D is a partial Borel directing of G without sources or sinks. Suppose also that every connected component of G contains at least one vertex that is incident to an edge of D. Then D can be extended to a total Borel directing D^* of G that has no sources or sinks.*

Proof. Suppose that $(x_0, x_1, \ldots, x_n)$ is a path in G such that x_0 and x_n are both incident to edges already in D. Then we can extend D by adding the edges from $(x_0, x_1, \ldots, x_n)$ that do not conflict with edges already in D; add (x_i, x_{i+1}) to D unless (x_{i+1}, x_i) is already in D. The property that D has no sources or sinks is preserved when we add paths in this way. Similarly, given a cycle, we can extend D using this cycle in the analogous way, while preserving the property that D has no sources or sinks.

Using Lemma 6.1.3, partition all the cycles of G into countably may discrete Borel sets $\{C_i\}_{i\in\omega}$. By iteratively extending D via all the cycles of each C_i, we may assume that all the edges that are elements of cycles have been directed in D.

Use Lemma 6.1.3 once again to partition all the paths of G into countably many discrete Borel pieces $\{P_i\}_{i\in\omega}$. Let $n_0, n_1, \ldots$ be a sequence that contains each element of ω infinitely many times. Let $D_0 = D$. Now define D_{i+1} from D_i by extending D_i via all the paths of P_{n_i} that start and end at vertices incident to at least one edge in D_i. Let $D_\infty = \bigcup_{i\in\omega} D_i$.

We finish by extending D_∞ to D^* by directing the remaining edges of G "away" from D_∞. More precisely, let x and y be distinct elements of X and suppose that neither (x, y) nor (y, x) are in D_∞. Then there must be a unique path $(x_0, \ldots, x_n)$ such that x_0 is incident to an edge in D_∞, and ends with (x_{n-1}, x_n) equal to (y, x) or (x, y). Extend D_∞ to D^* by adding all such (x_{n-1}, x_n). $\square$

Theorem 6.3.11. *The following are equivalent:*

1. *Question 6.0.1 has an affirmative answer.*

2. *If G is a locally countable Borel graph of degree at least 6, then there exists a pair of Borel antimatchings of G, f_0 and f_1, such that for all x, $f_0(x) \neq f_1(x)$.*

3. *If E and F are aperiodic countable Borel equivalence relations on a Polish space X, then there exists a Borel B such that B and $\overline{B}$ are complete sections for both E and F.*

4. *Every 3-regular Borel graph G can be directed in a Borel way so that it has no sinks or sources.*

5. *Every Borel bipartite 3-regular graph has a Borel edge coloring with 4 colors.*

Proof. (1) $\implies$ (2). Suppose G is a locally countable Borel graph of degree at least 6. Using uniformization, we can define a pair of Borel functions S and R which map each $x \in X$ to two disjoint sets $S(x)$ and $R(x)$ each consisting of 3 neighbors of x.

Consider the subgraph H of G where $(x, y) \in G$ if $x \in R(y)$ and $y \in R(x)$. Each vertex in H has degree between 0 and 3. We may extend H in a Borel way to a 3-regular graph H^* on the Polish space $X \sqcup 2^\omega$ so that $H^* \restriction X = H$. Let f be a Borel antimatching of H^*. From f we can define f_0, a Borel antimatching of H such that $f_0(x) \in R(x)$ for all x. Let $f_0(x) = f(x)$ if $f(x) \in X$. Otherwise, let $f_0(x) = z$, where z is the leftmost real in $R(x)$ such that $x \notin R(z)$.

Now repeat this process for S to obtain f_1: another Borel antimatching of G such that $f_1(x) \in S(x)$ for all x. Clearly $f_0(x) \neq f_1(x)$ for all x, since $R(x)$ and $S(x)$ are disjoint.

(2) $\implies$ (3). Let E and F be aperiodic countable Borel equivalence relations. Then as in the proof of (4) $\implies$ (2) in Theorem 6.3.2, we can take finite Borel subequivalence relations of E and F whose equivalence classes all have cardinality 6. The resulting intersection graph is 6-regular. If we use the antimatching f_R we defined above to define the disjoint complete sections B and $\overline{B}$ for E and F, as in Theorem 6.3.2, then $\overline{B}$ will also be a complete section for E and B will also be a complete section for F.

(3) $\implies$ (1) follows from Theorem 6.3.2.

(3) $\implies$ (4). Let G be a locally countable Borel graph with degree at least 3. We begin by letting $Y \subseteq [X]^2$ be a discrete (with respect to the intersection graph) Borel set of edges of G that contains at least one edge from each connected component of G. Note that the connected components of $G \setminus Y$ are all infinite since Y is discrete and G is 3-regular. We define two countable Borel equivalence relations E and F on Y as follows: R and S are related by E if their leftmost points are connected in $G \setminus Y$, and related by F if their rightmost points are connected in $G \setminus Y$.

We may assume that all the equivalence class of E and F are infinite. $G \setminus Y$ is smooth on the connected components corresponding to equivalence classes of E and F that are finite, and we can apply Lemma 6.3.10.

Now take a Borel B such that both B and $\overline{B}$ are complete sections for E and F. Let $D_0 = \{(x, y) : \{x, y\} \in B \text{ and } x \text{ is the leftmost of } x \text{ and } y\}$. Each connected component of Y therefore contains infinitely many x such that $(x, y) \in D_0$ for some y, and infinitely many y such that $(x, y) \in D_0$ for some x. We will extend D_0 to a total Borel directing of G without sinks or sources. Let $Z = \{x : \{x, y\} \in Y \text{ for some } y\}$.

Consider the set of paths in $G \setminus Y$ that start with some x_0 such that $(z, x_0) \in D$ for some z, and end with some x_n such that $(x_n, y) \in D$ for some y. We may use Lemma 6.1.3, to partition these paths into ω many Borel sets $\{P_i\}_{i \in \omega}$ such that each P_i is discrete. Now as in the proof of Lemma, for $i \in \omega$, extend each D_i to D_{i+1} by adding the edges from the paths of P_i which do not conflict with edges already in D_i. Let $D_\infty = \bigcup_{i \in \omega} D_i$. Then complete D_∞ to a total directing D by Lemma 6.3.10.

(4) $\implies$ (1). It is clear that such a directing can be used to define a Borel antimatching.

(4) $\implies$ (5). Suppose that G is a Borel bipartite 3-regular graph whose bipartiteness is witnessed by the Borel sets A and B. Suppose D is a Borel directing of G without sinks or sources. We can use D to write G as the disjoint union of two graphs H_0 and H_1 in the following way: the edges of H_0 are those directed by D from A to B, and the edges of H_1 are those directed by D from B to A. The vertices in H_0 and H_1 all have degree 1 or 2. Hence, each connected component of the H_i is finite, a ray (having exactly one vertex of degree 1), or a line (having no vertices of degree 1).

If all the connected components of H_0 and H_1 were finite or rays, then it would be trivial to construct a Borel edge coloring of G with four colors; we could simply edge color H_0 using the colors $\{0, 1\}$, edge color H_1 using the colors $\{2, 3\}$, and then take the union of these colorings. Our problem is that in general, we will need to use 3 colors in an edge coloring of an H_i containing lines.

Let $Y \subseteq [X]^2$ be a discrete set of edges from H_0 consisting of infinitely many edges from each line in H_0. Define the countable Borel equivalence relations F_0 and F_1 on Y where S and R are F_i related if there exist $x \in S$ and $y \in R$ that are in the same connected component of H_i. Clearly every equivalence class of F_0 is infinite, however, there may be equivalence classes of F_1 that are finite.

Now take a Borel set $C \subseteq Y$ that is a complete section for F_0, so that $\overline{C}$ meets every infinite equivalence class of F_1. Let H_0^* be the graph H_0 but with the edges from C removed, and let H_1^* be the graph H_1 but with the edges from C added. Clearly H_0^* has no lines. Further, all the lines that we have added to H_1^* must contain rays from H_1. Hence, H_1^* is smooth on these lines, which we can edge-color in a Borel way with 2 colors.

If we perform the same process again with the roles of H_0^* and H_1^* reversed, then we obtain graph H_0^{**} and H_1^{**} such that $G = H_0^{**} \sqcup H_1^{**}$ and both H_0 and H_1 have Borel edge colorings with 2 colors.

(5) $\implies$ (1). Let G be a Borel bipartite 3-regular graph, whose bipartiteness is witnessed by the Borel sets A and B. Suppose that G has a Borel edge coloring with 4 colors. We can use this coloring to define a Borel antimatching of G. We can partition the four colors into the sets $\{0, 1\}$ and $\{2, 3\}$. Notice that each vertex must be incident to at least one edge of color 0 or 1, and at least one edge of color 2 or 3. Thus, we can define a Borel antimatching by setting $f(x) = y$ if $x \in A$ and y is the leftmost neighbor of x such that (x, y) is colored 0 or 1, or if $x \in B$ and y is the leftmost neighbor of x such that (x, y) is colored 2 or 3. $\square$

Once again, all these proofs are local, and so the equivalent forms above are true in the measure and category context.

Corollary 6.3.12. *If G is a Borel bipartite 3-regular graph on a Polish space X, then G has a Baire measurable 4-coloring, and a μ-measurable 4-coloring with respect to any Borel probability measure μ on X.*

The balance of evidence presently suggests that Question 6.0.1 has a negative answer.

Conjecture 6.3.13. *Question 6.0.1 has a negative answer.*

If this conjecture is true, then it would have many interesting consequences for Borel combinatorics, as we have seen above. Perhaps most significantly, proving this conjecture would require the development of new techniques beyond measure and category. New tools of this kind would likely have a profound effect on the field, as well as in the area of countable Borel equivalence relations. For example, an affirmative answer to Question 4.4.6 also demands such new techniques.

We finish by remarking that if Question 6.0.1 has a negative answer, then it is plausible that Question 5.1.1 does as well. An even wilder possibility in this direction would be if recursive isomorphism on 2^ω is not a universal countable Borel equivalence relation. This would be a striking contrast to the universality of recursive isomorphism on 3^ω. It would also strongly refute Kechris' conjecture that Turing equivalence is a universal countable Borel equivalence relation.

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