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RESPONSE OF EQUIPMENT IN STRUCTURES SUBJECTED TO TRANSIENT EXCITATION

by

ALAN G. HERNRIED
and
JEROME L. SACKMAN

Report to Sponsor:
National Science Foundation

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DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA
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Chapter One

INTRODUCTION

A rapidly growing area of great concern is the response of relatively light equipment components. Coolant piping systems within nuclear power plants, sensitive scientific equipment used in hospitals and research laboratories, and control systems mounted on airplanes, missiles, or other space structures are a few examples of portions of an overall system that are not only expensive but highly essential. It is necessary, therefore, that these relatively light items be designed so as to withstand any adverse conditions imposed upon the structure within which they are housed. The types of structural excitations we will consider are short duration ground shock and impact as well as earthquake ground shaking.

The purpose of this work is to develop analytical results which can be easily used in the determination of equipment response. It is our hope that these results will serve as an alternative to costly standard numerical integration schemes as well as to various ad-hoc methods that are presently being employed.

The study begins with a description of classical perturbation theory - a powerful tool in the analysis of equipment-structure systems. Next the equipment-structure system is modeled as a two-degree-of-freedom discrete system, primarily to familiarize the reader with an application of perturbation theory as well as to repeat some important fundamental results previously obtained in a different manner. A multi-degree-of-freedom secondary system (multi-degree-of-freedom structure connected by rigid links to multi-degree-of-freedom equipment) is then explored. The response is determined in terms of the convolution of Green's function for the equipment with the excitation imposed on the structure. The Green's function is composed solely of properties of the individual sub-systems, the manner in which they are connected, and damped simple harmonic functions. The case where all natural frequencies of the equipment are well spaced from all natural frequencies of the structure (the grossly detuned case) as well as the more complicated problem of one natural frequency of the equipment close to a natural frequency of the structure (the tuned case) are discussed. Approximate results in terms of design response spectra are obtained. The discussion of secondary systems is concluded with the results of various numerical tests.

A further exploration into the mechanics of equipment-structure systems is provided by the analysis of a simple three-degree-of-freedom tertiary system (very light equipment attached to light equipment which in turn is attached to the structure). The work concludes with an investigation of multi-degree-of-freedom tertiary systems.

Chapter Two

MATHEMATICAL PRELIMINARIES

The following sections discuss some of the basic mathematical techniques used in the analysis of equipment-structure systems. The discussion of classical perturbation theory as given in Sections 2.1, 2.2, and 2.3 follows Butkov [1] closely. However, the presentation in the text below is more detailed in certain areas and specialized to the types of systems we will be analyzing. Treatments of the subject similar to Butkov's can be found in Kemble [9], Schiff [16], or Morse and Feschbach [10] as well as numerous other sources.

2.1 Classical Perturbation: First Order Non-Degenerate Theory

We wish to solve the following eigenvalue problem

$$\left(\mathbf{H} + \mathbf{W}\right)\boldsymbol{\psi}_i = \lambda_i \boldsymbol{\psi}_i \qquad (2.1.1)$$

for the eigenvalues λ_i and eigenvectors $\boldsymbol{\psi}_i$. We will assume that $\mathbf{H}$ and $\mathbf{W}$ are real, symmetric matrices and that $\mathbf{W}$ can be considered a perturbation of $\mathbf{H}$ (i.e. given the elements of $\mathbf{H}$ are of order one, then the elements of $\mathbf{W}$ are of order α where $\alpha \ll 1$).

We now expand λ and $\boldsymbol{\psi}$ as power series in α;

$$\lambda = \lambda^{(0)} + \lambda^{(1)} + \lambda^{(2)} + \ldots$$

$$\boldsymbol{\psi} = \boldsymbol{\psi}^{(0)} + \boldsymbol{\psi}^{(1)} + \boldsymbol{\psi}^{(2)} + \ldots$$

where the superscript indicates the order of that term in the series (i.e. $\lambda^{(0)}$ is of the order of α^0, $\lambda^{(1)}$ is of the order of α^1, etc.) For the first order theory we retain terms only up to the order of α^1. Substituting the resulting expressions for λ and $\boldsymbol{\psi}$ into our original equation and discarding second order terms we are left with

$$\mathbf{H}\boldsymbol{\psi}_i^{(0)} + \mathbf{H}\boldsymbol{\psi}_i^{(1)} + \mathbf{W}\boldsymbol{\psi}_i^{(0)} = \lambda_i^{(0)}\boldsymbol{\psi}_i^{(0)} + \lambda_i^{(0)}\boldsymbol{\psi}_i^{(1)} + \lambda_i^{(1)}\boldsymbol{\psi}_i^{(0)} \qquad (2.1.2)$$

We now solve the zeroth order problem

$$\mathbf{H}\boldsymbol{\psi}_i^{(0)} = \lambda_i^{(0)}\boldsymbol{\psi}_i^{(0)} \qquad (2.1.3)$$

for the eigenvalues $\lambda_i^{(0)}$ and the eigenvectors $\boldsymbol{\psi}_i^{(0)}$. It will be assumed that *all* the eigenvalues are distinct - this is what we mean by a *non-degenerate* problem. Since the eigenvectors are only determined within an arbitrary scale factor, we will further require

$$\boldsymbol{\psi}_i^{(0)T}\boldsymbol{\psi}_j^{(0)} = \delta_{ij} \qquad (2.1.4)$$

(where δ_{ij} is the Kronecker delta and ^{T} denotes matrix transposition). This simplifies the subsequent results. Once we have satisfied the zeroth order problem, (2.1.2) reduces to

$$\mathbf{H}\boldsymbol{\psi}_i^{(1)} + \mathbf{W}\boldsymbol{\psi}_i^{(0)} = \lambda_i^{(0)}\boldsymbol{\psi}_i^{(1)} + \lambda_i^{(1)}\boldsymbol{\psi}_i^{(0)} \qquad (2.1.5)$$

We now seek the first order perturbation of the eigenvalues. Pre-multiplying (2.1.5) by $\boldsymbol{\psi}_i^{(0)T}$ we get

$$\psi_i^{(0)T}\mathbf{H}\psi_i^{(1)} + \psi_i^{(0)T}\mathbf{W}\psi_i^{(0)} = \lambda_i^{(0)}\psi_i^{(0)T}\psi_i^{(1)} + \lambda_i^{(1)}\psi_i^{(0)T}\psi_i^{(0)} \tag{2.1.6}$$

Since $\mathbf{H} = \mathbf{H}^T$ and

$$\psi_i^{(0)T}\mathbf{H}\psi_i^{(1)} = scalar = \psi_i^{(1)T}\mathbf{H}\psi_i^{(0)} = \psi_i^{(1)T}\lambda_i^{(0)}\psi_i^{(0)} = \lambda_i^{(0)}\psi_i^{(0)T}\psi_i^{(1)}$$

(2.1.6) becomes

$$\lambda_i^{(1)} = \psi_i^{(0)T}\mathbf{W}\psi_i^{(0)} \tag{2.1.7}$$

where we have also used (2.1.4).

In order to determine the first order perturbation of the eigenvectors we express them as a linear combination of the unperturbed eigenvectors (the unperturbed eigenvectors form a basis for the space).

$$\psi_i^{(1)} = \sum_{j=1}^{N} C_{ij}^{(1)}\psi_j^{(0)} \tag{2.1.8}$$

where N is the dimension of $\mathbf{H}$. Substituting (2.1.8) into (2.1.5) we have

$$\sum_{j=1}^{N} C_{ij}^{(1)}\mathbf{H}\psi_j^{(0)} + \mathbf{W}\psi_i^{(0)} = \lambda_i^{(0)}\sum_{j=1}^{N} C_{ij}^{(1)}\psi_j^{(0)} + \lambda_i^{(1)}\psi_i^{(0)}$$

Pre-multiplying the above by $\psi_l^{(0)T}$ where $l \neq i$ and using (2.1.3) and (2.1.4) we have

$$C_{il}^{(1)}\lambda_l^{(0)} + \psi_l^{(0)T}\mathbf{W}\psi_i^{(0)} = \lambda_i^{(0)}C_{il}^{(1)};\ l \neq i$$

or

$$C_{il}^{(1)} = \frac{W_{li}}{\lambda_i^{(0)} - \lambda_l^{(0)}}\ ;\ l \neq i \tag{2.1.9}$$

where

$$W_{li} = \psi_l^{(0)T}\mathbf{W}\psi_i^{(0)} \tag{2.1.10}$$

In order to determine the $C_{ii}^{(1)}$ we will require the approximate eigenvectors $\psi_i \approx \psi_i^{(0)} + \psi_i^{(1)}$ to be normalized (to first order). It is readily shown that the approximate eigenvectors are already orthogonal, thus giving no new information on $C_{ii}^{(1)}$. The orthonormality requirement states

$$\psi_i^T\psi_i = 1 \tag{2.1.11}$$

Recall

$$\psi_i^T\psi_i \approx \psi_i^{(0)T}\psi_i^{(0)} + \psi_i^{(1)T}\psi_i^{(0)} + \psi_i^{(0)T}\psi_i^{(1)} + \psi_i^{(1)T}\psi_i^{(1)} \tag{2.1.12}$$

The last term in the above expression is of second order and will be dropped. Substituting (2.1.8) into (2.1.12) and using (2.1.4) and (2.1.11) gives

$$C_{ii}^{(1)} = 0 \tag{2.1.13}$$

We can summarize the results of the first order non-degenerate theory as follows:

1) Solve (2.1.3) for $\lambda_i^{(0)}$ and $\psi_i^{(0)}$ requiring (2.1.4).

2) Calculate $\lambda_i^{(1)}$ from (2.1.7).

3) Calculate $\psi_i^{(1)}$ from (2.1.8) using (2.1.9), (2.1.10), and (2.1.13).

4) The solution of (2.1.1) to first order is given by $\lambda_i = \lambda_i^{(0)} + \lambda_i^{(1)}$ and $\psi_i = \psi_i^{(0)} + \psi_i^{(1)}$.

2.2 Classical Perturbation: Second Order Non-Degenerate Theory

In most instances the solution of (2.1.1) by the first order theory for the non-degenerate problem as described in Section 2.1 gives results of acceptable accuracy. However, there are instances where the increased computatonal effort required to obtain second order perturbations is not only justified but required. Such will be the case in multi-degree-of-freedom tertiary systems.

The development is closely related to the methodology presented in the previous section. Naturally, the first order perturbations are exactly those described in Section 2.1. As a matter of expediency we will merely quote the results for second order perturbations which will be used in our later work. (The interested reader is referred to Butkov, pp. 658-661 for a more comprehensive treatment).

The necessary quantities will be

$$\psi_i^{(2)} = \sum_{j=1}^{N} C_{ij}^{(2)} \psi_j^{(0)} \tag{2.2.1}$$

where

$$C_{ij}^{(2)} = \sum_{\substack{l=1 \\ l\neq i}}^{N} \frac{\mathbf{W}_{jl}\mathbf{W}_{li}}{(\lambda_i^{(0)}-\lambda_j^{(0)})(\lambda_i^{(0)}-\lambda_l^{(0)})} - \frac{\mathbf{W}_{ji}\mathbf{W}_{ii}}{(\lambda_i^{(0)}-\lambda_j^{(0)})^2} \quad ; i \neq j \tag{2.2.2}$$

and

$$C_{ii}^{(2)} = -\frac{1}{2}\sum_{j=1}^{N} C_{ij}^{(1)2} \tag{2.2.3}$$

with W_{ji} and $C_{ij}^{(1)}$ given by (2.1.9), (2.1.10), and (2.1.13).

2.3 Classical Perturbation: Degenerate Theory

It was assumed in the previous two sections that all of the eigenvalues of the zeroth order problem were unique. Of course this will not always be the case. Let us assume that the unperturbed system (2.1.3) is *completely degenerate*. This means that all of the eigenvalues of (2.1.3) are the same. When some eigenvalues of the zeroth order problem are distinct while others are repeated will be called a *mixed problem*. The mixed problem will be discussed briefly in the subsequent section.

Let us return now to the problem of complete degeneracy. Unfortunately, we now have N linearly independent eigenvectors (where N is the dimension of $\mathbf{H}$) associated with a single eigenvalue. Of course any linear combination of the eigenvectors is also an eigenvector. This gives rise to a fundamental difficulty. How does one choose the unperturbed eigenvectors upon which our solution will be constructed?

We are at liberty to choose the N linearly independent eigenvectors associated with the degenerate eigenvalue. We select an orthonormal basis $\boldsymbol{\phi}_j^{(0)}$; j=1,2, . . . , N. The unperturbed eigenvectors must lie in this space and can therefore be written as some linear combination of our arbitrarily chosen basis, i.e.

$$\boldsymbol{\psi}_i^{(0)} = \sum_{j=1}^{N} C'_{ji}\boldsymbol{\phi}_j^{(0)} \tag{2.3.1}$$

The unknown coefficients C'_{ji} as well as the first order perturbations in the eigenvalues can be determined from the first order equation (2.1.5). In a manner similar to that discussed in Section 2.1 we are led to the following eigenvalue problem

$$\overline{\mathbf{W}}\mathbf{C}' = \lambda^{(1)}\mathbf{C}' \tag{2.3.2}$$

where

$$\overline{\mathbf{W}} = \left[\overline{W}_{ij}\right] \quad \overline{W}_{ij} = \boldsymbol{\phi}_j^{(0)\,T}\mathbf{W}\boldsymbol{\phi}_i^{(0)} \tag{2.3.3}$$

We solve (2.3.2) for the eigenvalues $\lambda^{(1)}$ and eigenvectors $\mathbf{C}'$. These vectors are collected in a matrix (see below) whose elements are the unknown coefficients of (2.3.1).

$$\left[C'_{ji}\right] = \left[\mathbf{C}_1\ \mathbf{C}_2\ \ldots\ \mathbf{C}_N\right] \tag{2.3.4}$$

If the solution of this eigenvalue problem gives distinct values of $\lambda^{(1)}$, then the degeneracy is "broken in the first order". Fortunately this will be the case for the systems we will be investigating.

In a manner similar to that used in Section 2.1 to establish $C_{ii}^{(1)} = 0$, we can show that in the completely degenerate problem

$$\boldsymbol{\psi}_i^{(1)} = 0 \tag{2.3.5}$$

A more detailed discussion of the material presented in this section can be found in the references mentioned at the beginning of this chapter.

A procedural summary for the completely degenerate problem is given below:

1) Solve (2.1.3) for the N-fold degenerate eigenvalue

$$\lambda_i^{(0)} = \lambda^{(0)} \quad ; \ i=1,2,\ldots,N$$

2) Choose an orthonormal basis $\boldsymbol{\phi}_j^{(0)}$. A convenient basis will be $\boldsymbol{\phi}_j^{(0)} = \mathbf{e}_j$, where $\mathbf{e}_j$ is a $N\times 1$ vector whose jth entry is one and all other entries are zero.

3) Calculate $\overline{\mathbf{W}}$ as given by (2.3.3).

4) Solve (2.3.2) for $\lambda_i^{(1)}$ and C'_{ji}.

5) Then $\lambda_i = \lambda_i^{(0)} + \lambda_i^{(1)}$ and $\boldsymbol{\psi}_i = \boldsymbol{\psi}_i^{(0)} = \sum_{j=1}^{N} C'_{ji}\boldsymbol{\phi}_j^{(0)}$.

2.4 Classical Perturbation: Mixed Theory

Although the mixed problem (when the zeroth order problem gives distinct as well as multiple eigenvalues) is the most general one, it is also the most complex. Few authors deal with the mixed problem in its entirety, but rather concentrate on the non-degenerate and degenerate cases previously described. The approach discussed in Section 4.2 allows us to reduce the mixed problem to a series of problems that are either entirely non-degenerate or completely degenerate.

Chapter Three

TWO-DEGREE-OF-FREEDOM SYSTEM

Let us illustrate the use of the theory described in Chapter Two by applying it to a simple two-degree-of-freedom secondary system (see Figure One). In Sections 3.1 and 3.2 we determine the eigenvalues and eigenvectors (to first order) of the undamped system. The equipment response when the structure is subjected to arbitrary short duration ground shock or impact is discussed in Section 3.3.

3.1 The Non-Degenerate Eigensystem

The eigenvalue problem associated with the undamped system of Figure One is

$$\mathbf{Kx} = \lambda \mathbf{Mx} \tag{3.1.1}$$

where

$$\mathbf{K} = \begin{bmatrix} k & -k \\ -k & (K+k) \end{bmatrix} \;;\; \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & M \end{bmatrix} \;;\; \mathbf{x} = \begin{bmatrix} x \\ X \end{bmatrix}$$

In order to get (3.1.1) in the form of (2.1.1) we perform the following transformation:

$$\mathbf{x} = \mathbf{T}\boldsymbol{\psi} \tag{3.1.2}$$

where

$$\mathbf{T} = \begin{bmatrix} m^{-1/2} & 0 \\ 0 & M^{-1/2} \end{bmatrix}$$

Notice that the diagonal entires of the transformation $\mathbf{T}$ are the normalized (with respect to mass) modal matrices (in this case each matrix is of dimension one by one) of the equipment alone and the structure alone.

When (3.1.2) is substituted into (3.1.1) and the resulting equation is premultiplied by $\mathbf{T}^T$ we get

$$\overline{\mathbf{K}}\boldsymbol{\psi} = \lambda\boldsymbol{\psi} \tag{3.1.3}$$

since

$$\mathbf{T}^T\mathbf{MT} = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix and where

$$\mathbf{T}^T\mathbf{KT} = \overline{\mathbf{K}}$$

For this problem

$$\overline{\mathbf{K}} = \begin{bmatrix} \omega^2 & -\gamma^{1/2}\omega^2 \\ -\gamma^{1/2}\omega^2 & (\Omega^2+\gamma\omega^2) \end{bmatrix} \tag{3.1.4}$$

We will assume that our perturbation parameter α of Section 2.1 is $\gamma^{1/2}$. If γ is of order $\epsilon \ll 1$, then $\gamma^{1/2}$ is of order $\epsilon^{1/2}$, also small compared to one. We will also assume that

ω^2 is not close to Ω^2, i.e. that their difference is of order one and not of order α (or higher). This is what is called complete detuning. Then in the notation of Section 2.1

$$\mathbf{H} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \Omega^2 \end{bmatrix} \quad \mathbf{W} = \begin{bmatrix} 0 & -\gamma^{1/2}\omega^2 \\ -\gamma^{1/2}\omega^2 & 0 \end{bmatrix} \tag{3.1.5}$$

Notice that the $\gamma\omega^2$ portion of the term in the second row and second column of the matrix in (3.1.4) gives a second order contribution.

Solving the zeroth order problem (2.1.3) we get

$$\lambda_1^{(0)} = \omega^2 \,, \; \lambda_2^{(0)} = \Omega^2$$

$$\psi_1^{(0)} = \mathbf{e}_1 \,, \; \psi_2^{(0)} = \mathbf{e}_2$$

where $\mathbf{e}_i$ is a 2×1 vector with one in row i and zero elsewhere.

Since ω^2 and Ω^2 are distinct, this is a non-degenerate problem and we shall proceed with the theory of Section 2.1. A direct application of (2.1.7) gives

$$\lambda_1^{(1)} = \lambda_2^{(1)} = 0$$

In other words, there is no frequency shift (to first order) in the frequencies of the unperturbed system **H** due to the perturbations **W**. There will, however, be a first order correction to the eigenvectors. These are found by applying (2.1.8) with (2.1.9), (2.1.10) and (2.1.13). The result is

$$\psi_1^{(1)} = \begin{bmatrix} 0 \\ \dfrac{\gamma^{1/2}}{(\Omega^2/\omega^2 - 1)} \end{bmatrix} \quad \psi_2^{(1)} = \begin{bmatrix} \dfrac{\gamma^{1/2}}{(1 - \Omega^2/\omega^2)} \\ 0 \end{bmatrix}$$

Therefore the eigenvalues and eigenvectors of our transformed system (3.1.3) are to first order

$$\lambda_1 = \omega^2 \,, \; \lambda_2 = \Omega^2$$

$$\psi_1 = \begin{bmatrix} 1 \\ \dfrac{\gamma^{1/2}}{(\Omega^2/\omega^2 - 1)} \end{bmatrix} , \; \psi_2 = \begin{bmatrix} \dfrac{\gamma^{1/2}}{(1 - \Omega^2/\omega^2)} \\ 1 \end{bmatrix} \tag{3.1.6}$$

Note that the above eigenvectors are orthonormal (to first order).

The eigenvalues of our original system (3.1.1) are given by (3.1.6). The reason for this is that the eigenvalues of a matrix are not changed if the matrix undergoes a similarity transformation (see [12]). The operations we performed to get (3.1.3) comprise a similarity transformation. This is not the case for the eigenvectors. In order to get the eigenvectors of the original system, we must apply the transformation $\mathbf{T}$ to the vectors of (3.1.6). This gives

$$\mathbf{x}_1 = \begin{bmatrix} m^{-1/2} \\ \dfrac{m^{1/2}}{M(\Omega^2/\omega^2 - 1)} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \dfrac{M^{-1/2}}{(1 - \Omega^2/\omega^2)} \\ M^{-1/2} \end{bmatrix} \tag{3.1.7}$$

Eigenvectors can be multiplied by an arbitrary scale factor. Multiplying $\mathbf{x}_1$ by $m^{1/2}$ and $\mathbf{x}_2$ by $M^{1/2}$, we see that both terms in $\mathbf{x}_2$ are of order one. However the second term of $\mathbf{x}_1$ is of order γ (second order) while the first term is of order one. This is not due to an inconsistency in the perturbation, but a result of the transformation back to real space of the perturbation result. In order to obtain a result for $\mathbf{x}_1$ which is correct to second order, one would have to use the theory of Section 2.2 and include those results in (3.1.6). This is unnecesary since the vectors in (3.1.7) are orthogonal with respect to the mass and stiffness matrices of (3.1.1) to the first order.

3.2 The Degenerate Eigensystem

We now assume that the structure natural frequency is close to the natural frequency of the equipment. That is

$$\Omega^2 = \omega^2(1+\alpha)$$

where α is the detuning parameter and is of the order of $\gamma^{1/2}$. Then instead of (3.1.5) we will have

$$\mathbf{H} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \quad \mathbf{W} = \begin{bmatrix} 0 & -\gamma^{1/2}\omega^2 \\ -\gamma^{1/2}\omega^2 & \alpha\omega^2 \end{bmatrix} \tag{3.2.1}$$

Using the above operator **H**, the zeroth order problem (2.1.3) gives

$$\lambda_1^{(0)} = \lambda_2^{(0)} = \omega^2$$

Since all the eigenvalues of the zeroth order problem are the same, we have a degenerate problem and must use the methodology of Section 2.3. We will choose an arbitrary orthonormal basis

$$\boldsymbol{\phi}_1^{(0)} = \mathbf{e}_1\,,\ \boldsymbol{\phi}_2^{(0)} = \mathbf{e}_2 \tag{3.2.2}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit vectors discussed in Section 3.1. Then according to (2.3.2) and (2.3.3) we must solve the following eigenvalue problem for $\lambda^{(1)}$ and $\mathbf{C}'$;

$$\overline{\mathbf{W}}\mathbf{C}' = \lambda^{(1)}\mathbf{C}'$$

where because of our choice of basis (3.2.2), $\overline{\mathbf{W}}$ is the same as **W** given in (3.2.1). We solve the eigenvalue problem above by a determinental approach. One can obtain a nontrivial solution only if

$$\det\left[\overline{\mathbf{W}} - \lambda^{(1)}\mathbf{I}\right] = 0$$

This gives

$$\lambda_{\substack{1\\2}}^{(1)} = \omega^2\left[\frac{\alpha}{2} \pm \mu\right] \tag{3.2.3}$$

where

$$\mu = \left\{\gamma + \frac{\alpha^2}{4}\right\}^{1/2} \tag{3.2.4}$$

Notice that the expression (3.2.3) is of first order. Now that the admissible values of $\lambda^{(1)}$ are known, we return to the eigenvalue problem given above and solve for its eigenvectors $\mathbf{C}'$. This gives

$$\mathbf{C}'_1 = \begin{bmatrix} 1 \\ \dfrac{\gamma^{1/2}}{(\alpha/2 - \mu)} \end{bmatrix} \qquad \mathbf{C}'_2 = \begin{bmatrix} \dfrac{\gamma^{1/2}}{(\mu - \alpha/2)} \\ 1 \end{bmatrix}$$

We form a matrix whose columns are the above vectors and by (2.3.4) and (2.3.1) calculate the zeroth order eigenvectors

$$\boldsymbol{\psi}_1^{(0)} = \begin{bmatrix} 1 \\ \dfrac{\gamma^{1/2}}{(\alpha/2 - \mu)} \end{bmatrix} \qquad \boldsymbol{\psi}_2^{(0)} = \begin{bmatrix} \dfrac{\gamma^{1/2}}{(\mu - \alpha/2)} \\ 1 \end{bmatrix} \tag{3.2.5}$$

By (2.3.5), the first order perturbations in the eigenvectors are zero. The eigenvectors of (3.1.3) (to first order) in the nearly tuned system are given by (3.2.5). In order to obtain the eigenvectors in real space we apply (3.1.2) to (3.2.5) which gives

$$\mathbf{x}_1 = \begin{bmatrix} m^{-1/2} \\ \dfrac{m^{1/2}}{M(\alpha/2 - \mu)} \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} \dfrac{M^{-1/2}}{(\mu - \alpha/2)} \\ M^{-1/2} \end{bmatrix} \tag{3.2.6}$$

The eigenvalues of (3.1.1) are (to first order)

$$\lambda_{\substack{1\\2}} = \omega^2\left[1 + \frac{\alpha}{2} \pm \mu\right] \tag{3.2.7}$$

where μ is defined in (3.2.4). The frequency shift for the nearly tuned system illustrated by (3.2.7) is discussed in Section 3.2 of [15]. The eigenvectors shown in (3.2.6) are specialized for the perfectly tuned system and presented in [8]. It is readily seen that in both mode shapes the structure component is small in comparison to the equipment component.

3.3 Equipment Response

Let us now use the results of the previous sections to determine the equipment response of the system in Figure One when the fixed base is subjected to a short duration ground shaking u_g in the direction of U (and hence u). The equations of motion are

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{C}\mathbf{R}\dot{u}_g + \mathbf{K}\mathbf{R}u_g \tag{3.3.1}$$

where $\mathbf{K}$ and $\mathbf{M}$ are given in (3.1.2) and

$$\mathbf{C} = 2\begin{bmatrix} \beta\omega m & -\beta\omega m \\ -\beta\omega m & (\beta\omega m + B\Omega M) \end{bmatrix} \; ; \; \mathbf{u} = \begin{bmatrix} u \\ U \end{bmatrix}$$

We assume that the damping ratios β and B are of order $\alpha << 1$. The vector $\mathbf{R}$ couples the ground motion to the degrees of freedom. For this problem, $\mathbf{R}$ is a 2×1 vector of ones.

We solve (3.3.1) by a standard modal solution. Let us first examine the completely detuned system. Since we are interested in the acceleration response, we take two time derivatives of (3.3.1) and substitute the expression below

$$\ddot{\mathbf{u}} = \mathbf{X}\mathbf{y} \tag{3.3.2}$$

where

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \; ; \; \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and $\mathbf{x}_1$, $\mathbf{x}_2$ are given in (3.1.7). Pre-multiplying by $\mathbf{X}^T$ leads to the following uncoupled (to first order) equations on the modal coordinates y_i ; i=1,2

$$\ddot{y}_1(t) + 2\beta\omega\dot{y}_1(t) + \omega^2 y_1(t) = A_1\frac{d}{dt}\Big[\ddot{u}_g(t)\Big] + B_1\ddot{u}_g(t)$$

$$\ddot{y}_2(t) + 2B\Omega\dot{y}_2(t) + \Omega^2 y_2(t) = A_2\frac{d}{dt}\Big[\ddot{u}_g(t)\Big] + B_2\ddot{u}_g(t) \tag{3.3.3}$$

where

$$A_i = \mathbf{x}_i^T\mathbf{C}\mathbf{R}/\mathbf{x}_i^T\mathbf{M}\mathbf{x}_i$$

$$B_i = \mathbf{x}_i^T\mathbf{K}\mathbf{R}/\mathbf{x}_i^T\mathbf{M}\mathbf{x}_i \; ; \; i=1,2$$

Since $A_i << B_i$, the contribution to the overall response from the A_i terms will be negligible. They will therefore be dropped. From (3.1.1)

$$\mathbf{x}_i^T\mathbf{K}\mathbf{R} = \lambda_i \mathbf{x}_i^T\mathbf{M}\mathbf{R}$$

The solution to (3.3.3) is found from ordinary differential equation theory as

$$y_1(t) = \Gamma_1 \omega e^{-\beta\omega t}\sin(\omega t) \; * \; \ddot{u}_g$$

$$y_2(t) = \Gamma_2 \Omega e^{-B\Omega t}\sin(\Omega t) \; * \; \ddot{u}_g \qquad (3.3.4)$$

where

$$f(t) \; * \; \ddot{u}_g = \int_0^t f(t-\tau)\,\ddot{u}_g(\tau)\,d\tau$$

and

$$\Gamma_i = \mathbf{x}_i^T\mathbf{M}\mathbf{R}/\mathbf{x}_i^T\mathbf{M}\mathbf{x}_i \quad ; \; i=1,2$$

The above expression (3.3.4) is not correct to first order as a solution of the differential equations (3.3.3) introduces the damped rather than the undamped natural frequencies in the sine terms of (3.3.4). Use of the damped frequency instead of the undamped natural frequency has a negligible effect on the response for the completely detuned case and is therefore not included. Substituting (3.3.4) into (3.3.2) and solving for equipment response we have (retaining dominant terms only)

$$\ddot{u}(t) = \left[\frac{1}{1-(\omega/\Omega)^2}\omega e^{-\beta\omega t}\sin(\omega t) + \frac{1}{1-(\Omega/\omega)^2}\Omega e^{-B\Omega t}\sin(\Omega t)\right] \; * \; \ddot{u}_g$$

This result was obtained by Sackman and Kelly in [15] by Laplace transform methods. A result for maximum equipment acceleration can then be obtained in terms of the ground pseudo-acceleration response spectrum. This is discussed in [6] or [15]. Also discussed in [6] are two cases of gross detuning: $\omega << \Omega$ and $\omega >> \Omega$. The tuned or nearly tuned system is best analyzed by Laplace transform methods. The reader is referred to [15] where this is done in a comprehensive manner. The results presented there are specialized for extremely short duration input in [6]. The results for gross detuning as well as slight or no detuning are then combined in [6] to form a "universal" result. This "universal" result is a single formula that can readily be used to determine maximum equipment response over a broad range of frequencies. The "universal" result developed there is replicated here for reference purposes:

$$\left|\ddot{u}\right|_{max} = \frac{e^{-\kappa'\left[\frac{(\omega\Omega)^{1/2}}{(\omega+\Omega)/2}\right]}}{(\gamma+\Psi^2+4\beta B)^{1/2}}\left|\frac{\frac{\Omega}{\omega}S_A(\omega,\zeta) + \frac{\omega}{\Omega}S_A(\Omega,\zeta)}{2}\right| \tag{3.3.5}$$

where $\kappa' = arctan(\zeta')/\zeta'$; $\zeta' = [\gamma+\Psi^2-(\beta-B)^2]^{1/2}/(\beta+B)$

$$\zeta = \left|\frac{\Omega-\omega}{B\Omega + \beta\omega}\right|\beta B \ ; \ \Psi = \frac{\Omega-\omega}{(\omega\Omega)^{1/2}}$$

and $S_A(\omega,\xi)$ is the ground pseudo-acceleration response spectrum evaluated at frequency ω and damping ξ.

The equipment response when the structure is subjected to impact loading can be determined in a similar manner. The right hand side of (3.3.1) becomes a two by one vector with first entry zero and second entry the arbitrary force $F(t)$. The constants multiplying the exponential and sine terms in (3.3.4) will be different, yet the rest of the solution remains the same. This problem is discussed in detail in [6] where a "universal" result for impact is developed. This formula is replicated below for reference purposes.

$$\left|\ddot{u}\right|_{max} = \frac{e^{-\kappa'}}{(\gamma+\Psi^2+4\beta B)^{1/2}}\left[\frac{2\omega(\omega\Omega)^{1/2}}{(\omega+\Omega)^2 + \left|\omega^2-\Omega^2\right|}\right]\left[\frac{\omega}{\Omega}S_A(\omega,\tilde{\zeta}) + S_A(\Omega,\tilde{Z})\right] \tag{3.3.6}$$

where

$$\tilde{\zeta} = \left|\frac{\omega-\Omega}{\omega+\Omega}\right|\beta \ ; \ \tilde{Z} = \left|\frac{\omega-\Omega}{\omega+\Omega}\right|B$$

and $S_A(\omega,\xi)$ is the pseudo-acceleration response spectrum for the input $F(t)/M$ evaluated at frequency ω and damping ξ.

A number of numerical tests were performed in [6] to determine the accuracy of (3.3.5) and (3.3.6). The performance of both equations over a large range of frequencies was extremely good.

Chapter Four

MULTI-DEGREE-OF-FREEDOM SECONDARY SYSTEMS

A multi-degree-of-freedom secondary system is depicted in a general way in Figure Two. The system is assumed to have a discrete number of degrees of freedom. The structure [denoted (1)] is fixed to the ground. Attached to various degrees of freedom in both the structure and the equipment [denoted (2)] are rigid links. The structure has $n^{(1)}$ degrees of freedom (which include the link attachment points). The equipment has $n^{(2)}$ "free" degrees of freedom (the link attachment points are not included in these degrees of freedom). Thus the "fixed base" equipment properties are based on the links being fixed. The properties of the fixed base structure are, of course, based on the fixed ground condition (without the equipment). In order for the system of Figure Two to be a secondary system, all the elements of the mass and stiffness matrices of the equipment must be small in comparison to all the elements of the mass and stiffness matrices of the structure.

In the subsequent sections, expressions for the acceleration response of the equipment degrees of freedom, in terms of the properties of the sub-systems and the excitation, are determined. The completely detuned system (where all the natural frequencies of the fixed base sub-systems are well spaced) is analyzed by a "direct approach" in Section 4.1. In order to analyze the tuned secondary system (when a natural frequency of the structure is close to a natural frequency of the equipment) a sub-problem solution procedure is introduced in Section 4.2 and directly employed in Section 4.3. The maximum acceleration of each equipment degree of freedom, $\left|\ddot{u}_z^{(2)}\right|_{max}$, is determined in terms of the pseudo-acceleration response spectrum description of the excitation and the system properties. The numerical tests discussed in Section 4.5 indicate that the results of this chapter are indeed promising.

4.1 The Completely Detuned System: Direct Approach

In order to analyze the system shown in Figure Two we must solve the following eigenproblem

$$\mathbf{K}\mathbf{x} = \lambda \mathbf{M}\mathbf{x} \tag{4.1.1}$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{m}^{(2)} & \mathbf{m}^{(21)} \\ \mathbf{m}^{(12)} & \mathbf{m}^{(1)^*} \end{bmatrix} \; ; \; \mathbf{K} = \begin{bmatrix} \mathbf{k}^{(2)} & \mathbf{k}^{(21)} \\ \mathbf{k}^{(12)} & \mathbf{k}^{(1)^*} \end{bmatrix} \; ; \; \mathbf{x} = \begin{bmatrix} \mathbf{x}^{(2)} \\ \mathbf{x}^{(1)} \end{bmatrix} \tag{4.1.2}$$

$$\mathbf{m}^{(1)^*} = \mathbf{m}^{(1)} + \tilde{\mathbf{m}}^{(1)} \; ; \; \mathbf{k}^{(1)^*} = \mathbf{k}^{(1)} + \tilde{\mathbf{k}}^{(1)}$$

and

$\mathbf{x}^{(2)}$, $\mathbf{x}^{(1)}$ are the displacement vectors associated with the equipment and structure respectively

$\mathbf{m}^{(2)}$, $\mathbf{k}^{(2)}$ and $\mathbf{m}^{(1)}$, $\mathbf{k}^{(1)}$ are the fixed base mass and stiffness for the equipment and structure respectively

$\mathbf{m}^{(12)} = \mathbf{m}^{(21)^T}$ are the "cross-coupling" mass matrices (these matrices are zero for a lumped mass formulation)

$\mathbf{k}^{(12)} = \mathbf{k}^{(21)^T}$ are "cross-coupling" stiffness matrices (the elements of these matrices come from the fixed base equipment stiffness $\mathbf{k}^{(2)}$)

$\tilde{\mathbf{m}}^{(1)}$ and $\tilde{\mathbf{k}}^{(1)}$ are the contributions to $\mathbf{m}^{(1)^*}$ and $\mathbf{k}^{(1)^*}$ from the equipment.

We will solve the eigenvalue problem of (4.1.1) by classical perturbation theory. To do so we transform to standard form by the transformation

$$\mathbf{x} = \mathbf{T}\psi \tag{4.1.3}$$

where

$$\mathbf{T} = \begin{bmatrix} \hat{\mathbf{\Phi}}^{(2)} & 0 \\ 0 & \hat{\mathbf{\Phi}}^{(1)} \end{bmatrix} \tag{4.1.4}$$

In the above $\hat{\mathbf{\Phi}}^{(2)}$ is the normalized (with respect to mass) fixed base equipment modal matrix [dimension $(n^{(2)} \times n^{(2)})$] and $\hat{\mathbf{\Phi}}^{(1)}$ is the normalized (with respect to mass) fixed base structure modal matrix [dimension $(n^{(1)} \times n^{(1)})$].

Substituting (4.1.3) into (4.1.1) and pre-multiplying by $\mathbf{T}^T$ we get

$$\bar{\mathbf{K}}\psi = \lambda\psi \tag{4.1.5}$$

since

$$\mathbf{T}^T\mathbf{M}\mathbf{T} = \mathbf{I}$$

and where

$$\bar{\mathbf{K}} = \mathbf{T}^T\mathbf{K}\mathbf{T}.$$

If $\mathbf{m}^{(2)}$ is of order ϵ and $\mathbf{m}^{(1)}$ is of order one, then retaining terms only up to $\epsilon^{1/2}$ in $\bar{\mathbf{K}}$ we get

$$\bar{\mathbf{K}} = \left(\mathbf{H} + \mathbf{W}\right)$$

where

$$\mathbf{H} = \begin{bmatrix} \omega^{(2)^2} & 0 \\ 0 & \omega^{(1)^2} \end{bmatrix} \; ; \; \mathbf{W} = \begin{bmatrix} 0 & \mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix}$$

$$\mathbf{A} = \hat{\mathbf{\Phi}}^{(1)^T}\mathbf{k}^{(12)}\hat{\mathbf{\Phi}}^{(2)}$$

and $\omega^{(2)^2}$, $\omega^{(1)^2}$ are diagonal matrices whose diagonal elements are the squared natural frequencies of the fixed base equipment and structural system respectively.

As in the two-degree-of-freedom system, $\tilde{\mathbf{k}}^{(1)}$ and $\tilde{\mathbf{m}}^{(1)}$ give second order contributions. We assume that all the frequencies in $\mathbf{H}$ are distinct. Furthermore $\mathbf{W}$ is symmetric. Thus we shall use the theory of Section 2.1.

The zeroth order problem (2.1.3) gives

$$\lambda_k^{(0)} = \omega_k^{(2)^2} \quad ; \; k=1,2, \ldots, n^{(2)}$$

$$\lambda_{l+n^{(2)}}^{(0)} = \omega_l^{(1)^2} \quad ; \; l=1,2, \ldots, n^{(1)} \tag{4.1.6}$$

and

$$\psi_i^{(0)} = \mathbf{e}_i \quad ; \; i=1,2, \ldots, n^{(2)}+n^{(1)}$$

where $\mathbf{e}_i$ is a $(n^{(2)}+n^{(1)}) \times 1$ vector whose ith entry is one and zero otherwise. In order to facilitate the subsequent discussion, we will assume that the range of the indices i, k, and l are those given above. It is easily verified that (2.1.7) gives for all i

$$\lambda_i^{(1)} = 0$$

Thus the eigenvalues of (4.1.1) are to first order given by (4.1.6). In other words, the natural frequencies of the completely detuned combined system are the same (to first order) as the natural frequencies of the individual sub-systems.

We now calculate (2.1.9) which gives the following non-zero values

$$W_{n^{(2)}+l,k} = W_{k,n^{(2)}+l} = A_{lk}$$

So by (2.1.8) we have

$$\psi_k^{(1)} = \begin{bmatrix} \tilde{0} \\ \mathbf{f}_k \end{bmatrix} \;;\; \psi_{n^{(2)}+l}^{(1)} = \begin{bmatrix} \mathbf{g}_l \\ \hat{0} \end{bmatrix}$$

where $\tilde{0}$ and $\hat{0}$ are zero vectors of dimension $n^{(2)}$ and $n^{(1)}$ respetively. For a particular value of k, the vector $\mathbf{f}_k$ [dimension $(n^{(2)}\times 1)$] has components l=1,2, . . . , $n^{(1)}$ given below

$$\frac{A_{lk}}{(\omega_k^{(2)^2} - \omega_l^{(1)^2})}$$

For a particular value of l, the vector $\mathbf{g}_l$ [dimension $(n^{(2)}\times 1)$] has components k=1,2, . . . , $n^{(2)}$ given below

$$\frac{A_{lk}}{(\omega_l^{(1)^2} - \omega_k^{(2)^2})}$$

The eigenvectors of (4.1.5) are to first order given by

$$\psi_k = \begin{bmatrix} \tilde{\mathbf{e}}_k \\ \mathbf{f}_k \end{bmatrix} \;;\; \psi_{n^{(2)}+l} = \begin{bmatrix} \mathbf{g}_l \\ \hat{\mathbf{e}}_l \end{bmatrix}$$

where $\tilde{\mathbf{e}}_k$ is a $n^{(2)} \times 1$ vector whose kth element is one and all other entries are zero and $\hat{\mathbf{e}}_l$ is a $n^{(1)} \times 1$ vector whose lth entry is one and all other entries are zero.

We now find the eigenvectors in real space by performing the transformation given in (4.1.3) which gives

$$\mathbf{x}_k = \begin{bmatrix} \hat{\boldsymbol{\phi}}_k^{(2)} \\ \mathbf{f}_k^{(1)} \end{bmatrix} \; ; \; \mathbf{x}_{n^{(2)}+l} = \begin{bmatrix} \mathbf{g}_l^{(2)} \\ \hat{\boldsymbol{\phi}}_l^{(1)} \end{bmatrix}$$

where for a particular value of k the vector $\mathbf{f}_k^{(1)}$ is given as

$$\mathbf{f}_k^{(1)} = \sum_{l=1}^{n^{(1)}} \frac{\hat{\boldsymbol{\phi}}_l^{(1)} A_{lk}}{(\omega_k^{(2)2} - \omega_l^{(1)2})}$$

and for a particular value of l the vector $\mathbf{g}_l^{(2)}$ is given as

$$\mathbf{g}_l^{(2)} = \sum_{k=1}^{n^{(2)}} \frac{\hat{\boldsymbol{\phi}}_k^{(2)} A_{lk}}{(\omega_l^{(1)2} - \omega_k^{(2)2})}$$

The eigenvectors are then collected to form the following $N \times N$ matrix, where $N = n^{(2)}+n^{(1)}$

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 \; \mathbf{x}_2 \; \cdots \; \mathbf{x}_{n^{(2)}+n^{(1)}} \end{bmatrix}$$

Let us now solve the ground shock problem; i.e. where the base of the structure is subjected to a short duration ground shaking. The equation of motion on the total displacement $\mathbf{u}$ is then

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{C}\mathbf{R}\dot{u}_g + \mathbf{K}\mathbf{R}u_g \tag{4.1.7}$$

where $\mathbf{M}$ and $\mathbf{K}$ are given in (4.1.2) and

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}^{(2)} \\ \mathbf{u}^{(1)} \end{bmatrix} \; ; \; \mathbf{R} = \begin{bmatrix} \mathbf{r}^{(2)} \\ \mathbf{r}^{(1)} \end{bmatrix} \tag{4.1.8}$$

The vector $\mathbf{u}^{(2)}$ contains the total displacements of the equipment degrees of freedom. Similarly $\mathbf{u}^{(1)}$ contains the total displacements of the structure degrees of freedom. If the ground is given a unit displacement in the direction of the ground shaking, the vector $\mathbf{r}^{(2)}$ represents the displacements of the equipment degrees of freedom, while the vector $\mathbf{r}^{(1)}$ represents the displacements of the structure degrees of freedom (pseudo-static motion vector). The combined damping matrix is

$$\mathbf{C} = \begin{bmatrix} \mathbf{c}^{(2)} & \mathbf{c}^{(12)\,T} \\ \mathbf{c}^{(12)} & \mathbf{c}^{(1)\,*} \end{bmatrix}$$

where the matrices in $\mathbf{C}$ are of the same form as the matrices in $\mathbf{K}$ given in (4.1.2). We will assume that the sub-systems are modally damped. That is

$$\hat{\mathbf{\Phi}}^{(2)}\mathbf{c}^{(2)}\hat{\mathbf{\Phi}}^{(2)} = 2\boldsymbol{\beta}^{(2)}\boldsymbol{\omega}^{(2)} \;\; ; \; \hat{\mathbf{\Phi}}^{(1)}\mathbf{c}^{(1)}\hat{\mathbf{\Phi}}^{(1)} = 2\boldsymbol{\beta}^{(1)}\boldsymbol{\omega}^{(1)} \tag{4.1.9}$$

where $\boldsymbol{\beta}^{(2)}\boldsymbol{\omega}^{(2)}$ and $\boldsymbol{\beta}^{(1)}\boldsymbol{\omega}^{(1)}$ are diagonal matrices whose entries are the product of modal damping (fraction of critical) and modal frequency. We further require the modal damping ratios to be small (of first order).

We solve (4.1.7) by first taking two derivatives with respect to time. A transformation to modal coordinates

$$\ddot{\mathbf{u}} = \mathbf{X}\mathbf{y} \tag{4.1.10}$$

and pre-multiplying the resulting equation by $\mathbf{X}^T$ gives a set of uncoupled equations on the modal coordinates y_i. The fact that this similarity transformation diagonalizes $\mathbf{K}$ and $\mathbf{M}$ (to first order) comes from the eigenvalue problem. That it also diagonalizes $\mathbf{C}$ (to first order) comes from (4.1.9) and the structure of $\mathbf{X}$. The contribution to the response from the damping term in the right hand side of (4.1.7) is negligible and will be dropped. Also we have from the eigenproperties

$$\mathbf{x}_i^T\mathbf{K}\mathbf{R} = \lambda_i \mathbf{x}_i^T\mathbf{M}\mathbf{R}$$

Using this we have for the modal coordinates

$$y_k(t) \;=\; \Gamma_k \omega_k^{(2)} e^{-\beta_k^{(2)}\omega_k^{(2)}t} \sin(\omega_k^{(2)}t) \;*\; \ddot{u}_g$$

and

$$y_{n^{(2)}+l}(t) \;=\; \Gamma_{n^{(2)}+l}\,\omega_l^{(1)} e^{-\beta_l^{(1)}\omega_l^{(1)}t} \sin(\omega_l^{(1)}t) \;*\; \ddot{u}_g$$

where

$$\Gamma_i = \mathbf{x}_i^T\mathbf{M}\mathbf{R}/\mathbf{x}_i^T\mathbf{M}\mathbf{x}_i$$

Solving for the participation factors given above we have (to dominant order)

$$\Gamma_k = P_k^{(2)} + \sum_{l=1}^{n^{(1)}} \frac{C_{lk}}{(\omega_k^{(2)^2} - \omega_l^{(1)^2})}$$

and

$$\Gamma_{n^{(2)}+l} = P_l^{(1)}$$

where

$$P_l^{(1)} = \hat{\boldsymbol{\phi}}_l^{(1)T} \mathbf{m}^{(1)} \mathbf{r}^{(1)}$$

$$P_k^{(2)} = \hat{\boldsymbol{\phi}}_k^{(2)T} \mathbf{m}^{(2)} \mathbf{r}^{(2)}$$

$$A_{lk} = \hat{\boldsymbol{\phi}}_l^{(1)T} \mathbf{k}^{(12)} \hat{\boldsymbol{\phi}}_k^{(2)}$$

$$C_{lk} = P_l^{(1)} A_{lk}$$

Using these results together with (4.1.10) we find the equipment response as

$$\ddot{u}_z^{(2)}(t) = \Bigg\{ \sum_{k=1}^{n^{(2)}} \hat{\phi}_{zk}^{(2)} P_k^{(2)} \omega_k^{(2)} e^{-\beta_k^{(2)}\omega_k^{(2)}t} \sin(\omega_k^{(2)} t)$$

$$+ \sum_{k=1}^{n^{(2)}} \sum_{l=1}^{n^{(1)}} \frac{\hat{\phi}_{zk}^{(2)} C_{lk}}{(\omega_k^{(2)^2} - \omega_l^{(1)^2})} \omega_k^{(2)} e^{-\beta_k^{(2)}\omega_k^{(2)}t} \sin(\omega_k^{(2)} t)$$

$$+ \sum_{k=1}^{n^{(2)}} \sum_{l=1}^{n^{(1)}} \frac{\hat{\phi}_{zk}^{(2)} C_{lk}}{(\omega_l^{(1)^2} - \omega_k^{(2)^2})} \omega_l^{(1)} e^{-\beta_l^{(1)}\omega_l^{(1)}t} \sin(\omega_l^{(1)} t) \Bigg\} * \ddot{u}_g \qquad (4.1.12)$$

for $z = 1,2, \ldots, n^{(2)}$. Notice that the expressions within the braces is the *Green's function* for the total acceleration of the equipment due to ground shaking of the system in Figure Two.

Let us now consider the impact case. The right hand side of (4.1.7) becomes $\mathbf{F}(t)$, the vector of nodal forces. We will assume that a particular force-time history is applied at a single structural degree of freedom or distributed over many structural degrees of freedom. This allows us to write the following

$$\mathbf{F} = \mathbf{S}F(t)$$

where

$$\mathbf{S} = \begin{bmatrix} 0 \\ \mathbf{S}^{(1)} \end{bmatrix}$$

and $\mathbf{S}^{(1)}$ is a $n^{(1)} \times 1$ vector whose entries are one (or a fraction thereof) if the force-time history is applied to this degree of freedom and zero otherwise. Letting

$$\mathbf{u} = \mathbf{X}\mathbf{y}$$

and pre-multiplying by $\mathbf{X}^T$ we are led to uncoupled ordinary differential equations on the modal coordinates y_i. Their solution is given by

$$y_k(t) = \frac{\overline{\Gamma}_k}{\omega_k^{(2)}} e^{-\beta_k^{(2)}\omega_k^{(2)}t} \sin(\omega_k^{(2)} t) \ * \ F$$

$$y_{n^{(2)}+l}(t) = \frac{\overline{\Gamma}_{n^{(2)}+l}}{\omega_l^{(1)}} e^{-\beta_l^{(1)}\omega_l^{(1)}t} \sin(\omega_l^{(1)} t) \ * \ F \tag{4.1.13}$$

where

$$f(t) * F = \int_0^t f(t-\tau) F(\tau) \, d\tau$$

$$\overline{\Gamma}_i = \mathbf{x}_i^T \mathbf{S} / \mathbf{x}_i^T \mathbf{M} \mathbf{x}_i$$

Since we are interested in acceleration response, we take two derivatives with respect to time of (4.1.13). This results in (dropping negligible terms)

$$\ddot{y}_k(t) = -\overline{\Gamma}_k \omega_k^{(2)} e^{-\beta_k^{(2)}\omega_k^{(2)}t} \sin(\omega_k^{(2)} t) \ * \ F$$

$$\ddot{y}_{n^{(2)}+l}(t) = -\overline{\Gamma}_{n^{(2)}+l} \omega_l^{(1)} e^{-\beta_l^{(1)}\omega_l^{(1)}t} \sin(\omega_l^{(1)} t) \ * \ F$$

When these results are combined in vector form and pre-multiplied by the matrix of eigenvectors $\mathbf{X}$, we have the equipment acceleration response for the case of complete detuning due to impact loading as

$$\ddot{u}_z^{(2)}(t) = -\left\{ \sum_{k=1}^{n^{(2)}} \sum_{l=1}^{n^{(1)}} \frac{\hat{\phi}_{zk}^{(2)} D_{lk}}{(\omega_k^{(2)2} - \omega_l^{(1)2})} \omega_k^{(2)} e^{-\beta_k^{(2)}\omega_k^{(2)}t} \sin(\omega_k^{(2)} t) \right.$$

$$\left. + \sum_{k=1}^{n^{(2)}} \sum_{l=1}^{n^{(1)}} \frac{\hat{\phi}_{zk}^{(2)} D_{lk}}{(\omega_l^{(1)2} - \omega_k^{(2)2})} \omega_l^{(1)} e^{-\beta_l^{(1)}\omega_l^{(1)}t} \sin(\omega_l^{(1)} t) \right\} * F \tag{4.1.14}$$

where

$$D_{lk} = \bar{P}_l^{(1)} A_{lk} \tag{4.1.15}$$

and

$$\bar{P}_l^{(1)} \equiv \bar{\Gamma}_l$$

4.2 The Completely Detuned System: Sub-Problem Approach

In this section, we obtain the results of the previous section by the solution of a series of sub-problems. This would certainly not be an efficient solution procedure for the completely detuned system. This method can be used to solve the tuned secondary system that will be examined in the next section.

Problem One

Instead of employing the entire transformation matrix $\mathbf{T}$ of Section 4.1, let us make the transformation

$$\mathbf{x} = \mathbf{T}_1 \boldsymbol{\psi}$$

where

$$\mathbf{T}_1 = \begin{bmatrix} \hat{\boldsymbol{\Phi}}^{(2)} & 0 \\ 0 & \hat{\boldsymbol{\Phi}}^{(1)-} \end{bmatrix}$$

where $\hat{\boldsymbol{\Phi}}^{(2)}$ is the equipment modal matrix described in Section 4.1 and $\hat{\boldsymbol{\Phi}}^{(1)-}$ is the normalized structural modal matrix without the mth mode [dimension $(n^{(1)} \times n^{(1)}-1)$].

When the matrix $\mathbf{T}_1$ instead of the matrix $\mathbf{T}$ of Section 4.1 is used in our perturbation solution of the eigenvalue problem (4.1.1), we are led to a reduced set of eigenvalues and eigenvectors. The eigenvalues will be those of Section 4.1 without the mth structure natural frequency. The matrix of eigenvectors will be a rectangular matrix [dimension $(n^{(2)}+n^{(1)} \times n^{(2)}+n^{(1)}-1)$] which, in essence, lacks the effect of the mth structure mode. When this rectangular matrix of eigenvectors is used in a modal solution of (4.1.7), the equipment response will be

$$\ddot{u}_z^{(2)}(t) = \left\{ \sum_{k=1}^{n^{(2)}} \hat{\phi}_{zk}^{(2)} P_k^{(2)} \omega_k^{(2)} e^{-\beta_k^{(2)} \omega_k^{(2)} t} \sin(\omega_k^{(2)} t) \right.$$

$$+ \sum_{k=1}^{n^{(2)}} \sum_{\substack{l=1 \\ l \neq m}}^{n^{(1)}} \frac{\hat{\phi}_{zk}^{(2)} C_{lk}}{(\omega_k^{(2)^2} - \omega_l^{(1)^2})} \omega_k^{(2)} e^{-\beta_k^{(2)} \omega_k^{(2)} t} \sin(\omega_k^{(2)} t)$$

$$+\sum_{k=1}^{n^{(2)}}\sum_{\substack{l=1 \\ l\neq m}}^{n^{(1)}} \frac{\hat{\phi}_{zk}^{(2)} C_{lk}}{(\omega_l^{(1)2} - \omega_k^{(2)2})} \omega_l^{(1)} e^{-\beta_l^{(1)}\omega_l^{(1)} t} \sin(\omega_l^{(1)} t) \Bigg\} * \ddot{u}_g \tag{4.2.1}$$

We notice that all the $P_k^{(2)}$ type terms present in the solution (4.1.12) appear here, while two series involving the C_{lk} are missing. The procedure described here is essentially a Ritz condensation procedure.

Problem Two

We now need to include the effect of the mth structure mode on the response. This can be done by introducing the transformation

$$\overline{\mathbf{T}} = \begin{bmatrix} \hat{\mathbf{\Phi}}^{(2)} & 0 \\ 0 & \hat{\boldsymbol{\phi}}_m^{(1)} \end{bmatrix}$$

It will be convenient in the tuned system to further subdivide the above matrix to

$$\mathbf{T}_2 = \begin{bmatrix} \hat{\mathbf{\Phi}}^{(2)-} & 0 \\ 0 & \hat{\boldsymbol{\phi}}_m^{(1)} \end{bmatrix}$$

where $\hat{\mathbf{\Phi}}^{(2)-}$ is the matrix of eigenvectors of the equipment system without the nth eigenvector [dimension $(n^{(2)} \times n^{(2)}-1)$] and $\hat{\boldsymbol{\phi}}_m^{(1)}$ is the mth structure eigenvector. The remaining effect (between the mth structure mode and the nth equipment mode) will be discussed in Problem Three.

Again when the matrix $\mathbf{T}_2$ instead of the matrix $\mathbf{T}$ of Section 4.1 is used in our perturbation solution of the eigenvalue problem (4.1.1) we are led to a reduced set of eigenvalues and eigenvectors. The eigenvalues will be all equipment natural frequencies except the nth frequency as well as the mth structural natural frequency. The matrix of eigenvectors will contain only the effect of the modes associated with the above frequencies on the system and will be a rectangular matrix [dimension $(n^{(2)}+n^{(1)} \times n^{(2)})$]. When this rectangular matrix of eigenvectors is used in a modal solution of (4.1.7), the equipment response will be

$$\ddot{u}_z^{(2)}(t) = \Bigg\{ \sum_{\substack{k=1 \\ k \neq n}}^{n^{(2)}} \hat{\phi}_{zk}^{(2)} P_k^{(2)} \omega_k^{(2)} e^{-\beta_k^{(2)} \omega_k^{(2)} t} \sin(\omega_k^{(2)} t)$$

$$+ \sum_{\substack{k=1 \\ k \neq n}}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} C_{mk}}{(\omega_k^{(2)2} - \omega_m^{(1)2})} \omega_k^{(2)} e^{-\beta_k^{(2)} \omega_k^{(2)} t} \sin(\omega_k^{(2)} t)$$

$$+ \sum_{\substack{k=1 \\ k \neq n}}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} C_{mk}}{(\omega_m^{(1)2} - \omega_k^{(2)2})} \omega_m^{(1)} e^{-\beta_m^{(1)} \omega_m^{(1)} t} \sin(\omega_m^{(1)} t) \Bigg\} * \ddot{u}_g$$

As we remarked at the end of Problem One all the $P_k^{(2)}$ type terms necessary for the complete solution are contained there. Thus the $P_k^{(2)}$ type terms appearing here are superfluous and must be dropped. The relevant result from Problem Two is

$$\ddot{u}_z^{(2)}(t) = \Bigg\{ \sum_{\substack{k=1 \\ k \neq n}}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} C_{mk}}{(\omega_k^{(2)2} - \omega_m^{(1)2})} \omega_k^{(2)} e^{-\beta_k^{(2)} \omega_k^{(2)} t} \sin(\omega_k^{(2)} t)$$

$$+ \sum_{\substack{k=1 \\ k \neq n}}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} C_{mk}}{(\omega_m^{(1)2} - \omega_k^{(2)2})} \omega_m^{(1)} e^{-\beta_m^{(1)} \omega_m^{(1)} t} \sin(\omega_m^{(1)} t) \Bigg\} * \ddot{u}_g \qquad (4.2.2)$$

Problem Three

The effect of the mth structure mode on the nth equipment mode is now sought.

Let

$$\mathbf{T}_3 = \begin{bmatrix} \hat{\boldsymbol{\phi}}_n^{(2)} & 0 \\ 0 & \hat{\boldsymbol{\phi}}_m^{(1)} \end{bmatrix} \qquad (4.2.3)$$

where the above matrix is of dimension $n^{(2)}+n^{(1)}$ by 2. Using $\mathbf{T}_3$ instead of $\mathbf{T}$ in the perturbation solution of the eigenvalue problem (4.1.1) leads to two frequencies and mode shapes. The eigenvalues are the nth equipment frequency and the mth structure frequency. The matrix of eigenvectors contains only the effect of the modes associated with these two frequencies on the system and will be a rectangular matrix [dimension $(n^{(2)}+n^{(1)} \times 2)$]. When this matrix of eigenvectors is used in a modal solution of (4.1.7), the equipment response will be

$$\ddot{u}_z^{(2)}(t) = \left\{ \hat{\phi}_{zn}^{(2)} \left[P_n^{(2)} + \frac{C_{mn}}{(\omega_n^{(2)2} - \omega_m^{(1)2})} \right] \omega_n^{(2)} e^{-\beta_n^{(2)} \omega_n^{(2)} t)} \sin(\omega_n^{(2)} t) \right.$$

$$\left. + \hat{\phi}_{zn}^{(2)} \frac{C_{mn}}{(\omega_m^{(1)2} - \omega_n^{(2)2})} \omega_m^{(1)} e^{-\beta_m^{(1)} \omega_m^{(1)} t} \sin(\omega_m^{(1)} t) \right\} * \ddot{u}_g \tag{4.2.4}$$

Again the term $P_n^{(2)}$ must be discarded resulting in

$$\ddot{u}_z^{(2)}(t) = \left\{ \frac{\hat{\phi}_{zn}^{(2)} C_{mn}}{(\omega_n^{(2)2} - \omega_m^{(1)2})} \omega_n^{(2)} e^{-\beta_n^{(2)} \omega_n^{(2)} t)} \sin(\omega_n^{(2)} t) \right.$$

$$\left. + \hat{\phi}_{zn}^{(2)} \frac{C_{mn}}{(\omega_m^{(1)2} - \omega_n^{(2)2})} \omega_m^{(1)} e^{-\beta_m^{(1)} \omega_m^{(1)} t} \sin(\omega_m^{(1)} t) \right\} * \ddot{u}_g \tag{4.2.5}$$

When the results from Problems One, Two, and Three [given in (4.2.1), (4.2.2), and (4.2.4)] are added, the solution to the original problem [given in (4.1.12)] is obtained.

The results of a sub-problem solution procedure in the *impact* case are given below.

Problem One

$$\ddot{u}_z^{(2)}(t) = -\left\{ \sum_{k=1}^{n^{(2)}} \sum_{\substack{l=1 \\ l \neq m}}^{n^{(1)}} \frac{\hat{\phi}_{zk}^{(2)} D_{lk}}{(\omega_k^{(2)2} - \omega_l^{(1)2})} \omega_k^{(2)} e^{-\beta_k^{(2)} \omega_k^{(2)} t} \sin(\omega_k^{(2)} t) \right.$$

$$\left. + \sum_{k=1}^{n^{(2)}} \sum_{\substack{l=1 \\ l \neq m}}^{n^{(1)}} \frac{\hat{\phi}_{zk}^{(2)} D_{lk}}{(\omega_l^{(1)2} - \omega_k^{(2)2})} \omega_l^{(1)} e^{-\beta_l^{(1)} \omega_l^{(1)} t} \sin(\omega_l^{(1)} t) \right\} * F$$

Problem Two

$$\ddot{u}_z^{(2)}(t) = -\left\{ \sum_{\substack{k=1 \\ k \neq n}}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} D_{mk}}{(\omega_k^{(2)2} - \omega_m^{(1)2})} \omega_k^{(2)} e^{-\beta_k^{(2)} \omega_k^{(2)} t} \sin(\omega_k^{(2)} t) \right.$$

$$+ \sum_{\substack{k=1 \\ k \neq n}}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} D_{mk}}{(\omega_m^{(1)^2} - \omega_k^{(2)^2})} \omega_m^{(1)} e^{-\beta_m^{(1)} \omega_m^{(1)} t} \sin(\omega_m^{(1)} t) \Bigg\} * F$$

Problem Three

$$\ddot{u}_z^{(2)}(t) = -\Bigg\{ \frac{\hat{\phi}_{zn}^{(2)} D_{mn}}{(\omega_n^{(2)^2} - \omega_m^{(1)^2})} \omega_n^{(2)} e^{-\beta_n^{(2)} \omega_n^{(2)} t)} \sin(\omega_n^{(2)} t)$$

$$+ \frac{\hat{\phi}_{zn}^{(2)} D_{mn}}{(\omega_m^{(1)^2} - \omega_n^{(2)^2})} \omega_m^{(1)} e^{-\beta_m^{(1)} \omega_m^{(1)} t} \sin(\omega_m^{(1)} t) \Bigg\} * F$$

Notice that in the sub-problem approach for impact, no terms need be discarded.

4.3 The Tuned System

Suppose now that one natural frequency of the structure ($\omega_m^{(1)}$) is close to a natural frequency of the equipment ($\omega_n^{(2)}$). That is

$$\omega_m^{(1)^2} = (1 + \alpha_{mn})\omega_n^{(2)^2}$$

where α_{mn} is the detuning parameter and is of first order. If one were to directly use the transformation **T** discussed in Section 4.1 in a perturbation solution of the eigenproblem (4.1.1), we would be led to a *mixed problem.* As we have mentioned before, such systems are very complex and are not usually treated in a general way. The methodology of the previous section will be used to resolve this problem.

Problems One and Two of Section 4.2 are unaffected by the introduction of tuning. In Problem Three, however, if $\omega_m^{(1)}$ is exactly $\omega_n^{(2)}$ (the case of $\alpha_{mn} = 0$), the response becomes indeterminate. Also the 2 × 2 eigenproblem associated with Problem Three in the case of tuning will be degenerate rather than non-degenerate as we had previously. Both of these facts force us to take an alternate solution approach in Problem Three, when there is tuning or slight detuning between the *m*th structure and *n*th equipment natural frequency.

Let

$$\mathbf{u} = \mathbf{T}_3\mathbf{Q} \tag{4.3.1}$$

where $\mathbf{T}_3$ is given in (4.2.3) and

$$\mathbf{Q} = \begin{bmatrix} q^{(2)} \\ q^{(1)} \end{bmatrix}$$

We explore the ground shock problem first. Substituting (4.3.1) into (4.1.7) and pre-multiplying by $\mathbf{T}_3^T$ results in

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{q}^{(2)} \\ \ddot{q}^{(1)} \end{bmatrix} + \begin{bmatrix} 2\beta_n^{(2)}\omega_n^{(2)} & B_{mn} \\ B_{mn} & 2\beta_m^{(1)}\omega_m^{(1)} \end{bmatrix}\begin{bmatrix} \dot{q}^{(2)} \\ \dot{q}^{(1)} \end{bmatrix} + \begin{bmatrix} \omega_n^{(2)^2} & A_{mn} \\ A_{mn} & \omega_m^{(1)^2} \end{bmatrix}\begin{bmatrix} q^{(2)} \\ q^{(1)} \end{bmatrix} = \mathbf{T}_3^T\mathbf{K}\mathbf{R}\,u_g + \mathbf{T}_3^T\mathbf{C}\mathbf{R}\,\dot{u}_g$$

where

$$B_{mn} = \hat{\boldsymbol{\phi}}_m^{(1)T}\mathbf{c}^{(12)}\hat{\boldsymbol{\phi}}_n^{(2)}$$

$$A_{mn} = \hat{\boldsymbol{\phi}}_m^{(1)T}\mathbf{k}^{(12)}\hat{\boldsymbol{\phi}}_n^{(2)}$$

We now solve this two-degree-of-freedom system by Laplace transforms. This results in

$$\bar{q}^{(2)} = \left[N(p)/D(p)\right]\bar{\ddot{u}}_g \tag{4.3.2}$$

where p is the Laplace transform parameter and a bar over a function denotes its Laplace transform. Since $\mathbf{T}_3^T\mathbf{CR} << \mathbf{T}_3^T\mathbf{KR}$ we shall neglect it. Then

$$N(p) = (p^2+2\beta_m^{(1)}\omega_m^{(1)}p+\omega_m^{(1)2})f^{(2)} - (B_{mn}p+A_{mn})f^{(1)}$$

$$D(p) = (p^2+2\beta_n^{(2)}\omega_n^{(2)}p+\omega_n^{(2)^2})(p^2+2\beta_m^{(1)}\omega_m^{(1)}p+\omega_m^{(1)^2}) - (B_{mn}p+A_{mn})^2$$

and

$$f^{(2)} = \hat{\boldsymbol{\phi}}_n^{(2)T}(\mathbf{k}^{(2)}\mathbf{r}^{(2)} + \mathbf{k}^{(21)}\mathbf{r}^{(1)})$$

$$f^{(1)} = \hat{\boldsymbol{\phi}}_m^{(1)T}(\mathbf{k}^{(12)}\mathbf{r}^{(2)} + \mathbf{k}^{(1)}\mathbf{r}^{(1)})$$

We now prove that $f^{(2)} = 0$. The vector $\mathbf{R}$ as given in (4.1.8) is the rigid body displacement field of the combined system when the ground is given a unit displacement. Since this is an unconstrained rigid body motion, no internal forces develop. We now fix the ground and impose the above displacement field $\mathbf{R}$ on the combined system. To do this would require forces applied to the structure degrees of freedom. No forces would be required in the equipment, however. Thus

$$\mathbf{KR} = \mathbf{F}$$

where

$$\mathbf{F} = \begin{bmatrix} 0 \\ \mathbf{F}^{(1)} \end{bmatrix}$$

Performing the matrix multiplication we have the following sets of equations

$$\begin{aligned} \mathbf{k}^{(2)}\mathbf{r}^{(2)} + \mathbf{k}^{(21)}\mathbf{r}^{(1)} &= 0 \\ \mathbf{k}^{(12)}\mathbf{r}^{(2)} + \mathbf{k}^{(1)*}\mathbf{r}^{(1)} &= \mathbf{F}^{(1)} \end{aligned} \tag{4.3.3}$$

One could calculate $\mathbf{F}^{(1)}$ from the second of equations (4.3.3) given the rigid body displacement field $\mathbf{R}$. From the first of equations (4.3.3) we see

$$f^{(2)} = 0$$

Let us examine $f^{(1)}$ further. In the expression for $f^{(1)}$, we see that the term $\mathbf{k}^{(12)}\mathbf{r}^{(2)}$ is of the order of $\mathbf{m}^{(2)}$ while the term $\mathbf{k}^{(1)}\mathbf{r}^{(1)}$ is of the order of $\mathbf{m}^{(1)}$. Since $\mathbf{m}^{(2)} << \mathbf{m}^{(1)}$ we will neglect this term and are left with

$$f^{(1)} \approx \hat{\boldsymbol{\phi}}_m^{(1)T}\mathbf{k}^{(1)}\mathbf{r}^{(1)} = \omega_m^{(1)^2}\hat{\boldsymbol{\phi}}_m^{(1)T}\mathbf{m}^{(1)}\mathbf{r}^{(1)} = \omega_m^{(1)^2}P_m^{(1)}$$

This equivalent two-degree-of-freedom secondary system will be analyzed by the approach used in [15]. First, the zeros of $D(p)$ are determined. The results are identical to those in Section 3 of [15] if we replace the ω, ξ, γ, β, and B of that report with $\omega_n^{(2)}$, $\alpha_{mn}/2$, γ_{mn}, $\beta_n^{(2)}$, and $\beta_m^{(1)}$ respectively. Where

$$\gamma_{mn} = A_{mn}^2/\omega_n^{(2)^4}$$

is an "effective" mass ratio. Of course, the zeros of $D(p)$ are closely related to the natural frequencies of this equivalent two-degree-of-freedom system. There is a first order shift in frequency here which was not present in Problems One and Two. An inversion of (4.3.2) by the approach used in [15] leads to a Green's function associated with equipment response, $\ddot{q}_G^{(2)}(t)$, of the type discussed in Section 4 of [15]. The expression for $\ddot{q}_G^{(2)}(t)$ is given by substituting $\ddot{q}_G^{(2)}(t)$ for the $\ddot{u}_G(t)$ of Section 4 of [15], as well as performing the substitutions previously mentioned above.

The solution to Problem Three when there is tuning is given below (for the case of ground shock)

$$\ddot{u}_z^{(2)}(t) = \left\{ -\hat{\phi}_{zn}^{(2)} \frac{C_{mn}}{\omega_n^{(2)^2}} \ddot{q}_G^{(2)}(t) \right\} * \ddot{u}_g \tag{4.3.4}$$

The comparable response for the impact case can be established in a similar manner and is given below.

$$\ddot{u}_z^{(2)}(t) = \left\{ \hat{\phi}_{zn}^{(2)} \frac{D_{mn}}{\omega_n^{(2)^2}} \ddot{q}_G^{(2)}(t) \right\} * F \tag{4.3.5}$$

The complete equipment response is given by adding the response above to the results of Problems One and Two previously obtained.

We summarize the results for equipment response as follows

Ground Shock

$$\begin{aligned}
\ddot{u}_z^{(2)}(t) = \Bigg\{ & \sum_{k=1}^{n^{(2)}} \hat{\phi}_{zk}^{(2)} P_k^{(2)} \omega_k^{(2)} e^{-\beta_k^{(2)}\omega_k^{(2)} t} \sin(\omega_k^{(2)} t) \\
& + \sum_{k=1}^{n^{(2)}} \sum_{\substack{l=1 \\ l \neq m}}^{n^{(1)}} \frac{\hat{\phi}_{zk}^{(2)} C_{lk}}{(\omega_k^{(2)^2} - \omega_l^{(1)^2})} \omega_k^{(2)} e^{-\beta_k^{(2)}\omega_k^{(2)} t} \sin(\omega_k^{(2)} t) \\
& + \sum_{k=1}^{n^{(2)}} \sum_{\substack{l=1 \\ l \neq m}}^{n^{(1)}} \frac{\hat{\phi}_{zk}^{(2)} C_{lk}}{(\omega_l^{(1)^2} - \omega_k^{(2)^2})} \omega_l^{(1)} e^{-\beta_l^{(1)}\omega_l^{(1)} t} \sin(\omega_l^{(1)} t) \\
& + \sum_{\substack{k=1 \\ k \neq n}}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} C_{mk}}{(\omega_k^{(2)^2} - \omega_m^{(1)^2})} \omega_k^{(2)} e^{-\beta_k^{(2)}\omega_k^{(2)} t} \sin(\omega_k^{(2)} t) \\
& + \sum_{\substack{k=1 \\ k \neq n}}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} C_{mk}}{(\omega_m^{(1)^2} - \omega_k^{(2)^2})} \omega_m^{(1)} e^{-\beta_m^{(1)}\omega_m^{(1)} t} \sin(\omega_m^{(1)} t) \\
& + \left[-\hat{\phi}_{zn}^{(2)} \frac{C_{mn}}{\omega_n^{(2)^2}} \ddot{q}_G^{(2)}(t) \right] \Bigg\} * \ddot{u}_g
\end{aligned} \tag{4.3.6}$$

Impact

$$\begin{aligned}
\ddot{u}_z^{(2)}(t) = -\Bigg\{ & \sum_{k=1}^{n^{(2)}} \sum_{\substack{l=1 \\ l \neq m}}^{n^{(1)}} \frac{\hat{\phi}_{zk}^{(2)} D_{lk}}{(\omega_k^{(2)^2} - \omega_l^{(1)^2})} \omega_k^{(2)} e^{-\beta_k^{(2)}\omega_k^{(2)} t} \sin(\omega_k^{(2)} t) \\
& + \sum_{k=1}^{n^{(2)}} \sum_{\substack{l=1 \\ l \neq m}}^{n^{(1)}} \frac{\hat{\phi}_{zk}^{(2)} D_{lk}}{(\omega_l^{(1)^2} - \omega_k^{(2)^2})} \omega_l^{(1)} e^{-\beta_l^{(1)}\omega_l^{(1)} t} \sin(\omega_l^{(1)} t)
\end{aligned}$$

$$+\sum_{\substack{k=1\\k\neq n}}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} D_{mk}}{(\omega_k^{(2)^2} - \omega_m^{(1)^2})} \omega_k^{(2)} e^{-\beta_k^{(2)}\omega_k^{(2)} t} \sin(\omega_k^{(2)} t)$$

$$+\sum_{\substack{k=1\\k\neq n}}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} D_{mk}}{(\omega_m^{(1)^2} - \omega_k^{(2)^2})} \omega_m^{(1)} e^{-\beta_m^{(1)}\omega_m^{(1)} t} \sin(\omega_m^{(1)} t)$$

$$+\left.\left[\hat{\phi}_{zn}^{(2)} \frac{D_{mn}}{\omega_n^{(2)^2}} \ddot{q}_G^{(2)}(t)\right]\right\} * F \qquad (4.3.7)$$

For completeness, we also list the Green's function $\ddot{u}_G(t)$ from [15].

$$\ddot{u}_G(t) = \frac{\omega}{\bar{\lambda}^2+\bar{\mu}^2} e^{-(\beta+B)\omega t/2}\left[\bar{\lambda}\sinh\frac{\bar{\mu}}{2}\omega t\cos\frac{\bar{\lambda}}{2}\omega t\sin\left(1+\frac{\xi}{2}\right)\omega t\right.$$

$$-\bar{\lambda}\cosh\frac{\bar{\mu}}{2}\omega t\sin\frac{\bar{\lambda}}{2}\omega t\cos\left(1+\frac{\xi}{2}\right)\omega t - \bar{\mu}\sinh\frac{\bar{\mu}}{2}\omega t\cos\frac{\bar{\lambda}}{2}\omega t\cos\left(1+\frac{\xi}{2}\right)\omega t$$

$$\left.-\bar{\mu}\cosh\frac{\bar{\mu}}{2}\omega t\sin\frac{\bar{\lambda}}{2}\omega t\sin\left(1+\frac{\xi}{2}\right)\omega t\right]$$

$$\bar{\lambda} = \frac{1}{\sqrt{2}}\{\{[\gamma+\xi^2-(\beta-B)^2]^2+4\xi^2(\beta-B)^2\}^{1/2}+[\gamma+\xi^2-(\beta-B)^2]\}^{1/2}$$

$$\bar{\mu} = \frac{1}{\sqrt{2}}\{\{[\gamma+\xi^2-(\beta-B)^2]^2+4\xi^2(\beta-B)^2\}^{1/2}-[\gamma+\xi^2-(\beta-B)^2]\}^{1/2}$$

For particular situations, e.g., the system is undamped or damped with perfect tuning, the expression for $\ddot{u}_G(t)$ greatly simplifies; see [15].

4.4 Response Spectrum Analysis

The results of the previous section, as given in (4.3.6) and (4.3.7), can be used in a time domain analysis. In most instances the quantity of interest is maximum equipment acceleration, not the equipment acceleration as a function of time. The purpose of this section is to express the maximum equipment acceleration in terms of properties of the system and ground motion (or impact). The results will be developed in terms of design spectra since these are the input characterizations most readily available to the structural designer.

Consider the equipment response of the tuned system when the structure is subjected to short duration ground shock or impact. In order for the impulse to be "short" we require its duration to be small in comparison to the period of the two close frequencies. The maximum equipment response in this case is governed by the beating phenomenum associated with the two close frequencies. Considering only this effect and using our "equivalent" two-degree-of-freedom system properties, we have the following result for maximum equipment acceleration when the structure is subjected to ground shock

$$\left|\ddot{u}_z^{(2)}\right|_{max} = \frac{\left|\hat{\phi}_{zn}^{(2)}\dfrac{C_{mn}}{\omega_n^{(2)2}}\right| e^{-\kappa_{mn}}}{(\gamma_{mn} + \alpha_{mn}^2/4 + 4\beta_n^{(2)}\beta_m^{(1)})^{1/2}} S_A\left(\frac{\omega_n^{(2)}+\omega_m^{(1)}}{2},0\right) \qquad (4.4.1)$$

where $S_A(\omega,\beta)$ is the pseudo-accleration response spectrum for the input ground motion $\ddot{u}_g(t)$ evaluated at frequency ω and damping factor β and

$$\kappa_{mn} = arctan(\zeta_{mn})/\zeta_{mn}$$

$$\zeta_{mn} = \left[\gamma_{mn} + \alpha_{mn}^2/4 - (\beta_n^{(2)} - \beta_m^{(1)})^2\right]^{1/2} / (\beta_n^{(2)} + \beta_m^{(1)})$$

The corresponding result for impact is

$$\left|\ddot{u}_z^{(2)}\right|_{max} = \frac{\left|\hat{\phi}_{zn}^{(2)}\dfrac{D_{mn}}{\omega_n^{(2)2}}\right| e^{-\kappa_{mn}}}{(\gamma_{mn} + \alpha_{mn}^2/4 + 4\beta_n^{(2)}\beta_m^{(1)})^{1/2}} \bar{S}_A\left(\frac{\omega_n^{(2)}+\omega_m^{(1)}}{2},0\right) \qquad (4.4.2)$$

where $\bar{S}_A(\omega,\beta)$ is the pseudo-acceleration response spectrum for the input $F(t)/M_n$

evaluated at frequency ω and damping factor β. The modal mass M_n is, throughout the development, unity for all n.

When the duration of the input is of the same order as the period of the two close frequencies, but the peak response occurs after the excitation has finished, an improved estimate of maximum equipment acceleration is obtained by including the average damping $(\beta_n^{(2)}+\beta_m^{(1)})/2$ in the above response spectrum.

Caution must be exercised in the use of the above formulas. Although there are large amplifications associated with beating, the beat envelope is scaled by the coefficients

$$\hat{\phi}_{zn}^{(2)} \frac{C_{mn}}{\omega_n^{(2)2}} \tag{4.4.3}$$

for ground shock and

$$\hat{\phi}_{zn}^{(2)} \frac{D_{mn}}{\omega_n^{(2)2}} \tag{4.4.4}$$

for impact. If these quantities happen to be small in comparison to similar terms in (4.3.6) and (4.3.7), then the above results (4.4.1) and (4.4.2) will underestimate the maximum response. In such instances, the complete expressions (4.3.6) and (4.3.7) should be evaluated and the maximum in each degree of freedom obtained by a numerical scheme. Alternatively, the convolution with the input can be replaced by an "effective impulse" for excitation that is primarily of delta function character. This "effective impulse" would be the area under the input-time history (a constant). The pseudo-acceleration response spectra used in (4.4.1) and (4.4.2) essentially determines the "effective impulse".

The coefficients in (4.4.3) and (4.4.4) can become "small" under a variety of conditions, as they depend upon symmetries in the structure or equipment as well as the manner in which the two are connected. How the input excites the structure also has an effect.

Assume now that the structure is subjected to earthquake excitation. This type of excitation is generally characterized as a stationary random process. In a random vibration analysis of response, a system with well spaced frequencies (the completely detuned system) exhibits negligible correlation between modes thus allowing the application of a standard summation rule. Applying the square root of the sum of the squares procedure to (4.1.12) results in

$$\left|\ddot{u}_z^{(2)}\right|_{max} = \left\{ \sum_{k=1}^{n^{(2)}} \left[\left(\hat{\phi}_{zk}^{(2)} P_k^{(2)} + \sum_{l=1}^{n^{(1)}} \frac{\hat{\phi}_{zk}^{(2)} C_{lk}}{(\omega_k^{(2)2} - \omega_l^{(1)2})} \right) S_A(\omega_k^{(2)}, \beta_k^{(2)}) \right]^2 + \sum_{l=1}^{n^{(1)}} \left[\frac{\hat{\phi}_{zk}^{(2)} C_{lk}}{(\omega_l^{(1)2} - \omega_k^{(2)2})} S_A(\omega_l^{(1)}, \beta_l^{(1)}) \right]^2 \right\}^{1/2} \tag{4.4.5}$$

Correlation between the two close frequencies of the tuned system is significant (see [4] or [5]) thus prohibiting the use of the square root of the sum of the squares procedure. Crandall and Mark [2] analyze a two-degree-of-freedom system and determine the response in terms of the system parameters and the power spectral density function of the stationary random excitation of the primary system.

The power spectral density S_0 can be interpreted in terms of a response spectrum. Assuming the root mean square response is the peak response (this is the commonly accepted practice of ignoring "peak factors"), we have from [2] eq. (2.44)

$$S_A^2(\omega, \zeta) \equiv E\left[\dot{y}^2\right] = \frac{\pi}{2} \frac{S_0}{\zeta} \omega$$

The maximum equipment response due to the two close frequencies can now be determined directly in terms of the response spectrum and our "equivalent" two-degree-of-freedom system properties. Specializing [2] eq. (2.57) for the parameters of our perfectly tuned two-degree-of-freedom system, we have the following result for the maximum equipment response due to the tuned frequencies

$$\left|\ddot{u}_z^{(2)}\right|_{max} = \frac{\left| \hat{\phi}_{zn}^{(2)} \dfrac{C_{mn}}{\omega_n^{(2)2}} \right|}{\sqrt{2}\left(\gamma_{mn} + 4\beta_n^{(2)}\beta_m^{(1)}\right)^{1/2}} S_A\left(\omega_n^{(2)}, \frac{\beta_n^{(2)} + \beta_m^{(1)}}{2}\right)$$

This result is then combined with the contribution to maximum equipment response from the non-tuned frequencies given below

$$\left|\ddot{u}_z^{(2)}\right|_{max} = \left\{ \sum_{\substack{k=1 \\ k \neq n}}^{n^{(2)}} \left[\hat{\phi}_{zk}^{(2)} \left(P_k^{(2)} + \sum_{l=1}^{n^{(1)}} \frac{C_{lk}}{(\omega_k^{(2)^2} - \omega_l^{(1)^2})} \right) S_A(\omega_k^{(2)}, \beta_k^{(2)}) \right]^2 \right.$$

$$+ \left[\hat{\phi}_{zn}^{(2)} \left(P_n^{(2)} + \sum_{\substack{l=1 \\ l \neq m}}^{n^{(1)}} \frac{C_{ln}}{(\omega_n^{(2)^2} - \omega_l^{(1)^2})} \right) \right]^2 S_A^2(\omega_n^{(2)}, \beta_n^{(2)})$$

$$+ \sum_{\substack{l=1 \\ l \neq m}}^{n^{(1)}} \left[\sum_{k=1}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} C_{lk}}{(\omega_l^{(1)^2} - \omega_k^{(2)^2})} S_A(\omega_l^{(1)}, \beta_l^{(1)}) \right]^2$$

$$\left. + \left[\sum_{\substack{k=1 \\ k \neq n}}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} C_{mk}}{(\omega_m^{(1)^2} - \omega_k^{(2)^2})} \right]^2 S_A^2(\omega_m^{(1)}, \beta_m^{(1)}) \right\}^{1/2}$$

by the square root of the sum of the squares procedure.

4.5 Numerical Studies

A variety of numerical tests to determine the validity of the method presented in the previous sections were performed on the equipment-structure system of Figure Three. The structure mass and stiffness were set to one and 100 respectively. The natural frequencies of the structure were then .708, 1.98, and 2.87 cycles per second (cps). The undamped system was checked for a variety of m/M ratios and tunings. The natural frequencies (ω_i^2) and mode shapes ($\mathbf{x}_i$) for the combined system were calculated using the CAL [17] computer program. The mode shapes were then used to calculate a "response matrix" associated with the undamped ground shaking problem. The response is given as

$$\ddot{\mathbf{u}} = \bar{\mathbf{R}}\mathbf{v} * \ddot{u}_g$$

where $\bar{\mathbf{R}}$ is the "response matrix" whose columns are the vectors $\mathbf{x}_i\Gamma_i$, $\Gamma_i = \mathbf{x}_i^T\mathbf{MR}/\mathbf{x}_i^T\mathbf{Mx}_i$ [(**M**, **R** are defined in (4.1.2) and (4.1.8)] and **v** is a vector whose ith entry is given by

$$v_i = \omega_i \sin(\omega_i t) \;\; ; \; i=1,2, \ldots, N$$

where N is the dimension of the combined system.

From the methodology of this chapter, the natural frequencies of the combined system are approximated by the natural frequencies of the sub-systems when these frequencies are well spaced. When a natural frequency of the equipment system is close to a natural frequency of the structure system, a first order shift occurs in the combined system frequencies which must be taken into account. For perfect tuning ($\omega_m^{(1)^2} = \omega_n^{(2)^2}$) of the undamped system the two close frequencies of the combined system are given by

$$\omega_{\pm}^2 = \omega_n^{(2)^2}(1 \pm \gamma_{mn}^{1/2})$$

These frequencies are then arranged in numerically increasing order and compared to the CAL program results. Tables One and Two show excellent correlation between the "method" and "exact" results. In Table One and subsequent tables $\epsilon = 10^{-1}\frac{m}{M}$, the percent difference is given by $\frac{(METHOD-CAL)}{CAL}\times 100$, and the starred numbers are associated with the two close frequencies. The contribution by a particular non-tuned frequency to the equipment part of the response matrix is readily seen from (4.3.6) to be

$$\sum_{k=1}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} C_{lk}}{(\omega_l^{(1)^2} - \omega_k^{(2)^2})}$$

for the non-tuned structure frequencies and

$$\hat{\phi}_{zk}^{(2)} \left[P_k^{(2)} + \sum_{l=1}^{n^{(1)}} \frac{C_{lk}}{(\omega_k^{(2)^2} - \omega_l^{(1)^2})} \right]$$

for the non-tuned equipment frequencies.

The contributions to the response matrix from the tuned frequency is more complicated. Contributions to the response matrix associated with $\omega_{\pm}$ can be found from a classical perturbation solution of the undamped Problem Three to be

$$\pm \frac{\hat{\phi}_{zn}^{(2)} C_{mn}}{\gamma_{mn}^{1/2} \omega_n^{(2)^2}}$$

From (4.3.6) we see there is also a non-dominant contribution associated with the tuned equipment frequency of

$$\hat{\phi}_{zn}^{(2)} \left[P_n^{(2)} + \sum_{\substack{l=1 \\ l \neq m}}^{n^{(1)}} \frac{C_{ln}}{(\omega_n^{(2)^2} - \omega_l^{(1)^2})} \right] \tag{4.5.1}$$

and associated with the tuned structure frequency

$$\sum_{\substack{k=1 \\ k \neq n}}^{n^{(2)}} \frac{\hat{\phi}_{zk}^{(2)} C_{mk}}{(\omega_m^{(1)^2} - \omega_k^{(2)^2})} \tag{4.5.2}$$

In the case of perfect tuning it is difficult to determine which is associated with ω_+ and which with ω_-. Let us assume the equipment and structure natural frequencies are slightly detuned. Then

$$\omega_{\pm} = \omega_n^{(2)^2}(1 + \alpha_{mn}/2 \pm \mu_{mn})$$

where

$$\mu_{mn} = (\gamma_{mn} + \frac{\alpha_{mn}^2}{4})^{1/2}$$

As discussed in [15] the tuned structure frequency shifts to ω_+ while the tuned equipment frequency shifts to ω_-. Thus in the limiting case of α_{mn} identically zero, the non-dominant contribution to the column in the response matrix associated with ω_+ is (4.5.2) while that asociated with ω_- is (4.5.1). The vectors calculated above when arranged according to numerically increasing frequency are compared to the equipment part of $\bar{\bar{\mathbf{R}}}$. Tables Three and Four show good correlation between the "method" and "exact" results. It is unfortunate that the quantities associated with the two close frequencies exhibit the greatest difference from the "exact" result since they combine to form the dominant response. The errors increase with increasing effective mass ratio γ_{mn}, as one would expect. For small γ_{mn}, the method approach certainly produces highly accurate frequencies and response matrices.

The equipment frequencies for the grossly detuned system are .942 and 1.63 cps.

Time history analyses by a Newmark integration scheme which solves directly for acceleration (see [6]) were then performed on the damped equipment-structure system of Figure Three. The following equipment-structure properties were taken from [13] and are considered typical in nuclear power plant construction

$$m = 1\times10^5 kg \qquad M = 2\times10^7 kg$$

$$k = 5.942\times10^7 kgf/m \qquad K = 6\times10^{10} kgf/m$$

$$\beta_j^{(2)} = .01 \qquad \beta_i^{(1)} = .04$$

for $j = 1,2$ and $i = 1,2,3$. The structure natural frequencies are 3.88, 10.9 and 15.7 cps. Choosing the equipment stiffness in this manner tunes the first structure natural frequency to the first equipment natural frequency. This particular tuning was chosen because it gives the largest equipment response and also allows the use of (4.4.1) and (4.4.2) instead of the convolution necessary in (4.3.6) or (4.3.7).

A Phantom F-4 jet impact force-time history, also taken from [13], was used as an acceleration ground shock history (maximum acceleration of 1-g.). The properties of the equipment-structure system given above produce an effective mass ratio (γ_{11}) of approximately .003. The equipment mass and stiffness were then scaled by 10 and 0.1 to achieve effective mass ratios of approximately .03 and .0003 respectively. The tuning is unaffected. The response spectrum values were computed using [11]. Table Five shows excellent correlation between the Newmark and Method results.

The same Phantom F-4 jet impact was applied to the first structure degree of freedom. The properties of the equipment-structure system were those first mentioned in (4.5.3). Again excellent correlation between the derived result (4.4.2) and the Newmark integration scheme were obtained (see Table Six).

The first 30 seconds of the El Centro record (see Figure Four) were used in a verification of the last results presented in Section 4.4. The equipment-structure system was that of Figure Three with

$$m = 1\times10^{-3}kg \qquad M = 1\,kg$$

$$k = 1.981\times10^{-2}kgf/m \qquad K = 1\times10^{2}kgf/m$$

$$\beta_j^{(2)} = .01 \qquad \beta_i^{(1)} = .04$$

These properties were chosen to reduce the cost of the Newmark integration scheme which in order to insure accurate results requires fifty time steps per minimum period. Excellent results were obtained (see Table Seven). The efficiency of the method is certainly realized given the requirements above on the Newmark scheme.

Chapter Five

THREE-DEGREE-OF-FREEDOM TERTIARY SYSTEMS

The system of Figure Five, where $m_1 << m_2 << m_3$ is analyzed in the subsequent sections. The governing equation for the response of the very light equipment item (m_3) when the system is subjected to an arbitrary ground acceleration $\ddot{u}_g(t)$ is given in transform space by

$$\bar{\ddot{u}}_3 = \left[N(p)/D(p)\right]\bar{\ddot{u}}_g \tag{5.0.1}$$

where p is the Laplace transform parameter and

$$\begin{aligned}
N(p) &= (2\beta_1\omega_1 p + \omega_1^2)(2\beta_2\omega_2 p + \omega_2^2)(2\beta_3\omega_3 p + \omega_3^2) \\
D(p) &= (p^2 + 2\beta_1\omega_1 p + \omega_1^2)(p^2 + 2\beta_2\omega_2 p + \omega_2^2)(p^2 + 2\beta_3\omega_3 p + \omega_3^2) \\
&\quad + \gamma_{21}p^2(p^2 + 2\beta_2\omega_2 p + \omega_2^2)(2\beta_3\omega_3 p + \omega_3^2) \\
&\quad + \gamma_{32}p^2(p^2 + 2\beta_1\omega_1 p + \omega_1^2)(2\beta_3\omega_3 p + \omega_3^2) \\
&\quad + \gamma_{31}p^2(2\beta_2\omega_2 p + \omega_2^2)(2\beta_3\omega_3 p + \omega_3^2)
\end{aligned} \tag{5.0.2}$$

The mass ratios are $\gamma_{31} = \dfrac{m_3}{m_1}$, $\gamma_{32} = \dfrac{m_3}{m_2}$, and $\gamma_{21} = \dfrac{m_2}{m_1}$. These quantities are not independent as $\gamma_{31} = \dfrac{\gamma_{32}}{\gamma_{21}}$.

5.1 The Uncoupled System

We begin our study of tertiary systems by examining the uncoupled system. Setting all mass ratios in (5.0.2) to zero, eliminates the interaction effect between the sub-systems. This is the conventional floor spectrum analysis procedure, where the primary system motion is determined as if the other systems were not present and used as the input to the secondary system, etc. An analysis of this type (a cascaded system analysis) will lead to an overestimation of equipment response when the system is tuned or nearly tuned, since energy transfer mechanisms are ignored.

Undamped, Slightly Detuned Case

For the undamped slightly detuned uncoupled system one sets all damping coefficients and mass ratios in the governing equation (5.0.2) to zero and defines the detuning parameters

$$\alpha_{12} = \frac{\omega_1^2 - \omega_2^2}{\omega_2^2} \quad , \quad \alpha_{32} = \frac{\omega_3^2 - \omega_2^2}{\omega_2^2}$$

If the detuning parameters are zero, one has a triple zero of $D(p)$ at

$$p^2 = -\omega_2^2$$

Since the detuning parameters are small, we assume the zeros of $D(p)$ will be slightly perturbed from the above. That is

$$p^2 = -\omega_2^2(1 + \eta)$$

where $\eta << 1$. Solving for the zeros of $D(p)$ we get the roots

$$\eta = 0, \ \alpha_{12}, \ \alpha_{32}$$

In order to invert (5.0.1), we first write the denominator as

$$D(p) = (p-p_1)(p-p'_1)(p-p_2)(p-p'_2)(p-p_3)(p-p'_3)$$

where

$$p_1 = i\omega_2$$

$$p_2 = i\omega_2(1 + \frac{\alpha_{12}}{2})$$

$$p_3 = i\omega_2(1 + \frac{\alpha_{32}}{2})$$

and a prime indicates the complex conjugate. We are able to do this because the zeros of $D(p)$ occur in complex conjugate pairs. By evaluating the residues of (5.0.1) at each pole and collecting complex conjugate terms in pairs, we obtain the following result, correct to dominant order

$$\ddot{u}_3(t) = \left\{ \frac{\omega_2}{\alpha_{12}\alpha_{32}\alpha_{13}} \left[\alpha_{13}\sin(\omega_2 t) + \alpha_{32}\sin\left[\omega_2(1 + \frac{\alpha_{12}}{2})t\right] - \alpha_{12}\sin\left[\omega_2(1 + \frac{\alpha_{32}}{2})t\right]\right]\right\} * \ddot{u}_g$$

where $\alpha_{13} = \alpha_{12} - \alpha_{32}$ and

$$f(t) * \ddot{u}_g = \int_0^t f(t-\tau)\, \ddot{u}_g(\tau)\, d\tau$$

We notice that Green's function (the expression in the braces) becomes indeterminate as one of the detuning parameters becomes zero.

Damped, Perfectly Tuned Case

Again we set the mass ratios in (5.0.2) to zero and make the further simplification that

$$\omega_1 = \omega_2 = \omega_3 \equiv \omega$$

If the damping factors were also set to zero, we would have again a triple zero at $p^2 = -\omega^2$. Since the damping factors are small compared to one we assume the zeros of $D(p)$ will be slightly perturbed from this triple zero and thus let

$$p = i\omega(1 + \delta)$$

where δ is small compared to one. Using this transformation we have for the zeros of $D(p)$

$$\delta = i\beta_1,\ i\beta_2,\ i\beta_3$$

and

$$p_1 = i\omega(1 + i\beta_1)$$
$$p_2 = i\omega(1 + i\beta_2) \qquad (5.1.1)$$
$$p_3 = i\omega(1 + i\beta_3)$$

The roots occur in complex conjugate pairs, which leads to the following result upon inversion

$$\ddot{u}_3(t) = \left\{ -\frac{\omega}{4}\sin(\omega t)\left[\frac{e^{-\beta_1\omega t}}{(\beta_1-\beta_2)(\beta_1-\beta_3)} + \frac{e^{-\beta_2\omega t}}{(\beta_2-\beta_1)(\beta_2-\beta_3)} + \frac{e^{-\beta_3\omega t}}{(\beta_3-\beta_1)(\beta_3-\beta_2)} \right]\right\} * \ddot{u}_g$$

Damped, Slightly Detuned Case

A similar analysis of the damped slightly detuned uncoupled system yields the following zeros of $D(p)$ and their complex conjugates (assuming the damping and detuning are of the same order and still small compared to one).

$$p_1 = i\omega_2(1 + \frac{\alpha_{12}}{2} + i\beta_1)$$
$$p_2 = i\omega_2(1 + i\beta_2)$$
$$p_3 = i\omega_2(1 + \frac{\alpha_{32}}{2} + i\beta_3)$$

These yield upon inversion (to dominant order)

$$\ddot{u}_3(t) = \left\{ \omega_2 e^{-\beta_1\omega_2 t}\left[\frac{a_1x_1 + b_1y_1}{a_1^2 + b_1^2} \right] + \omega_2 e^{-\beta_2\omega_2 t}\left[\frac{a_2x_2 + b_2y_2}{a_2^2 + b_2^2} \right] + \omega_2 e^{-\beta_3\omega_2 t}\left[\frac{a_3x_3 + b_3y_3}{a_3^2 + b_3^2} \right]\right\} * \ddot{u}_g$$

where

$$a_1 = -2\left[\alpha_{12}(\beta_1-\beta_3) + \alpha_{13}(\beta_1-\beta_2)\right]$$
$$a_2 = -2\left[\alpha_{12}(\beta_2-\beta_3) + \alpha_{32}(\beta_2-\beta_1)\right]$$

$$a_3 = -2\left[\alpha_{13}(\beta_2-\beta_3) + \alpha_{32}(\beta_3-\beta_2)\right]$$

$$b_1 = -\left[4(\beta_1-\beta_2)(\beta_1-\beta_3) + \alpha_{12}\alpha_{13}\right]$$

$$b_2 = -\left[4(\beta_1-\beta_2)(\beta_2-\beta_3) + \alpha_{12}\alpha_{13}\right]$$

$$b_3 = -\left[4(\beta_1-\beta_3)(\beta_2-\beta_3) + \alpha_{13}\alpha_{32}\right]$$

with

$$x_1 = \cos\left[\omega_2(1 + \frac{\alpha_{12}}{2})t\right] \;;\; y_1 = \sin\left[\omega_2(1 + \frac{\alpha_{12}}{2})t\right]$$

$$x_2 = \cos(\omega_2 t) \;;\; y_2 = \sin(\omega_2 t)$$

$$x_3 = \cos\left[\omega_2(1 + \frac{\alpha_{32}}{2})t\right] \;;\; y_3 = \sin\left[\omega_2(1 + \frac{\alpha_{32}}{2})t\right]$$

One sees that the expression for tertiary system response greatly increases in complexity as the number of parameters is increased. Thus only under limited conditions will we be able to obtain closed form expressions for response when interaction is included (the case of greater interest).

5.2 The Effect of Interaction

Undamped, Perfectly Tuned Case

For the undamped perfectly tuned system with tuning frequency ω, substituting

$$p^2 = -\omega^2(1+\eta) \quad ; \quad \eta << 1$$

into $D(p) = 0$ gives

$$D(\eta) = \eta^3 - (\gamma_{32} + \gamma_{21})\eta^2 - (\gamma_{32} + \gamma_{21} - \gamma_{31})\eta + \gamma_{31} = 0$$

If one assumes γ_{32} and γ_{21} are of order $\epsilon << 1$, then γ_{31} is of order ϵ^2. This leads to two zeros of $D(\eta)$ of order $\epsilon^{1/2}$ that are

$$\eta = \pm(\gamma')^{1/2} \tag{5.2.1}$$

where $\gamma' = \gamma_{32} + \gamma_{21}$ and a zero of $D(\eta)$ of order ϵ given by

$$\eta = \frac{\gamma_{31}}{\gamma'} \tag{5.2.2}$$

We will assume that the response is dominated by the frequency shift associated with the roots η that are of order $\epsilon^{1/2}$. This is equivalent to setting the small root to zero. Then

$$p_1 = i\omega\left[1 + \frac{(\gamma')^{1/2}}{2}\right]$$

$$p_2 = i\omega\left[1 - \frac{(\gamma')^{1/2}}{2}\right]$$

$$p_3 = i\omega$$

and their complex conjugates are the zeros of $D(p)$. An inversion of (5.0.1) by evaluating the residues at the poles and collecting complex conjugate terms in pairs, yields the result, correct to dominant order

$$\ddot{u}_3(t) = \left\{-\frac{2\omega}{\gamma'}\sin^2(\eta^* t)\sin(\omega t)\right\} * \ddot{u}_g \tag{5.2.3}$$

where

$$\eta^* = \omega(\gamma')^{1/2}/4.$$

The Green's function (expression in the braces above) exhibits a beating phenomena as described in [15] for the tuned secondary system. The beat envelope in this case however, is a sine squared envelope (see Figure Six) rather than the sine envelope discussed in [15]. If we had retained the shift of order ϵ given by (5.2.2), the response would take the following form

$$\ddot{u}_3(t) = \left\{ \frac{\omega}{2\gamma'} \left[\sin\omega(1+\frac{\sqrt{\gamma'}}{2})\,t + \sin\omega(1-\frac{\sqrt{\gamma'}}{2})\,t \right.\right.$$

$$\left.\left. - 2\sin\omega(1+\frac{\gamma_{31}}{2\gamma'})\,t \right]\right\} * \ddot{u}_g \tag{5.2.4}$$

A variety of numerical tests were done to determine the validity of the simplifying assumption that led to (5.2.3). The Green's function given in (5.2.4) was evaluated at specific time intervals, the maximum value and time at which this maximum occured were stored and compared to similar quantities derived from a numerical evaluation of (5.2.3). These tests were performed for a variety of mass ratios and are presented in Table Eight. The results certainly indicate that the response is not dependent on the frequency shift of order ϵ.

If we follow the procedure used in [15], we can develop from (5.2.3) the following expression for maximum equipment acceleration

$$\left|\ddot{u}_3\right|_{max} = \frac{2}{\gamma'} S_A(\omega,0)$$

where $S_A(\omega,0)$ is the pseudo-acceleration response spectrum for the input ground motion $\ddot{u}_g$ evaluated at frequency ω and zero damping. It is certainly not intuitively obvious that the amplification factor should take the form $2/\gamma'$.

Undamped, Slightly Detuned Case

For α_{12}, α_{13} of order $\epsilon^{1/2}$ and γ_{21}, γ_{32} of order ϵ, the zeros of $D(p)$ are

$$p^2 = -\omega_2^2(1 + \eta)$$

where η is given by solutions of the following equation

$$\eta^3 - (\alpha_{12} + \alpha_{32})\eta^2 - (\gamma' - \alpha_{12}\alpha_{32})\eta + (\alpha_{32}\gamma_{21} + \alpha_{12}\gamma_{32}) = 0 \tag{5.2.5}$$

In the undamped tuned case, one solution for η was of higher order than the others. This allowed us to express η directly in terms of the mass ratios [see (5.2.1) and (5.2.2)]. In general this will not be the case when detuning is introduced, as all solutions to (5.2.5) will be of order $\epsilon^{1/2}$. One special case which yields one root of order ϵ and two roots of order $\epsilon^{1/2}$ is when the mass ratios and detunings are of the order given above but combine in such a manner that $\alpha_{32}\gamma_{21} + \alpha_{12}\gamma_{32}$ is of order ϵ or higher. Since this is a very unlikely occurrence, it will not be explored further. Of course, if the detunings are of the order of γ' or higher, then they are small enough not to contribute a noticeable effect and the solution is given by (5.2.3). Similarly if γ' is of order higher than ϵ while the detunings are of order $\epsilon^{1/2}$, then the detuning dominates and the system behaves as if it were uncoupled (see Section 5.1). Given the ordering first mentioned, one could obtain a closed form solution for the three roots of (5.2.5) that are of order $\epsilon^{1/2}$ by the general solution procedure for a cubic equation (see [3]). This leads to unnecessarily complicated results, however, and it is best to solve (5.2.5) for the roots η_j j=1,2,3 by a numerical procedure given the numerical values of the mass ratios and detunings.

Once the roots η_j are obtained, we can construct Green's function in the manner previously described. The result is

$$\ddot{u}_G(t) = \omega_2\left[\frac{\sin\omega_2(1+\frac{\eta_1}{2})t}{(\eta_1-\eta_2)(\eta_1-\eta_3)} - \frac{\sin\omega_2(1+\frac{\eta_2}{2})t}{(\eta_1-\eta_2)(\eta_2-\eta_3)} - \frac{\sin\omega_2(1+\frac{\eta_3}{2})t}{(\eta_1-\eta_3)(\eta_2-\eta_3)}\right] \tag{5.2.6}$$

Plots of $\ddot{u}_G(t)$ for various mass ratios and detunings are shown in Figures Seven through Twelve. One notices that a beating phenomena is exhibited, yet it is of a highly irregular character.

One can obtain an upper bound to (5.2.6) as given below

$$\left|\ddot{u}_G\right|_{max} \leqslant \omega_2\left[\left|\frac{1}{(\eta_3-\eta_1)(\eta_1-\eta_2)}\right| + \left|\frac{1}{(\eta_1-\eta_2)(\eta_2-\eta_3)}\right| + \left|\frac{1}{(\eta_2-\eta_3)(\eta_3-\eta_2)}\right|\right]$$

An actual evalutaion of the maximum of (5.2.6) for various parameter values is compared to this upper bound in Table Nine.

Damped, Perfectly Tuned Case

For β_1, β_2, β_3 of order $\epsilon^{1/2}$ and γ_{21}, γ_{32}, of order ϵ, the zeros of $D(p)$ are

$$p_j = i\omega(1+\delta_j) \quad ; j=1,2,3$$

and the complex conjugates p'_j.

The δ_j are solutions of the following

equation:

$$\delta^3 - i\left[\beta_1+\beta_2+\beta_3\right]\delta^2 - \left[\gamma'/4 + (\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3)\right]\delta + i\left[\beta_1\beta_2\beta_3 + (\beta_1\gamma_{32}+\beta_3\gamma_{21})/4\right] = 0 \qquad (5.2.7)$$

We are again led to the difficulties previously discussed. In general, all three solutions of (5.2.7) will be of order $\epsilon^{1/2}$. If, given the above ordering, $\beta_1\beta_2\beta_3 + (\beta_1\gamma_{32}+\beta_3\gamma_{21})/4$ is of order ϵ, then two roots will be of order $\epsilon^{1/2}$ and the other root of order ϵ. The roots in this case are easily found, however the likelihood that the dampings and mass ratios will combine in such a way is slim. Of course if the β_i (i=1,2,3) are of the order of γ', then the mass ratio effect dominates and the roots are given by (to dominant order)

$$\delta_{\substack{1\\2}} = \pm\frac{(\gamma')^{1/2}}{2} \quad , \quad \delta_3 = 0$$

On the other hand if γ' is of third order in comparison to β_j, then the damping effect dominates and the roots δ_j are given by (5.1.1). Given the ordering first discussed, the most efficient solution procedure for the roots δ_j will be a numerical solution of (5.2.7) for specific numerical parameter values. Once the roots δ_j are found, where

$$\delta_j = u_j + iv_j$$

the Green's function is given by

$$\ddot{u}_3(t) = \frac{\omega_2}{4}\left\{ e^{-v_1\omega_2 t}\left[\frac{a_1x_1 + b_1y_1}{a_1^2 + b_1^2}\right] + e^{-v_2\omega_2 t}\left[\frac{a_2x_2 + b_2y_2}{a_2^2 + b_2^2}\right]\right.$$

$$\left. + e^{-v_3\omega_2 t}\left[\frac{a_3x_3 + b_3y_3}{a_3^2 + b_3^2}\right]\right\} * \ddot{u}_g \tag{5.2.8}$$

where

$$a_1 = -\Big[(v_1 - v_2)(u_1 - u_3) + (u_1 - u_2)(v_1 - v_3)\Big]$$

$$a_2 = -\Big[(v_2 - v_1)(u_2 - u_3) + (u_2 - u_1)(v_2 - v_3)\Big]$$

$$a_3 = -\Big[(v_3 - v_2)(u_3 - u_1) + (u_3 - u_2)(v_3 - v_1)\Big]$$

$$b_1 = -\Big[(u_1 - u_2)(u_1 - u_3) - (v_1 - v_2)(v_1 - v_3)\Big]$$

$$b_2 = -\Big[(u_2 - u_1)(u_2 - u_3) - (v_2 - v_1)(v_2 - v_3)\Big]$$

$$b_3 = -\Big[(u_3 - u_1)(u_3 - u_2) - (v_3 - v_1)(v_3 - v_2)\Big]$$

and

$$x_1 = \cos\Big[\omega_2(1 + u_1)t\Big] \;;\; y_1 = \sin\Big[\omega_2(1 + u_1)t\Big]$$

$$x_2 = \cos\Big[\omega_2(1 + u_2)t\Big] \;;\; y_2 = \sin\Big[\omega_2(1 + u_2)t\Big]$$

$$x_3 = \cos\Big[\omega_2(1 + u_3)t\Big] \;;\; y_3 = \sin\Big[\omega_2(1 + u_3)t\Big]$$

Damped, Slightly Detuned Case

Assume now that α_{12}, α_{32}, β_1, β_2, and β_3 are of order $\epsilon^{1/2}$ while γ_{21} and γ_{32} are of order ϵ. The zeros of $D(p)$ are

$$p_j = i\omega(1 + \delta_j) \;;\; j=1,2,3$$

and the complex conjugates p'_j. The δ_j are solutions of the following equation:

$$\Big[-8\Big]\delta^3 + \Big[8i(\beta_1+\beta_2+\beta_3)+4(\alpha_{12}+\alpha_{32})\Big]\delta^2$$

$$+\Big[8(\beta_2\beta_3+\beta_1\beta_3+\beta_1\beta_2)-4i(\beta_1+\beta_2)\alpha_{32}$$

$$-4i(\beta_2+\beta_3)\alpha_{12}-2\alpha_{12}\alpha_{32}+2\gamma'\Big]\delta+$$

$$\Big[-8i\beta_1\beta_2\beta_3-4\beta_1\beta_2\alpha_{32}-4\beta_2\beta_3\alpha_{12}+2i\beta_2\alpha_{32}\alpha_{12}$$

$$-2i\beta_1\gamma_{32}-2i\beta_3\gamma_{21}-\gamma_{32}\alpha_{12}-\gamma_{21}\alpha_{32}\Big]=0. \tag{5.2.9}$$

For this particular ordering, the solution of (5.2.9) will, in general, yield three roots of order $\epsilon^{1/2}$. These roots are best found by a numerical solution of (5.2.9) for specific numerical parameter values. Once the roots δ_j are found, where

$$\delta_j = u_j + iv_j$$

the Green's function is given by (5.2.8). If the relative orders of the detunings, dampings, and mass ratios are not as given above; then certain effects will dominate and the problem reduces to one of the particular cases previously discussed.

Chapter Six

MULTI-DEGREE-OF-FREEDOM TERTIARY SYSTEMS

A multi-degree-of-freedom tertiary system is depicted in a general way in Figure Thirteen. The system has a discrete number of degrees of freedom. The links joining the primary system [denoted (1)] to the secondary system [denoted (2)] as well as those joining the secondary system and tertiary system [denoted (3)] are rigid. The primary system has $n^{(1)}$ degrees of freedom (which include the link attachment points). The secondary system has $n^{(2)}$ degrees of freedom (which include the attachment points of the links to the tertiary system but not the attachment points of the links to the primary system). The tertiary system has $n^{(3)}$ degrees of freedom (which do not include the link attachment points). The fixed base properties of the primary system are obtained by fixing the ground and removing the secondary and tertiary systems. To obtain the "fixed base" secondary system properties, the links to the primary system are fixed and the tertiary system is removed. Similarly the "fixed base" properties of the tertiary system are based on the "fixed link" condition. We require that all elements of $\mathbf{m}^{(3)}$ are small in comparison to all elements of $\mathbf{m}^{(2)}$ which in turn are small in comparison to all elements of $\mathbf{m}^{(1)}$. A similar requirement is made of the stiffness matrices $\mathbf{k}^{(3)}$, $\mathbf{k}^{(2)}$, and $\mathbf{k}^{(1)}$ described in Section 6.1.

In the subsequent sections, expressions for the acceleration response of the very light equipment [system (3)] degrees of freedom, in terms of the properties of the sub-systems and the excitation, are determined. The completely detuned system (when all of the natural frequencies of the fixed base sub-systems are well spaced) is analyzed in Section 6.1. The tuned tertiary system (when a natural frequency of the primary system is close to a natural frequency of the secondary system which in turn is close to a natural frequency of the tertiary system) is discussed in Section 6.2. Insights gained from the sub-problem solution approach of Chapter Four are used to deduce the response of the tuned tertiary system.

Finally, the maximum acceleration of each very light equipment degree of freedom, $\left|\ddot{u}_z^{(3)}\right|_{max}$, is determined in terms of the pseudo-acceleration response spectrum description of the earthquake excitation and the system properties.

6.1 The Completely Detuned System

We begin our study of tertiary systems by analyzing the system shown in Figure Thirteen when all natural frequencies are well spaced. The basic equations of motion for earthquake ground shaking are

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{C}\mathbf{R}\dot{u}_g + \mathbf{C}\mathbf{R}u_g \tag{6.1.1}$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{m}^{(3)} & 0 & 0 \\ 0 & \mathbf{m}^{(2)^*} & 0 \\ 0 & 0 & \mathbf{m}^{(1)^*} \end{bmatrix} \; ; \; \mathbf{K} = \begin{bmatrix} \mathbf{k}^{(3)} & \mathbf{k}^{(32)} & 0 \\ \mathbf{k}^{(23)} & \mathbf{k}^{(2)^*} & \mathbf{k}^{(21)} \\ 0 & \mathbf{k}^{(12)} & \mathbf{k}^{(1)^*} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{c}^{(3)} & \mathbf{c}^{(32)} & 0 \\ \mathbf{c}^{(23)} & \mathbf{c}^{(2)^*} & \mathbf{c}^{(21)} \\ 0 & \mathbf{c}^{(12)} & \mathbf{c}^{(1)^*} \end{bmatrix} \; ; \; \mathbf{u} = \begin{bmatrix} \mathbf{u}^{(3)} \\ \mathbf{u}^{(2)} \\ \mathbf{u}^{(1)} \end{bmatrix} \; ; \; \mathbf{R} = \begin{bmatrix} \mathbf{r}^{(3)} \\ \mathbf{r}^{(2)} \\ \mathbf{r}^{(1)} \end{bmatrix}$$

$$\mathbf{m}^{(1)^*} = \mathbf{m}^{(1)} + \tilde{\mathbf{m}}^{(1)} \; ; \; \mathbf{k}^{(1)^*} = \mathbf{k}^{(1)} + \tilde{\mathbf{k}}^{(1)}$$

$$\mathbf{m}^{(2)^*} = \mathbf{m}^{(2)} + \tilde{\mathbf{m}}^{(2)} \; ; \; \mathbf{k}^{(2)^*} = \mathbf{k}^{(2)} + \tilde{\mathbf{k}}^{(2)}$$

and

$\mathbf{u}^{(3)}$, $\mathbf{u}^{(2)}$, $\mathbf{u}^{(1)}$ are the absolute displacements of the tertiary system, secondary system, and primary system respectively

$\mathbf{m}^{(3)}$, $\mathbf{k}^{(3)}$; $\mathbf{m}^{(2)}$, $\mathbf{k}^{(2)}$; and $\mathbf{m}^{(1)}$, $\mathbf{k}^{(1)}$ are the fixed base mass and stiffness for the tertiary, secondary, and primary system respectively

$\mathbf{k}^{(23)} = \mathbf{k}^{(32)^T}$ are "cross-coupling" stiffness matrices between the tertiary and secondary systems (the elements of these matrices come from the fixed base tertiary system stiffness)

$\mathbf{k}^{(12)} = \mathbf{k}^{(21)^T}$ are "cross-coupling" stiffness matrices (the elements of these matrices come from the fixed base secondary system stiffness)

$\tilde{\mathbf{m}}^{(2)}$ and $\tilde{\mathbf{k}}^{(2)}$ are the contributions to $\mathbf{m}^{(2)^*}$ and $\mathbf{k}^{(2)^*}$ from the tertiary system

$\tilde{\mathbf{m}}^{(1)}$ and $\tilde{\mathbf{k}}^{(1)}$ are the contributions to $\mathbf{m}^{(1)^*}$ and $\mathbf{k}^{(1)^*}$ from the secondary system

$\mathbf{r}^{(3)}$, $\mathbf{r}^{(2)}$, and $\mathbf{r}^{(1)}$ are the pseudo-static influence coefficients for the tertiary, secondary, and primary system due to the ground motion u_g.

The damping is of the same form as the stiffness. The zeros in the mass matrix are the result of a lumped mass formulation.

The associated eigenproblem is, of course, given by

$$\mathbf{K}\mathbf{x} = \lambda \mathbf{M}\mathbf{x} \tag{6.1.2}$$

where $\mathbf{x}$ is of the same form as $\mathbf{u}$. Let us perform the transformation

$$\mathbf{x} = \mathbf{T}\boldsymbol{\psi} \tag{6.1.3}$$

where

$$\mathbf{T} = \begin{bmatrix} \hat{\boldsymbol{\Phi}}^{(3)} & 0 & 0 \\ 0 & \hat{\boldsymbol{\Phi}}^{(2)} & 0 \\ 0 & 0 & \hat{\boldsymbol{\Phi}}^{(1)} \end{bmatrix}$$

and $\hat{\boldsymbol{\Phi}}^{(3)}$, $\hat{\boldsymbol{\Phi}}^{(2)}$, $\hat{\boldsymbol{\Phi}}^{(1)}$ are the fixed base modal matrices of the tertiary, secondary, and primary system respectively.

Pre-multiplying (6.1.2) by $\mathbf{T}^T$ results in

$$\overline{\mathbf{K}}\boldsymbol{\psi} = \lambda\boldsymbol{\psi}$$

where

$$\overline{\mathbf{K}} = \mathbf{T}^T\mathbf{K}\mathbf{T}$$

We solve the above by classical perturbation. Separating the zeroth and first order terms in $\overline{\mathbf{K}}$ and discarding second order terms we have

$$\overline{\mathbf{K}} = \mathbf{H} + \mathbf{W}$$

where

$$\mathbf{H} = \begin{bmatrix} \boldsymbol{\omega}^{(3)^2} & 0 & 0 \\ 0 & \boldsymbol{\omega}^{(2)^2} & 0 \\ 0 & 0 & \boldsymbol{\omega}^{(1)^2} \end{bmatrix} \; ; \; \mathbf{W} = \begin{bmatrix} 0 & \mathbf{A}^{(23)^T} & 0 \\ \mathbf{A}^{(23)} & 0 & \mathbf{A}^{(12)^T} \\ 0 & \mathbf{A}^{(12)} & 0 \end{bmatrix}$$

and $\boldsymbol{\omega}^{(3)^2}$, $\boldsymbol{\omega}^{(2)^2}$, $\boldsymbol{\omega}^{(1)^2}$ are diagonal matrices of the tertiary, secondary, and primary system squared natural frequencies respectively. Also

$$\mathbf{A}^{(23)} = \hat{\mathbf{\Phi}}^{(2)T}\mathbf{k}^{(23)}\hat{\mathbf{\Phi}}^{(3)}$$

$$\mathbf{A}^{(12)} = \hat{\mathbf{\Phi}}^{(1)T}\mathbf{k}^{(12)}\hat{\mathbf{\Phi}}^{(2)}$$

Since all frequencies are well spaced, we have a non-degenerate problem and we proceed with the theory of Section 2.1.

A solution of the zeroth order problem gives

$$\begin{aligned} \lambda_j^{(0)} &= \omega_j^{(3)^2} \quad ; \; j=1,2,\ldots,n^{(3)} \\ \lambda_{n^{(3)}+k}^{(0)} &= \omega_k^{(2)^2} \quad ; \; k=1,2,\ldots,n^{(2)} \\ \lambda_{n^{(3)}+n^{(2)}+l}^{(0)} &= \omega_l^{(1)^2} \quad ; \; l=1,2,\ldots,n^{(1)} \end{aligned} \tag{6.1.4}$$

and

$$\boldsymbol{\psi}_i^{(0)} = \mathbf{e}_i \quad ; \; i=1,2,\ldots,n^{(3)}+n^{(2)}+n^{(1)} \tag{6.1.5}$$

where $\mathbf{e}_i$ is a vector of dimension $n^{(3)}+n^{(2)}+n^{(1)} \times 1$ with a one in column i and zeros elsewhere. To facilitate the subsequent discussion, we will assume the range of the indices i, j, k, l will be that given above. By (2.1.7) we have

$$\lambda_i^{(1)} = 0$$

for all i.

Thus the eigenvalues of (6.1.2) to first order are given by (6.1.4). Calculating the first order perturbations in the eigenvectors we have from (2.1.10) the following non-zero values

$$W_{j,n^{(3)}+k} = W_{n^{(3)}+k,j} = A_{kj}^{(23)}$$

$$W_{n^{(3)}+k,n^{(3)}+n^{(2)}+l} = W_{n^{(3)}+n^{(2)}+l,n^{(3)}+k} = A_{lk}^{(12)}$$

and hence from (2.1.8)

$$\boldsymbol{\psi}_j^{(1)} = \begin{bmatrix} 0' \\ \mathbf{g}_j \\ \hat{0} \end{bmatrix} \quad ; \quad \boldsymbol{\psi}_{n^{(3)}+k}^{(1)} = \begin{bmatrix} \mathbf{h}_k \\ \tilde{0} \\ \mathbf{f}_k \end{bmatrix}$$

$$\psi^{(1)}_{n^{(3)}+n^{(2)}+l} = \begin{bmatrix} 0' \\ \mathbf{g}_l \\ \hat{0} \end{bmatrix}$$

where 0′, $\tilde{0}$, and $\hat{0}$ are zero vectors of dimension $n^{(3)}$, $n^{(2)}$, and $n^{(1)}$ respectively. For a particular value of j, the vector $\mathbf{g}_j$ [dimension $(n^{(2)}\times 1)$] has components $k=1,2,\ldots,n^{(2)}$ given below as

$$\frac{A^{(23)}_{kj}}{(\omega^{(3)^2}_j - \omega^{(2)^2}_k)}$$

For a particular value of k; the vector $\mathbf{h}_k$ [dimension $(n^{(3)}\times 1)$] has components $j=1,2,\ldots,n^{(3)}$ given below as

$$\frac{A^{(23)}_{kj}}{(\omega^{(2)^2}_k - \omega^{(3)^2}_j)}$$

while the vector $\mathbf{f}_k$ [dimension $(n^{(1)}\times 1)$] has components $l=1,2,\ldots,n^{(1)}$ given below as

$$\frac{A^{(12)}_{lk}}{(\omega^{(2)^2}_k - \omega^{(1)^2}_l)}$$

For a particular value of l, the vector $\mathbf{g}^*_l$ [dimension $(n^{(2)}\times 1)$] has components $k=1,2,\ldots,n^{(2)}$ given below as

$$\frac{A^{(12)}_{lk}}{(\omega^{(1)^2}_l - \omega^{(2)^2}_k)}$$

These first order perturbations are added to the zeroth order perturbations given in (6.1.5) and transformed back to real space by the transformation given in (6.1.3). The result is

$$\mathbf{x}_j = \begin{bmatrix} \hat{\boldsymbol{\phi}}^{(3)}_j \\ \mathbf{g}^{(2)}_j \\ \hat{0} \end{bmatrix} \; ; \; \mathbf{x}_{n^{(3)}+k} = \begin{bmatrix} \mathbf{h}^{(3)}_k \\ \hat{\boldsymbol{\phi}}^{(2)}_k \\ \mathbf{f}^{(1)}_k \end{bmatrix}$$

$$\mathbf{x}_{n^{(3)}+n^{(2)}+l} = \begin{bmatrix} 0' \\ \mathbf{g}^{*(2)}_l \\ \hat{\boldsymbol{\phi}}^{(1)}_l \end{bmatrix}$$

For a particular value of j, the vector $\mathbf{g}^{(2)}_j$ [dimension $(n^{(2)}\times 1)$] is

$$\mathbf{g}_j^{(2)} = \sum_{k=1}^{n^{(2)}} \frac{\hat{\boldsymbol{\phi}}_k^{(2)} A_{kj}^{(23)}}{(\omega_j^{(3)^2} - \omega_k^{(2)^2})}$$

For a particular value of k, the vector $\mathbf{h}_k^{(3)}$ [dimension $(n^{(3)}\times 1)$] is

$$\mathbf{h}_k^{(3)} = \sum_{j=1}^{n^{(3)}} \frac{\hat{\boldsymbol{\phi}}_j^{(3)} A_{kj}^{(23)}}{(\omega_k^{(2)^2} - \omega_j^{(3)^2})}$$

while the vector $\mathbf{f}_k^{(1)}$ [dimension $(n^{(1)}\times 1)$] is

$$\mathbf{f}_k^{(1)} = \sum_{l=1}^{n^{(1)}} \frac{\hat{\boldsymbol{\phi}}_l^{(1)} A_{lk}^{(12)}}{(\omega_k^{(2)^2} - \omega_l^{(1)^2})}$$

For a particular value of l, the vector $\mathbf{g}_l^{*(2)}$ [dimension $(n^{(2)}\times 1)$] is

$$\mathbf{g}_l^{*(2)} = \sum_{k=1}^{n^{(2)}} \frac{\hat{\boldsymbol{\phi}}_k^{(2)} A_{lk}^{(12)}}{(\omega_l^{(1)^2} - \omega_k^{(2)^2})}$$

The presence of the zero vector in both $\mathbf{x}_j$ and $\mathbf{x}_{n^{(3)}+n^{(2)}+l}$ is highly suspicious. In a modal approach, the tertiary system response would be a function of the convolution of oscillatory harmonics whose frequencies are the natural frequencies of the secondary and primary system with the ground motion. Although it is a possibility that there are no harmonics associated with the tertiary system natural frequencies contributing to the response, it is not a likelihood. We therefore resort to higher order (second order) perturbation theory for the eigenvectors. Increasing the accuracy of the eigenvalues is not necessary. By the theory of Section 2.2 we have

$$\boldsymbol{\psi}_j^{(2)} = \begin{bmatrix} \mathbf{h}_j^* \\ \tilde{0} \\ \mathbf{f}_j^* \end{bmatrix} \quad ; \quad \boldsymbol{\psi}_{n^{(3)}+k}^{(2)} = \begin{bmatrix} 0' \\ \mathbf{g}_k^{**} \\ \hat{0} \end{bmatrix}$$

$$\boldsymbol{\psi}_{n^{(3)}+n^{(2)}+l}^{(2)} = \begin{bmatrix} \mathbf{h}_l^{**} \\ \tilde{0} \\ \mathbf{f}_l^{**} \end{bmatrix}$$

For a particular value of j, the vector $\mathbf{h}_j^*$ has components $r=1,2, \ldots, n^{(3)}$ given as

$$\sum_{k=1}^{n^{(2)}} \frac{A_{kr}^{(23)} A_{kj}^{(23)}}{(\omega_j^{(3)^2} - \omega_r^{(3)^2})(\omega_j^{(3)^2} - \omega_k^{(2)^2})}$$

for $r \neq j$ and whose jth component is

$$-\frac{1}{2}\sum_{k=1}^{n^{(2)}}\left[\frac{A_{kj}^{(23)}}{(\omega_j^{(3)2}-\omega_k^{(2)2})}\right]^2$$

while the vector $\mathbf{f}_j^*$ has components $l=1,2,\ldots,n^{(1)}$ given as

$$\sum_{k=1}^{n^{(2)}}\frac{A_{lk}^{(12)}A_{kj}^{(23)}}{(\omega_j^{(3)2}-\omega_l^{(1)2})(\omega_j^{(3)2}-\omega_k^{(2)2})}$$

For a particular value of k, the vector $\mathbf{g}_k^{**}$ has components $s=1,2,\ldots,n^{(2)}$ given as

$$\sum_{j=1}^{n^{(3)}}\frac{A_{sj}^{(23)2}}{(\omega_k^{(2)2}-\omega_s^{(2)2})(\omega_k^{(2)2}-\omega_j^{(3)2})}$$

for $s \neq k$ and whose kth component is

$$-\frac{1}{2}\left\{\sum_{j=1}^{n^{(3)}}\left[\frac{A_{kj}^{(23)}}{(\omega_k^{(2)2}-\omega_j^{(3)2})}\right]^2+\sum_{l=1}^{n^{(1)}}\left[\frac{A_{lk}^{(12)}}{(\omega_k^{(2)2}-\omega_l^{(1)2})}\right]^2\right\}$$

For a particular value of l, the vector $\mathbf{h}_l^{**}$ has components $j=1,2,\ldots,n^{(3)}$ given as

$$\sum_{k=1}^{n^{(2)}}\frac{A_{lk}^{(12)}A_{kj}^{(23)}}{(\omega_l^{(1)2}-\omega_j^{(3)2})(\omega_l^{(1)2}-\omega_k^{(2)2})}$$

while the vector $\mathbf{f}_l^{**}$ has components $t=1,2,\ldots,n^{(1)}$ given as

$$\sum_{k=1}^{n^{(2)}}\frac{A_{tk}^{(12)2}}{(\omega_l^{(1)2}-\omega_t^{(1)2})(\omega_l^{(1)2}-\omega_k^{(2)2})}$$

for $t \neq l$ and whose lth component is

$$-\frac{1}{2}\sum_{k=1}^{n^{(2)}}\left[\frac{A_{lk}^{(12)}}{(\omega_l^{(1)2}-\omega_k^{(2)2})}\right]^2$$

We include the second order corrections above only where they give a dominant contribution (i.e. in place of $\hat{0}$ in $\psi_j^{(1)}$ and $0'$ in $\psi_{n^{(3)}+n^{(2)}+l}^{(1)}$). Then by (6.1.3) we have

$$\mathbf{x}_j=\begin{bmatrix}\hat{\boldsymbol{\phi}}_j^{(3)}\\ \mathbf{g}_j^{(2)}\\ \mathbf{g}_j^{*(1)}\end{bmatrix}\quad;\quad \mathbf{x}_{n^{(3)}+k}=\begin{bmatrix}\mathbf{h}_k^{(3)}\\ \hat{\boldsymbol{\phi}}_k^{(2)}\\ \mathbf{f}_k^{(1)}\end{bmatrix}$$

$$\mathbf{x}_{n^{(3)}+n^{(2)}+l} = \begin{bmatrix} \mathbf{h}_l^{**(3)} \\ \mathbf{g}_l^{*(2)} \\ \hat{\boldsymbol{\phi}}_l^{(1)}) \end{bmatrix}$$

where the only new quantities are $\mathbf{g}_j^{*(1)}$ and $\mathbf{h}_l^{**(3)}$. For a particular value of j, the vector $\mathbf{g}_j^{*(1)}$ is

$$\mathbf{g}_j^{*(1)} = \sum_{l=1}^{n^{(1)}} \sum_{k=1}^{n^{(2)}} \frac{\hat{\boldsymbol{\phi}}_l^{(1)} A_{lk}^{(12)} A_{kj}^{(23)}}{(\omega_j^{(3)^2} - \omega_l^{(1)^2})(\omega_j^{(3)^2} - \omega_k^{(2)^2})}$$

For a particular value of l, the vector $\mathbf{h}_l^{**(3)}$ is

$$\mathbf{h}_l^{**(3)} = \sum_{k=1}^{n^{(2)}} \sum_{j=1}^{n^{(3)}} \frac{\hat{\boldsymbol{\phi}}_j^{(3)} A_{lk}^{(12)} A_{kj}^{(23)}}{(\omega_l^{(1)^2} - \omega_j^{(3)^2})(\omega_l^{(1)^2} - \omega_k^{(2)^2})}$$

A straightforward modal solution of (6.1.1) yields

$$\ddot{\mathbf{u}} = \mathbf{X}\mathbf{y} \tag{6.1.6}$$

where

$$\mathbf{X} = \left[\mathbf{x}_i\right] \quad ; \ i=1,2,\ldots, n^{(3)}+n^{(2)}+n^{(1)}$$

and $\mathbf{y}$ is a vector whose rows are given by

$$y_i = \left\{\Gamma_i \sqrt{\lambda_i} e^{-\beta_i \sqrt{\lambda_i} t} \sin(\sqrt{\lambda_i} t)\right\} * \ddot{u}_g$$

where

$$\Gamma_i = \mathbf{x}_i^T \mathbf{M}\mathbf{R} / \mathbf{x}_i^T \mathbf{M}\mathbf{x}_i$$

Calculating the participation factors we have (to dominant order)

$$\Gamma_j = P_j^{(3)} + \sum_{k=1}^{n^{(2)}} \frac{C_{kj}^{(23)}}{(\omega_j^{(3)^2} - \omega_k^{(2)^2})} + \sum_{l=1}^{n^{(1)}} \sum_{k=1}^{n^{(2)}} \frac{C_{lk}^{(12)} A_{kj}^{(23)}}{(\omega_j^{(3)^2} - \omega_l^{(1)^2})(\omega_j^{(3)^2} - \omega_k^{(2)^2})}$$

$$\Gamma_{n^{(3)}+k} = P_k^{(2)} + \sum_{l=1}^{n^{(1)}} \frac{C_{lk}^{(12)}}{(\omega_k^{(2)^2} - \omega_l^{(1)^2})}$$

$$\Gamma_{n^{(3)}+n^{(2)}+l} = P_l^{(1)}$$

where

$$P_l^{(1)} = \hat{\boldsymbol{\phi}}_l^{(1)T} \mathbf{m}^{(1)} \mathbf{r}^{(1)} \quad , \quad P_k^{(2)} = \hat{\boldsymbol{\phi}}_k^{(2)T} \mathbf{m}^{(2)} \mathbf{r}^{(2)} \quad , \quad P_j^{(3)} = \hat{\boldsymbol{\phi}}_j^{(3)T} \mathbf{m}^{(3)} \mathbf{r}^{(3)}$$

$$C_{lk}^{(12)} = P_l^{(1)} A_{lk}^{(12)} \quad , \quad C_{kj}^{(23)} = P_k^{(2)} A_{kj}^{(23)}$$

From (6.1.6) we obtain the following expression for tertiary system response for the completely detuned system

$$\ddot{u}_z^{(3)}(t) = \left\{ \sum_{j=1}^{n^{(3)}} \hat{\phi}_{zj}^{(3)} P_j^{(3)} \omega_j^{(3)} e^{-\beta_j^{(3)} \omega_j^{(3)} t} \sin(\omega_j^{(3)} t) \right.$$

$$+ \sum_{j=1}^{n^{(3)}} \sum_{k=1}^{n^{(2)}} \frac{\hat{\phi}_{zj}^{(3)} C_{kj}^{(23)}}{(\omega_j^{(3)2} - \omega_k^{(2)2})} \omega_j^{(3)} e^{-\beta_j^{(3)} \omega_j^{(3)} t} \sin(\omega_j^{(3)} t)$$

$$+ \sum_{j=1}^{n^{(3)}} \sum_{k=1}^{n^{(2)}} \sum_{l=1}^{n^{(1)}} \frac{\hat{\phi}_{zj}^{(3)} C_{lk}^{(12)} A_{kj}^{(23)}}{(\omega_j^{(3)2} - \omega_l^{(1)2})(\omega_j^{(3)2} - \omega_k^{(2)2})} \omega_j^{(3)} e^{-\beta_j^{(3)} \omega_j^{(3)} t} \sin(\omega_j^{(3)} t)$$

$$+ \sum_{k=1}^{n^{(2)}} \sum_{j=1}^{n^{(3)}} \frac{\hat{\phi}_{zj}^{(3)} C_{kj}^{(23)}}{(\omega_k^{(2)2} - \omega_j^{(3)2})} \omega_k^{(2)} e^{-\beta_k^{(2)} \omega_k^{(2)} t} \sin(\omega_k^{(2)} t)$$

$$+ \sum_{k=1}^{n^{(2)}} \sum_{l=1}^{n^{(1)}} \sum_{j=1}^{n^{(3)}} \frac{\hat{\phi}_{zj}^{(3)} C_{lk}^{(12)} A_{kj}^{(23)}}{(\omega_k^{(2)2} - \omega_j^{(3)2})(\omega_k^{(2)2} - \omega_l^{(1)2})} \omega_k^{(2)} e^{-\beta_k^{(2)} \omega_k^{(2)} t} \sin(\omega_k^{(2)} t)$$

$$\left. + \sum_{l=1}^{n^{(1)}} \sum_{k=1}^{n^{(2)}} \sum_{j=1}^{n^{(3)}} \frac{\hat{\phi}_{zj}^{(3)} C_{lk}^{(12)} A_{kj}^{(23)}}{(\omega_l^{(1)2} - \omega_j^{(3)2})(\omega_l^{(1)2} - \omega_k^{(2)2})} \omega_l^{(1)} e^{-\beta_l^{(1)} \omega_l^{(1)} t} \sin(\omega_l^{(1)} t) \right\} * \ddot{u}_g \qquad (6.1.7)$$

for $z = 1,2, \ldots, n^{(3)}$.

6.2 The Case of Tuning

Of course a straightforward application of the transformation **T** to the eigensystem (6.1.1) in the case of tuning

$$\omega_m^{(1)^2} = (1 + \alpha_{mn}^{(12)})\omega_n^{(2)^2}$$

$$\omega_o^{(3)^2} = (1 + \alpha_{on}^{(23)})\omega_n^{(2)^2}$$

where $\alpha_{mn}^{(12)}$ and $\alpha_{on}^{(23)}$ are of first order, results in the mixed problem discussed in Section 2.4. We therefore attempt to decompose the system into a non-degenerate and degenerate part. Let us first review the procedure used for the multi-degree-of-freedom secondary system.

In Problem Three of Section 4.2, three terms arose, two of which became singular under the condition of perfect tuning [see (4.2.4)]. It was these two terms that combined to form the dominant response described in Section 4.3. We therefore could have solved the problem of Section 4.3 in a slightly different manner. That is to realize that the response (4.3.6) is found by beginning with the completely detuned system result (4.1.12), subtracting the terms of Problem Three that become singular at perfect tuning, and adding the Green's function which results from a combination of these "singular" terms. We shall take this approach here.

We begin by introducing the transformation

$$\mathbf{x} = \mathbf{T}_3\boldsymbol{\psi}$$

where

$$\mathbf{T}_3 = \begin{bmatrix} \hat{\boldsymbol{\phi}}_o^{(3)} & 0 & 0 \\ 0 & \hat{\boldsymbol{\phi}}_n^{(2)} & 0 \\ 0 & 0 & \hat{\boldsymbol{\phi}}_m^{(1)} \end{bmatrix} \tag{6.2.1}$$

Using this transformation instead of the transformation **T** of Section 6.1 results in the following response

$$\ddot{u}_z^{(3)}(t) = \Bigg\{ \hat{\phi}_{zo}^{(3)} P_o^{(3)} \omega_o^{(3)} e^{-\beta_o^{(3)}\omega_o^{(3)}t} \sin(\omega_o^{(3)}t)$$

$$+ \frac{\hat{\phi}_{zo}^{(3)} C_{no}^{(23)}}{(\omega_o^{(3)^2} - \omega_n^{(2)^2})} \omega_o^{(3)} e^{-\beta_o^{(3)}\omega_o^{(3)}t} \sin(\omega_o^{(3)}t)$$

$$+ \frac{\hat{\phi}_{zo}^{(3)} C_{mn}^{(12)} A_{no}^{(23)}}{(\omega_o^{(3)^2} - \omega_m^{(1)^2})(\omega_o^{(3)^2} - \omega_n^{(2)^2})} \omega_o^{(3)} e^{-\beta_o^{(3)}\omega_o^{(3)}t} \sin(\omega_o^{(3)}t)$$

$$+ \frac{\hat{\phi}_{zo}^{(3)} C_{no}^{(23)}}{(\omega_n^{(2)^2} - \omega_o^{(3)^2})} \omega_n^{(2)} e^{-\beta_n^{(2)}\omega_n^{(2)}t} \sin(\omega_n^{(2)}t)$$

$$+ \frac{\hat{\phi}_{zo}^{(3)} C_{mn}^{(12)} A_{no}^{(23)}}{(\omega_n^{(2)^2} - \omega_o^{(3)^2})(\omega_n^{(2)^2} - \omega_m^{(1)^2})} \omega_n^{(2)} e^{-\beta_n^{(2)}\omega_n^{(2)}t} \sin(\omega_n^{(2)}t)$$

$$+ \frac{\hat{\phi}_{zo}^{(3)} C_{mn}^{(12)} A_{no}^{(23)}}{(\omega_m^{(1)^2} - \omega_o^{(3)^2})(\omega_m^{(1)^2} - \omega_n^{(2)^2})} \omega_m^{(1)} e^{-\beta_m^{(1)}\omega_m^{(1)}t} \sin(\omega_m^{(1)}t) \Bigg\} * \ddot{u}_g \tag{6.2.2}$$

We see that all terms of (6.2.2) become singular except the first when we have perfect tuning. We therefore subtract these terms from (6.1.7).

We now combine the singular terms of (6.2.2). Notice that the second and fourth terms of (6.2.2) are of the same type as the "singular" terms of (4.2.4). As the frequency $\omega_o^{(3)}$ approaches $\omega_n^{(2)}$ these terms will combine in a similar manner as they did in Section 4.3. The result can therefore be written directly as

$$\ddot{u}_z^{(3)}(t) = \left\{ -\hat{\phi}_{zo}^{(3)} \frac{C_{no}^{(23)}}{\omega_n^{(2)^2}} \ddot{q}_{G_S}^{(3)}(t) \right\} * \ddot{u}_g \tag{6.2.3}$$

where $\ddot{q}_{G_S}^{(3)}(t)$ is the same as the $\ddot{q}_G^{(2)}(t)$ of Section 4.3 if one replaces the $\alpha_{mn}^{(12)}$, $\gamma_{mn}^{(12)}$, $\beta_n^{(2)}$, $\beta_m^{(1)}$ of $\ddot{q}_G^{(2)}(t)$ with $-\alpha_{on}^{(23)}$, $\gamma_{no}^{(23)}$, $\beta_o^{(3)}$, $\beta_n^{(2)}$ respectively.

Finally we see how the third, fifth, and sixth terms of (6.2.2) combine. This is done by letting

$$\mathbf{u} = \mathbf{T}_3\mathbf{Q}$$

where $\mathbf{T}_3$ is given in (6.2.1) and

$$\mathbf{Q} = \begin{bmatrix} q^{(3)} \\ q^{(2)} \\ q^{(1)} \end{bmatrix}$$

When the above is substituted into (6.1.1) and premultiplied by $\mathbf{T}_3^T$ an "equivalent" three-degree-of-freedom tertiary system results. This system is solved by Laplace transform methods. The governing equation is

$$\bar{\ddot{q}}^{(3)} = [N(p)/D(p)]\bar{\ddot{u}}_g \tag{6.2.4}$$

where

$$N(p) = f^{(3)}\Big[(p^2+2\beta_n^{(2)}\omega_n^{(2)}p+\omega_n^{(2)^2})(p^2+2\beta_m^{(1)}\omega_m^{(1)}p+\omega_m^{(1)^2})$$

$$-\,(B_{mn}^{(12)}+A_{mn}^{(12)})^2\Big] - f^{(2)}\Big[(p^2+2\beta_m^{(1)}\omega_m^{(1)}p+\omega_m^{(1)^2})(B_{no}^{(23)}p+A_{no}^{(23)})\Big]$$

$$+\,f^{(1)}\Big[(B_{mn}^{(12)}+A_{mn}^{(12)})^2\Big]$$

$$D(p) = (p^2+2\beta_o^{(3)}\omega_o^{(3)}p+\omega_o^{(3)^2})\Big[(p^2+2\beta_n^{(2)}\omega_n^{(2)}p+\omega_n^{(2)^2})\cdot$$

$$(p^2+2\beta_m^{(1)}\omega_m^{(1)}p+\omega_m^{(1)^2}) - (B_{mn}^{(12)}p+A_{mn}^{(12)})^2\Big] - (B_{no}^{(23)}p+A_{no}^{(23)})\cdot$$

$$\Big[(p^2+2\beta_m^{(1)}\omega_m^{(1)}p+\omega_m^{(1)^2})(B_{no}^{(23)}p+A_{no}^{(23)})\Big]$$

and

$$B_{mn}^{(12)} = \hat{\boldsymbol{\phi}}_m^{(1)^T}\mathbf{c}^{(12)}\hat{\boldsymbol{\phi}}_n^{(2)} \;,\quad B_{no}^{(23)} = \hat{\boldsymbol{\phi}}_n^{(2)^T}\mathbf{c}^{(23)}\hat{\boldsymbol{\phi}}_o^{(3)}$$

$$A_{mn}^{(12)} = \hat{\boldsymbol{\phi}}_m^{(1)^T}\mathbf{k}^{(12)}\hat{\boldsymbol{\phi}}_n^{(2)} \;,\quad A_{no}^{(23)} = \hat{\boldsymbol{\phi}}_n^{(2)^T}\mathbf{k}^{(23)}\hat{\boldsymbol{\phi}}_o^{(3)}$$

In a manner similar to that used in Section 4.3, one can establish that

$$f^{(2)} = f^{(3)} \equiv 0$$

We proceed with an inversion of (6.2.4) by the methodology of Section 4.3. It is readily apparent that the solution will be given as

$$\ddot{u}_z^{(3)}(t) = \left\{\hat{\phi}_{zo}^{(3)}\frac{C_{mn}^{(12)}A_{no}^{(23)}}{\omega_n^{(2)^4}}\ddot{q}_{G_T}^{(3)}(t)\right\} * \ddot{u}_g \tag{6.2.5}$$

where $\ddot{q}_{G_T}^{(3)}(t)$ is the Green's function of Section 5.2 with the α_{12}, α_{32}, γ_{21}, γ_{32}, β_1, β_2, β_3 of that section replaced with $\alpha_{mn}^{(12)}$, $\alpha_{on}^{(23)}$, $\gamma_{mn}^{(12)}$, $\gamma_{no}^{(23)}$, $\beta_m^{(1)}$, $\beta_n^{(2)}$, $\beta_o^{(3)}$ respectively. The quantities

$$\gamma_{mn}^{(12)} = \left[\frac{A_{mn}^{(12)}}{\omega_n^{(2)2}} \right]^2 \quad , \quad \gamma_{no}^{(23)} = \left[\frac{A_{no}^{(23)}}{\omega_n^{(2)2}} \right]^2$$

are "effective" mass ratios.

The total response for the tuned tertiary system is then given by adding (6.2.3) and (6.2.5) to the completely detuned result (6.1.7) minus the singular terms of (6.2.2).

6.3 Response Spectrum Analysis

Since all frequencies of the completely detuned system are well spaced, a conventional summation rule can be applied to the results of Section 6.1 when the primary system is subjected to long duration earthquake ground shaking. Using the square root of the sum of the squares procedure, we obtain

$$\left|\ddot{u}_z^{(3)}\right|_{max} = \left\{ \sum_{j=1}^{n^{(3)}} \left[\hat{\phi}_{zj}^{(3)} P_j^{(3)} + \sum_{k=1}^{n^{(2)}} \left(\frac{\hat{\phi}_{zj}^{(3)} C_{kj}^{(23)}}{(\omega_j^{(3)2} - \omega_k^{(2)2})} \right. \right. \right.$$

$$\left. \left. + \sum_{l=1}^{n^{(1)}} \frac{\hat{\phi}_{zj}^{(3)} C_{lk}^{(12)} A_{kj}^{(23)}}{(\omega_j^{(3)2} - \omega_l^{(1)2})(\omega_j^{(3)2} - \omega_k^{(2)2})} \right) \right]^2 S_A^2(\omega_j^{(3)}, \beta_j^{(3)})$$

$$+ \sum_{k=1}^{n^{(2)}} \left[\sum_{j=1}^{n^{(3)}} \left(\hat{\phi}_{zj}^{(3)} \frac{C_{kj}^{(23)}}{(\omega_k^{(2)2} - \omega_j^{(3)2})} \right. \right.$$

$$\left. \left. + \sum_{l=1}^{n^{(1)}} \frac{\hat{\phi}_{zj}^{(3)} C_{lk}^{(12)} A_{kj}^{(23)}}{(\omega_k^{(2)2} - \omega_j^{(3)2})(\omega_k^{(2)2} - \omega_l^{(1)2})} \right) \right]^2 S_A^2(\omega_k^{(2)}, \beta_k^{(2)})$$

$$\left. + \sum_{l=1}^{n^{(1)}} \left[\sum_{k=1}^{n^{(2)}} \sum_{j=1}^{n^{(3)}} \frac{\hat{\phi}_{zj}^{(3)} C_{lk}^{(12)} A_{kj}^{(23)}}{(\omega_l^{(1)2} - \omega_j^{(3)2})(\omega_l^{(1)2} - \omega_k^{(2)2})} \right]^2 S_A^2(\omega_l^{(1)}, \beta_l^{(1)}) \right\}^{1/2}$$

where $S_A(\omega,\beta)$ is the pseudo-acceleration response spectrum evaluated at frequency ω and damping factor β. Displacement and velocity response can be obtained by substituting appropriate design spectra in the above.

For short duration ground excitation, the dominant part of the response, in the case of tuning, is given by the terms involving $\ddot{q}_{G_T}^{(3)}(t)$ and $\ddot{q}_{G_S}^{(3)}(t)$ given below (since these exhibit a beating phenomenum) provided the coefficients multiplying these time functions are not so small as to markedly reduce the beating affect (see the discussion in Section 4.4). Assuming this is not the case, then the maximum equipment response is best found by a numerical evaluation of the following expression

$$\ddot{u}_z^{(3)}(t) = \left\{ \left[-\hat{\phi}_{zo}^{(3)} \frac{C_{no}^{(23)}}{\omega_n^{(2)^2}} \ddot{q}_{G_S}^{(3)}(t) \right] + \left[\hat{\phi}_{zo}^{(3)} \frac{C_{mn}^{(12)} A_{no}^{(23)}}{\omega_n^{(2)^4}} \ddot{q}_{G_T}^{(3)}(t) \right] \right\} * \ddot{u}_g$$

Chapter Seven

FURTHER RESEARCH

The problem of tuning in multi-degree-of-freedom secondary systems and in multi-degree-of-freedom tertiary systems was discussed herein. An important problem which deserves attention is that of clustering. An example in the multi-degree-of-freedom secondary system would be a natural frequency of the equipment that is close to more than one natural frequency of the structure (i.e. the structure has a number of natural frequencies that are close to each other). It was found in the case of simple tuning that large amplifications of equipment response are possible. It is expected that clustering will significantly affect such amplifications. Of course, the equipment could also have a number of close frequencies as well. Clustering is also possible in the multi-degree-of-freedom tertiary system.

The sub-problem approach of Section 4.2 might be used to analyze systems with clusters.

Many of the results of this work can be used in a more comprehensive random vibration analysis of equipment response.

Finally, a greater variety of numerical tests should be performed.

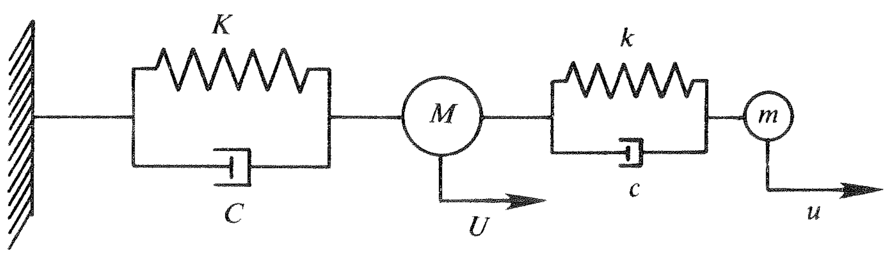
Chapter Eight

CONCLUSION

Various equipment-structure models are explored in this work. Results for equipment response are derived through the use of perturbation methods. The significant interaction effect present in tuned or nearly tuned systems is considered and treated in a rational manner. Expensive numerical integrations and ad-hoc methods are avoided by the use of the results portrayed herein that depend solely on data that is already available to the structural designer: the fixed base properties of the equipment alone, the structure alone; the manner in which they are connected; and the characterization of the excitation - either in terms of a response spectrum or as a given time-varying function.

Figure One

THE TWO-DEGREE-OF-FREEDOM SYSTEM



$$C = 2BM\Omega \; ; \; K = M\Omega^2$$

$$c = 2\beta m\omega \; ; \; k = m\omega^2$$

$$\gamma = \frac{m}{M}$$

Figure Two

MULTI-DEGREE-OF-FREEDOM SECONDARY SYSTEM

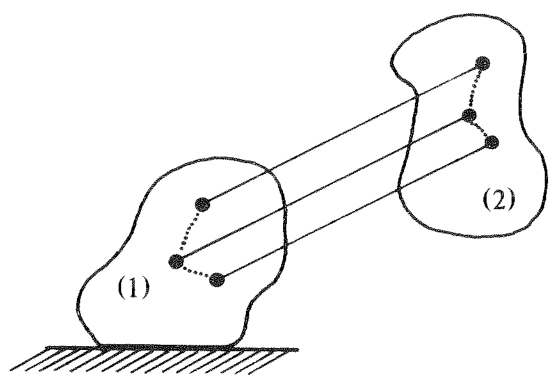
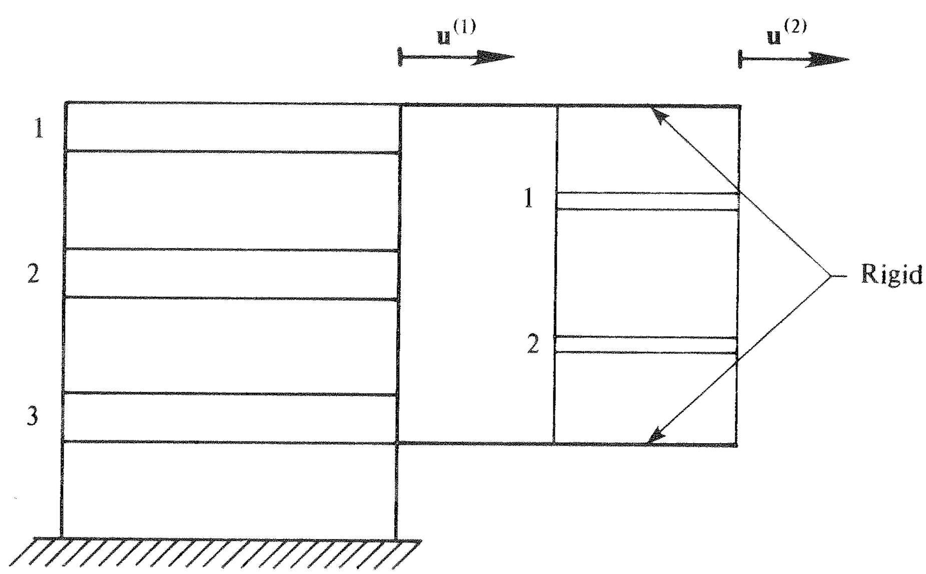


Figure Three

EQUIPMENT-STRUCTURE SYSTEM USED FOR NUMERICAL EXPERIMENTS



Shear Building Equipment-Structure System

Inter-Story Stiffnesses

Structure : K

Equipment : k

Floor Masses

Structure : M

Equipment : m

Figure Four

EL CENTRO RECORD

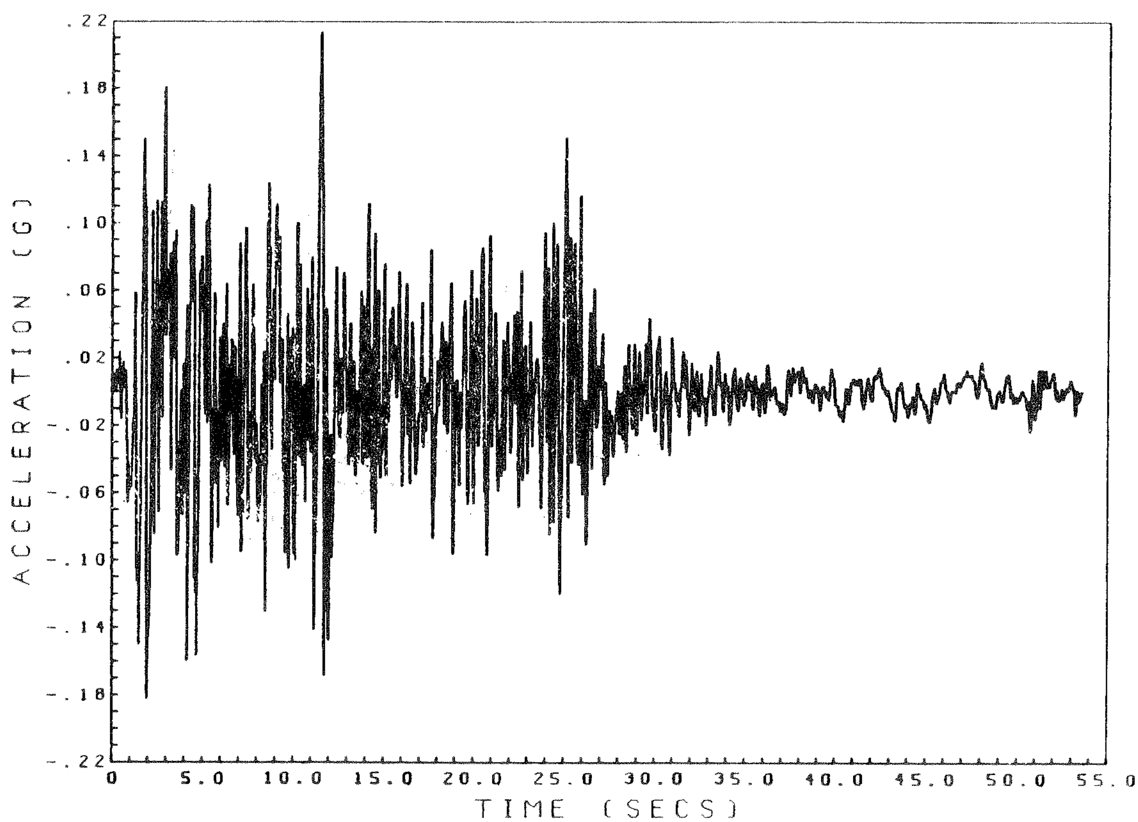
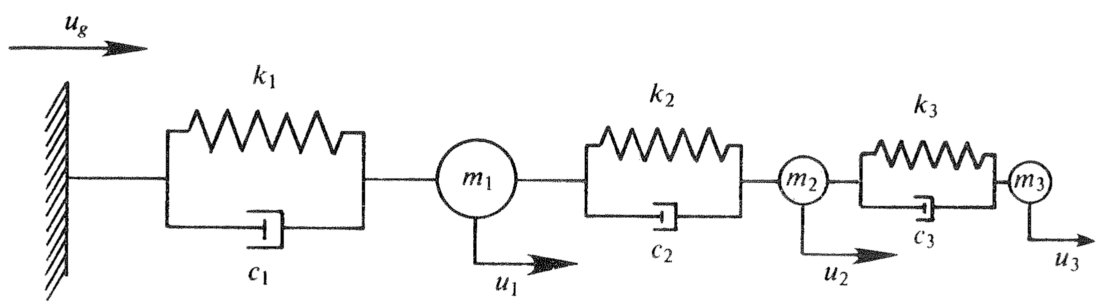


Figure Five

THREE-DEGREE-OF-FREEDOM TERTIARY SYSTEM



$$k_i = m_i \omega_i^2$$

$$c_i = 2\beta_i m_i \omega_i \;; \quad i=1,2,3$$

Figure Six

A PORTION OF THE SINE-SQUARED ENVELOPE

$$\left(\gamma_{32} = \gamma_{21} = 10^{-3}\right)$$

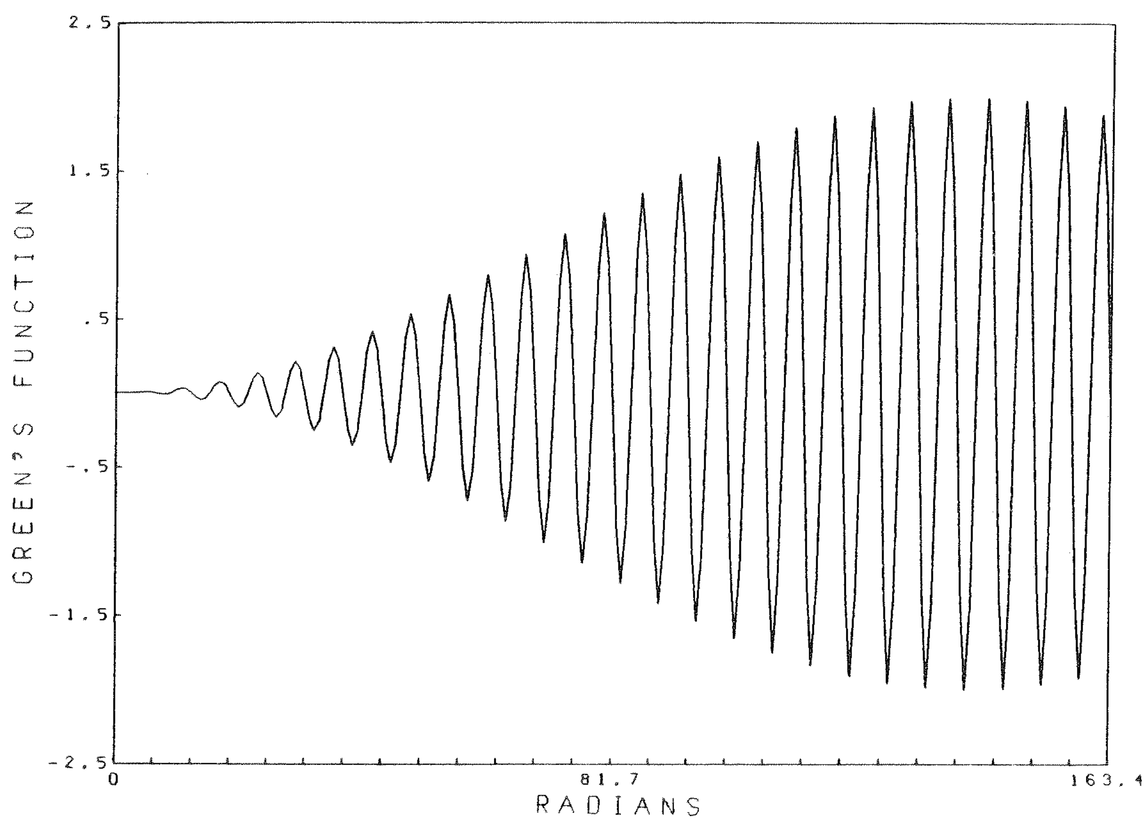


Figure Seven

$$\left(\epsilon = 10^{-3},\ \alpha_{12} = \sqrt{\epsilon},\ \alpha_{32} = \sqrt{\epsilon},\ \gamma_{21} = \epsilon,\ \gamma_{32} = \epsilon\right)$$

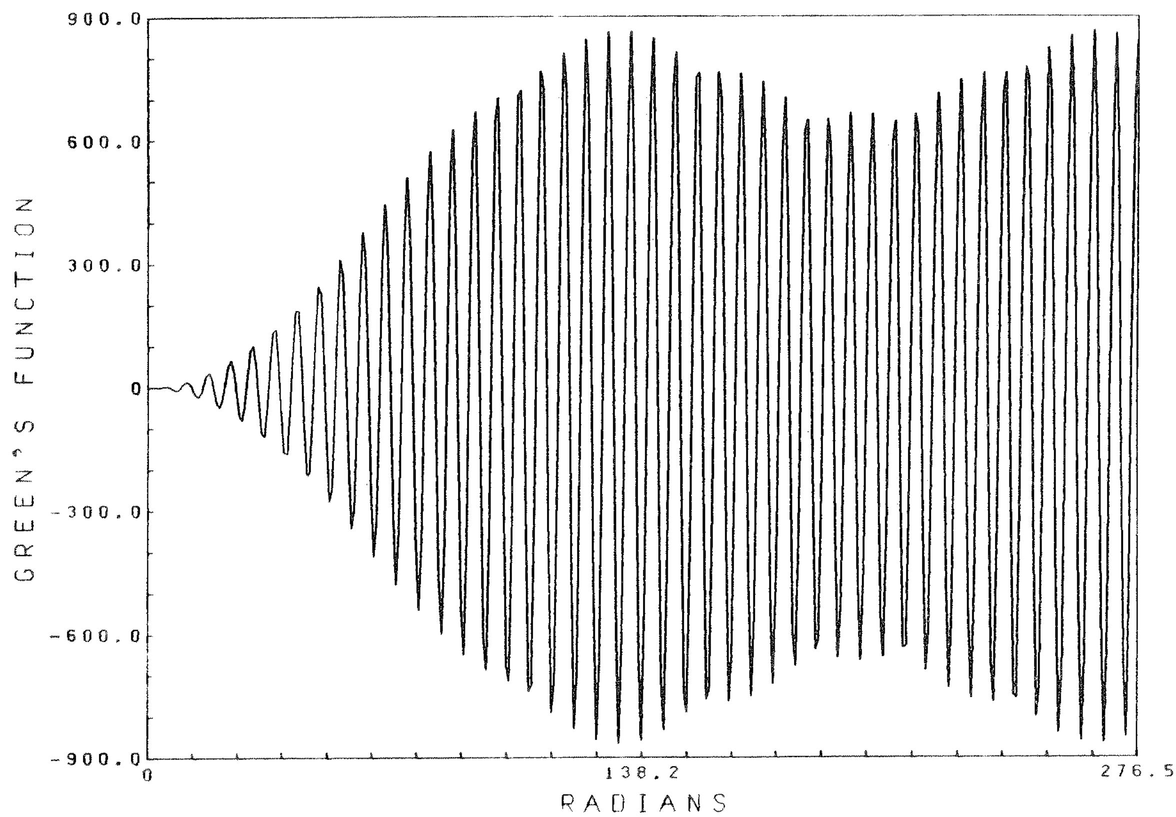


Figure Eight

$$\left(\epsilon = 10^{-3},\ \alpha_{12} = \sqrt{\epsilon},\ \alpha_{32} = \sqrt{\epsilon},\ \gamma_{21} = \epsilon,\ \gamma_{32} = 6\epsilon\right)$$

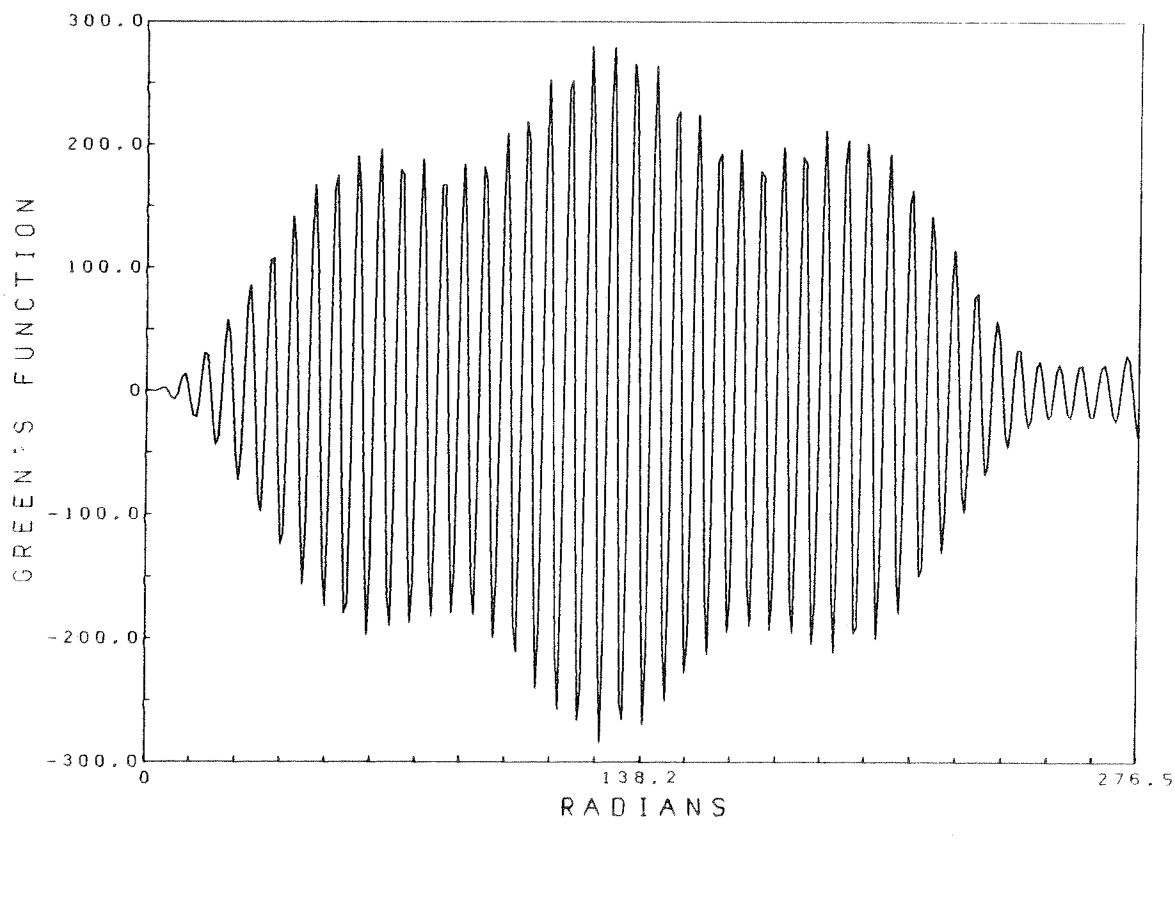


Figure Nine

$$\left(\epsilon = 10^{-3},\ \alpha_{12} = 3\sqrt{\epsilon},\ \alpha_{32} = \sqrt{\epsilon},\ \gamma_{21} = \epsilon,\ \gamma_{32} = \epsilon\right)$$

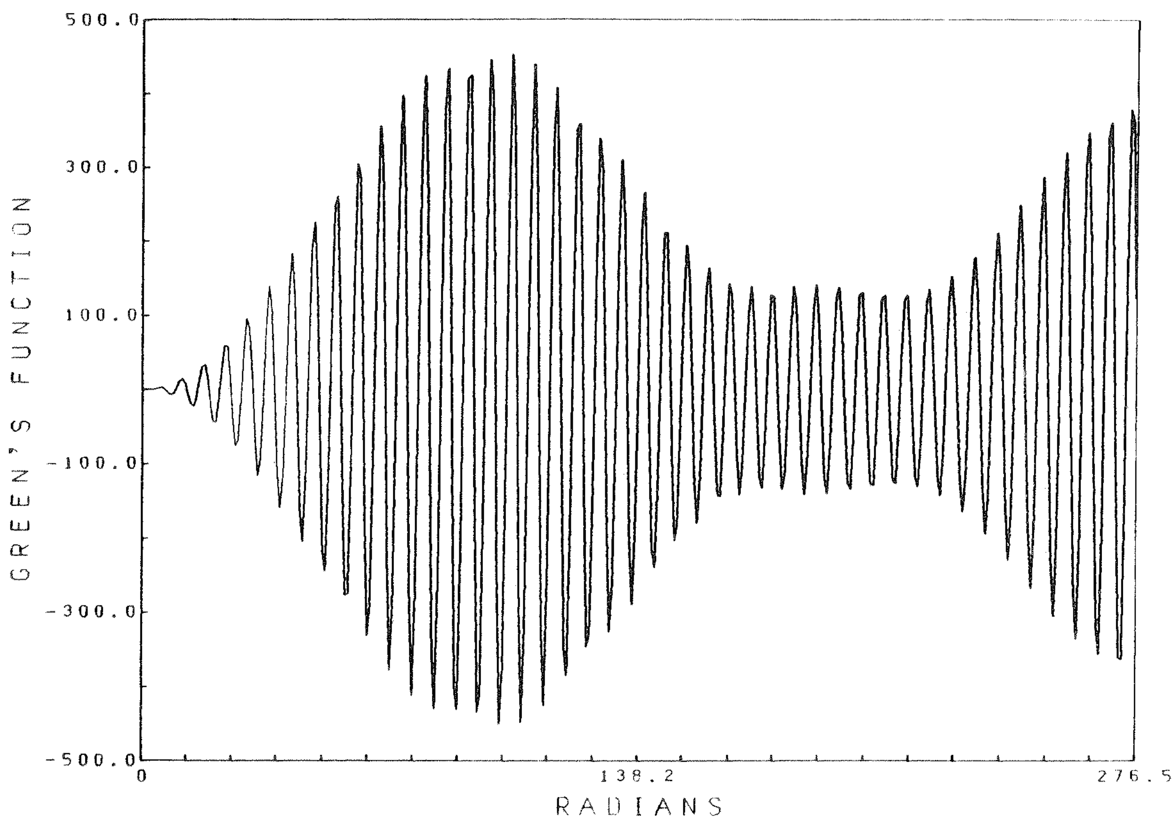


Figure Ten

$$\left(\epsilon = 10^{-3},\ \alpha_{12} = 3\sqrt{\epsilon},\ \alpha_{32} = \sqrt{\epsilon},\ \gamma_{21} = \epsilon,\ \gamma_{32} = 6\epsilon\right)$$

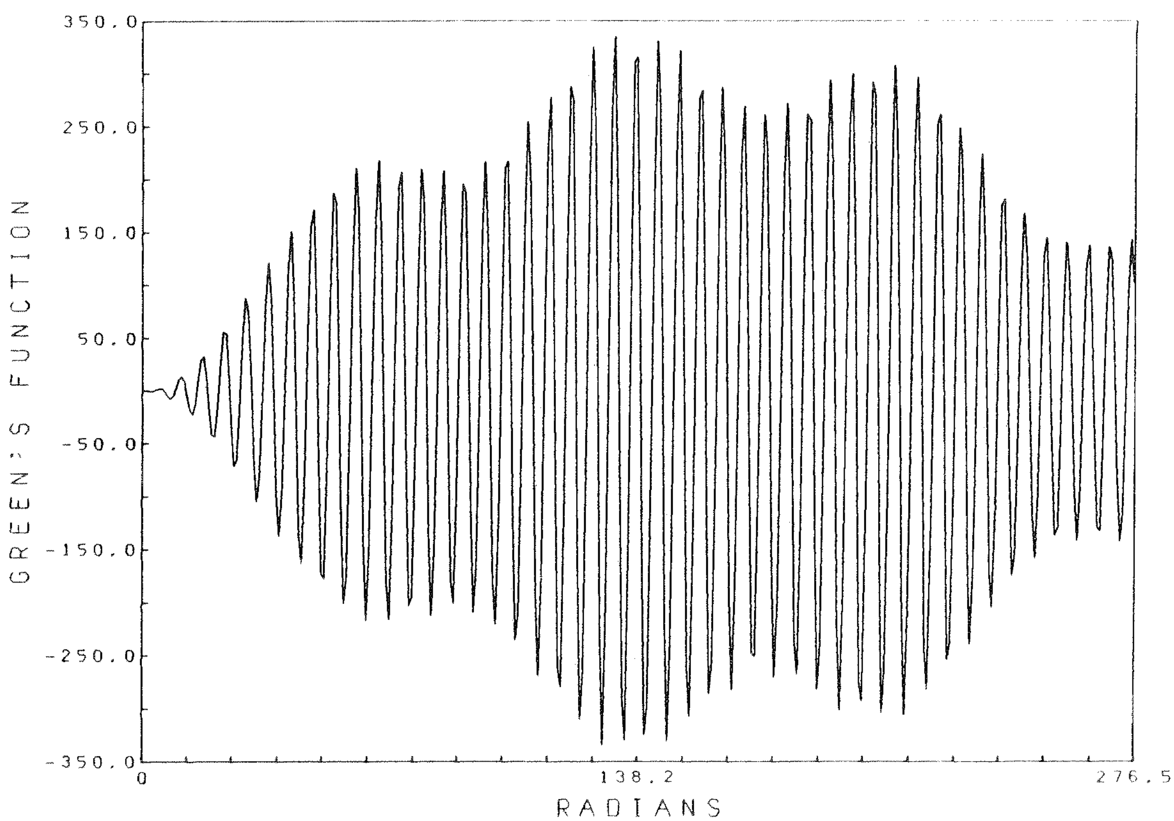


Figure Eleven

$$\left(\epsilon = 10^{-3},\ \alpha_{12} = 3\sqrt{\epsilon},\ \alpha_{32} = 0,\ \gamma_{21} = \epsilon,\ \gamma_{32} = \epsilon\right)$$

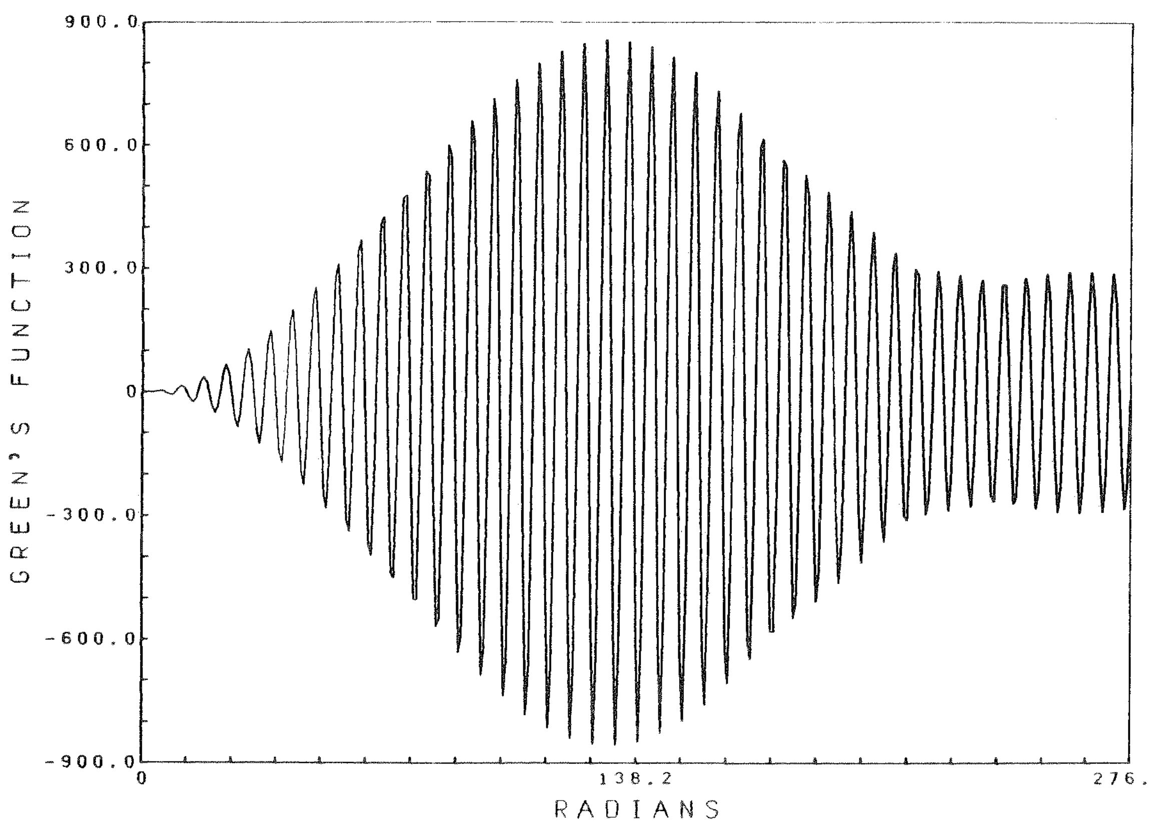


Figure Twelve

$$\left(\epsilon = 10^{-3},\ \alpha_{12} = 3\sqrt{\epsilon},\ \alpha_{32} = 0,\ \gamma_{21} = \epsilon,\ \gamma_{32} = 6\epsilon\right)$$

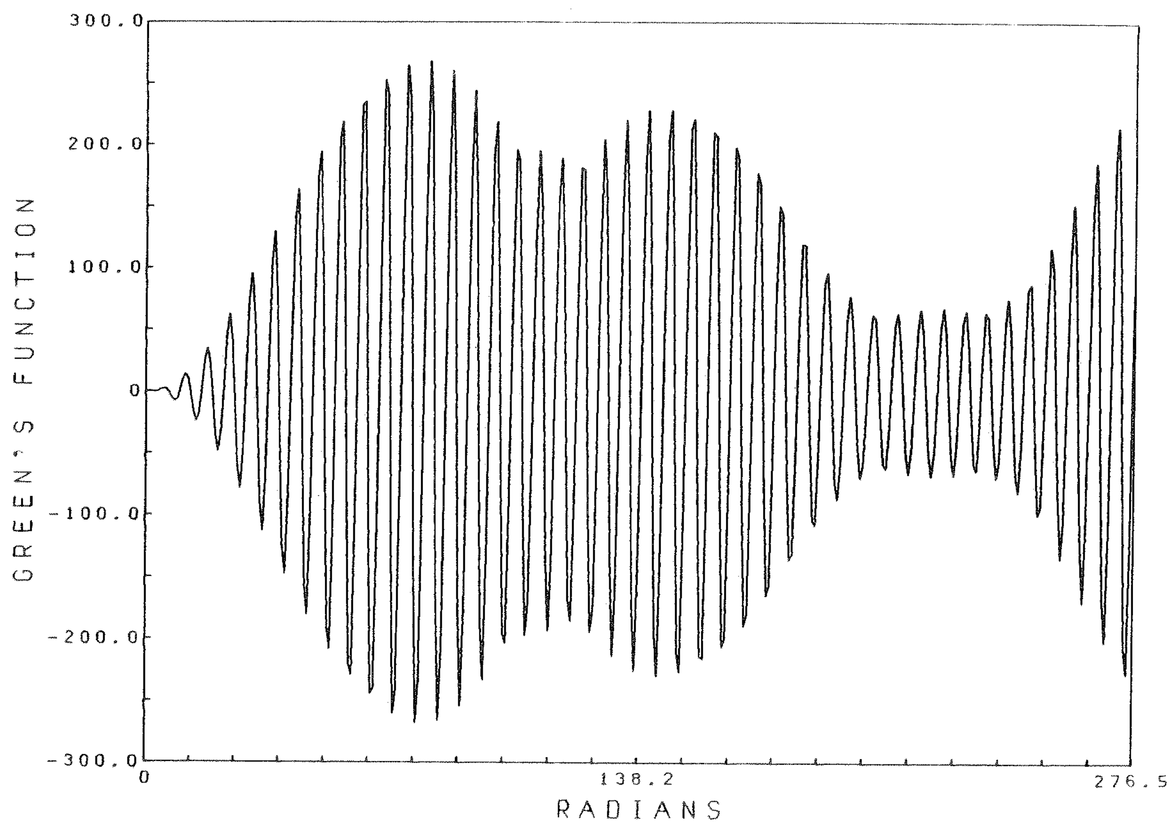


Figure Thirteen

THE MULTI-DEGREE-OF-FREEDOM TERTIARY SYSTEM

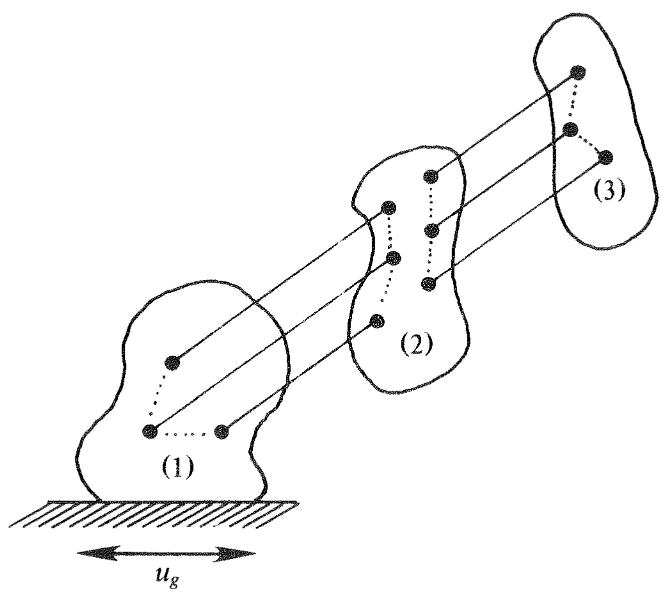


Table One

FREQUENCY COMPARISON FOR PERFECT TUNING

$$\left(\omega_m^{(1)^2} = \omega_n^{(2)^2}\right)$$

DIFFERENCE IN PERCENT				
	m=1, n=1	m=1, n=2	m=2, n=1	
$\gamma_{mn} \times \epsilon$	5.67	.093	.107	
Freqs. of combined system (num. incr. order)	-.026* -.030* +.005 -.013 -.003	+.030 -.014* -.013* -.001 .000	+.021 -.035* -.037* -.038 -.016	$\epsilon = 10^{-4}$
Freqs. of combined system (num. incr. order)	-.307* -.271* +.048 -.134 -.029	+.293 -.138* -.160* -.037 -.009	+.223 -.375* -.335* -.385 -.169	$\epsilon = 10^{-3}$
Freqs. of combined system (num. incr. order)	-3.49* -2.37* +.459 -1.34 -.297	+2.88 -1.11* -1.75* -.393 -.094	+2.44 -3.87* -2.81* -3.78 -1.71	$\epsilon = 10^{-2}$

Table One (Continued)

FREQUENCY COMPARISON FOR PERFECT TUNING

$$\left(\omega_m^{(1)^2} = \omega_n^{(2)^2}\right)$$

DIFFERENCE IN PERCENT				
	$m=2, n=2$	$m=3, n=1$	$m=3, n=2$	
$\gamma_{mn} \times \epsilon$	.980	4.22	.038	
Freqs. of combined system (num. incr. order)	+.077 -.001 -.013* -.018* -.008	-.031 -.106 -.052* -.053* -.014	+.039 -.048 -.022 -.021* -.022*	$\epsilon = 10^{-4}$
Freqs. of combined system (num. incr. order)	+.777 -.821 -.154* -.153* -.087	-.285 -1.08 -.546* -.513* -.132	+.398 -.458 -.241 -.208* -.211*	$\epsilon = 10^{-3}$
Freqs. of combined system (num. incr. order)	+7.43 -7.34 -1.57* -1.46* -.876	-1.63 -9.85 -5.63* -4.65* -1.33	+4.00 -4.29 -2.28 -2.09* -2.11*	$\epsilon = 10^{-2}$

Table Two

FREQUENCY COMPARISON FOR COMPLETE DETUNING

DIFFERENCE IN PERCENT			
	$\epsilon = 10^{-4}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-2}$
Freqs. of	+.122	+1.20	+10.7
combined	-.133	-1.22	-9.95
system	+.019	+.192	+1.78
(num. incr.	-.031	-.334	-3.16
order)	-.005	-.055	-.554

Table Three

RESPONSE MATRIX COMPARISON FOR PERFECT TUNING

$$\left(\omega_m^{(1)2} = \omega_n^{(2)2}\right)$$

DIFFERENCE IN PERCENT					
$\gamma_{mn} \times \epsilon$	$m=1, n=1$		$m=1, n=2$		
	5.67		.093		
	DOF 1	DOF 2	DOF 1	DOF 2	
Freqs. of combined system (num. incr. order)	+3.65* -2.32* .000 +.040 -.005	+4.57* -3.23* +.013 -.049 +.004	+.028 +1.28* -1.26* +.007 +.016	+.035 +.980* -1.26* -.012 .000	$\epsilon = 10^{-4}$
Freqs. of combined system (num. incr. order)	+1.63* -7.68* -.004 +.406 -.003	+9.13* -10.5* +.144 -.468 +.047	+.229 +5.09* -4.76* +.080 +.008	+.333 +1.46* -1.66* -.113 +.018	$\epsilon = 10^{-3}$
Freqs. of combined system (num. incr. order)	+14.9* -28.6* +.025 +4.02 -.006	+23.3* -36.6* +1.41 -5.16 +.456	+2.17 +26.9* -22.5* +1.35 -.107	+7.78 -9.82* +9.32* -1.12 -.151	$\epsilon = 10^{-2}$

DOF 1 = First equipment degree of freedom

DOF 2 = Second equipment degree of freedom

Table Three (Continued)

RESPONSE MATRIX COMPARISON FOR PERFECT TUNING

$$\left(\omega_m^{(1)^2} = \omega_n^{(2)^2}\right)$$

DIFFERENCE IN PERCENT					
	m=2, n=1		m=2, n=2		
$\gamma_{mn} \times \epsilon$	.107		.980		
	DOF 1	DOF 2	DOF 1	DOF 2	
Freqs. of combined system (num. incr. order)	.000 +2.28* -2.24* -.029 +.012	-.021 -.665* +.701* -.099 -.003	+.084 -.321 -1.12* +1.16* -.007	+.077 -.275 +1.47* -1.50* +.168	$\epsilon = 10^{-4}$
Freqs. of combined system (num. incr. order)	-.009 +8.41* -7.94* -.350 +.131	-.197 -.848* +1.12* -1.00 -.024	+.829 -3.31 -3.52* +3.85* -.067	+.831 -2.87 +4.72* -5.05* +.120	$\epsilon = 10^{-3}$
Freqs. of combined system (num. incr. order)	+.027 +43.9* -35.3* -4.10 +1.53	-1.69 +11.5* -12.3* -11.1 -.411	+7.12 -34.5 -11.1* +14.3* -.724	+7.07 -28.9 +16.1* -19.7* +1.18	$\epsilon = 10^{-2}$

DOF 1 = First equipment degree of freedom

DOF 2 = Second equipment degree of freedom

Table Three (Continued)

RESPONSE MATRIX COMPARISON FOR PERFECT TUNING

$$\left(\omega_m^{(1)^2} = \omega_n^{(2)^2}\right)$$

DIFFERENCE IN PERCENT					
	m=3, n=1		m=3, n=2		
$\gamma_{mn} \times \epsilon$	4.22		.038		
	DOF 1	DOF 2	DOF 1	DOF 2	
Freqs. of combined system (num. incr. order)	-.009 +.327 +.237* -.240* +.005	-.055 +.262 -.354* +.350* .000	+.008 +.384 -.096 +2.99* -2.94*	-.001 +.329 -.016 -.978* +.921*	$\epsilon = 10^{-4}$
Freqs. of combined system (num. incr. order)	-.123 +3.13 +1.27* -1.33* +.037	-.522 +2.62 -.626* +.539* -.014	+.100 +2.76 -.936 +9.30* -8.70*	-.019 +2.19 -.164 -3.25* +2.62*	$\epsilon = 10^{-3}$
Freqs. of combined system (num. incr. order)	-.874 +20.6 +7.76* -10.3* +.400	-4.27 +29.5 +.558* -4.31* -.139	+.985 +29.3 +29.3* +29.3* -22.8*	-.158 +22.5 -13.3* -13.3* +5.37*	$\epsilon = 10^{-2}$

DOF 1 = First equipment degree of freedom

DOF 2 = Second equipment degree of freedom

Table Four

RESPONSE MATRIX COMPARISON FOR COMPLETE DETUNING

DIFFERENCE IN PERCENT				
	$\epsilon = 10^{-4}$		$\epsilon = 10^{-3}$	
	DOF 1	DOF 2	DOF 1	DOF 2
Freqs. of combined system (num. incr. order)	+.296	+.307	+2.86	+2.99
	-.637	-.587	-6.34	-5.77
	-.111	+.128	-1.08	+1.26
	+.193	-.193	+2.36	-1.93
	+.005	+.014	-.014	+.079
	$\epsilon = 10^{-2}$			
	DOF 1	DOF 2		
Freqs. of combined system (num. incr. order)	+21.9	+23.1		
	+83.7	+84.5		
	-9.95	+11.9		
	+18.4	-19.1		
	-.170	+.805		

DOF 1 = First equipment degree of freedom

DOF 2 = Second equipment degree of freedom

Table Five

PHANTOM EARTHQUAKE

γ_{11} (approx.)	DOF	NEWMARK (max. acc. in g)	METHOD (max. acc. in g)	% DIFF
.03	1	2.9480	2.8709	2.62
	2	3.1169	2.8709	7.89
.003	1	4.9708	4.8314	1.19
	2	4.9375	4.8314	1.24
.0003	1	5.7240	5.6207	1.80
	2	5.7706	5.6207	2.60

DOF = Equipment degree of freedom

% DIFF = Percent difference

Table Six

PHANTOM FORCE

γ_{11} (approx.)	DOF	NEWMARK (max. acc. in g)	METHOD (max. acc. in g)	% DIFF
.003	1	1.292	1.295	0.23
	2	1.294	1.295	0.08

Table Seven

EL CENTRO

γ_{11} (approx.)	DOF	NEWMARK (max. acc. in g)	METHOD (max. acc. in g)	% DIFF
.003	1	3.326	3.565	6.29
	2	3.341	3.565	6.70

Table Eight

GREEN'S FUNCTION COMPARISON

γ_{21}	γ_{31}	$\lvert GF_1 \rvert_{max}$	LOCATION (rads.)	$\lvert GF_2 \rvert_{max}$	LOCATION (rads.)
10^{-3}	10^{-6}	1.999	142.9	1.998	142.9
10^{-3}	6×10^{-6}	1.999	73.8	1.998	73.8
10^{-2}	10^{-4}	1.990	42.4	1.984	42.4
10^{-2}	6×10^{-4}	2.000	23.6	1.995	23.6
10^{-1}	10^{-2}	1.776	11.0	1.738	11.0

$\lvert GF_{\substack{1\\2}} \rvert_{max}$ = Maximum of the Green's function given in equations (5.2.3) and (5.2.4) respectively.

Table Nine

UPPER BOUND COMPARISON

γ_{31}	α_{12}	α_{32}	$\lvert \ddot{u}_G \rvert_{max}$	UPPER BOUND	% DIFF
ϵ^2	$\sqrt{\epsilon}$	$\sqrt{\epsilon}$	865.5	1000.	15.5
ϵ^2	$3\sqrt{\epsilon}$	$\sqrt{\epsilon}$	452.5	467.7	3.4
ϵ^2	$\sqrt{\epsilon}$	0	857.3	871.2	1.6
$6\epsilon^2$	$\sqrt{\epsilon}$	$\sqrt{\epsilon}$	280.4	285.5	1.8
$6\epsilon^2$	$3\sqrt{\epsilon}$	$\sqrt{\epsilon}$	335.4	350.0	4.4
$6\epsilon^2$	$\sqrt{\epsilon}$	0	268.1	304.9	13.7

$\lvert \ddot{u}_G \rvert_{max}$ = Maximum of equation (5.2.6)

$\epsilon = 10^{-3}$, $\gamma_{21} = \epsilon$.

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