SPACE CHARGE AND EQUILIBRIUM EMITTANCES IN DAMPING RINGS*

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Abstract

We present a model of dynamics to account for the possible impact of space charge on the equilibrium emittances in storage rings and apply the model to study the current design of the International Linear Collider (ILC) damping rings.

INTRODUCTION

Direct space charge effects have the potential to be of some relevance in the ILC damping rings because of long ring circumference and small transverse emittance. Space charge was extensively studied in the reference lattices considered for the baseline recommendation [1]. It was found that at 5 GeV the 'dogbone' lattices with about 16-17 km length could be vulnerable to space charge but some effects were also noticed in the shorter lattices, especially when lattice errors were considered. In those studies a weak-strong model was used to determine rms emittance degradation by tracking macroparticles over a number of turns comparable to the damping time. We started from a beam with the design emittances, and looked for the excitation of dangerous resonances. Radiation effects were not included during tracking and the reference values for the equilibrium emittances (which determined the initial beam sizes at the start of tracking), were those estimated in the absence of any collective effect. Far from the harmful resonances identified in that study, the question can be raised of the possible impact of space charge on the beam equilibrium and in particular the vertical emittance. In lepton storage rings the beam equilibrium is determined by the balance between radiation damping and quantum excitations. In a non-ideal lattice, space charge can modify the x/y coupling and alter the determination of the vertical equilibrium emittance – horizontally we do not expect any noticeable effect as the emittance is much larger and space-charge effects much smaller than in the vertical direction. In [1] a preliminary application of the envelope formalism [2] was carried out by K. Oide by making a simple linear approximation of the transverse space charge force valid in a small neighborhood of the bunch center. Sizeable effects were found for the dogbone lattices. In this paper we revisit the model yielding that result and argue that the approximation there employed would overestimate the impact of space charge on the beam equilibrium. We base our contention on Sacherer's observation [4] that under certain conditions, which appear to be naturally met in damping rings operating far from resonances, the evolution of the beam envelopes in the presence of space charge can indeed be reproduced using a linear approximation of the space charge forces but with a value for the effective charge density used to estimate those forces that is only a fraction of the charge density of the actual bunches.

MODEL OF BEAM DYNAMICS

We consider a model of beam dynamics described by the Vlasov-Fokker-Planck equation

$$\frac{\partial f}{\partial s} + [f, H] = \frac{\partial}{\partial z_i}(A_{ij} z_j f) + \frac{1}{2} B_{ij} \frac{\partial^2}{\partial z_i \partial z_j} f, \qquad (1)$$

where $\boldsymbol{z} = (x, p_x, y, p_y, z, \delta)$ is the vector of the dynamical variables, f the single-bunch beam distribution in phase space. The RHS of the above equation describes the dynamical effect of radiation emission with the $s-$dependent matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ modelling radiation damping and quantum excitation respectively. In the LHS, $[\cdot, \cdot]$ are the Poisson brackets and H the Hamiltonian generating the symplectic part of the dynamics (q, m are the particle charge and mass; γ_0 the relativistic factor):

$$H = \frac{1}{2}(p_x^2 + k_x x^2 + p_y^2 + k_y y^2 + p_z^2 + k_z z^2) + \alpha\psi. \quad (2)$$

We assume the linear approximation for the external forces. The self-force is described by the potential ψ giving rise to the electric field $\boldsymbol{E} = \nabla\psi$. The coefficient $\alpha = q/mv_0^2\gamma_0^3$ multiplying ψ in (2) accounts for the partial cancellation of the transverse self-force in the ultra-relativistic regime that is of interest here. In the same regime the self-force in the lab frame appears mostly transverse and we will neglect its longitudinal component in the equations of motion. As for the transverse components of the electric field at location z along the bunch, they can be determined as if the bunch was infinitely long and longitudinally uniform with transverse density equal to the density at z.

First consider the case where space-charge forces are completely negligible, in which case H is a quadratic form $H = (\boldsymbol{z}, \boldsymbol{S}_H \boldsymbol{z})/2$, with $\boldsymbol{S}_H$ being a symmetric matrix. Because the external forces are purely linear, it can be shown

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that study of Eq. (1) can be replaced with that of the equations for the first and second moments of f. Following conventional notation we write the second moments as the entries of the symmetric 6×6 'sigma' matrix $\Sigma_{ij} = \langle z_i z_j\rangle$. These obey the differential equations [2]

$$\boldsymbol{\Sigma}' = \boldsymbol{L}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\boldsymbol{L}^T + \boldsymbol{B}, \tag{3}$$

with $\boldsymbol{L} = \boldsymbol{J}\boldsymbol{S}_H - \boldsymbol{A}$ (where $\boldsymbol{J}$ is the fundamental symplectic matrix) and the prime $'$ indicating differentiation with respect to s. Devising an integration method for these equations is straightforward. A simple first-order scheme consists of introducing a splitting between non-diffusive ($\boldsymbol{L}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\boldsymbol{L}^T$) and diffusive ($\boldsymbol{B}$) parts in the RHS of (3) and appling the latter in the kick approximation. If the diffusive kicks are separated by the small interval Δs integrating through a kick at s yields $\boldsymbol{\Sigma}(s^+) = \boldsymbol{\Sigma}(s^-) + \Delta s \boldsymbol{B}(s)$. Between these kicks propagation of the Σ-matrix under the deterministic part of the dynamics can be done exactly in terms of the transfer map (including damping) $\boldsymbol{M}(s', s)$ with $s \rightarrow s' = s + \Delta s$

$$\boldsymbol{\Sigma}(s') = \boldsymbol{M}(s', s)\boldsymbol{\Sigma}(s)\boldsymbol{M}^T(s', s). \tag{4}$$

Because of damping, solutions of (3) starting from arbitrary initial conditions will relax to the equilibrium beam envelopes $\boldsymbol{\Sigma}_{eq}$ of a gaussian distribution in phase space. The corresponding rms (eigen-)emittances can then be determined as the imaginary part of the eigenvalues of the matrix $\boldsymbol{\Sigma}\boldsymbol{J}$ [3]. As the dynamics is not symplectic the emittances so defined will not be exact invariant through the lattice but violation of the invariance is small. Although straightforward this method is not very efficient because of the slow relaxation to equilibrium. Alternately, the equilibrium beam envelopes or the emittances can be determined at once using the method developed in [2]. If we define the transformed diffusion matrix

$$\overline{\boldsymbol{B}}(s) = \int_s^{s+C} d\tau \boldsymbol{M}(s + C, \tau)\boldsymbol{B}(\tau)\boldsymbol{M}^T(s + C, \tau) \tag{5}$$

where C is the machine circumference, the beam envelope at equilibrium is then given by solving for $\boldsymbol{\Sigma}$ the following algebraic linear equation

$$\boldsymbol{\Sigma}(s) = \boldsymbol{M}(s)\boldsymbol{\Sigma}(s)\boldsymbol{M}^T(s) + \overline{\boldsymbol{B}}(s). \tag{6}$$

When space-charge effects are present but are small, as is the case for the ILC damping rings, it seems reasonable to predict that an equilibrium may still exist and that the form of the beam density at equilibrium will remain close to gaussian. If this is the case we can make use of Sacherer's observation [4]that for beams with charge density displaying elliptical symmetry (including in particular gaussian beams) the envelope equations in the presence of space charge form a closed set of equations provided that the beam rms emittance is conserved (or known a priori). To illustrate Sacherer's statement consider the envelope equations for the horizontal motion in differential form

$$\begin{aligned} \langle x^2\rangle' &= 2\langle xp_x\rangle, & (7)\\ \langle xp_x\rangle' &= \langle p_x^2\rangle - k_x\langle x^2\rangle + \alpha\langle xE_x\rangle, & (8)\\ \langle p_x^2\rangle' &= -2k_x\langle xp_x\rangle + 2\langle p_x E_x\rangle. & (9) \end{aligned}$$

Suppose we write the beam distribution in phase space as $f = f_z(z, \delta)f_\perp(x, p_x, y, p_y)$. In the ultrarelativistic regime the transverse components of the electric field obey the 2D Poisson equation

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = qN\lambda\frac{\rho}{\varepsilon_0} \tag{10}$$

where $\lambda(z) = \int d\delta f_z$ is the longitudinal linear density and $\rho(x, y)$ the transverse density with an elliptic symmetry *i.e.* $\rho = \rho(x^2/a^2 + y^2/b^2)$, normalized to unity. Sacherer showed that for these beams the expectation value $\langle xE_x\rangle_5$ only depends on the rms transverse sizes $\sigma_x^2 = \langle x^2\rangle$ and $\sigma_y^2 = \langle y^2\rangle$:

$$\langle xE_x\rangle_5 = \frac{qN\lambda}{4\pi\varepsilon_0}\frac{\sigma_x}{\sigma_x + \sigma_y}. \tag{11}$$

In the above $\langle\cdot\rangle_5$ refers to the average taken over all phase space except the z direction. If we consider a density gaussian in z, $\lambda(z) = \exp(-z^2/2\sigma_z^2)/\sqrt{2\pi}\sigma_z$ and average over the longitudinal variable as well we find for the last term in Eq. (8)

$$\langle xE_x\rangle = \frac{qN}{4\pi\varepsilon_0}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\pi}\sigma_z}\frac{\sigma_x}{\sigma_x + \sigma_y}, \tag{12}$$

with a similar expression holding in the vertical plane. Now, assuming rms emittance conservation we can use the equation $\varepsilon_x^2 = \langle p_x^2\rangle\langle x^2\rangle - \langle xp_x\rangle^2$ to find $\langle p_x^2\rangle$ in terms of $\langle x^2\rangle$ and $\langle xp_x\rangle$, thus replacing Eq. (9), and are left with a closed set of equations for the second moments. Next, observe that Eq. (12) is equivalent to the expression we would obtain if we used an effective purely linear transverse E-field of the form

$$E_x^{\rm eff} = \frac{qN}{4\pi\varepsilon_0}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\pi}\sigma_z}\frac{x}{\sigma_x(\sigma_x + \sigma_y)}. \tag{13}$$

Under Sacherer's assumptions the evolution of the beam envelopes described by Eqs. (7-9) is undistinguishable from that resulting from using this purely linear field – emittance conservation, in particular, is a direct consequence of the linearity of (13). As a result, Eq. (3) is still valid provided that we allow for the matrix $\boldsymbol{L} = \boldsymbol{L}(\boldsymbol{\Sigma})$ a dependence on the envelopes to account for space charge in accordance to (13). The integration scheme outlined below Eq. (3) can be trivially extended to include space charge also in the kick approximation. Eq. (6) for the equilibrium emittances is also still valid. However determining its solution is more complicated as the transfer matrices $\boldsymbol{M}(s', s)$ will also depend on the envelope matrices $\boldsymbol{\Sigma}$ turning (6) into a nonlinear equation. Finally, one can make the interesting observation that compared with the linearized field close to the center of a gaussian bunch, the effective field (13) turns out to be $1/2^{3/2}$ smaller (for the same N).

APPLICATION TO THE ILC DR'S

We applied these ideas to study the equilibrium emittances for the current design of the 6.7 km ILC damping rings. To this end we added further functionality to the MaryLie/Impact (MLI) [5] code, previously used in the study [1], to include treatment of synchrotron radiation. We studied a non ideal lattice with random vertical displacement of the sextupoles to create a finite vertical emittance close to the $\varepsilon_y = 2$ pm target in the design specifications. In the absence of space charge we varied the variance $\sigma_{\rm sxt}$ of the sextupole displacement and determined the dependence of the vertical equilibrium emittance. For each $\sigma_{\rm sxt}$ we created 1000 lattice error realizations. The $5\% - 95\%$ percentile ranges of the resulting ε_y are reported in Fig. 1 as vertical bars, together with the expected average according to theory (dashed line) [6].

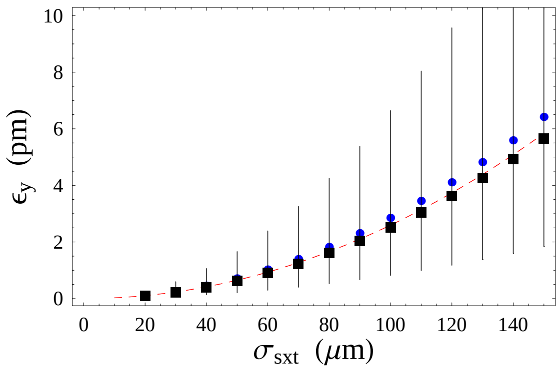

Figure 1: Equilibrium vertical emittance as a function of a random rms vertical displacement of the sextupoles. Boxes are the average values over the $5\% - 95\%$ percentile range of the emittances found; the dots the average over the full range. Space charge not included.

For the case of a non-ideal lattice with space charge perturbed with a single vertically displaced sextupole we calculated the equilibrium vertical emittance by numerical solving Eq. (3) ('slow method', boxes in Fig. 2) and by solving the envelope equation (6) for equilibrium ('fast method', dots in Fig. (2). To solve (6) we adopted a simplified iteration scheme expected to converge to the actual solution only in the limit of vanishing space-charge. Comparison between the results obtained with the two methods provides a range where the fast method can be used, see Fig 2. Finally in Fig. 3 we show an example of frequency distribution of vertical emittances over 200 lattice error realizations with and without space charge (obtained using the fast method). A large value of the bunch population ($N = 5.6 \times 10^{10}$ *vs.* the $N = 2 \times 10^{10}$ design value) was deliberately used here to enhance the effect.

Our conclusion on the basis of the model discussed in this paper is that the impact of space charge on the equilibrium emittances far from resonances for the current design of the 6.7 km ILC damping rings should be extremely mild. Only by increasing the bunch population and experimenting with emittances somewhat smaller than design specification (hence boosting space charge) we could find a significant effect.

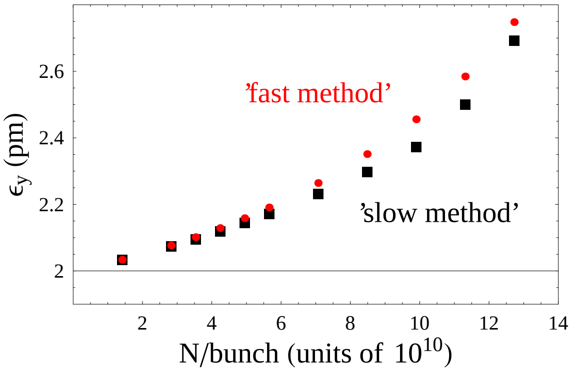

Figure 2: Equilibrium vertical emittance *vs.* bunch population.

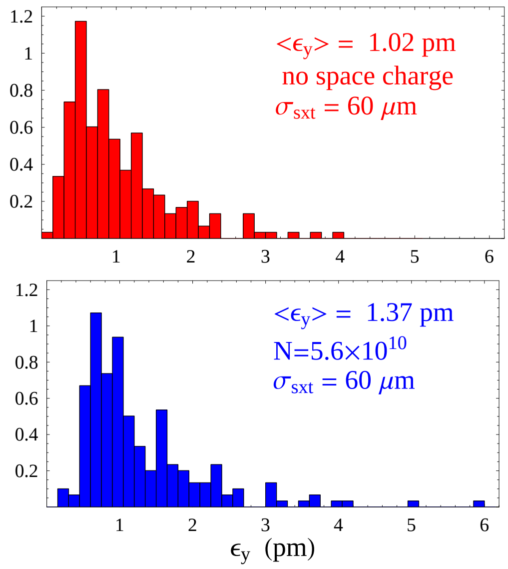

Figure 3: Frequency plot of equilibrium vertical emittance with and without space charge.

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