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Generalized solutions and non-uniqueness in the Einstein constraint equations : some unresolved issues with the conformal formulation

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Generalized Solutions and Non-uniqueness in the Einstein Constraint Equations: Some Unresolved Issues with the Conformal Formulation

A dissertation submitted in partial satisfaction of the
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by

Caleb Meier

Committee in charge:

Professor Michael Holst, Chair
Professor Jim Agler
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The dissertation of Caleb Meier is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2012

DEDICATION

To my beautiful wife Leslie. Without your love and support over the years, none of this would be possible.

EPIGRAPH

Imagination is more important than knowledge.

—Albert Einstein

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Chapter 3, in full, is currently being prepared for submission for publication of the material. M. Holst and C. Meier. The dissertation author was the primary investigator and author of this paper.

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ABSTRACT OF THE DISSERTATION

Generalized Solutions and Non-uniqueness in the Einstein Constraint Equations: Some Unresolved Issues with the Conformal Formulation

by

Caleb Meier

Doctor of Philosophy in Mathematics

University of California, San Diego, 2012

Professor Michael Holst, Chair

In this thesis we consider the problem of determining solutions to the conformal formulation of the Einstein constraint equations with low regularity coefficients and then discuss certain non-uniqueness properties of the conformal formulation of the constraints on a closed manifold. We first investigate the existence of a solution to a semi-linear, elliptic, partial differential equation with distributional coefficients and boundary conditions that models the problem that one encounters when studying the Einstein constraint equations with low regularity data and constant mean curvature. Our method for solving this problem consists of solving a net of regularized, semi-linear problems with data obtained by smoothing the original, distributional coefficients. We then obtain a net of solutions and show that this net satisfies certain decay estimates by

determining estimates for sub- and super-solutions and utilizing classical, *a priori* elliptic estimates. The estimates for this net of solutions allow us to regard this collection of functions as a solution in a Colombeau-type algebra. Following our analysis of this low regularity problem, we consider whether or not solutions to the conformal formulation of the constraints with an unscaled matter source are unique. For positive, constant scalar curvature and constant mean curvature, we first demonstrate the existence of a critical energy density for the Hamiltonian constraint. We then show that for this choice of energy density, the linearization of the elliptic system develops a one-dimensional kernel in both the constant mean curvature and non-constant mean curvature cases. Using a Liapunov-Schmidt reduction and standard techniques from bifurcation theory, we demonstrate that solutions to the conformal formulation with an unscaled data source are non-unique by determining an explicit solution curve and analyzing its behavior in the neighborhood of a particular solution.

Chapter 1

Introduction

1.1 Background and Overview of Research

The Einstein field equation $G_{\mu\nu} = \kappa T_{\mu\nu}$ can be formulated as an initial value (or Cauchy) problem where the initial data consists of a Riemannian metric $\hat{g}_{ab}$ and a symmetric tensor $\hat{k}_{ab}$ on a specified 3-dimensional manifold $\mathcal{M}$ [9, 28]. However, one is not able to freely specify such initial data. Like Maxwell's equations, the initial data $\hat{g}_{ab}$ and $\hat{k}_{ab}$ must satisfy constraint equations, where the constraints take the form

$$\hat{R} + \hat{k}^{ab}\hat{k}_{ab} + \hat{k}^2 = 2\kappa\hat{\rho}, \tag{1.1.1}$$

$$\hat{D}_b\hat{k}^{ab} - \hat{D}^a\hat{k} = \kappa\hat{j}^a. \tag{1.1.2}$$

Here $\hat{R}$ and $\hat{D}$ are the scalar curvature and covariant derivative associated with $\hat{g}_{ab}$, $\hat{k}$ is the trace of $\hat{k}_{ab}$ and $\hat{\rho}$ and $\hat{j}^a$ are matter terms obtained by contracting $T_{\mu\nu}$ with a vector field normal to $\mathcal{M}$. As the Cauchy formulation of the Einstein field equations is one of the most important means of modeling and studying astrophysical phenomena, knowledge of the constraint equations is very important because of the influence that solutions to these equations has on solutions to the evolution problem.

Equation (1.1.1) is known as the Hamiltonian constraint while (1.1.2) is known as the momentum constraint, and collectively the two expressions are known as the Einstein constraint equations. These equations form an underdetermined system of four equations to be solved for twelve unknowns $\hat{g}_{ab}$ and $\hat{k}_{ab}$. In order to transform the constraint equations into a determined system, one divides the unknowns into freely specifiable data and determined data by using what is known as the conformal method. In this method introduced by Lichnerowicz [19] and York [30], we assume that the metric $\hat{g}_{ab}$ is known up to a conformal factor and that the trace $\hat{k}$ and a term proportional to a trace-free divergence-free part of $\hat{k}_{ab}$ is known. Therefore the determined data in this formulation of the constraints is the conformal factor and a vector field whose symmetrized derivative represents the undetermined portion of $\hat{k}_{ab}$. The resulting system forms a determined, coupled nonlinear system of elliptic equations which is referred to as the conformal, transverse, traceless (CTT) formulation of the constraints. However, for simplicity we will refer to this system as the conformal formulation (cf. [3] for further discussion).

In this thesis we consider what David Maxwell has referred to as the ”warts” of the conformal formulation [11] by addressing unresolved issues with the rough solution theory and certain non-uniqueness phenomena associated with this system. We seek to extend the current rough solution theory for the conformal formulation of the constraints [12, 13, 20, 21] by considering a semilinear elliptic PDE that is a generalization of the conformally reformulated Hamiltonian constraint with distributional coefficients. In Chapter 2 we develop a framework to solve this semilinear equation in a generalized function space that contains the space of distributions. The larger generalized function space, or so-called Colombeau algebra, has a well defined notion of multiplication that provides a means of multiplying arbitrary distributions. As we will see when we discuss the rough solution theory later, this extrinsically defined form of distributional multiplication is very beneficial when working with low regularity problems. Following our discussion of this semilinear problem, we address non-uniqueness phenomena associated with the conformal formulation. In Chapter 3 we use standard techniques from bifurcation theory to show that solutions to the conformal formulation with an unscaled energy density are non-unique. We explicitly determine a solution curve of a certain one parameter family of nonlinear elliptic systems and show that there exists a value of the parameter for which at least two solutions must exist.

In the following section we discuss the derivation of the conformal formulation from the constraint equations (1.1.1) and (1.1.2) and then give a broad overview of the solution theory of the conformally rescaled constraint equations.

1.1.1 The Conformal Formulation of the Constraints

In the conformal formulation of the Einstein constraint equations we assume that the spatial metric $\hat{g}_{ab}$ is determined up to a conformal factor on each hypersurface of the foliation of spacetime. That is, we assume that $\hat{g}_{ab} = \phi^4 g_{ab}$, where ϕ is a positive function and g_{ab} is a freely specified symmetric, positive definite (SPD) metric. Additionally, we decompose the extrinsic curvature tensor $\hat{k}_{ab} = \hat{l}_{ab} + \frac{1}{3}\hat{g}_{ab}\hat{\tau}$, where $\hat{\tau} = \hat{g}^{ab}\hat{k}_{ab}$ is the trace and $\hat{l}_{ab}$ is the traceless part (i.e. $\hat{g}^{ab}\hat{l}_{ab} = 0$) of the tensor $\hat{k}_{ab}$. We then conformally rescale the data in the following way

$$\hat{l}^{ab} = \phi^{-10} l^{ab}, \qquad \hat{\tau} = \tau, \qquad\qquad (1.1.3)$$
$$\hat{j}^{a} = \phi^{-10} j^{a}, \qquad \hat{\rho} = \phi^{-8}\rho.$$

The powers in the re-scaling are carefully chosen so that after a rather involved calculation, the constraint equations (1.1.1)-(1.1.2) transform into the following equations

$$-8\Delta\phi + R\phi + \frac{2}{3}\tau^2\phi^5 - l_{ab}l^{ab}\phi^{-7} - 2\kappa\rho\phi^{-3} = 0, \qquad (1.1.4)$$
$$-D_b l^{ab} + \frac{2}{3}D^a\tau\phi^6 + \kappa j^a = 0.$$

See the appendix of [13] for a more complete discussion. In the above expression, Δ and D are the Laplace-Beltrami operator and covariant derivative associated with g_{ab}. The function R is the scalar curvature associated with g_{ab}, where the well-known formula, defining the Yamabe problem for ϕ,

$$\hat{R} = \phi^{-5}(R\phi - 8\Delta\phi) \qquad (1.1.5)$$

allows one to express the scalar curvature $\hat{R}$ associated with $\hat{g}_{ab}$ in terms of the scalar curvature of the conformally related metric g_{ab}.

The final step in converting the original constraint equation into a determined elliptic system is to decompose the traceless tensor $l^{ab} = \sigma^{ab} + (\mathcal{L}w)^{ab}$. Here σ_{ab} is a divergence-free tensor satisfying $D_a\sigma^{ab} = 0$, w^a is a vector field and $\mathcal{L}$ is the conformal killing operator defined by

$$(\mathcal{L}w)^{ab} = D^a w^b + D^b w^a - \frac{2}{3}(D_c w^c)g^{ab}. \qquad (1.1.6)$$

Incorporating this decomposition into (1.1.4) yields the standard CTT or conformal formulation of the constraints

$$-8\Delta\phi + R\phi + \frac{2}{3}\tau^2\phi^5 - [\sigma_{ab} + (\mathcal{L}w)_{ab}][\sigma^{ab} + (\mathcal{L}w)^{ab}]\phi^{-7} - 2\kappa\rho\phi^{-3} = 0, \quad (1.1.7)$$
$$-D_b(\mathcal{L}w)^{ab} + \frac{2}{3}D^a\tau\phi^6 + \kappa j^a = 0. \quad (1.1.8)$$

Equation (1.1.7) is the conformally rescaled Hamiltonian constraint and (1.1.8) is the conformally rescaled momentum constraint.

Remark 1.1.1. *When we demonstrate non-uniqueness properties of the conformal formulation of the constraints, we will consider Eqs* (1.1.7)-(1.1.8) *with unscaled matter fields. Therefore, we do not apply the rescaling in* (1.1.3) *to* $\hat{\rho}$ *and* $\hat{j}^a$. *The conformally rescaled equations with unscaled matter fields are of the form*

$$-8\Delta\phi + R\phi + \frac{2}{3}\tau^2\phi^5 - [\sigma_{ab} + (\mathcal{L}w)_{ab}][\sigma^{ab} + (\mathcal{L}w)^{ab}]\phi^{-7} - 2\kappa\hat{\rho}\phi^5 = 0, \quad (1.1.9)$$
$$-D_b(\mathcal{L}w)^{ab} + \frac{2}{3}D^a\tau\phi^6 + \kappa\hat{j}^a\phi^{10} = 0. \quad (1.1.10)$$

The freely specifiable data in (1.1.7)-(1.1.8) is the mean curvature τ, the divergence free tensor σ_{ab}, the metric g_{ab} (which determines R), and the matter fields ρ and j^a. The determined quantities in the this elliptic system are then the conformal factor ϕ and the vector field w^a. Given a solution ϕ and w^a of Eqs. (1.1.7)-(1.1.8), one obtains the physical metric and extrinsic curvature on the specified surface $\mathcal{M}$ by

$$\hat{g}_{ab} = \phi^4 g_{ab}, \quad \hat{k}_{ab} = \phi^{-10}[\sigma^{ab} + (\mathcal{L}w)^{ab}] + \frac{1}{3}\phi^{-4}\tau g^{ab}. \quad (1.1.11)$$

The matter fields are recovered by using Eq (1.1.3).

One views the conformal method as a means of parametrizing physical solutions to the constraint equations (1.1.1)-(1.1.2). Eqs (1.1.7)-(1.1.8) map a choice of conformal metric g_{ab}, divergence-free tensor $\mathcal{M}_{ab}$, mean curvature τ, and matter fields to a physical metric $\hat{g}_{ab}$ and extrinsic curvature $\hat{k}_{ab}$ satisfying the Einstein constraint equations with physical matter fields transformed according to (1.1.3). In order to understand this parametrization, one must fully understand the solution theory for the system of coupled, nonlinear equations (1.1.7)-(1.1.8).

1.1.2 Existence of Solutions to the Conformal Formulation

The solution theory for the conformally rescaled Einstein constraint equations can be roughly classified according to the manifold $\mathcal{M}$ specified, the Yamabe class of the given metric g_{ab}, the properties of τ (the mean extrinsic curvature) and the regularity of the initial data. The manifolds specified are possibly asymptotically Euclidean, asymptotically hyperbolic, closed, compact with boundary, and Euclidean with interior boundary conditions. As we will be focusing exclusively on closed manifolds, we will only discuss the solution theory for this class. See the survey article [3] for a complete discussion. The Yamabe class of the specified metric g_{ab} is an equivalence class that is determined by whether the metric is conformally equivalent to a metric with constant positive scalar curvature, zero scalar curvature, or negative scalar curvature, in which case we say that the metric is positive Yamabe, zero Yamabe, or negative Yamabe respectively. A well-known result of geometric analysis is that a given metric g_{ab} with scalar curvature R is in the positive (resp. zero, negative) Yamabe class if and only if g_{ab} and R satisfy the Yamabe Eq. (1.1.5) for some metric $\hat{g}_{ab}$ with constant positive (resp. zero, negative) scalar curvature $\hat{R}$ [2]. The properties of the mean curvature refer to whether the function is constant,"near-constant", or non-constant and the regularity of the initial data refers to the smoothness of the intial data (i.e. analytic, smooth, Sobolev and Hölder classes, or distributional).

The mean curvature plays perhaps the largest role in the solution theory of the conformal formulation. If the mean curvature is constant, then the analysis of the conformal formulation simplifies greatly because the Hamiltonian constraint and the momentum constraint decouple, leaving a single semilinear elliptic PDE to analyze. Most of the existence results are based on the assumption that the mean curvature is constant, which we will refer to as the CMC case. For closed manifolds (compact without boundary) with C^2 metrics, the classical solution theory for the conformal formulation is determined for all Yamabe classes and is summarized in [15]. The solution theory is based on the method of sub- and super-solutions, which we will discuss later. In all but one trivial case, the solutions for the cases outlined in [15] are unique. The rough solution theory is also well developed in the CMC case. In [5], Choquet-Bruhat extended the solution theory of the conformal formulation to allow metrics in $W^{2,p}$ where $p > \frac{3}{2}$.

In [20, 21], Maxwell developed a rough solution framework for metrics in H^s where $s > \frac{3}{2}$, and in [12, 13] Holst et al. generalized the rough solution framework to allow for metrics in $W^{s,p}$, with with any pair s, p such that $s > \frac{3}{p}$.

Much less is known about the existence of solutions in the non-constant mean curvature, or non-CMC setting. The first progress made in this area was in [16], where Isenberg and Moncrief assumed that the mean curvature was "nearly constant" (or near-CMC) and that the given metric was C^3 and in the negative Yamabe class. The near-CMC assumption still places serious restrictions on the mean curvature τ in that the function is not allowed to vanish and must have a bounded derivative. However, in the near-CMC case the conformally scaled constraints remain coupled. To analyze the coupled system, Moncrief and Isenberg used the near-CMC assumption to construct constant sub- and super-solutions that served as lower and upper bounds of an iterative sequence of solutions to the Hamiltonian constraint. These sub- and super-solutions allowed them to show that the iteration was a contraction mapping and thereby obtained a unique solution to the conformal system through the Banach Fixed-Point Theorem. In [1], Isenberg et. al determined near-CMC results for smooth, zero Yamabe and positive Yamabe metrics by using a similar construction. Then in [12, 13], Holst et. al utilized a more general fixed-point framework to obtain rough, non-CMC solutions for given metrics in the positive Yamabe class. Here the authors employed a coupled version of the Schauder fixed-point Theorem and utilized a technique to construct sub- and super-solutions that was independent of the near-CMC assumption. This approach produced the first non-CMC existence result that did not require the near-CMC condition and extended the rough solution theory of Maxwell to the non-CMC case. However, given that the result utilized a more general fixed-point construction, the so called far-from-CMC solutions obtained in [12] are not necessarily unique.

1.2 Rough Solutions to the Conformal Formulation

Now that we've given a general overview of the solution theory of the conformal formulation on closed manifolds, we will discuss the motivation behind constructing low regularity solutions to the constraints and some of the difficulties one encounters in the process. This will provide some motivation for the generalized solution framework that we develop in Chapter 2. We first introduce some notation that will be useful in the following discussion and throughout the rest of the paper.

1.2.1 Function Spaces

Let $\mathcal{M}$ be a three-dimensional manifold and let g_{ab} be a fixed SPD background metric. Let $v^{a_1,\cdots,a_r}_{b_1,\cdots,b_s}$ be a tensor of type $r+s$. Then at a given point $x \in \mathcal{M}$ we define its magnitude to be

$$|v| = (v^{a_1,\cdots,b_s} v_{a_1,\cdots,b_s})^{\frac{1}{2}}, \tag{1.2.1}$$

where the indices of v are raised and lowered with respect to g_{ab}. We then define the Banach space of k-differentiable functions $C^k(\mathcal{M})$ with norm $\|\cdot\|_k$ to be those functions u satisfying

$$\|u\|_k = \sum_{j=0}^{k} \sup_{x\in\mathcal{M}} |D^j u| < \infty,$$

where D is the covariant derivative associated with g_{ab}. Similarly, we define the space $C^k(\mathcal{T}^r_s\mathcal{M})$ of k-times differentiable (r,s) tensor fields to be those tensors v satisfying $\|v\|_k < \infty$.

We will also make use of the Sobolev spaces $W^{k,p}(\mathcal{M})$ and $W^{k,p}(\mathcal{T}^r_s\mathcal{M})$ where we assume $k \in \mathbb{N}$ $(k \geq 0)$ and $p \in \mathbb{R}$, $(p \geq 1)$. If dV_g denotes the volume form associated with g_{ab}, then the L^p norm of an (r,s) tensor is defined to be

$$\|v\|_p = \left(\int_{\mathcal{M}} |v|^p dV_g \right)^{\frac{1}{p}}. \tag{1.2.2}$$

We can then define the Banach space $W^{k,p}(\mathcal{M})$ (resp. $W^{k,p}(\mathcal{T}^r_s\mathcal{M})$) to be those functions (resp. (r,s) tensors) v satisfying

$$\|v\|_{k,p} = \left(\sum_{j=0}^{k} \|D^j v\|_p^p \right)^{\frac{1}{p}} < \infty.$$

We will denote the Hilbert spaces $W^{k,2}(\mathcal{M})$ (resp. $W^{k,2}(\mathcal{T}^r_s\mathcal{M})$) by $H^k(\mathcal{M})$ (resp.

$H^k(\mathcal{T}^r_s\mathcal{M})$). The above spaces are related by the usual Sobolev embedding theorems. For example, if $u \in W^{k,p}(\mathcal{M})$ and $k - \frac{3}{p} > r$, then $u \in C^r(\mathcal{M})$. See [25, 10, 2] for a more in depth discussion of the above norms and spaces.

An important feature of Sobolev spaces is that in general they are not closed under multiplication. With the exception of the Hilbert spaces $H^k(\mathcal{M})$ with $k > \frac{d}{2}$ (where d is the dimension of the space), the product of two Sobolev functions in a given space will not in general lie in that space. This greatly limits the Sobolev spaces that one considers when attempting to develop a weak formulation of a given elliptic partial differential equation. The following theorem taken from [13] summarizes the conditions imposed on the Sobolev indices so that pointwise multiplication is a bounded map between given Sobolev spaces.

Theorem 1.2.1. *Let $s_i \geq s$ with $s_1 + s_2 \geq 0$, and $1 \leq p, p_i \leq \infty \ \ (i = 1, 2)$ be real numbers satisfying*

$$s_i - s \geq n\left(\frac{1}{p_i} - \frac{1}{p}\right), \qquad s_1 + s_2 - s > n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right). \tag{1.2.3}$$

In case $\min(s_1, s_2) < 0$, in addition let

$$s_1 + s_2 \geq n\left(\frac{1}{p_1} + \frac{1}{p_2} - 1\right). \tag{1.2.4}$$

Then the pointwise multiplication of functions extends uniquely to a continuous bilinear map

$$W^{s_1,p_1}(\mathcal{M}) \otimes W^{s_2,p_2}(\mathcal{M}) \to W^{s,p}(\mathcal{M}). \tag{1.2.5}$$

Remark 1.2.2. *Note that the above theorem allows for the Sobolev exponent s to be in $\mathbb{R}$ whereas we have only provided a definition for $s \in \mathbb{N}, \ (s \geq 0)$. Using the Fourier transform, one may extend the definition of the Sobolev spaces $W^{s,p}$ to allow for $s \in \mathbb{R}$. See [12, 13] for a complete discussion.*

Theorem 1.2.1 can be interpreted as a statement about distributional multiplication. In general, the multiplication of arbitrary distributions is not well-defined [8]. However, the above theorem states that for certain combinations of Sobolev functions (distributions), multiplication is a well-defined operation that produces another Sobolev function (in a possibly different Sobolev space). With the above theorem in hand, we are ready to discuss the rough solution theory of the constraints in a bit more detail.

1.2.2 Solutions of Constraint Equations with Low Regularity Metrics

We've already given a general overview of the history of rough solutions to the conformal formulation on closed manifolds. Here we describe the motivation for developing a solution theory with low regularity metrics and then discuss the rough solution results in [20, 21, 12, 13] in more detail.

There is an incentive to develop a low regularity solution framework for the Einstein field equations to model plausible astronomical phenomena such as cosmic strings and gravitational waves [8]. The solutions to the constraint equations not only place a restriction on which metrics and extrinsic curvature tensors can be considered as initial data, but they also determine the function spaces of maximally globally hyperbolic solutions to the evolution problem [3]. Therefore it is necessary that the solution theory for the constraint equations keep pace with (or even move further than) the solution theory for the evolution equations, because otherwise existence results for rough solutions of the Einstein field equation cannot be determined. Historically, the rough solution theory of the constraints has lagged behind that of the evolution problem. The local well-posedness result for quasilinear hyperbolic systems in [14] allows for initial data (g, K) in $H^s \times H^{s-1}$ for $s > \frac{5}{2}$; however it was not until [5, 20, 21] that solutions of this regularity existed to the constraint equations. Low regularity solutions became increasingly important when Klainerman and Rodnianski developed *a priori* estimates in [18] for the time existence of solutions to the vacuum Einstein equations in terms of the $H^{s-1} \times H^{s-1}$ norm of (Dg, K), again with $s > 2$. This prompted Maxwell's work in [20] and [21] and Holst's et al. work in [12, 13].

The current rough solution theory outlined in [12, 13] requires that if $p \in (1, \infty)$, then the metric g_{ab} must lie in the Sobolev space $W^{s,p}$ where $s > \frac{3}{p} + 1$ in the non-CMC case and $s > \frac{3}{p}$ in the CMC case. When $p = 2$, we see that the current CMC solution theory allows for lower regularity metrics than the $H^s \times H^{s-1}$ $(s > 2)$ well-posedness result of Klainerman and Rodnianski. However, Klainerman's and Rodnianski's work allows for lower regularity initial data than that produced by the non-CMC rough solution theory. Therefore there is room for improvement in this area.

One of the difficulties associated with obtaining rough solutions to the conformal formulation is that the choice of metric in a given Sobolev class restricts the spaces for which the Laplace-Beltrami operator and the operator $D_b(\mathcal{L}w)^{ab}$ are well-defined. Indeed, the Laplace-Beltrami operator in local coordinates

$$\Delta_g u = \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}\, g^{ij}\partial_j u\right), \tag{1.2.6}$$

has the form $a^{ij}\partial_i\partial_j + b^i\partial_i$, where $a^{ij} \in W^{s,p}$ and $b^i \in W^{s-1,p}$ [20]. So for example, if $p = 2$, then Theorem 1.2.1 implies that $\Delta_g : H^\sigma \to H^{\sigma-2}$ is only well-defined for $\sigma \in [2-s, s]$. Similarly, if we regard the operator $D_b(\mathcal{L}w)^{ab} : W^{e,q} \to W^{e-2,q}$ as a map between Sobolev spaces and $g_{ab} \in W^{s,p}$ with $s \in (\frac{3}{p}, \infty)$, $q \in (1, \infty)$, then we have the rather limiting condition that $e \in (2-s, s] \cap (-s + \frac{3}{p} - 1 + \frac{3}{q}, s - \frac{3}{p} + \frac{3}{q}]$.

In addition to affecting to the domain of the second order elliptic operators Δ and $D_b(\mathcal{L}w)^{ab}$, the regularity of the metric also determines the regularity of the other initial data. If

$$\Delta\phi : W^{s,p} \to W^{s-2,p},$$

then in order for the Hamiltonian constraint to be regarded as an operator between these spaces it is necessary that the pointwise multiplication operator

$$\begin{aligned} &f(\phi, \mathbf{w}) = R\phi + \frac{2}{3}\tau^2\phi^5 - a_{\mathbf{w}}\phi^{-7} - 2\kappa\rho\phi^{-3}, \\ &f(\phi, \mathbf{w}) : W^{s,p} \to W^{s-2,p} \end{aligned} \tag{1.2.7}$$

be well-defined. Given that $\phi \in W^{s,p}$, Eq. (1.2.7) and Theorem 1.2.1 place regularity restrictions on τ and

$$a_{\mathbf{w}} = [\sigma_{ab} + (\mathcal{L}w)_{ab}][\sigma^{ab} + (\mathcal{L}w)^{ab}],$$

which in turn places restrictions on σ_{ab} and the solution to the momentum constraint $\mathbf{w}$.

Furthermore, in the non-CMC case one needs to use *a priori* estimates to control the norm $\|\mathcal{L}\mathbf{w}\|_\infty$ in terms of $D^a\tau$, ϕ, and j^a to manage the coupling of the Hamiltonian constraint with the momentum constraint. This requires the use of the Sobolev embedding theorem and therefore imposes additional restrictions on the Sobolev indices. These restrictions are the reason that $s > 1 + \frac{3}{p}$ in the non-CMC case.

Our work with Colombeau algebras in this paper is motivated by the restrictions described above and by the need to reconcile the low regularity existence results obtained in [22] with the positive Yamabe framework established in [12, 13]. In [22], Maxwell considers the two problems that are at the center of our discussion–existence of solutions with low regularity data and non-uniqueness–and he investigates them in the non-CMC setting. He shows that for a certain family of low regularity mean curvature functions that violate the near-CMC assumption that solutions to the conformal formulation are non-unique. However, Maxwell considers this problem in the Yamabe zero setting with mean curvature functions that are less regular than those allowed in [12, 13]. Part of the goal of our research is to extend the low regularity results of Holst et. al in [12, 13] to include the results of Maxwell in [22] so that the relationship between low regularity data, far-from-CMC mean curvature assumptions and non-uniqueness can be more clearly understood.

We seek to develop a framework for solving semilinear PDE with distributional coefficients within an abstract function algebra that is free of the limitations of distributional (Sobolev) multiplication. However, this approach is not without other limitations. Once one obtains a solution in the larger generalized function space, it is then desirable to relate the solution to some sort of distributional solution via weak convergence. In the case that the elliptic PDE is linear and the coefficients are sufficiently regular, then the generalized solution will converge weakly to the classical solution [24]. In the event that the PDE is nonlinear, it has proved difficult to show a clear relationship between generalized solutions and classical solutions if they exist. We discuss this point more in Section 2.2 in Chapter 2.

1.3 Examples of Non-uniqueness of Solutions to the Constraints

We now turn our attention from the rough solution theory described in the previous section to the other main topic of this thesis–non-uniqueness of solutions. In Chapter 3 we demonstrate non-uniqueness of solutions to the conformal formulation with unscaled matter fields. Here we provide a little history and motivation for considering this problem.

We have already mentioned a few instances where solutions to the conformal formulation are not necessarily unique. The far-from-CMC results obtained in [12, 13] did not utilize a contraction mapping argument, but instead employed a more general topological fixed-point theorem. Unlike the contraction mapping argument, this approach does not guarantee uniqueness of solutions. Maxwell's work in [22] showed that solutions to the conformal formulation are not unique if the metric is in the zero Yamabe class and the mean curvature τ has sufficiently low regularity and fails to satisfy the near-CMC assumptions.

There has also been a great deal of research done on the non-uniqueness of solutions to alternative formulations of the constraints known as the conformal thin sandwich method (CTS method) and the extended conformal thin sandwich method (XCTS method). These two formulations are extremely popular amongst numerical relativists and the first evidence that solutions to the these formulations were non-unique was demonstrated in [26]. In this paper, Pfieffer and York obtained numerical evidence that the solution curve of the XCTS system contained two distinct branches of solutions. Then in [4], O'Murchadha et al. constructed a simplified model of a spherical constant density star and explicitly showed that solutions to the Hamiltonian constraint with an unscaled matter source were non-unique. This analysis was conducted to show that certain terms with the wrong sign that appear in the nonlinear XCTS system caused the non-uniqueness phenomena observed by York in [26]. Finally, in [29], Walsh generalized O'Murchadha's work by applying basic techniques from bifurcation theory to give a partial analysis of the non-uniquess behavior of the XCTS system on an asymptotically Euclidean manifold. His analysis, however, relied on the key assumption that the linearization of the system developed a one-dimensional kernel at a particular solution, and he also assumed that certain terms in his representation of the solution curve did not vanish.

In Chapter 3 we expand on Walsh's work and the work above by considering the conformal formulation with unscaled matter fields on a closed manifold. We explicitly show that the linearization of a one parameter family of elliptic systems develops a one dimensional kernel at a particular solution. This allows us to rigorously apply a Liapunov-Schmidt reduction to determine an explicit solution curve of the family of problems considered. Analysis of the curve reveals that if the parameter is sufficiently small then solutions to the system are non-unique. Now we give a more precise statement of the problems that we consider in this paper.

1.4 Problems Considered

The problems that we consider in this thesis address two of the most important, unresolved issues associated with the conformal method–the need for a more general low regularity solution framework, and the non-uniqueness behavior associated with solutions. In the following sections we give a more explicit statement of the problems that we consider that are meant to partially address these issues. We first give a rough outline of our method to solve a certain family of semilinear PDEs with distributional coefficients. Then we describe our process to prove non-uniqueness of solutions to the conformal formulation on a closed manifold.

1.4.1 Generalized Solutions of a Semilinear Elliptic PDE with Distributional Coefficients

In Chapter 2 we consider the following problem

$$-\sum_{i,j=1}^{N} D_i(a^{ij}D_ju) + \sum_{i=1}^{K} b^i u^{n_i} = 0 \quad \text{in } \Omega, \tag{1.4.1}$$

$$u|_{\partial\Omega} = \rho,$$

on an open, bounded domain $\Omega \subset \mathbb{R}^n$, where $n_i \in \mathbb{Z}$ and the coefficients a^{ij}, b^i and boundary function ρ are distributional with certain restrictions. This semilinear elliptic equation (1.4.1) is a generalization of the local formulation of the Hamiltonian constraint on a domain in $\mathbb{R}^n$. We seek to develop a low regularity solution framework for this class of problems.

Problems Associated with Distributional Multiplication

The difficultly with considering the problem (1.4.1) is that the elliptic operator (1.4.1) is not a well-defined operator even on the space of distributions. Recall that if $C_0^\infty(\Omega)$ denotes the spaces of infinitely differentiable functions with compact support in Ω, then the space of distributions, $\mathcal{D}'(\Omega)$, is the collection of linear functionals acting on $C_0^\infty(\Omega)$. Even though multiplication of a distribution by a C^∞ function is a well-defined procedure, there is no "well-behaved" form of distributional multiplication that extends this definition. By well-behaved, we mean that any attempt at defining an intrinsic form of distributional multiplication (i.e. resulting product is a distribution) leads to an operation that is either not associative, commutative or is inconsistent with the properties of generalized differentiation. See [8] for more details. Consider the following example, which is typical of the pathologies that result from attempts at defining distributional multiplication.

Example 1.4.1. *Let $\delta(x)$ denote the dirac delta function and let $vp\frac{1}{x}$ denote the principle value of $\frac{1}{x}$. Suppose that a well-defined notion of distributional multiplication exists that extends the definition of multiplication by a smooth function. Then we have*

$$0 = (\delta(x) \cdot x) \cdot vp\frac{1}{x} \neq \delta(x) \cdot \left(x \cdot vp\frac{1}{x} \right) = \delta(x). \tag{1.4.2}$$

This example shows that if any notion of multiplication exists within the space of distributions that is consistent with the definition of distributional multiplication by a C^∞ function, then it cannot be associative. See [8] for other examples of pathologies. In the following discussion, which is taken from [8], we present one approach to resolve some of the issues described.

One solution to the problem of distributional multiplication is to seek an embedding of $\mathcal{D}'(\Omega)$ into some larger algebra. More specifically, one seeks an embedding and an algebra $(\mathcal{A}(\Omega), +, \circ)$ such that

(i) $\mathcal{D}'(\Omega)$ is linearly embedded into $\mathcal{A}(\Omega)$ and $f(x) \equiv 1$ is the unity in $\mathcal{A}(\Omega)$.

(ii) There exist derivation operators $\partial_i : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ $(i = 1, \cdots, n)$ that are linear and satisfy the Leibniz rule.

(iii) $\partial|_{\mathcal{D}'(\Omega)}$ is the usual partial derivative $(i = 1, \cdots, n)$.

(iv) $\circ|_{\mathcal{C}(\Omega)\times\mathcal{C}(\Omega)}$ coincides with the pointwise product of functions.

However, in his famous impossibility result, L. Schwartz showed in [27] that there is no associative, commutative algebra satisfying (i)-(iv). However, if one weakens (iv) to

(v) $\circ|_{\mathcal{C}^\infty(\Omega)\times\mathcal{C}^\infty(\Omega)}$ coincides with the pointwise product of functions,

then commutative, associative algebras satisfying (i)-(iii) and (v) exist. These are the algebras constructed by Colombeau in [7] and we will use these so-called Colombeau algebras to solve (1.4.1). We give a very brief introduction of the Colombeau algebra construction on a bounded open set Ω and then describe how to formulate (1.4.1) as a problem within this generalized function space.

Colombeau Algebras

The following discussion provides a brief overview of the construction of the so-called special algebra of Colombeau and is taken from [8].

It is well known that any distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is the weak limit of a net of $C^\infty(\mathbb{R}^n)$ functions $(u_\epsilon)_\epsilon$. This follows by considering the convolution $(u * \phi_\epsilon)_\epsilon$, where $\epsilon \in (0, 1]$ and

$$\phi \in C_0^\infty(\mathbb{R}^n), \qquad \int \phi \, dx = 1, \quad \text{and} \quad \phi_\epsilon = \epsilon^{-n}\phi\left(\frac{x}{\epsilon}\right). \tag{1.4.3}$$

With this in mind, we consider the following subsets of $C^\infty(\mathbb{R}^n)^I$, where $I = (0, 1]$:

$$\mathcal{V} := \{(u_\epsilon)_\epsilon \in C^\infty(\mathbb{R}^n)^I \mid \exists u \in \mathcal{D}'(\mathbb{R}^n) \ \ with \ \ u_\epsilon \to u \text{ in } \mathcal{D}'\}, \tag{1.4.4}$$

$$\mathcal{V}_0 := \{(u_\epsilon)_\epsilon \in C^\infty(\mathbb{R}^n)^I \mid u_\epsilon \to 0 \text{ in } \mathcal{D}'(\mathbb{R}^n)\}. \tag{1.4.5}$$

Then by the regularization property of $\mathcal{D}'(\mathbb{R}^n)$ mentioned above, it follows that the linear map

$$\begin{aligned} \Psi: \ \mathcal{V}/\mathcal{V}_0 \quad &\rightarrow \quad \mathcal{D}'(\mathbb{R}^n), \\ (u_\epsilon)_\epsilon + \mathcal{V}_0 \quad &\rightarrow \quad \lim_{\epsilon \to 0} u_\epsilon, \end{aligned} \tag{1.4.6}$$

is an isomorphism, where the above limit is taken in the weak sense. This heuristic provides the motivation for the construction of the special algebra.

The significance of the space $C^\infty(\mathbb{R}^n)^I$ in the above construction is that it is a differential algebra when endowed with componentwise operations (vector space operations, multiplication, partial differentiation)[8]. Therefore, in order to construct an associative, commutative algebra in which we can embed $\mathcal{D}'(\mathbb{R}^n)$, the previous discussion suggests that we choose an appropriate ideal $\mathcal{I}$ of $C^\infty(\mathbb{R}^n)^I$ such that $\mathcal{I} \subset \mathcal{V}_0$ [8]. Then we can include $\mathcal{D}'$ by convolving with a mollifier ϕ satisfying (1.4.3) by using the following embedding:

$$\begin{aligned} &\mathcal{D}'(\mathbb{R}^n) \hookrightarrow \mathcal{C}^\infty(\mathbb{R}^n)^I/\mathcal{I}, \\ &u \rightarrow (u * \phi_\epsilon)_\epsilon + \mathcal{I}. \end{aligned} \tag{1.4.7}$$

We now observe that for any ideal $\mathcal{I} \subset \mathcal{V}_0$, Eq. (1.4.7) will be an embedding of $\mathcal{D}'$ into an associative, commutative differential algebra. However, we want the embedding to satisfy the compatibility condition (v), namely that multiplication in the algebra coincide with multiplication of C^∞ functions. The ideal $\mathcal{I}$ is constructed to meet this requirement. More specifically, one can embed $f \in C^\infty(\mathbb{R}^n)$ by $f \rightarrow (f)_\epsilon + \mathcal{I}$. This will ensure that the compatibility conditions are met. On the other hand, our means of embedding distributions implies that $f \rightarrow (f * \phi_\epsilon)_\epsilon + \mathcal{I}$ must hold. Therefore we construct the ideal $\mathcal{I}$ so that these two expressions coincide in the quotient space [8]. By considering the difference

$$(f * \phi_\epsilon - f)(x) = \int (f(x-y) - f(x))\phi_\epsilon(y)dy \tag{1.4.8}$$
$$= \int \sum_{k=1}^{m} \frac{(-\epsilon y)^k}{k!} f^{(k)}(x)\phi(y)dy + \int \frac{(-\epsilon y)^{m+1}}{(m+1)!} f^{(m+1)}(x - \theta\epsilon y)\phi(y)dy,$$

and additionally assuming that $\int \phi(x)x^k dx = 0$ for all $k \geq 1$, we see that (1.4.8) will tend to 0 if and only if u_ϵ and all derivatives vanish faster than any power of ϵ on compact sets. We consider $\mathcal{I}$ to be the subset of $C^\infty(\mathbb{R}^n)^I$ containing nets that satisfy these properties. However, $C^\infty(\mathbb{R}^n)$ contains nets of arbitrarily fast growth (exponential) growth in $\frac{1}{\epsilon}$ [8]. To ensure that $\mathcal{I}$ is an ideal, we restrict ourselves to a subalgebra–the space of moderate elements–which consists of those nets $(u_\epsilon)_\epsilon$ whose derivatives $\partial^\alpha u_\epsilon$ blow up at a rate of $O(\epsilon^{-N(\alpha)})$ for some $N(\alpha) \in \mathbb{N}$. We summarize the above discussion with the following definition.

Definition 1.4.2. *Let*

$$\mathcal{E}^s(\Omega) \;:=\; (C^\infty(\Omega))^I, \tag{1.4.9}$$
$$\mathcal{E}^s_M(\Omega) \;:=\; \{(u_\epsilon)_\epsilon \in \mathcal{E}^s(\Omega) \mid \forall K \subset\subset \Omega \;\; \forall \alpha \in \mathbb{N}_0^\alpha \;\; \exists N \in \mathbb{N} \text{ with } \sup_{x \in K} |\partial^\alpha u_\epsilon(x)| = O(\epsilon^{-N}) \text{ as } \epsilon \to 0\}, \tag{1.4.10}$$
$$\mathcal{N}^s(\Omega) \;:=\; \{(u_\epsilon)_\epsilon \in \mathcal{E}^s(\Omega) \mid \forall K \subset\subset \Omega \;\; \forall \alpha \in \mathbb{N}_0^\alpha \;\; \forall m \in \mathbb{N} \; : \sup_{x \in K} |\partial^\alpha u_\epsilon(x)| = O(\epsilon^{m}) \text{ as } \epsilon \to 0\}. \tag{1.4.11}$$

Elements of $\mathcal{E}^s_M(\Omega)$ are called moderate elements and elements of $\mathcal{N}^s(\Omega)$ are referred to as negligible elements. The quotient algebra $\mathcal{G}^s(\Omega) = \mathcal{E}^s_M(\Omega)/\mathcal{N}^s(\Omega)$ is the special Colombeau algebra on Ω.

Solution of a Semilinear Problem in $\mathcal{G}^s(\Omega)$

Now that we have introduced the special algebra $G^s(\Omega)$, we are ready to discuss our approach for solving a semilinear elliptic PDE in this generalized function space.

An important property of the algebra $G^s(\Omega)$ is that it is a fine sheaf (i.e. $G^s(-)$ is well behaved with respect to open coverings and partitions of unity) [8]. This property allows one to construct an embedding

$$i: \ \mathcal{D}'(\Omega) \to \mathcal{G}^s(\Omega), \qquad (1.4.12)$$
$$i(u) \to (u_\epsilon)_\epsilon + \mathcal{N}^s(\Omega),$$

where we omit the details of the embedding due to its complicated form.

By embedding the distributions into $\mathcal{G}^s(\Omega)$, we are mapping a distribution u to an equivalence class $[(u_\epsilon)_\epsilon]$. Therefore, in order to solve (1.4.1) in $G^s(\Omega)$, we embed the distributional coefficients a^{ij}, b^i and ρ into $\mathcal{G}^s(\Omega)$ and choose certain representatives $(a^{ij}_\epsilon)_\epsilon$, $(b^i_\epsilon)_\epsilon$ and $(\rho_\epsilon)_\epsilon$ of their respective equivalence classes to obtain a net of operators

$$A_\epsilon u_\epsilon = -D_i(a^{ij}_\epsilon D_j u_\epsilon) + \sum_i^K b^i_\epsilon (u_\epsilon)^{n_i}. \qquad (1.4.13)$$

We then show that A_ϵ is independent of the choice of representatives, thereby obtaining an equivalence class of operators $[A_\epsilon]$ which represents a well-defined differential operator on $G^s(\Omega)$.

For each fixed ϵ, we solve the net of semilinear problems (1.4.13) by using a technique known as method of sub- and super-solutions. This approach requires determining a net of constants $(\alpha_\epsilon)_\epsilon$ and $(\beta_\epsilon)_\epsilon$ such that for each $\epsilon > 0$, $\alpha_\epsilon < \beta_\epsilon$ and

$$A_\epsilon(\alpha_\epsilon) \le 0, \qquad (1.4.14)$$
$$A_\epsilon(\beta_\epsilon) \ge 0.$$

The sub- and super-solution Theorem proved in Chapter 2 then implies the existence of a solution $\alpha_\epsilon \le u_\epsilon \le \beta_\epsilon$ for each $\epsilon \in (0, 1]$. This generates a net of solutions $(u_\epsilon)_\epsilon$ which we then verify is a representative of an element of $\mathcal{G}^s(\Omega)$ by utilizing classical Schauder estimates.

The framework described above is the general approach that we will use in Chapter 2 to solve the low-regularity, semilinear problem (1.4.1) in $\mathcal{G}^s(\Omega)$. Now we discuss the method by which we prove non-uniqueness in more detail.

1.4.2 Non-uniqueness of the Conformal Formulations with an Unscaled Matter Field

The non-uniqueness results of Chapter 3 pertain to the following one parameter family of problems

$$-\Delta\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - a_{\mathbf{w}}\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5 = 0, \qquad (1.4.15)$$
$$\mathbb{L}\mathbf{w} + \lambda b^a_\tau \phi^6 = 0.$$

Here we assume that g_{ab} is a given SPD metric with no conformal killing fields that has constant, positive scalar curvature. We let D_a and Δ denote the covariant derivative and the Laplace-Beltrami operator associated with g_{ab} and let

$$\mathbb{L}\mathbf{w} = D_b(\mathcal{L}\mathbf{w})^{ab}$$

denote the divergence of the conformal killing operator associated with g_{ab}. Finally, we define

$$a_R = \frac{1}{8}R, \qquad a_\tau = \frac{1}{12}\tau^2, \qquad (1.4.16)$$
$$a_{\mathbf{w}} = \frac{1}{8}[\sigma_{ab} + (\mathcal{L}\mathbf{w})_{ab}][\sigma^{ab} + (\mathcal{L}\mathbf{w})^{ab}], \qquad b_\tau = \frac{2}{3}D^a\tau.$$

We assume that R is a positive constant and that $|\sigma| = (\sigma_{ab}\sigma^{ab})^{\frac{1}{2}}$ is also a nonzero constant. Notice that (1.4.15) has the form of the conformally rescaled constraints (1.1.9)-(1.1.10) with unscaled matter sources ρ, j^a and initial data depending on λ where

$$\tau_\lambda = \lambda\tau, \quad \rho_\lambda = e^{-\lambda}\rho \quad \text{and} \quad \mathbf{j}_\lambda = \mathbf{0}.$$

We show that in both the CMC and non-CMC cases that solutions to (1.4.15) are non-unique. Our method for doing this is to apply a standard technique from bifurcation theory known as the Liapunov-Schmidt reduction. This technique consists of formulating our one-parameter family of problems (1.4.15) as a nonlinear operator problem between Banach spaces. Then, the problem is reduced to a finite-dimensional problem by using the Implicit function theorem and other basic tools from functional analysis. We first review these basic tools, then discuss the reduction process.

Some Basic Functional Analysis

Here we follow the notation in [31]. Recall that a Banach space X is a complete, normed vector space and a Hilbert space $\mathcal{H}$ is a Banach space whose norm is induced by an inner product. Given two Banach spaces X_1 and X_2, with norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$, one can construct a new Banach space by forming the direct sum $X_1 \oplus X_2$ which consists of elements in $X_1 \times X_2$ with the norm

$$\|\cdot\|_{X_1 \oplus X_2} = \sqrt{\|\cdot\|_{X_1}^2 + \|\cdot\|_{X_2}^2}.$$

One can do the same for Hilbert spaces by taking the sum of the inner products associated with each space to obtain a new Hilbert space. See [31] for a complete discussion of these topics.

In our non-uniqueness analysis of the one-parameter family (1.4.15), we formulate the problem in the form

$$F(x, \lambda) = 0, \tag{1.4.17}$$

where $F : U \times V \to Z$ is a nonlinear operator between Banach spaces. The most important theorem in functional analysis for analyzing such a problem is the Implicit Function Theorem. This theorem plays an integral role in our non-uniqueness analysis and we state it here without proof for completeness. The following statement is taken from [6]

Theorem 1.4.3 (Implicit Function Theorem). *Suppose X, Y, Z are Banach spaces, $U \subset X$, $V \subset Y$ are open sets, $F : U \times V \to Z$ is continuously differentiable, $(x_0, y_0) \in U \times V$, $F(x_0, y_0) = 0$ and $D_x F(x_0, y_0)$ has a bounded inverse. Then there is a neighborhood $U_1 \times V_1 \subset U \times V$ of (x_0, y_0) and a function $f : V_1 \to U_1$, $f(y_0) = x_0$ such that $F(x, y) = 0$ for $(x, y) \in U_1 \times V_1$ if and only if $x = f(y)$. If $F \in C^k(U \times V, Z)$, $k \geq 1$ or analytic in a neighborhood of (x_0, y_0), then $f \in C^k(V_1, X)$ or is analytic in a neighborhood of y_0.*

The concepts of the adjoint and a Fredholm operator will also be important in our work. Indeed, these concepts are essential for reducing the nonlinear operator problem (1.4.17) into a finite-dimensional problem. Given an inner product $\langle \cdot, \cdot \rangle$ associated with a particular Hilbert space $\mathcal{H}$ and a linear operator $A : \mathcal{H} \to \mathcal{H}$, the adjoint $A^* : \mathcal{H} \to \mathcal{H}$ associated with $\mathcal{H}$ is the unique operator satisfying

$$\langle Au, v \rangle = \langle u, A^* v \rangle,$$

for all $u, v \in \mathcal{H}$. The definition of the adjoint can be extended to operators $A : X \to Y$ where the Banach spaces X and Y are dense subspaces of some Hilbert space. An operator $A : X \to Y$ is a Fredholm operator if dim ker$(A) < \infty$, dim ker$(A^*) < \infty$ and $R(A)$ is a closed subspace of Y. Finally, we say a nonlinear operator $F : X \to Y$ between Banach spaces is a nonlinear Fredholm operator if the Fréchet derivative $D_x F(x_0) : X \to Y$ is a Fredholm operator for each $x_0 \in X$.

Finally, we also need to consider projection operators on Hilbert spaces and Banach spaces. Given a Banach space X and a subspace $V \subset X$, the projection P onto V is a bounded linear operator $P : X \to V$ that satisfies $P^2 = P$. In particular, if V is a finite-dimensional subspace spanned by the orthonormal basis $\hat{v}_1, \cdots, \hat{v}_n$ and X is Hilbert space, then we can easily construct the projection onto V by the formula

$$Pu : \sum_{i=1}^{n} \langle u, \hat{v}_i \rangle \hat{v}_i, \tag{1.4.18}$$

where $u \in X$ and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$. Note that if $X \subset \mathcal{H}$ is a Banach space and $V \subset X$, we can define a projection operator $P : X \to V$ by restricting the projection operator on $\mathcal{H}$ to X.

We now have everything we need to discuss the Liapunov-Schmidt reduction.

Liapunov-Schmidt Reduction

The following treatment is taken from [17]. Let X, Λ and Z be Banach spaces and assume that $U \subset X$, $V \subset \Lambda$. For $\lambda = \lambda_0$, we require that the mapping $F : U \times V \to Z$ be a nonlinear Fredholm operator with respect to x; i.e. the linearization $D_xF(\cdot,\lambda_0)$ of $F(\cdot,\lambda_0) : U \to Z$ is a Fredholm operator. Assume that F also satisfies the following assumptions:

$$F(x_0,\lambda_0) = 0 \quad \text{for some } (x_0,\lambda_0) \in U \times V, \tag{1.4.19}$$
$$\text{dim ker}(D_xF(x_0,\lambda_0)) = \text{dim ker}(D_xF(x_0,\lambda_0)^*) = 1.$$

Given that $D_xF(x_0,\lambda_0)$ has a one-dimensional kernel, there exists a projection operator $P : X \to X_1 = \text{ker}D_xF(x_0,\lambda_0)$. Similarly, one has the projection operator $Q : Y \to Y_2 = \text{ker}(D_xF(x_0,\lambda_0)^*)$. This allows us to decompose $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ where $Y_1 = R(D_XF(x_0,\lambda_0))$. We will refer to this decomposition as the **Liapunov-Schmidt decomposition**, and we see that $F(x,\lambda) = 0$ if and only if the following two equations are satisfied

$$QF(x,\lambda) = 0, \tag{1.4.20}$$
$$(I-Q)F(x,\lambda) = 0.$$

For any $x \in X$, we can write $x = v + w$, where $v = Px$ and $w = (I-P)x$. This defines a map $G : U_1 \times W_1 \times V_1 \to Y_1$ by

$$G(v,w,\lambda) = (I-Q)F(v+w,\lambda) \quad \text{where} \tag{1.4.21}$$
$$U_1 \subset X_1, \quad W_1 \subset X_2, \quad V_1 \subset \mathbb{R} \quad \text{and}$$
$$v_0 = Px_0 \in U_1, \quad w_0 = (I-P)x_0 \in W_1.$$

The definition of $G(v,w,\lambda)$ implies that $G(v_0,w_0,\lambda_0) = 0$, and our choice of function spaces allows us to apply the Implicit Function Theorem to $G(v,w,\lambda)$ to conclude that there exists a continuously differentiable function

$$\psi : U_2 \times V_2 \to W_2 \quad \text{such that all solutions to } G(v,w,\lambda) = 0, \tag{1.4.22}$$
$$\text{in } U_2 \times W_2 \times V_2 \quad \text{are of the form } G(v,\psi(v,\lambda),\lambda) = 0.$$

Insertion of the function $\psi(v,\lambda)$ into the second equation of (1.4.20) yields a finite-dimensional problem

$$\Phi(v,\lambda) = QF(v+\psi(v,\lambda),\lambda) = 0. \tag{1.4.23}$$

We observe that finding solutions (v,λ) to (1.4.23) is equivalent to finding solutions to $F(x,\lambda) = 0$ in a neighborhood of (x_0,λ_0). We will refer to the finite-dimensional problem (1.4.23) as the Liapunov-Schmidt reduction of the nonlinear problem $F(x,\lambda) = 0$.

Given that ker$(D_xF(x_0,\lambda_0))$ is spanned by $\hat{v}_0$, then we can write $v = s\hat{v}_0 + v_0$. Substituting this into (1.4.23) we obtain

$$\Phi(s,\lambda) = QF(s\hat{v}_0 + v_0 + \psi(s\hat{v}_0 + v_0,\lambda),\lambda) = 0. \tag{1.4.24}$$

With additional assumptions on the operator $F(x,\lambda)$, we can apply the Implicit Function Theorem again to Eq. (1.4.24) to conclude that in a neighborhood of λ_0, $\lambda = \rho(s)$ for some continuously differentiable function $\rho(s)$. We then observe that solutions to $F(x,\lambda) = 0$ in a neighborhood of (x_0,λ_0) must lie on the parametrized curve

$$\begin{aligned} x(s) &= s\hat{v}_0 + v_0 + \psi(s\hat{v}_0 + v_0,\rho(s)), \\ \lambda(s) &= \rho(s). \end{aligned} \tag{1.4.25}$$

With additional assumptions on the operator $F(x,\lambda)$, we can determine the behavior of $\psi(s\hat{v}_0 + \hat{v}_0,\rho(s))$ and $\rho(s)$ in the above expression in a neighborhood of $s = 0$.

Non-uniqueness of Solutions to a One-parameter Family of Problems

Here we outline our approach for proving that solutions to Eq. (1.4.15) are non-unique. Given that our approach for both the CMC and non-CMC cases are similar, we only outline our strategy for the non-CMC case.

Define

$$F((\phi,\mathbf{w}),\lambda) = \begin{bmatrix} -\Delta\phi + a_R\phi + \lambda^2 a_\tau\phi^5 - a_{\mathbf{w}}\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5 \\ \mathbb{L}\mathbf{w} + \lambda b_\tau^a\phi^6 \end{bmatrix}. \tag{1.4.26}$$

We view (1.4.26) as a nonlinear operator between Banach spaces

$$F((\phi, \mathbf{w}), \lambda) : C^{k,\alpha}(\mathcal{M}) \oplus C^{k,\alpha}(\mathcal{TM}) \oplus \mathbb{R} \to C^{k-2,\alpha}(\mathcal{M}) \oplus C^{k-2,\alpha}(\mathcal{TM}), \quad (k \geq 2).$$

We also observe that if $F((\phi, \mathbf{w}), \lambda) = 0$, then $((\phi, \mathbf{w}), \lambda)$ solves the system (1.4.15).

In Chapter 3 we apply the Liapunov-Schmidt reduction outlined above to the nonlinear operator defined in Eq. (1.4.26). We first show that there is critical density $\rho = \rho_c$ and solution $((\phi_c, \mathbf{0}), 0)$ to (1.4.15) such that the linearization $D_X F((\phi_c, \mathbf{0}), 0)$ has a one-dimensional kernel when $\rho = \rho_c$ in (1.4.26). This assumption allows us to apply the reduction to determine a parametrization of the solution curve of (1.4.15) in a neighborhood of $((\phi_c, \mathbf{0}), 0)$. We then determine a second order Taylor series expansion of $\psi(s\hat{v}_0 + \hat{v}_0, \rho(s))$ and $\rho(s)$ in a neighborhood of $s = 0$. This expansion combined with Eq. (1.4.25) imply that all solutions to Eq. (1.4.15) in a neighborhood of $((\phi_c, \mathbf{0}), 0)$ have the form

$$\phi(s) = \phi_c + s + \frac{1}{2}\ddot{\lambda}(0)u(x)s^2 + O(s^3), \tag{1.4.27}$$

$$\mathbf{w}(s) = \frac{1}{2}\ddot{\lambda}(0)\mathbf{v}(x)s^2 + O(s^3), \tag{1.4.28}$$

$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \tag{1.4.29}$$

where $\ddot{\lambda}(0) \neq 0$ and $\mathbf{v}(x) \neq 0$. The non-uniqueness of solutions to Eq. (1.4.15) then follows from the fact that the lowest order term in $\lambda(s)$ and $\mathbf{w}(s)$ is quadratic.

The discussion above outlines the approach we take to prove non-uniqueness of solutions to (1.4.15) in both the CMC and non-CMC cases. Now that we have described the general solution procedure for both the rough solution problem and the non-uniqueness problem, we outline the remainder of the paper.

1.5 Summary of the Papers

The remainder of the thesis consists of two chapters, each of which is comprised of a self-contained paper. The paper that makes up Chapter 2 develops the rough solution framework that we discussed in Section 1.4.1 and the article comprising Chapter 3 discusses the non-uniqueness of the conformal formulation on a closed manifold discussed in 1.4.2. Here we provide an outline of each paper, starting with Chapter 2.

Chapter 2 is structured as follows. In Section 2.2 we provide additional motivation for the rough solution framework by proving the existence of a solution to an ill-posed semilinear problem (2.1.3). In Section 2.3 we state a number of preliminary results and develop the technical tools required to solve our semilinear problem (1.4.1). Among these tools and results are refined Schauder estimates found in [23] and a description of the Colombeau framework in which the coefficients and data will be embedded. The main result of Chapter 2 is stated in Section 2.4 in Theorem 2.4.1. This Theorem states conditions under which a semilinear problem has a solution in the special Colombeau algebra. In this same section, we also give a statement and proof of the method of sub- and super-solutions in Theorem 2.4.3, and then give an outline of the method of proof of Theorem 2.4.1. Following our discussion of elliptic problems in Colombeau algebras, we discuss a method to embed (1.4.1) into the algebra to apply our Colombeau existence theory. The remainder of the chapter is dedicated to developing the tools to prove Theorem 2.4.1. In Section 2.5 we determine *a priori* L^∞ bounds of solutions to our semilinear problem and a net of sub- and super-solutions satisfying explicit ϵ-decay estimates. Finally, in Section 2.6 we utilize the results from Section 2.5 to prove the main result outlined in Section 2.4.

Chapter 3 is organized as follows. In section 3.2 we introduce the function spaces that we will use and some basic concepts from functional analysis. Then we discuss the Liapunov-Schmidt reduction in greater detail. A statement of the main, non-uniqueness results of Chapter 3 can be found in section 3.3. The remainder of the chapter is then devoted to proving these results. The foundation for our argument is developed in sections 3.4 and 3.5. In section 3.4 we demonstrate the existence of a critical, constant density ρ_c such that for τ and $|\sigma|$ constant and $\mathbf{j}^a = 0$, the Hamiltonian constraint (3.1.6) will have a positive solution if $\rho \le \rho_c$ and will have no positive solution if $\rho > \rho_c$. Then in section 3.5 we use the properties of ρ_c to show that there exists a function ϕ_c at which the linearizations of the uncoupled Hamiltonian operator (CMC case) and coupled system (non-CMC case) have one-dimensional kernels. The existence of a one-dimensional kernel then allows us to apply the Liapunov-Schmidt reduction in section 3.7 in the CMC case and in section 3.8 in the non-CMC case. In particular, in section 3.7 we determine an explicit solution curve for (3.1.6) that goes through the point $(\phi_c, 0)$ in the CMC case. An analysis of this curve then implies the non-uniqueness of solutions to (3.1.6) when the mean curvature is constant. Similarly, in section 3.8 we also determine an explicit solution curve for the full, uncoupled system (3.1.6) through a point of the form $((\phi_c, \mathbf{0}), 0)$.

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Chapter 2

Generalized Solutions to the Lichnerowicz Equation

GENERALIZED SOLUTIONS TO SEMILINEAR ELLIPTIC PDE WITH APPLICATIONS TO THE LICHNEROWICZ EQUATION

MICHAEL HOLST AND CALEB MEIER

ABSTRACT. In this article we investigate the existence of solutions to a semi-linear, elliptic, partial differential equation with distributional coefficients and data. The problem we consider is a generalization of the Lichnerowicz equation that one encounters in studying the constraint equations in general relativity. Our method for solving this problem consists of solving a net of regularized, semi-linear problems with data obtained by smoothing the original, distributional coefficients. In order to solve these regularized problems, we develop *a priori* L^∞-bounds and sub- and super-solutions to apply a fixed point argument. We then show that the net of solutions obtained through this process satisfies certain decay estimates by determining estimates for the sub- and super-solutions and utilizing classical, *a priori* elliptic estimates. The estimates for this net of solutions allow us to regard this collection of functions as a solution in a Colombeau-type algebra. We motivate this Colombeau algebra framework by first solving an ill-posed critical exponent problem. To solve this ill-posed problem, we use a collection of smooth, "approximating" problems and then use the resulting sequence of solutions and a compactness argument to obtain a solution to the original problem. This approach is modeled after the more general Colombeau framework that we develop, and it conveys the potential that solutions in these abstract spaces have for obtaining classical solutions to ill-posed nonlinear problems with irregular data.

2.1 Introduction: Semilinear Problems and Critical Nonlinearities

In this paper we consider a family of elliptic, semilinear Dirichlet problems that are of the form

$$-\sum_{i,j=1}^{N} D_i(a^{ij}D_ju) + \sum_{i=1}^{K} b^i u^{n_i} = 0 \quad \text{in } \Omega, \tag{2.1.1}$$

$$u|_{\partial\Omega} = \rho, \tag{2.1.2}$$

where a^{ij}, b^i and ρ are potentially distributional and $n_i \in \mathbb{Z}$ for each i. These problems are a generalization of the the Lichnerowicz Equation that appears in the study of the constraint equations of General Relativity. The need to understand an equation of this form with rough data arises if one attempts to study equations such as the Lichnerowicz equation when the metric of the embedded hypersurface is not smooth. From a physical perspective, such problems are interesting because distributional metrics correspond to the initial data for physically plausible spacetimes generated by strings and gravitational waves [4, 6]. From a mathematical point of view, developing results for solutions to the Lichnerowicz equation under such low regularity conditions is of interest in that it extends the current "rough metric" existence theory, as described in [7, 8, 11, 10, 2]. The hope is to eventually extend the solution theory to cover rough data examples such as those studied by Maxwell in [12].

The main contributions of this article are an existence result for (2.1.1) in a Colombeau-type algebra, and an existence result in $W^{1,2}(\Omega)$ for an ill-posed, critical exponent problem of the form

$$-\Delta u + au^5 + bu^i = 0 \quad \text{in } \Omega, \tag{2.1.3}$$

$$u|_{\partial\Omega} = \rho,$$

where $1 \le i \le 4$ is in $\mathbb{N}$, $\Omega \subset \mathbb{R}^3$, $a \in L^p(\Omega)$ and $b \in L^q$ with $\frac{6}{5} \le p \le q$. The framework we use to prove existence for (2.1.1) consists of embedding the singular data and coefficients into a Colombeau-type algebra so that multiplication of the distributional coefficients is well-defined. To solve (2.1.3), we do not explicitly require the Colombeau machinery that we develop to solve (2.1.1), but we use similar ideas to produce a sequence of functions that converge to a solution of (2.1.3) in $W^{1,2}(\Omega)$.

The Colombeau solution framework for this paper is based mainly on the ideas found in [13]. Here we extend the work done by Mitrovic and Pilipovic in [13] to include a certain collection of semilinear problems. While Pilipovic and Scarpalezos solved a divergent type, quasilinear problem in a Colombeau type algebra in [14], the class of nonlinear problems we consider here does not fit naturally into that framework. Here we provide a solution method that is distinct from those posed in [14] and [13] that is better suited for the class of semilinear problems that we are interested in solving. The set up of our problem is completely similar to the set-up in [13]: given the semilinear Dirichlet problem in (2.1.1), we consider the family of problems

$$\begin{aligned} P_\epsilon(x,D)u_\epsilon &= f_\epsilon(x,u_\epsilon) \quad \text{on } \Omega, \\ u_\epsilon|_{\partial\Omega} &= \rho_\epsilon, \end{aligned} \tag{2.1.4}$$

where f_ϵ, h_ϵ, and $P_\epsilon(x,D)$ are obtained by convolving the data and coefficients of (2.1.1) with a certain mollifier. Therefore, a solution to our problem (2.1.1) in a Colombeau algebra is a net of solutions to the above family satisfying certain decay estimates in ϵ. This is discussed in detail in Sections 2.3.2 and 2.3.4. This basic concept underlies both the solution process in our paper and in [13] and [14]. However, it is our solution process in the Colombeau algebra that is quite distinct from that laid out in [13], where the authors used linear elliptic theory to determine a family of solutions and then classical elliptic, *a priori* estimates to prove certain decay estimates. Most notably, the authors developed a precise maximum principle-type argument necessary to obtain the decay estimates required to find a solution. Our strategy for solving (2.1.1) differs in a number of ways. First, in Section 2.5.1 we develop a family of *a priori* L^∞ bounds to the family of problems (2.1.4). Then in Section 2.5.2 we show that these estimates determine sub- and super-solutions to (2.1.4). We then employ the method of sub- and super-solutions in Section 2.4.1 to determine a family of solutions. Finally, ϵ-decay estimates on the sub- and super-solutions are established in Section 2.5.2, and in Section 2.6 these estimates are used in conjunction with the *a priori* estimates in Section 2.3.1 to prove the necessary

ϵ-decay estimates on our family of solutions.

This paper can be broken down into two distinct, but related parts. The first part is dedicated to solving (2.1.3). Our solution to this problem does not explicitly require the techniques that we develop to solve problems with distributional data in Colombeau algebras and only relies on standard elliptic PDE theory. However, the ideas that we use to solve the problem are closely related: we obtain a solution by solving a family of problems similar to (2.1.4) and then show that these solutions converge to a function in $W^{1,2}(\Omega)$. Therefore, we present our existence result for (2.1.3) first to convey the benefit that the more general Colombeau solution strategy has, not only for solving problems in the Colombeau Algebra, but also for obtaining solutions in more classical spaces. The remainder of the paper is dedicated to developing the Colombeau framework described in the preceding paragraph. This consists of defining an algebra appropriate for a Dirichlet problem and properly defining a semilinear elliptic problem in the algebra. Once a well-posed elliptic problem in the Colombeau algebra has been formed, we discuss the conditions under which the problem has a solution in the algebra and finally, describe how to translate a given problem of the form (2.1.1) into a problem that can be solved in the algebra. It should be noted that while the intention is to find solutions to (2.1.1), the main result pertaining to Colombeau algebras in this paper is Theorem 2.4.1, which is the main solution result for semilinear problems in our particular Colombeau algebra.

Outline of the paper. The remainder of the paper is structured as follows: In Section 2.2 we motivate this article by proving the existence of a solution to (2.1.3). In Section 2.3 we state a number of preliminary results and develop the technical tools required to solve (2.1.1). Among these tools and results are the explicit *a priori* estimates found in [13] and a description of the Colombeau framework in which the coefficients and data will be embedded. Then in Section 2.4 we state the main existence result in Theorem 2.4.1, give a statement and proof of the method of sub- and super-solutions in Theorem 2.4.3, and then give an outline of the method of proof of Theorem 2.4.1. Following our discussion of elliptic problems in Colombeau algebras, we discuss a method to embed (2.1.1) into the algebra to apply our Colombeau existence theory. The remainder of the paper is dedicated to developing the tools to prove Theorem 2.4.1. In Section 2.5 we determine *a priori* L^∞ bounds of solutions to our semilinear problem and a net of sub- and super-solutions satisfying explicit ϵ-decay estimates. Finally, in Section 2.6 we utilize the results from Section 2.5 to prove the main result outlined in Section 2.4.

2.2 Solution Construction using a Sequence of Approximate Problems

If $\Omega \subset \mathbb{R}^3$, the Sobolev embedding theorem tells us that $W^{1,2}(\Omega)$ will compactly embed into $L^p(\Omega)$ for $1 \leq p < 6$ and continuously embed for $1 \leq p \leq 6$. Given functions $u, v \in W^{1,2}(\Omega)$, this upper bound on p places a constraint on the values of i that allow for the product $u^i v$ to be integrable. In particular, Sobolev embedding and standard Hölder inequalities imply that this product will be integrable for arbitrary elements of $W^{1,2}(\Omega)$ only if $1 \leq i \leq 5$. More generally, if $a \in L^\infty(\Omega)$, the term au^5v will also be integrable. However, if a is an unbounded function in $L^p(\Omega)$ for some $p \geq 1$, then this product is not necessarily integrable without some sort of *a priori* bounds on a, u, and v. Therefore, the following problem does not have a well-defined weak formulation in $W^{1,2}(\Omega)$:

$$-\Delta u + au^5 + bu^i = 0 \quad \text{in } \Omega, \tag{2.2.1}$$
$$u|_{\partial\Omega} = \rho,$$

where $1 \leq i \leq 4$ is in $\mathbb{N}$, $\rho \in W^{1,2}(\Omega')$, $a \in L^p(\Omega')$, $b \in L^q(\Omega')$ for $\frac{6}{4} \leq p \leq q$. Here we also assume that $\Omega \subset\subset \Omega'$ and Ω' is a bounded domain in $\mathbb{R}^3$.

The objective of this section is to find a solution to the above problem. In order to solve (2.2.1), we solve a sequence of approximate, smooth problems and use a compactness argument to obtain a convergent subsequence. We first define necessary notation and then present the statements of two theorems that will be necessary for our discussion in this section. Then we prove the existence of a solution to (2.2.1). Finally, we show that if a solution exists, then under certain conditions we can construct a net of problems whose solutions converge to the given solution.

2.2.1 Overview of Spaces and Results for the Critical Exponent Problem

For the remainder of the paper, for a fixed domain $\Omega \subset \mathbb{R}^n$, we denote the standard Sobolev norms on Ω by

$$\|u\|_{L^p} = \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}, \tag{2.2.2}$$
$$\|u\|_{W^{k,p}} = \left(\sum_{i=0}^{k} \|D^i u\|_{L^p}^p \right)^{\frac{1}{p}}.$$

Furthermore, let

$$\text{ess sup } u = \hat{u}, \tag{2.2.3}$$
$$\text{ess inf } u = \check{u}.$$

In our subsequent work we will also require regularity conditions on the domain Ω and its boundary. Therefore, we will need the following definition taken from [5]:

Definition 2.2.1. *A bounded domain $\Omega \subset \mathbb{R}^n$ and its boundary are of class $C^{k,\alpha}$, $0 \leq \alpha \leq 1$, if for each $x_0 \in \partial\Omega$ there is a ball $B(x_0)$ and a one-to-one mapping Ψ of B onto $D \subset \mathbb{R}^n$ such that:*

1. $\Psi(B \cap \Omega) \subset \mathbb{R}^n_+$,
2. $\Psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$,
3. $\Psi \in C^{k,\alpha}(B), \quad \Psi^{-1} \in C^{k,\alpha}(D)$.

We say that a domain Ω is of class C^∞ if for a fixed $0 \leq \alpha \leq 1$ it is of class $C^{k,\alpha}$ for each $k \in \mathbb{N}$. Additionally, for this section and the next we will require the following Theorem and Proposition. The first result, Theorem 2.2.2, is a simplified statement of the sub- and super-solution theorem that we prove later in the paper and is essential for solving our semilinear problem.

Theorem 2.2.2. *Suppose* $\Omega \subset \mathbb{R}^n$ *is a* C^∞ *domain and assume* $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ *is in* $C^\infty(\overline{\Omega} \times \mathbb{R}^+)$ *and* $\rho \in C^\infty(\overline{\Omega})$*. Let* L *be an elliptic operator of the form*

$$Lu = -D_i(a^{ij} D_j u) + cu, \quad \textit{and} \quad a^{ij}, c \in C^\infty(\overline{\Omega}). \tag{2.2.4}$$

Suppose that there exist sub- and super-solutions $u_- : \overline{\Omega} \to \mathbb{R}$ *and* $u_+ : \overline{\Omega} \to \mathbb{R}$ *of* L *such that the following hold:*

1. $u_-, u_+ \in C^\infty(\overline{\Omega})$,
2. $0 < u_-(x) < u_+(x) \quad \forall x \in \overline{\Omega}$.

Then there exists a solution $u \in C^\infty(\overline{\Omega})$ *to*

$$Lu = f(x, u) \quad \textit{on } \Omega, \tag{2.2.5}$$

$$u|_{\partial\Omega} = \rho, \tag{2.2.6}$$

such that $u_-(x) \leq u(x) \leq u_+(x)$.

The following Proposition allows us to determine *a priori estimates* for solutions to our semilinear problem. These estimates will serve as sub- and super-solutions and will allow us to apply the sub- and super-solution theorem to solve (2.2.1).

Proposition 2.2.3. *Let* u *be a solution to a semilinear equation of the form*

$$-\sum_{i,j}^{N} D_i(a^{ij} D_j u) + \sum_{i=1}^{K} b^i u^{n_i} = 0 \quad \textit{in } \Omega, \tag{2.2.7}$$

$$u|_{\partial\Omega} = \rho, \quad \rho(x) > 0 \ \textit{ on } \ \partial\Omega,$$

where $a^{ij}, b^i, \ \rho \in C^\infty(\overline{\Omega})$ *and* $n_i \in \mathbb{Z}$ *for* $1 \leq i \leq K$*. Suppose that the semilinear operator in* (2.2.7) *has the property that* $n_i > 0$ *for some* $1 \leq i \leq K$*. Let* n_K *be the largest positive exponent and suppose that* $b_K(x) > 0$ *in* $\overline{\Omega}$*. Additionally, assume that* *one of the following two cases holds:*

$$\textit{(1) } n_i < 0 \textit{ for some } 1 \le i < K \textit{ and if } n_1 = \min\{n_i : n_i < 0\}, \quad \text{then } b_1(x) < 0 \text{ in } \overline{\Omega}. \tag{2.2.8}$$

$$\textit{(2) } n_K \textit{ is odd and } 0 < n_i \textit{ for all } 1 \le i \le K. \tag{2.2.9}$$

If case (2.2.8) *holds, define*

$$\alpha' = \sup_{c \in \mathbb{R}_+} \left\{ \sum_{i=1}^{K} \sup_{x \in \Omega} b_i(x) b^{n_i} < 0 \quad \forall b \in (0, c) \right\}, \tag{2.2.10}$$

and let $\alpha = \min\{\alpha', \inf_{x \in \partial\Omega} \rho(x)\}$. *If case* (2.2.8) *or case* (2.2.9) *holds, define*

$$\beta' = \inf_{c \in \mathbb{R}} \left\{ \sum_{i=1}^{K} \inf_{x \in \Omega} b_i(x) b^{n_i} > 0 \quad \forall b \in (c, \infty) \right\}, \tag{2.2.11}$$

and let $\beta = \max\{\beta', \sup_{x \in \partial\Omega} \rho(x)\}$.

Under these assumptions and definitions, if case (2.2.8) *holds, then* $0 < \alpha \le u \le \beta$. *Otherwise, if case* (2.2.9) *holds, then* $-\beta' \le u \le \beta$.

For a more detailed statement of Theorem 2.2.2 and its proof, see Section 2.4.1. The proof of Proposition 2.2.3 can be found in Section 2.5. Now that we have all of the tools we need, we shall now prove the existence of a solution to the semilinear elliptic problem (2.2.1).

2.2.2 Existence of a Solution to an Ill-Posed Critical Exponent Problem

For the following discussion, let $\Omega \subset \mathbb{R}^3$ be a closed and bounded domain of C^∞-class. We assume that $\Omega \subset\subset \Omega' \subset \mathbb{R}^3$, with Ω' open and bounded. Here we prove the existence of a solution to the problem

$$-\Delta u + au^5 + bu^i = 0 \quad \text{in } \Omega, \tag{2.2.12}$$

$$u|_{\partial\Omega} = \rho,$$

where $1 \leq i \leq 4$ is in $\mathbb{N}$,

$$a \in L^p(\Omega'), \quad b \in L^q(\Omega') \cap L^\infty(\Omega'), \quad \frac{6}{4} < p \leq q < \infty, \quad \rho \in W^{1,2}(\Omega'), \quad (2.2.13)$$

and

$$\check{a} > 0, \quad \hat{b} < 0, \quad \text{and} \quad \check{\rho} > 0. \quad (2.2.14)$$

Proposition 2.2.4. *The semilinear problem* (2.2.12) *has a solution* $u \in W^{1,2}(\Omega)$ *if* $a, b,$ *and* ρ *satisfy the conditions in* (2.2.13)-(2.2.14).

Proof. To determine a solution to (2.2.12), we consider the sequence of solutions to the approximate problems

$$-\Delta u_n + a_n(u_n)^5 + b_n(u_n)^i = 0 \quad \text{in } \Omega, \quad (2.2.15)$$
$$u_n|_{\partial\Omega} = \rho_n,$$

where $a_n = a * \phi_n$, $b_n = b * \phi_n$, and $\rho_n = \rho * \phi_n$ and $\phi_n = n^3\phi(nx)$ is a positive mollifier where $\int \phi(x)\, dx = 1$. Given that ϕ is a positive mollifier, Eq. (2.2.14) implies that for each $n \in \mathbb{N}$,

$$\check{a}_n > 0, \quad \hat{b}_n < 0 \quad \text{and} \quad \check{\rho}_n > 0.$$

We first verify that the sequence of problems (2.2.15) has a solution for each n. To do this, we will utilize Theorem 2.2.2 and Proposition 2.2.3. Let β'_n and β_n have the same properties as β' and β in Proposition 2.2.3 for the sequence of problems (2.2.15). Then using the notation in Proposition 2.2.3, we can write explicit expressions for β for (2.2.12), β'_n and β_n. It is not hard to show that

$$\beta = \max\left\{\left(-\frac{\check{b}}{\check{a}}\right)^{\frac{1}{5-i}}, \hat{\rho}\right\},$$

and

$$\beta'_n = \left(-\frac{\check{b}_n}{\check{a}_n}\right)^{\frac{1}{5-i}} \quad \text{and} \quad \beta_n = \max\left\{\beta'_n, \hat{\rho}_n\right\}.$$

By Proposition 2.2.3, β'_n and β_n are *a priori* bounds for the approximate problems.

Furthermore, it is not difficult to see that for each $n \in \mathbb{N}$ that $-\beta'_n$ and β_n are sub- and super-solutions for (2.2.15). See Section 2.5.2 and Theorem 2.4.3 for more details. Therefore, given that $\rho_n, a_n, b_n \in C^\infty(\overline{\Omega})$ for each n, we have that $u_n \in C^\infty(\overline{\Omega})$ is a solution to (2.2.15) for each n by Theorem 2.2.2.

Now observe that for each $n \in \mathbb{N}$, $\beta_n \le \beta$, which follows from the fact that

$$-b_n(x) = \int (-b(y))\phi_n(x-y)\,dy \le \int (-\check{b})\phi_n(x-y) = -\check{b}, \tag{2.2.16}$$

and $a_n(x) \ge \check{a}$, which is verified by a similar calculation. Therefore, by standard L^p elliptic regularity theory

$$\begin{aligned} \|u_n\|_{W^{2,p}} \le & C(\| - a_n(u_n)^5 - b_n(u_n)^i\|_{L^p} + \|u_n\|_{L^p}) \\ \le & C(\beta_n^5\|a_n\|_{L^p} + \beta_n^i\|b_n\|_{L^p} + \beta_n) < M < \infty, \end{aligned} \tag{2.2.17}$$

where M is independent of n given that $\beta_n \le \beta$, $a_n \to a$ in L^p, $b_n \to b$ in L^q and $p \le q$. Because $p > \frac{6}{5}$ and Ω is of C^∞-class, $W^{2,p}(\Omega)$ embeds compactly into $W^{1,2}(\Omega)$. Therefore, there exists a convergent subsequence $u_{n_j} \to u$ in $W^{1,2}(\Omega)$. We claim now that u satisfies the following two properties:

1. $-\beta \le u \le \beta$ almost everywhere,
2. u solves (2.2.12).

The inequality $-\beta \le u \le \beta$ $a.e.$ follows from the fact the $u_{n_j} \to u$ in $W^{1,2}(\Omega)$ and

$$-\beta \le -\beta_{n_j} \le -\beta'_{n_j} \le u_{n_j} \le \beta_{n_j} \le \beta.$$

Indeed, if we assume that $u > \beta$ on some set of nonzero measure, then for some n the set $A_n = \{x \in \Omega : u(x) > \beta + \frac{1}{n}\}$ has positive measure. Then for all $j \in \mathbb{N}$, we have that

$$\int |u_{n_j} - u|^2\,dx \ge \int_{A_n} |u_{n_j} - u|^2\,dx \ge \frac{1}{n^2}\mu(A_n) > 0.$$

But this clearly contradicts the fact that $u_{n_j} \to u$ in $W^{1,2}(\Omega)$. A similar argument shows that $u \ge -\beta,\ a.e$ in Ω.

Finally, we want to show that u solves (2.2.12). Let $\epsilon > 0$. Then we have that

$$\left|\int \left(\nabla u \cdot \nabla v + au^5 v + bu^i v\right)\, dx\right| = \left|\int \left(\nabla u \cdot \nabla v + au^5 v + bu^i v\right)\, dx \right. \tag{2.2.18}$$
$$\left. - \int \left(\nabla u_{n_j} \cdot \nabla v + a_{n_j}(u_{n_j})^5 v + b_{n_j}(u_{n_j})^i v\right)\, dx\right|,$$

given that u_{n_j} solves (2.2.15). Then expanding the second line of the above equation we find that

$$\left|\int \nabla u \cdot \nabla v + au^5 v + bu^i v \, dx\right| \tag{2.2.19}$$
$$\leq \int \left|\nabla u \cdot \nabla v - \nabla u_{n_j} \cdot \nabla v\right|\, dx + \int \left|au^5 v - a_{n_j}(u_{n_j})^5 v\right|\, dx$$
$$+ \int \left|bu^i v - b_{n_j}(u_{n_j})^i v\right|\, dx \tag{2.2.20}$$
$$\leq \int \left|\nabla u \cdot \nabla v - \nabla u_{n_j} \cdot \nabla v\right|\, dx + \int \left|au^5 v - a(u_{n_j})^5 v\right|\, dx$$
$$+ \int \left|a(u_{n_j})^5 v - a_{n_j}(u_{n_j})^5 v\right|\, dx + \int \left|bu^i v - b(u_{n_j})^i v\right|\, dx$$
$$+ \int \left|b(u_{n_j})^i v - b_{n_j}(u_{n_j})^i v\right|\, dx. \tag{2.2.21}$$

Every term in (2.2.21) tends to 0 given that $u_{n_j} \to u$ in $W^{1,2}(\Omega)$, $a_{n_j} \to a$ in $L^p(\Omega)$, $b_{n_j} \to b$ in $L^q(\Omega)$ and $-\beta \leq u \leq \beta$. To show that the expression

$$\int \left|au^5 v - a(u_{n_j})^5 v\right|\, dx \to 0,$$

we can use a power series expansion to obtain

$$\begin{aligned}
\int \left|au^5 v - a(u_{n_j})^5 v\right|\, dx &= \int \left|av \sum_{i=1}^{5} \binom{5}{i} (u_{n_j})^{5-i}(u - u_{n_j})^i\right|\, dx \\
&\leq C(5, \beta^4) \sum_{i=1}^{5} \int \left|(u - u_{n_j})av\right|\, dx \\
&\leq C\|a\|_{L^p}\|u - u_{n_j}\|_{L^{\frac{2p}{p-1}}}\|v\|_{L^{\frac{2p}{p-1}}} \\
&\leq C\|a\|_{L^p}\|u - u_{n_j}\|_{W^{1,2}}\|v\|_{W^{1,2}},
\end{aligned}$$

where the last inequality follows from the fact that $W^{1,2}(\Omega)$ embeds into $L^{\frac{2p}{p-1}}(\Omega)$ if $p > \frac{3}{2}$. Finally, by definition of the trace Tu, we have that,

$$Tu = \lim_{j\to\infty} u_{n_j}|_{\partial\Omega} = \lim_{j\to\infty} \rho_{n_j} = \rho,$$

where the limit is taken in $L^2(\partial\Omega)$. Therefore $u - \rho \in W_0^{1,2}(\Omega)$ and so u solves (2.2.12).

□

2.2.3 Convergence of Approximate Solutions to an Existing Solution

In this section we again assume $\Omega \subset \mathbb{R}^3$ is of $C^\infty(\Omega)$-class and that $\Omega \subset\subset \Omega'$, with $\Omega' \subset \mathbb{R}^3$ open and bounded. We also consider the same semilinear problem as in the previous section:

$$\begin{aligned} -\Delta u + au^5 + bu^i = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = \rho, \end{aligned} \tag{2.2.22}$$

and assume that $a \in L^p(\Omega')$, $b \in L^q(\Omega')$, $\frac{6}{5} \geq p, q < \infty$, $\rho \in W^{1,2}(\Omega')$ and

$$\check{a} > 0, \quad \hat{b} < 0, \quad \text{and} \quad \check{\rho} > 0.$$

Now we assume that a solution $u \in W^{1,2}(\Omega)$ to (2.2.22) exists, and we consider the convergence of solutions to the following net of approximate problems to u:

$$\begin{aligned} -\Delta u_\epsilon + a_\epsilon(u_\epsilon)^5 + b_\epsilon(u_\epsilon)^i = 0 \quad \text{in } \Omega, \\ u_\epsilon|_{\partial\Omega} = \rho_\epsilon, \end{aligned} \tag{2.2.23}$$

where $a_\epsilon = a * \phi_\epsilon$, $b_\epsilon = b * \phi_\epsilon$, $\rho_\epsilon = \rho * \phi_\epsilon$, $\phi_\epsilon(x) = \frac{1}{\epsilon^3}\phi(\frac{x}{\epsilon})$ and ϕ is a positive mollifier such that $\int \phi(x)\, dx = 1$. Again, given that ϕ is positive, for each $\epsilon < 1$

$$\check{a}_\epsilon > 0, \quad \check{b}_\epsilon < 0, \quad \text{and} \quad \check{\rho}_\epsilon > 0.$$

Because u is a solution to (2.2.22), Proposition 2.2.3 implies that

$$-\beta \leq -\left(-\frac{\breve{b}}{\breve{a}}\right)^{\frac{1}{5-i}} \leq u \leq \max\left\{\hat{\rho}, \left(-\frac{\breve{b}}{\breve{a}}\right)^{\frac{1}{5-i}}\right\} = \beta.$$

Similarly, for each $\epsilon \in (0,1)$ we have that

$$-\beta_\epsilon \leq u_\epsilon \leq \max\left\{\hat{\rho}_\epsilon, \left(-\frac{\breve{b}_\epsilon}{\breve{a}_\epsilon}\right)^{\frac{1}{5-i}}\right\} = \beta_\epsilon.$$

Given that ϕ_ϵ is a positive mollifier, $\beta_\epsilon \leq \beta$ for all $\epsilon \in (0,1)$. This follows from (2.2.16). Therefore $-\beta \leq u_\epsilon \leq \beta$ for all $\epsilon \in (0,1)$.

Proposition 2.2.5. *If a solution u to* (2.2.22) *exists in $W^{1,2}(\Omega)$, then the approximate solutions u_ϵ of* (2.2.23) *converge to u in $W^{1,2}(\Omega)$ provided that*

$$\max\left\{\hat{\rho}, \left(-\frac{\breve{b}}{\breve{a}}\right)^{\frac{1}{5-i}}\right\} << 1. \tag{2.2.24}$$

Proof. Define $\overline{u}_\epsilon$ to be the solution to the following auxiliary problem:

$$\begin{aligned} -\Delta\overline{u}_\epsilon &= -a_\epsilon u^5 - b_\epsilon u^i \quad \text{in } \Omega, \\ \overline{u}_\epsilon|_{\partial\Omega} &= \rho_\epsilon. \end{aligned} \tag{2.2.25}$$

Note that $\overline{u}_\epsilon$ exists by standard linear, elliptic solution theory because of the bounds on u and the assumption that $p, q \geq \frac{6}{5}$ implies that

$$-a_\epsilon u^5 - b_\epsilon u^i \in H^{-1}(\Omega).$$

Now consider the following string of inequalities:

$$\begin{aligned} \|u - u_\epsilon\|_{W^{1,2}} &\leq \|u - \overline{u}_\epsilon\|_{W^{1,2}} + \|\overline{u}_\epsilon - u_\epsilon\|_{W^{1,2}} \\ &\leq C\left(\|(a_\epsilon - a)u^5 + (b_\epsilon - b)u^i\|_{H^{-1}}\right. \\ &\quad \left.+C\|a_\epsilon(u_\epsilon)^5 + b_\epsilon(u_\epsilon)^i - a_\epsilon u^5 - b_\epsilon u^4\|_{H^{-1}}\right). \end{aligned} \tag{2.2.26}$$

We observe that

$$\|(a_\epsilon - a)u^5 + (b_\epsilon - b)u^i\|_{H^{-1}} \le \beta^5\|a_\epsilon - a\|_{L^p} + \beta^i\|b_\epsilon - b\|_{L^q} \to 0, \qquad (2.2.27)$$

for any $1 \le p, q < \infty$.

Furthermore, we may rewrite the second term in the inequality using a power series expansion to obtain the following string of inequalities:

$$\begin{aligned} &\|a_\epsilon(u_\epsilon)^5 + b_\epsilon(u_\epsilon)^i - a_\epsilon u^5 - b_\epsilon u^i\|_{H^{-1}} \qquad (2.2.28)\\ &\quad \le \|a_\epsilon \sum_{j=1}^{5}\binom{5}{j}(u_\epsilon)^{5-j}(u-u_\epsilon)^j\|_{H^{-1}} + \|b_\epsilon \sum_{j=1}^{i}\binom{i}{j}(u_\epsilon)^{i-j}(u-u_\epsilon)^j\|_{H^{-1}}. \end{aligned}$$

Given that $|u - u_\epsilon| \le 2\beta \; a.e.$, for $p' = \min\{\frac{7}{6}, \frac{6p}{6+p}\}$, we have

$$\begin{aligned} \|a_\epsilon \sum_{j=1}^{5}\binom{5}{j}(u_\epsilon)^{5-j}(u-u_\epsilon)^j\|_{H^{-1}} &\le \|a_\epsilon \sum_{j=1}^{5}\binom{5}{j}(u_\epsilon)^{5-j}(u-u_\epsilon)^j\|_{L^{p'}}\\ &\le 2^4\beta^4 C(5,\Omega)\|a_\epsilon(u-u_\epsilon)\|_{L^{p'}}\\ &\le C(\Omega,5)\beta^4\|a_\epsilon\|_{L^p}\|u-u_\epsilon\|_{L^{\frac{pp'}{p-p'}}}\\ &\le C(\Omega,5)\beta^4\|a_\epsilon\|_{L^p(\Omega)}\|u-u_\epsilon\|_{W^{1,2}}, \qquad (2.2.29) \end{aligned}$$

provided that $p \ge \frac{6}{5}$. Similarly, if $q \ge \frac{6}{5}$, we have that

$$\|b_\epsilon \sum_{j=1}^{i}\binom{i}{j}(u_\epsilon)^{i-j}(u-u_\epsilon)^j\|_{H^{-1}} \le C(4,\Omega)\beta^{i-1}\|b_\epsilon\|_{L^q}\|u-u_\epsilon\|_{W^{1,2}}. \qquad (2.2.30)$$

Therefore, equations (2.2.26) - (2.2.30) imply that

$$\begin{aligned} \|u-u_\epsilon\|_{W^{1,2}} \le \beta^4\|a_\epsilon - a\|_{L^p} + \beta^4\|b_\epsilon - b\|_{L^q} + C(\Omega,5)\beta^4\|a_\epsilon\|_{L^p}\|u-u_\epsilon\|_{W^{1,2}}\\ + C(4,\Omega)\beta^{i-1}\|b_\epsilon\|_{L^q}\|u-u_\epsilon\|_{W^{1,2}}. \qquad (2.2.31) \end{aligned}$$

Given that $\|a_\epsilon - a\|_{L^p} \to 0$ and $\|b_\epsilon - b\|_{L^q} \to 0$, if

$$\beta^4\|a\|_{L^p} << 1, \quad \text{and} \quad \beta^{i-1}\|b\|_{L^q} << 1,$$

we will have that $\|u - u_\epsilon\|_{W^{1,2}} \to 0$. But (2.2.24) implies the above condition provided that

$$\max\left\{\hat{\rho}, \left(-\frac{\breve{b}}{\breve{a}}\right)^{\frac{1}{5-i}}\right\} << 1,$$

is sufficiently small. □

Now that we have demonstrated the potential that sequences of solutions to approximate problems have in solving classical semilinear equations, we begin to develop our Colombeau Algebra framework. Within this framework, one embeds a problem with rough data into the algebra and obtains a net of solutions. If the net satisfies certain decay estimates, it is regarded as a member of the algebra. Then ideally one can relate the net of solutions to a classical solution using techniques like those discussed in section 2.2.2 and 2.2.3.

2.3 Preliminary Material: Hölder Spaces and Colombeau Algebras

We now begin to develop the Colombeau Algebra framework that will be used to solve (2.1.1). We first define Hölder Spaces and state precise versions of the classical Schauder estimates given in [13]. The definition of the Colombeau Algebra in which we will be working and these classical elliptic regularity estimates make these spaces the most natural choice in which to do our analysis. Therefore we will work almost exclusively with Hölder spaces for the remainder of the paper. Following our discussion of function spaces, we define the Colombeau algebra in which we will work and then formulate an elliptic, semilinear problem in this space.

2.3.1 Function Spaces and Norms

In this paper we will make frequent use of Schauder estimates on Hölder spaces defined on an open set $\Omega \subset \mathbb{R}^n$. Here we give notation for the Hölder norms and then state the regularity estimates that will be used.

All notation and results are taken from [5]. Assume that $\Omega \subset \mathbb{R}^n$ is open, connected and bounded. Then define the following norms and seminorms:

$$[u]_{\alpha;\Omega} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}, \tag{2.3.1}$$

$$[u]_{k,0;\Omega} = \sup_{|\beta|=k} \sup_{x\in\Omega} |D^\beta u|, \tag{2.3.2}$$

$$[u]_{k,\alpha;\Omega} = \sup_{|\beta|=k} [D^\beta u]_{\alpha;\Omega}, \tag{2.3.3}$$

$$\|u\|_{C^k(\overline{\Omega})} = |u|_{k;\Omega} = \sum_{j=0}^{k} [u]_{j,0;\Omega}, \tag{2.3.4}$$

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} = |u|_{k,\alpha;\Omega} = |u|_{k;\Omega} + [u]_{k,\alpha;\Omega}. \tag{2.3.5}$$

We interpret $C^{k,\alpha}(\overline{\Omega})$ as the subspace of functions $f \in C^k(\overline{\Omega})$ such that $f^{(k)}$ is α-Hölder continuous. Also, we view the subspace $C^{k,\alpha}(\Omega)$ as the subspace of functions $f \in C^k(\Omega)$ such that $f^{(k)}$ is locally $\alpha-$Hölder continuous (over compact sets $K \subset\subset \Omega$).

Now we consider the equation

$$Lu = a^{ij}D_{ij}u + b^i u D_i u + cu = f \quad \text{in } \Omega, \tag{2.3.6}$$

$$u = \rho \quad \text{on } \partial\Omega, \tag{2.3.7}$$

where L is a strictly elliptic operator satisfying

$$a^{ij} = a^{ji} \quad \text{and} \quad a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n.$$

The following regularity theorems can be found in [5] and [13]. See [5] for proofs. Note that the constant C in the following theorems has no dependence on Λ or λ.

Theorem 2.3.1. *Assume that Ω is a $C^{2,\alpha}$-class domain in $\mathbb{R}^n$ and that $u \in C^{2,\alpha}(\overline{\Omega})$ is a solution* (2.3.6)*, where $f \in C^\alpha(\overline{\Omega})$ and $\rho \in C^{2,\alpha}(\overline{\Omega})$. Additionally assume that*

$$|a^{ij}|_{0,\alpha;\Omega}, |b^i|_{0,\alpha;\Omega}, |c|_{0,\alpha;\Omega} \leq \Lambda.$$

Then there exists $C > 0$ such that

$$|u|_{2,\alpha;\Omega} \leq C \left(\frac{\Lambda}{\lambda}\right)^3 (|u|_{0;\Omega} + |\rho|_{2,\alpha;\Omega} + |f|_{0,\alpha;\Omega}).$$

This theorem can then be extended to higher order derivatives if one inserts the derivative of a solution u into (2.3.6), rearranges the equation and repeatedly applies Theorem 2.3.1. See [13] for details. We summarize this result in the next theorem.

Theorem 2.3.2. *Let Ω be a $C^{k+2,\alpha}$-class domain and $u \in C^2\Omega \cap C^0(\overline{\Omega})$ be a solution of* (2.3.6)*, where $f \in C^{k,\alpha}(\overline{\Omega})$ and $\rho \in C^{k+2,\alpha}(\overline{\Omega})$. Additionally assume that*

$$|a^{ij}|_{k,\alpha;\Omega}, |b^i|_{k,\alpha;\Omega}, |c|_{k,\alpha;\Omega} \leq \Lambda.$$

Then $u \in C^{k+2,\alpha;\Omega}(\overline{\Omega})$ and

$$|u|_{k+2,\alpha;\Omega} \leq C^{k+1} \left(\frac{\Lambda}{\lambda}\right)^{3(k+1)} (|u|_{0;\Omega} + |\rho|_{k+2,\alpha;\Omega} + |f|_{k,\alpha;\Omega}),$$

where C is the constant from Theorem 2.3.1.

2.3.2 Colombeau Algebras

Now that we have defined the basic function spaces that we will be working with and stated the regularity theorems that will be required to obtain necessary decay estimates, we are ready to define the Colombeau algebra with which we will be working and formulate our problem in this algebra. The following definition is taken from [13].

Let V be a topological vector space whose topology is given by an increasing family of seminorms μ_k. That is, for $u \in V$, $\mu_i(u) \leq \mu_j(u)$ if $i \leq j$. Then letting $I = (0, 1]$, we define the following:

$$\mathcal{E}_V = (V)^I \quad \text{where } u \in \mathcal{E}_V \text{ is a net } (u_\epsilon) \text{ of elements in } V \text{ with } \epsilon \in (0,1], \tag{2.3.8}$$

$$\mathcal{E}_{M,V} = \{(u_\epsilon) \in \mathcal{E}_V \mid \forall k \in \mathbb{N}\ \exists a \in \mathbb{R}\ :\ \mu_k(u_\epsilon) = \mathcal{O}(\epsilon^a) \text{ as } \epsilon \to 0\}, \tag{2.3.9}$$

$$\mathcal{N}_V = \{(u_\epsilon) \in \mathcal{E}_{V,M} \mid \forall k \in \mathbb{N}\ \forall a \in \mathbb{R}\ :\ \mu_k(u_\epsilon) = \mathcal{O}(\epsilon^a) \text{ as } \epsilon \to 0\}. \tag{2.3.10}$$

Then the polynomial generalized extension of V is formed by considering the quotient $\mathcal{G}_V = \mathcal{E}_{M,V}/\mathcal{N}_V$. Let's give a few examples, following [13, 6].

Definition 2.3.3. *If $V = \mathbb{C}$, $r \in \mathbb{C}$, $\mu_k(r) = |r|$, then one obtains $\overline{\mathbb{C}}$, the ring of generalized constants.*

Definition 2.3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $U_k \subset\subset \Omega$ an exhaustive sequence of compact sets and $\alpha \in \mathbb{N}_0^n$ a multi-index. Then if*

$$V = C^\infty(\Omega), \ \ f \in C^\infty(\Omega), \quad \mu_k(f) = \sup\{|D^\alpha f| \ : \ x \in U_k, \ \ |\alpha| \leq k\},$$

one obtains $\mathcal{G}^s(\Omega)$, the simplified Colombeau Algebra.

Definition 2.3.5. *If $V = C^\infty(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is bounded and*

$$\mu_k(f) = \sup\{|D^\alpha f| : |\alpha| \leq k, \ \ x \in \overline{\Omega}\},$$

we denote the generalized extension by $\mathcal{G}(\overline{\Omega})$. The set $\mathcal{E}_{M,C^\infty(\overline{\Omega})}$ will be denoted by $\mathcal{E}_M(\overline{\Omega})$ and be referred to as the space of moderate elements. The set $\mathcal{N}_{C^\infty(\overline{\Omega})}$ will be denoted by $\mathcal{N}(\overline{\Omega})$ and will be referred to as the space of null elements.

Both $\mathcal{G}^s(\Omega)$ and $\overline{\mathbb{C}}$ were developed by Colombeau and laid the basis for the more general construction described in (2.3.8)-(2.3.10). See [3] for more details. As in [13], for the purposes of this paper we are concerned with $\mathcal{G}(\overline{\Omega})$ given that we are interested in solving the Dirichlet problem and require a well-defined boundary value. If $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$ is a representative of an element $u \in \mathcal{G}(\overline{\Omega})$, we shall write $u = [(u_\epsilon)]$ to indicate that u is the equivalence class of (u_ϵ). At times we will drop the parentheses and simply write $[u_\epsilon]$. Addition and multiplication of elements in $\mathcal{G}(\overline{\Omega})$ is defined in terms of addition and multiplication of representatives. That is, if $u = [(u_\epsilon)]$ and $v = [(v_\epsilon)]$, then $uv = [(u_\epsilon v_\epsilon)]$ and $u + v = [(u_\epsilon + v_\epsilon)]$. Derivations are defined for $u = [(u_\epsilon)] \in \mathcal{G}(\overline{\Omega})$ by $\partial_{x_i} u = [(\partial_{x_i} u_\epsilon)]$.

Theorem 2.3.6. *With the above definitions of addition, multiplication and differentiation, $\mathcal{G}(\overline{\Omega})$ is a associative, commutative, differential algebra.*

Proof. This follows from the fact component-wise addition, multiplication, and differentiation makes $V^I = (C^\infty(\overline{\Omega}))^I$ into a differential algebra. By design, $\mathcal{E}_M(\overline{\Omega})$ is a the largest sub-algebra of $(C^\infty(\overline{\Omega}))^I$ that contains $\mathcal{N}(\overline{\Omega})$ as an ideal. Therefore $\mathcal{G}(\overline{\Omega})$ is a differential algebra as well. See [6]. □

Now that we have given the basic definition of a Colombeau algebra, we can discuss how distributions can be embedded into a space of this type.

2.3.3 Embedding Schwartz Distributions into Colombeau Algebras

While the algebras defined above are somewhat unwieldy, these spaces are well suited for analyzing problems with distributional data. The primary reason for this is that the Schwartz distributions $\mathcal{D}'(\Omega)$ can be linearly embedded into them. This allows one to define an *extrinsic* notion of distributional multiplication that is consistent with the pointwise product of C^∞ functions. For the purposes of this article, we require a means of embedding a particular subset of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\overline{\Omega})$. Therefore, we only discuss this simpler embedding here. For a full discussion of how to embed the entire space $\mathcal{D}'(\Omega)$ into $\mathcal{G}^s(\Omega)$, see [6].

The Schwartz distributions on an open set $\Omega \subset \mathbb{R}^n$, denoted $\mathcal{D}'(\Omega)$, are defined to be the dual of $\mathcal{D}(\Omega)$, the space of C^∞ functions with support contained in Ω. For a given $\varphi \in \mathcal{D}(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, the action of T on φ will be denoted by $\langle T, \varphi \rangle$.

We begin by letting $\psi \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz functions, be a function such that $\psi \equiv 1$ on some neighborhood of 0. Then define $\phi \in \mathcal{S}(\mathbb{R}^n)$ by $\phi = \mathcal{F}^{-1}[\psi]$, the inverse Fourier transform of ψ. It is easy to see that

$$\int_{\mathbb{R}^n} \phi \, dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} x^\alpha \phi \, dx = 0 \quad \forall |\alpha| \geq 1. \tag{2.3.11}$$

Let $\phi_\epsilon = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon})$ be the usual mollifier.

The properties of ϕ specified in (2.3.11) are extremely important. Suppose that $\Omega \subset\subset \Omega'$ where $\Omega' \subset \mathbb{R}^n$ is an open, bounded set and Ω is of C^∞-class. Given a distribution $u \in \mathcal{D}'(\Omega')$, define the restriction $u|_\Omega$ to be u restricted to test functions $\varphi \in \mathcal{D}(\Omega)$. Let $\mathcal{D}'(\Omega')|_\Omega \subset \mathcal{D}'(\Omega)$ be the subset of distributions obtained in this way. If $\xi \equiv 1$ in a neighborhood of $\overline{\Omega}$ and $\text{supp}(\xi) \subset \Omega'$, then any choice of ϕ satisfying

(2.3.11) as a mollifier makes following map well-defined:

$$\begin{aligned} i &: \mathcal{D}'(\Omega')|_{\Omega} \to \mathcal{G}(\overline{\Omega}), \\ i(u) &= (((\xi u) * \phi_\epsilon)|_{\overline{\Omega}}) + \mathcal{N}(\overline{\Omega}). \end{aligned} \tag{2.3.12}$$

Remark 2.3.7. *The properties of ϕ specified in Eq.* (2.3.11) *may seem a bit unnecessary given that any distribution u is the weak limit of $(u * \phi_\epsilon)$, where $\phi \in C_0^\infty(\Omega)$ only satisfies $\int \phi \, dx = 1$. The added property that $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\int x^\alpha \phi \, dx = 0$ for all $|\alpha| \geq 1$ is included so that i preserves multiplication of $C^\infty(\Omega)$ functions. That is, $i(fg) = i(f)i(g)$ for all $f, g \in C^\infty(\Omega)$. The map i was developed to satisfy this compatibility condition so that multiplication in $\mathcal{G}(\overline{\Omega})$ extends pointwise multiplication of $C^\infty(\overline{\Omega})$.*

We summarize the important properties of the map i in the following proposition:

Proposition 2.3.8. *Let i denote the embedding defined in Eq.* (2.3.12)*. Define the following map*

$$\begin{aligned} \sigma &: C^\infty(\overline{\Omega}) \to \mathcal{G}(\overline{\Omega}), \\ \sigma(f) &= (f), \end{aligned} \tag{2.3.13}$$

where (f) is a net (u_ϵ) such that $u_\epsilon = f$ for all $\epsilon \in (0, 1]$. Then i is a linear embedding such that $i|_{C^\infty(\overline{\Omega})} = \sigma$.

Proof. To verify that $i|_{C^\infty(\overline{\Omega})} = \sigma$, we use that fact that Ω is of C^∞-class, which implies that there exists a total extension operator E such that

$$E : C^\infty(\overline{\Omega}) \to C^\infty(\mathbb{R}^n), \tag{2.3.14}$$

$$(Ef)|_{\Omega} = f \quad \text{for each } f \in C^\infty(\Omega). \tag{2.3.15}$$

See [1] for details. Then for each $f \in C^\infty(\overline{\Omega})$, $f = (Ef)|_{\overline{\Omega}}$. Clearly $Ef \in \mathcal{D}'(\Omega')$, and therefore, $C^\infty(\overline{\Omega}) \subset \mathcal{D}'(\Omega')|_{\Omega}$. The fact that $i|_{C^\infty(\overline{\Omega})} = \sigma$ then follows from Eq. (1.4.8). By considering the difference $(((\xi f) * \phi_\epsilon) - f)$ on an arbitrary compact set

$K \subset\subset \Omega$ and using a Taylor expansion of this expression one is able to show that $(((\xi f) * \phi_\epsilon) - f) \in \mathcal{N}^s(\Omega)$.

The fact that i is linear and well-defined follows from the linearity of the convolution and our choice of $\phi \in \mathcal{S}(\mathbb{R}^n)$. To prove injectivity, suppose that $i(u) = 0$ for some $u \in \mathcal{D}'(\Omega')|_\Omega$. Then if ϕ_ϵ and ξ are the same as in Eq. 2.3.12, we have that

$$i(u) = (((\xi u) * \phi_\epsilon)|_\Omega) \in \mathcal{N}(\overline{\Omega}).$$

This implies that

$$((\xi u) * \phi_\epsilon) \to 0 \quad \text{uniformly on any } K \subset\subset \Omega.$$

Therefore, for any $\varphi \in \mathcal{D}(\Omega)$, we have that

$$\langle u, \varphi \rangle = \langle u, \xi\varphi \rangle = \langle \xi u, \varphi \rangle = \lim_{\epsilon \to 0} \langle (\xi u) * \phi_\epsilon, \varphi \rangle = 0. \tag{2.3.16}$$

So i is injective. □

We will require the map (2.3.12) when we discuss how to solve (2.1.1) in $\mathcal{G}(\overline{\Omega})$. The approach will be to reformulate the problem (2.1.1) as a semilinear elliptic PDE in $\mathcal{G}(\overline{\Omega})$ by utilizing Eq. (2.3.12). But before we discuss how to do this, we must first define what we mean by a partial differential equation is in $\mathcal{G}(\overline{\Omega})$.

2.3.4 Nets of Semilinear Differential Operators

We begin by defining a semilinear differential operator on $\mathcal{G}(\overline{\Omega})$. Our construction strongly resembles the construction by Mitrovic and Pilipovic in [13]. For $\epsilon < 1$, if (a^{ij}_ϵ), $(b^i_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$, we obtain a net of operators by defining A_ϵ to be

$$A_\epsilon u_\epsilon = -D_i(a^{ij}_\epsilon D_j u) + \sum_i^K b^i_\epsilon u^{n_i} = -a^{ij}_\epsilon D_i D_j u_\epsilon - (D_i a^{ij}_\epsilon)(D_j u_\epsilon) + \sum_{i=1}^K b^i_\epsilon (u_\epsilon)^{n_i},$$

where $n_i \in \mathbb{Z}$. Under certain conditions, we can view a net of operators of the above form as an operator on $\mathcal{G}(\overline{\Omega})$. Here we determine these conditions, which will guarantee that this net of operators is a well-defined operator on $\mathcal{G}(\overline{\Omega})$.

Given an element u in $\mathcal{G}(\overline{\Omega})$, we first need to ensure that $(A_\epsilon u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$. Based on how derivations and multiplication are defined in $\mathcal{G}(\overline{\Omega})$, the only serious obstacle to this is if $n_i < 0$ for some $i \leq K$. Therefore, we must guarantee that the element $((u_\epsilon)^{n_i})$ is a well-defined representative in $\mathcal{G}(\overline{\Omega})$ if $n_i < 0$. It suffices to ensure that $u = [(u_\epsilon)]$ has an inverse in $\mathcal{G}(\overline{\Omega})$. This is true if for each representative (u_ϵ) of u, there exists $\epsilon_0 \in (0, 1]$ and $m \in \mathbb{N}$ such that for all $\epsilon \in (0, \epsilon_0)$, $\inf_{x\in\overline{\Omega}} |u_\epsilon(x)| \geq C\epsilon^m$. See [6] for more details. So $u \in \mathcal{G}(\overline{\Omega})$ must possess this property in order for the above operator to have any chance of being well-defined. For the rest of this section we assume that u satisfies this condition.

Now must define our differential operators so that they are independent of the choice of representative coefficients $(a_\epsilon^{ij}), (b_\epsilon^i)$. Here we closely follow the discussion in [13]. Suppose $(\overline{a}_\epsilon^{ij})$, $(\overline{b}_\epsilon^i)$ in $\mathcal{E}_M(\overline{\Omega})$, and let

$$\overline{A}_\epsilon u = -\sum_{i,j=1}^{N} D_i(\overline{a}_\epsilon^{ij} D_j u) + \sum_{i}^{K} \overline{b}_\epsilon^i u^{n_i} = \overline{a}_\epsilon^{ij} D_i D_j u_\epsilon - (D_i \overline{a}_\epsilon^{ij})(D_j u_\epsilon) + \sum_{i=1}^{K} \overline{b}_\epsilon^i (u_\epsilon)^{n_i}.$$

We say that $(A_\epsilon) \sim (\overline{A}_\epsilon)$ if $(a_\epsilon^{ij} - \overline{a}_\epsilon^{ij}), (b_\epsilon^i - \overline{b}_\epsilon^i) \in \mathcal{N}^s(\overline{\Omega})$. Then $(A_\epsilon) \sim (\overline{A}_\epsilon)$ if and only if $(A_\epsilon u_\epsilon - \overline{A}_\epsilon u_\epsilon) \in \mathcal{N}(\overline{\Omega})$ for all $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$ due to the fact that the above operators are linear in (a_ϵ^{ij}) and (b_ϵ^i).

Let $\mathcal{A}$ be the family of nets of differential operators of the above form and define $\mathcal{A}_0 = \mathcal{A}/\sim$. Then for $A \in \mathcal{A}_0$ and $u \in \mathcal{E}_M(\overline{\Omega})$, define

$$A : \mathcal{G}(\overline{\Omega}) \to \mathcal{G}(\overline{\Omega}) \text{ by } Au = [A_\epsilon u_\epsilon],$$

where

$$[A_\epsilon u_\epsilon] = [-a_\epsilon^{ij}][D_i D_j u_\epsilon] + [-D_i a_\epsilon^{ij}][D_j u_\epsilon] + \sum_{i=1}^{K} [b_\epsilon^i][u_\epsilon^{n_i}]. \tag{2.3.17}$$

Using this definition, $A \in \mathcal{A}_0$ is a well-defined operator on $\mathcal{G}(\overline{\Omega})$. We summarize this statement in the following proposition.

Proposition 2.3.9. *$\mathcal{A}_0$ is a well-defined class of differential operators from $\mathcal{G}(\overline{\Omega})$ to $\mathcal{G}(\overline{\Omega})$.*

Proof. Based on the construction of $\mathcal{A}_0$, it is clear that for a given representative (u_ϵ) of $u \in \mathcal{G}(\overline{\Omega})$, $(A_\epsilon u_\epsilon)$ and $(\overline{A}_\epsilon u_\epsilon)$ represent the same element in $\mathcal{G}(\overline{\Omega})$. Furthermore, given a representative (A_ϵ) of $\mathcal{A}_0$, we also have that $[A_\epsilon u_\epsilon] = [A_\epsilon \overline{u}_\epsilon]$ for any two representatives of $u \in \mathcal{G}(\overline{\Omega})$. To see this, we first observe that for each ϵ, every term in $A_\epsilon u_\epsilon$ is linear except for the $(u_\epsilon)^{n_i}$ terms. So to verify the previous statement it suffices to show that for each $n_i \in \mathbb{Z}$, $((u_\epsilon)^{n_i}) = ((\overline{u}_\epsilon)^{n_i}) + (\overline{\eta}_\epsilon)$, where $(\overline{\eta}_\epsilon) \in \mathcal{N}(\overline{\Omega})$. Given that $[(u_\epsilon)] = [(\overline{u}_\epsilon)]$ in $\mathcal{G}(\overline{\Omega})$, we have $(\overline{u}_\epsilon) = (u_\epsilon) + (\eta_\epsilon)$ for $(\eta_\epsilon) \in \mathcal{N}(\overline{\Omega})$. For fixed ϵ, $n_i \in \mathbb{Z}^+$,

$$(\overline{u}_\epsilon)^{n_i} = (u_\epsilon + \eta_\epsilon)^{n_i} = \sum_{j=0}^{n_i} \binom{n_i}{j} (u_\epsilon)^j (\eta_\epsilon)^{n_i - j} = (u_\epsilon)^{n_i} + \overline{\eta}_\epsilon,$$

where $\overline{\eta}_\epsilon$ consists of the summands that each contain some nonzero power of η_ϵ. Clearly the net $(\overline{\eta}_\epsilon) \in \mathcal{N}(\overline{\Omega})$. If $n_i \in \mathbb{Z}^-$, then for a fixed ϵ,

$$(\overline{u}_\epsilon)^{n_i} = \frac{1}{(u_\epsilon + \eta_\epsilon)^{|n_i|}} = \frac{1}{\sum_{j=0}^{|n_i|} \binom{|n_i|}{j} (u_\epsilon)^j (\eta_\epsilon)^{|n_i| - j}} = \frac{1}{(u_\epsilon)^{|n_i|} + \overline{\eta}_\epsilon}.$$

By looking at the difference

$$(u_\epsilon)^{n_i} - \frac{1}{(u_\epsilon)^{|n_i|} + \overline{\eta}_\epsilon} = \frac{\overline{\eta}_\epsilon}{((u_\epsilon)^{|n_i|})((u_\epsilon)^{|n_i|} + \overline{\eta}_\epsilon)} = \hat{\eta}_\epsilon,$$

we see that the net $((u_\epsilon)^{n_i}) = ((\overline{u}_\epsilon)^{n_i}) + (\hat{\eta}_\epsilon)$, where $(\hat{\eta}_\epsilon) \in \mathcal{N}(\overline{\Omega})$. Therefore for any $u \in \mathcal{G}(\overline{\Omega})$ possessing an inverse, and any $A \in \mathcal{A}_0$, the expression $Au = [A_\epsilon u_\epsilon] \in \mathcal{G}(\overline{\Omega})$ is well-defined. □

2.3.5 The Dirichlet Problem in $\mathcal{G}(\overline{\Omega})$

Using the above definition of $\mathcal{A}$, we can now define our semilinear Dirichlet problem on $\mathcal{G}(\overline{\Omega})$. Let $u, \rho \in \mathcal{G}(\overline{\Omega})$ where $\Omega \subset \mathbb{R}^n$ is open, bounded and of C^∞-class. Then let E be a total extension operator of Ω such that for $f \in C^\infty(\Omega)$, $Ef \in C^\infty(\mathbb{R}^n)$ and $Ef|_\Omega = f$. See [1] for details. Using E we may may define $u|_{\partial\Omega} = \rho|_{\partial\Omega}$ for elements $u, \rho \in \mathcal{G}(\overline{\Omega})$ if there are representatives (u_ϵ) and (ρ_ϵ) such that

$$Eu_\epsilon|_{\partial\Omega} = E\rho_\epsilon|_{\partial\Omega} + n_\epsilon|_{\partial\Omega},$$

where n_ϵ is a net of C^∞ functions defined in a neighborhood of $\partial\Omega$ such that

$$\sup_{x\in\partial\Omega} |n_\epsilon(x)| = o(\epsilon^a) \quad \forall a \in \mathbb{R}. \tag{2.3.18}$$

This will ensure that $u|_{\partial\Omega} = \rho|_{\partial\Omega}$ does not depend on representatives [13]. From now on we will abbreviate $u|_{\partial\Omega} = \rho|_{\partial\Omega}$ by $u|_{\partial\Omega} = \rho$. With this definition of boundary equivalence, for a given operator $A \in \mathcal{A}_0$, the Dirichlet problem

$$\begin{aligned} Au &= 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= \rho, \end{aligned} \tag{2.3.19}$$

is well-defined in $\mathcal{G}(\overline{\Omega})$. Now we state the conditions under which the above problem can be solved in $\mathcal{G}(\overline{\Omega})$.

2.4 Overview of the Main Results

We begin this section by stating the main existence result for the Dirichlet problem (2.3.19). Let $A \in \mathcal{A}_0$ be an operator on $\mathcal{G}(\overline{\Omega})$ defined by (2.3.17). Also assume that the coefficients of A have representatives $(a_\epsilon^{ij}), (b_\epsilon^i) \in \mathcal{E}_M(\overline{\Omega})$ that satisfy the following properties for $\epsilon \in (0,1)$:

$$\begin{aligned} &a_\epsilon^{ij} = a_\epsilon^{ji}, \quad a_\epsilon^{ij}\xi_i\xi_j \geq \lambda_\epsilon|\xi|^2 \geq C_1\epsilon^a|\xi|^2, \\ &\text{for each } k \in \mathbb{N}, \quad |a_\epsilon^{ij}|_{k+1,\alpha;\Omega}, \ |b_\epsilon^i|_{k,\alpha;\Omega} \leq \Lambda_{k,\epsilon} \leq C_2\epsilon^{b(k)}, \\ &\{n_i : n_i < 0\} \neq \emptyset, \quad n_1 = \min\{n_i : n_i < 0\}, \quad b_\epsilon^1 \leq -C_3\epsilon^c \\ &\{n_i : n_i > 0\} \neq \emptyset, \quad n_K = \max\{n_i : n_i > 0\}, \quad b_\epsilon^K \geq C_4\epsilon^d, \end{aligned} \tag{2.4.1}$$

where C_1, C_2, C_3 and C_4 are positive constants independent of ϵ. Then the following Dirichlet problem has a solution in $\mathcal{G}(\overline{\Omega})$:

$$Au = [A_\epsilon u_\epsilon] = 0 \text{ in } \Omega, \tag{2.4.2}$$
$$u|_{\partial\Omega} = \rho.$$

We summarize this result in the following theorem, which will be the focus of the remainder of the paper.

Theorem 2.4.1. *Suppose that* $A : \mathcal{G}(\overline{\Omega}) \to \mathcal{G}(\overline{\Omega})$ *is in* $\mathcal{A}_0$ *and that the conditions of* (2.4.1) *hold. Assume that for each* $\epsilon \in (0, 1]$ *that for some* $1 \leq i \leq K$, b^i_ϵ *is non-constant. Furthermore, assume that* $\rho \in \mathcal{G}(\overline{\Omega})$ *has a representative* (ρ_ϵ) *such that for* $\epsilon < 1$, $\rho_\epsilon \geq C\epsilon^a$ *for some* $C > 0$ *and* $a \in \mathbb{R}$*. Then there exists a solution to the Dirichlet problem* (2.4.2) *in* $\mathcal{G}(\overline{\Omega})$.

Proof. The proof will be given in Section 2.6. □

Remark 2.4.2. *We can actually weaken the assumptions in* (2.4.1) *so that the conditions on the representatives* $(a^{ij}_\epsilon), (b^1_\epsilon), (b^K_\epsilon), (\rho_\epsilon)$ *only have to hold for all* $\epsilon \in (0, \epsilon_0)$ *for some* $\epsilon_0 \in (0, 1)$*. Suppose that this is the case, and that using these conditions we are able to show that for all* $\epsilon \in (0, \epsilon_0)$*, there exists* u_ϵ *that solves*

$$A_\epsilon u_\epsilon = 0 \quad \textit{in } \Omega, \tag{2.4.3}$$
$$u_\epsilon|_{\partial\Omega} = \rho_\epsilon.$$

If u_ϵ *satisfies the additional property that for all* $k \in \mathbb{N}$*, there exists some* $\epsilon_k \in (0, \epsilon_0)$, $C_k > 0$*, and* $a_k \in \mathbb{R}$ *such that for all* $\epsilon \in (0, \epsilon_k)$, $|u_\epsilon|_{k,\alpha} \leq C_k \epsilon^{a_k}$*, then we can form a solution* $(v_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$ *to* (2.4.2) *by defining* $v_\epsilon = u_\epsilon$ *for* $\epsilon \in (0, \epsilon_0)$ *and* $v_\epsilon = u_{\epsilon_0}$ *for* $\epsilon \in [\epsilon_0, 1]$*. The solution theory that we develop to prove Theorem 2.4.1 with the stronger conditions* (2.4.1) *will also imply the existence of the partial net* (u_ϵ) *of solutions to* (2.4.3) *in the event that the constraints outlined in* (2.4.1) *only hold for* $\epsilon \in (0, \epsilon_0) \subset (0, 1)$*. We will require this fact when we consider how to embed and solve* (2.1.1) *in* $\mathcal{G}(\overline{\Omega})$ *later on in Section 2.4.3.*

We begin assembling the tools we will need to prove Theorem 2.4.1. The first tool we need is a fixed-point theorem that will provide the basis for our solution method.

2.4.1 The Method of Sub- and Super-Solutions

In Theorem 2.4.3 below, we state a fixed-point result that will be essential in proving Theorem 2.4.1. This fixed-point result is known as the method of sub- and super-solutions due to the fact that for a given operator A, the method relies on finding a sub-solution u_- and super-solution u_+ such that $u_- < u_+$. A large part of this paper is devoted to finding a net of positive sub- and super-solutions for (2.4.2) and establishing growth conditions for them. In the proof below, let

$$Lu = -D_i(a^{ij}D_ju) + cu, \tag{2.4.4}$$

be an elliptic operator where

$$a^{ij} = a^{ji}, \quad a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{and} \quad a^{ij}, c \in C^\infty(\overline{\Omega}).$$

We now state and prove the sub- and super-solution fixed-point result for these assumptions. Here we closely follow the proof in [9].

Theorem 2.4.3. *Suppose* $\Omega \subset \mathbb{R}^n$ *is a* C^∞ *domain and assume* $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ *is in* $C^\infty(\overline{\Omega} \times \mathbb{R}^+)$ *and* $\rho \in C^\infty(\overline{\Omega})$. *Let* L *be of the form* (2.4.4). *Suppose that there exist functions* $u_- : \overline{\Omega} \to \mathbb{R}$ *and* $u_+ : \overline{\Omega} \to \mathbb{R}$ *such that the following hold:*

1. $u_-, u_+ \in C^\infty(\overline{\Omega})$,
2. $0 < u_-(x) \leq u_+(x) \quad \forall x \in \overline{\Omega}$,
3. $Lu_- \leq f(x, u_-)$,
4. $Lu_+ \geq f(x, u_+)$,
5. $u_- \leq \rho$ *on* $\partial\Omega$,
6. $u_+ \geq \rho$ *on* $\partial\Omega$.

Then there exists a solution u to

$$Lu = f(x,u) \quad \textit{on } \Omega, \qquad (2.4.5)$$
$$u|_{\partial\Omega} = \rho,$$

such that

(i) $u \in C^{\infty}(\overline{\Omega})$,

(ii) $u_{-}(x) \leq u(x) \leq u_{+}(x)$.

Proof. The general approach of the proof will be to construct a monotone sequence $\{u_n\}$ that is point-wise bounded above and below by our super- and sub-solutions, u_+ and u_-. We will then apply elliptic regularity estimates and the Arzela-Ascoli Theorem to conclude that the sequence $\{u_n\}$ has a $C^{\infty}(\overline{\Omega})$ limit u that is a solution to

$$Lu = f(x,u) \quad \text{on } \Omega, \qquad (2.4.6)$$
$$u|_{\partial\Omega} = \rho.$$

Given that $u_{-}(x), u_{+}(x) \in C^{\infty}(\overline{\Omega})$, the interval $[\min u_{-}(x), \max_{+} u_{+}(x)] \subset \mathbb{R}^{+}$ is well-defined. We then restrict the domain of the function f to the compact set $K = \overline{\Omega} \times [\min u_{-}(x), \max_{+} u_{+}(x)]$. Given that $f \in C^{\infty}(\overline{\Omega} \times \mathbb{R}^{+})$, it is clearly in $C^{\infty}(\overline{\Omega} \times [\min u_{-}(x), \max_{+} u_{+}(x)])$ and so the function $|\frac{\partial f(x,t)}{\partial t}|$ is continuous and attains a maximum on K. Denoting this maximum value by m, let $M = \max\{m, -\inf_{x\in\overline{\Omega}} c(x)\}$. Then consider the operator

$$Au = Lu + Mu,$$

and the function

$$F(x,t) = Mu + f(x,t).$$

Note that this choice of M ensures that $F(x,t)$ is an increasing function in t on K and that A is an invertible operator. Also, we clearly have the following:

$$A(u) = F(x,u) \Longleftrightarrow Lu = f(x,u), \tag{2.4.7}$$

$$A(u_-) \le F(x,u_-) \Longleftrightarrow L(u_-) \le f(x,u_-), \tag{2.4.8}$$

$$A(u_+) \ge F(x,u_+) \Longleftrightarrow L(u_+) \ge f(x,u_+). \tag{2.4.9}$$

The first step in the proof is to construct the sequence $\{u_n\}$ iteratively. Let u_1 satisfy the equation

$$\begin{aligned} &A(u_1) = F(x,u_-) \ \text{ on } \Omega, \\ &u_1|_{\partial\Omega} = \rho. \end{aligned} \tag{2.4.10}$$

We observe that for $u, v \in H_0^1(\Omega)$, the operator A satisfies

$$C_1\|u\|^2_{H^1(\Omega)} \le \langle Au, u\rangle, \qquad \text{and} \qquad \langle Au, v\rangle \le \|u\|^2_{H^1(\Omega)}\|v\|^2_{H^1(\Omega)},$$

where

$$\langle u, v\rangle = \int_\Omega uv dx, \quad \text{and} \quad \langle Lu, v\rangle = \int_\Omega (a^{ij}D_j u D_i v + cuv)dx.$$

Therefore the Lax-Milgram theorem implies that there exists a weak solution $u_1 \in H^1(\Omega)$ satisfying $u_1 - \rho \in H_0^1(\Omega)$. Given our assumptions on $F(x,t)$ and ρ, $F(x,u_-) \in H^m(\Omega)$ and $\rho \in H^m(\Omega)$ for all $m \in \mathbb{N}$. Therefore, by standard elliptic regularity arguments, $u_1 \in H^m(\Omega)$ for all $m \in \mathbb{N}$. This, the assumption that Ω is of C^∞-class and the assumption that $a^{ij}, c, \rho \in C^\infty(\overline{\Omega})$ imply that $u_1 \in C^\infty(\overline{\Omega})$ and $u_1 = \rho$ on $\partial\Omega$. Therefore, we may iteratively define the sequence $\{u_j\} \subset C^\infty(\overline{\Omega})$ where

$$\begin{aligned} &A(u_j) = F(x,u_{j-1}) \ \text{ on } \Omega, \\ &u_j|_{\partial\Omega} = \rho. \end{aligned} \tag{2.4.11}$$

The next step is to verify that the sequence $\{u_j\}$ is a monotonic increasing sequence satisfying $u_- \le u_1 \le \cdots \le u_{j-1} \le u_j \le \cdots \le u_+$. We prove this by induction.

First we observe that

$$A(u_- - u_1) \leq F(x, u_-) - F(x, u_-) = 0 \quad \text{on } \Omega, \tag{2.4.12}$$
$$(u_- - u_1)|_{\partial\Omega} \leq 0.$$

Therefore, by the weak maximum principle, $u_- \leq u_1$ on $\overline{\Omega}$. Now suppose that $u_{j-1} \leq u_j$. Then

$$A(u_j - u_{j+1}) = F(x, u_{j-1}) - F(x, u_j) \leq 0 \quad \text{on } \Omega, \tag{2.4.13}$$
$$(u_j - u_{j+1})|_{\partial\Omega} = 0,$$

given that $F(x,t)$ is an increasing function in the variable t and $u_{j-1} \leq u_j$. The weak maximum principle again implies that $u_j \leq u_{j+1}$, so by induction we have that $\{u_j\}$ is monotonic increasing sequence that is point-wise bounded below by $u_-(x)$. Now we show that our increasing sequence is point-wise bounded above by $u_+(x)$ by proceeding in a similar manner. Given that $u_- \leq u_+$ and u_+ is a super-solution, we have that

$$A(u_1 - u_+) \leq F(x, u_-) - F(x, u_+) \leq 0 \quad \text{on } \Omega, \tag{2.4.14}$$
$$(u_1 - u_+)|_{\partial\Omega} \leq 0.$$

The weak maximum principle implies that $u_1 \leq u_+$. Now assume that $u_j \leq u_+$. Then

$$A(u_{j+1} - u_+) \leq F(x, u_j) - F(x, u_+) \leq 0 \quad \text{on } \Omega, \tag{2.4.15}$$
$$(u_{j+1} - u_+)|_{\partial\Omega} \leq 0,$$

given that $F(x,t)$ is an increasing function and $u_j \leq u_+$. So by induction the sequence $\{u_j\}$ is a monotonic increasing sequence that is point-wise bounded above by $u_+(x)$ and point-wise bounded below by $u_-(x)$.

So far we have constructed a monotonic increasing sequence $\{u_j\} \subset C^\infty(\overline{\Omega})$ such that for each j, u_j satisfies the Dirichlet problem (2.4.11) and is point-wise bounded below by u_- and above by u_+. The next step will be to apply the Arzela-Ascoli theorem and a bootstrapping argument to conclude that this sequence converges to $u \in C^\infty(\overline{\Omega})$.

We first show that it converges to $u \in C(\overline{\Omega})$ by an application of the Arzela-Ascoli Theorem. Clearly the family of functions $\{u_j\}$ is point-wise bounded, so it is only necessary to establish the equicontinuity of the sequence. Given that each function u_j solves the problem (2.4.11), by standard L^p elliptic regularity estimates (cf. [5]) we have that

$$\|u_j\|_{W^{2,p}} \leq C(\|u_j\|_{L^p} + \|F(x, u_{j-1})\|_{L^p}).$$

The regularity of $F(x,t)$ and the sequence $\{u_j\}$ along with the above estimate and the compactness of $\overline{\Omega} \times [\inf u_-, \sup u_+]$ imply that there exists a constant N such that $\|F(x, u_{j-1})\|_{L^p} \leq N$ for all j. Therefore, if $p > 3$, the above bound and the fact that $u_- \leq u_j \leq u_+$ imply that for each $j \in \mathbb{N}$,

$$|u_j|_{1,\alpha;\Omega} \leq C\|u\|_{W^{2,p}} \leq \infty,$$

where $\alpha = 1 - \frac{3}{p}$. This implies that the sequence $\{u_j\}$ is equicontinuous. The Arzela-Ascoli Theorem then implies that there exists a $u \in C(\overline{\Omega})$ and a subsequence $\{u_{j_k}\}$ such that $u_{j_k} \to u$ uniformly. Furthermore, due to the fact that the sequence $\{u_j\}$ is monotonic increasing, we actually have that $u_j \to u$ uniformly on $\overline{\Omega}$. Once we have that $u_j \to u$ in $C(\overline{\Omega})$, we apply L^p regularity theory again to conclude that

$$\begin{aligned} |u_j - u_k|_{1,\alpha;\Omega} \leq & C\|u_j - u_k\|_{W^{2,p}} \\ \leq & C'(\|u_j - u_k\|_{L^p} + \|F(x, u_{j-1}) - F(x, u_{k-1})\|_{L^p}). \end{aligned} \tag{2.4.16}$$

Note that the above estimate follows from the fact that $u_j - u_k$ satisfies

$$\begin{aligned} A(u_j - u_k) = F(x, u_{j-1}) - F(x, u_{k-1}) \quad \text{on } \Omega, \\ (u_j - u_k)|_{\partial\Omega} = 0. \end{aligned} \tag{2.4.17}$$

Given that $u_j \to u$ in $C(\overline{\Omega})$, (2.4.16) implies that the sequence $\{u_j\}$ is a Cauchy sequence in $C^1(\overline{\Omega})$. The completeness of $C^1(\overline{\Omega})$ then implies that this sequence has a limit $v \in C^1(\overline{\Omega})$, and given that $u_j \to u$ in $C(\overline{\Omega})$, it follows that $u = v$. Similarly, by repeating the above argument and using higher order L^p estimates we have that

$$\begin{aligned} |u_j - u_k|_{2,\alpha;\Omega} \leq & C(\|u_j - u_k\|_{W^{3,p}}) \\ \leq & C'(\|u_j - u_k\|_{W^{1,p}} + \|F(x, u_{j-1}) - F(x, u_{k-1})\|_{W^{1,p}}), \end{aligned} \tag{2.4.18}$$

where $u_j \to u$ in $C^1(\overline{\Omega})$ as $k \to \infty$. Again, (2.4.18), the regularity of F and the fact that $u_j \to u$ in $C^1(\overline{\Omega})$ imply that the sequence $\{u_j\}$ is Cauchy in $C^2(\overline{\Omega})$. A simple induction argument then shows that $u \in C^\infty(\overline{\Omega})$.

The final step of the proof is to show that u is an actual solution to the problem (2.4.5). It suffices to show that u is a weak solution to the above problem. It is clear that $u = \rho$ on $\partial\Omega$, so we only need to show that u satisfies (2.4.5) on Ω. Fix $v \in H_0^1(\Omega)$. Then based on the definition of the sequence $\{u_j\}$, we have

$$\int_\Omega (a^{ij} D_j u_j D_i v + M u_j v) dx = \int_\Omega (f(x, u_{j-1}) + M u_{j-1}) v dx.$$

As $u_j \to u$ uniformly in $C(\overline{\Omega})$, we let let $j \to \infty$ to conclude that

$$\int_\Omega (a^{ij} D_j u D_i v + M u v) dx = \int_\Omega (f(x, u) + M u) v dx.$$

Upon canceling the term involving M from both sides, we find that u is a weak solution. □

2.4.2 Outline of the Proof of Theorem 2.4.1

Now that the sub- and super-solution fixed-point theorem is in place, we give an outline for how to prove Theorem 2.4.1.

Step 1: *Formulation of the problem.* We phrase (2.4.2) in a way that allows us to solve a net of semilinear elliptic problems. We assume that the coefficients of A and boundary data ρ have representatives (a_ϵ^{ij}), (b_ϵ^i), and (ρ_ϵ) in $\mathcal{E}_M(\overline{\Omega})$ satisfying the assumptions (2.4.1). Then for this particular choice of representatives, we solve the family of problems:

$$A_\epsilon u_\epsilon = -\sum_{i,j=1}^{N} D_i(a_\epsilon^{ij} D_j u_\epsilon) + \sum_{i}^{N} b_\epsilon^i u_\epsilon^{n_i} = 0 \quad \text{in } \Omega, \tag{2.4.19}$$

$$u_\epsilon|_{\partial\Omega} = \rho_\epsilon.$$

Then we must ensure that the net of solutions $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$ and that Eq. (2.4.19) is satisfied for other representatives of A, ρ, u.

Step 2: *Determine L^∞-estimates and a net of generalized constant sub-solutions and super-solutions*. We determine constant, *a priori* L^∞ bounds such that for a positive net of solutions (u_ϵ) of the semilinear problem (2.4.19), there exist constants $a_1, a_2 \in \mathbb{R}$, $C_1, C_2 > 0$ independent of $\epsilon \in (0,1)$ such that

$$C_1\epsilon^{a_1} < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon < C_2\epsilon^{a_2}.$$

These estimates are constructed in such a way that for each ϵ, the pair $\alpha_\epsilon, \beta_\epsilon$ are sub- and super-solutions for (2.4.19).

Step 3: *Apply fixed-point theorem to solve each semilinear problem in* (2.4.19). Using the sub- and super-solutions $\alpha_\epsilon, \beta_\epsilon$, we apply Theorem 2.4.3 to obtain a net of solutions $(u_\epsilon) \in C^\infty(\overline{\Omega})$.

Step 4: *Verify that the net of solutions* $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$. Here we show that the net of solutions satisfies the necessary growth conditions in ϵ using the growth conditions on the sub- and super-solutions and Theorem 2.3.1.

Step 5: *Verify that the solution is well-defined*. Once we've determined that the net of solutions $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$, we conclude that $[(u_\epsilon)] \in \mathcal{G}(\overline{\Omega})$ is a solution to the Dirichlet problem (2.4.2) by showing that the solution is independent of the representatives chosen. Note that most of the work for this step was done in Proposition 2.3.9.

We shall carry out the above steps in our proof of Theorem 2.4.1 in Section 2.6. We still need to determine a net of sub- and super-solutions for (2.4.1), which we do in Section 2.5. But before we move on to this and the other steps in the above outline, we briefly return to the motivating problem (2.1.1) by discussing how to solve a problem with distributional data into $\mathcal{G}(\overline{\Omega})$.

2.4.3 Embedding a Semilinear Elliptic PDE with Distributional Data into $\mathcal{G}(\overline{\Omega})$.

Now that we have defined what it means to solve a differential equation in $\mathcal{G}(\overline{\Omega})$, we are ready to return to the problem discussed at the beginning of the paper. We are interested in solving an elliptic, semilinear Dirichlet problem of the form

$$-\sum_{i,j=1}^{N} D_i(a^{ij} D_j u) + \sum_{i=1}^{K} b^i u^{n_i} = 0 \quad \text{in } \Omega, \tag{2.4.20}$$
$$u|_{\partial\Omega} = \rho,$$

where a^{ij}, b^i and ρ are potentially distributional and $n_i \in \mathbb{Z}$ for each i. If we can formulate this problem as a family of equations similar to (2.4.19), then it can readily be solved in $\mathcal{G}(\overline{\Omega})$ by Theorem 2.4.1. The key to this formulation will be the embedding defined in Eq. (2.3.12).

Suppose that we are given a problem of the form (2.4.20), where the terms a^{ij}, b^i and ρ are in $\mathcal{D}'(\Omega')$. Then we may use Proposition 2.3.8 to embed the coefficients a^{ij}, b^i and ρ into $\mathcal{G}(\overline{\Omega})$. We will denote a representative of the image of each these terms in $\mathcal{G}(\overline{\Omega})$ by $(a^{ij}_\epsilon), (b^i_\epsilon)$ and (ρ_ϵ). Then for a choice of representatives, we obtain a net of problems of the form (2.4.19).

In order to solve this net of problems by using Theorem 2.4.1, we need there to exist a choice of representatives $(a^{ij}_\epsilon), (b^i_\epsilon)$ and (ρ_ϵ) that satisfy the conditions specified in (2.4.1). While these conditions might seem exacting, this solution framework still admits a wide range of interesting problems. This is evident when one considers the following proposition:

Proposition 2.4.4. *Let $n_i \in \mathbb{Z}$ be a collection of integers for $1 \le i \le K$. Assume that there exist $1 \le i, j \le K$ such that $n_i < 0$ and $n_j > 0$ and let*

$$n_1 = \min\{n_i : \ n_i < 0\}, \quad \textit{and} \quad n_K = \max\{n_i : \ n_i > 0\}.$$

Suppose that $a^{ij}, b^1, b^K, \rho \in C(\Omega')$ and $b^2, \cdots, b^{K-1} \in \mathcal{D}'(\Omega')$, where $\Omega' \subset \mathbb{R}^n$ is an open and bounded set. Additionally assume that a^{ij} satisfies the symmetric, ellipticity condition and $\rho > 0$, $b_1 < 0$ and $b_K > 0$ in Ω'. Then if $\Omega \subset\subset \Omega'$ is of C^∞-class, the problem

$$-\sum_{i,j=1}^{N} D_i(a^{ij} D_j u) + \sum_{i=1}^{K} b^i u^{n_i} = 0 \quad \text{in } \Omega, \tag{2.4.21}$$
$$u|_{\partial\Omega} = \rho,$$

admits a solution in $\mathcal{G}(\overline{\Omega})$.

Proof. This follows from Proposition 2.3.8, Theorem 2.4.1, Remark 2.4.2 and the fact that
$(a^{ij} * \phi_\epsilon)$, $(b^1 * \phi_\epsilon)$, $(b^K * \phi_\epsilon)$ and $(\rho * \phi_\epsilon)$ converge uniformly to a^{ij}, b^1, b^K and ρ in $\overline{\Omega}$. For ϵ sufficiently small, the corresponding problem (2.4.19) in $\mathcal{G}(\overline{\Omega})$ will satisfy the conditions specified in (2.4.1). Therefore, Theorem 2.4.1 and Remark 2.4.2 imply the result. □

With the issue of solving (2.4.20) at least partially resolved, we return to the task of proving Theorem 2.4.1. We begin by establishing some *a priori* L^∞-bounds for a solution to our semilinear problem (2.4.21) if the given data is smooth.

2.5 Sub- and Super-Solution Construction and Estimates

Given an operator $A \in \mathcal{A}_0$ with coefficients satisfying (2.4.1), our solution strategy for the Dirichlet problem (2.4.2) is to solve the family of problems (2.4.19) and then establish the necessary decay estimates. In order for this to be a viable strategy, we first need to show that (2.4.19) has a solution for each $\epsilon \in (0, 1]$. Given that $n_i < 0$ for some $1 \leq i \leq K$, for each ϵ we must restrict the operator

$$A_\epsilon u_\epsilon = -\sum_{i,j=1}^{N} D_i(a_\epsilon^{ij} D_j u_\epsilon) + \sum_{i=1}^{K} b^i u_\epsilon^{n_i},$$

to a subset of functions in $C^{\infty}(\overline{\Omega})$ to guarantee that A_ϵ is well-defined. In particular, for each ϵ we consider functions $u_\epsilon \in C^{\infty}(\overline{\Omega})$ such that $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon < \infty$ for some choice of α_ϵ and β_ϵ. The first part of this section is dedicated to making judicious choices of α_ϵ and β_ϵ for each ϵ such that a solution u_ϵ to (2.4.19) exists that satisfies $\alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$.

Once a net of solutions (u_ϵ) is determined, it is necessary to show that if $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$, then an operator $A \in \mathcal{A}_0$ whose coefficients satisfy (2.4.1) is well-defined for (u_ϵ). Recall that A is only a well defined operator for elements $u \in \mathcal{G}(\overline{\Omega})$ satisfying $u_\epsilon \geq C\epsilon^a$ for $\epsilon \in (0, \epsilon_0) \subset (0, 1)$, $a \in \mathbb{R}$ and some constant C independent of ϵ. This will require us to establish certain ϵ-decay estimates on α_ϵ, which we do later in this section.

2.5.1 L^{∞} Bounds for the Semilinear Problem

We begin by determining the net of *a priori* bounds α_ϵ and β_ϵ described above. For now we disregard the ϵ notation. In the following proposition we determine *a priori* estimates for a positive solution u to a problem of the form

$$-\sum_{i,j}^{N} D_i(a^{ij} D_j u) + \sum_{i=1}^{K} b^i u^{n_i} = 0 \text{ in } \Omega, \tag{2.5.1}$$

$$u|_{\partial\Omega} = \rho,$$

where $\Omega \subset \mathbb{R}^n$ is connected, bounded, and of C^{∞}-class, and $a^{ij}, b^i, \rho \in C(\overline{\Omega})$ with $\rho > 0$ in $\overline{\Omega}$.

Proposition 2.5.1. *Suppose that the semilinear operator in* (2.5.1) *has the property that* $n_i > 0$ *for some* $1 \leq i \leq K$. *Let* n_K *be the largest positive exponent and suppose that* $b_K(x) > 0$ *in* $\overline{\Omega}$. *Additionally, assume that one of the following two cases holds:*

(1) $n_i < 0$ *for some* $1 \leq i \leq K$ *and if* $n_1 = \min\{n_i : n_i < 0\}$, (2.5.2)

then $b_1(x) < 0$ *in* $\overline{\Omega}$.

(2) n_K *is odd and* $0 < n_i$ *for all* $1 \leq i \leq K$. (2.5.3)

Then if case (2.5.2) *holds and* u *is a nonnegative solution to* (2.5.1)*, there exist constants* α *and* β *such that* $0 < \alpha \leq u \leq \beta$*. If case* (2.5.3) *holds and* u *is a nonnegative solution to* (2.5.1)*, there exists a constant* β *such that* $0 \leq u \leq \beta$*.*

Remark 2.5.2. *Note that for the purposes of proving Theorem 2.4.1, we are primarily concerned with case* (2.5.2)*. This is the case that we will focus on for the remainder of the paper. However, with a little extra work we could very easily generalize Theorem 2.4.1 to allow for* $n_i > 0$ *for all* $1 \leq i \leq K$ *and* $n_K > 0$ *odd. Then we could use case* (2.5.3) *to establish the necessary bounds.*

Proof. We first define α' and β' in the following way. If case (2.5.2) holds, then let

$$\alpha' = \sup_{c \in \mathbb{R}_+} \left\{ \sum_{i=1}^{K} \sup_{x \in \Omega} b_i(x) b^{n_i} < 0 \quad \forall b \in (0, c) \right\}. \tag{2.5.4}$$

If case (2.5.3) holds, let $\alpha' = 0$. In either case (2.5.2) or (2.5.3), define

$$\beta' = \inf_{c \in \mathbb{R}} \left\{ \sum_{i=1}^{K} \inf_{x \in \Omega} b_i(x) b^{n_i} > 0 \quad \forall b \in (c, \infty) \right\}. \tag{2.5.5}$$

Note that in all cases α' and β' are always well-defined given the conditions on $b^1(x)$ and $b^K(x)$. Then define

$$\alpha = \min\{\alpha', \inf_{x \in \partial\Omega} \rho(x)\}, \tag{2.5.6}$$

$$\beta = \max\{\beta', \sup_{x \in \partial\Omega} \rho(x)\}. \tag{2.5.7}$$

Based on these definitions of α and β, if u is a nonnegative solution to (2.5.1) then it is easy to verify that the functions $\overline{\phi} = (u - \beta)^+$ and $\underline{\phi} = (u - \alpha)^-$ are in $H_0^1(\Omega)$ if either (2.5.2) or (2.5.3) holds, and we define

$$\overline{\mathcal{Y}} = \left\{ x \in \overline{\Omega} \mid u \geq \beta \right\}.$$

If case (2.5.2) holds, let

$$\underline{\mathcal{Y}} = \{ x \in \overline{\Omega} \mid 0 < u \leq \alpha \},$$

if case (2.5.3) holds, let

$$\underline{\mathcal{Y}} = \{x \in \overline{\Omega} \mid u < \alpha\}.$$

Then if $u \in H^1(\Omega)^+$ is a weak solution to (2.5.1), supp($\overline{\phi}$) = $\overline{\mathcal{Y}}$ and supp($\underline{\phi}$) = $\underline{\mathcal{Y}}$. We have the following string of inequalities for $\underline{\phi}$:

$$\begin{aligned} C_2\|\underline{\phi}\|^2_{H^1(\Omega)} &\leq C_1\|\nabla((u-\alpha)^-)\|^2_{L^2(\Omega)} \\ &\leq \int_\Omega a^{ij}D_j((u-\alpha)^-)D_j((u-\alpha)^-)\,dx \\ &= \int_\Omega a^{ij}D_j(u-\alpha)D_j((u-\alpha)^-)\,dx \\ &= \int_{\underline{\mathcal{Y}}}(-\sum_{i=1}^K b_i(x)u^{n_i})(u-\alpha)\,dx \leq 0. \end{aligned} \tag{2.5.8}$$

Similarly, we have the following string of inequalities for $\overline{\phi}$:

$$\begin{aligned} C_2\|\overline{\phi}\|^2_{H^1(\Omega)} &\leq C_1\|\nabla((u-\beta)^+)\|^2_{L^2(\Omega)} \\ &\leq \int_\Omega a^{ij}D_j((u-\beta)^+)D_i((u-\beta)^+)\,dx \\ &= \int_\Omega a^{ij}D_j(u-\beta)D_i((u-\beta)^+)\,dx \\ &= \int_{\overline{\mathcal{Y}}}(-\sum_{i=1}^K b_i(x)u^{n_i})(u-\beta)\,dx \leq 0. \end{aligned} \tag{2.5.9}$$

The above inequalities imply that if u is a positive, weak solution to the semilinear (2.5.1), then $u \in [\alpha, \beta]$ where in case (2.5.2), $\alpha > 0$. □

Now that we've established L^∞-bounds for solutions to (2.5.1), we can apply these bounds for each fixed ϵ to determine a net of bounds for the following net of problems:

$$-\sum_{i,j}^{N} D_i a_\epsilon^{ij} D_j u_\epsilon + \sum_{i=1}^{K} b_\epsilon^i u_\epsilon^{n_i} = 0 \quad \text{in } \Omega \tag{2.5.10}$$
$$u_\epsilon|_{\partial\Omega} = \rho_\epsilon,$$

where $(a_\epsilon^{ij}), (b_\epsilon^i), (\rho_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$ satisfy the following conditions for all $\epsilon < 1$:

$$a_\epsilon^{ij} = a_\epsilon^{ji}, \quad a_\epsilon^{ij}\xi_i\xi_j \geq \lambda_\epsilon|\xi|^2 \geq C_1\epsilon^a|\xi|^2, \tag{2.5.11}$$
$$\text{for each } k \in \mathbb{N}, \quad |a_\epsilon^{ij}|_{k+1,\alpha;\Omega}, \ |b_\epsilon^i|_{k,\alpha;\Omega} \leq \Lambda_{k,\epsilon} \leq C_2\epsilon^{b(k)},$$
$$\{n_i : n_i < 0\} \neq \emptyset, \quad n_1 = \min\{n_i : n_i < 0\}, \quad b_\epsilon^1 \leq -C_3\epsilon^c$$
$$\{n_i : n_i > 0\} \neq \emptyset, \quad n_K = \max\{n_i : n_i > 0\}, \quad b_\epsilon^K \geq C_4\epsilon^d,$$
$$\rho_\epsilon \geq C_5\epsilon^e,$$

and $C_1, \cdots, C_5$ are independent of ϵ and $a, \cdots, e \in \mathbb{R}$.

Proposition 2.5.3. *Suppose that for each fixed $\epsilon \in (0,1]$, u_ϵ is a positive solution to* (2.5.10) *with coefficients satisfying* (2.5.11). *Then there exist L^∞-bounds α_ϵ and β_ϵ such that for each ϵ, $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$.*

Proof. For each fixed ϵ, if the assumptions in (2.5.11) hold, then case (2.5.2) of Proposition 2.5.1 is satisfied. Therefore, for each $\epsilon \in (0,1]$, there exists α_ϵ and β_ϵ such that $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$. □

2.5.2 Sub- and Super-Solutions

In the previous section we showed that if the data of (2.5.10) satisfies (2.5.11) and if $u_\epsilon \in C^\infty(\overline{\Omega})$ solves (2.5.10) for each ϵ, then $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$. Now, for each $\epsilon \in (0,1]$, we want to show that there actually exists a solution $u_\epsilon \in C^\infty(\overline{\Omega})$ satisfying $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$. The key to proving this result lies in the fact that α_ϵ and β_ϵ are sub- and super-solutions to (2.5.10) for each ϵ.

Proposition 2.5.4. *Let α_ϵ and β_ϵ be the bounds established in Proposition 2.5.3. Suppose that the coefficients in the net of problems* (2.5.10) *satisfy* (2.5.11) *and that for each*

ϵ some $b^i_\epsilon(x)$ is non-constant. Then there exists a net $(u_\epsilon) \in (C(\overline{\Omega}))^I$ such that for each ϵ, u_ϵ solves (2.5.10) *and* $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$.

Proof. To solve the above family of problems in (2.5.10), we show that the net of L^∞-bounds (α_ϵ) and (β_ϵ) found in Proposition 2.5.3 is a net of sub and super-solutions to (2.5.10). Then we verify that the interval $[\alpha_\epsilon, \beta_\epsilon]$ is nonempty and is a subset of $\mathbb{R}^+$. We will then be able to apply Theorem 2.4.3 to conclude that for each ϵ, there exists a solution $u_\epsilon \in C^\infty(\overline{\Omega})$.

Fix ϵ and let α'_ϵ and β'_ϵ be defined by (2.5.4) and (2.5.5) respectively, and let

$$\alpha_\epsilon = \min\{\alpha'_\epsilon, \inf_{\partial\Omega} \rho_\epsilon(x)\},$$
$$\beta_\epsilon = \max\{\beta'_\epsilon, \sup_{x\in\partial\Omega} \rho_\epsilon(x)\}.$$

By (2.5.4) and the fact that $\rho_\epsilon > 0$, we clearly have that $\alpha_\epsilon > 0$. Then (2.5.4) and the definition of α_ϵ imply that

$$A_\epsilon\alpha_\epsilon = \sum_{i=1}^{K} b^i_\epsilon(\alpha_\epsilon)^{n_i} \leq \sum_{i=1}^{K} \sup_{x\in\overline{\Omega}} b^i_\epsilon(\alpha_\epsilon)^{n_i} \leq 0, \tag{2.5.12}$$
$$\alpha_\epsilon \leq \inf_{x\in\partial\Omega} \rho_\epsilon(x) \leq \rho_\epsilon,$$

which shows that α_ϵ is sub-solution for each ϵ. Similarly, (2.5.5) and the definition of β'_ϵ imply that

$$A_\epsilon\beta_\epsilon = \sum_{i=1}^{K} b^i_\epsilon(\beta_\epsilon)^{n_i} \geq \sum_{i=1}^{K} \inf_{x\in\overline{\Omega}} b^i_\epsilon(\beta_\epsilon)^{n_i} \geq 0, \tag{2.5.13}$$
$$\beta_\epsilon \geq \sup_{x\in\partial\Omega} \rho_\epsilon \geq \rho_\epsilon,$$

which shows that β_ϵ is a super-solution for each ϵ.

Now that we've determined that the pair α_ϵ and β_ϵ are sub- and super-solutions of (2.5.10), we show that the interval $[\alpha_\epsilon, \beta_\epsilon]$ is nonempty. Given the definition of α_ϵ and β_ϵ, it suffices to show that $\alpha'_\epsilon < \beta'_\epsilon$. We claim that $\alpha'_\epsilon < \beta'_\epsilon$ provided that at least one

$b^i_\epsilon(x)$ is nonzero and non-constant. This follows from the fact that if we define

$$\gamma_\epsilon = \inf_{c\in\mathbb{R}}\{\sum_{i=1}^{K}\sup_{x\in\overline{\Omega}} b^i(x)d^{n_i} \geq 0 \quad \forall d \in (c,\infty)\},$$

then we have that $\alpha'_\epsilon \leq \gamma_\epsilon$ by (2.5.4) and (2.5.11). Furthermore, for a fixed ϵ, given the assumptions on $b^i_\epsilon(x)$,

$$\sum_{i=1}^{K}\inf_{x\in\overline{\Omega}} b^i_\epsilon(x)y^{n_i} < \sum_{i=1}^{K}\sup_{x\in\overline{\Omega}} b^i_\epsilon(x)y^{n_i} \quad \forall y \in \mathbb{R}.$$

But (2.5.5) and the above inequality clearly imply that $\gamma_\epsilon < \beta'_\epsilon$. Therefore $\alpha'_\epsilon < \beta'_\epsilon$ and the interval $[\alpha_\epsilon, \beta_\epsilon]$ is a nonempty subset of $\mathbb{R}^+$. For each $\epsilon \in (0,1]$, the hypotheses of Theorem 2.4.3 are satisfied for the elliptic problem (2.5.11), so we may conclude that there exists a net of solutions $(u_\epsilon) \in (C^\infty(\overline{\Omega}))^I$ that satisfy $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$ for each fixed ϵ. □

The final task in this section is to show that an operator $A \in \mathcal{A}_0$, with coefficients satisfying (2.5.11), is a well-defined operator on any element $u \in \mathcal{E}_M(\overline{\Omega})$ satisfying

$$\alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon \quad \forall\epsilon \in (0,1].$$

Recall that in Section 2.3.4 we determined that A is only well-defined for invertible $u \in \mathcal{G}(\overline{\Omega})$. Therefore, it suffices to show that $(\alpha_\epsilon), (\beta_\epsilon)$ and $(\frac{1}{\alpha_\epsilon}), (\frac{1}{\beta_\epsilon})$ are generalized constants (2.3.3), which we verify in the following lemma.

Lemma 2.5.5. *Let (α_ϵ) and (β_ϵ) be the net of sub- and super-solutions to* (2.5.10) *determined in Section 2.5.1. Suppose that the coefficients of* (2.5.10) *satisfy* (2.5.11). *Then (α_ϵ), (β_ϵ),$(\frac{1}{\alpha_\epsilon})$, and $(\frac{1}{\beta_\epsilon})$ are in $\overline{\mathbb{C}}$, the ring of generalized constants.*

Remark 2.5.6. *Note that if $(\frac{1}{\alpha_\epsilon}) \in \overline{\mathbb{C}}$, then this implies that there exists an $\epsilon_0 \in (0,1)$, some constant C independent of ϵ, and $a \in \mathbb{R}$ such that $\alpha_\epsilon \geq C\epsilon^a$ for all $\epsilon \in (0,\epsilon_0)$. Then if $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$ satisfies $\alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$ for each ϵ, $(\frac{1}{\alpha_\epsilon}) \in \overline{\mathbb{C}}$ implies that $u = [(u_\epsilon)]$ is invertible in $\mathcal{G}(\overline{\Omega})$. See Section 2.3.4 and [6] for more details.*

Proof. We need to show that there exists constants D_1, D_2 independent of ϵ and $\epsilon_0 \in (0,1]$ such that for all $\epsilon \in (0, \epsilon_0)$,

$$\alpha_\epsilon \geq D_1 \epsilon^{b_1} \quad \text{for some } b_1 \in \mathbb{R},$$
$$\beta_\epsilon \leq D_2 \epsilon^{b_2} \quad \text{for some } b_2 \in \mathbb{R}.$$

So it is necessary to verify that there exists constants D_1 and D_2 so that for ϵ sufficiently small

$$\alpha'_\epsilon \geq D_1 \epsilon^{b_1}, \quad \text{and} \quad \inf_{x \in \partial\Omega} \rho_\epsilon \geq D_1 \epsilon^{b_1},$$
$$\beta'_\epsilon \leq D_2 \epsilon^{b_2}, \quad \text{and} \quad \sup_{x \in \partial\Omega} \rho_\epsilon \leq D_2 \epsilon^{b_2}.$$

Given that $(\rho_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$,

$$\sup_{x \in \partial\Omega} \rho_\epsilon \leq \sup_{x \in \overline{\Omega}} \rho_\epsilon = \mathcal{O}(\epsilon^b),$$

for some $b \in \mathbb{R}$. This and the assumption on (ρ_ϵ) in (2.5.11) imply that we only need to obtain the necessary ϵ-bounds on α'_ϵ and β'_ϵ.

For now, drop the ϵ notation and consider α' defined in (2.5.4). For a given function f, define

$$\underline{\gamma}_f = \sup_{c \in \mathbb{R}_+} \left\{ f(b) \leq 0 \quad \forall b \in (0, c) \right\}.$$

Given that

$$\alpha' = \sup_{c \in \mathbb{R}_+} \left\{ \sum_{i=1}^{K} \sup_{x \in \Omega} b_i(x) b^{n_i} \leq 0 \quad \forall b \in (0, c) \right\},$$

it is clear that for another function $f(y)$ such that

$$f(y) \geq \sum_{i=1}^{K} \sup_{x \in \Omega} b_i(x) y^{n_i} \quad \text{on } (0, c),$$

if $\underline{\gamma}_f$ is defined and $\alpha' \in (0, c)$, it must hold that $\underline{\gamma}_f \leq \alpha'$. Let $C_1 = |\{n_i : n_i \geq 0\}|$ and $C_2 = |\{n_i : n_i < 0\}|$ and if $C_2 > 1$, let $n_{i_2} = \min\{n_i : n_1 < n_i < 0\}$. Note that $C_1, C_2 \geq 1$ based on the assumptions in (2.5.11). Then recalling that $b_1(x) < 0$,

$b_K(x) > 0$ correspond to the coefficients of the terms with the smallest negative and largest positive exponent of $\sum_i^K b_i(x)u^{n_i}$, if $\sup_{x\in\overline{\Omega}} |b_i(x)| \leq \Lambda$ for each i, the following must hold for $y \in (0,1)$:

$$\sum_{i=1}^{K} \sup_{x\in\Omega} b_i(x)y^{n_i} \leq \sup_{x\in\overline{\Omega}} b_1(x)y^{n_1} + C_1\Lambda + (C_2-1)\Lambda y^{n_{i_2}}. \tag{2.5.14}$$

Define

$$d = \left(\frac{-\sup_{x\in\overline{\Omega}}(b_1(x))}{2(C_2-1)\Lambda}\right)^{\frac{1}{n_{i_2}-n_1}}$$

if $C_2 > 1$ and let $d = 1$ if $C_2 = 1$. Then let $c = \min\{1, d\}$. The definition of c implies that

$$(C_2-1)\Lambda y^{n_{i_2}} \leq -\frac{\sup_{x\in\overline{\Omega}} b_1(x)}{2} y^{n_1},$$

for all $y \in (0,c)$. So for $y \in (0,c)$,

$$\sum_{i=1}^{K} \sup_{x\in\overline{\Omega}} b_i(x)y^{n_i} \leq \frac{\sup_{x\in\overline{\Omega}} b_1(x)}{2} y^{n_1} + C_1\Lambda = f(y).$$

Then if $\alpha' \in (0,c)$, $\alpha' \geq \underline{\gamma}_f$. Given that $f(y)$ is a monotone increasing function on $\mathbb{R}_+$, $\underline{\gamma}_f$ is the lone positive root of $f(y)$. Thus,

$$\underline{\gamma}_f = \left(\frac{-\sup_{x\in\overline{\Omega}} b_1(x)}{2C_1\Lambda}\right)^{\frac{1}{-n_1}},$$

which implies that if $\alpha' \in (0,c)$,

$$\alpha' \geq \left(\frac{-\sup_{x\in\Omega} b_1(x)}{2C_1\Lambda}\right)^{\frac{1}{-n_1}}.$$

Similarly, for a fixed $\epsilon \in (0,1)$, define

$$d_\epsilon = \left(\frac{-\sup_{x\in\overline{\Omega}}(b_\epsilon^1(x))}{2(C_2-1)\Lambda_\epsilon}\right)^{\frac{1}{n_{i_2}-n_1}},$$

if $C_2 > 1$ and let $d_\epsilon = 1$ if $C_2 = 1$. Let $c_\epsilon = \min\{1, d_\epsilon\}$. Then for $y \in (0, c_\epsilon)$, we have that

$$(C_2 - 1)\Lambda_\epsilon y^{n_{i_2}} \leq -\frac{\sup_{x\in\overline{\Omega}} b_\epsilon^1(x)}{2} y^{n_1}.$$

So the above arguments imply that if $\alpha'_\epsilon \in (0, c_\epsilon)$, then $\alpha'_\epsilon \geq \underline{\gamma}_{f,\epsilon}$ and

$$\alpha'_\epsilon \geq \left(\frac{-\sup_{x\in\overline{\Omega}} b_\epsilon^1(x)}{2C_1\Lambda_\epsilon}\right)^{\frac{1}{-n_1}}.$$

Given the assumptions on $b_\epsilon^1(x)$ and Λ_ϵ in (2.5.11), in this case we have that $\alpha'_\epsilon \geq C\epsilon^a$ for some constant $C > 0$, $a \in \mathbb{R}$ and ϵ sufficiently small. Now we must show that $c_\epsilon \geq C\epsilon^a$ for some constant $C > 0$, $a \in \mathbb{R}$ and ϵ sufficiently small in the event that $\alpha'_\epsilon \notin (0, c_\epsilon)$. It suffices to show that $d_\epsilon \geq C\epsilon^a$ in the event that $C_2 > 1$. But clearly, for ϵ sufficiently small

$$d_\epsilon = \left(-\frac{\sup_{x\in\overline{\Omega}} b_\epsilon^1(x)}{2(C_2-1)\Lambda_\epsilon}\right)^{\frac{1}{n_{i_2}-n_1}} \geq C\epsilon^a,$$

given the assumptions on b_ϵ^1 and Λ_ϵ in (2.5.11). Therefore $\alpha'_\epsilon \geq D_1\epsilon^a$ for some constant $D_1 > 0$, $a \in \mathbb{R}$ and ϵ sufficiently small.

Now we determine bounds on the net (β'_ϵ). Again, we temporarily drop the ϵ and only consider β'. Recall that

$$\beta' = \inf_{c\in\mathbb{R}} \left\{ \sum_{i=1}^{K} \inf_{x\in\Omega} b_i(x) b^{n_i} \geq 0 \quad \forall b \in (c, \infty) \right\}.$$

For a given function $f(y)$, define

$$\overline{\gamma}_f = \inf_{c\in\mathbb{R}} \left\{ f(b) \geq 0 \quad \forall b \in (c, \infty) \right\}.$$

Then if $f(y) \leq \sum_{i=1}^{K} \sup_{x\in\Omega} b_i(x) y^{n_i}$ on some interval (c, ∞) and $\beta' \in (c, \infty)$, it must hold that $\overline{\gamma}_f \geq \beta'$ if $\overline{\gamma}_f$ is defined. Let C_1, C_2 be as before and let $n_{i_1} = \max\{n_i : 0 \leq n_i < n_K\}$ if $C_1 > 1$. If $y > 1$, then

$$\sum_{i=1}^{K} \inf_{x\in\Omega} b_i(x) y^{n_i} \geq \inf_{x\in\overline{\Omega}} (b_K(x)) y^{n_K} - (C_1 - 1)\Lambda y^{n_{i_1}} - C_2\Lambda.$$

Now define

$$d = \left(\frac{2(C_1 - 1)\Lambda}{\inf_{x\in\overline{\Omega}}(b_K(x))} \right)^{\frac{1}{n_k - n_{i_1}}}$$

if $C_1 > 1$ and let $d = 1$ if $C_1 = 1$. Let $c = \max\{1, d\}$. Then our choice of d ensures that if $C_1 > 1$, then

$$-(C_1 - 1)\Lambda y^{n_{i_1}} \geq -\frac{\inf_{x\in\overline{\Omega}}(b_K(x))y^{n_K}}{2},$$

and that for $y \in (c, \infty)$,

$$\sum_{i=1}^{K} \sup_{x\in\Omega} b_i(x)y^{n_i} \geq \frac{\inf_{x\in\overline{\Omega}}(b_K(x))}{2} y^{n_K} - C_2\Lambda = f(y).$$

So if $\beta' \in (c, \infty)$, $\beta' \leq \overline{\gamma}_f$, where $\overline{\gamma}_f$ is the lone positive root of f on $\mathbb{R}_+$ given that f is monotone increasing on this interval. So if $\beta' \in (c, \infty)$,

$$\beta' \leq \overline{\gamma}_f = \left(\frac{2C_2\Lambda}{\inf_{x\in\overline{\Omega}}(b_K(x))} \right)^{\frac{1}{n_K}}.$$

By defining

$$d_\epsilon = \left(\frac{2(C_1 - 1)\Lambda_\epsilon}{\inf b_\epsilon^K(x)} \right)^{\frac{1}{n_k - n_{i_1}}} \quad \text{and} \quad c_\epsilon = \max\{1, d_\epsilon\}, \tag{2.5.15}$$

and applying the above argument for β' to the net (β'_ϵ) for each fixed ϵ, it is clear that if $\beta'_\epsilon \in (c_\epsilon, \infty)$, then

$$\beta'_\epsilon \leq \left(\frac{2C_2\Lambda_\epsilon}{\inf b_\epsilon^K(x)} \right)^{\frac{1}{n_K}} \leq C\epsilon^a$$

given the assumptions on b_ϵ^K and Λ_ϵ in (2.5.11).

Now assume that $\beta'_\epsilon \notin (c_\epsilon, \infty)$. Then it suffices to show that if $C_1 > 1$, then $d_\epsilon \leq C\epsilon^a$ for ϵ sufficiently small, some positive constant C and $a \in \mathbb{R}$. But again, this is clearly true given the assumptions (2.5.11) and the fact that

$$d_\epsilon = \left(\frac{2(C_1 - 1)\Lambda_\epsilon}{\inf b_\epsilon^K(x)} \right)^{\frac{1}{n_k - n_{i_1}}}.$$

□

2.6 Proof of the Main Results

We now prove Theorem 2.4.1 using the results from Section 2.5. For clarity, we break the proof up into the steps outlined in Section 2.4.2.

2.6.1 Proof of Theorem 2.4.1

Step 1: *Formulation of the problem.* For convenience, we restate the problem and the formulation that we will use to find a solution. Given an operator $A \in \mathcal{A}_0$ defined by (2.3.17), we want to solve the following Dirichlet problem in $\mathcal{G}(\overline{\Omega})$:

$$Au = 0 \quad \text{in } \Omega, \qquad (2.6.1)$$
$$u|_{\partial\Omega} = \rho.$$

We phrase (2.6.1) in a way that allows us to solve a net of semilinear elliptic problems. We assume that the coefficients of A and boundary data ρ have representatives (a_ϵ^{ij}), (b_ϵ^i), and (ρ_ϵ) in $\mathcal{E}_M(\overline{\Omega})$ satisfying the assumptions (2.4.1). Then for this particular choice of representatives, our strategy for solving (2.6.1) is to solve the family of problems

$$A_\epsilon u_\epsilon = -\sum_{i,j=1}^{N} D_i(a_\epsilon^{ij} D_j u_\epsilon) + \sum_{i}^{N} b_\epsilon^i u_\epsilon^{n_i} = 0 \quad \text{in } \Omega, \qquad (2.6.2)$$
$$u_\epsilon|_{\partial\Omega} = \rho_\epsilon,$$

and then show that the net of solutions $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$.

Step 2: *Determine L^∞-estimates and a net of sub-solutions and super-solutions*. In Section 2.5 we concluded that for each ϵ, the pair α_ϵ and β_ϵ determine sub- and super-solutions to (2.6.2) such that $0 < \alpha_\epsilon < \beta_\epsilon$. Furthermore, in Lemma 2.5.5 we concluded that there exist constants $C_1, C_2 > 0$ and $a_1, a_2 \in \mathbb{R}$ such that for ϵ sufficiently small, the nets (α_ϵ) and (β_ϵ) satisfy $C_1\epsilon^{a_1} \leq \alpha_\epsilon < \beta_\epsilon \leq C_2\epsilon^{a_2}$, thereby verifying that $(\alpha_\epsilon), (\beta_\epsilon), (\frac{1}{\alpha_\epsilon}), (\frac{1}{\beta_\epsilon}) \in \overline{\mathbb{C}}$, the ring of generalized constants.

Step 3: *Apply fixed-point theorem to solve each semilinear problem in* (2.4.19). This follows from Proposition 2.5.4. We briefly reiterate the proof here. We simply verify the hypotheses of Theorem 2.4.3. For each fixed ϵ we have sub- and super-solutions α_ϵ and β_ϵ satisfying $0 < \alpha_\epsilon < \beta_\epsilon$ and $a^{ij}_\epsilon, b^i_\epsilon, \rho_\epsilon \in C^\infty(\overline{\Omega})$ satisfying (2.5.11). Finally, Ω is of C^∞-class and the function

$$f(x,y) = -\sum_{i=1}^{K} b^i_\epsilon(x) y^{n_i} \in C^\infty(\overline{\Omega} \times \mathbb{R}^+),$$

so we may apply Theorem 2.4.3 to conclude that there exists a net of solutions $(u_\epsilon) \in C^\infty(\overline{\Omega})$ to (2.5.10) satisfying $0 < \alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$.

Step 4: *Verify that the net of solutions* $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$. Now that it is clear that a solution exists for (2.5.10) for each $\epsilon \in (0,1]$, it is necessary to establish estimates that show that the net of solutions (u_ϵ) is in $\mathcal{E}_M(\overline{\Omega})$. That is, we want to show that for each $k \in \mathbb{N}$ and all multi-indices $|\beta| \leq k$, there exists $a \in \mathbb{R}$ such that

$$\sup_{x \in \overline{\Omega}} \{|D^\beta u_\epsilon(x)|\} = \mathcal{O}(\epsilon^a).$$

By standard interpolation inequalities, it suffices to show that for $\gamma \in (0,1)$ and each $k \in \mathbb{N}$, there exists an $a \in \mathbb{R}$ such that

$$|u_\epsilon|_{k,\gamma;\Omega} = \mathcal{O}(\epsilon^a).$$

By Theorem 2.3.1, we have that if u_ϵ is a solution to (2.5.10) with coefficients satisfying (2.5.11), then

$$|u_\epsilon|_{2,\gamma;\Omega} \le C\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 (|u_\epsilon|_{0;\Omega} + |\rho_\epsilon|_{2,\gamma;\Omega} + \sum_{i=1}^{K} |b_\epsilon^i(u_\epsilon)^{n_i}|_{0,\gamma;\Omega}). \tag{2.6.3}$$

Observe that

$$|u_\epsilon^{n_i}|_{0,\gamma;\Omega} \le |u_\epsilon^{n_i}|_{0;\Omega} + n_i[u_\epsilon]_{0,\gamma;\Omega}|u_\epsilon|_{0;\Omega}^{n_i-1} \tag{2.6.4}$$

if $n_i > 0$, and

$$|u_\epsilon^{n_i}|_{0,\gamma;\Omega} \le |u_\epsilon^{n_i}|_{0;\Omega} + \frac{1}{|u_\epsilon^{-n_i}|_{0;\Omega}^2}(-n_i)[u_\epsilon]_{0,\gamma;\Omega}|u_\epsilon|_{0;\Omega}^{-n_i-1} \tag{2.6.5}$$

if $n_i < 0$. The above inequality implies that

$$\begin{aligned} |u_\epsilon|_{2,\gamma;\Omega} \le C\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 &(|u_\epsilon|_{0;\Omega} + |\rho_\epsilon|_{2,\gamma;\Omega} \\ &+ \sum_{i=1}^{K} |b_\epsilon^i(x)|_{0,\gamma;\Omega}(C_1(n_i,\alpha_\epsilon,\beta_\epsilon) + C_2(n_i,\alpha_\epsilon,\beta_\epsilon)|u_\epsilon|_{0,\gamma;\Omega})), \end{aligned} \tag{2.6.6}$$

where

$$C_1(n_i,\alpha_\epsilon,\beta_\epsilon) = \beta_\epsilon^{n_i} \quad \text{and} \quad C_2(n_i,\alpha_\epsilon,\beta_\epsilon) = n_i\beta_\epsilon^{n_i-1}, \quad \text{if} \quad n_i > 0 \quad \text{and}$$

$$C_1(n_i,\alpha_\epsilon,\beta_\epsilon) = \alpha_\epsilon^{n_i} \quad \text{and} \quad C_2(n_i,\alpha_\epsilon,\beta_\epsilon) = \frac{(-n_i)\beta_\epsilon^{-n_i-1}}{\alpha_\epsilon^{-2n_i}} \quad \text{if} \quad n_i < 0.$$

Application of the interpolation inequality

$$|u_\epsilon|_{0,\gamma} \le C(\delta_\epsilon^{-1}|u_\epsilon|_0 + \delta_\epsilon|u_\epsilon|_{2,\gamma}),$$

where δ_ϵ is arbitrarily small and C is independent of δ_ϵ, implies that

$$|u_\epsilon|_{2,\gamma;\Omega} \leq C\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 (|u_\epsilon|_{0;\Omega} + |\rho_\epsilon|_{2,\gamma;\Omega} + \sum_{i=1}^{K} |b_\epsilon^i(x)|_{0,\gamma;\Omega}(C_1(n_i,\alpha_\epsilon,\beta_\epsilon) + C_2(n_i,\alpha_\epsilon,\beta_\epsilon)(C(\delta_\epsilon^{-1}|u_\epsilon|_{0;\Omega} + \delta_\epsilon|u_\epsilon|_{2,\gamma;\Omega})))). \tag{2.6.7}$$

Therefore,

$$\left(1 - \delta_\epsilon\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)\sum_{i=1}^{K}|b_\epsilon^i(x)|_{0,\gamma;\Omega}C_2(n_i,\alpha_\epsilon,\beta_\epsilon)\right)|u_\epsilon|_{2,\gamma;\Omega} \leq C\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 (|u_\epsilon|_{0;\Omega} + |\rho_\epsilon|_{2,\gamma;\Omega} + \sum_{i=1}^{K}|b_\epsilon^i(x)|_{0,\gamma;\Omega}(C_1(n_i,\alpha_\epsilon,\beta_\epsilon) + C_2(n_i,\alpha_\epsilon,\beta_\epsilon)\delta_\epsilon^{-1}|u_\epsilon|_{0;\Omega})). \tag{2.6.8}$$

But given the assumptions on Λ_ϵ, λ_ϵ, the bounds previously established for the nets (α_ϵ) and (β_ϵ) in Lemma 2.5.5, and given that $(b_\epsilon^i(x)) \in \mathcal{E}_M(\overline{\Omega})$, there exists $\epsilon_0 \in (0,1)$, $a \in \mathbb{R}$ and $C > 0$ such that for all $\epsilon \in (0,\epsilon_0)$,

$$\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)\sum_{i=1}^{K}|b_\epsilon^i(x)|_{0,\gamma}C_2(n_i,\alpha_\epsilon,\beta_\epsilon) \leq C\epsilon^a.$$

Therefore, choosing

$$\delta_\epsilon = \frac{1}{2C\epsilon^a},$$

it follows that for $\epsilon \in (0,\epsilon_0)$,

$$|u_\epsilon|_{2,\gamma;\Omega} \leq C\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 (|u_\epsilon|_{0;\Omega} + |\rho_\epsilon|_{2,\gamma;\Omega} + \sum_{i=1}^{K}|b_\epsilon^i(x)|_{0,\gamma;\Omega}(C_1(n_i,\alpha_\epsilon,\beta_\epsilon) + C_2(n_i,\alpha_\epsilon,\beta_\epsilon,\epsilon^a)\delta_\epsilon^{-1}|u_\epsilon|_{0;\Omega})). \tag{2.6.9}$$

Given that (α_ϵ), $(\beta_\epsilon) \in \overline{\mathbb{C}}$, $\alpha_\epsilon \leq u_\epsilon \leq \beta_\epsilon$ and (ρ_ϵ), $(b_\epsilon^i) \in \mathcal{E}_M(\overline{\Omega})$, the above inequality implies that for some $a \in \mathbb{R}$,

$$|u_\epsilon|_{2,\gamma;\Omega} = \mathcal{O}(\epsilon^a).$$

Now we need to utilize the ϵ-growth conditions on $|u_\epsilon|_{2,\gamma;\Omega}$ and induction to show that for any $k > 2$ that

$$|u_\epsilon|_{k,\gamma;\Omega} = \mathcal{O}(\epsilon^a) \quad \text{for some } a \in \mathbb{R}. \tag{2.6.10}$$

Let (u_ϵ) be a smooth net of solutions to (2.6.2) and additionally assume that (2.6.10) holds for all $j \leq k$. Let ν be a multi-index of length $k-1$. Then by differentiating both sides of (2.6.2), we see that for each ϵ, u_ϵ satisfies the Dirichlet problem

$$\sum_{i,j=1}^{N} D^\nu(-D_i(a_\epsilon^{ij} D_j u_\epsilon)) = -\sum_{i=1}^{K} D^\nu(b_\epsilon^i u_\epsilon^{n_i}) \quad \text{in } \Omega \tag{2.6.11}$$
$$D^\nu u_\epsilon|_{\partial\Omega} = D^\nu \rho_\epsilon.$$

Rearranging the above equation and applying the product rule, where we use standard multi-index notation, we find that

$$\begin{aligned} \sum_{i,j=1}^{N} a_\epsilon^{ij} D_{ij}(D^\nu u_\epsilon) = &-\sum_{i,j=1}^{N} D^\nu((D_i a_\epsilon^{ij})(D_j u_\epsilon)) \\ &-\sum_{i,j=1}^{N} \sum_{\substack{\sigma+\mu=\nu \\ \sigma\neq\nu}} \frac{\nu!}{\sigma!\mu!}(D^\mu a_\epsilon^{ij})(D^\sigma D_{ij} u_\epsilon) \\ &+\sum_{i=1}^{K} \sum_{\sigma+\mu=\nu} \frac{\nu!}{\sigma!\mu!}(D^\mu b_\epsilon^i)(D^\sigma((u_\epsilon)^{n_i})). \end{aligned} \tag{2.6.12}$$

Therefore, we may apply Theorem 2.3.1 to Eq. (2.6.12) to conclude that for an arbitrary multi-index ν such that $|\nu| = k-1$,

$$
\begin{aligned}
|D^\nu u_\epsilon|_{2,\gamma;\Omega} \leq C\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 & (|D^\nu u_\epsilon|_{0;\Omega} + |D^\nu \rho_\epsilon|_{2,\gamma;\Omega} \\
& + |\sum_{i,j=1}^N D^\nu((D_i a_\epsilon^{ij})(D_j u_\epsilon))|_{0,\gamma;\Omega} \\
& + \sum_{i,j=1}^N \sum_{\substack{\sigma+\mu=\nu \\ \sigma\neq\nu}} \frac{\nu!}{\sigma!\mu!} |D^\mu a_\epsilon^{ij}|_{0,\gamma;\Omega} |D^\sigma D_{ij} u_\epsilon|_{0,\gamma;\Omega} \\
& + \sum_{i=1}^K \sum_{\sigma+\mu=\nu} \frac{\nu!}{\sigma!\mu!} |D^\mu b_\epsilon^i|_{0,\gamma;\Omega} |D^\sigma((u_\epsilon)^{n_i})|_{0,\gamma;\Omega}) \\
\leq C\left(\frac{\Lambda_\epsilon}{\lambda_\epsilon}\right)^3 & (|D^\nu u_\epsilon|_{0;\Omega} + |D^\nu \rho_\epsilon|_{2,\gamma;\Omega} \\
& + \sum_{i,j=1}^N \sum_{\sigma+\mu=\nu} \frac{\nu!}{\sigma!\mu!} |D^\mu (D_i a_\epsilon^{ij})|_{0,\gamma;\Omega} |D^\sigma (D_j u_\epsilon)|_{0,\gamma;\Omega} \\
& + \sum_{i,j=1}^N \sum_{\substack{\sigma+\mu=\nu \\ \sigma\neq\nu}} \frac{\nu!}{\sigma!\mu!} |D^\mu a_\epsilon^{ij}|_{0,\gamma;\Omega} |D^\sigma D_{ij} u_\epsilon|_{0,\gamma;\Omega} \\
& + \sum_{i=1}^K \sum_{\sigma+\mu=\nu} \frac{\nu!}{\sigma!\mu!} |D^\mu b_\epsilon^i|_{0,\gamma;\Omega} |D^\sigma((u_\epsilon)^{n_i})|_{0,\gamma;\Omega}).
\end{aligned}
\tag{2.6.13}
$$

By our inductive hypothesis and the assumptions on the coefficients, it is immediate that every term in the above expression is $\mathcal{O}(\epsilon^a)$ for some $a \in \mathbb{R}$ except for the last term. So to show

$$|D^\nu u_\epsilon|_{2,\gamma;\Omega} = \mathcal{O}(\epsilon^a) \quad \text{for some } a \in \mathbb{R},$$

it suffices to show that

$$\sum_{i=1}^K \sum_{\sigma+\mu=\nu} \frac{\nu!}{\sigma!\mu!} |D^\mu b_\epsilon^i|_{0,\gamma;\Omega} |D^\sigma((u_\epsilon)^{n_i})|_{0,\gamma;\Omega} = \mathcal{O}(\epsilon^a) \quad \text{for some } a \in \mathbb{R}.$$

Given that $b_\epsilon^i \in \mathcal{E}_M(\overline{\Omega})$ for each $1 \leq i \leq K$,

$$|D^\mu b_\epsilon^i|_{0,\gamma;\Omega} = \mathcal{O}(\epsilon^a) \quad \text{for some } a \in \mathbb{R}.$$

Therefore, it is really only necessary to show that for any multi-index σ, such that $|\sigma| = j \leq k-1$, that there exists an $a \in \mathbb{R}$ such that

$$|D^\sigma((u_\epsilon)^{n_i})|_{0,\gamma;\Omega} = \mathcal{O}(\epsilon^a).$$

But observe that $D^\sigma((u_\epsilon)^{n_i})$ is a sum of terms of the form

$$(u_\epsilon)^{n_i-m} D^{\sigma_1} u_\epsilon D^{\sigma_2} u_\epsilon \cdots D^{\sigma_m} u_\epsilon,$$

where $\sigma_1 + \sigma_2 + \cdots \sigma_m = \sigma$ and $m \leq j \leq k-1$. This follows immediately from the chain rule. Therefore we have the following bound:

$$\begin{aligned} |D^\sigma((u_\epsilon)^{n_i})|_{0,\gamma;\Omega} \leq (n_i)|(u_\epsilon)^{n_i-1}|_{0,\gamma;\Omega}|D^\sigma u_\epsilon|_{0,\gamma;\Omega} \\ + \sum_{\sigma_1+\sigma_2=\sigma} \frac{\sigma!}{\sigma_1!\sigma_2!}(n_i)(n_i-1)|(u_\epsilon)^{n_i-2}|_{0,\gamma;\Omega} \\ \cdot |D^{\sigma_1} u_\epsilon|_{0,\gamma;\Omega}|D^{\sigma_2} u_\epsilon|_{0,\gamma;\Omega} + \cdots \\ + \sum_{\sigma_1+\sigma_2+\cdots+\sigma_j=\sigma} \frac{\sigma!}{\sigma_1!\sigma_2!\cdots\sigma_j!}(n_i)(n_i-1) \\ \cdots (n_i - j)|(u_\epsilon)^{n_i-j}|_{0,\gamma;\Omega}|D^{\sigma_1} u_\epsilon|_{0,\gamma;\Omega} \\ \cdots |D^{\sigma_j} u_\epsilon|_{0,\gamma;\Omega}. \end{aligned} \tag{2.6.14}$$

Eqs. (2.6.4)-(2.6.5) allow us to bound the terms of the form $|(u_\epsilon)^{n_i-m}|_{0,\gamma;\Omega}$ for each $m \leq j$ by using the terms $|u_\epsilon|_{0,\gamma;\Omega}$, α'_ϵ and β'_ϵ along with some positive constant that is independent of ϵ. Then our inductive hypothesis and the growth conditions on (α'_ϵ) and (β'_ϵ) imply that

$$|D^\sigma((u_\epsilon)^{n_i})|_{0,\gamma;\Omega} = \mathcal{O}(\epsilon^a) \quad \text{for some } a \in \mathbb{R}.$$

This implies that

$$|D^\nu u_\epsilon|_{2,\gamma;\Omega} = \mathcal{O}(\epsilon^a) \quad \text{for some } a \in \mathbb{R}.$$

As ν was an arbitrary multi-index such that $|\nu| = k-1$, this implies there exists

$a \in \mathbb{R}$ such that

$$|u_\epsilon|_{k+1,\gamma;\Omega} = \mathcal{O}(\epsilon^a).$$

Therefore, $(u_\epsilon) \in \mathcal{E}_M(\overline{\Omega})$.

Step 5: *Verify that the solution is well-defined.* Proposition 2.3.9 and the definition of the Dirichlet problem in $\mathcal{G}(\overline{\Omega})$ given in Section 2.3.5 imply that $[(u_\epsilon)]$ is indeed a solution to the problem

$$\begin{aligned} Au &= 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= \rho, \end{aligned} \tag{2.6.15}$$

in $\mathcal{G}(\overline{\Omega})$. To see this, we consider other representatives $(\overline{a}_\epsilon^{ij})$, $(\overline{b}_\epsilon^i)$, $(\overline{\rho}_\epsilon)$, and $(\overline{u}_\epsilon)$ of $[(a_\epsilon^{ij})]$, $[(b_\epsilon^i)]$, $[(\rho_\epsilon)]$, and $[(u_\epsilon)]$. Then the proof of Proposition 2.3.9 clearly implies that

$$\begin{aligned} -\sum_{i,j=1}^N D_i(\overline{a}_\epsilon^{ij} D_j \overline{u}_\epsilon) + \sum_{i=1}^K \overline{b}_\epsilon^i (\overline{u}_\epsilon)^{n_i} &= \eta_\epsilon \quad \text{in } \Omega, \\ \overline{u_\epsilon}|_{\partial\Omega} &= \overline{\rho}_\epsilon + \overline{\eta}_\epsilon, \end{aligned} \tag{2.6.16}$$

where $\eta_\epsilon \in \mathcal{N}(\overline{\Omega})$ and $\overline{\eta}_\epsilon$ is a net of functions satisfying (2.3.18). But this implies that this choice of representatives also satisfies (2.6.15) in $\mathcal{G}(\overline{\Omega})$, so our solution $[(u_\epsilon)]$ is independent of the representatives used. □

This completes our proof of Theorem 2.4.1. We now conclude this article by giving a brief summary of everything that we have discussed.

2.7 Summary

We began the paper with an example to motivate the Colombeau Algebra method of solving the semilinear problem (2.1.1) with potentially distributional data. In particular, in Section 2.2 we proved the existence of a solution to an ill-posed critical exponent problem in Proposition 2.2.4. Our method of proving the existence of a solution to this problem consisted of mollifying the data of the original problem and solving a sequence of "approximate" problems with the smooth coefficients. We then obtained a sequence of solutions that yielded a convergent subsequence. The framework we used to obtain a solution to this problem was modeled on the more general Colombeau approach that we developed later in the paper, but required only basic elliptic PDE theory. Following this existence proof, we began to develop the more general Colombeau algebra framework. To this end, in Section 2.3.1 we introduced notation for Hölder norms and stated two *a priori* estimates from [5] that were made more precise by Mitrovic and Pilipovic in [13]. In Section 2.3.2 we then introduced the general framework for constructing Colombeau-type algebras and the Colombeau algebra $\mathcal{G}(\overline{\Omega})$ used in this paper. We then discussed a method used to embed the Schwartz distributions $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\overline{\Omega})$. We used this embedding to analyze a problem of the form (2.1.1) with distributional coefficients. In particular, in Proposition 2.4.4 we determined explicit conditions under which we could solve a semilinear problem of the form (2.1.1) with rough coefficients. Then we finished Section 2.3.2 by defining a class of semilinear operators on $\mathcal{G}(\overline{\Omega})$ in 2.3.4, and we then defined the Dirichlet problem for these operators.

Our main results for the Colombeau algebra framework were presented in Section 2.4, namely Theorem 2.4.1, which consists of an existence result for the semilinear problem (2.4.2) in $\mathcal{G}(\overline{\Omega})$. We then developed the necessary tools to analyze our semilinear problem in Section 2.5. We first determined a net of L^∞ bounds for positive solutions to our problem. Then, in Section 2.5.2 we showed that this net of L^∞ bounds was in fact a net of sub- and super-solutions contained in $\overline{\mathbb{C}}$, the ring of generalized constants described in Section 2.3.2. After developing our sub- and super-solutions, we proved Theorem 2.4.1 in Section 2.6. We set up our problem in a manner similar to that used by Mitrovic and Pilipovic in [13]. However, our approach to solving our semilinear problem was distinct from theirs; we first determined a net of solutions (u_ϵ) to the family of semilinear problems (2.6.2) by using the method of sub- and super-solutions (Theorem 2.4.3), and our net of sub- and super-solutions determined in Section 2.5.2. Once our net of solutions was determined, we then employed Theorems 2.3.1 and our net of sub- and super-solutions to show that our net of solutions was contained in $\mathcal{E}_M(\overline{\Omega})$.

In this article we have attempted to develop some basic tools to allow for a more general study of the Einstein constraint equations with distributional data. Our goal was to extend the current solution theory for scalar, critical exponent semilinear problems such as the Lichnerowicz equation, allowing for more irregular data than is currently covered by the existing solutions theories (cf. [7, 8] for a summary of the known results for the CMC, near-CMC, and Far-CMC cases through 2009). As a next step, we hope to use the tools developed in this article to extend the near-CMC and Far-CMC existence framework for rough metrics developed in [7, 10, 11, 2] to cover the rough data example studied by Maxwell in [12].

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Chapter 3

Non-uniqueness and the Conformal Formulation

NON-UNIQUENESS OF SOLUTIONS TO THE CONFORMAL FORMULATION OF THE EINSTEIN CONSTRAINTS

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ABSTRACT. In this article we investigate the uniqueness properties of solutions to the Einstein constraint equations on a closed manifold. In particular, we investigate whether or not solutions to the conformal formulation of the constraints with an unscaled data source are unique. For positive, constant scalar curvature and constant mean curvature, we first demonstrate the existence of a critical energy density for the Hamiltonian constraint. We then show that for this choice of energy density, the linearization of the elliptic system develops a one-dimensional kernel in both the constant mean curvature and non-constant mean curvature cases. Using a Liapunov-Schmidt reduction and standard techniques from bifurcation theory, we demonstrate that solutions to the conformal formulation with unscaled data source are non-unique by determining an explicit solution curve and analyzing its behavior in the neighborhood of a particular solution.

3.1 Introduction

In this paper we demonstrate that solutions to the Einstein constraint equations on a 3-dimensional closed manifold $(\mathcal{M}, \hat{g}_{ab})$ with no conformal killing field are non-unique. More specifically, we show that solutions to the conformal formulation of the constraint equations with an unscaled matter source on $(\mathcal{M}, \hat{g})$ exhibit non-uniqueness in the case that the scalar curvature is positive and constant. Letting $\hat{k}_{ab}$ be a $(0, 2)$ tensor and $\hat{R}$ and $\hat{D}$ be the scalar curvature and connection associated with $\hat{g}_{ab}$, the constraint equations take the form

$$\hat{R} + \hat{k}^2 - \hat{k}^{ab}\hat{k}_{ab} = 2\kappa\hat{\rho}, \tag{3.1.1}$$

$$\hat{D}^a\hat{k} + \hat{D}_b\hat{k}^{ab} + \kappa\hat{j}^a = 0. \tag{3.1.2}$$

Equation (3.1.1) is known as the **Hamiltonian Constraint** and (3.1.2) is known as the **momentum constraint**.

Equations (3.1.1) and (3.1.2) form a system of coupled elliptic partial differential equations. When one attempts to solve the constraint equations they are faced with the problem of having twelve pieces of initial data and only four constraints. One solution to this problem is to attempt to parametrize solutions to (3.1.1) and (3.1.2) by formulating the constraints so that eight pieces of initial data are freely specifiable while four are determined by (3.1.1)-(3.1.2). The conformal transverse traceless (CTT) decomposition and the conformal thin sandwich method (CTS method) are standard ways of doing this. The extended conformal thin sandwich method (XCTS method) is popular among numerical relativists and reformulates (3.1.1) and (3.1.2) as a coupled system of 5 elliptic equations. In the CTT method one decomposes $\hat{k}_{ab}$ into its trace or mean curvature and trace free part and then scales this trace free tensor, the metric $\hat{g}_{ab}$ and the source terms $\hat{\rho}$ and $\hat{j}$ by judicious choices of some power of a positive, smooth function ϕ. The choice of scaling power for each term is typically made to simplify the analysis of the resulting system. In particular, one chooses powers to eliminate terms involving $(D_a\phi)/\phi$ and so that the system decouples when the mean curvature is constant.

It is well known that solutions to the CTT formulation of the constraint equations with scaled data sources are unique in the event that the mean curvature is constant or near constant [10, 11, 1, 8, 9]. However, given that far-from-constant mean curvature solutions are constructed using a variation of the Schauder fixed-point technique as opposed to the contraction mapping theorem (cf. [9]), little is known about the uniqueness of far-from-CMC solutions. In fact, in [13] Maxwell demonstrated that solutions of the CTT formulation of the constraint equations are non-unique in the far-from-CMC case for certain families of low regularity mean curvatures. However, it should be noted that the discontinuous mean curvature functions considered by Maxwell in [13] fall outside of the current far-from-CMC solution framework presented by Holst et al. in [9].

In [15] York provided numerical evidence for non-uniqueness of the XCTS method on an asymptotically Euclidean manifold. In [4] O'Murchadha et al. conjectured that the non-uniqueness demonstrated by York was related to the fact that certain terms in the momentum constraint related to the lapse function have the "wrong sign" which prevents an application of the maximum principle. To support their claim, the authors of [4] analyzed a simplified system corresponding to a spherically symmetric constant density star and explicitly constructed two branches of solutions. In their analysis they proved that solutions to the Hamiltonian constraint (3.1.1) with an unscaled matter source are non-unique. Then in [17], Walsh generalized the work of O'Murchadha et al. by applying a Liapunov-Schmidt reduction to both the Hamiltonian constraint with an unscaled matter source and to the XCTS system on an asymptotically Euclidean manifold. However, Walsh relied on the assumption of the existence of a critical density for which the linearization of these two systems developed a one-dimensional kernel. Here we extend the work of Walsh by applying a Liapunov Schmidt reduction to the CTT formulation of the constraint equations on a closed manifold. We explicitly construct a critical, constant density in the event that the scalar curvature is positive and constant and the transverse traceless tensor has constant magnitude. For this particular density, we then show that solutions to the CTT formulation with an unscaled density are non-unique.

As in [4, 17], we consider a less standard conformal formulation of the constraints by allowing unscaled matter sources ρ and $\mathbf{j}$. However, as opposed to considering the CTS and XCTS formulations as in [15, 4, 17], we consider the CTT formulation.

By decomposing our initial data

$$\hat{k}_{ab} = \hat{l}_{ab} + \frac{1}{3}\hat{g}_{ab}\hat{\tau}, \tag{3.1.3}$$

where $\hat{\tau} = \hat{k}_{ab}\hat{g}^{ab}$ is the trace and $\hat{l}_{ab}$ is the traceless part, making the following conformal rescaling

$$\hat{g}_{ab} = \phi^4 g_{ab}, \quad \hat{l}_{ab} = \phi^{-10} l^{ab}, \quad \hat{\tau} = \tau, \tag{3.1.4}$$

and then decomposing

$$l_{ab} = (\sigma_{ab} + (\mathcal{L}\mathbf{w})_{ab}), \tag{3.1.5}$$

where $D_a\sigma^{ab} = 0$ and

$$(\mathcal{L}\mathbf{w})^{ab} = D^a w^b + D^b w^a - \frac{2}{3}(D_c w^c) g^{ab}$$

is the **conformal Killing operator**, we obtain the following unscaled conformal reformulation of (3.1.1) and (3.1.2) that we will analyze

$$\begin{aligned} -\Delta\phi + \frac{1}{8}R\phi + \frac{1}{12}\tau^2\phi^5 - \frac{1}{8}(\sigma_{ab} + (\mathcal{L}\mathbf{w})_{ab})(\sigma^{ab} + (\mathcal{L}\mathbf{w})^{ab})\phi^{-7} - 2\pi\rho\phi^5 = 0, \\ - D_b(\mathcal{L}\mathbf{w})^{ab} + \frac{2}{3}D^a\tau\phi^6 + \kappa j^a\phi^{10} = 0. \end{aligned} \tag{3.1.6}$$

Our non-uniqueness results for (3.1.6) are of interest for a number or reasons. Most immediately, our analysis shows that the formulation (3.1.6) is unfavorable due to the non-uniqueness of solutions. Therefore, for a given system, if the CTT formulation with a scaled matter source leads to a set of constraints that is suitable for analysis, which it usually does, then one should use the scaled formulation. However, it is not always the case that the conformal formulation with scaled matter sources is the ideal formulation for a given source. In the case of the Einstein-scalar field system, the conformal formulation that is most amenable to analysis takes on a form very similar to the system (3.1.6) [5]. In addition, it is the hope of the authors that these results will provide additional insight into the non-uniqueness phenomena associated with the CTT formulation in the far-from-CMC case [13] and with the non-uniqueness phenomena analyzed by York, Walsh and O'Murchadha et al. In particular, the analysis conducted in this article clearly demonstrates the effect that the terms with the "wrong sign" discussed by York have on the non-uniqueness of the conformal formulations of the constraints. In the case of (3.1.6), the negative sign in front of the term $2\pi\rho\phi^5$ is undesirable given that it prevents the semilinear portion of the Hamiltonian constraint from being monotone and the corresponding energy from being convex. By a maximum principle argument, we will see in section 3.4 that it is this term that directly contributes to the non-uniqueness properties of (3.1.6).

The rest of this paper is organized as follows. In section 3.2 we introduce the function spaces that we will use and some basic concepts from functional analysis. Then we discuss the Liapunov-Schmidt reduction that we use to prove non-uniqueness. The statements of the main results of this paper can be found in section 3.3. The remainder of this paper is then devoted to proving these results. The foundation for our argument is developed in sections 3.4 and 3.5. In we section 3.4 we demonstrate the existence of a critical, constant density ρ_c such that for τ and $|\sigma|$ constant and $\mathbf{j}^a = 0$ the Hamiltonian constraint in (3.1.6) will have a positive solution if $\rho \leq \rho_c$ and will have no positive solution if $\rho > \rho_c$. Then in section 3.5 we use the properties of ρ_c to show that there exists a function ϕ_c at which the linearizations of the uncoupled Hamiltonian operator (CMC case) and coupled system (non-CMC case) have one-dimensional kernels. The existence of a one-dimensional kernel then allows us to apply the Liapunov-Schmidt reduction in section 3.7 in the CMC case and in section 3.8 in the non-CMC case. In particular, in section 3.7 we determine an explicit solution curve for (3.1.6) that goes through the point $(\phi_c, 0)$ in the CMC case. An analysis of this curve then implies the non-uniqueness of solutions to (3.1.6) when the mean curvature is constant. Similarly, in section 3.8 we also determine an explicit solution curve for the full, uncoupled system (3.1.6) through a point of the form $((\phi_c, \mathbf{0}), 0)$. Again, an analysis of this curve reveals non-uniqueness in the event that the mean curvature is non-constant.

3.2 Preliminary Material

In this section we give a brief definition of the function spaces, norms and notation that we will use in this article and then discuss some basic concepts from functional analysis and bifurcation theory that will be necessary going forward.

3.2.1 Banach Spaces, Hilbert Spaces and Direct Sums

We introduce the fundamental properties of the function spaces with which we will be working. We will primarily be working with Banach spaces, however at times we will need to consider these spaces as subspaces of a Hilbert space. For convenience, we present the basic definitions of these general spaces and define the direct sum of two vector spaces, which will be necessary in our non-uniqueness analysis.

The basic space that we will be working with is a Banach space, where a **Banach space** X is a complete, normed vector space. If the norm $\|\cdot\|$ on X is induced by an inner product, we say that X is a **Hilbert Space**. One can form new Banach spaces and Hilbert space from two preexisting spaces by considering the direct sum.

Definition 3.2.1. *Suppose that X_1 and X_2 are Banach spaces with norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$. Then the direct sum $X_1 \oplus X_2$ is the vector space of ordered pairs (x, y) where $x \in X_1$, $y \in X_2$ and addition and scalar multiplication are carried out componentwise.*

We have the following proposition:

Proposition 3.2.2. *The vector space $X_1 \oplus X_2$ is a Banach space when given the norm*

$$\|(x, y)\|_{X_1 \oplus X_2} = \left(\|x\|_{X_1}^2 + \|y\|_{X_2}^2\right)^{\frac{1}{2}}. \tag{3.2.1}$$

Proof. This follows from the fact that $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$ are norms and the spaces X_1 and X_2 are complete with respect to these norms. □

We have a similar proposition for Hilbert spaces.

Proposition 3.2.3. *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}_2}$. Then the direct sum $H_1 \oplus H_2$ is a Hilbert space with inner product*

$$\langle (w, x), (y, z) \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \langle w, y \rangle_{\mathcal{H}_1} + \langle x, z \rangle_{\mathcal{H}_2}. \tag{3.2.2}$$

Proof. The fact that $\langle \cdot, \cdot \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2}$ is an inner product follows from the fact that $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}_2}$ are inner products. The expression

$$\|(u, v), (u, v)\|_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \sqrt{\langle (u, v), (u, v) \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2}},$$

is a norm on $\mathcal{H}_1 \oplus \mathcal{H}_2$ that coincides with the norm in Proposition 3.2.2 in the event that the norms on X_1 and X_2 are induced by inner products. □

See [18] for a more complete discussion about the direct sums of Banach spaces.

3.2.2 Function Spaces

Let E denote a given vector bundle over $\mathcal{M}$. In this paper we will consider the Sobolev spaces $W^{k,p}(E)$, the space of k-differentiable sections $C^k(E)$, and the Hölder spaces $C^{k,\alpha}(E)$ where $k \in \mathbb{N},\ p \geq 1,\ \alpha \in (0, 1)$ and E will either be the vector bundle $\mathcal{M} \times \mathbb{R}$ of scalar-valued functions or $\mathcal{T}^r_s\mathcal{M}$, the space of (r, s) tensors. Note that all of these spaces with the following norm definitions are Banach spaces and the space $W^{k,2}(E)$ is a Hilbert space for $k \in \mathbb{N}$.

Fix a smooth background metric g_{ab} and let $v^{a_1,\cdots,a_r}_{b_1,\cdots,b_s}$ be a tensor of type $r + s$. Then at a given point $x \in \mathcal{M}$, we define its magnitude to be

$$|v| = (v^{a_1,\cdots,b_s} v_{a_1,\cdots,b_s})^{\frac{1}{2}}, \tag{3.2.3}$$

where the indices of v are raised and lowered with respect to g_{ab}. We then define the Banach space of k-differentiable functions $C^k(\mathcal{M} \times \mathbb{R})$ with norm $\| \cdot \|_k$ to be those functions u satisfying

$$\|u\|_k = \sum_{j=0}^{k} \sup_{x \in \mathcal{M}} |D^j u| < \infty,$$

where D is the covariant derivative associated with g_{ab}. Similarly, we define the space $C^k(\mathcal{T}^r_s\mathcal{M})$ of k-times differentiable (r,s) tensor fields to be those tensors v satisfying $\|v\|_k < \infty$.

Given two points $x, y \in \mathcal{M}$, we define $d(x,y)$ to be the geodesic distance between them. Let $\alpha \in (0,1)$. Then we may define the $C^{0,\alpha}$ Hölder seminorm for a scalar-valued function u to be

$$[u]_{0,\alpha} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{(d(x,y))^{\alpha}}.$$

Using parallel transport, this definition can be extended to (r,s)-tensors v to obtain the $C^{k,\alpha}$ seminorm $[u]_{k,\alpha}$ [2]. This leads us to the following definition of the $C^{k,\alpha}(\mathcal{M}\times\mathbb{R})$ Hölder norm

$$\|u\|_{k,\alpha} = \|u\|_k + [u]_{k,\alpha}$$

for scalar-valued functions, and we may define the $C^{k,\alpha}(\mathcal{T}^r_s\mathcal{M})$ Hölder norm for (r,s) tensors in a similar fashion.

Finally, we will also make use of the Sobolev spaces $W^{k,p}(\mathcal{M}\times\mathbb{R})$ and $W^{k,p}(\mathcal{T}^r_s\mathcal{M})$ where we assume $k \in \mathbb{N}$ and $p \geq 1$. If dV_g denotes the volume form associated with g_{ab}, then the L^p norm of an (r,s) tensor is defined to be

$$\|v\|_p = \left(\int_{\mathcal{M}} |v|^p dV_g\right)^{\frac{1}{p}}. \tag{3.2.4}$$

We can then define the Banach space $W^{k,p}(\mathcal{M}\times\mathbb{R})$ (resp. $W^{k,p}(\mathcal{T}^r_s\mathcal{M})$) to be those functions (resp. (r,s) tensors) v satisfying

$$\|v\|_{k,p} = \left(\sum_{j=0}^{k} \|D^j v\|_p^p\right)^{\frac{1}{p}} < \infty.$$

The above norms are independent of the background metric chosen. Indeed, given any two metrics g_{ab} and $\hat{g}_{ab}$, one can show that the norms induced by the two metrics are equivalent. For example, if D and $\hat{D}$ are the derivatives induced by g_{ab} and

$\hat{g}_{ab}$ respectively, then then there exist constants C_1 and C_2 such that

$$C_1\|u\|_{k,\hat{g}} \leq \|u\|_{k,g} \leq C_2\|u\|_{k,\hat{g}},$$

where $\|\cdot\|_{k,g}$ denotes the $C^k(\mathcal{M})$ norm with respect to g. This holds for the $W^{k,p}$ and $C^{k,\alpha}$ norms as well. We also note that the above norms are related through the Sobolev embedding theorem. In particular, the spaces $C^{k,\alpha}$ and $W^{l,p}$ are related in the sense that if n is the dimension of $\mathcal{M}$ and $u \in W^{l,p}$ and

$$k + \alpha < l - \frac{n}{p},$$

then $u \in C^{k,\alpha}$. See [2, 3, 7, 14] for a complete discussion of the Sobolev embedding Theorem, Banach spaces on manifolds, and the above norms.

3.2.3 Adjoints and Projection Operators

Solutions to the coupled system (3.1.6) satisfy

$$F(x, \mathbf{w}) = 0, \tag{3.2.5}$$

where $F : X \times Y \to Z$ is a nonlinear operator between Banach spaces. This allows us to use basic tools from functional analysis to analyze our problem. In particular, we will repeatedly need to consider the linearization $D_xF(x, \mathbf{w})$, its adjoint, and projections onto subspaces determined by these operators. Later on in the section when we introduce the Liapunov-Schmidt reduction, we will use the kernel of the linearization $D_xF(x_0, \mathbf{w}_0)$ at a point $(x_0, \mathbf{w}_0)$, the kernel of the adjoint, and projection operators onto these subspaces, to decompose X and Y in a manner that will greatly simplify our analysis. Here we briefly discuss the adjoint and projection operators. See and [18] for a more complete discussion of these topics and see Appendix 3.10.1 for a discussion of Fréchet derivatives.

The Adjoint and Properties

Suppose that $\mathcal{H}$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Then if $A : \mathcal{H} \to \mathcal{H}$ is a linear operator, the Riesz Representation Theorem implies that there exists a unique operator A^* that satisfies

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \text{for all } x, y \in \mathcal{H}. \tag{3.2.6}$$

If $R(A)$ denotes the range of A and $\ker(A)$ denotes the kernel, then the operator A^* satisfies the following properties:

$$\mathbf{1}) \quad \ker(A^*) = R(A)^{\perp} \tag{3.2.7}$$

$$\mathbf{2}) \quad (\ker(A^*))^{\perp} = \overline{R(A)}. \tag{3.2.8}$$

Projection Operators and Fredholm Operators

Now assume that $X \subset \mathcal{H}$ is a Banach space contained in a Hilbert space $\mathcal{H}$. Given a subspace $V \subset X$, the projection P onto V is a bounded linear operator $P : X \to V$ that satisfies $P^2 = P$. In particular, if V is a finite-dimensional subspace spanned by the orthonormal basis $\hat{v}_1, \cdots, \hat{v}_n$ then we can easily construct the projection onto V by the formula

$$Pu : \sum_{i=1}^{n} \langle u, \hat{v}_i \rangle \hat{v}_i, \tag{3.2.9}$$

where $u \in X$ and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$. Note that P is just the normal projection operator from $\mathcal{H}$ to V restricted to X.

We end the section by introducing one more definition that will be important in the following section. A **Fredholm operator** is a bounded linear operator $A : X \to Y$ where X and Y are Banach spaces such that dim $\ker(A)$ and dim $\ker(A^*)$ are finite-dimensional and $R(A)$ is closed. Given a nonlinear operator $F : U \to Y$ where $U \subset X$, we say that F is a **nonlinear Fredholm operator** if it is Fréchet differentiable on U and $D_x F(x)$ is a Fredholm operator.

Notice that if A is a Fredholm operator, then $\ker(A^*)^{\perp} = R(A)$ and furthermore, the fact that ker(A) and ker(A^*) are finite dimensional allows one to define projection operators P and Q onto ker(A) and ker(A^*) to decompose X and Y. As we will see, these properties make Fredholm operators ideal candidates for bifurcation analysis.

3.2.4 Elements of Bifurcation Theory

We now present some basic concepts from bifurcation theory that will be essential in obtaining our non-uniqueness results. In particular, we give a formal definition of a bifurcation point and then present the Liapunov-Schmidt reduction. This reduction allows one to reduce a nonlinear problem between infinite-dimensional Banach spaces to a finite-dimensional or even scalar-valued problem. Therefore it greatly simplifies the analysis and will serve as a basic tool for us going forward. The following treatment is taken from [12] and [6].

Suppose that $F : U \times V \to Z$ is a mapping with open sets $U \subset X, V \subset \Lambda$, where X and Z are Banach spaces and $\Lambda = \mathbb{R}$. We let $x \in X$ and $\lambda \in \Lambda$. Additionally assume that $F(x, \lambda)$ is Fréchet differentiable with respect to x and λ on $U \times V$. We are interested in solutions to the nonlinear problem

$$F(x, \lambda) = 0. \tag{3.2.10}$$

A solution of (3.2.10) is a point $(x, \lambda) \in X \times \Lambda$ such that (3.2.10) is satisfied.

Definition 3.2.4. *Suppose that* (x_0, λ_0) *is a solution to* (3.2.10). *We say that* λ_0 *is a* **bifurcation point** *if for any neighborhood* U *of* (x_0, λ_0) *there exists a* $\lambda \in \Lambda$ *and* $x_1, x_2 \in X$, $x_1 \neq x_2$ *such that* $(x_1, \lambda), (x_2, \lambda) \in U$ *and* (x_1, λ) *and* (x_2, λ) *are both solutions to* (3.2.10).

Given a solution (x_0, λ_0) to (3.2.10), we are interested in analyzing solutions to (3.2.10) in a neighborhood of (x_0, λ_0) to determine whether it is a bifurcation point. One of the most useful tools for this is the Implicit Function Theorem 3.10.5. This theorem asserts that if $D_x F(x_0, \lambda_0)$ is invertible, then there exists a neighborhood $U_1 \times V_1 \subset U \times V$ and a continuous function $f : V_1 \to U_1$ such that all solutions to (3.2.10) in $U_1 \times V_1$ are of the form $(f(\lambda), \lambda)$. Therefore in order for a bifurcation to occur at (x_0, λ), it follows that $D_x F(x_0, \lambda_0)$ must not be invertible.

Liapunov-Schmidt Reduction

The following discussion is taken from [12]. Let X, Λ and Z be Banach spaces and assume that $U \subset X$, $V \subset \Lambda$. For $\lambda = \lambda_0$, we require that the mapping $F : U \times V \to Z$ be a nonlinear Fredholm operator with respect to x; i.e. the linearization $D_x F(\cdot, \lambda_0)$ of $F(\cdot, \lambda_0) : U \to Z$ is a Fredholm operator. Assume that F also satisfies the following assumptions:

$$F(x_0, \lambda_0) = 0 \quad \text{for some } (x_0, \lambda_0) \in U \times V, \tag{3.2.11}$$
$$\text{dim ker}(D_x F(x_0, \lambda_0)) = \text{dim ker}(D_x F(x_0, \lambda_0)^*) = 1.$$

Given that $D_x F(x_0, \lambda_0)$ has a one-dimensional kernel, there exists a projection operator $P : X \to X_1 = \text{ker}(D_x F(x_0, \lambda_0))$. Similarly, one has the projection operator $Q : Y \to Y_2 = \text{ker}(D_x F(x_0, \lambda_0)^*)$. This allows us to decompose $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ where $Y_1 = R(D_X F(x_0, \lambda_0))$. We will refer to the decomposition $X_1 \oplus X_2$ and $Y_1 \oplus Y_2$ induced by $D_x F(x_0, \lambda_0)$ as the **Liapunov decomposition**, and we see that $F(x, \lambda) = 0$ if and only if the following two equations are satisfied

$$QF(x, \lambda) = 0, \tag{3.2.12}$$
$$(I - Q)F(x, \lambda) = 0.$$

For any $x \in X$, we can write $x = v + w$, where $v = Px$ and $w = (I - P)x$. Define $G : U_1 \times W_1 \times V_1 \to Y_1$ by

$$G(v, w, \lambda) = (I - Q)F(v + w, \lambda), \quad \text{where} \tag{3.2.13}$$
$$U_1 \subset X_1, \quad W_1 \subset X_2, \quad V_1 \subset \mathbb{R} \quad \text{and}$$
$$v_0 = Px_0 \in U_1, \quad w_0 = (I - P)x_0 \in W_1,$$

and U_1, W_1 are neighborhoods such that $U_1 + W_1 \subset U \subset X$.

Then the definition of $G(v, w, \lambda)$ implies that $G(v_0, w_0, \lambda_0) = 0$ and our choice of function spaces ensures that

$$D_w G(v_0, w_0, \lambda_0) = (I - Q)D_x F(x_0, \lambda_0) : X_2 \to Y_1,$$

is bijective. The Implicit Function Theorem then implies that there exist neighborhoods $U_2 \subset U_1, W_2 \subset W_1$ and $V_2 \subset V_1$ and a continuous function

$$\begin{aligned} &\psi : U_2 \times V_2 \to W_2 \quad \text{such that all solutions to } G(v, w, \lambda) = 0 \\ &\text{in } U_2 \times W_2 \times V_2 \quad \text{are of the form } G(v, \psi(v, \lambda), \lambda) = 0. \end{aligned} \tag{3.2.14}$$

Insertion of the function $\psi(v, \lambda)$ into the second equation of (3.2.12) yields a finite-dimensional problem

$$\Phi(v, \lambda) = QF(v + \psi(v, \lambda), \lambda) = 0. \tag{3.2.15}$$

We observe that finding solutions (v, λ) to (3.2.15) is equivalent to finding solutions to $F(x, \lambda) = 0$ in a neighborhood of (x_0, λ_0). We will refer to the finite-dimensional problem (3.2.15) as the **Liapunov-Schmidt reduction** of (3.2.10).

Given that ker$(D_xF(x_0, \lambda_0))$ is spanned by $\hat{v}_0$, then we can write $v = s\hat{v}_0 + v_0$. Substituting this into (3.2.15) we obtain

$$\Phi(s, \lambda) = QF(s\hat{v}_0 + v_0 + \psi(s\hat{v}_0 + v_0, \lambda), \lambda) = 0. \tag{3.2.16}$$

Using the reduction (3.2.16) and another application of the Implicit Function Theorem, one obtains the following theorem taken from [12], which allows us to determine a unique solution curve through the point (x_0, λ_0). We also include the proof for completeness.

Theorem 3.2.5. *Assume that $F : U \times V \to Z$ is continuously differentiable on $U \times V \subset X \times \mathbb{R}$ and that assumptions* (3.2.11) *hold. Additionally we assume that*

$$D_\lambda F(x_0, \lambda_0) \notin R(D_xF(x_0, \lambda_0)). \tag{3.2.17}$$

Then there is a continuously differentiable curve through (x_0, λ_0); that is, there exists

$$\{(x(s), \lambda(s)) \mid s \in (-\delta, \delta),\ (x(0), \lambda(0)) = (x_0, \lambda_0)\}, \tag{3.2.18}$$

such that

$$F(x(s), \lambda(s)) = 0 \quad \textit{for } s \in (-\delta, \delta), \tag{3.2.19}$$

and all solutions of $F(x, \lambda) = 0$ *in a neighborhood of* (x_0, λ_0) *belong to the curve* (3.2.18).

Proof. Let $x_0 = v_0 + w_0 = v_0 + \psi(v_0, \lambda_0)$. Differentiating (3.2.15) with respect to λ we obtain

$$\begin{aligned} &D_\lambda \Phi(v_0, \lambda_0) = \\ &QD_xF(x_0, \lambda_0)D_\lambda\psi(v_0, \lambda_0) + QD_\lambda F(x_0, \lambda_0) = QD_\lambda F(x_0, \lambda_0) \neq 0, \end{aligned} \tag{3.2.20}$$

where (3.2.20) is nonzero due to the extra assumption (3.2.17). The above expression simplifies due to the fact that that

$$D_xF(x_0, \lambda_0)D_\lambda\psi(v_0, \lambda_0) \in R(D_xF(x_0, \lambda_0)),$$

and Q is the projection onto ker$(D_XF(x_0, \lambda_0)^*)$.

The fact that $D_\lambda\Phi(v_0, \lambda_0) \neq 0$ and that X_1, Y_2 and $\mathbb{R}$ are one-dimensional implies that we may apply the Implicit Function Theorem to $\Phi(v, \lambda)$ to conclude that there exists a continuously differentiable $\gamma : U_2 \to V_2 \subset \mathbb{R}$ such that

$$\gamma(v_0) = \lambda_0 \quad \text{and} \quad \Phi(v, \gamma(v)) = 0 \quad \text{for all} \quad v \in U_2 \subset X_1. \tag{3.2.21}$$

Therefore our reduced equation (3.2.15) becomes

$$\Phi(v, \gamma(v)) = QF(v + \psi(v, \gamma(v)), \gamma(v)) = 0, \tag{3.2.22}$$

where solutions to (3.2.22) are of the form

$$x(v) = v + \psi(v, \gamma(v)) \quad \text{and} \quad \lambda(v) = \gamma(v). \tag{3.2.23}$$

By writing $v = s\hat{v}_0 + v_0$ as in (3.2.16) and inserting this into (3.2.23), we obtain our solution curve

$$x(s) = v_0 + s\hat{v}_0 + \psi(v_0 + s\hat{v}_0, \gamma(v_0 + s\hat{v}_0)), \tag{3.2.24}$$

$$\lambda(s) = \gamma(v_0 + s\hat{v}_0). \tag{3.2.25}$$

□

Now we compile some useful properties of the maps $\Phi(v, \lambda)$, $\psi(v, \lambda)$ and $\gamma(v)$ defined in the (3.2.15), (3.2.21) and (3.2.33). These results, along with their proofs, are taken from [12].

Proposition 3.2.6. *Let the assumptions of Theorem 3.2.5 be in effect and let the operators $\Phi(v, \lambda)$, $\psi(v, \lambda)$ and $\gamma(v)$ be defined as in* (3.2.15)*,* (3.2.21) *and* (3.2.33) *and let λ_0 and $x_0 = v_0 + w_0$ be as in the previous discussion. Then*

$$D_v\Phi(v_0, \lambda_0) = 0, \quad D_v\psi(v_0, \lambda_0) = 0, \quad \text{and} \quad D_v\gamma(v_0) = 0, \tag{3.2.26}$$

and each of these operators has the same order of differentiability as $F(x, \lambda)$.

Proof. The fact that $\Phi(v, \lambda)$, $\psi(v, \lambda)$ and $\gamma(v)$ all have the same order of differentiability as $F(x, \lambda)$ follows from the definition of $\Phi(v, \lambda)$ and the Implicit Function Theorem 3.10.5. By differentiating $(I - Q)F(v + \psi(v, \lambda), \lambda) = 0$ with respect to v we obtain

$$(I - Q)D_xF(v + \psi(v, \lambda), \lambda)(I_{X_1} + D_v\psi(v, \lambda)) = 0, \tag{3.2.27}$$

where I_{X_1} denotes the identity on $X_1 = \text{ker}(D_xF(x_0, \lambda_0))$. By evaluating at (v_0, λ_0), where $x_0 = v_0 + w_0$, we obtain

$$(I - Q)D_xF(x_0, \lambda_0)D_v\psi(v_0, \lambda_0) = 0. \tag{3.2.28}$$

Given that $D_v\psi(v_0, \lambda_0)$ maps onto X_2 and $(I - Q)D_XF(x_0, \lambda_0)$ is an invertible operator from X_2 to Y_1, we have that $D_v\psi(x_0, \lambda_0) = 0$.

Then if we differentiate $\Phi(v, \lambda) = QF(v + \psi(v, \lambda), \lambda) = 0$ with respect to v and evaluate at (v_0, λ_0), we obtain

$$D_v\Phi(v_0, \lambda_0) = QD_xF(x_0, \lambda_0)I_{X_1} = 0. \tag{3.2.29}$$

By differentiating (3.2.22) with respect to v and utilizing (3.2.29), we have

$$D_\lambda\Phi(v_0, \lambda_0)D_v\gamma(v_0) = 0.$$

The assumption that $D_\lambda\Phi(v_0, \lambda_0) \neq 0$ implies that

$$D_v\gamma(v_0) = 0. \tag{3.2.30}$$

□

Once we've obtained a unique solution curve $(x(s), \lambda(s))$ through (x_0, λ_0), we analyze $\ddot{\lambda}(0)$ (where $\dot{} = \frac{d}{ds}$) to determine additional information about the solution curve. In particular, we can determine whether or not a **saddle node bifurcation** or fold occurs at (x_0, λ_0). This type of bifurcation occurs when the solution curve $\{x(s), \lambda(s)\}$ has a turning point at (x_0, λ_0). The next proposition, taken from [12], provides us with a method to determine information about $\ddot{\lambda}(0)$.

Proposition 3.2.7. *Let the assumptions of Theorem 3.2.5 be in effect. Additionally assume that $ker(D_XF(x_0, \lambda_0))$ is spanned by $\hat{v}_0$. Then*

$$\frac{d}{ds}F(x(s), \lambda(s))\bigg|_{s=0} = \tag{3.2.31}$$

$$D_xF(x_0, \lambda_0)\dot{x}(0) + D_\lambda F(x_0, \lambda_0)\dot{\lambda}(0) = D_xF(x_0, \lambda_0)\hat{v}_0 = 0$$

$$\frac{d^2}{ds^2}F(x(s), \lambda(s))\bigg|_{s=0} = \tag{3.2.32}$$

$$D^2_{xx}F(x_0, \lambda_0)[\hat{v}_0, \hat{v}_0] + D_xF(x_0, \lambda_0)\ddot{x}(0) + D_\lambda F(x_0, \lambda_0)\ddot{\lambda}(0) = 0.$$

In particular, an application of the projection operator Q defined in (3.2.12) *to* (3.2.32) *yields*

$$QD^2_{xx}F(x_0, \lambda_0)[\hat{v}_0, \hat{v}_0] + QD_\lambda F(x_0, \lambda_0)\ddot{\lambda}(0) = 0. \tag{3.2.33}$$

This implies that if $D_\lambda F(x_0, \lambda_0) \notin R(D_x F(x_0, \lambda_0))$ *and*

$$D^2_{xx} F(x_0, \lambda_0)[\hat{v}_0, \hat{v}_0] \notin R(D_x F(x_0, \lambda_0)),$$

then $\ddot{\lambda}(0) \neq 0$.

Proof. Let $\{x(s), \lambda(s)\}$ be the solution curves for $F(x, \lambda) = 0$ defined by (3.2.24) and (3.2.25). Differentiating these curves we obtain

$$\frac{d}{ds} x(s)\bigg|_{s=0} = \hat{v}_0 + D_v \psi(v_0, \lambda_0)\hat{v}_0 + D_\lambda \psi(v_0, \lambda_0) D_v \gamma(v_0)\hat{v}_0 = \hat{v}_0, \tag{3.2.34}$$

$$\frac{d}{ds} \lambda(s)\bigg|_{s=0} = D_v \gamma(v_0)\hat{v}_0 = 0, \tag{3.2.35}$$

where the above expressions simplify as a result of Proposition 3.2.6. Differentiating the expression $F(x(s), \lambda(s)) = 0$ twice and again using Proposition 3.2.6 to simplify, we obtain

$$\frac{d^2}{ds^2} F(x(s), \lambda(s))\bigg|_{s=0} = \tag{3.2.36}$$

$$D^2_{xx} F(x_0, \lambda_0)[\hat{v}_0, \hat{v}_0] + D_x F(x_0, \lambda_0)\ddot{x}(0) + D_\lambda F(x_0, \lambda_0)\ddot{\lambda}(0) = 0,$$

where

$$\ddot{\lambda}(0) = D^2_{vv}\gamma(v_0)[\hat{v}_0, \hat{v}_0] \quad \text{and} \quad \ddot{x}(0) = D^2_{vv}\psi(v_0, \lambda_0)[\hat{v}_0, \hat{v}_0],$$

by differentiating (3.2.34) and (3.2.35) once more with respect to s. Applying the projection operator Q to (3.2.36) yields (3.2.33). Then the assumptions that $D_\lambda F(x_0, \lambda_0) \notin R(D_x F(x_0, \lambda_0))$ and $D^2_{xx} F(x_0, \lambda_0)[\hat{v}_0, \hat{v}_0] \notin R(D_x F(x_0, \lambda_0))$ imply that $\ddot{\lambda}(0) \neq 0$. □

The significance of Proposition 3.2.7 is that it gives explicit conditions that allow us to determine whether or not $\ddot{\lambda}(0)$ is nonzero. Heuristically, the fact that $\ddot{\lambda}(0) \neq 0$ means that $\lambda(s)$ has a turning point at $s = 0$. This means that the graph of $\{x(s), \lambda(s)\}$ looks like a parabola and that a saddle node bifurcation occurs at $s = 0$ (cf. [12]). If we assume that $F(x, \lambda)$ is at least 3-times differentiable we may expand the operators $\psi(v_0 + s\hat{v}_0, \gamma(v_0 + s\hat{v}_0))$ and $\gamma(v_0 + s\hat{v}_0)$ about $s = 0$ as a second order Taylor series and use (3.2.24) and (3.2.25) to obtain second order representations of our solutions

$\{x(s), \lambda(s)\}$. This is the solution approach we take to prove non-uniqueness in both the CMC and non-CMC cases.

3.3 Main Results

The main results of this article pertain to the following one parameter family of problems

$$\begin{aligned} -\Delta\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - a_{\mathbf{w}}\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5 = 0, \\ \mathbb{L}\mathbf{w} + \lambda b_\tau^a \phi^6 = 0. \end{aligned} \tag{3.3.1}$$

Here we assume that g_{ab} is a given SPD metric with no conformal killing fields that has constant, positive scalar curvature. The expressions D_a and Δ denote the derivative and the Laplace-Beltrami operator associated with g_{ab} and

$$\mathbb{L}\mathbf{w} = -D_b(\mathcal{L}\mathbf{w})^{ab},$$

denotes the divergence of the conformal killing operator associated with g_{ab}. Finally, we define

$$\begin{aligned} a_R &= \frac{1}{8}R, & a_\tau &= \frac{1}{12}\tau^2, \\ a_{\mathbf{w}} &= \frac{1}{8}(\sigma + \mathcal{L}\mathbf{w})_{ab}(\sigma + \mathcal{L}\mathbf{w})^{ab}, & b_\tau &= \frac{2}{3}D^a\tau. \end{aligned} \tag{3.3.2}$$

In general, we assume that $\tau \in C^{1,\alpha}(\mathcal{M})$, however when we prove our CMC results we will additionally require that τ be constant. For the remainder of this paper we assume that R is a positive constant and that $|\sigma| = (\sigma_{ab}\sigma^{ab})^{\frac{1}{2}}$ is also a nonzero constant. Notice that (3.3.1) has the form of (3.1.6) with initial data depending on λ where

$$\tau_\lambda = \lambda\tau, \quad \rho_\lambda = e^{-\lambda}\rho \quad \text{and} \quad \mathbf{j}_\lambda = 0.$$

We show that in both the CMC and non-CMC cases that solutions to (3.3.2) are non-unique. Our method for doing this is to apply the bifurcation theory outlined in

Section 3.2.4. The first step in doing this is to formulate (3.3.1) in a way that allows us to utilize the framework outlined in Section 3.2.4.

3.3.1 Set Up of Problem

We now formulate (3.3.1) so that we can apply the Liapunov-Schmidt reduction. Define $F((\phi, \mathbf{w}), \lambda)$ by

$$F((\phi, \mathbf{w}), \lambda) = \begin{bmatrix} -\Delta\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - a_{\mathbf{w}}\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5 \\ \mathbb{L}\mathbf{w} + \lambda b_\tau^a \phi^6 \end{bmatrix}, \tag{3.3.3}$$

and in the event that τ is constant, define

$$G(\phi, \lambda) = -\Delta\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - \frac{1}{8}\sigma^2\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5. \tag{3.3.4}$$

If $F((\phi, \mathbf{w}), \lambda) = 0$ (resp. $G(\phi, \lambda) = 0$) for a given λ, then $((\phi, \mathbf{w}), \lambda)$ (resp. (ϕ, λ)) solves Eq. (3.3.1) (resp. Eq. (3.3.4)).

We view (3.3.3) and (3.3.4) as nonlinear operators between the Banach spaces

$$F((\phi, \mathbf{w}), \lambda) : C^{k,\alpha}(\mathcal{M}) \oplus C^{k,\alpha}(\mathcal{TM}) \times \mathbb{R} \to C^{k-2,\alpha}(\mathcal{M}) \oplus C^{k-2,\alpha}(\mathcal{TM}),$$

$$G(\phi, \lambda) : C^{k,\alpha}(\mathcal{M}) \times \mathbb{R} \to C^{k-2,\alpha}(\mathcal{M}).$$

where $k \geq 2$. For $\phi \neq 0$ and $X = (\phi, \mathbf{w})$, the first order Fréchet derivatives $D_\phi G(\phi, \lambda)$, $D_\lambda G(\phi, \lambda)$, $D_X F((\phi, \mathbf{w}), \lambda)$ and $D_\lambda F((\phi, \mathbf{w}), \lambda)$ all exist. In fact, both F and G are k-differentiable for any $k \in \mathbb{N}$ provided that $\phi \neq 0$. See the Appendix 3.10.1 for more information regarding Fréchet derivatives.

Now we are ready to state the main results of this paper. The first two results state that there is a critical density $\rho = \rho_c$ such that there exists a constant ϕ_c where the linearizations $D_\phi G(\phi_c, 0)$ and $D_X F((\phi_c, \mathbf{0}), 0)$ have a kernel of dimension one. This provides the basis for our final two main results where we determine explicit solution curves $\{\phi(s), \lambda(s)\}$ and $\{(\phi(s), \mathbf{w}(s)), \lambda(s)\}$ to obtain our non-uniqueness results.

3.3.2 Existence of ρ_c such that dim ker$(D_X F((\phi_c, \mathbf{0}, 0)) = 1$

The two results in this section pertain to the existence of a critical energy density $\rho = \rho_c$ at which the linearizations of the operators F and G develop a one-dimensional kernel. These results allow us to apply the Liapunov-Schmidt reduction outlined in Section 3.2.4 to analyze solutions in a neighborhood of $((\phi_c, \mathbf{0}), 0)$ and $(\phi_c, 0)$. We present the theorems here without proof and postpone them until Section 3.5.

Theorem 3.3.1 (CMC). *Let $D_\phi G(\phi, \lambda)$ denote the Fréchet derivative of* (3.3.4) *with respect to ϕ. Then there exists a critical value of $\rho = \rho_c$ and a constant ϕ_c such that when $\rho = \rho_c$, Eq.* (3.3.4) *has a solution if and only if $\lambda \geq 0$. Furthermore, dim ker$(D_\phi G(\phi_c, 0))) = 1$ and it is spanned by the constant function $\phi = 1$. Moreover, we can determine the explicit values of ρ_c and ϕ_c, which are*

$$\rho_c = \frac{R^{\frac{3}{2}}}{24\sqrt{3}\pi|\sigma|} \quad \textit{and} \quad \phi_c = \left(\frac{R}{24\pi\rho}\right)^{\frac{1}{4}}. \tag{3.3.5}$$

Proof. We present the proof in Section 3.5. □

Theorem 3.3.2 (non-CMC). *Let $D_X F((\phi, \mathbf{w}), \lambda)$ denote the Fréchet derivative of* (3.3.3) *with respect to $X = (\phi, \mathbf{w})$ and let ρ_c and ϕ_c be as in Theorem 3.3.1. Then when $\rho = \rho_c$, dim ker$(D_X F((\phi_c, \mathbf{0}), 0))) = 1$ and it is spanned by the constant vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.*

Proof. We present the proof in Section 3.5. □

3.3.3 Non-unique Solutions to $F((\phi, \mathbf{w}), \lambda) = 0$ when $\rho = \rho_c$

The two Theorems in this section pertain to the non-uniqueness of solutions to the nonlinear problems (3.3.3) and (3.3.4). Theorem 3.3.3 provides the explicit form of solutions to (3.3.4) in a neighborhood of the point $(\phi_c, 0)$ in the CMC case. The form of this solution curve implies that a saddle node bifurcation occurs at $(\phi_c, 0)$ and that solutions are non-unique in a neighborhood of this point. Theorem 3.3.4 provides analogous results in the non-CMC case for the point $((\phi_c, \mathbf{0}), 0)$.

Theorem 3.3.3 (CMC)**.** *Suppose that* τ *is constant. Then* (3.3.3) *reduces to the scalar problem*

$$-\Delta\phi + a_R\phi + (\lambda^2 a_\tau - 2\pi\rho e^{-\lambda})\phi^5 - \frac{1}{8}\sigma^2\phi^{-7} = 0. \tag{3.3.6}$$

When $\rho = \rho_c$*, with* ρ_c *as in Theorem 3.3.2, there exists a neighborhood of* $(\phi_c, 0)$ *such that all solutions to* (3.3.6) *in this neighborhood lie on a smooth solution curve* $\{\phi(s), \lambda(s)\}$ *that has the form*

$$\phi(s) = \phi_c + s + O(s^2), \tag{3.3.7}$$

$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3). \tag{3.3.8}$$

In particular, there exists a $\delta > 0$ *such that for all* $0 < \lambda < \delta$ *there exist at least two distinct solutions* $\phi_{1,\lambda} \neq \phi_{2,\lambda}$ *to* (3.3.6)*.*

Proof. We postpone the proof until Section 3.7. □

Theorem 3.3.4 (non-CMC)**.** *Suppose that* $\tau \in C^{1,\alpha}(\mathcal{M})$ *is non-constant and let* $F((\phi, \mathbf{w}), \lambda)$ *be defined as in* (3.3.3)*. Then if* ρ_c *and* ϕ_c *are defined as in Theorem 3.3.1 and* $\rho = \rho_c$*, there exists a neighborhood of* $((\phi_c, \mathbf{w}), 0)$ *such that all solutions to* $F((\phi, \mathbf{w}), \lambda) = 0$ *in this neighborhood lie on a smooth curve of the form*

$$\begin{aligned} \phi(s) &= \phi_c + s + \frac{1}{2}\ddot{\lambda}(0)u(x)s^2 + O(s^3), \\ \mathbf{w}(s) &= \frac{1}{2}\ddot{\lambda}(0)\mathbf{v}(x)s^2 + O(s^3), \\ \lambda(s) &= \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \end{aligned} \tag{3.3.9}$$

where $u(x) \in C^{2,\alpha}(\mathcal{M})$ *and* $\mathbf{w}(x) \in C^{2,\alpha}(\mathcal{TM}) \neq \mathbf{0}$*. In particular, there exists a* $\delta > 0$ *such that for all* $0 < \lambda < \delta$ *there exist elements* $(\phi_{1,\lambda}, \mathbf{w}_{1,\lambda}), (\phi_{2,\lambda}, \mathbf{w}_{2,\lambda}) \in C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$ *such that*

$$F((\phi_{i,\lambda}, \mathbf{w}_{i,\lambda}), \lambda) = 0, \ \textit{for} \ \ i \in \{1, 2\}, \ \ \textit{and} \ \ (\phi_{1,\lambda}, \mathbf{w}_{1,\lambda}) \neq (\phi_{2,\lambda}, \mathbf{w}_{2,\lambda}).$$

Proof. We present the proof in Section 3.8. □

3.4 Existence of a Critical Value ρ_c

In this section we lay the foundation for proving Theorems 3.3.1 and 3.3.2. As in [15], we seek a critical density ρ_c where our elliptic problem goes from having positive solutions to having no positive solutions. In particular, what we seek is a value ρ_c such that when $\lambda = 0$, then (3.3.3) will have no solution for $\rho > \rho_c$ and will have a solution for $\rho \le \rho_c$.

When $\lambda = 0$, the assumption that g_{ab} admits no conformal killing fields implies that

$$F((\phi, \mathbf{w}), 0) = F((\phi, \mathbf{0}), 0) = \begin{bmatrix} -\Delta\phi + a_R\phi - \frac{\sigma^2}{8}\phi^{-7} - 2\pi\rho\phi^5 = 0 \\ \mathbf{w} = 0 \end{bmatrix}. \quad (3.4.1)$$

Define

$$q(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\rho_c\chi^5, \quad (3.4.2)$$

where ρ_c is a constant to be determined. The objective will be to determine ρ_c so that $q(\chi)$ has a single, positive, multiple root and then use the maximum principle discussed in Appendix 3.10.6 to conclude that if $\rho > \rho_c$, then (3.4.1) will have no solution. This leads us to the following proposition.

Proposition 3.4.1. *Let* $q(\chi)$ *be defined as in* (3.4.2)*. Then there exists constants* $\rho_c > 0$ *and* $\phi_c > 0$ *such that* $q(\chi) \le 0$ *for all* $\chi > 0$ *and the only positive root of* $q(\chi)$ *is* ϕ_c*.*

Proof. To determine ρ_c, we observe that because a_R and σ^2 are constants, we simply need to analyze the roots of (3.4.2) as ρ_c varies. We seek ρ_c such that $q(\chi)$ has a single, positive, multiple root. We observe that $q(\chi) = 0$ if and only if

$$p(\chi) = a_R\chi^8 - \frac{1}{8}\sigma^2 - 2\pi\rho_c\chi^{12} = 0.$$

Furthermore, it is clear that each pair of roots $\{-\chi_0, \chi_0\}$ of the even polynomial $p(\chi)$ is in direct correspondence with each positive root of $p(\gamma) = a_R\gamma^2 - \frac{1}{8}\sigma^2 - 2\pi\rho_c\gamma^3$, where $\gamma = \chi^4$. Therefore, we simply need to choose ρ_c such that $p(\gamma)$ has a single positive root. To accomplish this, we find the lone, local maximum of $p(\gamma)$ and require it to be a root of $p(\gamma)$. We have that

$$0 = p'(\gamma) = 2a_R\gamma - 6\pi\rho_c\gamma^2 \implies \gamma_c = \frac{a_R}{3\pi\rho_c} \quad \text{is a local max,}$$

and

$$\begin{aligned} 0 = p(\gamma_c) &= a_R\left(\frac{a_R}{3\pi\rho_c}\right)^2 - \frac{1}{8}\sigma^2 - 2\pi\rho_c\left(\frac{a_R}{3\pi\rho_c}\right) \\ &= \frac{a_R^3 - \frac{1}{8}\sigma^2(27\pi^2\rho_c^2)}{27\pi^2\rho_c^2} \implies \rho_c = \frac{R^{\frac{3}{2}}}{24\sqrt{3}|\sigma|\pi}. \end{aligned} \tag{3.4.3}$$

□

The next result follows immediately from the previous analysis but will be useful going forward.

Corollary 3.4.2. *Define the constants*

$$\rho_c = \frac{R^{\frac{3}{2}}}{24\sqrt{3}|\sigma|\pi} \quad \textit{and} \quad \phi_c = \left(\frac{a_R}{3\pi\rho_c}\right)^{\frac{1}{4}}. \tag{3.4.4}$$

Then if

$$q(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\rho_c\chi^5,$$

it follows that $q(\phi_c) = q'(\phi_c) = 0$.

Proof. This follows immediately from the proof of Proposition 3.4.1 or by direct computation. □

Now we show that ρ_c is a critical value of (3.4.1).

Proposition 3.4.3. *Let* $\rho(x) \in C(\mathcal{M})$. *Then the constant* ρ_c *defined in Corollary 3.4.2 has the property that Eq.* (3.4.1) *has a positive solution if* $0 < \rho \leq \rho_c$ *and has no positive solution if* $\rho > \rho_c$.

Proof. Let $q(\chi)$ be defined as in Corollary 3.4.2. If $\phi > 0$ solves (3.4.1), then

$$\Delta\phi = a_R\phi - \frac{1}{8}\sigma^2\phi^{-7} - 2\pi\rho\phi^5 = f(x,\phi). \tag{3.4.5}$$

We observe that if $\rho > \rho_c$, then $\check{\rho} = \inf_{x\in\mathcal{M}} \rho > \rho_c$ and for $\chi > 0$,

$$f(x,\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\rho\chi^5 \leq a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\check{\rho}\chi^5 < q(\chi). \tag{3.4.6}$$

Therefore if $\rho > \rho_c$, (3.4.5) and (3.4.6) imply that any positive solution ϕ to (3.4.1) satisfies

$$\Delta\phi = f(x,\phi) < q(\phi) \leq 0.$$

Therefore an application of the maximum principle (3.10.6) implies that if $\rho > \rho_c$, then (3.4.1) has no solution.

To verify that (3.4.1) has a solution if $\rho \leq \rho_c$, first observe that Corollary 3.4.2 implies that

$$\phi_c = \left(\frac{a_R}{3\pi\rho_c}\right)^{\frac{1}{4}} = \left(\frac{R}{24\pi\rho_c}\right)^{\frac{1}{4}}, \tag{3.4.7}$$

solves Eq. (3.4.1) when $\rho = \rho_c$. If $\rho < \rho_c$, the properties of $q(\chi)$ imply that the polynomial

$$q_1(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\hat{\rho}\chi^5, \qquad \hat{\rho} = \sup_{x\in\mathcal{M}} \rho(x),$$

will have two positive roots $\chi_1 < \chi_2$. Therefore, any ϕ_+ satisfying $0 < \chi_1 < \phi_+ < \chi_2$ will be a positive super-solution to (3.4.1) given that

$$f(x,\chi) > q_1(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\hat{\rho}\chi^5.$$

Similarly, we may choose a positive sub-solution $\phi_- < \phi_+$ to (3.4.1) by choosing any sufficiently small ϕ_- satisfying $0 < \phi_- < \chi_3$, where χ_3 is the lone positive root of

$$q_2(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7}.$$

We can then apply the method of sub- and super-solutions outlined in Section 3.10.2 to solve (3.4.1). □

The next result extends Proposition 3.4.3 to the case when $\lambda \neq 0$ and indicates that ρ_c is also a critical value for the decoupled problem (3.3.4).

Corollary 3.4.4. *Let $\rho(x) \in C(\mathcal{M})$ and suppose that τ is a constant and that*

$$\rho_c = \frac{R^{\frac{3}{2}}}{24\sqrt{3}|\sigma|\pi}.$$

There exists an $\epsilon > 0$ such that there is no positive solution to (3.3.4) *if $\rho > \rho_c$ and $-\epsilon < \lambda < 0$, and there exists a positive solution to* (3.3.4) *if $0 < \rho \leq \rho_c$ and $0 \leq \lambda < \epsilon$. Finally, if $\rho = \rho_c$ and λ is sufficiently small, then* (3.3.4) *has a solution if and only if $\lambda \geq 0$.*

Proof. Again, we observe that if $\phi > 0$ solves (3.3.4), then

$$\Delta\phi = a_R\phi + \lambda^2 a_\tau \phi^5 - \frac{1}{8}\sigma^2\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5 = f(x, \phi, \lambda). \tag{3.4.8}$$

Let $q(\chi)$ be as in Corollary 3.4.2 and define

$$p_1(\chi, \lambda) = a_R\chi + \lambda^2 a_\tau \chi^5 - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\check{\rho}e^{-\lambda}\chi^5,$$

where $\check{\rho} = \inf_{x\in\mathcal{M}} \rho(x)$. It is clear that $f(x, \phi, \lambda) \leq p_1(\phi, \lambda)$ for any $\phi > 0$, and for $\lambda < 0$ and $\rho > \rho_c$ we have that

$$\begin{aligned} p_1(\chi, \lambda) &= a_R\chi + \lambda^2 a_\tau \chi^5 - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\check{\rho}e^{-\lambda}\chi^5 \\ &\leq a_R\chi + (\lambda^2 a_\tau - 2\pi\rho_c + 2\pi\rho_c\lambda + o(\lambda^2))\chi^5 - \frac{1}{8}\sigma^2\chi^{-7} \\ &= q(\chi) + (\lambda^2 a_\tau + 2\pi\rho_c\lambda + o(\lambda^2))\chi^5 = q(\chi) + g(\lambda)\chi^5. \end{aligned} \tag{3.4.9}$$

Here we observe that $g(\lambda) \to 0$ as $\lambda \to 0$, and for $|\lambda|$ sufficiently small, $g(\lambda) < 0$ if $\lambda < 0$. By Proposition 3.4.1, we know that if $\chi > 0$ then $q(\chi) \leq 0$. So Eq. (3.4.9) implies that if $\rho > \rho_c$ and $\lambda < 0$ is sufficiently small, then $f(x, \chi, \lambda) \leq p_1(\chi, \lambda) < 0$, and the maximum principle then implies that (3.3.4) will have no solution.

If $\rho \leq \rho_c$, then define

$$p_2(\chi, \lambda) = a_R\chi + \lambda^2 a_\tau \chi^5 - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\hat{\rho}e^{-\lambda}\chi^5,$$

where $\hat{\rho} = \sup_{x\in\mathcal{M}} \rho(x)$. It is clear that $f(x, \chi, \lambda) \geq p_2(\chi, \lambda)$ for all $\chi > 0$, and for

$\lambda \leq 0$ we have

$$p_2(\chi,\lambda) = a_R\chi + \lambda^2 a_\tau \chi^5 - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\hat{\rho}e^{-\lambda}\chi^5 \tag{3.4.10}$$
$$\geq a_R\chi + (\lambda^2 a_\tau - 2\pi\rho_c + 2\pi\rho_c\lambda + o(\lambda^2))\chi^5 - \frac{1}{8}\sigma^2\chi^{-7}$$
$$= q(\chi) + (\lambda^2 a_\tau + 2\pi\rho_c\lambda + o(\lambda^2))\chi^5 = q(\chi) + g(\lambda)\chi^5.$$

Again, $g(\lambda) \to 0$ as $\lambda \to 0$ and $g(\lambda) > 0$ for $\lambda > 0$ sufficiently small. Therefore if $\chi > 0$, Eq. (3.4.10) implies that $f(x,\chi,\lambda) > p_2(\chi,\lambda) \geq q(\chi)$ if $\lambda \geq 0$. The properties of $q(\chi)$ specified in Proposition 3.4.1 imply that that for any $\lambda > 0$, either $p_2(\chi,\lambda)$ has a single positive root χ_0 and $p_2(\chi,\lambda) > 0$ for all $\chi > \chi_0$, or $p_2(\chi,\lambda)$ has two distinct positive roots. This implies that if $\lambda > 0$ we can find a positive super-solution ϕ_+ to (3.3.4). If $\lambda = 0$ we take $\phi_+ = \phi_c$ to be a super-solution where ϕ_c is defined in Corollary 3.4.2. Similarly, we can also find a positive sub-solution ϕ_- satisfying $\phi_- < \phi_+$ by choosing any sufficiently small $0 < \phi_- < \chi_0$, where χ_0 is the unique positive root of

$$r(\chi,\lambda) = a_R\chi + \lambda^2 a_\tau\chi^5 - \frac{1}{8}\sigma^2\chi^{-7}.$$

The method of sub-and super-solutions outlined in Section 3.10.2 then implies that if $\rho \leq \rho_c$ and $\lambda \geq 0$, then (3.3.4) has a solution.

Finally, we observe that if $\rho = \rho_c$, then we have that

$$f(x,\chi,\lambda) = q(\chi) + g(\lambda)\chi^5,$$

where f and g are the same as above. Therefore, when λ is small and $\rho = \rho_c$, we can apply the above analysis to conclude that (3.3.4) will have a solution if and only if $\lambda \geq 0$. □

Remark 3.4.5. *We note that the negative sign in front of the term $2\pi\rho\chi^5$ in the polynomial*

$$q(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\rho\chi^5,$$

played an essential role in allowing us to determine our critical density ρ_c and critical solution ϕ_c. If this term were positive, then $q(\chi)$ would be monotonic increasing for

$\chi > 0$, and we would not be able to find a positive ϕ_c and ρ_c so that $q(\phi_c) = 0$ and $q'(\phi_c) = 0$. As we saw in Corollary 3.4.4 and Proposition 3.4.3, these properties of $q(\chi)$ played an important role in the existence of solutions to Eq. (3.3.4) *and Eq.* (3.4.1)*. Later in this article, we will also see that these properties of $q(\chi)$ play an important role in our non-uniqueness analysis by allowing for the kernel of the linearization of $F((\phi, \mathbf{w}), \lambda)$ and $G(\phi, \lambda)$ to be one-dimensional. These facts further emphasize the role that terms with the "wrong sign" (cf. [15]) have in the non-uniqueness phenomena associated with CTS, CTT and XCTS formulations of the Einstein constraint equations.*

3.5 Existence of a One Dimensional kernel of $D_X F((\phi_c, \mathbf{0}), 0)$ when $\rho = \rho_c$

In the previous section we proved the existence of a critical density ρ_c that affected whether Eq. (3.4.1) and Eq. (3.3.4) had positive solutions. We now show that when $\rho = \rho_c$, the linearization of both (3.3.4) and (3.3.3) develops a one-dimensional kernel.

We first calculate the Frëchet derivatives $D_X F((\phi, \mathbf{w}), \lambda)$ and $D_\phi G(\phi, \lambda)$. To compute these derivatives, we need only compute the Gâteaux derivatives given that the G-derivatives are continuous in a neighborhood of $((\phi_c, \mathbf{0}, 0)$. See [18] and Remark 3.10.2. Therefore,

$$D_X F((\phi_c, \mathbf{0}), 0) = \frac{d}{dt} F((\phi_c + t\phi, t\mathbf{w}), 0)\bigg|_{t=0},$$

where $(\phi, \mathbf{w}) \in C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$ satisfies $\|(\phi, \mathbf{w})\|_{C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})} = 1$.

So for a given $((\phi, \mathbf{w}), \lambda)$, the Frëchet derivative

$$D_X F((\phi, \mathbf{w}), \lambda) : C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM}) \to C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM}),$$

is a block matrix of operators where the first column consists of derivatives of $F((\phi, \mathbf{w}), \lambda)$ with respect to ϕ and the second column consists of derivatives with respect to $\mathbf{w}$. This implies that

$$D_X F((\phi,\mathbf{w}),\lambda) = \begin{bmatrix} -\Delta + a_R + 5\lambda^2 a_\tau \phi^4 + 7a_{\mathbf{w}}\phi^{-8} - 10\pi\rho_c e^{-\lambda}\phi^4 & \overline{\mathbb{L}} \\ 6\lambda b_\tau^a \phi^5 & \mathbb{L} \end{bmatrix}, \tag{3.5.1}$$

where

$$\overline{\mathbb{L}}h = \overline{\mathbb{L}}(\phi,\mathbf{w})h = -\frac{1}{4}\phi^{-7}\left((\mathcal{L}w)_{ab}(\mathcal{L}h)^{ab} + \sigma_{ab}(\mathcal{L}h)^{ab}\right), \tag{3.5.2}$$

and $\mathcal{L}$ is the conformal Killing operator. Similarly, in the CMC case the map

$$D_\phi G(\phi,\lambda) : C^{2,\alpha}(\mathcal{M}) \to C^{0,\alpha}(\mathcal{M}),$$

has the form

$$D_\phi G(\phi,\lambda) = -\Delta + a_R + 5\lambda^2 a_\tau \phi^4 + \frac{7}{8}\sigma^2\phi^{-8} - 10\pi\rho_c e^{-\lambda}\phi^4. \tag{3.5.3}$$

We now make some key observations about (3.5.1).

Proposition 3.5.1. *Let ϕ_c be as in Corollary 3.4.2. Then $F((\phi_c,\mathbf{0}),0) = 0$ and $D_X F((\phi_c,\mathbf{0}),0)$ has the form*

$$D_X F(\phi_c,\mathbf{0},0) = \begin{bmatrix} -\Delta & \tilde{\mathbb{L}} \\ 0 & \mathbb{L} \end{bmatrix}, \tag{3.5.4}$$

where $\tilde{\mathbb{L}} : C^k(\mathcal{TM}) \to C^{k-1}(\mathcal{M})$ is defined by

$$\overline{\mathbb{L}}(\phi_c,\mathbf{0})h = \tilde{\mathbb{L}}h = -\frac{1}{4}\phi_c^{-7}\sigma_{ab}(\mathcal{L}h)^{ab},$$

and $\mathcal{L}$ is the conformal killing operator.

Proof. By Corollary 3.4.2 it follows that ϕ_c is a root of the polynomial

$$q(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\rho_c\chi^5,$$

and also a root of

$$q'(\chi) = a_R + \frac{7}{8}\sigma^2\chi^{-8} - 10\pi\rho_c\chi^4. \tag{3.5.5}$$

This implies that $F((\phi_c, \mathbf{0}), 0) = 0$ and that Eq. (3.5.1) reduces to (3.5.4) when $((\phi, \mathbf{w}), \lambda) = ((\phi_c, \mathbf{0}), 0)$. □

Remark 3.5.2. *Corollary 3.4.2 implies that* (3.5.3) *reduces to*

$$D_\phi G(\phi_c, 0) = -\Delta, \tag{3.5.6}$$

in the CMC case. Therefore dim ker$(D_\phi G(\phi_c, 0))) = 1$ *and it is spanned by the constant function* $\phi = 1$.

Corollary 3.5.3. *Letting* $\mathcal{H}_1 = L^2(\mathcal{M})$ *and* $\mathcal{H}_2 = L^2(\mathcal{TM})$*, the* $\mathcal{H}_1 \oplus \mathcal{H}_2$*-adjoint of* $D_X F((\phi_c, \mathbf{0}), 0)$ *has the form*

$$(D_X F(\phi_c, \mathbf{0}, 0))^* = \begin{bmatrix} -\Delta & 0 \\ \hat{\mathbb{L}} & \mathbb{L} \end{bmatrix}, \tag{3.5.7}$$

where $\hat{\mathbb{L}} : C^{k,\alpha}(\mathcal{M}) \to C^{k-1,\alpha}(\mathcal{TM})$ *is defined by*

$$\hat{\mathbb{L}}u = D^b(\frac{1}{4}\phi_c^{-7}u\sigma_{ab}). \tag{3.5.8}$$

Proof. Let $(u_1, \mathbf{v}_1)$ and $(u_2, \mathbf{v}_2)$ both be elements of $C^2(\mathcal{M}) \oplus C^2(\mathcal{TM})$. Then given that both $-\Delta$ and $\mathbb{L} = -D_b(\mathcal{L})^{ab}$ are self-adjoint with respect to the $L^2(\mathcal{M})$ and $L^2(\mathcal{TM})$ inner products, it follows that

$$\left\langle D_X F((\phi_c, \mathbf{0}), 0) \begin{bmatrix} u_1 \\ \mathbf{v}_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ \mathbf{v}_2 \end{bmatrix} \right\rangle = \int_{\mathcal{M}} (-u_1\Delta u_2 + \mathbf{v}_1 \cdot \mathbb{L}\mathbf{v}_2 + \tilde{\mathbb{L}}\mathbf{v}_1 u_2) dV_g, \tag{3.5.9}$$

where dV_g is the volume element associated with g_{ab} and $\tilde{\mathbb{L}}\mathbf{v}_1 = -\frac{1}{4}\phi_c^{-7}\sigma_{ab}(\mathcal{L}\mathbf{v}_1)^{ab}$. Given that the negative divergence of a $(0, 2)$ tensor and the conformal killing operator

$\mathcal{L}$ are formal adjoints (see [18]), we have that

$$\int_{\mathcal{M}} \tilde{\mathbb{L}}\mathbf{v}_1 u_2 dV_g = \int_{\mathcal{M}} \left(-\frac{1}{4}u_2\phi_c^{-7}\sigma_{ab}(\mathcal{L}\mathbf{v}_1)^{ab}\right) dV_g \tag{3.5.10}$$
$$= \int_{\mathcal{M}} \left(D^b(\frac{1}{4}u_2\phi_c^{-7}\sigma_{ab}) \cdot \mathbf{v}_1\right) dV_g = \int_{\mathcal{M}} \hat{\mathbb{L}}u_2 \cdot \mathbf{v}_1 dV_g.$$

Therefore,

$$\left\langle D_X F((\phi_c, \mathbf{0}), 0) \begin{bmatrix} u_1 \\ \mathbf{v}_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ \mathbf{v}_2 \end{bmatrix} \right\rangle = \tag{3.5.11}$$
$$\int_{\mathcal{M}} (-u_1\Delta u_2 + \mathbf{v}_1 \cdot \mathbb{L}\mathbf{v}_2 + \hat{\mathbb{L}}u_2 \cdot \mathbf{v}_1) dV_g = \left\langle \begin{bmatrix} u_1 \\ \mathbf{v}_1 \end{bmatrix}, \begin{bmatrix} -\Delta & 0 \\ \hat{\mathbb{L}} & \mathbb{L} \end{bmatrix} \begin{bmatrix} u_2 \\ \mathbf{v}_2 \end{bmatrix} \right\rangle.$$

□

Corollary 3.5.4. *$D_X F((\phi_c, \mathbf{0}, 0)$ has a kernel of dimension 1 that is spanned by $\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$, and $(D_X F(\phi_c, \mathbf{0}, 0))^*$ also has a kernel of dimension one that is spanned by $\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$.*

Proof. We solve for $\begin{bmatrix} u \\ \mathbf{v} \end{bmatrix} \in C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$ such that

$$D_X F((\phi_c, \mathbf{0}), 0) \begin{bmatrix} u \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} -\Delta & \tilde{\mathbb{L}} \\ 0 & \mathbb{L} \end{bmatrix} \begin{bmatrix} u \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}.$$

Given that g_{ab} admits no conformal killing fields, we must have that $\mathbf{v} = 0$. This implies that

$$0 = -\Delta u - \frac{1}{4}\phi_c^{-7}(\sigma_{ab}(\mathcal{L}\mathbf{v})^{ab}) = -\Delta u \Longrightarrow u \quad \text{is a constant.}$$

Therefore $\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$ spans $\ker(D_X F((\phi_c, \mathbf{0}), 0)$.

Similarly, we solve for $\begin{bmatrix} u \\ \mathbf{v} \end{bmatrix}$ such that

$$(D_X F((\phi_c, \mathbf{0}), 0))^* \begin{bmatrix} u \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} -\Delta & 0 \\ \hat{\mathbb{L}} & \mathbb{L} \end{bmatrix} \begin{bmatrix} u \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}.$$

This implies that u is a constant and that

$$0 = \hat{\mathbb{L}}u + \mathbb{L}\mathbf{v} = \nabla^b(\frac{1}{4}\phi_c u\sigma_{ab}) + \mathbb{L}\mathbf{v} = \frac{1}{4}\phi_c u\nabla^b\sigma_{ab} + \mathbb{L}\mathbf{v}.$$

Given that σ_{ab} is divergence free, we have that $\nabla^b\sigma_{ab} = 0$, which implies that $\mathbf{v} = 0$. Therefore $\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$ spans $\ker(D_X F((\phi_c, \mathbf{0}), 0)^*)$. □

We can now prove Theorems 3.3.1 and 3.3.2. The proofs are an immediate consequence of the preceding results, but we summarize them here in the proof for convenience.

3.5.1 Proof of Theorems 3.3.1 and 3.3.2

Proposition 3.4.3 implies the existence of critical values

$$\rho_c = \left(\frac{R}{24\pi\rho_c}\right)^{\frac{1}{4}} \quad \text{and} \quad \phi_c = \left(\frac{a_R}{3\pi\rho_c}\right)^{\frac{1}{4}},$$

such that if

$$q(\chi) = a_R\chi - \frac{1}{8}\sigma^2\chi^{-7} - 2\pi\rho_c\chi^5,$$

then $q(\phi_c) = q'(\phi_c) = 0$. By Remark 3.5.2 we have that the linearization (3.5.3) in the CMC case reduces to $-\Delta$. This proves Theorem 3.3.1. Similarly, in Proposition 3.5.1 we explicitly determined $D_X F((\phi_c, \mathbf{0}), 0)$, and in Corollary 3.4.2 we showed that it has a kernel spanned by the constant vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This proves Theorem 3.3.2.

3.6 Fredholm properties of the operators $D_X F((\phi_c, \mathbf{0}), 0)$ and $D_\phi G(\phi_c, 0)$

Now that we have shown that the linearizations $D_X F((\phi_c, \mathbf{0}), 0)$ and $D_\phi G(\phi_c, 0)$ have one-dimensional kernels, we are almost ready to apply the Liapunov-Schmidt reduction. Recall from section 3.2 that a key assumption in this reduction was that the operator be a nonlinear Fredholm operator. Therefore, to apply this reduction in the CMC and non-CMC cases we must show that the operators $D_\phi G(\phi_c, 0)$ and $D_X F((\phi_c, \mathbf{0}), 0)$

are Fredholm operators between the spaces on which they are defined. In particular, we need to show that $D_\phi G(\phi_c, 0)$ is a Fredholm operator between the spaces $C^{2,\alpha}(\mathcal{M})$ and $C^{0,\alpha}(\mathcal{M})$ and that the operator $D_X F((\phi_c, \mathbf{0}), 0)$ is a Fredholm operator between $C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$ and $C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM})$.

In the CMC case, we have that $D_\phi G(\phi_c, 0) = -\Delta$. It is well known that this operator is a Fredholm operator between the Hilbert spaces $H^2(\mathcal{M})$ and $L^2(\mathcal{M})$ [9]. Furthermore, $-\Delta$ is a Fredholm operator between the subspaces $C^{2,\alpha}(\mathcal{M})$ and $C^{0,\alpha}(\mathcal{M})$ because of the regularity properties of the the Laplacian and the fact that these spaces continuously embed into the Hilbert spaces $H^2(\mathcal{M})$ and $L^2(\mathcal{M})$. See Appendix 3.10.2 for a more detailed discussion of these facts.

Letting $L = -\Delta$, we regard $L = L^*$ as operators from $H^2(\mathcal{M}) \to L^2(\mathcal{M})$. The Fredholm properties of these operators allow us to make the following decompositions that are orthogonal with respect to the L^2-inner product:

$$\begin{aligned} L^2(\mathcal{M}) &= R(L^*) \oplus \ker(L) \\ L^2(\mathcal{M}) &= R(L) \oplus \ker(L^*). \end{aligned} \tag{3.6.1}$$

In this case, these decompositions are the same given that L is self-adjoint. Therefore if we regard $C^{2,\alpha}(\mathcal{M})$ and $C^{0,\alpha}(\mathcal{M})$ as subspaces of $L^2(\mathcal{M})$, then we may use (3.6.1) to obtain the following decompositions

$$\begin{aligned} C^{2,\alpha}(\mathcal{M}) &= (R(L^*) \cap C^{2,\alpha}(\mathcal{M})) \oplus \ker(L), \\ C^{0,\alpha}(\mathcal{M}) &= (R(L) \cap C^{0,\alpha}(\mathcal{M})) \oplus \ker(L^*), \end{aligned} \tag{3.6.2}$$

which are also orthogonal with respect to the L^2-inner product. See Appendix 3.10.2 for further details.

It is not as clear that the operator $D_X F((\phi_c, \mathbf{0}), 0)$ is a Fredholm operator between the spaces $C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$ and $C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM})$. For the sake of completeness, we briefly discuss this point. As in Appendix 3.10.2, we first show that $D_X F(\phi_c, \mathbf{0}), 0)$ is a Fredholm operator from the Hilbert space $L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$ to itself, where we consider the domain of definition of $D_X F((\phi_c, \mathbf{0}), 0)$ to be

$H^2(\mathcal{M}) \oplus H^2(\mathcal{TM})$. Indeed, the operator $D_X F((\phi_c, \mathbf{0}), 0)$ induces the bilinear form

$$B((u_1, \mathbf{v}_1), (u_2, \mathbf{v}_2)) : (H^1(\mathcal{M}) \oplus H^1(\mathcal{TM})) \times (H^1(\mathcal{M}) \oplus H^1(\mathcal{TM})) \to \mathbb{R},$$

where $\langle \cdot, \cdot \rangle$ is the inner product associated with $L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$ and

$$B((u_1, \mathbf{v}_1), (u_2, \mathbf{v}_2)) = \left\langle \begin{bmatrix} -\Delta & \tilde{\mathbb{L}} \\ \mathbf{0} & \mathbb{L} \end{bmatrix} \begin{bmatrix} u_1 \\ \mathbf{v}_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ \mathbf{v}_2 \end{bmatrix} \right\rangle. \tag{3.6.3}$$

Paralleling the discussion in 3.10.2, we first show there exists constants $C, c > 0$ such that

$$B((u, \mathbf{v}), (u, \mathbf{v})) + c\langle (u, \mathbf{v}), (u, \mathbf{v}) \rangle \geq C \|(u, \mathbf{v})\|^2_{H^1(\mathcal{M}) \oplus H^1(\mathcal{TM})}.$$

Let $c > 0$ be a constant to be determined. Then

$$\begin{aligned} &B((u, \mathbf{v}), (u, \mathbf{v})) + c\langle (u, \mathbf{v}), (u, \mathbf{v}) \rangle \qquad (3.6.4) \\ &= \int_{\mathcal{M}} \left(D^a u D_a u - \frac{1}{4} u \phi_c^{-7} \sigma_{ab} (\mathcal{L}v)^{ab} + (\mathcal{L}v)^{ab} (\mathcal{L}v)_{ab} + cu^2 + cv^a v_a \right) dV_g \\ &\geq \int_{\mathcal{M}} \left(D^a u D_a u - \frac{1}{16c\epsilon} u^2 - \epsilon \phi_c^{-14} (\sigma_{ab} (\mathcal{L}v)^{ab})^2 + (\mathcal{L}v)^{ab} (\mathcal{L}v)_{ab} + cu^2 + cv^a v_a \right) dV_g, \end{aligned}$$

where the above inequality follows from an application of Young's inequality. The Schwartz inequality and the definition of $\mathcal{L}$ then imply that

$$\sigma_{ab} (\mathcal{L}v)^{ab} = \langle \sigma, \mathcal{L}\mathbf{v} \rangle_g \leq C |\sigma| |D\mathbf{v}|.$$

Therefore

$$\int_{\mathcal{M}} \epsilon \phi_c^{-14} (\sigma_{ab} (\mathcal{L}v)^{ab})^2 \leq c(\epsilon) \|\mathbf{v}\|^2_{1,2}, \tag{3.6.5}$$

where $c(\epsilon) \to 0$ as $\epsilon \to 0$. Combining (3.6.4) and (3.6.5) we have that

$$\begin{aligned} &B((u, \mathbf{v}), (u, \mathbf{v})) + c\langle (u, \mathbf{v}), (u, \mathbf{v}) \rangle \geq \qquad (3.6.6) \\ &(1 - c(\epsilon)) \|\mathbf{v}\|^2_{1,2} + \|Du\|^2_{0,2} + (c - \frac{1}{16\epsilon}) \|u\|^2_{0,2} \geq C(\|\mathbf{v}\|^2_{1,2} + \|u\|^2_{1,2}), \end{aligned}$$

where the final inequality holds by choosing ϵ sufficiently small and c sufficiently large.

The above discussion tells us that the bilinear form

$$B((u,\mathbf{v}),(u,\mathbf{v})) + c\langle (u,\mathbf{v}),(u,\mathbf{v})\rangle$$

is coercive on $H^1(\mathcal{M})\oplus H^1(\mathcal{TM})$. The Lax-Milgram theorem implies that the problem

$$(D_X F((\phi_c,\mathbf{0}),0) + cI)\begin{bmatrix} u \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} f \\ \mathbf{g} \end{bmatrix}$$

has a unique weak solution $(u,\mathbf{v}) \in H^1(\mathcal{M})\oplus H^1(\mathcal{TM})$ for each $(f,\mathsf{g}) \in L^2(\mathcal{M})\oplus L^2(\mathcal{TM})$, and elliptic regularity gives us that $(u,\mathbf{v}) \in H^2(\mathcal{M})\oplus H^2(\mathcal{TM})$. Therefore we conclude that the operator $D_X F((\phi_c,\mathbf{0}),0) + cI$ is a bijection between $H^2(\mathcal{M})\oplus H^2(\mathcal{TM})$ and $L^2(\mathcal{M})\oplus L^2(\mathcal{TM})$. We are the able to conclude that

$$(D_X F((\phi_c,\mathbf{0}),0) + cI)^{-1} \quad \text{exists and is compact.}$$

Paralleling the discussion in Appendix 3.10.2, we can then conclude that the operator $D_X F((\phi_c,\mathbf{0}),0)$ is a Fredholm operator between $H^2(\mathcal{M})\oplus H^2(\mathcal{TM})$ and $L^2(\mathcal{M})\oplus L^2(\mathcal{TM})$. Using the fact that $C^{0,\alpha}(\mathcal{M})\oplus C^{0,\alpha}(\mathcal{TM})$ embeds continuously into $L^2(\mathcal{M})\oplus L^2(\mathcal{TM})$ and invoking classical Schauder estimates, an argument similar to the argument in 3.10.2 implies that $D_X F((\phi_c,\mathbf{0}),0)$ is Fredholm operator between the spaces $C^{2,\alpha}(\mathcal{M})\oplus C^{2,\alpha}(\mathcal{TM})$ and $C^{0,\alpha}(\mathcal{M})\oplus C^{0,\alpha}(\mathcal{TM})$. By applying the same argument to $D_X F((\phi_c,\mathbf{0}),0)^*$, we can also conclude that this operator is a Fredholm operator between $C^{2,\alpha}(\mathcal{M})\oplus C^{2,\alpha}(\mathcal{TM})$ and $C^{0,\alpha}(\mathcal{M})\oplus C^{0,\alpha}(\mathcal{TM})$.

If $L = D_X F((\phi_c,\mathbf{0}),0)$, then the fact that both L, L^* are Fredholm operators from $H^2(\mathcal{M})\oplus H^2(\mathcal{TM}) \to L^2(\mathcal{M})\oplus L^2(\mathcal{TM})$ allows us to decompose $L^2(\mathcal{M})\oplus L^2(\mathcal{TM})$ as in (3.6.1). Therefore, regarding $C^{2,\alpha}(\mathcal{M})\oplus C^{2,\alpha}(\mathcal{TM})$ and $C^{0,\alpha}(\mathcal{M})\oplus C^{0,\alpha}(\mathcal{TM})$ as subspaces of $L^2(\mathcal{M})\oplus L^2(\mathcal{TM})$, we obtain the following decompositions that are orthogonal with respect to the $L^2(\mathcal{M})\oplus L^2(\mathcal{TM})$- inner product:

$$C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM}) = \text{ker}(L) \oplus (R(L^*) \cap (C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM}))), \quad (3.6.7)$$
$$C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM}) = \text{ker}(L^*) \oplus (R(L) \cap (C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM}))).$$

In the above decomposition, $L = D_X F((\phi_c, \mathbf{0}), 0)$ and ker(L), $R(L)$, ker(L^*) and $R(L^*)$ are all regarded as subspaces of $L^2(\mathcal{M}) \oplus L^2(\mathcal{TM})$.

3.7 Bifurcation and non-uniqueness in the CMC case

We are now ready to prove Theorem 3.3.3. In the CMC case, our system (3.3.1) with $\rho = \rho_c$ reduces to

$$G(\phi, \lambda) = -\Delta\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - \frac{1}{8}\sigma^2\phi^{-7} - 2\pi\rho_c e^{-\lambda}\phi^5. \quad (3.7.1)$$

To prove that solutions to (3.7.1) are non-unique, we will apply the Liapunov-Schmidt reduction outlined in Section 3.2.4 and then invoke Theorem 3.2.5 and Proposition 3.2.7.

3.7.1 Proof of Theorem 3.3.3

By Theorem 3.3.1 and Remark 3.5.2, we know that $D_\phi G(\phi_c, 0) = -\Delta$. It follows that dim ker$(D_\phi G(\phi_c, 0))$ = dim ker$(D_\phi G(\phi_c, 0)^*) = 1$, where both spaces are spanned by $\phi = 1$.

Using the notation from Section 3.2.4, we can apply the Liapunov-Schmidt Reduction, where $\hat{v}_0 = 1$ is a basis of ker$(D_\phi G(\phi_c, 0))$ = ker$(D_\phi G(\phi_c, 0)^*)$. By the discussion in Section 3.6 and appendix 3.10.2, we can decompose $X = C^{2,\alpha}(\mathcal{M}) = X_1 \oplus X_2$ and $Y = C^{0,\alpha}(\mathcal{M}) = Y_1 \oplus Y_2$, where

$$X_1 = \text{ker}(D_\phi G(\phi_c, 0)), \quad X_2 = R(D_\phi G(\phi_c, 0)^*) \cap C^{2,\alpha}(\mathcal{M}), \quad (3.7.2)$$
$$Y_1 = R(D_\phi G(\phi_c, 0)) \cap C^{0,\alpha}(\mathcal{M}), \quad \text{and} \quad Y_2 = \text{ker}(D_\phi G(\phi_c, 0)^*).$$

Letting $P : X \to X_1$ and $Q : Y \to Y_2$ be projection operators as in Section 3.2.4, and writing $\phi = P\phi + (I - P)\phi = v + w$, the Implicit Function Theorem applied to

$$(I - Q)G(v + w, \lambda) = 0, \tag{3.7.3}$$

implies that $w = \psi(v, \lambda)$ in a neighborhood of $(\phi_c, 0)$ and $0 = \psi(\phi_c, 0)$. Plugging $\psi(v, \lambda)$ into

$$QG(v + w, \lambda) = 0,$$

we obtain

$$\Phi(v, \lambda) = QG(v + \psi(v, \lambda), \lambda) = 0. \tag{3.7.4}$$

All solutions to $G(\phi, \lambda) = 0$ in a neighborhood of $(\phi_c, 0)$ must satisfy Eq. (3.7.4).

We now observe that $D_\lambda G(\phi_c, 0) = 2\pi\rho_c\phi_c^5 \neq 0$. This implies that

$$D_\lambda \Phi(\phi_c, 0) = QD_\lambda G(\phi_c, 0) = 2\pi\rho_c\phi_c^5 \neq 0, \tag{3.7.5}$$

given that Q is the projection onto Y_2 and Y_2 is spanned by the constant function 1. The Implicit Function Theorem applied to Eq. (3.7.4) implies that there exists a function $\gamma : U_1 \to V_1$ such that $U_1 \subset X_1$, $V_1 \subset \mathbb{R}$ and $\gamma(v) = \lambda$ in a neighborhood of ϕ_c with $\gamma(\phi_c) = 0$.

Therefore (3.7.4) becomes

$$g(v) = QG(v + \psi(v, \gamma(v)), \gamma(v)), \tag{3.7.6}$$

and by writing $v = s + \phi_c$, which we can do for $s \in (-\delta, \delta)$ with $\delta > 0$ sufficiently small, we obtain

$$g(s) = QG(s + \phi_c + \psi(s + \phi_c, \gamma(s + \phi_c)), \gamma(s + \phi_c)) = 0. \tag{3.7.7}$$

This implies that solutions to $G(\phi, \lambda) = 0$ are given by $g(s) = 0$ in a neighborhood of

$(\phi_c, 0)$, where

$$\begin{aligned}\phi(s) &= s + \phi_c + \psi(s+\phi_c, \gamma(s+\phi_c)), \\ \lambda(s) &= \gamma(s+\phi_c)\end{aligned} \tag{3.7.8}$$

determine a differentiable solution curve through $(\phi_c, 0)$.

Equation (3.7.8) gives us a fairly explicit representation of the continuously differentiable curve $\{\phi(s), \lambda(s)\}$ provided by Theorem 3.2.5. However, by applying Proposition 3.2.7 we can determine that $\ddot{\lambda}(0) \neq 0$ to obtain even more information about $\{\phi(s), \lambda(s)\}$. We observe that

$$D^2_{\phi\phi}G(\phi_c, 0)[\hat{v}_0, \hat{v}_0] = -7\sigma^2\phi_c^{-9} - 40\pi\rho_c\phi_c^3 \neq 0. \tag{3.7.9}$$

Therefore

$$-7\sigma^2\phi_c^{-9} - 40\pi\rho_c\phi_c^3 \in Y_2 \Longrightarrow D^2_{\phi\phi}G(\phi_c, 0)[\hat{v}_0, \hat{v}_0] \notin R(D_\phi G(\phi_c, 0)) = Y_1,$$

given that $Y_1 \perp Y_2$. Proposition 3.2.7 implies that $\ddot{\lambda}(0) \neq 0$ and that a saddle node bifurcation occurs at $(\phi_c, 0)$.

We now combine (3.7.8) and the fact that $\ddot{\lambda}(0) \neq 0$ to obtain a more explicit representation to the solution curve $\{\phi(s), \lambda(s)\}$ in a neighborhood of $(\phi_c, 0)$. Define the function

$$f(s) = \psi(s+\phi_c, \gamma(s+\phi_c)). \tag{3.7.10}$$

Then by Propositions 3.2.6 and 3.2.7 we have that

$$\begin{aligned} & f(0) = 0, \quad \text{and} \quad \lambda(0) = \gamma(\phi_c) = 0, \\ & \dot{\lambda}(0) = \frac{d}{ds}\lambda(s)\bigg|_{s=0} = D_v\gamma(\phi_c) = 0, \\ & \dot{f}(0) = \frac{d}{ds}f(s)\bigg|_{s=0} = D_v\psi(\phi_c, 0) + D_\lambda\psi(\phi_c, 0)D_v\gamma(\phi_c) = 0. \end{aligned} \tag{3.7.11}$$

Therefore the function $f(s) = O(s^2)$. By computing a Taylor expansion of $\lambda(s)$ about

$s = 0$ and using Eq. (3.7.11) and Eq. (3.7.8), we find that for $s \in (-\delta, \delta)$,

$$\begin{aligned} \phi(s) &= \phi_c + s + O(s^2), \\ \lambda(s) &= \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \end{aligned} \tag{3.7.12}$$

where $\ddot{\lambda}(0) \neq 0$.

Based on the form of $\phi(s)$ and $\lambda(s)$ in Eq. (3.7.12), there exists a $0 < \delta' < \delta$ such that $\phi(s) < 0$, $\lambda(s) > 0$ for all $s \in [-\delta', 0)$, and $\phi(s) > 0$, $\lambda(s) > 0$ for all $s \in (0, \delta']$. Letting $M = \max\{M_1, M_2\}$, where

$$M_1 = \sup_{s \in [-\delta', 0]} \lambda(s) \quad \text{and} \quad M_2 = \sup_{s \in [0, \delta']} \lambda(s),$$

the Intermediate Value Theorem then implies that for all $\lambda_0 \in (0, M)$, there exists $s_1, s_2 \in [-\delta', \delta']$, $s_1 \neq s_2$, such that $\lambda(s_1) = \lambda(s_2) = \lambda_0$. Based on how we chose δ', we also have that $\phi(s_1) \neq \phi(s_2)$. This completes the proof of Theorem 3.3.3.

3.8 Bifurcation and non-uniqueness in the non-CMC case

In this section we will show that solutions to $F((\phi, \mathbf{w}), 0) = 0$ for the full system

$$F((\phi, \mathbf{w}), \lambda) = \begin{bmatrix} -\Delta\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - a_{\mathbf{w}}\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5 \\ \mathbb{L}\mathbf{w} + \lambda b_\tau^a \phi^6 \end{bmatrix} \tag{3.8.1}$$

are non-unique, where $\tau \in C^{1,\alpha}(\mathcal{M})$ is a non-constant function. Our approach is similar to that of the CMC case: we apply a Liapunov-Schmidt reduction to Eq. (3.8.1) to determine an explicit solution curve through the point $((\phi_c, \mathbf{0}), 0)$. The form of this curve will imply that solutions to the system (3.8.1) are non-unique.

3.8.1 Proof of Theorem 3.3.4

By Proposition 3.5.1 we know that ker$D_X F((\phi_c, \mathbf{0}), 0)$ takes the form

$$D_X F(\phi_c, \mathbf{0}, 0) = \begin{bmatrix} -\Delta & \tilde{\mathbb{L}} \\ 0 & \mathbb{L} \end{bmatrix},$$

where $\tilde{\mathbb{L}}h = -\frac{1}{4}\phi_c^{-7}\sigma_{ab}(\mathcal{L}h)^{ab}$. Corollary 3.5.4 gives us that ker$(D_X F((\phi_c, \mathbf{0}), 0))$ and ker$(D_X F((\phi_c, \mathbf{0}), 0)^*)$ are spanned by $\hat{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Using the notation from Section 3.2.4, we apply the Liapunov-Schmidt Reduction. By the decomposition (3.6.7), we have that

$$X = C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM}) = X_1 \oplus X_2,$$

and

$$Y = C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM}) = Y_1 \oplus Y_2,$$

where

$$X_1 = \text{ker}(D_X F((\phi_c, \mathbf{0}), 0)), \tag{3.8.2}$$

$$X_2 = R(D_X F((\phi_c, \mathbf{0}), 0)^*) \cap (C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})), \tag{3.8.3}$$

$$Y_1 = R(D_X F((\phi_c, \mathbf{0}), 0)) \cap (C^{0,\alpha}(\mathcal{M}) \oplus C^{0,\alpha}(\mathcal{TM})), \tag{3.8.4}$$

$$Y_2 = \text{ker}(D_X F((\phi_c, \mathbf{0}), 0)^*). \tag{3.8.5}$$

Let $P : X \to X_1$ and $Q : Y \to Y_2$ be the projection operators defined using $\hat{v}_0$ as in Section 3.2.4. Then by writing

$$\begin{bmatrix} \phi \\ \mathbf{w} \end{bmatrix} = P\begin{bmatrix} \phi \\ \mathbf{w} \end{bmatrix} + (I - P)\begin{bmatrix} \phi \\ \mathbf{w} \end{bmatrix} = v + y,$$

the Implicit Function Theorem applied to

$$(I - Q)F(v + y, \lambda) = 0, \tag{3.8.6}$$

implies that solutions to $F((\phi, \mathbf{w}), 0) = 0$ satisfy

$$\Phi(v, \lambda) = QF(v + \psi(v, \lambda), \lambda) = 0 \tag{3.8.7}$$

in a neighborhood of $((\phi_c, \mathbf{0}), 0)$, where $y = \psi(v, \lambda)$ in this neighborhood and $(0, \mathbf{0}) = \psi((\phi_c, \mathbf{0}), 0)$.

We now observe that

$$D_\lambda F((\phi_c, \mathbf{0}), 0) = \begin{bmatrix} 2\pi\rho_c\phi_c^5 \\ b_\tau^a\phi_c^6 \end{bmatrix} \notin Y_1,$$

due to the fact that

$$\begin{bmatrix} 2\pi\rho_c\phi_c^5 \\ 0 \end{bmatrix} \in Y_2 \quad \text{and} \quad Y_1 \perp Y_2.$$

This implies that

$$D_\lambda \Phi((\phi_c, \mathbf{0}), 0) = QD_\lambda F((\phi_c, \mathbf{0}), 0) = \begin{bmatrix} 2\pi\rho_c\phi_c^5 \\ 0 \end{bmatrix} \neq 0, \tag{3.8.8}$$

given that Q is the projection onto Y_2. The Implicit Function Theorem again implies that there exists a function $\gamma : U_1 \to V_1$, where $(\phi_c, \mathbf{0}) \in U_1 \subset X_1$, $V_1 \subset \mathbb{R}$ and $\gamma(v) = \lambda$ in U_1 with $\gamma(\phi_c, \mathbf{0}) = 0$. Using this fact, Eq. (3.8.7) becomes

$$g(v) = QF(v + \psi(v, \gamma(v)), \gamma(v)) = 0, \tag{3.8.9}$$

and by writing

$$v = (s + \phi_c)\hat{v}_0 = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix},$$

for $s \in (-\delta, \delta)$ with $\delta > 0$ sufficiently small, we then obtain

$$g(s) = QF\left(s\hat{v}_0 + \phi_c\hat{v}_0 + \psi(s\hat{v}_0 + \phi_c\hat{v}_0, \gamma(s\hat{v}_0 + \phi_c\hat{v}_0)), \gamma(s\hat{v}_0 + \phi_c\hat{v}_0)\right) = 0. \tag{3.8.10}$$

This implies that solutions to $F((\phi, \mathbf{0}), \lambda) = 0$ in a neighborhood of $((\phi_c, \mathbf{0}), 0)$ satisfy

$g(s) = 0$, where

$$\begin{bmatrix} \phi(s) \\ \mathbf{w}(s) \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix} + \psi \left(s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix}, \gamma \left(s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix} \right) \right), \tag{3.8.11}$$

$$\lambda(s) = \gamma \left(s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_c \\ 0 \end{bmatrix} \right),$$

determine a smooth solution curve through $((\phi_c, \mathbf{0}), 0)$.

As in the CMC case, we seek additional information so that we can further analyze the solution curve (3.8.11). Now we apply Proposition 3.2.7 to determine information about $\ddot{\lambda}(0)$, and then we will expand the function

$$f(s) = \psi((s + \phi_c)\hat{v}_0, \gamma((s + \phi_c)\hat{v}_0)) \tag{3.8.12}$$

as a Taylor series to obtain a more explicit representation of $\{(\phi(s), \mathbf{w}(s)), \lambda(s)\}$.

Taking the second derivative of $F((\phi, \mathbf{w}), \lambda)$, we have that

$$D^2_{XX} F((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] = \begin{bmatrix} -7\sigma^2 \phi_c^{-9} - 40\pi \rho_c \phi_c^3 \\ \mathbf{0} \end{bmatrix} \in Y_2. \tag{3.8.13}$$

Given that the vector (3.8.13) lies in Y_2 and $Y_1 \perp Y_2$,

$$D^2_{XX} F((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] \notin Y_1.$$

We can therefore apply Proposition 3.2.7 to conclude that $\ddot{\lambda}(0) \neq 0$.

Our next goal is to expand the function $f(s)$ as a Taylor series about 0. In order to do this, we use (3.8.11), Proposition 3.2.6 and the fact that $\ddot{\lambda}(0) \neq 0$ to obtain information about coefficients in this expansion. In particular, the objective is to determine information about the coefficient of the second order term in the expansion of $f(s)$.

By differentiating

$$(I - Q)F(v + \psi(v, \lambda), \lambda) = 0,$$

with respect to λ and evaluating the resulting expression at $((\phi_c, \mathbf{0}), 0)$, we obtain

$$(I-Q)D_X F((\phi_c, \mathbf{0}), 0) D_\lambda \psi((\phi_c, \mathbf{0}), 0) + (I-Q) D_\lambda F((\phi_c, \mathbf{0}), 0) = 0. \quad (3.8.14)$$

Given that

$$D_\lambda F((\phi_c, \mathbf{0}), 0) = \begin{bmatrix} 2\pi \rho_c \phi_c^5 \\ b_\tau^a \phi_c^6 \end{bmatrix},$$

and Q is the projection operator onto Y_2, which is spanned by $\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$, we have that

$$(I-Q) D_\lambda F((\phi_c, \mathbf{0}), 0) = \begin{bmatrix} 0 \\ b_\tau^a \phi_c^6 \end{bmatrix}. \quad (3.8.15)$$

Equations (3.8.15) and (3.8.14) imply that

$$(I-Q)D_X F((\phi_c, \mathbf{0}), 0) D_\lambda \psi((\phi_c, \mathbf{0}), 0) = -\begin{bmatrix} 0 \\ b_\tau^a \phi_c^6 \end{bmatrix}. \quad (3.8.16)$$

Given that $D_X F((\phi_c, \mathbf{0}), 0)$ has the form (3.5.4) and the operator $\mathbb{L}$ is invertible, Eq. (3.8.16) implies that

$$D_\lambda \psi((\phi_c, \mathbf{0}), 0) = \begin{bmatrix} u(x) \\ \mathbf{v}(x) \end{bmatrix}, \quad \text{with} \quad \mathbf{v}(x) \neq \mathbf{0}. \quad (3.8.17)$$

As we shall see, this fact implies that $\mathbf{w}(s)$ has quadratic terms in s.

We have one last piece of data left to determine the coefficient of the second order term in the Taylor expansion of $f(s)$. Differentiating $(I-Q)F(v+\psi(v,\lambda),\lambda) = 0$ twice with respect to v, evaluating at $((\phi_c, \mathbf{0}), 0)$ and applying the resulting bilinear form to $\hat{v}_0$, we obtain

$$\begin{aligned}&(I-Q)D_{XX}^2 F((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] + \\ &(I-Q)D_X F((\phi_c, \mathbf{0}), 0) D_{vv}^2 \psi((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] = 0.\end{aligned} \quad (3.8.18)$$

By Eq. (3.8.13) we know that $D^2_{XX}F((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] \in Y_2$. Because $(I - Q)$ projects onto Y_1 and $Y_1 \perp Y_2$, we have that

$$(I - Q)D^2_{XX}F((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] = 0. \tag{3.8.19}$$

Equations (3.8.19) and (3.8.18) and the invertibility of $(I - Q)D_X F((\phi_c, \mathbf{0}), 0)$ as an operator from X_2 to Y_1 imply that

$$D^2_{vv}\psi((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] = 0. \tag{3.8.20}$$

This was the final piece of information that we needed to to determine the second order expansion of $f(s)$.

We now expand the function $f(s)$ in Eq. (3.8.12) about $s = 0$. We have that

$$f(0) = \psi((\phi_c, \mathbf{0}), 0) = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}, \tag{3.8.21}$$

$$\dot{f}(0) = D_v\psi((\phi_c, \mathbf{0}), 0)\hat{v}_0 + D_\lambda\psi((\phi_c, \mathbf{0}), 0))D_v\gamma(\phi_c, \mathbf{0})\hat{v}_0 = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix},$$

$$\begin{aligned}\ddot{f}(0) &= D^2_{vv}\psi((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] + D^2_{v\lambda}\psi((\phi_c, \mathbf{0}), 0)[\hat{v}_0, D_v\gamma(\phi_c, \mathbf{0})\hat{v}_0] \\ &+ D^2_{\lambda v}\psi((\phi_c, \mathbf{0}), 0)[D_v\gamma(\phi_c, \mathbf{0})\hat{v}_0, \hat{v}_0] + D_\lambda\psi((\phi_c, \mathbf{0}), 0)D^2_{vv}\gamma(\phi_c, \mathbf{0})[\hat{v}_0, \hat{v}_0] \\ &+ D^2_{\lambda\lambda}\psi((\phi_c, \mathbf{0}), 0)[D_v\gamma(\phi_c, \mathbf{0})\hat{v}_0, D_v\gamma(\phi_c, \mathbf{0})\hat{v}_0] \\ &= D_\lambda\psi((\phi_c, \mathbf{0}), 0)D^2_{vv}\gamma(\phi_c, \mathbf{0})[\hat{v}_0, \hat{v}_0] = D_\lambda\psi((\phi_c, \mathbf{0}), 0)\ddot{\lambda}(0) \neq \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix},\end{aligned}$$

where $\ddot{f}(0)$ simplifies as a result of Proposition 3.2.7, Eq. (3.8.17) and Eq. (3.8.20), which imply

$$\begin{aligned} &D_v\psi((\phi_c, \mathbf{0}), 0) = 0, && D_v\gamma(\phi_c, \mathbf{0}) = 0, \\ &D^2_{vv}\psi((\phi_c, \mathbf{0}), 0)[\hat{v}_0, \hat{v}_0] = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}, && D_\lambda\psi((\phi_c, \mathbf{0}), 0) \neq \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}. \end{aligned} \tag{3.8.22}$$

Therefore it follows that

$$f(s) = \frac{1}{2}(D_\lambda\psi(\phi_c\hat{v}_0, \gamma(\phi_c\hat{v}_0))\ddot{\lambda}(0))s^2 + O(s^3) = \begin{bmatrix} \frac{1}{2}u(x)\ddot{\lambda}(0) \\ \frac{1}{2}\mathbf{v}(x)\ddot{\lambda}(0) \end{bmatrix} s^2 + O(s^3), \tag{3.8.23}$$

where we identify $D_\lambda\psi((\phi_c, \mathbf{0}), 0)$ with the vector $\begin{bmatrix} u(x) \\ \mathbf{v}(x) \end{bmatrix}$ in $C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$. By Eq. (3.8.17) we have that $\mathbf{v}(x) \neq 0$ and expanding out $\lambda(s)$ as a second order Taylor series about $s = 0$ we obtain

$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3). \tag{3.8.24}$$

Putting together (3.8.11), (3.8.23) and (3.8.24) we find that solutions to $F((\phi, \mathbf{w}), \lambda) = 0$ in a neighborhood of $((\phi_c, \mathbf{0}), 0)$ take the form

$$\phi(s) = \phi_c + s + \frac{1}{2}\ddot{\lambda}(0)u(x)s^2 + O(s^3), \tag{3.8.25}$$

$$\mathbf{w}(s) = \frac{1}{2}\ddot{\lambda}(0)\mathbf{v}(x)s^2 + O(s^3), \tag{3.8.26}$$

$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \tag{3.8.27}$$

where $s \in (-\delta, \delta)$ for sufficiently small $\delta > 0$.

By analyzing the solution curve (3.8.25)-(3.8.27) as we did for the curve (3.7.12) in the proof of Theorem 3.3.3, we can conclude that solutions to the system (3.8.1) are non-unique. This completes the proof of Theorem 3.3.4.

3.9 Summary

We began in Section 3.2 by introducing our notation for function spaces and presenting the basic concepts from functional analysis and bifurcation theory that we used throughout this paper. In particular, we gave an outline of the Liapunov-Schmidt reduction that was the basis of our non-uniqueness arguments. Then in Section 3.3 we presented our main results, which consisted of the existence of a critical solution where the linearizations of our system

$$F((\phi,\mathbf{w}),\lambda) = \begin{bmatrix} -\Delta\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - a_{\mathbf{w}}\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5 \\ \mathbb{L}\mathbf{w} + \lambda b_\tau^a \phi^6 \end{bmatrix}, \tag{3.9.1}$$

developed a one-dimensional kernel and non-uniqueness results for solutions to $F((\phi,\mathbf{w}),0) = 0$ in both the CMC and non-CMC cases. We then set about proving these results in the following sections. In Section 3.4 we showed that in the CMC case there exists a critical density ρ_c for the operator

$$G(\phi,\lambda) = -\Delta\phi + a_R\phi + \lambda^2 a_\tau - a_{\mathbf{w}}\phi^{-7} - 2\pi\rho e^{-\lambda}\phi^5. \tag{3.9.2}$$

This density satisfied the property that if $|\lambda|$ was sufficiently small, then $\rho > \rho_c$ and $\lambda < 0$ implied that there was no solution to $G(\phi,\lambda) = 0$, and if $\rho \leq \rho_c$ and $\lambda \geq 0$ then there was a solution. This result provided the foundation in Section 3.5 for showing that the linearization of (3.9.1) developed a one-dimensional kernel. Then in Section 3.6 we briefly discussed the Fredholm properties of the linearized operators $D_X F((\phi_c,\mathbf{0}),0)$ and $D_\phi G(\phi_c,0)$ on the Banach spaces on which they are defined.

In Section 3.7 we proved the first of our non-uniqueness results. We showed that in the event that the mean curvature was constant, the decoupled system (3.9.2) exhibited non-uniqueness. This was indicated by the fact that the solution curve through the point $(\phi_c,0)$ had the form

$$\begin{aligned} \phi(s) &= \phi_c + s + O(s^2), \\ \lambda(s) &= \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \end{aligned} \tag{3.9.3}$$

which implied that a saddle-node bifurcation occurred at the point $(\phi_c,0)$. We were able to determine the explicit form of the solution curve (3.9.3) by applying a Liapunov-Schmidt to (3.9.2) at the point $(\phi_c,0)$, which was possible given that the operator $D_\phi G(\phi_c,0)$ had a one-dimensional kernel. Similarly, in Section 3.8 we showed that when the mean curvature τ was an arbitrary, continuously differentiable function, solutions to $F((\phi,\mathbf{w}),\lambda) = 0$ were non-unique. Again, this followed because we explicitly computed the solution curve through the point $((\phi_c, \mathbf{0}), 0)$. In Section 3.8 we found that the solution curve through $((\phi_c, \mathbf{0}), 0)$ had the form

$$\phi(s) = \phi_c + s + \frac{1}{2}\ddot{\lambda}(0)u(x)s^2 + O(s^3), \tag{3.9.4}$$

$$\mathbf{w}(s) = \frac{1}{2}\ddot{\lambda}(0)\mathbf{v}(x)s^2 + O(s^3), \tag{3.9.5}$$

$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \tag{3.9.6}$$

which we demonstrated by applying a Liapunov-Schmidt reduction to the system (3.9.1) at the point $((\phi_c, \mathbf{0}), 0)$. Again, this was possible because of our work in Section 3.4 where we showed that the linearization $D_X F((\phi_c, \mathbf{0}), 0)$ had a one-dimensional kernel.

The importance of these non-uniqueness results is that they demonstrate first and foremost that the conformal formulation with unscaled source terms is undesirable given that solutions for this formulation will not allow us to uniquely parametrize physical solutions to the Einstein constraint equations. Additionally, this paper helps build on the work of Walsh in [17] by expanding the understanding of how bifurcation techniques can be applied to the various conformal formulations of the constraint equations. This work is also interesting in that the analysis conducted here helps clarify the ideas of York et. al. in [15] by showing how terms with "the wrong sign" that contribute to the non-monotonicity (non-convexity of the corresponding energy) of the nonlinearity in the Hamiltonian constraint directly contribute to the non-uniqueness of solutions. Finally, it is hope of the authors that this work will also help to lay the foundation for future analysis of the uniqueness properties of the Conformal Thin Sandwich method and the far-from-CMC solution framework established in [8, 9].

3.10 Appendix

3.10.1 Banach Calculus and the Implicit Function Theorem

Here we give a brief review of some basic tools from functional analysis. The following results are presented without proof and are taken from [18]. We begin with some notation.

Suppose that X and Y are Banach spaces and $U \subset X$ is a neighborhood of 0. For a given map $f : U \subset X \to Y$, we say that

$$f(x) = o(\|x\|), \ \ x \to 0 \quad \text{iff} \quad r(x)/\|x\| \to 0 \ \text{ as } \ x \to 0.$$

We write $L(X, Y)$ for the class of continuous linear maps between the Banach spaces X and Y.

Definition 3.10.1. *Let $U \subset X$ be a neighborhood of x and suppose that X and Y are Banach spaces.*

(1) We say that a map $f : U \to Y$ is **F-differentiable** *or* **Fréchet differentiable** *at x iff there exists a map $T \in L(X, Y)$ such that*

$$f(x + h) - f(x) = Th + o(\|h\|), \quad \text{as } \ h \to 0,$$

for all h in some neighborhood of zero. If it exists, T is called the **F-derivative** *or* **Fréchet derivative** *of f and we define $f'(x) = T$. If f is Fréchet differentiable for all $x \in U$ we say that f is Fréchet differentiable in U. Finally, we define the* **F-differential** *at x to be $df(x; h) = f'(x)h$.*

(2) The map f is **G-differentiable** *or* **Gâteaux differentiable** *at x iff there exists a map $T \in L(X, Y)$ such that*

$$f(x + tk) - f(x) = tTk + o(t), \quad \text{as } \ t \to 0,$$

for all k with $\|k\| = 1$ and all real numbers t in some neighborhood of zero. If it exists, T is called the **G-derivative** *or* **Gâteaux derivative** *of f and we define $f'(x) = T$. If f is G-differential for all $x \in U$ we say that f is Gâteaux differentiable in U. The* **G-differential** *at x is defined to be $d_G f(x; h) = f'(x)h$.*

Remark 3.10.2. *Clearly if an operator is F-differentiable, then it must also be G-differentiable. Moreover, if the G-derivative f' exists in some neighborhood of x and f' is continuous at x, then $f'(x)$ is also the F-derivative. This fact is quite useful for* *computing F-derivatives given that G-derivatives are easier to compute. See [18] for a complete discussion.*

We view F-derivatives and G-derivatives as linear maps $f'(x) : U \to L(X, Y)$. More generally, we may consider higher order derivatives of f. For example, $f''(x) : U \to L(X, L(X, Y))$ is a bilinear form. We now state some basic properties of F-derivatives. All of the following properties also hold for G-derivatives.

The Fréchet derivative satisfies many of the usual properties that we are accustomed to by doing calculus in $\mathbb{R}^n$. For example, we have the chain rule.

Proposition 3.10.3 (Chain Rule)**.** *Suppose that X, Y and Z are Banach spaces and assume that $f : U \subset X \to Y$ and $g : V \subset Y \to Z$ are differentiable on U and V resp. and that $f(U) \subset V$. Then the function $H(x) = g \circ f$, i.e. $H(x) = g(f(x))$, is differentiable where*

$$H'(x) = g'(f(x))f'(x)$$

where we write $g'(f(x))f'(x)$ for $g'(f(x)) \circ f'(x)$.

Given an operator $f : X \times Y \to Z$, we can also consider the partial derivative of f with respect to either x or y. If we fix the variable y and define $g(x) = f(x, y) : X \to Z$ and $g(x)$ is Fréchet differentiable at x, then the **partial derivative** of f with respect to x at (x, y) is $f_x(x, y) = g'(x)$. We can a make a similar definition for $f_y(x, y)$. Finally, we observe that we can express the F-differential of $f'(x, y)$ in terms of the partials by using the following formula:

$$f'(x, y)(h, k) = f_x(x, y)h + f_y(x, y)k. \tag{3.10.1}$$

We have the following relationship between the partial derivatives and the Fréchet derivative.

Proposition 3.10.4. *Suppose that $f : X \times Y \to Z$ is F-differentiable at (x, y). Then the partial F-derivatives f_x and f_y exist at (x, y) and they satisfy* (3.10.1)*. Moreover, if f_x and f_y both exist and are continuous in a neighborhood of (x, y) then $f'(x, y)$ exists as an F-derivative and* (3.10.1) *holds.*

Implicit Function Theorem

Suppose that $F : U \times V \to Z$ is a mapping with $U \subset X, V \subset Y$ and X, Y, Z are real Banach spaces. The **Implicit Function Theorem** is an extremely important tool in analyzing the nonlinear problem

$$F(x, y) = 0. \tag{3.10.2}$$

We present the statement of the Theorem here, the form of which is taken from [12]. For a proof see [18, 6].

Theorem 3.10.5. *Let* (3.10.2) *have a solution* $(x_0, y_0) \in U \times V$ *such that the Fréchet derivative of* F *with respect to* x *at* (x_0, y_0) *is bijective:*

$$\begin{aligned} &F(x_0, y_0) = 0, \\ &D_x F(x_0, y_0) :\to Z \quad \textit{is bounded (continuous)} \\ &\textit{with bounded inverse.} \end{aligned} \tag{3.10.3}$$

Assume also that F *and* $D_x F$ *are continuous:*

$$\begin{aligned} &F \in C(U \times V, Z), \\ &D_x F \in C(U \times V, L(X, Z)), \quad \textit{where } L(X, Z) \\ &\textit{denotes the Banach space of bounded linear operators} \\ &\textit{from } X \textit{ into } Z \textit{ endowed with the operator norm.} \end{aligned} \tag{3.10.4}$$

Then there is a neighborhood $U_1 \times V_1 \subset U \times V$ *of* (x_0, y_0) *and a mapping* $f : V_1 \to U_1 \subset X$ *such that*

$$\begin{aligned} &f(y_0) = x_0, \\ &F(f(y), y) = 0 \quad \textit{for all } y \in V_1. \end{aligned} \tag{3.10.5}$$

Furthermore, $f \in C(V_1, X)$ *and every solution to* (3.10.2) *in* $U_1 \times V_1$ *is of the form* $(f(y), y)$*. Finally, if* F *is* k*-times differentiable, then* f *is* k*-times differentiable.*

3.10.2 Elliptic PDE tools

Here we assemble some useful tools for working with nonlinear elliptic partial differential equations. Throughout this section we will assume that $\mathcal{M}$ is a closed manifold with a smooth SPD metric g_{ab} and that Δ is the associated Laplace-Beltrami operator.

Maximum Principle

In this section we present a version of the maximum principle on closed manifolds. The following result is well-known, but we present it here for completeness.

Theorem 3.10.6. *Let $u \in C^2(\mathcal{M})$. Then if*

$$\Delta u \geq 0 \quad or \quad \Delta u = 0 \quad or \quad \Delta u \leq 0, \tag{3.10.6}$$

then u must be a constant. In particular, the problem

$$\Delta u = f(x, u),$$

has no solution if $f(x,u) \geq 0$ or $f(x,u) \leq 0$ unless $f(x,u) \equiv 0$.

Proof. See [16] for a proof. □

Method of Sub- and Super-Solutions

Here we present a theorem that provides a method to solve an elliptic problem of the form

$$Lu = f(x, u), \tag{3.10.7}$$

where

$$Lu = -\Delta u + c(x)u, \quad c(x) \in C(\mathcal{M} \times \mathbb{R}), \quad c(x) > 0 \tag{3.10.8}$$

and the function $f(x, y)$ is nonlinear in the variable y.

Theorem 3.10.7. *Suppose that* $f : \mathcal{M} \times \mathbb{R}^+ \to \mathbb{R}$ *is in* $C^k(\mathcal{M} \times \mathbb{R}^+)$*. Let* L *be of the form* (3.10.8) *and suppose that there exist functions* $u_- : \mathcal{M} \to \mathbb{R}$ *and* $u_+ : \mathcal{M} \to \mathbb{R}$ *such that the following hold:*

1. $u_-, u_+ \in C^k(\mathcal{M})$,
2. $0 < u_-(x) \leq u_+(x) \quad \forall x \in \mathcal{M}$,
3. $Lu_- \leq f(x, u_-)$,
4. $Lu_+ \geq f(x, u_+)$.

Then there exists a solution u *to*

$$Lu = f(x,u) \quad \text{on } \mathcal{M}, \tag{3.10.9}$$

such that

(i) $u \in C^k(\mathcal{M})$,

(ii) $u_-(x) \leq u(x) \leq u_+(x)$.

Proof. See [10] for a proof. □

Fredholm Properties and Liapunov-Schmidt Decompositions for Elliptic Operators

In this appendix we discuss the Fredholm properties of linear elliptic operators on a closed manifold. We use these properties to form Liapunov-Schmidt decompositions for a given elliptic operator L between certain Banach spaces. The following treatment is taken from [12].

Let $u \in C^{2,\alpha}(\mathcal{M})$ and define the elliptic operator $L : C^{2,\alpha}(\mathcal{M}) \to C^{0,\alpha}(\mathcal{M})$ by

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x)u, \tag{3.10.10}$$

where a_{ij}, b_i and c are smooth, bounded coefficients where $a_{ij} = a_{ji}$. We also assume that the a_{ij} satisfy the standard elliptic property

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq d\|\xi\|^2,$$

where $d > 0$ is constant and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

The operator (3.10.10) has an associated bilinear form

$$B(u,u) = \langle Lu, u\rangle = \langle u, L^*u\rangle, \tag{3.10.11}$$

where $\langle\cdot,\cdot\rangle$ is the $L^2(\mathcal{M})$ inner product and L^* is the L^2-adjoint defined by

$$L^*u = -\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i})_{x_j} - \sum_{i=1}^{n}(b_i(x)u)_{x_i} + c(x)u. \tag{3.10.12}$$

Using the bilinear form $B(u,u)$, the elliptic operator (3.10.10) defines an elliptic operator

$$L : L^2(\mathcal{M}) \to L^2(\mathcal{M}), \quad \text{with domain of definition } D(L) = H^2(\mathcal{M}). \tag{3.10.13}$$

It is a standard argument in linear elliptic PDE to show that there exists a $c > 0$ such the operator $L + cI : H^2(\mathcal{M}) \to L^2(\mathcal{M})$ is bounded and bijective. In particular, one shows that there exists a $c > 0$ such that the associated bilinear form $B(u,u) + c\|u\|_2$ is coercive and then applies the Lax-Milgram Theorem to conclude that there exists a unique weak solution $u \in H^1(\mathcal{M})$ to

$$Lu - cu = f \quad \text{for every } f \in L^2(\mathcal{M}).$$

Standard elliptic regularity theory implies that $u \in H^2(\mathcal{M})$ and the norm $\|\cdot\|_{2,2}$ makes $D(L)$ a Hilbert space. An application of the Open Mapping Theorem (Bounded Inverse Theorem) then implies that

$$(L + cI)^{-1} : L^2(\mathcal{M}) \to D(L),$$

is continuous. This implies that the operator $L + cI$ is closed and that the operator $(L + cI) - cI = L$ is closed. In addition, the operator

$$K_c = (L + cI)^{-1} \in L(L^2(\mathcal{M}), L^2(\mathcal{M})) \quad \text{is compact}$$

given that the embedding $H^2(\mathcal{M}) \subset L^2(\mathcal{M})$ is compact. For $f \in L^2(\mathcal{M})$, we have the equivalence

$$Lu = f, \quad u \in H^2(\mathcal{M}) \Leftrightarrow \tag{3.10.14}$$

$$u - cK_c u = K_c f, \quad u \in L^2(\mathcal{M}). \tag{3.10.15}$$

Riesz-Schauder theory implies that $(I - cK_c)$ is a Fredholm operator and the equivalence (3.10.14) implies that L is a Fredholm operator.

Because L is a Fredholm operator of index zero, we have that $R(L)$ is closed. Therefore we may write

$$L^2(\mathcal{M}) = R(L) \oplus Z_0,$$

where $Z_0 = R(L)^{\perp}$ is the orthogonal complement with respect to the L^2-inner product. Because $D(L)$ is dense in $L^2(\mathcal{M})$ and L is closed, may apply the Closed Range Theorem to conclude that

$$R(L) = \{f \in L^2(\mathcal{M}) \mid \langle f, u \rangle = 0 \quad \text{for all } u \in N(L^*)\} \tag{3.10.16}$$

and that $Z_0 = N(L^*)$, where $L^* : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is induced by (3.10.12). Therefore

$$L^2(\mathcal{M}) = R(L) \oplus N(L^*),$$

and if $D(L^*) = H^2(\mathcal{M})$, the above arguments imply that L^* is Fredholm operator. So we have the following decomposition of the codomain of L^*:

$$L^2(\mathcal{M}) = R(L^*) \oplus N(L). \tag{3.10.17}$$

Finally, given that $N(L) \subset D(L) = H^2(\mathcal{M}) \subset L^2(\mathcal{M})$, the decomposition (3.10.17) allows us to obtain the following Liapunov-Schmidt decomposition for the linear problem $L : H^2(\mathcal{M}) \to L^2(\mathcal{M})$:

$$H^2(\mathcal{M}) = N(L) \oplus (R(L^*) \cap H^2(\mathcal{M})), \tag{3.10.18}$$

$$L^2(\mathcal{M}) = R(L) \oplus N(L^*). \tag{3.10.19}$$

Now we observe that the Fredholm properties of linear elliptic operators derived on Hilbert spaces hold for subspaces that are only Banach spaces. We then use these Fredholm properties to derive Liapunov-Schmidt decompositions for these Banach spaces.

Suppose that the Banach space $Z \subset L^2(\mathcal{M})$ is continuously embedded and that the domain of definition $X \subset Z$ with a given norm is a Banach space that satisfies the following conditions:

$$L : X \to Z \quad \text{is continuous ,} \tag{3.10.20}$$

$$Lu = f \quad \text{for } u \in D(L) = H^2(\mathcal{M}), f \in Z \Rightarrow u \in X.$$

Equation (3.10.20) is an elliptic regularity condition and is satisfied for a variety of spaces, most notably $X = W^{2,p}(\mathcal{M}), Z = L^p(\mathcal{M})$ and $X = C^{2,\alpha}(\mathcal{M}), Z = C^{0,\alpha}(\mathcal{M})$ with the standard norms. Then for X and Z satisfying (3.10.18) and (3.10.20) we have that

$$N(L) = N(L|_Z) \subset X, \quad \text{and} \tag{3.10.21}$$

$$R(L) \cap Z = R(L|_Z) \quad \text{is closed in } Z, \tag{3.10.22}$$

given that $Z \subset L^2(\mathcal{M})$ is continuously embedded and $R(L)$ is closed in $L^2(\mathcal{M})$. The ellipticity property (3.10.20) also holds for the adjoint L^* and implies that

$$N(L^*) \subset X, \quad \text{where } D(L^*) = D(L) = X.$$

Applying the decomposition (3.10.19), we may write any $z \in Z$ as

$$\begin{aligned} z &= Lu + u^*, \quad \text{where } u \in D(L), u^* \in N(L^*), \\ Lu &= z - u^* \in Z \Rightarrow u \in X, \quad \text{therefore} \\ Z &= R(L|_Z) \oplus N(L^*). \end{aligned} \tag{3.10.23}$$

Finally, we have that dim$N(L|_Z) =$ dim$N(L) =$ dim$N(L^*)$ and that

$$L : X \to Z, \;\; X = D(L|_Z), \;\; \text{is a Fredholm operator of index zero.} \tag{3.10.24}$$

The decomposition (3.10.18) then implies that

$$X = N(L|_Z) \oplus (R(L^*) \cap X), \tag{3.10.25}$$

and so (3.10.23) and (3.10.25) constitute a Liapunov-Schmidt decomposition of the spaces X and Z with respect to a given linear, elliptic operator L.

Remark 3.10.8. *As noted in [12], we may regard the spaces $W^{2,p}(\mathcal{M}) \subset L^p(\mathcal{M}) \subset L^2(\mathcal{M})$ for $p > 2$, and we can then apply the above discussion to conclude that a linear elliptic operator $L : W^{2,p}(\mathcal{M}) \to L^p(\mathcal{M})$ is Fredholm and use this fact to obtain a Liapunov-Schmidt decomposition of $X = W^{2,p}(\mathcal{M})$ and $Z = L^p(\mathcal{M})$. Similarly, $C^{2,\alpha}(\mathcal{M}) \subset C^{0,\alpha}(\mathcal{M}) \subset L^2(\mathcal{M})$ for $\alpha \in (0,1)$, so $L : C^{2,\alpha}(\mathcal{M}) \to C^{0,\alpha}(\mathcal{M})$ is Fredholm and we may also obtain a Liapunov-Schmidt decomposition of $X = C^{2,\alpha}(\mathcal{M})$ and $Z = C^{0,\alpha}(\mathcal{M})$ using* (3.10.23) *and* (3.10.25).

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