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Journal

Ergodic Theory and Dynamical Systems, 39(4)

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Publication Date

2019-04-01

Data Availability

The data associated with this publication are available upon request.

Peer reviewed

Zero sets of Lie algebras of analytic vector fields on real and complex two-dimensional manifolds

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Abstract. Let M be an analytic connected 2-manifold with empty boundary, over the ground field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let Y and X denote differentiable vector fields on M. We say that Y tracks X if $[Y, X] = f X$ for some continuous function $f : M \to \mathbb{F}$. A subset K of the zero set $\mathsf{Z}(X)$ is an essential block for X if it is non-empty, compact and open in $\mathsf{Z}(X)$, and the Poincaré–Hopf index $i_K(X)$ is non-zero. Let $\mathcal{G}$ be a finite-dimensional Lie algebra of analytic vector fields that tracks a non-trivial analytic vector field X. Let $K \subset \mathsf{Z}(X)$ be an essential block. Assume that if M is complex and $i_K(X)$ is a positive even integer, no quotient of $\mathcal{G}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Then $\mathcal{G}$ has a zero in K (main result). As a consequence, if X and Y are analytic, X is non-trivial, and Y tracks X, then every essential component of $\mathsf{Z}(X)$ meets $\mathsf{Z}(Y)$. Fixed-point theorems for certain types of transformation groups are proved. Several illustrative examples are given.

Contents

1. *Introduction*

A fundamental issue in dynamical systems is deciding whether a vector field on a manifold has a zero. When the manifold is compact with empty boundary and non-vanishing Euler characteristic, a positive answer is given by the celebrated Poincaré–Hopf theorem.

Determining whether a set of vector fields have a common zero is more challenging. This problem is closely related to the question of which non-compact transformation groups have fixed points (see [**1**, **4**, **5**, **10**, **14**, **24**, **30**, **33**, **35**–**37**]).

Throughout this paper manifolds are real or complex with the corresponding ground field denoted by $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Each manifold N has a distinguished analytic structure, holomorphic in the complex case, and empty boundary ∂N unless the contrary is indicated. Objects associated to N are assumed analytic unless the contrary is mentioned.

The dimension of N over $\mathbb{F}$ is denoted by $\dim_{\mathbb{F}} N$ or by $\dim N$ when the ground field is clear from the context. $\mathcal{V}(N)$ is the vector space over $\mathbb{F}$ of continuous vector fields on N, with the compact open topology, while $\mathcal{V}^k(N)$, $k \in \{1, \ldots, \infty\}$ (respectively, $\mathcal{V}^\omega(N)$) denotes the subalgebra of vector fields that are C^k-differentiable (respectively, analytic over $\mathbb{F}$). Of course $\mathcal{V}^1(N) = \mathcal{V}^\omega(N)$ when N is complex.

Consider a subset $\mathcal{A} \subset \mathcal{V}(N)$. The set of their common zeros is $\mathsf{Z}(\mathcal{A}) := \bigcap_{X \in \mathcal{A}} \mathsf{Z}(X)$, where $\mathsf{Z}(X)$ is the set of zeros of X. A set $S \subset P$ is *X-invariant* if it contains the orbits under X of its points. When this holds for all X in $\mathcal{A}$, we say that S is *$\mathcal{A}$-invariant*.

Suppose that $X \in \mathcal{V}(N)$, $\partial N =$ (the empty set), $U \subset N$ is open with compact closure $\overline{U}$ and $\mathsf{Z}(X) \cap (\overline{U} \setminus U) = \varnothing$. The *index* of X on U, denoted by $\mathsf{i}(X, U) \in \mathbb{Z}$ (the group of integers), is defined as the Poincaré–Hopf index of any sufficiently close approximation $X' \in \mathcal{V}(U)$ to $X|U$ (in the compact open topology) such that $\mathsf{Z}(X')$ is finite. Equivalently: $\mathsf{i}(X, U)$ is the intersection number of $X|U$ with the zero section of the tangent bundle (see Bonatti [**3**]). This number is independent of the approximation, and is stable under perturbation of X and replacement of U by smaller open sets containing $\mathsf{Z}(X) \cap U$.

When X is C^1 and generates the local flow ϕ on M, for sufficiently small $t > 0$ the index $\mathsf{i}(X, U)$ equals the fixed-point index $I(\phi_t|U)$ defined by Dold [**7**].

A compact set $K \subset \mathsf{Z}(X)$ is a *block* of zeros for X (or an *X-block*) provided K is non-empty and relatively open in $\mathsf{Z}(X)$, that is to say, provided K is non-empty and $\mathsf{Z}(X) \setminus K$ is closed in M. Observe that a non-empty compact $K \subset \mathsf{Z}(X)$ is an *X*-block if and only if it has a precompact open neighborhood $U \subset N$, called *isolating* for (X, K), such that $\mathsf{Z}(X) \cap \overline{U} = K$ (manifolds are normal spaces). This implies that $\mathsf{i}(X, U)$ is determined by X and K, and does not depend on the choice of U. The *index of X at K* is $\mathsf{i}_K(X) := \mathsf{i}(X, U)$. The *X*-block K is *essential* provided $\mathsf{i}_K(X) \neq 0$, which implies that $K \neq \varnothing$.

The notions of 'block' and 'index' are well defined for holomorphic vector fields on a complex manifold, since these are also vector fields (sections of the tangent bundle) on the underlying real manifold.

If N is compact with empty boundary, it is isolating for every vector field on N and its set of zeros.

THEOREM. (Poincaré–Hopf [**18**, **31**]) *Suppose that N is compact, $\partial N = \varnothing$ and $X \in \mathcal{V}(N)$. Then* $\mathsf{i}_{\mathsf{Z}(X)}(X) = \mathsf{i}(X, N) = \chi(N)$.

For calculations of the index in more general settings, see Morse [**28**], Pugh [**32**], Gottlieb [**8**] and Jubin [**21**].

This paper was inspired by a remarkable result of Bonatti, which does not require compactness of N†.

THEOREM. (Bonatti [**3**]) *Assume that N is a real manifold of dimension ≤ 4 and X, Y are analytic vector fields on N such that $[X, Y] = 0$. Then $\mathsf{Z}(Y)$ meets every essential X-block‡.*

Here is the fundamental new concept in this paper.

> *Y tracks X* provided Y and X are C^1 vector fields on a real or complex manifold N, and $[Y, X] = fX$ for some continuous function $f \colon N \to \mathbb{F}$, referred to as the *tracking function*.
>
> A set $\mathcal{A}$ of vector fields tracks X provided each element of $\mathcal{A}$ tracks X.

Suppose that X is non-trivial on each connected component of N. In the complex case it it is easily seen that the tracking function f is holomorphic. But in the real case f need not even be smooth (see §4).

Example. If X spans a one-dimensional ideal of a Lie algebra $\mathcal{G} \subset \mathcal{V}^1(N)$ (meaning a Lie subalgebra), then $\mathcal{G}$ tracks X. But the converse does not always hold, even for finite-dimensional $\mathcal{G}$ (see Example 3.1).

In the rest of this section we postulate that:
- M is a real or complex two-dimensional connected manifold with empty boundary;
- $X \in \mathcal{V}^\omega(M)$ is non-trivial;
- K is an essential X-block;
- $\mathcal{G} \subset \mathcal{V}^\omega(M)$ is a Lie algebra that is finite dimensional over the ground field;
- $\mathcal{G}$ tracks X.

This is our main result.

THEOREM 1.1. (Main) *Assume that:*
(*)   *if M is complex and $\mathfrak{i}_K(X)$ is positive and even, no quotient of $\mathcal{G}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.*
Then $\mathsf{Z}(\mathcal{G}) \cap K \neq \varnothing$.

The proof is given in §6.

---

† 'The demonstration of this result involves a beautiful and quite difficult local study of the set of zeros of X, as an analytic Y-invariant set.': see Molino [**25**].
‡ In [**3**], this is stated for $\dim(N) = 3$ or 4. If $\dim(N) = 2$, the same conclusion is obtained by applying the three-dimensional case to the vector fields $X \times t\partial/\partial t$, $Y \times t\partial/\partial t$ on $N \times \mathbb{R}$.

$Z(\mathcal{G}) \cap K$ can be empty when hypothesis (*) is omitted (Example 3.1 and Theorem 2.6). But if M is compact and connected with non-zero Euler characteristic, there are strict limitations on M and $\mathcal{G}$, as shown in Theorem 2.6 and Remark 6.7. Summarizing these two results, we have the following theorem.

THEOREM 1.2. *Assume that M is compact and complex, $\chi(M) \neq 0$ and $Z(\mathcal{G}) \cap Z(X) = \emptyset$. Then M is a holomorphic $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$ and $\mathcal{G}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{gl}(2, \mathbb{C})$ or the product of $\mathfrak{sl}(2, \mathbb{C})$ with the affine algebra of $\mathbb{C}$.*

The first step in the proof of Theorem 1.1 is the special case $\mathcal{G} = \mathbb{F}$ (Lemma 6.1). This result is interesting in itself.

COROLLARY 1.3. *If $Y \in \mathcal{V}^\omega(M)$ tracks X, then $Z(Y) \cap K \neq \emptyset$.*

Analyticity is crucial here: there are C^∞ counterexamples (see [1, 24, 30, 39]). Nevertheless, Corollary 1.3 holds if X and Y are C^1 and $[X, Y] = 0$ (see Bonatti [2, Proposition 11])†.

COROLLARY 1.4. *If $\mathcal{G} \subset \mathcal{V}^\omega(M)$ is a solvable Lie algebra tracking X, then $Z(\mathcal{G}) \cap K \neq \emptyset$.*

Proof. This follows from Theorem 1.1 because solvability of $\mathcal{G}$ validates hypothesis (*).
                                                                                          □

THEOREM 1.5. *Consider a compact complex 2-manifold M with $\chi(M) \neq 0$. If $\mathcal{G} \subset \mathcal{V}^\omega(M)$ is a solvable Lie algebra, then $Z(\mathcal{G}) \neq \emptyset$.*

Proof. $\mathcal{G}$ is isomorphic to a Lie algebra of upper triangular matrices by Lie's theorem [20]. If $\mathcal{G} \neq \{0\}$, some $X \in \mathcal{G}$ spans a one-dimensional ideal and is thus tracked by $\mathcal{G}$. Since $Z(X)$ is an essential X-block by the Poincaré–Hopf theorem, the conclusion follows from Theorem 1.1 applied to the essential X-block $K := Z(X)$.   □

Remark 1.6. The analog of Theorem 1.5 for real manifolds is not true: the vector fields

$$\partial/\partial x, \quad \partial/\partial y, \quad -y(\partial/\partial x) + x(\partial/\partial y)$$

on $\mathbb{R}^2$ extend over the real projective plane $\mathbb{R}P^2$ to span a three-dimensional solvable Lie algebra $\mathcal{G} \subset \mathcal{V}^\omega(\mathbb{R}P^2)$ with $Z(\mathcal{G}) = \emptyset$. But the real analog holds provided $\mathcal{G}$ is *supersolvable*: faithfully represented by upper triangular real matrices (see Hirsch and Weinstein [17]).

1.1. *Lie group actions.*   Let G denote a Lie group over the same ground field $\mathbb{F}$ as M. An *action* of G on M is an $\mathbb{F}$-analytic map

$$\alpha : G \times M \to M$$

such that the map

$$g^\alpha : p \to \alpha(g, p)$$

† This can also be proved using the methods of Molino and Turiel [26, 27].

is a homomorphism from G to the group of $\mathbb{F}$-analytic diffeomorphisms of M (see Palais [**29**]). This action is also denoted by (α, G, M) or simply by (G, M). Its *fixed-point set* is

$$\mathsf{Fix}(\alpha) = \mathsf{Fix}((G, M)) := \{p \in M : g^{\alpha}(p) = p \quad (g \in G)\}.$$

The action is *effective* if its kernel is trivial, and *almost effective* if its kernel is discrete.

An analytic action (α, G, M) gives rise to a homomorphism d_{α} from the Lie algebra $\mathcal{G}$ of G onto a subalgebra $\mathcal{G}^{\alpha} \subset \mathcal{V}^{\omega}(M)$; this is the *infinitesimal action* determined by α. Note that d_{α} is injective if and only if α is almost effective. When G is connected, $\mathsf{Fix}(\alpha) = \mathsf{Z}(\mathcal{G}^{\alpha})$.

- *In the next two results G is connected and the action (α, G, M) is analytic.*

THEOREM 1.7. *Assume that:*
(a)   *M is compact and $\chi(M) \neq 0$;*
(b)   *G contains a one-dimensional normal subgroup;*
(c)   *if M is complex and $\chi(M)$ is positive and even, then the Lie algebra of G does not have $\mathfrak{sl}(2, \mathbb{C})$ as a quotient;*
(d)   *the action (α, G, M) is almost effective.*
Then $\mathsf{Fix}(\alpha) \neq \varnothing$.

Proof. By Hypothesis (b), some $X \in \mathcal{G}^{\alpha}$ spans a one-dimensional ideal. Because $\mathcal{G}^{\alpha}$ tracks X, and the X-block $\mathsf{Z}(X)$ is essential by Poincaré–Hopf, the conclusion follows from Theorem 1.1. $\square$

Applying Theorem 1.5 to the Lie algebra $\mathcal{G}^{\alpha} \subset \mathcal{V}^{\omega}(M)$ yields the following corollary.

COROLLARY 1.8. *Assume that:*
(i)   *M is complex and compact, and $\chi(M) \neq 0$;*
(ii)   *G is solvable.*
Then $\mathsf{Fix}(\alpha) \neq \varnothing$.

The real analog of Corollary 1.8 is not generally true (see Remark 1.6), but for supersolvable Lie groups it follows from Theorem 1.7 and was first proved by Hirsch and Weinstein [**17**].

Corollary 1.8 is reminiscent of the celebrated fixed-point theorem of Borel [**5**, **19**] for algebraic actions of solvable algebraic groups on projective varieties over an algebraically closed field, and its extension to holomorphic actions on Kaehler manifolds by Sommese [**35**]. While these theorems have strong algebraic hypotheses, they make no assumptions on dimensions or Euler characteristics.

1.2.   *Earlier results.*   Turiel [**38**] listed all the Lie groups having fixed-point-free analytic actions on compact connected real surfaces of non-zero Euler characteristic.

The existence of fixed points for continuous actions on compact real surfaces with non-zero Euler characteristic was proved by Lima [**24**] for the group $\mathbb{R}^{n}$. This was extended by Plante [**30**] to connected nilpotent Lie groups and by Hirsch [**12**] to nilpotent local actions. Lima [**24**] and Plante proved that every compact surface supports a continuous fixed-point-free action by the orientation-preserving affine group $Aff_{+}(\mathbb{R})$. It belongs to

the folklore that this kind of action can be taken smooth (see Belliart [**1**]; see Turiel [**39**] for an elementary construction).

Related results are in the articles [**4**, **10**–**14**, **33**, **36**, **37**]. Preliminary versions of this paper were published as [**15**, **16**].

2. *Lie algebras on compact connected complex 2-manifolds without common zeros*

Our purpose in this section is to describe all the compact cases in which Theorem 1.1 fails when hypothesis (*) is deleted. As we will see, essentially there are only two models.

Before constructing them, let us recall some well-known facts.

One starts with the notion of index transverse to a compact submanifold. Consider a compact one-codimensional submanifold P of a real or complex manifold N and a vector field X on N (our objects are supposed to be analytic, but in the real case smooth suffices). Assume the existence of an open set $P \subset A \subset N$ such that $Z(X) \cap A = P$.

Now consider $p \in P$. Suppose that there exists a one-dimensional foliation $\mathcal{F}$ defined on an open set $p \in B \subset A$ such that:

- X is tangent to $\mathcal{F}$, that is to say, $X(x) \in T_x\mathcal{F}$ for each $x \in B$, where $T\mathcal{F}$ denotes the involutive distribution associated to $\mathcal{F}$;
- $T_q N = T_q \mathcal{F} \oplus T_q P$ for every $q \in B \cap P$.

Note that if $\mathcal{F}$ exists, it is unique because this foliation is defined by X on $B - P$.

Let L_p denote the leaf of p. Since L_p is a leaf of $\mathcal{F}$, and so a submanifold that is transverse to P, $X|L_p$ may be seen as an analytic vector field on the manifold L_p whose singularities are isolated.

By definition, *the index of X at p transverse to P* is that of p as a singularity of $X|L_p$ in L_p.

Stability of the Poincaré–Hopf index can be used to prove that:

- the index of $X|L_p$ at p is independent of B and locally constant;
- when P is connected and the index is defined at every point, it is also independent of p and we call it *the index of X transverse to P*.

Recall that the sphere S^2 admits a unique complex structure up diffeomorphism, usually represented by $\mathbb{C}P^1$. Besides, the group of diffeomorphisms (biholomorphic maps) of $\mathbb{C}P^1$ is the projective group $PGL(2, \mathbb{C})$, which is the quotient of $SL(2, \mathbb{C})$ by $\{\mathbf{I}, -\mathbf{I}\}$, where $\mathbf{I} \in SL(2, \mathbb{C})$ denotes the identity. Thus, $\mathcal{V}^\omega(\mathbb{C}P^1)$ equals the algebra of projective vector fields, which is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Note that the structure group of holomorphic fibre bundles over $\mathbb{C}P^1$ with fibre $\mathbb{C}P^1$ is $PGL(2, \mathbb{C})$. Since $\mathbb{Z}_2$ is the fundamental group of $PGL(2, \mathbb{C})$, from the real point of view, there only exist two fibre bundles over $\mathbb{C}P^1$ with fibre $\mathbb{C}P^1$; more exactly, $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P}^2$, which is the result of blowing up a point of $\mathbb{C}P^2$.

Actually in the C^∞-category there are only two fibre bundles over S^2 with fibre S^2 ($\mathrm{Diff}_+(S^2)$ strongly retracts onto $SO(3)$ (see Smale [**34**])).

From the complex viewpoint things are different because a holomorphic map from an open set $A \subset \mathbb{C}$, which includes S^1, into $SL(2, \mathbb{C})$ extends to D^2 in the C^∞-category but not always like a holomorphic map.

Model 2.1. ($\mathsf{Z}(X)$ is connected) On $\mathbb{C}P^1$ consider the projective vector field $\widetilde{X}$, which on $\mathbb{C} \subset \mathbb{C}P^1$ is written as $z^2 \partial/\partial z$. Set $M = \mathbb{C}P^1 \times \mathbb{C}P^1$; let π_1, π_2 be the canonical projections. On the other hand, let $\mathcal{G} \subset \mathcal{V}^\omega(M)$ be the Lie algebra tangent to the first factor and isomorphic by $(\pi_1)_*$ to $\mathcal{V}^\omega(\mathbb{C}P^1)$, and X the vector field on M tangent to the second factor whose projection by $(\pi_2)_*$ equals $a\widetilde{X}$, $a \in \mathbb{C} \setminus \{0\}$. Clearly, $[X, \mathcal{G}] = 0$.

Moreover, $\mathsf{Z}(\mathcal{G}) = \varnothing$, so $\mathsf{Z}(X) \cap \mathsf{Z}(\mathcal{G}) = \varnothing$, while $\mathsf{Z}(X)$ is a 1-submanifold diffeomorphic to $\mathbb{C}P^1$ and transversely to it the index of X equals 2.

Remark 2.2. (A compactification construction) Since $\mathbb{C}^k \subset \mathbb{C}P^k$, any linear transformation of $\mathbb{C}^k$ is the restriction of a projective transformation of $\mathbb{C}P^k$ and $\mathrm{GL}(k, \mathbb{C})$ can be regarded as a subgroup of $P\mathrm{GL}(k + 1, \mathbb{C})$ in a natural way. Now consider a holomorphic line bundle $\pi : E \to N$. Completing each fibre with its own infinity point gives rise to a new holomorphic fibre bundle $\pi : Q \to N$ with fibre $\mathbb{C}P^1$ (for sake of simplicity the projection map is still denoted π).

More exactly, if $\{U_\lambda\}_{\lambda \in L}$ is a trivializing open covering of N with transition functions $g_{\lambda\mu} : U_\lambda \cap U_\mu \to \mathrm{GL}(1, \mathbb{C})$ associated to $\pi : E \to N$, then at the same time it is associated to $\pi : Q \to N$ if $\mathrm{GL}(1, \mathbb{C})$ is seen as a subgroup of $P\mathrm{GL}(2, \mathbb{C})$ and, therefore, every $g_{\lambda\mu}$ takes its values in $P\mathrm{GL}(2, \mathbb{C})$.

Let Q_0 be (the image of) the zero section of E and set $Q_\infty : = Q \setminus E$. Clearly, Q_∞ is a complex submanifold, which we will call the infinity section, and $\pi : Q_\infty \to N$ is a diffeomorphism.

The radial vector field R of E extends to a vector field on Q still called R since the diffeomorphism $z \in \mathbb{C} \setminus \{0\} \to z^{-1} \in \mathbb{C} \setminus \{0\}$ transforms $z(\partial/\partial z)$ into $-z(\partial/\partial z)$. Besides, $\mathsf{Z}(R) = Q_0 \cup Q_\infty$ and transversely to Q_0 and Q_∞ the index of R equals 1.

Set $E' : = Q - Q_0$. As any diffeomorphism $\rho : \mathbb{C} \to \mathbb{C}$ which preserves $-z(\partial/\partial z)$ is a linear automorphism, $\pi : E' \to N$ has a natural structure of a holomorphic line bundle with zero section Q_∞ and radial vector field $-R$. With respect to the open covering of N given before, its transition functions $g'_{\lambda\mu}$ are $g'_{\lambda\mu} = g^{-1}_{\lambda\mu}$. Thus, $c_1(E') = -c_1(E)$, where c_1 denotes the first Chern class. Moreover, if one adds the infinity point to each fibre of E', one obtains $\pi : Q \to N$ again but this time Q_0 is the infinity section.

Of course the constructions above do not depend on the trivializing open covering of N.

Remark 2.3. Let $\pi : E \to N$ be a holomorphic vector bundle and R its radial vector field. Consider a diffeomorphism $f : E \to E$ that preserves R. Then f maps fibres into fibres, which induces a second diffeomorphism $\tilde{f} : N \to N$ such that $\tilde{f} \circ \pi = \pi \circ f$ and every $f : \pi^{-1}(q) \to \pi^{-1}(\tilde{f}(q))$, $q \in N$, is a linear isomorphism.

Indeed, f has to map $\mathsf{Z}(R)$, that is, the zero section of E, into itself. On the other hand, each fibre $\pi^{-1}(q)$ is the set of all the points of E that have the zero of $\pi^{-1}(q)$ as α-limit.

Moreover, if $\pi : E \to N$ is a holomorphic line bundle and $\pi : Q \to N$ its compactification given in Remark 2.2, then f extends to a diffeomorphism of Q (obvious since each $f : \pi^{-1}(q) \to \pi^{-1}(\tilde{f}(q))$ is linear and so projective). Thus, any complete vector field Y on E such that $[Y, R] = 0$ extends to a vector field on Q.

Indeed, let Φ_t be the flow of Y. Then every Φ_t preserves R, so maps fibres into fibres, which implies that Y is foliate with respect to fibres. Moreover, Y has to be tangent to $\mathsf{Z}(R)$, that is, to the zero section. Therefore, if U is a trivializing open set of N and one

identifies $\pi^{-1}(U)$ with $U \times \mathbb{C}$ endowed with variables (y, z), one has

$$Y(y, z) = \widetilde{Y}(y) + \varphi(y, z) \cdot z \frac{\partial}{\partial z},$$

where $\widetilde{Y}$ is a vector field on U.

But $[Y, R] = 0$ and $R = z(\partial/\partial z)$, so the function φ depends only on y. Since the compactification of $U \times \mathbb{C}$ is $U \times \mathbb{C}P^1$ and clearly $\widetilde{Y}$ and $\varphi \cdot z(\partial/\partial z)$ extend to $U \times \mathbb{C}P^1$, so does Y.

Model 2.4. ($\mathsf{Z}(X)$ is not connected) These kinds of examples are constructed on the compactification given in Remark 2.2 of holomorphic line bundles over $\mathbb{C}P^1$. As the Picard group of $\mathbb{C}P^1$ is $\mathbb{Z}$, these line bundles are *holomorphically* classified by their first Chern class [**9**]. We will need the following result.

LEMMA 2.5. *Let* $\pi : E \to \mathbb{C}P^1$ *be a holomorphic line bundle and* R *its radial vector field. Then there exists one and only one Lie algebra* $\mathcal{G} \subset \mathcal{V}^\omega(E)$ *such that:*
- $[R, \mathcal{G}] = 0$;
- $\mathcal{G}$ *is isomorphic by* π_* *to* $\mathcal{V}^\omega(\mathbb{C}P^1)$.

Moreover, $\mathcal{G}$ *comes from an action of* $\mathrm{SL}(2, \mathbb{C})$ *on* E. *Therefore, its elements are complete vector fields.*

Proof. Uniqueness: let $\mathcal{G}, \mathcal{H}$ be as in the lemma. Since the elements of $\mathcal{G}$ and $\mathcal{H}$ are tangent to the zero section and $\pi_* : \mathcal{G} \to \mathcal{V}^\omega(\mathbb{C}P^1)$, $\pi_* : \mathcal{H} \to \mathcal{V}^\omega(\mathbb{C}P^1)$ isomorphisms, every element of $\mathcal{H}$ may be written as $Y + a_Y R$, where $Y \in \mathcal{G}$ and $a_Y : E \to \mathbb{C}$ is holomorphic. Now $[R, Y + a_Y R] = 0$ implies that a_Y is constant along fibres. Therefore, $a_Y = b_Y \circ \pi$, where $b_Y : \mathbb{C}P^1 \to \mathbb{C}$ is a holomorphic function and so constant. In other words, $\mathcal{H} = \{Y + a_Y R : Y \in \mathcal{G}, a_Y \in \mathbb{C}\}$. But $\mathcal{I} = \{Y \in \mathcal{G} : a_Y = 0\}$ is a non-zero ideal, so $a_Y = 0$ for whichever $Y \in \mathcal{G}$ and $\mathcal{H} = \mathcal{G}$.

Existence: given a holomorphic line bundle $\pi : E \to \mathbb{C}P^1$, it suffices to show the existence of an action $\alpha : \mathrm{SL}(2, \mathbb{C}) \times E \to E$ which is almost effective and fibre preserving. Recall that *fibre preserving* means the existence of a second action (the projected one on the basis $\mathbb{C}P^1$) $\beta : \mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}P^1 \to \mathbb{C}P^1$ such that $\pi(\alpha(g, e)) = \beta(g, \pi(e))$ and $\alpha(g,) : \pi^{-1}(p) \mapsto \pi^{-1}(\beta(g, p))$ is a linear isomorphism for any $g \in \mathrm{SL}(2, \mathbb{C})$, $e \in E$ and $p \in \mathbb{C}P^1$. For that, one will show the existence of such an action of $\mathrm{SL}(2, \mathbb{C})$ for any value of the first Chern class.

First observe that given two almost effective actions $(\alpha, \mathrm{SL}(2, \mathbb{C}), E)$ and $(\alpha', \mathrm{SL}(2, \mathbb{C}), E')$ related by π to actions $(\beta, \mathrm{SL}(2, \mathbb{C}), \mathbb{C}P^1)$ and $(\beta', \mathrm{SL}(2, \mathbb{C}), \mathbb{C}P^1)$, respectively, if $\beta = \beta'$, then there exists a natural almost effective action of $\mathrm{SL}(2, \mathbb{C})$ on $E \otimes E'$ related by π to $\beta = \beta'$. Moreover, $c_1(E \otimes E') = c_1(E) + c_1(E')$.

On the other hand, an action of $\mathrm{SL}(2, \mathbb{C})$ on E induces an action on its dual vector bundle E^*, both over the same action on $\mathbb{C}P^1$ (recall that $c_1(E^*) = -c_1(E)$).

Thus, it suffices to construct it on the canonical line bundle E_1 over $\mathbb{C}P^1$. But it is well known that the natural action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}^2$ induces such an action on E_1 first by setting $g \cdot (v, w) = (g \cdot v, g \cdot w)$ on $\widetilde{F} := \{(v, w) \in (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}^2 : v \wedge w = 0\}$ and then by considering the induced action on the quotient E_1 of $\widetilde{F}$ under the equivalence relation $(v, w)\mathcal{R}(v', w') \Leftrightarrow v \wedge v' = 0$ and $w = w'$. $\qquad \square$

Consider a holomorphic line bundle $\pi \colon E \to \mathbb{C}P^1$ and its compactification given in Remark 2.2. Depending on the context, denote by $\mathcal{G}$ the Lie algebra of Lemma 2.5 or its extension to M, which exists because $\mathcal{G}$ on E consists of complete vector fields (Remark 2.3). Analogously, R will be the radial vector field on E or its extension to M.

Always on M note that $[\mathcal{G}, R] = 0$ and $\mathcal{G}$ is isomorphic by π_* to $\mathcal{V}^\omega(\mathbb{C}P^1)$, so $\mathsf{Z}(\mathcal{G}) = \emptyset$ since $\mathsf{Z}(\mathcal{V}^\omega(\mathbb{C}P^1)) = \emptyset$. Therefore, $(M, X, \mathcal{G})$, where $X = aR$, $a \in \mathbb{C} \setminus \{0\}$, is an example in which Theorem 1.1 fails, and $\mathsf{Z}(X) = M_0 \cup M_\infty$, where M_0 is the zero section and M_∞ the infinity one (Q_0 and Q_∞ in the notation of Remark 2.2). Obviously, $\mathsf{Z}(X)$ possesses two connected components.

Clearly, $\pi \colon E \to \mathbb{C}P^1$ and $\pi \colon E' \to \mathbb{C}P^1$, where $E' := M - M_0$, give rise to the same example (up to a non-zero coefficient multiplying X), so these examples depend only on the absolute value of the first Chern class.

When $|c_1(E)| \neq 0$, this number equals the order of the fundamental group of $M - \mathsf{Z}(X) = M - (M_0 \cup M_\infty)$. If $|c_1(E)| = 0$, then the fundamental group of $M - \mathsf{Z}(X)$ is $\mathbb{Z}$. Thus, $|c_1(E)|$ is an invariant which classifies this kind of example up to a non-zero coefficient multiplying X.

In compact complex 2-manifolds with non-vanishing Euler characteristic, Models 2.1 and 2.4 are the only ways for constructing Lie algebras tracking non-trivial vector fields with no common zero. More exactly, we have the following theorem.

THEOREM 2.6. *Let M be a compact connected complex 2-manifold with $\chi(M) \neq 0$. Assume that $\mathcal{G} \subset \mathcal{V}^\omega(M)$ is a finite-dimensional Lie algebra that tracks a non-trivial vector field $X \in \mathcal{V}^\omega(M)$. If $\mathsf{Z}(X) \cap \mathsf{Z}(\mathcal{G}) = \varnothing$, then the following conditions hold.*

(a)   *M is a holomorphic fibre bundle over $\mathbb{C}P^1$ with fibre $\mathbb{C}P^1$ and hence M is simply connected and $\chi(M) = 4$.*

(b)   *$\mathcal{G}$ contains a subalgebra $\mathcal{A}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ with $\mathsf{Z}(\mathcal{A}) = \varnothing$.*

(c)   *$(M, X, \mathcal{A})$ is holomorphically equivalent to the example of Model 2.1 or to any of the examples of Model 2.4.*

This result will be proved in §6 (see Remark 6.7 for more details on the algebra $\mathcal{G}$).

Example 2.7. Let M be a compact connected complex 2-manifold, Assume that:

- $\mathcal{G} \subset \mathcal{V}^\omega(M)$ is a subalgebra isomorphic to $\mathfrak{gl}(2, \mathbb{C})$;
- $\mathcal{A} \subset \mathcal{G}$ is the unique subalgebra of $\mathcal{G}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ [20] and $X \in \mathcal{G}$ spans the center of $\mathcal{G}$;
- $\mathsf{Z}(\mathcal{G}) \neq \varnothing$. (The case $\mathsf{Z}(\mathcal{G}) = \varnothing$ is described by Theorem 2.6 when $\chi(M) \neq 0$.)

Lemma 5.4, applied to X and $\mathcal{A}$, shows that $\mathsf{Z}(\mathcal{G})$ is a finite set, $\{p_1, \ldots, p_r\}$, $r \geq 1$ and $\mathsf{Z}(\mathcal{G}) = \mathsf{Z}(\mathcal{A})$. Blowing up all these points gives rise to another compact connected complex 2-manifold M', a vector field $X' \in \mathcal{V}^\omega(M')$ coming from X and a Lie algebra $\mathcal{A}' \subset \mathcal{V}^\omega(M')$ isomorphic to $\mathcal{A}$ (and hence to $\mathfrak{sl}(2, \mathbb{C})$). Moreover, $\mathsf{Z}(\mathcal{A}') = \varnothing$; indeed, as $\mathcal{A}$ is simple, $\mathcal{A}_0(p_k)/\mathcal{A}_1(p_k)$ equals the special linear algebra $\mathfrak{sl}(T_{p_k}M)$ for each $k = 1, \ldots, r$ (see §5 for definitions) and this zero of $\mathcal{A}$ is deleted by the blowup process. Indeed, after blowing up p_k, the action of $\mathcal{A}$ at this point, more exactly that of $\mathcal{A}_0(p_k)/\mathcal{A}_1(p_k)$ on $T_{p_k}M$, becomes the natural action of $\mathfrak{sl}(T_{p_k}M)$ on the complex

projective space $\mathbb{C}P(T_{p_k}M)$ of vectorial lines of $T_{p_k}M$ and this one has no zero on $\mathbb{C}P(T_{p_k}M)$ (see [**6**, pp. 19–20] and [**9**]).

LEMMA 2.8. *The linear part of X at p_k equals $a_k\mathbf{Id}$, $a_k \in \mathbb{C} \setminus \{0\}$ and $\chi(M') > 0$.*

Proof. As $\mathsf{Z}(\mathcal{A}') = \varnothing$, $\dim \mathcal{A}'(p) \geq 1$ for whichever $p \in \mathsf{Z}(X')$. If, for some p in this last set, $\dim \mathcal{A}'(p) = 2$, then X' vanishes near p and, by analyticity, $X' = 0$ on M', which is not the case. Hence, $\dim \mathcal{A}'(p) = 1$ for every $p \in \mathsf{Z}(X')$. By (b) of Lemma 5.6, this implies that $\mathsf{Z}(X')$ is a compact one-dimensional submanifold of M' consisting of a finite number $N_1, \ldots, N_\ell$ of compact 1-orbits of $\mathcal{A}'$ each of them diffeomorphic to $\mathbb{C}P^1$. Moreover, the index r_j of X' transverse to N_j is positive. An elementary computation shows that $\chi(M') = 2(r_1 + \cdots + r_\ell)$, which is a positive integer provided $\mathsf{Z}(X')$ is not empty. But $\mathsf{Z}(X') \neq \emptyset$ because at least it contains the 1-submanifolds added by the blowup of $\{p_1, \ldots, p_r\}$. Indeed, the linear part of X at p_k has to be $a_k\mathbf{Id}$, $a_k \in \mathbb{C}$, otherwise $\mathcal{A}_0(p_k)/\mathcal{A}_1(p_k) = \mathfrak{sl}(T_{p_k}M)$ does not commute with it and $[X, \mathcal{A}] \neq 0$.

On the other hand, if $a_k = 0$, as $[X, \mathcal{A}] = 0$ and $\mathcal{A}_0(p_k)/\mathcal{A}_1(p_k) = \mathfrak{sl}(T_{p_k}M)$, all ℓ-jets of X at p_k vanish and $X = 0$. (Recall that any homogeneous polynomial vector field on $\mathbb{C}^2$ of degree ≥ 2 that commutes with all the linear vector fields whose trace equals zero vanishes, and apply this property to the first non-zero term of the Taylor's expansion of X when $p_k \equiv 0$.) $\qquad\square$

It follows from the blowup construction that M and M' have the same fundamental group and $\chi(M') = \chi(M) + r$. Theorem 2.6 shows that M' is simply connected and $\chi(M') = 4$. Therefore, M is also simply connected and $\chi(M) \leq 3$. Since S^4 has no complex structure, topologically M is $\mathbb{C}P^2$ and $r = 1$.

As the linear part of X at p_1 equals $a_1\mathbf{Id}$, $a_1 \in \mathbb{C} \setminus \{0\}$, transversely to $\mathbb{C}P^1 \equiv M' \setminus (M \setminus \{p_1\})$ the index of X' equals 1 and $(M', X', \mathcal{A}')$ follows Model 2.4 for the canonical line bundle E_1 since the normal vector bundle of $M' \setminus (M \setminus \{p_1\})$ is isomorphic to E_1.

As the examples of Model 2.4 are determined by the absolute value of their first Chern class, if one considers a second manifold N, $\mathcal{H} \subset \mathcal{V}^\omega(N)$, $\mathcal{B} \subset \mathcal{H}$ and $Y \in \mathcal{H}$ under the same conditions as M, $\mathcal{G}$, $\mathcal{A}$ and X, and one blows up the only singular point q_1, then there is a (holomorphic) diffeomorphism $\varphi \colon M' \to N'$ which transforms $\mathcal{A}'$ into $\mathcal{B}'$ and X' into aY' for some $a \in \mathbb{C} \setminus \{0\}$. Besides, $\varphi(M' \setminus (M \setminus \{p_1\})) = N' \setminus (N \setminus \{q_1\})$ because the first Chern classes of their respective normal vector bundles are the same, that of E_1 (notice that the first Chern class of the other component of $\mathsf{Z}(X')$, or of $\mathsf{Z}(Y')$, is the opposite one).

Now, crushing $M' \setminus (M \setminus \{p_1\})$ and $N' \setminus (N \setminus \{q_1\})$ respectively into a point gives rise to a homeomorphism $\psi \colon M \to N$ with $\psi(p_1) = q_1$. Clearly, $\psi \colon M \setminus \{p_1\} \to N \setminus \{q_1\}$ is biholomorphic, so $\psi \colon M \to N$ is biholomorphic too and transforms $\mathcal{G}$ into $\mathcal{H}$.

In other words, *up to isomorphism there is only one example of a manifold M and a subalgebra $\mathcal{G}$ as above*. For instance: $M = \mathbb{C}P^2$ and $\mathcal{G}$ the subalgebra of those projective vector fields that on $\mathbb{C}^2 \subset \mathbb{C}P^2$ are linear.

Let G be a connected Lie group and let N, P be two compact connected manifolds (both real or both complex). Given two actions $\alpha \colon G \times N \to N$ and $\beta \colon G \times P \to P$,

we will say that they are *equivalent* if there exist a diffeomorphism $\psi \colon N \to P$ and an automorphism $\lambda \colon G \to G$ such that

$$\alpha(g, p) = \psi^{-1}(\beta(\lambda(g), \psi(p)))$$

for any $(g, p) \in G \times N$. Assume that α and β are effective. From [**29**], it follows that if there is a diffeomorphism $\psi \colon N \to P$ which transforms Im d_α into Im d_β, then α and β are equivalent. Therefore, we have the following theorem.

THEOREM 2.9. *Any effective holomorphic action of* $\mathrm{GL}(2, \mathbb{C})$ *on a connected compact complex 2-manifold M which possesses a fixed point is holomorphically equivalent to the natural action of* $\mathrm{GL}(2, \mathbb{C})$ *on* $\mathbb{C}P^2$.

3. Other examples

In this section we give several examples of finite-dimensional Lie algebras on surfaces tracking non-trivial vector fields, focusing on complex 2-manifolds, the most difficult case. Let us start with a short description of these examples.

Example 3.1 studies the trace on $\mathbb{F}^2 \subset \mathbb{F}P^2$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$, of the Lie algebra $\mathcal{G}$ of projective vector fields that vanish at the origin. The radial vector field X is tracked by $\mathcal{G}$ and the tracking functions are linear. The origin is an essential block of X of index one.

In the complex case, by blowing up the origin, one constructs an example with no common zero in which hypothesis (*) fails.

In Example 3.2 a similar Lie algebra is considered but this time the homogeneous vector fields of degree two are replaced by homogeneous vector fields of degree $n + 1$. That allows us to take the dimension of the Lie algebra arbitrarily large.

Example 3.3 is constructed on the 'sphere' $S^2_{\mathbb{F}}$ given by the equation $x_1^2 + x_2^2 + x_3^2 = 1$. In the real case X is the orthogonal projection of $\partial/\partial x_3$ on the sphere. Since the expressions of X and the Lie algebra $\mathcal{G}$ are polynomial, they can be formally extended to the complex sphere. In this example X possesses two essential blocks in $S^2_{\mathbb{F}}$ of index one, the two poles.

Until here there is no difference between real and complex, but the blowup process of the poles splits up real and complex cases.

In Example 3.4 a 3-solvable Lie subalgebra of dimension five of the Lie algebra of projective vector fields on $\mathbb{C}P^2$ is exhibited and then by means of the blowup process transferred to a complex 2-manifold, compact and simply connected, of Euler characteristic arbitrarily large.

Finally in Example 3.5 a compact connected one-dimensional complex manifold N and its cotangent bundle $\pi \colon T^*N \to N$ are considered. By means of the Liouville symplectic form on T^*N to any holomorphic 1-form α on N, one associates a vertical vector field X_α, which is constant on each fibre.

If $\pi \colon Q \to N$ is the compactification given by Remark 2.2, then X_α extends to a vector field $\hat{X}_\alpha$ on Q. In the same way the radial vector field R on T^*N extends to a vector field $\hat{R}$ on Q. Moreover, $[\hat{X}_\alpha, \hat{R}] = \hat{X}_\alpha$.

By this reason, taking into account R, any vector space V of dimension k consisting of holomorphic 1-forms on N gives rise to an algebra $\mathcal{G}$ of vector fields on Q of dimension

$k + 1$, which is not abelian but possesses an abelian ideal of codimension one. Recall that there always exists a vector space V whose dimension equals the genus of N.

Example 3.1. Recall that in dimension two a projective vector field Y means a fundamental vector field of the natural action of $SL(3, \mathbb{F})$ on $\mathbb{F}P^2$, which is effective if $\mathbb{F} = \mathbb{R}$ and almost effective when $\mathbb{F} = \mathbb{C}$ (its kernel equals $\{a\mathbf{I} : (a \in \mathbb{C}, \ a^3 = 1)\}$, where $\mathbf{I} \in SL(3, \mathbb{C})$ denotes the identity). The restriction of Y to $\mathbb{F}^2 \subset \mathbb{F}P^2$ can be written as

$$a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \sum_{k,r=1,2} b_{kr} x_k \frac{\partial}{\partial x_r} + (c_1 x_1 + c_2 x_2) \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \quad (1)$$

with $a_1, a_2, b_{kr}, c_k \in \mathbb{F}$.

Now on $\mathbb{F}^2$ consider the Lie algebra $\mathcal{G}$ corresponding to the projective vector fields on $\mathbb{F}P^2$ that vanish at the origin. This algebra tracks $X = x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2)$, itself belonging to $\mathcal{G}$ since $[Y, X] = -(c_1 x_1 + c_2 x_2)X$. Note that $\mathcal{G}$ has no ideal of dimension one, but the notion of tracking will allow us to bridge this gap. Note also that X and $\mathcal{G}$ extend to $\mathbb{F}P^2$, but the tracking functions do not.

Observe that $\mathsf{Z}(X) = \mathsf{Z}(\mathcal{G}) = \{(0, 0)\}$, and this is an essential X-block.

From $\mathbb{F}P^2$ we construct a new 2-manifold M' over $\mathbb{F}$ by blowing up the origin in $\mathbb{F}^2$, and on it a vector field X' and a Lie algebra $\mathcal{G}' \subset \mathcal{V}^\omega(M')$, isomorphic to $\mathcal{G}$, which tracks X' (see [**9**]). Note that the blowup of the origin is $\mathsf{Z}(X')$, an X'-block diffeomorphic to $\mathbb{F}P^1$. As the linear part of X at the origin is the identity, the index of X' transverse to $\mathsf{Z}(X')$ equals one and hence $\mathsf{i}_{\mathsf{Z}(X')}(X') = \chi(\mathsf{Z}(X')) = \chi(\mathbb{F}P^1)$. Thus, in the real case, this block is inessential (for X') while it is essential with index two in the complex one. Moreover, $\mathfrak{sl}(2, \mathbb{F})$ is a quotient of $\mathcal{G}'$ because clearly it is a quotient of $\mathcal{G}$.

From the blowup construction it follows that $\mathcal{G}'$ acts transitively on $\mathsf{Z}(X')$ since any linear vector field on $\mathbb{F}^2$ belongs to $\mathcal{G}$, so $\mathsf{Z}(\mathcal{G}') = \varnothing$. Therefore:

- *for the complex case of Theorem 1.1, the supplementary hypothesis* (*) *cannot be deleted even in the non-compact case.*

If we consider the solvable subalgebra $\mathcal{G}'_0$ of $\mathcal{G}'$, corresponding to $\mathcal{G}_0 \subset \mathcal{G}$ defined by setting $b_{21} = 0$ in Equation (1), then $\mathsf{Z}(X') \cap \mathsf{Z}(\mathcal{G}'_0) \neq \varnothing$ since $\mathcal{G}'_0$ vanishes at the point of $\mathbb{F}P^1$ associated to the second axis. In turn, blowing up this common zero gives rise to a new manifold endowed with a Lie algebra $\mathcal{G}''_0$, isomorphic to $\mathcal{G}_0$ and $\mathcal{G}'_0$, and a vector field X'' tracked by $\mathcal{G}''_0$. Now $\mathsf{Z}(X'')$ is again essential; more exactly, $\mathsf{i}_{\mathsf{Z}(X'')}(X'')$ equals -1 in the real case and 3 in the complex one. Therefore, $\mathsf{Z}(\mathcal{G}''_0) \cap \mathsf{Z}(X'') \neq \varnothing$.

For easily computing the index of X, X' and X'', notice that as a real vector field X is outwardly transverse to the spheres $S^1 \subset \mathbb{R}^2$ or $S^3 \subset \mathbb{C}^2$. Therefore, in each case this index equals the Euler characteristic of the ambient manifold.

Example 3.2. In a more general setting, let $\mathcal{P}_n$ be the vector space of homogeneous polynomials in x_1, x_2 of degree $n \geq 1$ over $\mathbb{F}$ and let $\mathcal{G}$ be the $(n + 5)$-dimensional Lie algebra of vector fields on $\mathbb{F}^2$ of the form

$$\sum_{k,r=1,2} b_{kr} x_k \frac{\partial}{\partial x_r} + \varphi \cdot \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right),$$

where $b_{kr} \in \mathbb{F}$ and $\varphi \in \mathcal{P}_n$.

As in Example 3.1, set $X = x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2)$. Then $\mathcal{G}$ tracks X.

Blowing up the origin in $\mathbb{F}^2$ provides a new 2-manifold M over $\mathbb{F}$ endowed with a Lie algebra $\mathcal{G}' \subset \mathcal{V}^\omega(M)$ and a vector field X' which is tracked by $\mathcal{G}'$. As before, $\mathsf{Z}(X') = \mathbb{F}P^1$ and $\mathsf{Z}(\mathcal{G}') = \varnothing$, so $\mathsf{Z}(\mathcal{G}') \cap \mathsf{Z}(X') = \varnothing$, while $i_{\mathsf{Z}(X')}(X')$ equals zero if $\mathbb{F} = \mathbb{R}$ and 2 when $\mathbb{F} = \mathbb{C}$.

Notice that the dimension of $\mathcal{G}'$ can be taken as large as desired. Thus:

- *in the non-compact complex case, when Theorem 1.1 fails the respective Lie algebra has $\mathfrak{sl}(2, \mathbb{C})$ as a quotient, but its dimension can be arbitrarily large*

(see Remark 6.7 for the compact case).

Of course one may consider the subalgebra $\mathcal{G}_0 \subset \mathcal{G}$ given by the condition $b_{21} = 0$ and do as in Example 3.1.

Example 3.3. In $\mathbb{F}^3$ with coordinates $x = (x_1, x_2, x_3)$, let $S^2_\mathbb{F}$ be the 'sphere' given by the equation $x_1^2 + x_2^2 + x_3^2 = 1$. This is the real sphere S^2 if $\mathbb{F} = \mathbb{R}$. When $\mathbb{F} = \mathbb{C}$, it is a non-compact complex 2-manifold whose underlying real manifold is the tangent vector bundle TS^2; but $S^2_\mathbb{C}$ is not biholomorphic to $T\mathbb{C}P^1$ because $\mathbb{C}P^1$ is never a complex submanifold of $\mathbb{C}^3$. On $S^2_\mathbb{F} \subset \mathbb{F}^3$, consider the tangent vector fields

$$X = \frac{\partial}{\partial x_3} - x_3\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right), \quad Y = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}.$$

Denote by $\mathcal{P}_k$ the space of homogeneous polynomials in x_1, x_2 of degree k. Set

$$\mathcal{G}_k = \{aY + \varphi X \colon a \in \mathbb{F}, \varphi \in \mathcal{P}_k\},$$

which is a $(k + 2)$-dimensional solvable Lie algebra that tracks X.

On $S^2_\mathbb{F}$, our X has just two singular points $(0, 0, \pm 1)$, each of them an essential block of index one. Indeed, first observe that the functions x_1, x_2 can be regarded as coordinates of $S^2_\mathbb{F}$ around $(0, 0, \pm 1)$, which we name (u_1, u_1). As $X \cdot x_k = -x_3 x_k$, $k = 1, 2$, up to sign the linear part of X at $(0, 0, \pm 1)$ equals $u_1(\partial/\partial u_1) + u_2(\partial/\partial u_2)$. Note that $\mathcal{G}_k$ vanishes at these points. Blowing up $(0, 0, 1)$ and $(0, 0, -1)$ gives rise to a 2-manifold M, a vector field X' with two isolated blocks K_1, K_2 associated to these points and a Lie algebra $\mathcal{G}'$ that is isomorphic to $\mathcal{G}$ and tracks X'.

- *But the behavior of the real and complex cases is quite different.*

Indeed, in the complex one K_1 and K_2 are diffeomorphic to $\mathbb{C}P^1$ and are thus essential blocks and $\mathcal{G}'$ vanishes somewhere in K_1 and in K_2. In the real case M is the Klein bottle, K_1, K_2 are S^1, and so non-essential blocks, and $\mathcal{G}'$ does not vanish at any point of M.

Example 3.4. Let $\mathcal{G}$ be the Lie algebra on $\mathbb{C}P^2$ of those projective vector fields Y that on $\mathbb{C}^2 \subset \mathbb{C}P^2$ are written as

$$Y = a_1\frac{\partial}{\partial z_1} + a_2\frac{\partial}{\partial z_2} + \sum_{k,r=1,2} b_{kr}z_k\frac{\partial}{\partial z_r},$$

where $a_k, b_{kr} \in \mathbb{C}$ and $b_{21} = 0$. Then $\mathcal{G}$ is a 3-solvable Lie algebra of dimension five and the vector field represented by $\partial/\partial z_2$ spans an ideal of dimension one. Moreover, $\mathcal{G}$ vanishes at the infinity point of the second axis (belonging to $\mathbb{C}P^2 - \mathbb{C}^2$).

Now by blowing up this point one constructs a second complex 2-manifold M_1, of Euler characteristic 4 and simply connected, endowed with a Lie algebra $\mathcal{G}_1$ isomorphic to $\mathcal{G}$.

Again there are some zero of $\mathcal{G}_1$, which can be blown up to construct a simply connected manifold M_2 of Euler characteristic 5, and a Lie algebra $\mathcal{G}_2 \subset \mathcal{V}^\omega(M_2)$ isomorphic to $\mathcal{G}$ and so on. Therefore:

- *for any $m \geq 3$ there exists a simply connected compact complex 2-manifold of characteristic m that supports a 3-solvable Lie algebra of vector fields.*

Let $\mathcal{A}$ be a two-dimensional non-commutative Lie algebra on $\mathbb{C}P^1$. A similar process can be started from the product manifold $\mathbb{C}P^1 \times \mathbb{C}P^1$ endowed with the four-dimensional Lie algebra $\mathcal{G}$ of those vector fields X such that $(\pi_1)_*(X)$ and $(\pi_2)_*(X)$ belong to $\mathcal{A}$.

Example 3.5. Here we show that for every integer m and every positive integer d, there are a compact complex 2-manifold M and a solvable Lie algebra $\mathcal{A} \subset \mathcal{V}^\omega(M)$ such that

$$\chi(M) = m, \quad \dim_{\mathbb{C}} \mathcal{A} = d, \quad \dim_{\mathbb{C}} \mathsf{Z}(\mathcal{A}) = 1.$$

First consider a real or complex n-manifold, N, with cotangent bundle $\pi : T^*N \to N$. On the manifold T^*N, define the *Liouville 1-form* ρ by

$$\rho(v) = \alpha(\pi_* v) \quad (\alpha \in T^*N, \ v \in T_\alpha(T^*N)).$$

The *Liouville symplectic form* on T^*N is the exterior 2-form $\omega = d\rho$. Given a 1-form β on the manifold T^*N, assumed analytic over the ground field, define the vector field

$$X_\beta \in \mathcal{V}^\omega(T^*N), \quad \iota_{X_\beta}\omega = \beta, \tag{2}$$

where $\iota_{X_\beta}\omega$ is the interior product of ω by X_β.

Now let N be a one-dimensional complex manifold defined by a orientable compact connected surface of genus $g \geq 1$ endowed with a Kähler structure. The Dolbeault cohomology group of N in dimension one, which is isomorphic to the singular cohomology group $H^1(N, \mathbb{R})$, has a basis represented by g holomorphic 1-forms

$$\alpha_j := \lambda_j + i\mu_j.$$

Using equation (2), set

$$X_j := X_{\pi^*\alpha_j} \in \mathcal{V}^\omega(T^*N) \quad (j = 1, \ldots, g)$$

and let X_{g+1} denote the radial vector field on T^*N. By means of coordinates, it is easily checked that $X_1, \ldots, X_g$ and X_{g+1} are tangent to the fibres T_p^*N ($p \in N$) with each X_j, $j = 1, \ldots, g$, constant and X_{g+1} linear. Moreover, $[X_j, X_{g+1}] = X_j$, $j = 1, \ldots, g$.

Let $\pi : Q \to N$ be the compactification of $\pi : T^*N \to N$ given by Remark 2.2. The vector fields X_k, $k = 1, \ldots, g+1$, extend to holomorphic vector fields $\hat{X}_k \in \mathcal{V}^\omega(Q)$ such that $Q_\infty \subset \mathsf{Z}(\hat{X}_k)$, where Q_∞ denotes the infinity section of Q. It is easy to see that $\hat{X}_1, \ldots, \hat{X}_{g+1}$ form a basis of a solvable complex Lie algebra $\mathcal{G} \subset \mathcal{V}^\omega(Q)$ of dimension $g + 1$. Evidently $\mathsf{Z}(\mathcal{G})$ is the union of the one-dimensional complex submanifold Q_∞ and the image by the zero section of the set of common zeros of $\alpha_1, \ldots, \alpha_g$. Holomorphy shows that this last set is always finite, so we can reasonably write $\dim_{\mathbb{C}} \mathsf{Z}(\mathcal{G}) = 1$.

Note that $\chi(Q) = 4(1 - g)$. By blowing up r zeros of $\mathcal{G}$, we obtain a compact complex 2-manifold M with $\chi(M) = 4(1 - g) + r$ and a solvable Lie algebra $\mathcal{G}' \subset \mathcal{V}^\omega(M)$ isomorphic to $\mathcal{G}$. Finally, take g and r such that $4(1 - g) + r = m$ and $g \geq d$ and a Lie subalgebra $\mathcal{A} \subset \mathcal{G}'$ of dimension d.

4. *Consequences of tracking*

Throughout this section we assume that:

- *P is a real or complex n-manifold with empty boundary;*
- *X, Y are differentiable vector fields on P and Y tracks X with tracking function f.*

When P is complex and X non-trivial on each connected component of P, our function f is holomorphic. Indeed, locally in coordinates f is meromorphic because $[Y, X]$ 'divided' by X equals f and so holomorphic since it is continuous. *Thus, if P is compact, connected and complex, the tracking function f is constant.*

We point out that the result above on the tracking function does not extend to the real case. The function f can be just continuous at some point. For instance, on $\mathbb{R}$ set $Y = x^4(\partial/\partial x)$ and $X = g(\partial/\partial x)$, where $g(x) = e^{-1/x}$ if $x > 0$, $g(x) = e^{-1/x^2}$ if $x < 0$ and $g(0) = 0$. A computation shows that $f(0) = 0$, $f(x) = x^2 - 4x^3$ if $x > 0$ and $f(x) = 2x - 4x^3$ if $x < 0$; hence, f is not differentiable at the origin.

If Y_1 is another differentiable vector field tracking X with function f_1, and f, f_1 are at least C^1, which automatically holds if $\mathbb{F} = \mathbb{C}$, the Jacobi identity implies that $[Y, Y_1]$ tracks X.

By definition, the *dependency set* of X and Y (over the ground field $\mathbb{F}$) is

$$\mathsf{D}(X, Y) := \{p \in M : (X \wedge_{\mathbb{F}} Y)(p) = 0\}.$$

PROPOSITION 4.1. *If Y tracks X, then* $\mathsf{Z}(X)$ *and* $\mathsf{D}(X, Y)$ *are X- and Y-invariant.*

Proof. Evidently $\mathsf{Z}(X)$ is X-invariant. Let us see its Y-invariance. Consider an integral curve $\gamma : A \to P$ of Y, where A is a connected open set $\mathbb{F}$. Suppose that $\gamma(t_0) \in \mathsf{Z}(X)$; then $\gamma(t) \in \mathsf{Z}(X)$ for any $t \in A$ sufficiently close to t_0. Indeed, if $Y(\gamma(t_0)) = 0$ it is obvious; otherwise as the statement is local by means of suitable coordinates we may assume that P is a product of intervals ($\mathbb{F} = \mathbb{R}$) or a polydisk ($\mathbb{F} = \mathbb{C}$) always centered at $\gamma(t_0) = (0, \ldots, 0)$, $Y = \partial/\partial x_1$ and $X = \sum_{k=1}^{n} g_k(\partial/\partial x_k)$. Now $[Y, X] = fX$ and hence

$$\frac{\partial g_k}{\partial x_1} = fg_k, \quad k = 1, \ldots, n. \tag{3}$$

Since f is continuous ($\mathbb{F} = \mathbb{R}$) or holomorphic ($\mathbb{F} = \mathbb{C}$), the general solution to equation (3) is

$$g_k(x) = h_k(x_2, \ldots, x_n)e^{\varphi}, \quad k = 1, \ldots, n,$$

where $\partial\varphi/\partial x_1 = f$ and $\varphi(x) = 0$ whenever $x_1 = 0$. But $\gamma(t_0) = 0$, so each $h_k(0, \ldots, 0) = 0$ and X vanishes along the first axis.

Therefore, the set $A' = \{t \in A : \gamma(t) \in \mathsf{Z}(X)\}$ is open and closed, so $A' = A$ or $A' = \varnothing$, which proves the Y-invariance of $\mathsf{Z}(X)$.

The X- and Y-invariance of $\mathsf{D}(X, Y)$ is proved in the same way by taking into account that $L_X(X \wedge_{\mathbb{F}} Y) = 0$ and letting $L_Y(X \wedge_{\mathbb{F}} Y) = fX \wedge_{\mathbb{F}} Y$. $\qquad\square$

5. *Vanishing of the index and other results*

In the first lemma of this section we assume that:

- *P is a real n-manifold;*
- $X \in \mathcal{V}^{\infty}(P)$;
- *K is an X-block and $U \subset P$ is isolating for (X, K).*

LEMMA 5.1. *Assume that:*

(a) *K is a compact submanifold;*

(b) $X_1, \ldots, X_r \in \mathcal{V}^\infty(U)$ *are tangent to K, and their values are linearly independent at each point of K;*

(c) $X = \sum_{j=1}^r f_j X_j$ *on U.*

Let $\mathcal{D} \subset \mathcal{V}(M)$ *be a neighborhood of X in the compact open topology. Then there exists* $X' \in \mathcal{D}$ *such that*

$$X' = X \text{ on } M \setminus U, \quad \mathsf{Z}(X') \cap U = \varnothing.$$

As a consequence, $\mathsf{i}_K(X) = 0$.

Proof. Take a tubular neighborhood W of K such that $W \subset U$ and identify it with an orthogonal vector bundle $\pi \colon E \to K$, with the norm in each fibre denoted by $\| \cdot \|$. Set $E_a := \{e \in E \colon \|e\| < a\}$ for each $a > 0$, so that $\overline{E}_a \subset U$. Let $\varphi_a \colon M \to \mathbb{R}_+$ be a non-negative function with support in E_a such that $\varphi_a^{-1}(0) \cap K = \varnothing$. If a is small enough, for each $e \in E_a$ the vector subspace spanned by $X_1(e), \ldots, X_r(e)$ is almost transverse to the fibre of e, so its intersection with $T_e(\pi^1((e)))$ equals $\{0\}$. Let R denote the radial vector field on E.

For sufficiently small $\epsilon > 0$, the vector field $X' \in \mathcal{V}(M)$ defined as

$$X' | U := X + \epsilon \varphi_a(X_1 + R), \quad X' | M \setminus W : X | M \setminus W$$

has the required properties. □

Remark 5.2. The reader not familiar with the Poincaré–Hopf index is referred to [3, §0].

The lemma above is a kind of extension of Bonatti [3, Lemma 1.c.1]. Observe that here analyticity is not needed. Nevertheless, in Hypothesis (b) the requirement of being tangent to K cannot be deleted, as is shown by the following example.

On the product $\mathbb{R} \times S^2$, consider the vector fields X, X_1 that are tangent to the first factor and whose projections on $\mathbb{R}$ respectively are written as $x(\partial/\partial x)$ and $\partial/\partial x$. Then $\mathsf{i}_{(\{0\} \times S^2)}(X) = 2$ but $X = x X_1$. Of course a similar example may be constructed on $\mathbb{R}^n \times S^2$ with vector fields $X, X_1, \ldots, X_n$.

5.1. *Jets of vector fields..*

Let us recall some well-known facts on jets of vector fields, useful later on. Consider a set $\mathcal{B}$ of vector fields on a manifold Q. Given $p \in Q$ and $k \geq 0$, set $\mathcal{B}_k(p) := \{Y \in \mathcal{B} \colon j_p^k Y = 0\}$, while $\mathcal{B}_{-1}(p) := \mathcal{B}$ and $\mathcal{B}(p) := \{Y(p) \colon Y \in \mathcal{B}\} \subset T_p Q$. Every $\mathcal{B}_{k-1}(p)/\mathcal{B}_k(p)$, $k \geq 0$, can be regarded as a subset of $T_p Q \otimes S^k(T_p^* Q)$ and $\mathcal{B}_{-1}(p)/\mathcal{B}_k(p)$ as a subset of the set of all polynomial vector fields on $T_p Q$ of degree $\leq k$. When $\mathcal{B}$ is a Lie algebra, $[\mathcal{B}_k(p), \mathcal{B}_r(p)] \subset \mathcal{B}_{k+r}(p)$ for $k + r \geq -1$. Therefore, each $\mathcal{B}_k(p)$ for $k \geq 0$ is a Lie algebra, and every $\mathcal{B}_{k+s}(p)$ for $k, s \geq 0$ is an ideal in $\mathcal{B}_k(p)$. If Q is connected and $\mathcal{B}$ a finite-dimensional analytic Lie algebra, then $\mathcal{B}_k(p)$ for $k \geq 1$ is nilpotent and $\mathcal{B}_r(p) = 0$ for some r.

A (piecewise-differentiable) *curve tangent to* $\mathcal{B}$ is a finite family $\mathcal{C}$ of integral curves $\gamma_j \colon A_j \to Q$, $j = 1, \ldots, k$, of elements of $\mathcal{B}$ defined on connected open subsets of $\mathbb{F}$

such that $A_r \cap A_{r+1} \neq \varnothing$, $r = 1, \ldots, k-1$. We say that $\mathcal{C}$ *joins* p to q if $p \in A_1$ and $q \in A_k$. The $\mathcal{B}$-*orbit* of p is the set of all points $q \in Q$ joined to p by some curve tangent to $\mathcal{B}$.

When $\mathcal{B}(p) = T_p Q$, it is easily seen that p belongs to the interior of its $\mathcal{B}$-orbit. If $\mathcal{B}$ is a finite-dimensional Lie algebra, then the $\mathcal{B}$-orbit of p is a submanifold of P, not always regular, whose dimension equals that of the vector subspace $\mathcal{B}(p) \subset T_p Q$ [**29**] (a submanifold of Q is called *embedded or regular* when its topology as a submanifold and that as a topological subspace of Q coincide).

From Lie's work [**23**], we have the following result.

LEMMA 5.3. *Let Q be a one-dimensional connected manifold and let $\mathcal{B} \subset \mathcal{V}^\omega(Q)$ be a non-zero finite-dimensional Lie algebra. Then $\mathcal{B}$ is isomorphic to $\mathbb{F}$, or the affine algebra of $\mathbb{F}$ or $\mathfrak{sl}(2, \mathbb{F})$.*

In the next three results assume that:
- *N is a connected complex 2-manifold;*
- *$\mathcal{A} \subset \mathcal{V}^\omega(N)$ is a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ and $X \in \mathcal{V}^\omega(N)$ is non-trivial.*

LEMMA 5.4. *The points of $\mathsf{Z}(\mathcal{A})$ are isolated and $\mathsf{Z}(\mathcal{A}) \subset \mathsf{Z}(X)$ when $[X, \mathcal{A}] = 0$.*

Proof. Fix $p \in \mathsf{Z}(\mathcal{A})$. The discussion of jets, above, shows that $\mathcal{A}_0(p) = \mathcal{A}$, and $\mathcal{A}_1(p)$ is a nilpotent ideal of $\mathcal{A}$ and so zero because $\mathcal{A}$ is simple. Therefore, $\mathcal{A}_0(p)/\mathcal{A}_1(p) \subset \mathfrak{gl}(2, \mathbb{C})$ is simple, which implies that $\mathcal{A}_0(p)/\mathcal{A}_1(p) = \mathfrak{sl}(2, \mathbb{C})$ and, consequently, the existence of $Y \in \mathcal{A}$ whose linear part at p is invertible. Hence, p is isolated in $\mathsf{Z}(Y)$, and *a fortiori* in $\mathsf{Z}(\mathcal{A})$.

Suppose that $[X, \mathcal{A}] = 0$. Then the isolated set $\mathsf{Z}(\mathcal{A})$ is invariant under the local flow of X, implying that X vanishes at every point of $\mathsf{Z}(\mathcal{A})$. $\qquad\square$

LEMMA 5.5. *Assume that $[X, \mathcal{A}] = 0$. Consider a point $p \in N$ such that $X(p) \neq 0$. Then around p there exist coordinates $z = (z_1, z_2)$, with $p \equiv 0$, such that the vector fields*

$$Y_1 = \frac{\partial}{\partial z_1}, \quad Y_2 = z_1 \frac{\partial}{\partial z_1} + a \frac{\partial}{\partial z_2}, \quad Y_3 = z_1^2 \frac{\partial}{\partial z_1} + 2az_1 \frac{\partial}{\partial z_2}, \quad a \in \mathbb{C},$$

are a basis of the restriction of $\mathcal{A}$ to the domain of coordinates, and $X = \partial/\partial z_2$.

Proof. As $\mathcal{A}$ is simple, its projection on the local quotient N' of N by the foliation associated to X is either zero or a Lie algebra $\mathcal{A}'$ isomorphic to $\mathcal{A}$. If zero, each $Y \in \mathcal{A}$ is proportional to X, which is incompatible with the hypothesis $[X, \mathcal{A}] = 0$. Therefore, as $\dim N' = 1$, there exists a coordinate z_1, around the projection p' of p in N', such that $z_1(p') = 0$ and

$$\frac{\partial}{\partial z_1}, \quad z_1 \frac{\partial}{\partial z_1}, \quad z_1^2 \frac{\partial}{\partial z_1}$$

span $\mathcal{A}'$.

This coordinate z_1 may be regarded as a function around p. Adding a new function z_2, such that $z_2(p) = 0$ and $X \cdot z_2 = 1$, leads to a system of coordinates $z = (z_1, z_2)$, defined

on a domain identified through z to a polydisk centered at the origin (shrink it if necessary), in which $p \equiv 0$, $X = \partial/\partial z_2$ and

$$Y_1 = \frac{\partial}{\partial z_1} + f_1 \frac{\partial}{\partial z_2}, \quad Y_2 = z_1 \frac{\partial}{\partial z_1} + f_2 \frac{\partial}{\partial z_2}, \quad Y_3 = z_1^2 \frac{\partial}{\partial z_1} + f_3 \frac{\partial}{\partial z_2},$$

for suitable functions f_1, f_2, f_3, span $\mathcal{A}$ (more exactly, its restriction to the domain of coordinates). Observe that each f_k depends only on z_1 because $[X, Y_k] = 0$.

Taking $z_2 + g(z_1)$ instead of z_2 for a suitable function $g(z_1)$ allows us to suppose that $Y_1 = \partial/\partial z_1$. Then, as $[Y_1, Y_2] = Y_1$, $[Y_1, Y_3] = 2Y_2$ and $[Y_2, Y_3] = Y_3$ (project into $\mathcal{A}'$ to see this), a straightforward computation shows that f_2, f_3 are as stated. $\qquad\square$

LEMMA 5.6. *Consider a point $p \in N$ such that $X(p) = 0$ and $\dim \mathcal{A}(p) = 1$.*

(a) *If $\mathcal{A}$ tracks X, then either all orbits of $\mathcal{A}$ near p have dimension one and X is tangent to them, or in a neighborhood W of p there exist coordinates $z = (z_1, z_2)$, with $p \equiv 0$, an integer $n \geq 1$ and functions $h(z_1, z_2)$, $f(z_2)$, $g(z_2)$ with $h(0, 0) \neq 0$ and $f(0) = g(0) = 0$ such that*

$$X = h(z_1, z_2) z_2^n \frac{\partial}{\partial z_2}$$

and the vector fields

$$Y_1 = \frac{\partial}{\partial z_1}, \quad Y_2 = z_1 \frac{\partial}{\partial z_1} + f(z_2) \frac{\partial}{\partial z_2}, \quad Y_3 = z_1^2 \frac{\partial}{\partial z_1} + \big(2z_1 f(z_2) + g(z_2)\big) \frac{\partial}{\partial z_2}$$

span $\mathcal{A}|W$.

(b) *If $[X, \mathcal{A}] = 0$, then about p there exist coordinates $z = (z_1, z_2)$, with $p \equiv 0$, an integer $n \geq 1$ and scalars $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$ such that*

$$X = a z_2^n \frac{\partial}{\partial z_2}$$

and the vector fields

$$Y_1 = \frac{\partial}{\partial z_1}, \quad Y_2 = z_1 \frac{\partial}{\partial z_1} + b z_2^n \frac{\partial}{\partial z_2}, \quad Y_3 = z_1^2 \frac{\partial}{\partial z_1} + 2b z_1 z_2^n \frac{\partial}{\partial z_2}$$

are a basis of $\mathcal{A}$ (under restriction).

Proof. The $\mathcal{A}$-orbit of p is a submanifold N' of dimension one and $\mathcal{A}$ is tangent to it. Therefore, there exist a coordinate u, defined on a small open set $p \in N'' \subset N'$ with $u(p) = 0$, and three vector fields Y_1, Y_2, Y_3 which span $\mathcal{A}$ such that under restriction to N'' respectively are written as

$$Y_1 = \frac{\partial}{\partial u}, \quad Y_2 = u \frac{\partial}{\partial u}, \quad Y_3 = u^2 \frac{\partial}{\partial u}.$$

Thus, $[Y_1, Y_2] = Y_1$, $[Y_1, Y_3] = 2Y_2$ and $[Y_2, Y_3] = Y_3$.

Around p in N our u can be extended to a function z_1 such that $Y_1 \cdot z_1 = 1$. Take a second function z_2 vanishing at p such that $Y_1 \cdot z_2 = 0$ and $(dz_1 \wedge dz_2)(p) \neq 0$; then $z = (z_1, z_2)$ near $p \equiv 0$ is a system of coordinates with domain of polydisk type (shrink it if necessary) such that $Y_1 = \partial/\partial z_1$. Note that $z_2 = 0$ defines an open set of N', which

includes p. Besides, as $[\partial/\partial z_1, Y_2] = [Y_1, Y_2] = Y_1 = \partial/\partial z_1$, $[\partial/\partial z_1, Y_3] = [Y_1, Y_3] = 2Y_2$ and $Y_2(p) = Y_3(p) = 0$, one has

$$Y_2 = (z_1 + f_1(z_2))\frac{\partial}{\partial z_1} + f_2(z_2)\frac{\partial}{\partial z_2},$$

$$Y_3 = (z_1^2 + 2z_1 f_1(z_2) + g_1(z_2))\frac{\partial}{\partial z_1} + (2z_1 f_2(z_2) + g_2(z_2))\frac{\partial}{\partial z_2},$$

where $f_1(0) = f_2(0) = g_1(0) = g_2(0) = 0$.

In case (b), $[Y_1, X] = 0$. In case (a), since Y_1 tracks X and $Y_1(p) \neq 0$, there always exists a function λ, defined around p, with no zero such that $[Y_1, \lambda X] = 0$. Therefore, as $\mathcal{A}$ tracks λX too and it suffices to prove the result for λX, we may assume that $[Y_1, X] = 0$ without loss of generality. Thus, $X = h_1(z_2)(\partial/\partial z_1) + h_2(z_2)(\partial/\partial z_2)$ with $h_1(0) = h_2(0) = 0$. Since X is non-trivial, there exists an integer $n \geq 1$ such that $X = z_2^n(\varphi_1(z_2)(\partial/\partial z_1) + \varphi_2(z_2)(\partial/\partial z_2))$, where at least $\varphi_1(0) \neq 0$ or $\varphi_2(0) \neq 0$.

Assume that $\varphi_1(0) \neq 0$ and $\varphi_2(0) = 0$. Set $\widetilde{X} = \varphi_1(z_2)(\partial/\partial z_1) + \varphi_2(z_2)(\partial/\partial z_2)$; then $\widetilde{X} \wedge [\widetilde{X}, \mathcal{A}] = 0$ because $\mathcal{A}$ tracks X. Let Q be the quotient 1-manifold, around p, of N by $\widetilde{X}$ and let q be the projection of p. Then $\mathcal{A}$ projects onto a Lie algebra $\mathcal{B}$ of vector fields on Q, which is either zero or simple. But $\mathcal{B}(q) = 0$ since $(\widetilde{X} \wedge Y_1)(p) = 0$ and $Y_2(p) = Y_3(p) = 0$, so $\mathcal{B} = 0$. In other words, $\mathcal{A}$ is tangent to $\widetilde{X}$ and $Y_1 \wedge \widetilde{X} = 0$ everywhere, which proves the first possibility in the case (a). In the case (b), one has $X = h_1(z_2)(\partial/\partial z_1)$ and $Y_2 = (z_1 + f_1(z_2))(\partial/\partial z_1)$, so $[X, Y_2] \neq 0$, which is a contradiction.

In short we may suppose that $\varphi_2(0) \neq 0$. Now taking $z_1 + g(z_2)$ instead of z_1 for a suitable function $g(z_2)$ allows us to suppose that $\widetilde{X} = \varphi_2(z_2)(\partial/\partial z_2)$ and $X = z_2^n\varphi_2(z_2)(\partial/\partial z_2)$. On the other hand, it is well known that z_2 can be modified in such a way that $X = az_2^n(\partial/\partial z_2)$, $a \in \mathbb{C} \setminus \{0\}$.

The remainder of the proof easily follows from the fact that $\mathcal{A}$ tracks X (case (a)) or $[X, \mathcal{A}] = 0$ (case (b)) and $[Y_2, Y_3] = Y_3$, and it is left to the reader. □

6. *Proofs of the main theorems*

Proof of Theorem 1.1.　　Let us sketch this proof. First one shows that we may assume that M is connected and K is a connected analytic 1-manifold. Then one proves that $\mathsf{Z}(Y) \cap K \neq \varnothing$ for all $Y \in \mathcal{G}$ (Lemma 6.1), by analyzing the structure of the dependence set $\mathsf{D}(X, Y)$ near K. It follows that K is either S^1 or $\mathbb{C}P^1$.

Finally, one studies the image $\mathcal{H}$ of $\mathcal{G}$ in $\mathcal{V}^\omega(K)$, which is a Lie algebra of dimension ≤ 3. The case that $\dim \mathcal{H} \leq 2$ is routine, and so is the case that $\mathcal{H}$ is real and three dimensional. When $\mathcal{H}$ is complex and three dimensional, a more sophisticated argument is needed.

And now the full proof.

Some component K_0 of the compact analytic variety K is an essential block for X, with negative or odd index if that of K was negative or odd, because the index is additive over components. Shrinking U if necessary, some connected component U_0 of U is isolating for (X, K), and K is an essential block for $X|U_0$. As it suffices to prove that $\mathsf{Z}(\mathcal{G}|U_0) \cap K \neq \varnothing$, we assume henceforth that

M and K are connected.

Proceeding by contradiction, we assume that

$$\mathsf{Z}(\mathcal{G}) \cap K = \varnothing. \tag{4}$$

This implies that:

(A) $\dim K = 1$. For the analytic manifold M is connected and two dimensional (over $\mathbb{F}$), and the analytic subvariety K is proper because X is not the trivial vector field;

(B) *if $p \in K$, its orbit $\mathcal{G}p$ under $\mathcal{G}$ has dimension ≤ 1.* This follows from (A) because the $\mathcal{G}$-invariance of K (Proposition 4.1) implies that $\dim \mathcal{G}p \leq \dim K$;

(C) *K is an analytic regular submanifold.* The variety K is $\mathcal{G}$-invariant (Proposition 4.1). Its singular variety K_{sing} is also $\mathcal{G}$-invariant because the vector fields in $\mathcal{G}$ are analytic. Since $\dim K_{\mathrm{sing}} < \dim K$, from (A) we infer that K_{sing} is discrete and contained in $\mathsf{Z}(\mathcal{G})$. Therefore, $K_{\mathrm{sing}} = \varnothing$ by equation (4).

See Remark 6.2 at the end of this proof as well.

LEMMA 6.1. $\mathsf{Z}(Y) \cap K \neq \varnothing$ *for every $Y \in \mathcal{G}$.*

Proof. Assume that *per contra:* $Y \in \mathcal{G}$, $\mathsf{Z}(Y) \cap K = \varnothing$.

Case (i): The dependency set $\mathsf{D}(X, Y) = M$. If $\mathbb{F} = \mathbb{R}$, we reach the contradiction $\mathsf{i}_K(X) = 0$ from Lemma 5.1, setting $r = 1$, $X_1 = Y$. If $\mathbb{F} = \mathbb{C}$, the same contradiction is reached by setting $r = 2$, $X_1 = Y$, $X_2 = iY$.

Case (ii): $\mathsf{D}(X, Y) \neq M$. Since $\mathsf{D}(X, Y) \setminus \mathsf{Z}(Y)$ is Y-invariant (Proposition 4.1), reasoning as in (C) shows that $\mathsf{D}(X, Y) \setminus \mathsf{Z}(Y)$ is a one-dimensional analytic regular submanifold, evidently containing K. Fix an open set $U \subset M$ such that $\mathsf{Z}(X) \cap U = K$ and $U \cap \mathsf{D}(X, Y) = K$. Let $\phi : M \to \mathbb{R}$ be a continuous function with compact support included in U such that $\phi(K) = 1$. The vector fields $X^\epsilon := X + \epsilon \phi Y$, $\epsilon > 0$, have no zeros in U. Since they approximate X and $\mathsf{i}_K(X)$ is stable under perturbation, we have the contradiction $\mathsf{i}_K(X) = 0$. □

It follows that we can assume that $K = S^1$ in the real case.

In the complex case we can assume that $K = \mathbb{C}P^1$. For there always exists $Y \in \mathcal{G}$ whose restriction to K does not vanish identically; but this restriction has zeros, all of them of positive index because of holomorphy, so $\chi(K) > 0$.

Define

$$\mathcal{I} := \{Y \in \mathcal{G} \colon Y|K = 0\},$$

which is an ideal in $\mathcal{G}$. The image of $\mathcal{G}$ in $\mathcal{V}^\omega(K)$ maps $\mathcal{G}/\mathcal{I}$ isomorphically onto a subalgebra $\mathcal{H} \subset \mathcal{V}^\omega(K)$.

Each element of $\mathcal{H}$ vanishes somewhere and $\mathcal{H}$ is transitive, that is, $\dim \mathcal{H}(p) = 1$ for any $p \in K$, because $\mathsf{Z}(\mathcal{G}) \cap K = \varnothing$.

As $\dim K = 1$, from Lemma 5.3 it follows that up to isomorphism $\mathcal{H}$ has to be the trivial algebra $\{0\}$, a one-dimensional algebra, the affine algebra of $\mathbb{F}$ or $\mathfrak{sl}(2, \mathbb{F})$.

First assume that *per contra:* $\mathbb{F} = \mathbb{C}$, $\mathcal{H} = \mathfrak{sl}(2, \mathbb{C})$ and $\mathsf{i}_K(X)$ is negative or odd. Since $\mathfrak{sl}(2, \mathbb{C})$ is simple, there exists a subalgebra $\mathcal{A} \subset \mathcal{G}$ isomorphic to $\mathcal{H}$ (see Jacobson [20]).

Consider a point $p \in K$. If near p all orbits of $\mathcal{A}$ have dimension one and X is tangent to them, by analyticity since K is an $\mathcal{A}$-orbit of dimension one and X is tangent to it there

is an open neighborhood D of K such that any $\mathcal{A}$-orbit on D has dimension one and X is tangent to it. Thus, on D the Lie algebra $\mathcal{A}$ defines a 1-foliation $\mathcal{F}$ to which X is tangent. But K is a compact simply connected leaf of $\mathcal{F}$, so near it the foliation $\mathcal{F}$ is a product.

Let $L \neq K$ be a compact leaf sufficiently close to K. Then X, which is tangent to L, does not vanish on it, so $\chi(L) = 0$. But topologically L is $\mathbb{C}P^1$, which is a contradiction.

Therefore, from (a) of Lemma 5.6 applied to X and $\mathcal{A}$ (we have just seen that the first alternative of (a) is forbidden) it follows that $\mathsf{i}_K(X)$ equals $2n > 0$ since transversely to K the index of X is n. But by hypothesis (*) $\mathsf{i}_K(X)$ is negative or odd, a contradiction again. In short: $\mathcal{H} \neq \mathfrak{sl}(2, \mathbb{C})$.

Now assume that $\mathbb{F} = \mathbb{R}$ and $\mathcal{H} = \mathfrak{sl}(2, \mathbb{R})$. Then there is $T \in \mathcal{H}$ which does not belong to any 2-subalgebra; for instance if ad_T has some non-real eigenvalue [20]. This means that T never vanishes on K, otherwise if $T(q) = 0$ for some $q \in K$, then T belongs to the two-dimensional subalgebra $\mathcal{H}_0(q)$. Therefore, $\mathfrak{sl}(2, \mathbb{R})$ is excluded as well.

Finally, if $\mathcal{H}$ is the affine algebra of $\mathbb{F}$, there exists a basis $\{T_1, T_2\}$ of $\mathcal{H}$ such that $[T_1, T_2] = T_2$. But $T_2(q) = 0$ for some $q \in K$, so $T_1(q) = 0$ too, otherwise $[T_1, T_2] \neq T_2$. In other words, the affine algebra is not transitive.

Summing up: there is no way for choosing the subalgebra $\mathcal{H}$, so assuming that $\mathsf{Z}(\mathcal{G}) \cap K = \varnothing$ leads to a contradiction. $\qquad\square$

Remark 6.2. Readers not familiarized with varieties can reason as follows. Start by assuming that $\mathsf{Z}(\mathcal{G}) \cap K = \varnothing$; obviously $\dim \mathcal{G}(p) \geq 1$, $p \in K$. If $\dim \mathcal{G}(q) = 2$ for some $q \in K$, then its $\mathcal{G}$-orbit is an open set included in K (see Proposition 4.1), and $X = 0$, which is a contradiction. Given $p \in K$, take $Y \in \mathcal{G}$ with $Y(p) \neq 0$ and consider coordinates (A, x_1, x_2) around $p \equiv (0, 0)$ as in the proof of Proposition 4.1; let T be the transversal to Y defined by $x_1 = 0$. Then $K \cap T$ is the set of simultaneous zeros of two analytic functions of one variable, one of them at least non-trivial; so it is isolated in T and consists of a single point if A is sufficiently small. In this last case $A \cap K$ is given by the equation $x_2 = 0$, which shows that K is a regular 1-submanifold.

Some component K_0 of K is an essential block for X, with negative or odd index if that of K was negative or odd, etc.

Proof of Theorem 2.6. First recall some elementary facts about $\mathfrak{sl}(2, \mathbb{C})$.

Remark 6.3. Given $\varphi \in \mathfrak{sl}(2, \mathbb{C}) \setminus \{0\}$, one can find a basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ such that:

- $\varphi = a(e_1 \otimes e_1^* - e_2 \otimes e_2^*)$, $a \in \mathbb{C} \setminus \{0\}$, if φ is invertible. In this case the connected subgroup of $SL(2, \mathbb{C})$ whose subalgebra is spanned by φ is closed and isomorphic to the multiplicative group $\mathbb{C} \setminus \{0\}$;
- $\varphi = e_2 \otimes e_1^*$ if φ is not invertible. Now the connected subgroup determined by φ is closed and isomorphic to $\mathbb{C}$.

Therefore, the projective vector field Y_φ associated to φ can be identified, under conjugation by $PGL(2, \mathbb{C})$, to that whose restriction to $\mathbb{C} \subset \mathbb{C}P^1$ is written as:

- $2az(\partial/\partial z)$, $a \in \mathbb{C} \setminus \{0\}$, if φ is invertible. Then the vector field Y_φ possesses two singularities on $\mathbb{C}P^1$ both of them of index one and eigenvalues of its linear part $\pm 2(-\det\varphi)^{1/2}$;

- $\partial/\partial z$ if φ is not invertible. Observe that $\partial/\partial z$ and $z^2(\partial/\partial z)$ are $PGL(2, \mathbb{C})$ conjugated, so Y_φ can be represented by $z^2(\partial/\partial z)$ too.

And now the proof of Theorem 2.6.

Let us give a sketch of the proof. One starts by proving the existence of a subalgebra $\mathcal{A}$ of $\mathcal{G}$ which is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ such that $[X, \mathcal{A}] = 0$. Then one shows that $\mathsf{Z}(\mathcal{A}) = \emptyset$ and each connected component of $\mathsf{Z}(X)$ is an orbit of $\mathcal{A}$ diffeomorphic to $\mathbb{C}P^1$ (Lemma 6.4).

On the other hand, on $M \backslash \mathsf{Z}(X)$ the vector field X defines a complex 1-foliation $\mathcal{F}$ whose leaves are planes, tori or cylinders. Moreover, each leaf of $\mathcal{F}$ is closed in $M \backslash \mathsf{Z}(X)$ (Lemma 6.5) and the action of $SL(2, \mathbb{C})$ associated to $\mathcal{A}$ is transitive on the set of leaves of $\mathcal{F}$.

After observing that the leaves of $\mathcal{F}$ are never compact, one completes each of them with one or two points of $\mathsf{Z}(X)$ for giving rise to a second foliation $\mathcal{F}'$ on M with all of its leaves diffeomorphic to $\mathbb{C}P^1$. Even more, $\mathcal{F}'$ is given by a fibration $\pi \colon M \to \mathbb{C}P^1$. Besides, one has two possibilities.

- Any leaf of $\mathcal{F}$ is a cylinder and is completed with two points of $\mathsf{Z}(X)$.
- Every leaf of $\mathcal{F}$ is a plane and is completed with one point of $\mathsf{Z}(X)$.

In the first case $\mathsf{Z}(X)$ possesses two connected components P_1, P_2 and $\pi \colon M \backslash P_2 \to \mathbb{C}P^1$ is a line fibre bundle, which implies that X and $\mathcal{A}$ follow Model 2.4.

In the second one the orbits of the action of $SL(2, \mathbb{C})$ on M have dimension one (Lemmas 6.4 and 6.6), so this action defines a second complex 1-foliation $\mathcal{F}''$, which is transverse to $\mathcal{F}'$. Moreover, $\mathcal{F}''$ is defined by a fibration $\pi' \colon M \to \mathbb{C}P^1$. Thus, $\pi \times \pi' \colon M \to \mathbb{C}P^1 \times \mathbb{C}P^1$ is a diffeomorphism and X and $\mathcal{A}$ follow Model 2.1.

Now we are ready to go into details.

First recall that if Q is a connected complex manifold of dimension one and $\mathcal{B} \subset V^\omega(Q)$ a finite-dimensional Lie algebra such that $\mathsf{Z}(\mathcal{B}) \neq \varnothing$, then $\mathcal{B}$ is solvable (see Lemma 5.3).

Now assume that $\mathsf{Z}(X) \cap \mathsf{Z}(\mathcal{G}) = \varnothing$. Reasoning as in the proof of Theorem 1.1 shows that $\mathsf{Z}(X)$ is a compact 1-submanifold of M, obviously non-empty since $\chi(M) \neq 0$. Moreover, there has to exist a component K of $\mathsf{Z}(X)$ diffeomorphic to $\mathbb{C}P^1$ such that the image $\mathcal{H}$ of $\mathcal{G}$ in $V^\omega(K)$ under the restriction is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$; otherwise $K \cap \mathsf{Z}(\mathcal{G}) \neq \varnothing$.

As $\mathcal{H}$ is simple, there is a subalgebra $\mathcal{A}$ of $\mathcal{G}$ isomorphic under restriction to $\mathcal{H}$. This algebra $\mathcal{A}$ tracks X, so $[Y, X] = a_Y X$, $a_Y \in \mathbb{C}$, for any $Y \in \mathcal{A}$. But $\{Y \in \mathcal{A} : a_Y = 0\}$ is a non-zero ideal of $\mathcal{A}$; therefore, every $a_Y = 0$ and $[X, \mathcal{A}] = 0$.

LEMMA 6.4. $\mathsf{Z}(\mathcal{A}) = \varnothing$, *each connected component of* $\mathsf{Z}(X)$ *is diffeomorphic to* $\mathbb{C}P^1$ *and* $\mathcal{A}$ *acts transitively on it.*

Proof. Let P be a component of $\mathsf{Z}(X)$. If $P \cap \mathsf{Z}(\mathcal{A}) \neq \varnothing$, then the restriction of $\mathcal{A}$ to P has to be solvable and so zero. That is, $\mathsf{Z}(\mathcal{A}) \supset P$, which contradicts Lemma 5.4. In short $\mathsf{Z}(\mathcal{A}) \cap \mathsf{Z}(X) = \varnothing$, so $\mathsf{Z}(\mathcal{A}) = \varnothing$ since again by Lemma 5.4 $\mathsf{Z}(\mathcal{A}) \subset \mathsf{Z}(X)$, and $\mathcal{A}$ acts transitively on every component of $\mathsf{Z}(X)$. All these components are spheres, that is, $\mathbb{C}P^1$, because on each of them some $Y \in \mathcal{A} \setminus \{0\}$ has a zero. $\square$

On the connected open set $M \setminus Z(X)$ the vector field X defines a complex one-dimensional foliation $\mathcal{F}$, which is the real 2-foliation associated to the commuting vector fields X, iX. Thus, its leaves are planes, cylinders or tori because X is complete on $M \setminus Z(X)$.

LEMMA 6.5. *Each leaf L of $\mathcal{F}$ is closed in $M \setminus Z(X)$.*

Proof. Given $p \in L$, Lemma 5.5 shows the existence of $Y \in \mathcal{A} \setminus \{0\}$ and an open set $p \in D \subset M \setminus Z(X)$ such that the connected component of $L \cap D$ relative to p is included in $Z(Y) \cap D$. Therefore, $L \subset Z(Y) \cap (M \setminus Z(X))$ since $[X, Y] = 0$ implies that $Z(Y)$ is X-invariant; even more, $Z(Y) \cap (M \setminus Z(X))$ is a union of leaves of $\mathcal{F}$.

But the same lemma shows that $Z(Y) \cap (M \setminus Z(X))$ is a closed regular 1-submanifold of $M \setminus Z(X)$. Since different leaves of $\mathcal{F}$ are disjoint, it follows that L is a component of $Z(Y) \cap (M \setminus Z(X))$; in other words, L is an open and closed subset of $Z(Y) \cap (M \setminus Z(X))$ and so closed in $M \setminus Z(X)$. $\qquad \square$

On the other hand, since M is compact, the group $\mathrm{SL}(2, \mathbb{C})$ acts on M, with infinitesimal action $\mathcal{A}$, and on $M \setminus Z(X)$ as well. Observe that this action is $\mathcal{F}$-foliate and transversely transitive. Therefore, given $L_1, L_2 \in \mathcal{F}$, there always exists $g \in \mathrm{SL}(2, \mathbb{C})$ such that $g \cdot L_1 = L_2$.

Now take $p \in Z(X)$ and consider coordinates $z = (z_1, z_2)$ like in (b) of Lemma 5.6 with domain A of polydisk type. Then the trace of $\mathcal{F}$ on A is given by the slices $z_1 = constant$, and $Z(X) \cap A$ by $z_2 = 0$. Thus, the set defined by the conditions $z_1 = 0$ and $z \neq (0, 0)$ is included in a leaf L of $\mathcal{F}$. Therefore, L and any leaf of $\mathcal{F}$ are non-compact, that is, they are (real) planes or cylinders. Moreover, $L \cup \{p\}$ as a real surface has one end fewer than L. In this way adding the points of $Z(X)$ to the leaves of $\mathcal{F}$ gives rise to a new complex 1-foliation $\mathcal{F}'$ on M, whose trace on $M \setminus Z(X)$ is $\mathcal{F}$, and the action of $\mathrm{SL}(2, \mathbb{C})$ on the set of its leaves is still transitive.

Notice that any leaf $\tilde{L}$ of $\mathcal{F}$ at most intersects two slices of A; indeed, if S is a slice of A and $S \cap \tilde{L} \neq \varnothing$, then this non-empty intersection defines an end of $\tilde{L}$. Therefore, since the leaves of $\mathcal{F}$ are closed in $M \setminus Z(X)$, those of $\mathcal{F}'$ are closed, and so compact, in M.

The procedure above kills the ends of every leaf of $\mathcal{F}$, so each leaf of $\mathcal{F}'$ is topologically the sphere S^2, so $\mathbb{C}P^1$. Since the action of $\mathrm{SL}(2, \mathbb{C})$ is transversely transitive, the foliation $\mathcal{F}'$ is given by a (complex) fibre bundle $\pi: M \to Q$, where Q is a compact connected complex 1-manifold.

Finally, from $[X, \mathcal{A}] = 0$ it follows that $\mathcal{A}$ projects onto a Lie algebra $\mathcal{A}' \subset \mathcal{V}^\omega(Q)$ which is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Therefore, Q is the projective line, so from now on we will write $\pi: M \to \mathbb{C}P^1$. Observe that $\chi(M) = 4$.

Let P be a component of $Z(X)$. By construction of the foliation, P is transverse to $\mathcal{F}'$ and hence $\pi: P \to \mathbb{C}P^1$ is a local diffeomorphism. But P is compact, so $\pi: P \to \mathbb{C}P^1$ is a covering space. Since $\mathbb{C}P^1$ is simply connected and P is connected, $\pi: P \to \mathbb{C}P^1$ has to be a diffeomorphism. Finally, as $\mathcal{F}'$ is given by $\pi: M \to \mathbb{C}P^1$, one concludes that every leaf of $\mathcal{F}'$ intersects P once.

From Lemma 5.6, it follows that transversely to P the index of X equals n. Since $\chi(M) = 4$, we have just two possibilities.

- $Z(X)$ has two components and $n = 1$, that is, the leaves of $\mathcal{F}$ are cylinders.
- $Z(X)$ is connected and $n = 2$, that is, the leaves of $\mathcal{F}$ are (real) planes.

First assume that the leaves of $\mathcal{F}$ are cylinders.

Set $Z(X) = P_1 \cup P_2$ as the union of its components. The eigenvalue of the linear part of X transversely to P_1 is a holomorphic function on P_1 and so constant. Thus, considering aX instead of X for a suitable $a \in \mathbb{C} \setminus \{0\}$ allows assuming that this eigenvalue equals one. Therefore, for each $\pi^{-1}(q)$, $q \in \mathbb{C}P^1$, the projective vector field X has two singularities of index one and the eigenvalue of its linear part at $\pi^{-1}(q) \cap P_1$ equals one. By Remark 6.3, $(\pi^{-1}(q) \cap (M - P_2), X)$ is diffeomorphic to $(\mathbb{C}, z\partial/\partial z)$.

Since any diffeomorphism $\rho \colon \mathbb{C} \to \mathbb{C}$ which preserves $z\partial/\partial z$ is a linear automorphism and the action of $SL(2, \mathbb{C})$ associated to $\mathcal{A}$ preserves X, it follows that $\pi \colon M - P_2 \to \mathbb{C}P^1$ is a line fibre bundle endowed with a fibre action of $SL(2, \mathbb{C})$. Now it is obvious that X and $\mathcal{A}$ follow the construction of Model 2.4.

Now assume that the leaves of $\mathcal{F}$ are planes.

Then $\pi \colon M \backslash Z(X) \to \mathbb{C}P^1$ is a homotopy equivalence. As $[X, \mathcal{A}] = 0$ and X is transverse to the orbits of the action of $SL(2, \mathbb{C})$ on $M \backslash Z(X)$, they are diffeomorphic and so with the same dimension.

LEMMA 6.6. *The orbits of the action of* $SL(2, \mathbb{C})$ *on* $M \backslash Z(X)$ *have dimension one.*

Proof. If this dimension equals two, then $M \backslash Z(X)$ is an orbit of the action of $SL(2, \mathbb{C})$; indeed, orbits in $M \backslash Z(X)$ are open and $M \backslash Z(X)$ is connected. Take $p \in M \backslash Z(X)$; by Lemma 5.5, any $Y \in \mathcal{A} \setminus \{0\}$ such that $Y(p) = 0$ is the fundamental vector field associated to some $\varphi \in \mathfrak{sl}(2, \mathbb{C}) \setminus \{0\}$ with $\det\varphi = 0$ (consider the first variable). Therefore, $M \backslash Z(X)$ is, as homogeneous space, the quotient of $SL(2, \mathbb{C})$ by a closed subgroup H (the isotropy group of p) whose identity component is isomorphic to $\mathbb{C}$. Now the homotopy sequence of the fibre bundle

$$H \to SL(2, \mathbb{C}) \to M \backslash Z(X)$$

shows that $\pi_2(M \backslash Z(X)) = 0$ (topologically $SL(2, \mathbb{C})$ is $S^3 \times \mathbb{R}^3$). But $\pi_2(M \backslash Z(X)) = \pi_2(\mathbb{C}P^1) = \mathbb{Z}$, which is a contradiction. $\qquad\square$

Thus, the action of $SL(2, \mathbb{C})$ on M defines a second foliation of dimension one, $\mathcal{F}''$, transverse to $\mathcal{F}'$. Observe that $Z(X)$ is a compact leaf of $\mathcal{F}''$ diffeomorphic to $\mathbb{C}P^1$ and so simply connected. Therefore, near $Z(X)$ the foliation $\mathcal{F}''$ is a product.

As X is transverse to $\mathcal{F}''$ on $M \backslash Z(X)$, then all the leaves of $\mathcal{F}''$ are $\mathbb{C}P^1$ and $\mathcal{F}''$ is defined by a fibre bundle $\pi' \colon M \to Q'$, where Q' is a compact connected complex 1-manifold. But X projects onto Q' in a non-trivial vector field, so Q' is $\mathbb{C}P^1$ and the fibration becomes $\pi' \colon M \to \mathbb{C}P^1$.

In short $\pi \times \pi' \colon M \to \mathbb{C}P^1 \times \mathbb{C}P^1$ is a local diffeomorphism and so a covering and, finally, a diffeomorphism because $\mathbb{C}P^1 \times \mathbb{C}P^1$ is simply connected. Thus, $\pi \times \pi'$ identifies M and $\mathbb{C}P^1 \times \mathbb{C}P^1$ in such a way that $\mathcal{F}'$ is the foliation associated to the second factor and $\mathcal{F}''$ that given by the first factor. Now it is obvious that X and $\mathcal{A}$ are constructed like in Model 2.1. This finishes the proof of Theorem 2.6. $\qquad\square$

Remark 6.7. (On the algebra $\mathcal{G}$ of Theorem 2.6) Consider $\mathcal{G}$ as in Theorem 2.6 and the fibre bundle $\pi\colon M \to \mathbb{C}P^1$ given in this result. Let $\mathcal{A}_m \subset \mathcal{V}^\omega(M)$ be the maximal Lie algebra which includes $\mathcal{A}$ and tracks X (actually it normalizes X because M is compact). Clearly, $\mathcal{G}$ is a subalgebra of $\mathcal{A}_m$, and $\mathcal{A}_m$ projects onto the Lie algebra of projective vector fields of $\mathbb{C}P^1$. Set

$$\mathcal{I}_m := \{Y \in \mathcal{A}_m : \pi_*(Y) = 0\},$$

which is an ideal of $\mathcal{A}_m$.

For every $q \in \mathbb{C}P^1$, $\mathcal{I}_m|\pi^{-1}(q)$ is included in the normalizer of $X|\pi^{-1}(q)$ and, as $\pi^{-1}(q) = \mathbb{C}P^1$, it follows that $\mathcal{I}_m|\pi^{-1}(q)$ is a subalgebra of the Lie algebra of projective vector fields, with dimension one if $\mathsf{Z}(X)$ consists of two connected components and two if $\mathsf{Z}(X)$ is connected (see the proof of Theorem 2.6).

In the first case every $Y \in \mathcal{I}_m$ is written as $Y = fX$, where f is a holomorphic function and so constant. That is to say, $\mathcal{I}_m$ equals $\mathbb{C}\{X\} := \{aX : a \in \mathbb{C}\}$, which implies that $\mathcal{A}_m$ is the direct product of $\mathcal{I}_m$ and $\mathcal{A}$. Therefore, $\mathcal{G}$ is $\mathcal{A}$ or $\mathcal{A}_m$.

Now assume that $\mathsf{Z}(X)$ is connected. Then $M = \mathbb{C}P^1 \times \mathbb{C}P^1$, π is the first projection, $\mathcal{A}$ can be seen as the Lie algebra of projective vector fields on the first factor and X as a vector field on the second factor (see the proof of Theorem 2.6 again). Hence, one may choose a vector field $\widehat{X}$ tangent to the second factor such that $[\widehat{X}, X] = X$, $[\widehat{X}, \mathcal{A}] = 0$ and $\{X|\pi^{-1}(q), \widehat{X}|\pi^{-1}(q)\}$ is a basis of the normalizer of $X|\pi^{-1}(q)$, $q \in \mathbb{C}P^1$.

Therefore, if $Y \in \mathcal{I}_m$, then $Y = aX + \hat{a}\widehat{X}$, $a, \hat{a} \in \mathbb{C}$ (coefficients have to be holomorphic functions and so constant). Thus, $\mathcal{A}_m$ is the direct product of $\mathcal{A}$ and the two-dimensional Lie algebra spanned by $X, \widehat{X}$, while $\mathcal{G}$ equals $\mathcal{A}_m$, $\mathcal{A}$ or the direct product of $\mathbb{C}\{X\}$ and $\mathcal{A}$.

Summing up, in compact connected complex 2-manifolds, Theorem 1.1 only fails with three Lie algebras: $\mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{gl}(2, \mathbb{C})$ and the product of $\mathfrak{sl}(2, \mathbb{C})$ with the affine algebra of $\mathbb{C}$ (compare this fact to Example 3.2).

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