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Abstract

In recent years a number of empirical results indicate that humans tend to use mental models during the reasoning process. This theory claims that humans represent and reason about spatial information by constructing, inspecting, and generating alternative models to check for a putative conclusion. New results about preferred models and local transformations made it necessary to refine the classical theory. From a formal perspective, however, a number of technical questions w.r.t. the explicit details of the theory are still unanswered. In this paper we formally ground the preferred mental model theory with the help of formal logic. This formalization allows to show inherent differences between formal and human reasoning and allows to identify a source of human errors in reasoning.

Keywords: Cognitive Modeling; Qualitative Reasoning

Introduction

Imagine a friend (at an airshow) tells[1] you:

(1) The red plane is north-west of the yellow plane.
The orange plane is east of the yellow plane.
The yellow plane is west of the green plane.
The green plane is west of the blue plane.

Can you (easily) infer if the blue plane must be (necessarily) east of the orange plane? The question on how humans solve such deduction problems is at the core of qualitative reasoning. In other words, how do we infer new knowledge (a *conclusion*)from given knowledge, and moreover, what are the differences to formal approaches in artificial intelligence? Formally, there are two main approaches in AI on how such reasoning problems can be solved: By the application of (transitivity) rules or by the construction and inspection of models. Principally, both approaches are equivalent (Russell & Norvig, 2003), i.e. it is not possible to derive more information with each of these methods. This equivalence, however, makes it harder to distinguish which method(s) is applied by humans while solving such problems. Nonetheless, a number of empirical studies investigates this research question by psychological means. The most prominent and best supported theory with respect to the number of effects that can be accounted for is the theory of mental models (MMT) (Johnson-Laird & Byrne, 1991) (to name only a few: the indeterminacy effect (Johnson-Laird & Byrne, 1991), the form of premises and the figural effect (Knauff, Rauh, Schlieder, & Strube, 1998), the wording of conclusions (Henst & Schaeken, 2007), etc.). According to the MMT, linguistic processes are relevant to transfer information from the premises into a spatial array and back again, but the reasoning process itself relies on model manipulation only. A *mental model* is an internal representation of objects and relations in spatial working memory, which matches the state of affairs given in the premises. The semantic theory of mental models is based on the mathematical definition of deduction, i.e. a propositional statement φ is a consequence of a set of premises $\mathcal{P}$, written $\mathcal{P} \models \varphi$, if in each model $\mathcal{A}$ of $\mathcal{P}$, the conclusion φ is true.

[1] We assume he uses the cardinal direction relations to describe positions, i.e. N, NE, E, SE, S, SW, W, NW, EQ

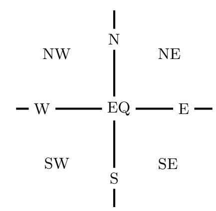

Figure 1: The nine cardinal direction relations.

In recent years, however, more and more empirical investigations show gaps in this theory. For example, the classical theory is not able to explain the preference effect, i.e. in multiple model cases (nearly always) one preferred model is constructed from participants and used as a reference for the deduction process (Rauh et al., 2005). Further findings showed that during the validation phase alternative models are constructed by small modifications to the initially constructed model. This was the reason why the mental model theory was precised within the framework of preferred mental models (Knauff, 2007).

In this paper we formally ground the theory of preferred mental models. With the help of this formalization we can work out consequences which can then be empirically tested, and additionally, based on this, differences between human and formal reasoning can be identified. This leads to an explanation for a source of errors in human reasoning: the "transformational distance" between the counter-example and the preferred model.

State of the Art

There are a number of different approaches of how humans solve such problems in the literature. We will briefly introduce three important ones:

Theory of Mental Logic

The central idea of this approach can be characterized as follows: "Reasoning consists in the application of mental inference rules to the premises and conclusion of an argument. The sequence of applied rules forms a mental proof or derivation of the conclusion from the premises, where these im-

1. $West(x, y)$ & $North(z, x) \rightarrow West(z, y)$
2. $West(x, y)$ & $North(z, y) \rightarrow West(x, z)$
3. $West(x, y)$ & $West(y, z) \rightarrow West(x, z)$
4. $West(x, y) \leftrightarrow East(y, x)$
5. $(West(y, x)$ & $West(z, x)) \rightarrow (West(y, z)$ *or* $West(z, y))$
6. $(West(y, z)$ *or* $West(z, y))$ & $North(w, z) \rightarrow$ $(West(y, w)$ *or* $West(w, y))$

Figure 2: Set of (incomplete) inference rules specified for spatial reasoning adapted from Van der Henst (2002).

plicit proofs are analogous to the explicit proofs of elementary logic" (Rips, 1994, p. 40). Hagert (1984) defined a first set of spatial inference rules (cf. Fig. 2). This set of rules has been extended by two additional rules (cf. the rules 5 and 6 in Fig. 2) to deal with indeterminacy by Van der Henst (2002). The rules in Fig. 2 are successively applied to the premises of a problem description.

Theory of Mental Models

The mental model theory assumes that the human reasoning process consists of three distinct phases: The *model generation phase*, in which a first model is constructed out of the premises, an *inspection phase*, in which the model is inspected to check if a putative conclusion is consistent with the current model. In the *validation phase*, finally, alternative models are generated from the premises that refute this putative conclusion. In our example presented above, the exact relation between "Y" and "G" is not specified and leads to multiple-model cases like R Y O G B, R Y G O B, and R Y G B O (by projecting R on the horizontal line). This indeterminacy effect is mainly responsible for human difficulty in reasoning (Johnson-Laird, 2001).

Theory of Preferred Mental Models

The classical MMT is not able to explain a phenomenon encountered in multiple-model cases, namely that humans generally tend to construct a *preferred mental model* (PMM). Compared to all other possible models, the PMM is easier to construct and easier to maintain in working memory (Knauff et al., 1998). The principle of economicity is the determining factor in explaining human preferences (Manktelow, 1999). This principle also explains that a model is constructed incrementally from its premises. Such a model construction process saves working memory capacities because each bit of information is immediately processed and integrated into the model (Johnson-Laird & Byrne, 1991). In the model variation phase, this PMM is varied to find alternative interpretations of the premises (Rauh et al., 2005). From a formal point of view, however, this theory has not been formalized yet and is therefore not fully specified in terms of necessary operations to process such simple problems as were described above. In other words, the use, construction, and inspection of mental models have been handled in a rather implicit and vague way (Johnson-Laird, 2001; Vandierendonck, Dierckx, & Vooght, 2004).

A Formal Investigation

In this section we compare the rule-based approaches to the mental model theory approaches (Ragni & Knauff, 2008). Assume we want to derive for the above example that the blue plane is necessarily to the eastern to the red plane. To solve this problem it is sufficient to apply the third and the fourth rule of Fig. 2 only (we interprete north-west as north & west, and so on). Namely,

$$West(R, Y) \,\&\, West(Y, G) \rightarrow West(R, G)$$

we derive that the red plane (R) is west of the green plane (G)

$$West(R, G) \,\&\, West(G, B) \rightarrow West(R, B)$$

the red plane (R) is west of the blue plane (B), and finally

$$West(R, B) \leftrightarrow East(B, R)$$

The blue plane is east of the red plane. In other words, a minimal solution needs three inference steps. Throughout the reasoning process a large number of relations have to be stored, namely all premise information and all inferred relations, which can lead to a high load on the working memory. This is the first difference to reasoning with models. A mental model is an *integrated representation* of relational information, i.e. the relation do not have to be stored but are implicitly represented by the relation between the objects.

The second point is the high number of rules to be specified for each relation to infer new information. It is possible to show that a relational system consisting of two natural relations like *adjacent* left and *distant* left on a discrete structure is not closed under composition making an in-cognitive high-number of rules necessary to prove all valid conclusions (Ragni, 2003). In the mental model theory it is sufficient to generate and alternate models – information about how to infer new relational information is not necessary.

The third point ist the way human reasoning difficulty is explained: According to rule-based theories it depends mainly on two things: On the number of rules that are applied to derive a conclusion and on the sort of inference rules. If we assume that the rules are equally difficult, then the main difficulty depends on the number of rules that are to be applied. That is, the more inference steps are necessary the more difficult is a conclusion (Johnson-Laird & Byrne, 1991; Henst, 2002). This idea, however, implicitly assumes that humans use some kind of search procedure which involves necessarily an optimal search strategy. A search strategy is called *optimal* if it always finds the minimal solution for a problem. For instance, breadth-search is an optimal search strategy while depth-first is not (Russell & Norvig, 2003). Obviously, the search strategy is an important factor in determining the complexity of a problem, but there is, so far, no data that indicates which kind of search procedure is applied by humans. To make this point more precise: Assume a set of premise $\mathcal{P}$ and two conclusions ϕ_1 and ϕ_2 are given, where the inference of ϕ_1 from $\mathcal{P}$ takes 4 steps and to infer ϕ_2

from $\mathcal{P}$ takes 5 steps. If it cannot be generally assumed that reasoners necessarily choose the minimal derivation steps to derive a putative conclusion, then $(\mathcal{P}, \phi_1)$ is not always easier than $(\mathcal{P}, \phi_2)$. Otherwise, the reasoner would always have to know which inference has been drawn in the next step–a property normally assumed with nondeterministic rather than deterministic systems.

Nonetheless, from a formal perspective both theories–the theory of mental logic and of mental models–have the same explanatory power (Russell & Norvig, 2003). That is why both theories must be distinguished by analyzing empirical data of behavioral experiments.

The fourth point is the number of models vs. the number of rules: One of the very first experiments conducted (Johnson-Laird & Byrne, 1991) investigated determinate and indeterminate problems. A determinate variant (only one model is consistent) of the indeterminate problem can be achieved if the third premise is replaced with "The green plane is east of the orange plane". If we apply the rules 1-4 of Fig. 2 to the determinate and indeterminate problem description, the conclusion can be derived in the same number of steps (Henst, 2002). So from a pure rule-based perspective, both problems have the same difficulty. Nonetheless, empirical studies show that determinate problems are significantly easier to solve than indeterminate problems (Johnson-Laird & Byrne, 1991). Van der Henst (2002) argues to extend the rules with the rules 5 and 6. But these rules are in some sense artificial and do not necessarily derive the conclusion. Therefore it seems to be a post-hoc alteration of the specified set of rules to explain the data (Goodwin & Johnson-Laird, 2005). The mental model theory explains this phenomenon by the help of the model relation. To check if the putative conclusion "R is to the west of G" can be derived, all consistent models have to be checked. Since in the one-model case there is only one model consistent, the effort to check the conclusion is smaller than for checking two models.

The fifth point is the existence of preferred mental models: Recent experiments also supply arguments in favor of the mental model theory as they show that humans generally tend to construct in multiple model cases a preferred mental model (Knauff, 2007) and tend to vary this model according to the principle of local transformations (Rauh et al., 2005). The principles for constructing the preferred models adhere to reorientation and transposition operations, which are purely model based operations. These preference strategies are not limited to Allen's Interval Calculus but also hold for other calculi (Ragni, Tseden, & Knauff, 2007). This shows that the rule-based theories are so far incomplete, since they cannot explain the relations that are finally preferred by participants. It is even more remarkable that these relations adhere to very general and model-based principles.

Finally, participants received in another experiment (Ragni, Fangmeier, Webber, & Knauff, 2006) indeterminate premises and questions of different relational complexity (Halford, Wilson, & Phillips, 1998) of the form: Is D as near to C as E is near to A? The participants had to decide if the query is consistent or not. The results show that binary relations are easier to process than ternary and quaternary relations. This may be explained in the relation procesing than with the model based reasoning. But even more remarkable, participants received a model description with right, front, etc., but were able to solve a question with the relation *near*. In other words, participants received description of relations different to the relations they were asked about. It is not only remarkable that they had to translate one kind of relation into another, but that they were even able to give an answer requiring positional information. Typical transitivity rules maintain the relation of the premises, i.e. the relation of the condition is used in the consequence (cf. Fig. 2). Instead of using classical rules, it is possible to describe the "nearness" by using some kind of default logic rules. However, it is much simpler to explain such phenomena on a basis of a model-based approach.

Formalization

In this section we ground the intuitively used theories formally (and mathematically) and analyze them with respect to their reasoning power.

A *relational structure* is a tuple $(D, (R_i)_{(i \in I)})$ consisting of a domain D (sometimes called discourse universe) and a set of (usually binary) relations R_i (Russell & Norvig, 2003). For example, geographic knowledge like *New York is north-east of Washington* can be expressed by cardinal direction relations $N, NE, E, SE, \ldots$ over the domain of cities. More complex expressions can be formed by using connectives like conjunctions (New York is north-east of Washington *and* New York is in the U.S.) and disjunctions (*... or ...*). By allowing negations, we have a propositional relational language $\mathcal{L}$ over cardinal direction relations. Such relational structures can be used to describe *knowledge representation*. But how can new information be derived? The theory of mental logic (Rips, 1994) assumes that we use (transitivity) rules to draw conclusions, whereas the classical model theory argues that we use models for this inference process. The classical mental model theory (Byrne & Johnson-Laird, 1989) claims that in multiple model cases (i.e. more than one model is consistent with the premises) other models are inspected (but not which of them). A model $\mathcal{A}$ is called *consistent* with a set of premises Φ over a relational language $\mathcal{L}$ (mathematically $\mathcal{A} \models \Phi$) if all expressions of Φ are true in $\mathcal{A}$. Then a conclusion Ψ can be derived from the premise set Φ (mathematically $\Phi \models \Psi$, whereby $\models$ is called the *consequence relation*) if

$$\begin{aligned} \Phi \models \Psi \quad &\Leftrightarrow \quad \text{All models of } \Phi \text{ are models of } \Psi. \\ &\Leftrightarrow \quad \text{There is no model } \mathcal{A} \text{ with } \mathcal{A} \models \Phi \text{ and} \\ &\qquad\ \ \mathcal{A} \models \neg\Psi. \end{aligned}$$

A model $\mathcal{A}$ with the property $\mathcal{A} \models \Phi$ and $\mathcal{A} \models \neg\Psi$ is called *counter-example*. It follows if there is a counter-example to Φ and Ψ then $\Phi \models \Psi$ cannot hold.

This classical (mathematical) consequence relation, however, does not explain how initial mental models are constructed and varied (Rauh et al., 2005). Since there is a huge empirical evidence supporting the preferred mental model theory for different calculi (Rauh et al., 2005; Ragni, Fangmeier, Webber, & Knauff, 2007; Ragni, Tseden, & Knauff, 2007) it seems worth to ground this theory mathematically.

Preferred Models & Local Transformations

Before we are able to define the consequence relation for the preferred mental model theory, we introduce some notions and concepts. The empirical found preferred model PMM for a premise set Φ is written as $\mathsf{PMM}(\Phi)$. The preferred mental model theory assumes (Rauh et al., 2005) that alternative models are generated by applying local transformations (Rauh, Hagen, Schlieder, Strube, & Knauff, 2000) on the model at hand. These transformations can be in general described by the application of a mathematical operator (e.g. exchanging adjacent objects (Ragni, Fangmeier, et al., 2007)) or moving one or two objects according to a continuous transformation concept (Ragni, Tseden, & Knauff, 2007). The set of applicable operators O is domain specific since topological objects like regions have other characteristics then pure point based representation. Common to all operators, however, is that they change the relation of one object to another object w.r.t. local transformations: Such continuous transformations can be conceptualized by transitions in so-called generalized neighborhood graphs (Ragni & Wölfl, 2005). We assume all operators in the following adhere to such neighborhood changes. The *image* of a model $\mathcal{A}$ w.r.t. an operator application o is

$$img_o(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{B} = o(\mathcal{A})\}.$$

We set $\text{Transf}^1(\mathsf{PMM}(\Phi), o, \Phi) = \{\mathcal{B} \mid \mathcal{B} \models \Phi$ and $\mathcal{B} = img_o(\mathcal{A})\}$. That is the set of all models (of Φ) which can be derived by applying operator o at model $\mathcal{A}$. $\text{Transf}^k(\mathsf{PMM}(\Phi), o, \Phi)$ describes the sequence of k applications of the operator o at model Φ. It is now possible to formally define the preferred consequence relation:

Definition 1 *For a set of premises Φ, a set of operators O and preferred model $\mathsf{PMM}(\Phi)$:*

$$\begin{array}{lcl} \Phi \models_p \Psi & \Leftrightarrow & \mathsf{PMM}(\Phi) \models \Psi \textit{ and for each } \mathcal{B} \models \Phi \textit{ with} \\ & & \mathcal{B} \in \textit{Transf}^1(\mathsf{PMM}(\Phi), O) \textit{ holds } \mathcal{B} \models \Psi \\ \Phi \models_{kp} \Psi & \Leftrightarrow & \mathsf{PMM}(\Phi) \models \Psi \textit{ and for each } \mathcal{B} \models \Phi \textit{ with} \\ & & \mathcal{B} \in \textit{Transf}^k(\mathsf{PMM}(\Phi), O) \textit{ holds } \mathcal{B} \models \Psi \end{array}$$

The *restricted consequence relation* $\models_{kp}$ reflects that humans might apply only a fixed number of operations and generate only a restricted number of alternative models. We analyze now the expressibility and deductive power of these notions.

Consequence Relation

It can be shown that for the case of one-dimensional non-negated problem descriptions (without disjunction) the equivalence

$$\Phi \models \Psi \Longleftrightarrow \Phi \models_p \Psi$$

holds. In the following, we show that the equivalence does not hold for two dimensional problems by constructing a counter-example, i.e. $\Phi \models_p \Psi$ holds but not $\Phi \models \Psi$. For finite relational vocabulary (like cardinal directions) holds:

$$\neg\, a\, W\, b \Longleftrightarrow a\, \{N, NE, E, SE, S, SW, NW, EQ\}\, b.$$

that is: if a is not west of b then it can be north-east or east or south-east or south, and so on. This disjunction is (as common) abbreviated by $\{N, NE, E, SE, S, SW, NW, EQ\}$. A premise set of the form:

(2) b is north of c.
a is not south of b.
a is not north of c.

can then be written as $a\ \{SE, E, NE, N, NW, W\}\ b$ and $a\ \{NE, E, SE, S, SW, W, NW\}\ c$. There are several consistent models, for instance, a model where object a is east of b and c (cf. Fig. 3). Then there is no transformation (by small movements of a) such that a can be west of b and c, because a cannot surround b northerly and c southerly at the same time (since b is north of c). This is called *gate*.

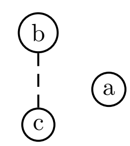

Figure 3: A gate. The premises of task (2) form an impassable gate b and c for a (i.e. $b\ N\ c$, $a\ \{SE, E, NE, N, NW, W\}\ b$, and $a\ \{NE, E, SE, S, SW, W, NW\}\ c$), since $b\ N\ c$ allows no movement of b or c and a cannot be at the same time norther of c and south of b.

Impassable gates (Fig. 3) are only possible by using negated (or disjunctive) expressions. How can gates be used to show that there are differences in reasoning with the consequence relations $\models$ and $\models_p$? This is now demonstrated by a task (consisting of a set of premises Φ and a putative conclusion Ψ) where the $\mathsf{PMM}(\Phi)$ and the counter-example are separated by a gate (in the transformation process). This shows then that there is no continuous transformation of $\mathsf{PMM}(\Phi)$ to a counter-example (due to the gate).

(3) a is west of g.
f is west of g.
b is north-west of f.
d is south-west of f.
b is north of d.
b is north of h.
d is south-east of h.
a is not south of b.
a is not north of d.

Does a is east of h follows?

The preferred mental model is constructed by the successive processing of the first two premises via the so-called first-

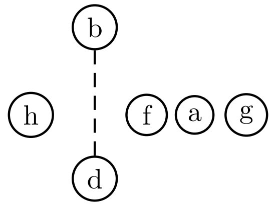

Figure 4: The consequence relations $\models$ and $\models_p$ lead to different conclusions. The model is the PMM to (3). The objects b and d form a gate for a. There is no continuous transformation of the actual position of a, such that $a\,\{NW, W, SW\}\,h$ could hold, so in all reachable models of the PMM holds $a\,\{NO, O, SO\}\,h$ and according to the definition $\Phi \models_p a\,\{NO, O, SO\}\,h$ and since there is a counter-example $\Phi \not\models a\,\{NO, O, SO\}\,h$ holds.

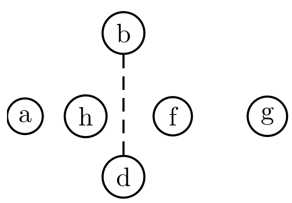

Figure 5: A consistent model to task (3) where $\Phi \not\models a\,\{NO, O, SO\}\,h$ holds.

free-fit-principle (fff) (Ragni, Fangmeier, et al., 2007). After the processing of the first two premises we get the PMM(Φ)

The negated two premises lead to an impassable gate for a (cf. Fig. 3). This task shows that in all "reachable" models (of the PMM(Φ)) only the disjunction $a\,\{NO, O, SO\}\,h$ holds (cf. Fig. 4). According to the definition of the consequence relation holds $\Phi \models_p a\,\{NO, O, SO\}\,h$. Since there is a counter-example in the classical consequence relation consistent with the premises (cf. Fig. 5). From that follows that the preferred and classical consequence relations are not equivalent. But since certain models cannot be generated, and, whenever there is no continuous transformation from the preferred model to a counter-example, a wrong conclusion is drawn (by $\models_p$). So we get the

Theorem 1 *For $\Phi \in \mathcal{L}(CD)$ holds:*

$$\Phi \models \Psi \not\Longleftrightarrow \Phi \models_p \Psi$$

That is, if humans reason according to the theory of preferred mental models (with local transformations), then certain errors are inescapable. Since the number of models are restricted for $\models_p$ as for the classical $\models$ and by using this principle the number of models for $\models_{kp}$ is even more restricted, we get a hierarchy of the consequence relations:

$$\models \;\subsetneq\; \models_p \;\subsetneq\; \models_{kp}$$

That is: The more limited the preference relation is, the more (incorrect) conclusions are drawn. An important question is if we can determine the number k (the number of models we inspect during the reasoning process). This purely cognitive question is investigated in the next section.

Transformation Distance, Complexity, and other Empirical Data

In this section the performance of this framework is very briefly compared to some experimental findings with human subjects. First, the preferred consequence relation is able to reconstruct the manipulation of models (Rauh et al., 2000) by the principle of local transformation (Schlieder, 1998). Many studies could show that participants inspect the model at hand and neglect certain models (which cannot be continuously generated from the preferred model) (Rauh et al., 2000). This is exactly what can be explained by the preferred consequence relation ($\models_p$) (which is based on the generalized neighborhood graph). Another well-known effect is that indeterminate problems are more difficult than determinate ones (Byrne & Johnson-Laird, 1989). This is reflected by the difference between the consequence relations $\models$, $\models_{kp}$, $\models_p$.

More empirical evidence comes from a study by Ragni, Fangmeier, et al. (2007). In this study, we directly tested the acceptance of different models against each other. Twenty-two students of the University of Freiburg served as paid participants. The reasoning problems consisted of four premises each that referred to horizontal one-dimensional layouts of four objects. The premises were consistent with three different arrangements. The problems were displayed on the computer screen, and the presentation was self-paced. Each trial began with the successive presentation of each premises. When participants pressed the space-key, the next premise replaced the current premise. After the presentation of the premises, a consistent or inconsistent representation appeared together with a prompt "Is this representation consistent with the premises?". The participants had to respond with yes or no with one of the response keys. The results of this study clearly support the continuous model variation and that they make only one alternation that is that they use $\models_{p1}$ as a consequence relation. That is, participants typically construct the preferred model and make one alternation (by local transformations). If they have not found a counter-example (to a premise set and a putative conclusion) in the neighborhood of the PMM they accept it (and make a wrong deduction).

General Discussion

The formalization of the preferred mental model theory by a consequence relation allows to make precise predictions about which kind of conclusion(s) are drawn (from a given set of premises) and which are neglected. Additionally, only by a formalization it is possible to compare human reasoning to approaches in AI. This also shows the limits of the human reasoning process. What are the limits of this formalization? One could object that the consequence relation cannot model that humans tend tend to forget objects during reasoning. This is true, but would make an embedding in a cognitive architecture like ACT-R necessary. Nonetheless, this approach is able to explain that if *even* humans were able to have a total recall of all objects (which is for a small domain still true) then there are inherent difficulties and differences to a formal reasoning process. Moreover, the "transformation distance" allows to explain why, for instance, experts are better in reasoning about their expert domain than novices (their search-process is more goal oriented). What are the differences to existing computational models for (spatial) reasoning? Our formalization is in some sense more abstract than for instance the UNICORE model developed by Bara, Bucciarelli, and Lombardo (2001) and is in some sense better suited for modelling "indeterminacy", i.e. if several models are consistent. This formalization also allows to model transitive and intransitive relations, as well as topological and point-based representations. In this sense it is a lot more "universal" than others. And contrary to the SRM-model (Ragni & Steffenhagen, 2007) it explains difficulty on models and not on operations and is–in this sense–completely in the spirit of the mental model theory.

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References

Bara, B., Bucciarelli, M., & Lombardo, V. (2001). Model theory of deduction: a unified computational approach. *Cognitive Science*, *25*, 839-901.

Byrne, R. M., & Johnson-Laird, P. N. (1989). Spatial reasoning. *Journal of Memory & Language*, *28*(5).

Goodwin, G. P., & Johnson-Laird, P. N. (2005). Reasoning about relations. *Psychological Review*, *112*(2).

Hagert, G. (1984). Modeling mental models: Experiments in cognitive modeling spatial reasoning. In T. O'Shea (Ed.), *Advances in artificial intelligence.*

Halford, G. S., Wilson, W. H., & Phillips, S. (1998). Processing capacity defined by relational complexity: implications for comparative, developmental, and cognitive psychology. *Behavioural Brain Science*, *21*.

Henst, J. Van der. (2002). Mental model theory versus the inference rule approach in relational reasoning. *Thinking and Reasoning*, *8*, 193-205.

Henst, J. Van der, & Schaeken, W. (2007). The wording of conclusions in relational reasoning. *Cognition*.

Johnson-Laird, P. N. (2001). Mental models and deduction. *Trends in Cognitive Sciences*, *5*(10).

Johnson-Laird, P. N., & Byrne, R. M. J. (1991). *Deduction*. Hove (UK): Erlbaum.

Knauff, M. (2007). Deduktives Denken. In J. Funke (Ed.), *Denken und Problemlösen. Enzyklopädie der Psychologie.* Göttingen: Hogrefe.

Knauff, M., Rauh, R., Schlieder, R., & Strube, G. (1998). Continuity effect and figural bias in spatial relational inference. In *Proceedings of the 20th Cognitive Science Conference* (pp. 573–578). Mahwah: Erlbaum.

Manktelow, K. (1999). *Reasoning and Thinking*. Hove: Psychology Press.

Ragni, M. (2003). An Arrangement Calculus, Its Complexity and Algorithmic Properties. In A. Günter, R. Kruse, & B. Neumann (Eds.), *KI 2003: Advances in AI, 26th German Conference on AI* (p. 580-590). Berlin: Springer.

Ragni, M., Fangmeier, T., Webber, L., & Knauff, M. (2006). Complexity in spatial reasoning. In *Proceedings of the 28th Annual Cognitive Science Conference.* Erlbaum.

Ragni, M., Fangmeier, T., Webber, L., & Knauff, M. (2007). Preferred mental models: How and why they are so important in human spatial reasoning. In T. Barkowsky, M. Knauff, D. Montello, & G. Ligozat (Eds.), *Spatial cognition v.* Berlin: Springer.

Ragni, M., & Knauff, M. (2008). Deductive spatial reasoning: A computational and a cognitive perspective. *Künstliche Intelligenz*, *1*.

Ragni, M., & Steffenhagen, F. (2007). A cognitive computational model for spatial reasoning. In *AAAI Spring Symposium 2007.* AAAI Press; Menlo Park, CA.

Ragni, M., Tseden, B., & Knauff, M. (2007). Cross cultural similarities in topological reasoning. In *Spatial information theory: Cosit 2007.* Springer.

Ragni, M., & Wölfl, S. (2005). Temporalizing spatial calculi: On generalized neighborhood graphs. In U. Furbach (Ed.), *KI 2005: Advances in AI, 28th Annual German Conference on AI* (pp. 64–78). Berlin: Springer.

Rauh, R., Hagen, C., Knauff, M., Kuss, T., Schlieder, C., & Strube, G. (2005). Preferred and Alternative Mental Models in Spatial Reasoning. *Spatial Cog. and Comp.*, *5*.

Rauh, R., Hagen, C., Schlieder, C., Strube, G., & Knauff, M. (2000). Searching for Alternatives in Spatial Reasoning: Local Transformations and Beyond. In *Proceedings of the 22nd annual conference of the cognitive science society.* Mahwah, NJ: Lawrence Erlbaum Associates.

Rips, L. J. (1994). *The psychology of proof: Deductive reasoning in human thinking*. Cambridge, MA: MIT Press.

Russell, S., & Norvig, P. (2003). *Artificial Intelligence: A Modern Approach* (2 ed.). Prentice-Hall.

Schlieder, C. (1998). Diagrammatic Transformation Processes on Two-dimensional Relational Maps. *Journal of Visual Languages & Computing*, *9*, 45-59.

Vandierendonck, A., Dierckx, V., & Vooght, G. D. (2004). Mental model construction in linear reasoning: Evidence for the construction of initial annotated models. *Quarterly Journal of Experimental Psychology*, *57A*, 1369-1391.