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On the Convergence of Local Orbital Integrals in Bad Characteristic

by

Thomas Browning

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Sug Woo Shin, Chair
Professor Martin Olsson
Professor Ken Ribet

Summer 2026

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Thomas Browning

Abstract

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We develop a general framework for proving convergence of local orbital integrals in bad characteristic. We provide technical improvements at several points, and provide concrete examples to illustrate where the difficulties remain. Finally, we apply this framework to a few specific classical groups, including symplectic groups in characteristic not equal to 2.

To the triune God, to whom I owe my life.

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Chapter 1

Introduction

1.1 Orbital Integrals

Orbital integrals are integrals over conjugacy classes in certain topological groups, typically matrix groups over a topological field. One notable setting in which orbital integrals appear is in the Arthur–Selberg trace formula, an equality between orbital integrals and traces of representations that generalizes the Poisson summation formula [1]. In the Langlands program, the Arthur–Selberg trace formula has become a central bridge between harmonic analysis and representation theory. For example, the foundational work of Jacquet–Langlands on automorphic forms on GL_2 relies crucially on the trace formula [18]. However, despite their importance, orbital integrals remain poorly understood. Before continuing further, we must first give a more precise definition of the orbital integrals that we will be considering.

Let F be a non-archimedean local field. Recall that this means that F is isomorphic to either a finite extension of the p-adic numbers $\mathbb{Q}_p$ (in characteristic zero) or a field of formal Laurent series $\mathbb{F}_q((t))$ (in positive characteristic). In either case, F is a locally compact totally disconnected topological field. Let G be a (connected) reductive group over F. Examples of reductive groups include the general linear groups GL_n, the special linear groups SL_n, the symplectic groups Sp_{2n}, as well as orthogonal and unitary groups.

The topology on F gives rise to a topology on the group of F-points $G(F)$. The group $G(F)$ is a locally compact totally disconnected topological group. One consequence of this is that $G(F)$ admits an invariant measure dg, also known as a Haar measure. In general, one must distinguish between left Haar measures that are invariant under multiplication on the left and right Haar measures that are invariant under multiplication on the right. The discrepancy between left Haar measures and right Haar measures is measured by the modular character $\delta_{G(F)} \colon G(F) \to \mathbb{R}^\times_{>0}$. But for reductive groups G, the group $G(F)$ is known to be unimodular, meaning that $\delta_{G(F)} = 1$, so the Haar measure dg is simultaneously a left Haar measure and a right Haar measure.

Now fix an element $\gamma \in G(F)$ with centralizer subgroup $G_\gamma(F) \leq G(F)$. We can identify the orbit $G(F) \cdot \gamma$ of γ in $G(F)$ with the coset space $G(F)/G_\gamma(F)$. The orbital integral of

a function f on $G(F)$ is the integral of $f(g\gamma g^{-1})$ over this coset space $G(F)/G_\gamma(F)$ with respect to a measure dg/dg_γ on $G(F)/G_\gamma(F)$ invariant under multiplication on the left by elements of $G(F)$. Such an invariant measure exists if and only if the modular character of $G(F)$ restricts to the modular character of the subgroup $G_\gamma(F)$. In our setting $G(F)$ is unimodular, so we require the centralizer subgroup $G_\gamma(F)$ to also be unimodular.

Definition 1.1. If the centralizer subgroup $G_\gamma(F)$ is unimodular, then the orbital integral of a continuous compactly supported function $f \in C_c(G(F))$ is defined to be the integral

$$\int_{G(F)/G_\gamma(F)} f(g\gamma g^{-1}) \, \frac{dg}{dg_\gamma}.$$

The most basic question one can ask about orbital integrals is whether they are even well defined, i.e., whether $G_\gamma(F)$ is unimodular and whether the resulting integral converges. This is still an open question in general.

Conjecture 1.2. *Let G be a (connected) reductive group over a non-archimedean local field F, and let $\gamma \in G(F)$. Then the centralizer $G_\gamma(F)$ is unimodular and the orbital integrals*

$$\int_{G(F)/G_\gamma(F)} f(g\gamma g^{-1}) \, \frac{dg}{dg_\gamma}$$

converge for all $f \in C_c(G(F))$.

The difficulty of Conjecture 1.2 depends greatly on the nature of γ. The two extremes are when γ is semisimple (conjugate to a diagonal matrix) and when γ is unipotent (conjugate to an upper triangular matrix with 1's along the diagonal). In general, the Jordan–Chevalley decomposition decomposes γ uniquely as a commuting product $\gamma = \gamma_s\gamma_u = \gamma_u\gamma_s$ of a semisimple part γ_s and a unipotent part γ_u.

If γ is semisimple, then unimodularity of $G_\gamma(F)$ follows from reductivity of G_γ, and convergence of the orbital integral follows from the orbit $G(F) \cdot \gamma$ being a closed subset of $G(F)$ (so that the compact support of f restricts to a compact subset of the orbit). In contrast, when γ is unipotent the situation is much less clear. Not only will G_γ fail to be reductive, but the orbit $G(F) \cdot \gamma$ will fail to be a closed subset of $G(F)$, so the compact support of f may restrict to a non-compact subset of the orbit. Then convergence of the orbital integral becomes a delicate question depending crucially on the invariant measure on the coset space $G(F)/G_\gamma(F)$.

Example 1.3. Let $G = \mathrm{SL}_2$, and let

$$\gamma = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}.$$

The centralizer G_γ consists of matrices of the form

$$\begin{bmatrix} x & y \\ & x \end{bmatrix}$$

with $x^2 = 1$. As an algebraic group, G_γ is isomorphic to $\mathbb{G}_a \times \mu_2$ which is not reductive. Despite this, $G_\gamma(F)$ is still unimodular. The full orbit $G(F) \cdot \gamma$ is complicated, so we will restrict our attention to the orbit $P(F) \cdot \gamma$ for the parabolic subgroup P of G consisting of matrices of the form

$$\begin{bmatrix} x & y \\ & x^{-1} \end{bmatrix}.$$

Then the orbit $P(F) \cdot \gamma$ consists of matrices of the form

$$\begin{bmatrix} 1 & x^2 \\ & 1 \end{bmatrix}$$

with $x \neq 0$. Notice that the orbit of γ accumulates to the identity matrix 1. In particular, if the support of f contains 1, then the orbital integral will behave like an infinite geometric series rather than a finite sum [22, §5.6].

1.2 Proving Convergence

The first progress on Conjecture 1.2 for non-semisimple γ was due to Deligne and Ranga Rao who proved convergence in characteristic zero [27]. More recently, McNinch extended their method to prove convergence in characteristic p for p good enough [25]. The exact conditions on p are technical, but the following table lists some of the cases covered by McNinch:

Reductive group	GL_n	SL_n	Sp_{2n}
Unipotent orbital integrals	all p	$p \nmid n$	$p > 2$
Arbitrary orbital integrals	$p > n$	$p > n$	$p > n + 1$

The key insight of this method employed by Deligne, Ranga Rao, and McNinch is that one can reduce to a situation in which the orbit is so well behaved that the desired invariant measure can be constructed directly as a modification of a Haar measure, at which point convergence is readily verified. Their reduction to such a nice situation involves three steps:

1. Use the Jordan–Chevalley decomposition of γ to reduce to the case where γ is unipotent.
2. Use the exponential map to pass from $\gamma \in G(F)$ to $X \in \mathfrak{g}$, the Lie algebra of G.
3. Pass from the full orbit $G(F) \cdot X$ to the orbit $P(F) \cdot X$ under a specific parabolic subgroup P.

Now the limitations of this method start to become clear. The exponential map is only well defined when the characteristic of F is good enough, and the Jordan–Chevalley decomposition of γ can easily fail to be defined over F in positive characteristic. For example, even in GL_2 over $\mathbb{F}_2((t))$ we have the non-rational Jordan decomposition

$$\begin{bmatrix} & t \\ 1 & \end{bmatrix} = \begin{bmatrix} \sqrt{t} & \\ & \sqrt{t} \end{bmatrix} \begin{bmatrix} & \sqrt{t} \\ 1/\sqrt{t} & \end{bmatrix}$$

where the two matrices commute and the latter is unipotent since

$$\begin{bmatrix} \sqrt{t} & \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} & \sqrt{t} \\ 1/\sqrt{t} & \end{bmatrix} \begin{bmatrix} \sqrt{t} & \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{t} & \\ 1/\sqrt{t} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{t} & \sqrt{t} \\ 1 & \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}.$$

We will study Conjecture 1.2 when the characteristic of F is arbitrary. In this setting, we have neither the Jordan–Chevalley decomposition nor the exponential map at our disposal. Instead, our strategy will be closer in spirit to Laumon's approach in Chapter 4 of [23] following Appendix 1 of [9]. Laumon proves Conjecture 1.2 for $\mathrm{GL}_n(F)$ by passing from the full orbit $\mathrm{GL}_n(F) \cdot \gamma$ to the orbit $P(F) \cdot \gamma$ for a specific parabolic subgroup P of GL_n satisfying $G_\gamma(F) \leq P(F)$. In other words, Laumon skips the first two reduction steps and instead deploys the third step directly on the group G rather than on the Lie algebra of G. And then like before, the orbit $P(F) \cdot \gamma$ is so well behaved that the desired invariant measure can be constructed directly as a modification of a Haar measure, at which point convergence is readily verified.

Our first main result is an extension of Laumon's approach to a general framework for reducing Conjecture 1.2 to the existence of a suitable parabolic subgroup P of G.

Theorem 2.9. *Let G be a (connected) reductive group over a non-archimedean local field F. Let P be a parabolic subgroup of G with Levi decomposition $P = MN$. Let $\gamma \in P(F)$ with image $\overline{\gamma} \in M(F)$, let $H = M_{\overline{\gamma}}N$, and let $\iota\colon H(F)/H_\gamma(F) \hookrightarrow N(F)$ be defined by $\iota(h) = h\gamma h^{-1}\overline{\gamma}^{-1}$. Suppose that the parabolic subgroup P satisfies*

(H1) *$G_\gamma(F) \leq P(F)$,*

(H2a) *$M_{\overline{\gamma}}(F)$ is unimodular,*

(H2b) *The $M(F)$-orbit of $\overline{\gamma}$ is closed,*

(H3) *There exists a nonzero Radon measure dq on $H(F)/H_\gamma(F)$ satisfying*

$$\int_{H(F)/H_\gamma(F)} f(h_0 h)\, dq = \delta_H^{-1}(h_0) \int_{H(F)/H_\gamma(F)} f(h)\, dq \tag{H3a}$$

for all $f \in C_c(H(F)/H_\gamma(F))$ and all $h_0 \in H(F)$, and

$$\int_{H(F)/H_\gamma(F)} f(\iota(h))\, dq < \infty \tag{H3b}$$

for all $f \in C_c(N(F))$.

Then $G_\gamma(F)$ is unimodular and the orbital integrals

$$\int_{G(F)/G_\gamma(F)} f(g\gamma g^{-1})\, \frac{dg}{dg_\gamma}$$

converge for all $f \in C_c(G(F))$.

The idea behind Theorem 2.9 is that the Iwasawa decomposition $G(F) = KM(F)N(F)$ allows us to break up an orbital integral in $G(F)$ into three parts: an integral over the maximal compact subgroup K, an orbital integral in $M(F)$ capturing the semisimple part of γ, and an integral in $N(F)$ capturing the unipotent part of γ. Roughly speaking, if $\overline{\gamma} \in M(F)$ is the projection of $\gamma \in P(F)$, and if we set $H = M_{\overline{\gamma}}N$, then we can decompose

$$\int_{G(F)/G_\gamma(F)} \text{“}=\text{”} \int_{M(F)/M_{\overline{\gamma}}(F)} \int_{H(F)/H_\gamma(F)} \int_K .$$

We can also view the three hypotheses (H1), (H2), and (H3) of Theorem 2.9 as corresponding to the three factors K, $M(F)$, and $N(F)$ in the Iwasawa decomposition. See Remark 2.10 for a more precise explanation of this perspective.

All known approaches to Conjecture 1.2 roughly follow the framework of Theorem 2.9, either directly on the group G as in Laumon or on the Lie algebra of G as in Deligne, Ranga Rao, and McNinch. In particular, once the parabolic subgroup P has been selected, progress on Conjecture 1.2 can be measured in terms of the five hypotheses of Theorem 2.9. In this light, our next main results pertain to hypotheses (H2b) and (H3a).

For hypothesis (H2b) of Theorem 2.9, we prove a rational Hilbert–Mumford theorem that allows us to relate being closed in the F-analytic topology to the notion of cocharacter-closed from [6]. The notion of cocharacter-closure does not require the field F to have a topology and is well studied due to is close relation to G-complete reducibility.

Theorem 2.17. *Let $G \times V \to V$ be an action of a (connected) reductive group G on an affine algebraic variety V, with G, V, and the action all defined over a fixed non-archimedean local field F. Let $v \in V(F)$. If $w \in \overline{G(F) \cdot v}$ (the closure in the F-analytic topology on $V(F)$), then there exists a cocharacter $\lambda \colon \mathbb{G}_m \to G$ and an element $g \in G(F)$ such that*

$$\lim_{t \to 0} \lambda(t) \cdot v = \lim_{t \to \infty} \lambda(t) \cdot g \cdot w.$$

Corollary 2.19. *Let $G \times V \to V$ be an action of a (connected) reductive group G on an affine algebraic variety V, with G, V, and the action all defined over a fixed non-archimedean local field F. Let $v \in V(F)$. If $G(F) \cdot v$ is cocharacter-closed, then $G(F) \cdot v$ is closed in the F-analytic topology on $V(F)$.*

We also demonstrate how recent work on G-complete reducibility provides parabolic subgroups P satisfying hypotheses (H1) and (H2b) of Theorem 2.9.

Proposition 2.25. *Let G be a (connected) reductive group over a non-archimedean local field F, and let $\gamma \in G(F)$. Then there exists a parabolic subgroup P of G such that $G_\gamma(F) \leq P(F)$ and such that the $G(F)$-orbit of the Levi quotient $\overline{\gamma}$ is cocharacter-closed.*

For hypothesis (H3a) of Theorem 2.9, our strategy will be to descend a suitable measure on the stable orbit $(H/H_\gamma)(F)$ to a measure on the rational orbit $H(F)/H_\gamma(F)$. We prove this descent under the technical assumptions that the stable orbit is an open subset of a

subgroup and that the space of rational orbits within the stable orbit is Hausdorff. This allows for the possibility of infinitely many rational orbits within the stable orbit.

Proposition 2.31. *Let H be a group with an action on a second countable locally compact totally disconnected topological group G. Assume that for each $h \in H$, if gK is a left coset of a compact open subgroup K of G, then $(gK)^h$ is also a left coset of a compact open subgroup of G. Let $\chi\colon H \to \mathbb{R}^\times_{>0}$ be a homomorphism. Let X be an H-invariant open subset of G. Assume that the quotient space X/H is Hausdorff. Suppose that there exists a strictly positive Radon measure μ_G on G satisfying $\mu_G(S^h) = \chi(h)\mu_G(S)$ for all $h \in H$. Then for each H-orbit $\mathcal{O}$ in X, there exists a nonzero Radon measure $\mu_\mathcal{O}$ on $\mathcal{O}$ satisfying $\mu_\mathcal{O}(S^h) = \chi(h)\mu_\mathcal{O}(S)$ for all $h \in H$ and Borel $S \subseteq \mathcal{O}$.*

1.3 Optimal Parabolic Subgroups

We now turn to the question of the selection of the parabolic subgroup P in Theorem 2.9. The most demanding condition on P in Theorem 2.9 is hypothesis (H3) which is equivalent to the existence of a suitable invariant measure on the orbit $P(F) \cdot \gamma$. Constructing this measure requires the orbit $P(F) \cdot \gamma$ to be sufficiently well behaved. For Laumon, if γ is unipotent then the orbit $P(F) \cdot \gamma$ is open dense in the unipotent radical $N(F)$. In this case, the restriction of a Haar measure on $N(F)$ to the open dense orbit $P(F) \cdot \gamma$ will be the desired invariant measure.

The primary obstacle to extending Laumon's method beyond GL_n is that Laumon's choice of parabolic subgroup P is specific to GL_n. Laumon constructs P from the action of γ on the vector space F^n, and this construction fails to generalize beyond GL_n. Indeed, beyond GL_n, it is too much to ask for the $P(F)$-orbit of a unipotent γ to be open dense in the unipotent radical $N(F)$.

Example 1.4. Let $G = \mathrm{Sp}_4$ with symplectic form $\omega(e_1, e_4) = \omega(e_2, e_3) = 1$ and let

$$\gamma = \begin{bmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

The centralizer G_γ consists of matrices of the form

$$\begin{bmatrix} x & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & x \end{bmatrix} \in \mathrm{Sp}_4\,.$$

The additional constraints of membership in Sp_4 (e.g., $x^2 = 1$) give $\dim G_\gamma = 6$. If P is a parabolic subgroup of G with $G_\gamma \leq P$, then the P-orbit of γ will have dimension

$$\dim P/P_\gamma = \dim P - \dim G_\gamma = \dim M + \dim N - 6.$$

But Sp_4 has no Levi subgroup of dimension 6, so it is impossible for the orbit $P \cdot \gamma$ to be open dense in N. The most natural choice of parabolic subgroup of G satisfying $G_\gamma(F) \leq P(F)$ would be the parabolic subgroup P consisting of matrices of the form

$$\begin{bmatrix} x & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & x^{-1} \end{bmatrix} \in \mathrm{Sp}_4$$

stabilizing the isotropic subspace $\langle e_1 \rangle$ and its symplectic complement $\langle e_1, e_2, e_3 \rangle$. For this choice of parabolic subgroup, the orbit $P(F) \cdot \gamma$ consists of matrices of the form

$$\begin{bmatrix} 1 & & & x^2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

with $x \neq 0$. This is far from being open dense in $N(F)$, especially for $F = \mathbb{F}_2((t))$ where the subgroup of squares $(F^\times)^2$ has infinite index in $F^\times$. The best we can say is that the stable orbit $(P \cdot \gamma)(F)$ consisting of matrices of the form

$$\begin{bmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

with $x \neq 0$ is at least open dense in a subgroup of $N(F)$.

This phenomenon of the stable orbit $(P \cdot \gamma)(F)$ being open dense in a subgroup of $N(F)$ is what we will exploit. But not every parabolic subgroup satisfying $G_\gamma(F) \leq P(F)$ will have this property so we must still be careful in our choice of parabolic subgroup.

Example 1.5. Let $G = \mathrm{GL}_5$ and let

$$\gamma = \begin{bmatrix} 1 & & 1 & & \\ & 1 & & 1 & \\ & & 1 & & 1 \\ & & & 1 & \\ & & & & 1 \end{bmatrix}.$$

The centralizer G_γ consists of matrices of the form

$$\begin{bmatrix} a & c & e & * & * \\ & b & d & * & * \\ & & a & c & e \\ & & & b & d \\ & & & & a \end{bmatrix}$$

with $a, b \neq 0$. Here are three candidate parabolic subgroups satisfying $G_\gamma(F) \leq P(F)$:

$$P_1 = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & & * & * & * \\ & & * & * & * \\ & & & & * \end{bmatrix}, \qquad P_2 = \begin{bmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & * & * & * \\ & & & & * \end{bmatrix}, \qquad P_3 = \begin{bmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{bmatrix},$$

$$P_1 \cdot \gamma \subseteq \begin{bmatrix} 1 & & * & * & * \\ & 1 & * & * & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & 1 \end{bmatrix}, \quad P_2 \cdot \gamma \subseteq \begin{bmatrix} 1 & & * & * & * \\ & 1 & * & * & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & 1 \end{bmatrix}, \quad P_3 \cdot \gamma \subseteq \begin{bmatrix} 1 & & * & * & * \\ & 1 & & * & * \\ & & 1 & & * \\ & & & 1 & \\ & & & & 1 \end{bmatrix}.$$

The first parabolic subgroup P_1 is the parabolic subgroup constructed by Laumon. Its orbit $P_1 \cdot \gamma$ has dimension $\dim P_1 - \dim G_\gamma = 17 - 9 = 8$, filling out the unipotent radical N of dimension 8. The second orbit $P_2 \cdot \gamma$ has dimension $\dim P_2 - \dim G_\gamma = 16 - 9 = 7$, failing to fill out the subgroup of dimension 8. The third parabolic subgroup P_3 is the Kempf-Rousseau optimal parabolic subgroup. Its orbit $P_3 \cdot \gamma$ has dimension $\dim P_3 - \dim G_\gamma = 15 - 9 = 6$, filling out the subgroup of dimension 6.

The optimal parabolic subgroup, originally due to Kempf [19] and Rousseau [28] and further developed by Hesselink [17] and Bate-Martin-Röhrle-Tange [6], is defined in general for a reductive group acting on an affine variety. In the case of G acting on itself by conjugation, the optimal parabolic subgroup automatically satisfies $G_\gamma(F) \leq P(F)$, so it is naturally suited to the study of orbital integrals.

We will opt to use the optimal parabolic subgroup P. For a reductive group G acting on an affine variety V over a field F, a point $v \in V(F)$, and a target closed subvariety S of V, the optimal parabolic subgroup is constructed from the optimal cocharacters which move v to S as "efficiently" as possible [6, §4]. For our application to orbital integrals, we would like to move $\gamma \in G(F)$ to a cocharacter-closed $G(F)$-orbit. However, the requirement that the target S be a closed subvariety will pose a key limitation. For example, if $G \cdot \gamma$ is closed, then the optimal cocharacters will not move γ at all, even if the $G(F)$-orbit of γ is not yet cocharacter-closed. This difficulty is also discussed in the introduction of [6].

Because of this difficulty, we will simplify matters by only applying the theory of optimal parabolic subgroups to the case of unipotent orbital integrals. In this setting, hypotheses (H1) in Theorem 2.9 is a property of optimal parabolic subgroups and hypothesis (H2) become automatic. Proposition 2.31 reduces hypothesis (H3a) to a dimension equality for a specific algebraic subgroup N_1 of N that arises in the theory of optimal parabolic subgroups. This dimension equality can be checked case by case for specific classical groups.

Conjecture 3.15. *Let G be a (connected) reductive group over a non-archimedean local field F, and let $\gamma \in G(F)$ be unipotent containing 1 in its cocharacter-closure. Let P be the optimal parabolic subgroup for γ. Then* $\dim N_1 = \dim P - \dim P_\gamma$.

The last remaining hypothesis of Theorem 2.9 left to handle is hypothesis (H3b) which states that the measure on the rational orbit $P(F)/P_\gamma(F)$ descended from the measure on the stable orbit $(P/P_\gamma)(F)$ remains finite on compact subsets of $N(F)$. Unfortunately, our lack of progress on hypothesis (H3b) in the case where there are infinitely many rational orbits within the stable orbit forces us to assume finiteness of the orbit space $(P/P_\gamma)(F)/P(F)$. Putting all of this together along with some additional technical hypotheses gives the following result.

Theorem 3.19. *Let G be a reductive group over a non-archimedean local field F. Let $\gamma \in G(F)$ be a unipotent element containing 1 in its cocharacter-closure. Assume that Conjecture 3.15 holds for γ. Let λ be an optimal cocharacter for γ, and let P be the associated optimal parabolic subgroup for γ. Assume that 2λ is integral and that $G_\gamma \leq P$. Assume that G_γ is smooth and that the orbit space $(P/P_\gamma)(F)/P(F)$ is finite. Then $G_\gamma(F)$ is unimodular and the orbital integrals*

$$\int_{G(F)/G_\gamma(F)} f(g\gamma g^{-1}) \frac{dg}{dg_\gamma}$$

converge for all $f \in C_c(G(F))$.

Applying Theorem 3.19 to GL_n, SL_n and Sp_{2n} recovers the results on McNinch on unipotent orbital integrals in this cases.

Corollary 4.3. *Let F be a non-archimedean local field, and let $\gamma \in \mathrm{GL}_n(F)$ be unipotent. Then the centralizer of γ in $\mathrm{GL}_n(F)$ is unimodular and orbital integrals of γ in $\mathrm{GL}_n(F)$ converge.*

Corollary 4.6. *Let F be a non-archimedean local field, and let $\gamma \in \mathrm{SL}_n(F)$ be unipotent. Assume that the characteristic of F does not divide n. Then the centralizer of γ in $\mathrm{SL}_n(F)$ is unimodular and orbital integrals of γ in $\mathrm{SL}_n(F)$ converge.*

Corollary 4.8. *Let F be a non-archimedean local field of characteristic not equal to 2, and let $\gamma \in \mathrm{Sp}_{2n}(F)$ be unipotent. Then the centralizer of γ in $\mathrm{Sp}_{2n}(F)$ is unimodular and orbital integrals of γ in $\mathrm{Sp}_{2n}(F)$ converge.*

1.4 Comparison and Future Directions

The most recent progress on convergence of local orbital integrals is due to Lemaire [24] and Kim and Park [20]. Kim and Park prove convergence of local orbital integrals in unitary groups by adapting Laumon's method. In particular, they construct the parabolic subgroup as the stabilizer of a flag. As they remark, a similar method could in principle be applied to other classical groups. But a uniform proof across all reductive groups requires a more general source of parabolic subgroups.

Lemaire uses optimal parabolic subgroups to prove convergence of unipotent orbital integrals in general reductive groups assuming unimodularity of the centralizer and the existence of a suitable "slice". Since Lemaire's approach is the closest to ours, we briefly highlight

our main innovations relative to Lemaire's work. First, the general framework presented in Theorem 2.23 applies to arbitrary elements as opposed to merely unipotent elements. And second, the descent of measure in Proposition 2.31 allows unimodularity of $G_\gamma(F)$ to be reduced to unimodularity of $M_{\overline{\gamma}}(F)$ in many cases. Our focus on cocharacter-closed orbits and our rational Hilbert–Mumford theorem in Theorem 2.17 also appear to be novel.

However, our work still has several major limitations. First, we do not know how to prove unimodularity of $G_{\overline{\gamma}}(F)$ when $G_{\overline{\gamma}}$ is not smooth (Conjecture 2.21). And second, the descent of measure in Proposition 2.31 fundamentally requires the image of $\iota\colon (H/H_\gamma)(F) \hookrightarrow N(F)$ to be an open subset of a subgroup. But Proposition 2.38 gives an example where this fails in an orthogonal group in bad characteristic. And third, while the descent of measure in Proposition 2.31 allows for infinitely many rational orbits within a stable orbit, we do not know how to prove finiteness of the descended measure in this situation. For this reason, our final applications in Chapter 4 must still retain some assumptions on the characteristic. And fourth, we end up only applying the theory of optimal parabolic subgroups to unipotent elements due to the difficulty with the target needing to be a subvariety that we discussed in the previous section. However, Proposition 2.25 demonstrates that the Tits center conjecture gives an alternate source of parabolic subgroups in all characteristics. So going forward it may be possible to bypass this difficulty by switching over from optimal parabolic subgroups to this alternate source of parabolic subgroups. This would require a sufficient understanding of the image of the embedding $\iota\colon (H/H_\gamma)(F) \hookrightarrow N(F)$ in this new setting in order to equip the image with a suitable measure.

Chapter 2

Measure-Theoretic Reductions

In this chapter we will reduce convergence of orbital integrals to the construction of a suitable parabolic subgroup. We will begin by reviewing some harmonic analysis. Note however that our formulation of invariant measures on homogeneous spaces in Proposition 2.4 will be slightly nonstandard because we will require measures on coset spaces like $H(F)/H_\gamma(F)$ where $H_\gamma(F)$ is unimodular but $H(F)$ is not. We will then review the theory of parabolic subgroups, following the dynamic description of parabolic subgroups in terms of cocharacters. Given a suitable parabolic subgroup P of our reductive group G, the theory of invariant measures will be applied to the Iwasawa decomposition $G(F) = KP(F) = KM(F)N(F)$, allowing us to deduce convergence of orbital integrals in $G(F)$. We will also discuss the notion of cocharacter-closure which will help clarify the requirements on P.

2.1 Invariant Measures

Let G be a locally compact Hausdorff topological group. We will denote the left Haar measure on G by dg. If $\delta_G \colon G \to \mathbb{R}^\times_{>0}$ denotes the modular character, then $d_r g = \delta_G(g)\, dg$ is a right Haar measure on G. These measures are only unique up to scalar multiple, but we will assume that we have fixed such a measure once and for all on each locally compact Hausdorff group that we consider. The right Haar measure $d_r g$ can also be characterized by how it transforms under left translations.

Lemma 2.1. *Let dh be a nonzero Radon measure on G satisfying*

$$\int_G f(gh)\, dh = \delta_G^{-1}(g) \int_G f(h)\, dh$$

for all $f \in C_c(G)$ and all $g \in G$. Then dh is a scalar multiple of the right Haar measure $d_r g$.

Proof. Since dh is nonzero, we can fix a function $f_0 \in C_c(G)$ satisfying

$$\int_G f_0(h)\,dh = 1.$$

Let $f \in C_c(G)$. Then

$$\begin{aligned}
\int_G f(g)\,d_r g &= \int_G f_0(h)\,dh \int_G f(g)\,d_r g \\
&= \int_G \int_G f_0(h) f(g)\,d_r g\,dh \\
\text{(multiply } g \text{ on the right by } h) \qquad &= \int_G \int_G f_0(h) f(gh)\,d_r g\,dh \\
\text{(multiply } h \text{ on the left by } g^{-1}) \qquad &= \int_G \int_G \delta_G^{-1}(g) f_0(g^{-1}h) f(h)\,d_r g\,dh \\
\text{(invert } g) \qquad &= \int_G \int_G f_0(gh) f(h)\,d_r g\,dh \\
\text{(multiply } g \text{ on the right by } h^{-1}) \qquad &= \int_G \int_G f_0(g) f(h)\,d_r g\,dh \\
&= \int_G f_0(g)\,d_r g \int_G f(h)\,dh.
\end{aligned}$$

Thus, $d_r g$ is a scalar multiple of dh. □

Now let H be a closed subgroup of G. There is a projection map $\pi\colon C_c(G) \to C_c(G/H)$ defined by

$$\pi(f)(g) = \int_H f(gh)\,dh.$$

We will need one technical lemma from [10].

Lemma 2.2 ([10, Lemma 2.49])**.** *Let $S \subseteq G/H$ be compact. Then there exists a $\phi \in C_c(G)$ such that $\pi(\phi) = 1$ on S.*

Lemma 2.3 ([10, Proposition 2.50])**.** *The projection map $\pi\colon C_c(G) \to C_c(G/H)$ is surjective.*

Proof. Let $f \in C_c(G/H)$. By Lemma 2.2, we can find a $\phi \in C_c(G)$ such that $\pi(\phi) = 1$ on the support of f. Let $\widetilde{f} \in C_c(G)$ be defined by $\widetilde{f}(g) = f(g)\phi(g)$. Then

$$\pi(\widetilde{f})(g) = \int_H \widetilde{f}(gh)\,dh = \int_H f(gh)\phi(gh)\,dh = \int_H f(g)\phi(gh)\,dh = f(g)\pi(\phi)(g) = f(g). \quad \square$$

We now give our formulation of invariant measures on homogeneous spaces. As noted at the start of this chapter, this formulation is slightly nonstandard since we will require measures on coset spaces like $H(F)/H_\gamma(F)$ where $H_\gamma(F)$ is unimodular but $H(F)$ is not.

Proposition 2.4.

1. *If H is unimodular, then there exists a unique Radon measure dg/dh on G/H satisfying*

$$\int_G f(g)\, d_r g = \int_{G/H} \int_H f(gh)\, dh\, \frac{dg}{dh} \tag{2.1}$$

   for all $f \in C_c(G)$.

2. *If a Radon measure dq on G/H satisfies* (2.1)*, then the Radon measure dq also satisfies*

$$\int_{G/H} f(g_0 g)\, dq = \delta_G^{-1}(g_0) \int_{G/H} f(g)\, dq \tag{2.2}$$

   for all $f \in C_c(G/H)$ and all $g_0 \in G$.

3. *If some nonzero Radon measure dq on G/H satisfies* (2.2)*, then H is unimodular.*

4. *Any Radon measure dq on G/H satisfying* (2.2) *is a scalar multiple of dg/dh.*

Proof. We will prove each part separately.

1. Suppose that H is unimodular. In light of the Riesz representation theorem, we must show that the positive linear functional I on $C_c(G)$ corresponding to $d_r g$ descends to a positive linear functional on $C_c(G/H)$ making the diagram

$$\begin{array}{ccc} C_c(G) & & \\ {\scriptstyle\pi}\big\downarrow & \searrow^{I} & \\ C_c(G/H) & \dashrightarrow & \mathbb{C} \end{array}$$

   commute. Then it suffices to show that $\ker \pi \leq \ker I$. Let $f \in \ker \pi$. By Lemma 2.2, we can find a $\phi \in C_c(G)$ such that $\pi(\phi) = 1$ on the support of f. Then

$$I(f) = \int_G f(g)\, d_r g = \int_G \pi(\phi)(g) f(g)\, d_r g = \int_G \int_H \phi(gh) f(g)\, dh\, d_r g.$$

   Now we can multiply g on the right by h^{-1} (after swapping the integrals) to get

$$I(f) = \int_G \int_H \phi(g) f(gh^{-1})\, dh\, d_r g.$$

   But H is unimodular, so we can invert h to get

$$I(f) = \int_G \int_H \phi(g) f(gh)\, dh\, d_r g = \int_G \phi(g) \pi(f)(g)\, d_r g = 0.$$

2. Suppose that dq is a Radon measure on G/H satisfying (2.1). Let $f \in C_c(G/H)$ and let $g_0 \in G$. By Lemma 2.3, we can write $f = \pi(\widetilde{f})$ for some $\widetilde{f} \in C_c(G)$. Then

$$\int_{G/H} f(g)\,dq = \int_{G/H}\int_H \widetilde{f}(gh)\,dh\,dq = \int_G \widetilde{f}(g)\,d_r g$$

and likewise

$$\int_{G/H} f(g_0 g)\,dq = \int_{G/H}\int_H \widetilde{f}(g_0 gh)\,dh\,dq = \int_G \widetilde{f}(g_0 g)\,d_r g = \delta_G^{-1}(g_0)\int_G \widetilde{f}(g)\,d_r g.$$

3. Suppose that dq is a nonzero Radon measure on G/H satisfying (2.2). Then for any $f \in C_c(G)$ and $g_0 \in G$, we have

$$\int_{G/H}\int_H f(g_0 gh)\,dh\,dq = \delta_G^{-1}(g_0)\int_{G/H}\int_H f(gh)\,dh\,dq.$$

By Lemma 2.1, we must have

$$\int_{G/H}\int_H f(gh)\,dh\,dq = \lambda\int_G f(g)\,d_r g$$

for some constant $\lambda > 0$ and all $f \in C_c(G)$. If we replace $f(g)$ by $f(gh_0)$ for $h_0 \in H$, then the right side will remain unchanged, but the left side will be scaled by $\delta_H(h_0)$. Thus, we must have $\delta_H(h_0) = 1$.

4. Suppose that dq is a Radon measure on G/H satisfying (2.2). As above, we must have

$$\int_{G/H}\int_H f(gh)\,dh\,dq = \lambda\int_G f(g)\,d_r g = \lambda\int_{G/H}\int_H f(gh)\,dh\,\frac{dg}{dh}$$

for some constant $\lambda \geq 0$ and all $f \in C_c(G)$. By Lemma 2.3, we have

$$\int_{G/H} f(g)\,dq = \lambda\int_{G/H} f(g)\,\frac{dg}{dh}$$

for all $f \in C_c(G/H)$. Thus, dq is a scalar multiple of dg/dh. □

Finally, we will need to know how Haar measures decompose when the group factors as a product of subgroups. We will prove this as an application of the theory of invariant measures on homogeneous spaces developed in Proposition 2.4. One useful observation that is needed in the proof is that if K is any compact subgroup of G, then the modular character δ_G is necessarily trivial on K. This is because $\delta_G\colon G \to \mathbb{R}^\times_{>0}$ is a continuous homomorphism, so the image of K would be a compact subgroup of $\mathbb{R}^\times_{>0}$, but any such subgroup is trivial.

Proposition 2.5 ([14, Proposition 3.2.2])**.** *Suppose that $G = ST$ for closed subgroups S and T of G with compact intersection. Then one can normalize the Haar measures so that*

$$\int_G f(g)\, d_r g = \int_{S\times T} f(st)\, \delta_G(s)\, ds\, d_r t$$

for all $f \in C_c(G)$.

Proof. The action $(s,t)\cdot g = sgt^{-1}$ makes G into a homogeneous space for $S\times T$. Let $H \leq S\times T$ be the stabilizer subgroup of $1\in G$. Then H is homeomorphic to the compact intersection $S\cap T$. In particular, H is compact so H is unimodular and we can normalize the Haar measure dh to have total volume 1. We can identify G with the coset space $(S\times T)/H$ via $(s,t)\mapsto (s,t)\cdot 1 = st^{-1}$. Then Proposition 2.4 gives a measure dq on the homogeneous space $(S\times T)/H$ satisfying

$$\int_{S\times T} f(s,t)\, d_r s\, d_r t = \int_{(S\times T)/H}\int_H f(qh)\, dh\, dq$$

for all $f\in C_c(S\times T)$. Restricting to functions of the form $f(st^{-1})$ for $f\in C_c(G)$ and identifying $(S\times T)/H$ with G gives a measure dq on the homogeneous space G satisfying

$$\int_G f(q)\, dq = \int_{S\times T} f(st^{-1})\, d_r s\, d_r t = \int_{S\times T} f(st)\, \delta_S(s)\, \delta_T^{-1}(t)\, ds\, d_r t$$

for all $f\in C_c(G)$. Now observe that the measure $\delta_G^{-1}(s)\,\delta_S(s)\,\delta_T^{-1}(t)\, d_r g$ is a well defined measure on the homogeneous space G (since s and t are unique up to the compact intersection $S\cap T$ on which all the modular characters are trivial) and satisfies

$$\begin{aligned}\int_G f(s_0 g t_0^{-1})\,\delta_G^{-1}(s)\,\delta_S(s)\,\delta_T^{-1}(t)\, d_r g &= \delta_G^{-1}(s_0)\int_G f(g)\,\delta_G^{-1}(s_0^{-1}s)\,\delta_S(s_0^{-1}s)\,\delta_T^{-1}(tt_0)\, d_r g\\ &= \delta_{S\times T}^{-1}(s_0,t_0)\int_G f(g)\,\delta_G^{-1}(s)\,\delta_S(s)\,\delta_T^{-1}(t)\, d_r g.\end{aligned}$$

By the fourth part of Proposition 2.4, we must have

$$\int_G f(g)\,\delta_G^{-1}(s)\,\delta_S(s)\,\delta_T^{-1}(t)\, d_r g = \int_G f(q)\, dq = \int_{S\times T} f(st)\,\delta_S(s)\,\delta_T^{-1}(t)\, ds\, d_r t$$

up to scalar multiple. Then modifying f by $\delta_G(s)\,\delta_S^{-1}(s)\,\delta_T(t)$ gives

$$\int_G f(g)\, d_r g = \int_{S\times T} f(st)\,\delta_G(s)\, ds\, d_r t$$

as desired. □

2.2 Parabolic Subgroups

In this section we will review the theory of parabolic subgroups, following the dynamic description of parabolic subgroups in terms of cocharacters. A cocharacter of a reductive group G over a field F is an algebraic homomorphism $\lambda\colon \mathbb{G}_m \to G$ defined over F.

Given a cocharacter $\lambda\colon \mathbb{G}_m \to G$, we can construct several subgroups as follows:

- The Levi subgroup $M(\lambda)$ assigns to each F-algebra R the set of all $g \in G(R)$ that commute with $\lambda_R\colon \mathbb{G}_{m,R} \to G_R$.
- The parabolic subgroup $P(\lambda)$ assigns to each F-algebra R the set of all $g \in G(R)$ such that the limit $\lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$ exists in the sense that the morphism $\mathbb{G}_{m,R} \to G_R$ defined by $t \mapsto \lambda(t)g\lambda(t)^{-1}$ factors through $\mathbb{A}^1_R$.

$$\begin{array}{ccc} \mathbb{G}_{m,R} & \xrightarrow{t\mapsto \lambda(t)g\lambda(t)^{-1}} & G_R \\ \downarrow & \nearrow & \\ \mathbb{A}^1_R & & \end{array}$$

- The unipotent radical $N(\lambda)$ assigns to each F-algebra R the set of all $g \in G(R)$ such that the limit $\lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$ exists and equals 1.

Example 2.6. The cocharacter $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_5$ defined by

$$\lambda(t) = \begin{bmatrix} t^2 & & & & \\ & t^2 & & & \\ & & t^{-1} & & \\ & & & t^{-1} & \\ & & & & t^{-2} \end{bmatrix}$$

gives rise to the following Levi subgroup, parabolic subgroup, and unipotent radical:

$$M(\lambda) = \begin{bmatrix} * & * & & & \\ * & * & & & \\ & & * & * & \\ & & * & * & \\ & & & & * \end{bmatrix}, \quad P(\lambda) = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & & * & * & * \\ & & * & * & * \\ & & & & * \end{bmatrix}, \quad N(\lambda) = \begin{bmatrix} 1 & & * & * & * \\ & 1 & * & * & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & 1 \end{bmatrix}.$$

In general, the Levi subgroup M is reductive, the unipotent radical N is unipotent, and the parabolic subgroup P will factor as the semidirect product $M \ltimes N$. This is called the Levi decomposition of P. If F is a local field then there is also the Iwasawa decomposition which writes $G(F) = KP(F)$ for some maximal compact subgroup K of $G(F)$.

2.3 Iwasawa Decomposition

Now let G be a (connected) reductive group over a non-archimedean local field F. Fix a parabolic subgroup P with Iwasawa decomposition $G(F) = KP(F)$ and Levi decomposition $P(F) = M(F)N(F)$. In this setting $G(F)$, $M(F)$, $N(F)$, and K are all unimodular (G and M are reductive, N is unipotent, and K is compact), but $P(F)$ is not necessarily unimodular. We now use the results of the previous section to relate orbital integrals in $G(F)$ to orbital integrals in $M(F)$.

Proposition 2.7. *Let $\gamma \in G(F)$. If $G_\gamma(F)$ is unimodular and $G_\gamma(F) \leq P(F)$, then*

$$\int_{G(F)/G_\gamma(F)} f(g)\,\frac{dg}{dg_\gamma} = \int_{P(F)/P_\gamma(F)} \int_K f(kp)\,dk\,\frac{dp}{dp_\gamma}$$

for all $f \in C_c(G(F)/G_\gamma(F))$.

Proof. By the surjectivity of $C_c(G(F)) \to C_c(G(F)/G_\gamma(F))$ from Lemma 2.3, it is enough to show that

$$\int_{G(F)/G_\gamma(F)} \int_{G_\gamma(F)} f(gg_\gamma)\,dg_\gamma\,\frac{dg}{dg_\gamma} = \int_{P(F)/P_\gamma(F)} \int_K \int_{P_\gamma(F)} f(kpp_\gamma)\,dp_\gamma\,dk\,\frac{dp}{dp_\gamma}$$

for all $f \in C_c(G(F))$. By equation (2.1), this reduces to showing that

$$\int_{G(F)} f(g)\,dg = \int_{P(F)} \int_K f(kp)\,dk\,d_rp$$

which follows from Proposition 2.5. □

Proposition 2.8. *Let $\gamma \in P(F)$ with image $\overline{\gamma} \in M(F)$, and let $H = M_{\overline{\gamma}}N$. If $P_\gamma(F)$ and $M_{\overline{\gamma}}(F)$ are unimodular, then*

$$\int_{P(F)/P_\gamma(F)} f(p)\,\frac{dp}{dp_\gamma} = \int_{M(F)/M_{\overline{\gamma}}(F)} \int_{H(F)/H_\gamma(F)} f(mh)\,\delta_P(m)\,\frac{dh}{dh_\gamma}\,\frac{dm}{dm_{\overline{\gamma}}}$$

for all $f \in C_c(P(F)/P_\gamma(F))$.

Proof. The inner integral is invariant under multiplication on the right by $m_{\overline{\gamma}} \in M_{\overline{\gamma}}(F)$ since

$$\int_{H(F)/H_\gamma(F)} f(mm_{\overline{\gamma}}h)\,\delta_P(mm_{\overline{\gamma}})\,\frac{dh}{dh_\gamma} = \int_{H(F)/H_\gamma(F)} f(mh)\,\delta_H^{-1}(m_{\overline{\gamma}})\,\delta_P(m_{\overline{\gamma}})\,\delta_P(m)\,\frac{dh}{dh_\gamma}$$

where $\delta_H(m_{\overline{\gamma}}) = \delta_P(m_{\overline{\gamma}})$ since both come from the action of $m_{\overline{\gamma}}$ on N. By the surjectivity of $C_c(P(F)) \to C_c(P(F)/P_\gamma(F))$ from Lemma 2.3, it is enough to show that

$$\int_{P(F)/P_\gamma(F)} \int_{P_\gamma(F)} f(pp_\gamma)\,dp_\gamma\,\frac{dp}{dp_\gamma} = \int_{M(F)/M_{\overline{\gamma}}(F)} \int_{H(F)/H_\gamma(F)} \int_{H_\gamma(F)} f(mhp_\gamma)\,\delta_P(m)\,dp_\gamma\,\frac{dh}{dh_\gamma}\,\frac{dm}{dm_{\overline{\gamma}}}$$

for all $f \in C_c(P(F))$. By equation (2.1), this reduces to showing that

$$\int_{P(F)} f(p)\, d_r p = \int_{M(F)/M_{\overline{\gamma}}(F)} \int_{H(F)} f(mh)\, \delta_P(m)\, d_r h\, \frac{dm}{dm_{\overline{\gamma}}}.$$

But Proposition 2.5 and equation (2.1) let us compute

$$\begin{aligned}
&\int_{M(F)/M_{\overline{\gamma}}(F)} \int_{H(F)} f(mh)\, \delta_P(m)\, d_r h\, \frac{dm}{dm_{\overline{\gamma}}} \\
&= \int_{M(F)/M_{\overline{\gamma}}(F)} \int_{M_{\overline{\gamma}}(F)} \int_{N(F)} f(mm_{\overline{\gamma}}n)\, \delta_H(m_{\overline{\gamma}})\, \delta_P(m)\, dn\, dm_{\overline{\gamma}}\, \frac{dm}{dm_{\overline{\gamma}}} \\
&= \int_{M(F)/M_{\overline{\gamma}}(F)} \int_{M_{\overline{\gamma}}(F)} \int_{N(F)} f(mm_{\overline{\gamma}}n)\, \delta_P(mm_{\overline{\gamma}})\, dn\, dm_{\overline{\gamma}}\, \frac{dm}{dm_{\overline{\gamma}}} \\
&= \int_{M(F)} \int_{N(F)} f(mn)\, \delta_P(m)\, dn\, dm \\
&= \int_{P(F)} f(p)\, d_r p. \qquad \square
\end{aligned}$$

We can now give sufficient conditions on P for convergence of orbital integrals in $G(F)$.

Theorem 2.9. *Let $\gamma \in P(F)$, let $H = M_{\overline{\gamma}}N$, and let $\iota\colon H(F)/H_\gamma(F) \hookrightarrow N(F)$ be defined by $\iota(h) = h\gamma h^{-1}\overline{\gamma}^{-1}$. Suppose that the parabolic subgroup P satisfies*

(H1) *$G_\gamma(F) \leq P(F)$,*

(H2a) *$M_{\overline{\gamma}}(F)$ is unimodular,*

(H2b) *The $M(F)$-orbit of $\overline{\gamma}$ is closed,*

(H3) *There exists a nonzero Radon measure dq on $H(F)/H_\gamma(F)$ satisfying*

$$\int_{H(F)/H_\gamma(F)} f(h_0 h)\, dq = \delta_H^{-1}(h_0) \int_{H(F)/H_\gamma(F)} f(h)\, dq \tag{H3a}$$

for all $f \in C_c(H(F)/H_\gamma(F))$ and all $h_0 \in H(F)$, and

$$\int_{H(F)/H_\gamma(F)} f(\iota(h))\, dq < \infty \tag{H3b}$$

for all $f \in C_c(N(F))$.

Then $G_\gamma(F)$ is unimodular and the orbital integrals

$$\int_{G(F)/G_\gamma(F)} f(g\gamma g^{-1})\, \frac{dg}{dg_\gamma}$$

converge for all $f \in C_c(G(F))$.

Proof. Hypothesis (H3a) implies unimodularity of $G_\gamma(F)$ by Proposition 2.4. For proving convergence of orbital integrals in $G(F)$, it is enough to consider nonnegative $f \in C_c(G(F))$. So let $f \in C_c(G(F))$ be nonnegative, and let $\tilde{f} \in C_c(P(F))$ be defined by

$$\tilde{f}(p) = \int_K f(kpk^{-1})\,dk.$$

Then Propositions 2.7 and 2.8 let us write

$$\begin{aligned}
\int_{G(F)/G_\gamma(F)} f(g\gamma g^{-1})\,\frac{dg}{dg_\gamma} &= \int_{P(F)/P_\gamma(F)} \int_K f(kp\gamma p^{-1}k^{-1})\,dk\,\frac{dp}{dp_\gamma} \\
&= \int_{P(F)/P_\gamma(F)} \tilde{f}(p\gamma p^{-1})\,\frac{dp}{dp_\gamma} \\
&= \int_{M(F)/M_{\overline{\gamma}}(F)} \int_{H(F)/H_\gamma(F)} \tilde{f}(mh\gamma h^{-1}m^{-1})\,\delta_P(m)\,\frac{dh}{dh_\gamma}\,\frac{dm}{dm_{\overline{\gamma}}} \\
&= \int_{M(F)/M_{\overline{\gamma}}(F)} \int_{H(F)/H_\gamma(F)} \tilde{f}(m\iota(h)m^{-1}\cdot m\overline{\gamma}m^{-1})\,\delta_P(m)\,\frac{dh}{dh_\gamma}\,\frac{dm}{dm_{\overline{\gamma}}}
\end{aligned}$$

where $m\iota(h)m^{-1} \in N(F)$ and $m\overline{\gamma}m^{-1} \in M(F)$. Let $S_P = \operatorname{supp}\tilde{f}$ be the compact support of $\tilde{f}$ in $P(F)$. Then $S_P \subseteq S_N \times S_M$ for compact sets $S_N \subseteq N(F)$ and $S_M \subseteq M(F)$. The key observation is that the inner integral will vanish whenever $m\overline{\gamma}m^{-1} \notin S_M$, so we only have to bound the inner integral when $m\overline{\gamma}m^{-1} \in S_M$.

Since the orbit $M(F)\cdot\overline{\gamma}$ is closed by hypothesis (H2b), the intersection $S_M \cap M(F)\cdot\overline{\gamma}$ will be a compact subset of $M(F)\cdot\overline{\gamma}$ corresponding to a compact set $S'_M \subseteq M(F)/M_{\overline{\gamma}}(F)$. Then it suffices to bound the inner integral for $m \in S'_M$.

But the compact set $S'_M \subseteq M(F)/M_{\overline{\gamma}}(F)$ will be covered by a compact set $S''_M \subseteq M(F)$ under the projection $M(F) \twoheadrightarrow M(F)/M_{\overline{\gamma}}(F)$ (e.g., take $S''_M = \operatorname{supp}\phi$ from Lemma 2.2). Then it is enough to bound the inner integral for $m \in S''_M$.

Let $S'_N \subseteq N(F)$ be the image of the compact set $S''_M \times S_N$ under the conjugation map $(m,n) \mapsto m^{-1}nm$. Then for $m \in S''_M$, we can bound

$$\widetilde{f}(m\iota(h)m^{-1}\cdot m\overline{\gamma}m^{-1})\,\delta_P(m) \le \max_{p\in S_P}\widetilde{f}(p)\max_{m\in S''_M}\delta_P(m)\,\mathbb{1}_{S'_N}(\iota(h)).$$

Calling this constant C, we obtain the bound

$$\int_{G(F)/G_\gamma(F)} f(g\gamma g^{-1})\,\frac{dg}{dg_\gamma} \le C\int_{H(F)/H_\gamma(F)} \mathbb{1}_{S'_N}(\iota(h))\,\frac{dh}{dh_\gamma}\int_{M(F)/M_{\overline{\gamma}}(F)} \mathbb{1}_{S'_M}(m)\,\frac{dm}{dm_{\overline{\gamma}}}.$$

The first integral is finite by (H3b) and the second integral is finite by compactness of S'_M. □

Remark 2.10. We can view the three hypotheses of Theorem 2.9 as corresponding to the three factors in the Iwasawa decomposition $G(F) = KM(F)N(F)$. Hypothesis (H1) allows us to pass from $G(F)$ to $P(F)$, thereby handling the contribution from K. After this

reduction to orbital integrals in $P(F)$, we should think of the problem as consisting of two parts: constructing an invariant measure (i.e., proving unimodularity of the centralizer), and proving that it is finite on compact subsets of $P(F)$ (i.e., proving convergence of the orbital integral). Hypothesis (H2a) gives an invariant measure on the $M(F)$ part of the $P(F)$-orbit, and hypothesis (H2b) implies finiteness on compact subsets of $M(F)$. Hypothesis (H3a) gives an invariant measure on the $N(F)$ part of the $P(F)$-orbit, and hypothesis (H3b) assumes finiteness on compact subsets of $N(F)$.

2.4 Cocharacter Closure

Let $G \times V \to V$ be an action of a (connected) reductive group G on an affine algebraic variety V, with G, V, and the action all defined over a fixed non-archimedean local field F.

Definition 2.11 ([4, Definition 1.2])**.** A subset $X \subseteq V(F)$ is cocharacter-closed if for every $v \in X$ and every cocharacter $\lambda\colon \mathbb{G}_m \to G$, if the limit $\lim_{t\to 0} \lambda(t)\cdot v$ exists, then the limit lies in X. The cocharacter-closure of X, denoted $\overline{X}^{\mathrm{c}}$ is the smallest cocharacter-closed subset of $V(F)$ containing X.

Note that for X stable under the action of $G(F)$, being closed in the F-analytic topology implies cocharacter-closed. Corollary 2.19 below will show that the converse holds for $G(F)$-orbits. First we will need the following technical result on $G(F)$-orbits being locally closed.

Proposition 2.12 ([2, Proposition 2.2])**.** *Let $G \times V \to V$ be an action of an algebraic group G on an algebraic variety V, with G, V, and the action all defined over a fixed non-archimedean local field F. Then $G(F)$-orbits in $V(F)$ are locally closed in the F-analytic topology on $V(F)$.*

Proof. This was first proved in [7] as a combination of Proposition 6.8(c) and Theorem A of Section 6.15 proved in the appendix. Another proof is given in [13, §1.5]. □

Corollary 2.13 ([3, Proposition 3.2])**.** *Let $f\colon G \to H$ be a homomorphism of algebraic groups, with G, H, and f all defined over a fixed non-archimedean local field F. Then the image $f(G(F))$ is a closed subgroup of $H(F)$ in the F-analytic topology on $H(F)$.*

Proof. Applying Proposition 2.12 to the multiplication action $G \times H \to H$ tells us that $f(G(F))$ is a locally closed subgroup of $H(F)$ in the F-analytic topology on $H(F)$. But a locally closed subgroup of a topological group is automatically closed (it is an open subgroup of its closure, but open subgroups are automatically closed). □

Remark 2.14. We will only need Corollary 2.13 in the case of split tori $S \to T$. In this case, the corollary can be seen directly from Smith normal form and the fact that the nth powers $(F^\times)^n$ are a closed subgroup of $F^\times$.

We will also need the following formulation of Motzkin's transposition theorem. It says roughly that a system of equalities and inequalities has an integer solution if and only if there does not exist an algebraic obstruction.

Lemma 2.15. *Let $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_\ell \colon \mathbb{Z}^n \to \mathbb{Z}$ be linear maps. The system of constraints*

$$\alpha_1(v) = 0, \dots, \alpha_k(v) = 0, \beta_1(v) > 0, \dots, \beta_\ell(v) > 0$$

has a solution $v \in \mathbb{Z}^n$ if and only if there does not exist a linear dependence

$$c_1\alpha_1 + \cdots + c_k\alpha_k + d_1\beta_1 + \cdots + d_\ell\beta_\ell = 0$$

for constants $c_1, \dots, c_k, d_1, \dots, d_\ell \in \mathbb{Z}$ with $d_1, \dots, d_\ell \geq 0$ and at least one $d_j \neq 0$.

Proof. The constraints are homogeneous, so it is equivalent to work over $\mathbb{Q}$. Then this is exactly Motzkin's transposition theorem [26], usually stated over $\mathbb{R}$, but the standard proof using Fourier-Motzkin elimination works equally well over any ordered field. □

Finally, we will need to know that our affine algebraic variety V can be embedded as a closed subvariety of a finite-dimensional representation of G over F.

Lemma 2.16 ([19, Lemma 1.1(a)])**.** *There exists a G-equivariant isomorphism between V and a closed subvariety of a finite-dimensional representation of G over F.*

We can now prove a rational Hilbert–Mumford criterion that will imply Corollary 2.19.

Theorem 2.17. *Let $v \in V(F)$. If $w \in \overline{G(F) \cdot v}$ (the closure in the F-analytic topology on $V(F)$), then there exists a cocharacter $\lambda \colon \mathbb{G}_m \to G$ and an element $g \in G(F)$ such that*

$$\lim_{t \to 0} \lambda(t) \cdot v = \lim_{t \to \infty} \lambda(t) \cdot g \cdot w.$$

Remark 2.18. This formulation of the Hilbert–Mumford criterion is taken from Proposition 6 of [8]. Other formulations such as Theorem 1.4 of [19] simply state that $\lim_{t \to 0} \lambda(t) \cdot v$ exists and lies in the closure of the orbit of w.

Proof of Theorem 2.17. By Lemma 2.16, we can assume that V is a finite-dimensional representation of G over F. If the theorem is true for v and w with

$$\lim_{t \to 0} \lambda(t) \cdot v = \lim_{t \to \infty} \lambda(t) \cdot g \cdot w,$$

then for any $g_1, g_2 \in G(F)$ we have

$$\lim_{t \to 0} g_1\lambda(t)g_1^{-1} \cdot g_1 \cdot v = \lim_{t \to \infty} g_1\lambda(t)g_1^{-1} \cdot g_1 g g_2^{-1} \cdot g_2 \cdot w$$

so the theorem is also true for $g_1 \cdot v$ and $g_2 \cdot w$. We can use this fact to modify v and w by elements of $G(F)$.

Let $T \leq G$ be a maximal split torus. Let $M = C_G(T)$ be the corresponding Levi subgroup. Then we have the Cartan decomposition $G(F) = KM(F)K$ for some maximal compact subgroup $K \leq G(F)$. Now write $k_i m_i \ell_i \cdot v \to w$ for $k_i, \ell_i \in K$ and $m_i \in M(F)$. The quotient M/T is anisotropic, so the group $(M/T)(F) = M(F)/T(F)$ is compact. By passing to a subsequence, we may assume that the sequences (k_i), (ℓ_i), and $(m_i T(F))$ all converge. If $m_i T(F) \to m_\infty T(F)$, then we can find $t_i \in T(F)$ such that $m_i t_i^{-1} \to m_\infty$. Then $k_i m_i t_i^{-1} t_i \ell_i \cdot v \to w$ where the sequences (ℓ_i) and $(k_i m_i t_i^{-1})$ both converge. By modifying v and w by the limiting values, we get $k_i t_i \ell_i \cdot v \to w$ for new sequences (k_i) and (ℓ_i) in $G(F)$ both converging to 1. But in fact we even have $t_i \ell_i \cdot v \to w$ since

$$|t_i \ell_i v - w| \leq \left|k_i^{-1}(k_i t_i \ell_i v - w)\right| + \left|k_i^{-1} w - w\right| \to 0.$$

We can decompose $V = \bigoplus_\alpha V_\alpha$ where T acts on V_α via $\alpha \colon T \to \mathbb{G}_m$. Now write $v = \sum_\alpha v_\alpha$, $w = \sum_\alpha w_\alpha$, and $\ell_i v = \sum_\alpha v_{\alpha,i}$. Then we have $v_{\alpha,i} \to v_\alpha$ and

$$t_i \ell_i v = \sum_\alpha \alpha(t_i) v_{\alpha,i} \to \sum_\alpha w_\alpha.$$

If $\alpha \colon T \to \mathbb{G}_m$ is a character with the property that $v_\alpha \neq 0$ and $w_\alpha \neq 0$, then the sequence $(\alpha(t_i))$ must converge to a limiting value in $F^\times$. Let $\alpha_1, \ldots, \alpha_k \colon T \to \mathbb{G}_m$ be all of the characters with the property that $v_\alpha \neq 0$ and $w_\alpha \neq 0$. Corollary 2.13 tells us that the map $(\alpha_1, \ldots, \alpha_k) \colon T(F) \to (F^\times)^k$ has closed image, so there exists a $t_\infty \in T(F)$ such that $\alpha_j(t_i) \to \alpha_j(t_\infty)$ for each j. By modifying w and each t_i by t_∞, we may assume that $\alpha_j(t_i) \to 1$ for each j. Our goal is to find a cocharacter $\lambda \colon \mathbb{G}_m \to G$ such that

$$\lim_{t \to 0} \sum_\alpha t^{\langle \alpha, \lambda \rangle} v_\alpha = \lim_{t \to \infty} \sum_\alpha t^{\langle \alpha, \lambda \rangle} w_\alpha.$$

This equation is equivalent to the following constraints on λ:

1. For α with $v_\alpha \neq 0$ and $w_\alpha \neq 0$, we have $\alpha(t_i) \to 1$ and need λ to satisfy $\langle \alpha, \lambda \rangle = 0$.
2. For α with $v_\alpha \neq 0$ and $w_\alpha = 0$, we have $\alpha(t_i) \to 0$ and need λ to satisfy $\langle \alpha, \lambda \rangle > 0$.
3. For α with $v_\alpha = 0$ and $w_\alpha \neq 0$, we have $\alpha(t_i) \to \infty$ and need λ to satisfy $\langle \alpha, \lambda \rangle < 0$.

If no such λ exists, then Lemma 2.15 gives us a linear dependence $\prod_\alpha \alpha^{c_\alpha} = 1$ for constants $c_\alpha \in \mathbb{Z}$ with $c_\alpha \geq 0$ in the second case, $c_\alpha \leq 0$ in the third case, and at least one $c_\alpha \neq 0$ in the second or third case. But then evaluating on t_i gives a contradiction $1 = \prod_\alpha \alpha(t_i)^{c_\alpha} \to 0$. □

Corollary 2.19. *Let $v \in V(F)$. If $G(F) \cdot v$ is cocharacter-closed, then $G(F) \cdot v$ is closed in the F-analytic topology on $V(F)$.*

Proof. We will prove the contrapositive. Let $\mathcal{O} = G(F) \cdot v$. Suppose that $w \in \overline{\mathcal{O}} \setminus \mathcal{O}$. Then Theorem 2.17 gives a cocharacter $\lambda \colon \mathbb{G}_m \to G$ and an element $g \in G(F)$ such that

$$\lim_{t \to 0} \lambda(t) \cdot v = \lim_{t \to \infty} \lambda(t) \cdot g \cdot w.$$

We know that $\mathcal{O} = G(F) \cdot v$ is locally closed by Proposition 2.12. Then $\overline{\mathcal{O}} \setminus \mathcal{O}$ is closed, so $\lim_{t\to\infty} \lambda(t) \cdot g \cdot w \in \overline{\mathcal{O}} \setminus \mathcal{O}$. But then $\lim_{t\to 0} \lambda(t) \cdot v \notin \mathcal{O}$, so $\mathcal{O}$ is not cocharacter-closed. □

Thus, we can recast assumption (H2b) of Theorem 2.9 as stating that the orbit $M(F) \cdot \overline{\gamma}$ is cocharacter-closed. But in fact this is equivalent to $G(F) \cdot \overline{\gamma}$ being cocharacter-closed.

Proposition 2.20 ([4, Theorem 5.4(ii)])**.** *Let G be a reductive group over a field F, let M be a Levi subgroup of G, and let $\gamma \in M(F)$. Then $G(F) \cdot \gamma$ is cocharacter-closed if and only if $M(F) \cdot \gamma$ is cocharacter-closed.*

As for assumption (H2a) of Theorem 2.9, we will leave it as a conjecture for now.

Conjecture 2.21. *Let G be a reductive group over a non-archimedean local field F and let $\gamma \in G(F)$. If $G(F) \cdot \gamma$ is cocharacter-closed, then $G_\gamma(F)$ is unimodular.*

Remark 2.22. When the centralizer G_γ is smooth, the modular character of $G_\gamma(F)$ is algebraic, so unimodularity of $G_\gamma(F)$ can be checked over a field extension.

Theorem 2.23. *Let G be a reductive group over a non-archimedean local field F. Let P be a parabolic subgroup of G with Levi decomposition $P = MN$. Let $\gamma \in P(F)$ with image $\overline{\gamma} \in M(F)$, let $H = M_{\overline{\gamma}} N$, and let $\iota\colon H(F)/H_\gamma(F) \hookrightarrow N(F)$ be defined by $\iota(h) = h\gamma h^{-1}\overline{\gamma}^{-1}$. Assume that Conjecture 2.21 holds for $\overline{\gamma} \in M(F)$. Suppose that the parabolic subgroup P satisfies*

(H1) *$G_\gamma(F) \leq P(F)$,*

(H2) *The $G(F)$-orbit of $\overline{\gamma}$ is cocharacter-closed,*

(H3) *There exists a nonzero Radon measure dq on $H(F)/H_\gamma(F)$ satisfying*

$$\int_{H(F)/H_\gamma(F)} f(h_0 h)\, dq = \delta_H^{-1}(h_0) \int_{H(F)/H_\gamma(F)} f(h)\, dq \tag{H3a}$$

for all $f \in C_c(H(F)/H_\gamma(F))$ and all $h_0 \in H(F)$, and

$$\int_{H(F)/H_\gamma(F)} f(\iota(h))\, dq < \infty \tag{H3b}$$

for all $f \in C_c(N(F))$.

Then $G_\gamma(F)$ is unimodular and the orbital integrals

$$\int_{G(F)/G_\gamma(F)} f(g\gamma g^{-1})\, \frac{dg}{dg_\gamma}$$

converge for all $f \in C_c(G(F))$.

Proof. The $M(F)$-orbit of $\overline{\gamma}$ is cocharacter-closed by Proposition 2.20. Then hypotheses (H2a) and (H2b) of Theorem 2.9 follow from Conjecture 2.21 and Corollary 2.19. □

Let $P = MN$ be a Levi decomposition of a parabolic subgroup, and let $\lambda\colon \mathbb{G}_m \to G$ be a cocharacter of G giving rise to this Levi decomposition. If $p \in P(F)$ factors as $p = mn$ for $m \in M(F)$ and $n \in N(F)$, then $\lim_{t\to 0} \lambda(t)m\lambda(t)^{-1} = m$ and $\lim_{t\to 0} \lambda(t)n\lambda(t)^{-1} = 1$, so we have $\lim_{t\to 0} \lambda(t)p\lambda(t)^{-1} = m$. In the situation of Theorem 2.23, this means that the image $\overline{\gamma} \in M(F)$ is exactly the limit $\lim_{t\to 0} \lambda(t)\gamma\lambda(t)^{-1}$ of γ under the action of λ. In other words, Theorem 2.23 requires the existence of a single cocharacter λ pushing γ all the way to a cocharacter-closed orbit. The following result tells us that this is always possible.

Theorem 2.24 ([4, Theorem 1.3])**.** *Let $v \in V(F)$. Then there is a unique cocharacter-closed $G(F)$-orbit $\mathcal{O}$ inside $\overline{G(F)\cdot v}^c$. Moreover, there exists a cocharacter $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot v$ exists and lies in $\mathcal{O}$.*

Theorem 2.24 gives a parabolic subgroup P satisfying hypothesis (H2) of Theorem 2.23. More recent work on G-complete reducibility can handle (H1) at the same time.

Proposition 2.25. *Let $\gamma \in G(F)$. Then there exists a parabolic subgroup P of G such that $G_\gamma(F) \leq P(F)$ and such that the $G(F)$-orbit of the Levi quotient $\overline{\gamma}$ is cocharacter-closed.*

Proof. If the $G(F)$-orbit of γ is cocharacter-closed, then we can take $P = G$. Now suppose that the $G(F)$-orbit of γ is not cocharacter-closed. Let $H = \overline{\langle\gamma\rangle}$ be the algebraic subgroup of G generated by γ. By Theorem 5.9 of [6], H is not G-completely reducible over F. By Lemma 5.1 of [5], there is a parabolic subgroup P of G yielding an F-semisimplification of H. This means that the image $H' = \overline{\langle\overline{\gamma}\rangle}$ of H in the Levi quotient is G-completely reducible. By Theorem 5.9 of [6], the $G(F)$-orbit of $\overline{\gamma}$ is cocharacter-closed.

This particular parabolic subgroup P arises from the Tits center conjecture which implies that it is fixed by any F-automorphism of G that stabilizes H (see the sentence immediately prior to the statement of Lemma 5.1 in [5]). In particular, conjugation by $g \in G_\gamma(F)$ fixes P. But parabolic subgroups are self-normalizing, so $G_\gamma(F) \leq P(F)$. □

2.5 From Stable to Rational

We would like to construct the measure on the rational orbit $H(F)/H_\gamma(F)$ from a measure on the stable orbit $(H/H_\gamma)(F)$. When the characteristic of F is good enough, the stable orbit breaks into finitely many rational orbits, each of which is an open subset of the stable orbit. In this case, we can simply restrict the measure on $(H/H_\gamma)(F)$ to the open subset $H(F)/H_\gamma(F)$. But when the characteristic of F is bad, the stable orbit may break into infinitely many rational orbits, and these rational orbits may no longer be open in the stable orbit. In this case, our precise conjecture is the following.

Conjecture 2.26. *Let G be a (connected) reductive group over a non-archimedean local field F, and let P be a parabolic subgroup of G with Levi decomposition $P = MN$. Let $\gamma \in P(F)$ with image $\overline{\gamma} \in M(F)$, let $H = M_{\overline{\gamma}}N$, and let $\iota\colon H/H_\gamma \hookrightarrow N$ be defined by $\iota(h) = h\gamma h^{-1}\overline{\gamma}^{-1}$. Suppose that the image of ι is a Zariski open subset of U for some algebraic subgroup $U \leq N$ normalized by H, and that there exists a strictly positive Radon measure du on $U(F)$ satisfying*

$$\int_{U(F)} f(h_0 u h_0^{-1} \cdot h_0 \overline{\gamma} h_0^{-1} \overline{\gamma}^{-1})\, du = \delta_H^{-1}(h_0) \int_{U(F)} f(u)\, du$$

for all $f \in C_c(U(F))$ and all $h_0 \in H(F)$. Then there exists a strictly positive Radon measure dq on $H(F)/H_\gamma(F)$ satisfying

$$\int_{H(F)/H_\gamma(F)} f(h_0 h)\, dq = \delta_H^{-1}(h_0) \int_{H(F)/H_\gamma(F)} f(h)\, dq \tag{a}$$

for all $f \in C_c(H(F)/H_\gamma(F))$ and all $h_0 \in H(F)$, and

$$\int_{H(F)/H_\gamma(F)} f(\iota(h))\, dq < \infty \tag{b}$$

for all $f \in C_c(N(F))$.

Remark 2.27. By irreducibility, the hypothesis that the image of ι is a Zariski open subset of U will hold provided that the image of ι is contained in U and of the same dimension.

Remark 2.28. Conjecture 2.26 naturally breaks up into two parts. Part (a) states that the invariant measure on the stable orbit $(H/H_\gamma)(F)$ descends to an invariant measure on the rational orbit $H(F)/H_\gamma(F)$. Part (b) states that this descended invariant measure on the rational orbit remains finite on compact subsets of $N(F)$. Proposition 2.31 will give a proof of part (a) when the orbit space $(H/H_\gamma)(F)/H(F)$ is Hausdorff. But for part (b), we are still unable to handle the situation when the orbit space $(H/H_\gamma)(F)/H(F)$ is infinite.

Assuming Conjecture 2.26, we can reduce convergence of orbital integrals in $G(F)$ to the existence of a suitable measure on the subgroup $U(F)$.

Theorem 2.29. *Let G be a reductive group over a non-archimedean local field F. Let P be a parabolic subgroup of G with Levi decomposition $P = MN$. Let $\gamma \in P(F)$ with image $\overline{\gamma} \in M(F)$, let $H = M_{\overline{\gamma}}N$, and let $\iota\colon H/H_\gamma \hookrightarrow N$ be defined by $\iota(h) = h\gamma h^{-1}\overline{\gamma}^{-1}$. Assume that Conjecture 2.26 holds for this setup. Assume that Conjecture 2.21 holds for $\overline{\gamma} \in M(F)$. Suppose that the parabolic subgroup P satisfies*

(H1) *$G_\gamma(F) \leq P(F)$,*

(H2) *The $G(F)$-orbit of $\overline{\gamma}$ is cocharacter-closed,*

(H3) *The image of ι is a Zariski open subset of U for some algebraic subgroup $U \leq N$ normalized by H, and there exists a strictly positive Radon measure du on $U(F)$ satisfying*

$$\int_{U(F)} f(h_0 u h_0^{-1} \cdot h_0 \overline{\gamma} h_0^{-1} \overline{\gamma}^{-1})\, du = \delta_H^{-1}(h_0) \int_{U(F)} f(u)\, du \tag{H3}$$

for all $f \in C_c(U(F))$ and all $h_0 \in H(F)$.

Then $G_\gamma(F)$ is unimodular and the orbital integrals

$$\int_{G(F)/G_\gamma(F)} f(g\gamma g^{-1})\, \frac{dg}{dg_\gamma}$$

converge for all $f \in C_c(G(F))$.

Proof. Conjecture 2.26 applied to (H3) exactly gives (H3a) and (H3b) of Theorem 2.23. □

2.6 Descent of Measure

While we do not have a full proof of Conjecture 2.26 in general, we can at least prove part (a) when the stabilizer subgroup P_γ is smooth. This will require some technical measure theory. Recall that a *ring of sets* means a nonempty collection of subsets closed under pairwise unions $A \cup B$ and relative complements $A \setminus B$.

Lemma 2.30. *Let G be a group with an action on a topological space X. Let $\chi\colon G \to \mathbb{R}_{>0}^\times$ be a homomorphism. Let $\mathcal{C}$ be a G-invariant ring of compact open subsets of X which generates the Borel σ-algebra on X. Fix $S_0 \in \mathcal{C}$. Suppose that for every finite subset $\mathcal{A} \subseteq \mathcal{C}$, there exists a function $\mu_\mathcal{A}\colon \mathcal{A} \to \mathbb{R}_{\geq 0}$ satisfying*

1. *$\mu_\mathcal{A}(S_0) > 0$ whenever $S_0 \in \mathcal{A}$,*
2. *$\mu_\mathcal{A}(S^g) = \chi(g)\mu_\mathcal{A}(S)$ for all $g \in G$ and $S \in \mathcal{A}$ with $S^g \in \mathcal{A}$,*
3. *$\mu_\mathcal{A}(S \cup T) = \mu_\mathcal{A}(S) + \mu_\mathcal{A}(T)$ for all disjoint $S, T \in \mathcal{A}$ with $S \cup T \in \mathcal{A}$.*

Then there exists a Borel measure μ on X satisfying $\mu(S_0) = 1$ and $\mu(S^g) = \chi(g)\mu(S)$ for all $g \in G$.

Proof. Note that the third assumption on $\mu_\mathcal{A}$ implies that $\mu_\mathcal{A}(\varnothing) = 0$ whenever $\varnothing \in \mathcal{A}$. By renormalizing the functions $\mu_\mathcal{A}$, we may assume that $\mu_\mathcal{A}(S_0) = 1$ whenever $S_0 \in \mathcal{A}$. Now arbitrarily extend the functions $\mu_\mathcal{A}\colon \mathcal{A} \to \mathbb{R}_{\geq 0}$ to be defined on all of $\mathcal{C}$. Let I be the directed set of all finite subsets of $\mathcal{C}$. The product $\prod_{S \in \mathcal{C}} [0, \infty]$ is compact, so the net $I \to \prod_{S \in \mathcal{C}} [0, \infty]$ defined by $\mathcal{A} \mapsto (S \mapsto \mu_\mathcal{A}(S))$ has a subnet $J \to I$ converging to a function $\mu\colon \mathcal{C} \to [0, \infty]$ satisfying $\mu(\varnothing) = 0$, $\mu(S_0) = 1$, $\mu(S^g) = \chi(g)\mu(S)$ for all $g \in G$ and $S \in \mathcal{C}$, and $\mu(S \cup T) = \mu(S) + \mu(T)$ for all disjoint $S, T \in \mathcal{C}$. Then μ is a finitely

additive premeasure on $\mathcal{C}$, but since the sets in $\mathcal{C}$ are compact open, μ vacuously satisfies σ-additivity. By Caratheodory's extension theorem, μ extends to a Borel measure on X satisfying $\mu(S_0) = 1$ and $\mu(S^g) = \chi(g)\mu(S)$. □

Proposition 2.31. *Let H be a group with an action on a second countable locally compact totally disconnected topological group G. Assume that for each $h \in H$, if gK is a left coset of a compact open subgroup K of G, then $(gK)^h$ is also a left coset of a compact open subgroup of G. Let $\chi\colon H \to \mathbb{R}^\times_{>0}$ be a homomorphism. Let X be an H-invariant open subset of G. Assume that the quotient space X/H is Hausdorff. Suppose that there exists a strictly positive Radon measure μ_G on G satisfying $\mu_G(S^h) = \chi(h)\mu_G(S)$ for all $h \in H$. Then for each H-orbit $\mathcal{O}$ in X, there exists a nonzero Radon measure $\mu_\mathcal{O}$ on $\mathcal{O}$ satisfying $\mu_\mathcal{O}(S^h) = \chi(h)\mu_\mathcal{O}(S)$ for all $h \in H$ and Borel $S \subseteq \mathcal{O}$.*

Proof. We will apply Lemma 2.30. Let $\mathcal{C}$ be the collection of subsets of $\mathcal{O}$ of the form $SK \cap \mathcal{O}$ where $SK \subseteq X$ is a union of finitely many left cosets of a compact open subgroup K of G. Since the orbit $\mathcal{O}$ is closed, the intersection $SK \cap \mathcal{O}$ remains a compact subset of $\mathcal{O}$. Then $\mathcal{C}$ is an H-invariant ring of compact open subsets of $\mathcal{O}$ generating the Borel σ-algebra on $\mathcal{O}$.

Let $\mathcal{A}$ be a finite subset of $\mathcal{C}$. By passing to the intersection of the compact open subgroups, we may write $\mathcal{A} = \{S_1K \cap \mathcal{O}, \dots, S_nK \cap \mathcal{O}\}$. By replacing S_i by $S_iK \cap \mathcal{O}$, we may assume that $S_i = S_iK \cap \mathcal{O}$, so $\mathcal{A} = \{S_1, \dots, S_n\}$. Let V be a small neighborhood of $\mathcal{O}$ in X/H. Set $\mu_\mathcal{A}(S_i) = \mu_G(S_iK \cap \pi^{-1}(V))$ where $\pi\colon X \to X/H$ is the quotient map. Suppose that $S_i \cup S_j = S_k$ is a disjoint union. Then $S_iK \cup S_jK = S_kK$ is a disjoint union, so

$$(S_iK \cap \pi^{-1}(V)) \cup (S_jK \cap \pi^{-1}(V)) = S_kK \cap \pi^{-1}(V)$$

is a disjoint union, and

$$\mu_\mathcal{A}(S_k) = \mu_G(S_kK \cap \pi^{-1}(V)) = \mu_G(S_iK \cap \pi^{-1}(V)) + \mu_G(S_jK \cap \pi^{-1}(V)) = \mu_\mathcal{A}(S_i) + \mu_\mathcal{A}(S_j).$$

Now suppose that $S_i^h = S_j$. Then $(S_iK)^h$ is a union of finitely many left cosets of a compact open subgroup K_0 of G. We may assume that $K_0 \leq K$. We at least have $S_jK_0 \subseteq (S_iK)^h$, but there may be other left cosets of K_0 that lie in $(S_iK)^h$ but do not meet $\mathcal{O}$. Likewise, we at least have $S_jK_0 \subseteq S_jK$, but there may be other left cosets of K_0 that lie in S_jK but do not meet $\mathcal{O}$. The image of these cosets of K_0 that do not meet $\mathcal{O}$ will be a compact subset of X/H that does not meet $\mathcal{O}$. Since the quotient X/H is assumed to be Hausdorff, we know that for sufficiently small V, these cosets of K_0 that do not meet $\mathcal{O}$ will also be disjoint from $\pi^{-1}(V)$. Then for such sufficiently small V, we will have

$$S_jK \cap \pi^{-1}(V) = S_jK_0 \cap \pi^{-1}(V) = (S_iK)^h \cap \pi^{-1}(V) = (S_iK \cap \pi^{-1}(V))^h$$

and

$$\mu_\mathcal{A}(S_j) = \mu_G(S_jK \cap \pi^{-1}(V)) = \mu_G((S_iK \cap \pi^{-1}(V))^h) = \chi(h)\mu_G(S_iK \cap \pi^{-1}(V)) = \chi(h)\mu_\mathcal{A}(S_i).$$

If some other $h' \in H$ satisfies $S_i^{h'} = S_j$, then taking a sufficiently small V for both h and h' will give $\mu_{\mathcal{A}}(S_j) = \chi(h)\mu_{\mathcal{A}}(S_i)$ and $\mu_{\mathcal{A}}(S_j) = \chi(h')\mu_{\mathcal{A}}(S_i)$, hence $\chi(h) = \chi(h')$. What this shows is that if V is sufficiently small for h, then $\mu_{\mathcal{A}}(S_j) = \chi(h')\mu_{\mathcal{A}}(S_i)$ holds automatically for all $h' \in H$ satisfying $S_i^{h'} = S_j$. Thus, by taking V sufficiently small to accommodate one choice of $h \in H$ for each of the finitely many pairs (S_i, S_j) satisfying $S_i^h = S_j$, we can ensure that $\mu_{\mathcal{A}}(S_j) = \chi(h)\mu_{\mathcal{A}}(S_i)$ for all occurrences of $S_i^h = S_j$.

If we also fix some nonempty $S_0 \in \mathcal{C}$, then Lemma 2.30 gives a Borel measure $\mu_{\mathcal{O}}$ on $\mathcal{O}$ satisfying $\mu_{\mathcal{O}}(S_0) = 1$ and $\mu_{\mathcal{O}}(S^h) = \chi(h)\mu_{\mathcal{O}}(S)$ for all $h \in H$. Then $\mu_{\mathcal{O}}$ is finite on compact sets (cover by translates of S_0). By [11, Theorem 7.8], $\mu_{\mathcal{O}}$ is Radon by our second countability assumption. □

Remark 2.32. Proposition 2.31 proves part (a) of Conjecture 2.26 whenever the orbit space is Hausdorff. If P_γ is smooth, then the orbit space is parameterized by profinite Galois cohomology and is thus guaranteed to be Hausdorff.

2.7 Symplectic Example

To illustrate the difficulty of Conjecture 2.26 in general, we present a pathological example in Sp_6 where the orbit space fails to be Hausdorff. Consider the symplectic group Sp_6 in characteristic 2 defined by

$$\mathrm{Sp}_6(F) = \{M \in \mathrm{GL}_6(F) : M^T \Omega M = \Omega\}$$

for the fixed choice $\Omega = \begin{bmatrix} & I \\ -I & \end{bmatrix}$. And consider an element $\gamma = \begin{bmatrix} I & X \\ & I \end{bmatrix} \in \mathrm{Sp}_6(F)$ with $X \in \mathrm{GL}_3(F)$ (necessarily symmetric). Note that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & X \\ & I \end{bmatrix} = \begin{bmatrix} A & AX+B \\ C & CX+D \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} I & X \\ & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A+XC & B+XD \\ C & D \end{bmatrix}.$$

But X is invertible, so the centralizer of γ in GL_6 is defined by $C = 0$ and $AX = XD$. And note that

$$\begin{bmatrix} A & B \\ & D \end{bmatrix}^T \begin{bmatrix} & I \\ -I & \end{bmatrix} \begin{bmatrix} A & B \\ & D \end{bmatrix} = \begin{bmatrix} A^T & \\ B^T & D^T \end{bmatrix} \begin{bmatrix} & D \\ -A & -B \end{bmatrix} = \begin{bmatrix} & A^TD \\ -D^TA & B^TD - D^TB \end{bmatrix}$$

so the centralizer of γ in Sp_6 is further defined by $A^TD = I$ and $B^TD = D^TB$. In particular, the centralizer

$$C_{\mathrm{Sp}_6}(\gamma) = \left\{ \begin{bmatrix} A & B \\ & D \end{bmatrix} : AX = XD \text{ and } A^TD = I \text{ and } B^TD = D^TB \right\}$$

is contained in the parabolic subgroup

$$P = \left\{ \begin{bmatrix} A & B \\ & D \end{bmatrix} : A^TD = I \text{ and } B^TD = D^TB \right\} \leq \mathrm{Sp}_6$$

with Levi decomposition

$$P = M \ltimes N = \left\{ \begin{bmatrix} A & \\ & D \end{bmatrix} : A^T D = I \right\} \ltimes \left\{ \begin{bmatrix} I & B \\ & I \end{bmatrix} : B^T = B \right\}.$$

The centralizer $C_{\mathrm{Sp}_6}(\gamma)$ contains its own projection to M and also contains N, and thus factors as

$$C_{\mathrm{Sp}_6}(\gamma) = C_M(\gamma) \ltimes N = \left\{ \begin{bmatrix} A & \\ & D \end{bmatrix} : AX = XD \text{ and } A^T D = I \right\} \ltimes \left\{ \begin{bmatrix} I & B \\ & I \end{bmatrix} : B^T = B \right\}.$$

The P-orbit of γ is $O_P(\gamma) = O_M(\gamma)$ which we can identify with $O_{\mathrm{GL}_3}(X)$ via

$$\begin{bmatrix} A & \\ & A^{-T} \end{bmatrix} \begin{bmatrix} I & X \\ & I \end{bmatrix} \begin{bmatrix} A & \\ & A^{-T} \end{bmatrix}^{-1} = \begin{bmatrix} A & AX \\ & A^{-T} \end{bmatrix} \begin{bmatrix} A^{-1} & \\ & A^T \end{bmatrix} = \begin{bmatrix} I & AXA^T \\ & I \end{bmatrix}.$$

Now fix $F = \mathbb{F}_2((t))$ and set

$$X_t = \begin{bmatrix} 1 & & \\ & t & 1 \\ & 1 & \end{bmatrix} \in \mathrm{GL}_3 .$$

More generally, we can consider

$$X_f = \begin{bmatrix} 1 & & \\ & f & 1 \\ & 1 & \end{bmatrix} \in \mathrm{GL}_3$$

for any $f \in F$. We can decompose f into terms of even degree and terms of odd degree, thereby expressing $f = a^2 + b^2 t$. If f is not a perfect square in F, then $b \neq 0$ and X_f can be seen to be in the F-orbit of X_t via

$$\begin{bmatrix} 1 & & -a/b \\ a & b & \\ & & 1/b \end{bmatrix} \begin{bmatrix} 1 & & \\ & t & 1 \\ & 1 & \end{bmatrix} \begin{bmatrix} 1 & & -a/b \\ a & b & \\ & & 1/b \end{bmatrix}^T = \begin{bmatrix} 1 & & \\ & a^2 + b^2 t & 1 \\ & 1 & \end{bmatrix}.$$

But the matrices X_f with f a perfect square in F form a separate F-orbit. We can see this by expanding

$$\begin{aligned} AX_0A^T &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T \\ &= \begin{bmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}^2 & a_{11}a_{21} + a_{13}a_{22} + a_{12}a_{23} & a_{11}a_{31} + a_{13}a_{32} + a_{12}a_{33} \\ a_{21}a_{11} + a_{23}a_{12} + a_{22}a_{13} & a_{21}^2 & a_{21}a_{31} + a_{23}a_{32} + a_{22}a_{33} \\ a_{31}a_{11} + a_{33}a_{12} + a_{32}a_{13} & a_{31}a_{21} + a_{33}a_{22} + a_{32}a_{23} & a_{31}^2 \end{bmatrix}. \end{aligned}$$

Here the central entry a_{21}^2 is a perfect square, so AX_0A^T can only equal X_f if f is a perfect square in F. And this can always be achieved by

$$\begin{bmatrix} 1 & \sqrt{f} & \\ \sqrt{f} & & 1 \\ & 1 & \end{bmatrix} \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \begin{bmatrix} 1 & \sqrt{f} & \\ \sqrt{f} & & 1 \\ & 1 & \end{bmatrix}^T = \begin{bmatrix} 1 & & \sqrt{f} \\ \sqrt{f} & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{f} & \\ \sqrt{f} & & 1 \\ & 1 & \end{bmatrix} = \begin{bmatrix} 1 & & \\ & f & 1 \\ & 1 & \end{bmatrix}.$$

Note that setting $f = t$ shows that X_t is in the $\overline{F}$-orbit of X_0. To compute the dimension of the $\overline{F}$-orbit of X_0, we can compute the stabilizer of X_0. In order for the matrix AX_0A^T to equal X_0, we must have $a_{11} = 1$ and $a_{21} = a_{31} = 0$, giving

$$AX_0A^T = \begin{bmatrix} 1 & a_{13}a_{22} + a_{12}a_{23} & a_{13}a_{32} + a_{12}a_{33} \\ a_{23}a_{12} + a_{22}a_{13} & & a_{23}a_{32} + a_{22}a_{33} \\ a_{33}a_{12} + a_{32}a_{13} & a_{33}a_{22} + a_{32}a_{23} & \end{bmatrix}.$$

But now the constraint

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix} \in \mathrm{GL}_3,$$

implies that the vectors $\langle a_{22}, a_{32}\rangle$ and $\langle a_{23}, a_{33}\rangle$ are linearly independent. In particular, the constraints $a_{13}a_{22} + a_{12}a_{23} = a_{13}a_{32} + a_{12}a_{33} = 0$ force $a_{12} = a_{13} = 0$. Then the reduced stabilizer of X_0 is $\mathrm{SL}_1 \times \mathrm{SL}_2 \leq \mathrm{GL}_3$, so the $\overline{F}$-orbit has dimension $9 - 3 = 6$. Note that this is exactly the dimension of N, so the $\overline{F}$-orbit is Zariski open in N by Remark 2.27.

This example is already exhibiting some interesting behavior. Note that the F-orbit of X_0 is contained in the closure of the F-orbit of X_t. So one might say that the F-orbit of X_0 is somehow "embedded" inside the F-orbit of X_t (not in the algebraic geometry sense). And despite being stably conjugate, the points X_0 and X_t have markedly different stabilizers. We have already seen that the reduced stabilizer of X_0 is $\mathrm{SL}_1 \times \mathrm{SL}_2 \leq \mathrm{GL}_3$. In contrast, we can compute the F-points of the stabilizer of X_t by expanding

$$\begin{aligned} AX_tA^T &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & t & 1 \\ & 1 & \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T \\ &= \begin{bmatrix} a_{11} & a_{12}t + a_{13} & a_{12} \\ a_{21} & a_{22}t + a_{23} & a_{22} \\ a_{31} & a_{32}t + a_{33} & a_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}^2 + a_{12}^2 t & \cdots & \cdots \\ \cdots & a_{21}^2 + a_{22}^2 t & \cdots \\ \cdots & \cdots & a_{31}^2 + a_{32}^2 t \end{bmatrix}. \end{aligned}$$

In order for this to equal X_t, we must have $a_{11} = a_{22} = 1$ and $a_{12} = a_{21} = a_{31} = a_{32} = 0$. Then we have

$$AX_tA^T = \begin{bmatrix} 1 & & a_{13} \\ & 1 & a_{23} \\ & & a_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & t & 1 \\ & 1 & \end{bmatrix} \begin{bmatrix} 1 & & a_{13} \\ & 1 & a_{23} \\ & & a_{33} \end{bmatrix}^T = \begin{bmatrix} 1 & a_{13} & \\ a_{13} & t & a_{33} \\ & a_{33} & \end{bmatrix}$$

so we must further have $a_{13} = 0$ and $a_{33} = 1$. Thus, the F-points of the stabilizer of X_t consists of the elements

$$A = \begin{bmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{bmatrix} \in \mathrm{GL}_3(F).$$

Perhaps the most crucial difference between the stabilizer of X_0 and the stabilizer of X_t is that the stabilizer of X_0 has split rank 1 whereas the stabilizer of X_t has split rank 0. Heuristically, this means that some of the split rank of GL_3 is absorbed by the stabilizer of X_0, so the F-orbit of X_0 has fewer limiting directions and thus better convergence.

We can summarize our discussion so far with the following proposition.

Proposition 2.33. *The matrices X_0 and X_t are in different F-orbits, but are in the same $\overline{F}$-orbit. This $\overline{F}$-orbit is Zariski open in the unipotent radical N of P. The F-orbit of X_0 is contained in the closure of the F-orbit of X_t. The stabilizer of X_0 has split rank 1 whereas the stabilizer of X_t has split rank 0.*

We now determine the F-orbit of X_0 exactly. We have seen that every matrix AX_0A^T in the F-orbit of X_0 is a symmetric invertible matrix whose diagonal entries are perfect squares. Conversely, let X be a symmetric invertible matrix whose diagonal entries are perfect squares u_i^2. Then the matrix $X' = X - \vec{u}\vec{u}^T$ is a symmetric matrix whose diagonal entries are zero. Now write

$$X' = \begin{bmatrix} & x_3 & x_2 \\ x_3 & & x_1 \\ x_2 & x_1 & \end{bmatrix}$$

and write $\vec{x} = \vec{v} \times \vec{w}$ for vectors $\vec{v}$ and $\vec{w}$. The triple product expansion gives

$$X'\vec{y} = \vec{x} \times \vec{y} = (\vec{v} \times \vec{w}) \times \vec{y} = (\vec{v} \cdot \vec{y})\vec{w} + (\vec{w} \cdot \vec{y})\vec{v} = (\vec{w}\vec{v}^T + \vec{v}\vec{w}^T)\vec{y}$$

so $X' = \vec{w}\vec{v}^T + \vec{v}\vec{w}^T$ and we have

$$\begin{bmatrix} | & | & | \\ u & v & w \\ | & | & | \end{bmatrix} \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \begin{bmatrix} | & | & | \\ u & v & w \\ | & | & | \end{bmatrix}^T = \begin{bmatrix} | & | & | \\ u & w & v \\ | & | & | \end{bmatrix} \begin{bmatrix} — & u & — \\ — & v & — \\ — & w & — \end{bmatrix} = \vec{u}\vec{u}^T + \vec{w}\vec{v}^T + \vec{v}\vec{w}^T = X.$$

Thus, the F-orbit of X_0 is exactly the set of invertible matrices of the form

$$\begin{bmatrix} y_1^2 & x_3 & x_2 \\ x_3 & y_2^2 & x_1 \\ x_2 & x_1 & y_3^2 \end{bmatrix}.$$

We will compute the action of $A \in \mathrm{GL}_3$ on the F-orbit of X_0 by decomposing

$$A \cdot \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix} = A \begin{bmatrix} y_1^2 & x_3 & x_2 \\ x_3 & y_2^2 & x_1 \\ x_2 & x_1 & y_3^2 \end{bmatrix} A^T = A \begin{bmatrix} & x_3 & x_2 \\ x_3 & & x_1 \\ x_2 & x_1 & \end{bmatrix} A^T + A \begin{bmatrix} y_1^2 & & \\ & y_2^2 & \\ & & y_3^2 \end{bmatrix} A^T = A \cdot \begin{bmatrix} \vec{x} \\ \vec{0} \end{bmatrix} + A \cdot \begin{bmatrix} \vec{0} \\ \vec{y} \end{bmatrix}.$$

For the action of A on the $\vec{x}$-component, recall that the matrix

$$X = \begin{bmatrix} & x_3 & x_2 \\ x_3 & & x_1 \\ x_2 & x_1 & \end{bmatrix}$$

acts by $X\vec{v} = \vec{x} \times \vec{v}$. Then the matrix AXA^T acts by

$$AXA^T\vec{v} = A(\vec{x} \times A^T\vec{v}) = A(A^TA^{-T}\vec{x} \times A^T\vec{v}) = A(\det A)A^{-1}(A^{-T}\vec{x} \times \vec{v}) = (\det A)A^{-T}\vec{x} \times \vec{v}$$

which shows that

$$A \cdot \begin{bmatrix} \vec{x} \\ \vec{0} \end{bmatrix} = \begin{bmatrix} (\det A)A^{-T}\vec{x} \\ \vec{0} \end{bmatrix}.$$

For the action of A on the $\vec{y}$-component, we can directly compute

$$\begin{aligned} A \cdot \begin{bmatrix} \vec{0} \\ \vec{y} \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} y_1^2 & & \\ & y_2^2 & \\ & & y_3^2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T \\ &= \begin{bmatrix} a_{11}y_1^2 & a_{12}y_2^2 & a_{13}y_3^2 \\ a_{21}y_1^2 & a_{22}y_2^2 & a_{23}y_3^2 \\ a_{31}y_1^2 & a_{32}y_2^2 & a_{33}y_3^2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \\ &= \begin{bmatrix} (a_{11}y_1 + a_{12}y_2 + a_{13}y_3)^2 & \cdots & \cdots \\ \cdots & (a_{21}y_1 + a_{22}y_2 + a_{23}y_3)^2 & \cdots \\ \cdots & \cdots & (a_{31}y_1 + a_{32}y_2 + a_{33}y_3)^2 \end{bmatrix} \\ &= \begin{bmatrix} \cdots \\ A\vec{y} \end{bmatrix}. \end{aligned}$$

Thus, A acts on the F-orbit of X_0 by the matrix

$$\begin{bmatrix} (\det A)A^{-T} & \cdots \\ & A \end{bmatrix}$$

with determinant $(\det A)^3$. This shows that the conjugation action of $P(F)$ on this F-orbit scales the volume $d\vec{x}\,d\vec{y}$ by $|\det A|_F^3$.

We can perform a similar analysis for the action of A on the unipotent radical $N(F)$.

For the action of A on the $\vec{y}$-component, we can directly compute

$$\begin{aligned} A \cdot \begin{bmatrix} \vec{0} \\ \vec{y} \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T \\ &= \begin{bmatrix} a_{11}y_1 & a_{12}y_2 & a_{13}y_3 \\ a_{21}y_1 & a_{22}y_2 & a_{23}y_3 \\ a_{31}y_1 & a_{32}y_2 & a_{33}y_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}^2y_1 + a_{12}^2y_2 + a_{13}^2y_3 & \cdots & \cdots \\ \cdots & a_{21}^2y_1 + a_{22}^2y_2 + a_{23}^2y_3 & \cdots \\ \cdots & \cdots & a_{31}^2y_1 + a_{32}^2y_2 + a_{33}^2y_3 \end{bmatrix} \\ &= \begin{bmatrix} \cdots \\ A^{(2)}\vec{y} \end{bmatrix} \end{aligned}$$

where

$$A^{(2)} = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 \\ a_{21}^2 & a_{22}^2 & a_{23}^2 \\ a_{31}^2 & a_{32}^2 & a_{33}^2 \end{bmatrix}.$$

Thus, A acts on the unipotent radical $N(F)$ linearly by the matrix

$$\begin{bmatrix} (\det A)A^{-T} & \cdots \\ & A^{(2)} \end{bmatrix}$$

with determinant $(\det A)^4$. This shows that the conjugation action of $P(F)$ on $N(F)$ scales the volume $d\vec{x}\,d\vec{y}$ by $|\det A|_F^4$. Note that this is exactly the modular character of $P(F)$. Then Conjecture 2.26 predicts that the F-orbit of X_0 admits a nonzero Radon measure that also transforms under the action of $P(F)$ by the modular character of $P(F)$ and furthermore that this nonzero Radon measure is finite on compact subsets of $N(F)$.

Proposition 2.34. *Conjecture 2.26 holds for the $P(F)$-orbit of X_0.*

Proof. Observe that

$$\det \begin{bmatrix} y_1^2 & x_3 & x_2 \\ x_3 & y_2^2 & x_1 \\ x_2 & x_1 & y_3^2 \end{bmatrix} = (x_1y_1 + x_2y_2 + x_3y_3 + y_1y_2y_3)^2$$

scales by $(\det A)^2$ under the action of $A \in \mathrm{GL}_3$, so the squareroot

$$c(\vec{x}, \vec{y}) = x_1y_1 + x_2y_2 + x_3y_3 + y_1y_2y_3$$

scales by $\det A$ under the action of $A \in \mathrm{GL}_3$. Then the desired measure is $|c(\vec{x}, \vec{y})|\,d\vec{x}\,d\vec{y}$. Since $|c(\vec{x}, \vec{y})|$ is continuous, this measure remains finite on compact subsets of $N(F)$. □

In summary, even in this bad situation where the F-orbit of X_0 is "embedded" inside the F-orbit of X_t, the presence of $|c(\vec{x}, \vec{y})|$ only improves the convergence of the orbital integrals.

2.8 Orthogonal Example

We conclude this chapter with an example demonstrating the limitations of Theorem 2.29. This example was found with the help of ChatGPT 5.6, but is already studied in [29]. Let $F = \mathbb{F}_2((t))$, let $V = F^8$, and consider the quadratic form

$$q(\vec{x}) = x_1x_7 + x_2x_8 + x_3^2 + x_3x_5 + x_5^2 + t(x_4^2 + x_4x_6 + x_6^2).$$

We then define $G = SO(V, q)$, the identity component of the orthogonal group of linear transformations of V preserving the quadratic form q. The matrices

$$\gamma_s = \begin{bmatrix} 1 & & & & s & & s^2 & \\ & 1 & & & & st & & s^2t \\ & & 1 & & & & s & \\ & & & 1 & & & & s \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

can be verified to preserve the quadratic form q by expanding

$$\begin{aligned} q(\gamma_s\vec{x}) &= q(x_1 + sx_5 + s^2x_7, x_2 + stx_6 + s^2tx_8, x_3 + sx_7, x_4 + sx_8, x_5, x_6, x_7, x_8) \\ &= (x_1 + sx_5 + s^2x_7)x_7 + (x_2 + stx_6 + s^2tx_8)x_8 + (x_3 + sx_7)^2 + (x_3 + sx_7)x_5 + x_5^2 \\ &\quad + t((x_4 + sx_8)^2 + (x_4 + sx_8)x_6 + x_6^2) \\ &= x_1x_7 + x_2x_8 + x_3^2 + x_3x_5 + x_5^2 + t(x_4^2 + x_4x_6 + x_6^2) = q(\vec{x}). \end{aligned}$$

Since $\gamma_0 = 1 \in G(F)$, this verifies that $\gamma_s \in G(F)$. Now set $\gamma = \gamma_1 \in G(F)$, and let $T\colon V \to V$ be the linear transformation $T = \gamma - 1$ given by $T(v) = \gamma v - v$. Then $\ker T = \langle e_1, e_2, e_3, e_4\rangle$. Let $V_{12} = \langle e_1, e_2\rangle$.

Lemma 2.35. *If $v \in \ker T$ and $q(v) = 0$, then $v \in V_{12}$.*

Proof. Let $v = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$. Then $q(v) = x_3^2 + tx_4^2$. If $q(v) = 0$, then $x_3^2 = tx_4^2$. But then comparing valuations forces $x_3 = x_4 = 0$, so $v \in V_{12}$. □

Recall that a subspace $W \subseteq V$ is said to be isotropic if $q|_W = 0$.

Lemma 2.36. *If an isotropic subspace $W \subseteq V$ is γ-stable, then $W \subseteq V_{12}$.*

Proof. Let $v \in W$. Since W is γ-stable, we have $T(v) \in W$. Since W is isotropic, we have $q(T(v)) = 0$. Since $T^2 = 0$, we also have $T(v) \in \ker T$. Then $T(v) \in V_{12}$ by Lemma 2.35. Writing $v = \sum x_ie_i$, this implies that $x_7 = x_8 = 0$. Then $x_3^2 + x_3x_5 + x_5^2 = t(x_4^2 + x_4x_6 + x_6^2)$. But then comparing valuations forces $x_3 = x_4 = x_5 = x_6 = 0$, so $v \in V_{12}$. □

Lemma 2.37. *Let P be a parabolic subgroup of G with $\gamma \in P(F)$. If $g \in G(F)$ fixes V_{12} pointwise, then $g \in P(F)$.*

Proof. The parabolic subgroup P is the stabilizer of some isotropic flag. Since $\gamma \in P(F)$, these isotropic subspaces must be γ-stable. By Lemma 2.36, these isotropic subspaces are contained in V_{12}. Since g fixes V_{12} pointwise, g stabilizes the isotropic flag, so $g \in P(F)$. □

Proposition 2.38. *Let $\gamma = \gamma_1 \in G(F)$. There does not exist a parabolic subgroup P of G with $\gamma \in P(F)$ and such that the P-orbit of γ is Zariski open in an algebraic subgroup.*

Proof. Since γ is unipotent with $\gamma^2 = 1$, Zariski density implies that the algebraic subgroup would be unipotent of exponent 2. In particular, this unipotent algebraic subgroup is necessarily abelian, so we will have $p\gamma p^{-1} \in G_\gamma(F)$ for all $p \in P(F)$.

Now Lemma 2.37 tells us that every element of $G(F)$ that fixes V_{12} pointwise lies in $P(F)$. Let $a = 1/(1+t)$ and $b = t/(1+t)$. The matrix

$$p = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & & & 1 & & \\ & & a & & 1 & b & & \\ & & a & 1 & & b & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

can be verified to preserve the quadratic form q by expanding

$$\begin{aligned} q(p\vec{x}) &= q(x_1, x_2, x_3, x_6, ax_3 + x_5 + tax_6, ax_3 + x_4 + tax_6, x_7, x_8) \\ &= q(\vec{x}) + x_3(ax_3 + bx_6) + (ax_3 + bx_6)^2 + t(x_6(ax_3 + bx_6) + (ax_3 + bx_6)^2) \\ &= q(\vec{x}) + a(1 + a + ta)x_3^2 + (ta + b)x_3x_6 + b(t + b + tb)x_6^2 = q(\vec{x}). \end{aligned}$$

And then $p \in G(F)$ since the Dickson invariant vanishes ($p - 1$ has rank 2). Now p fixes V_{12} pointwise, so $p \in P(F)$. But this is a contradiction since the matrix

$$p\gamma p^{-1} = \begin{bmatrix} 1 & & a & b & 1 & & 1 & \\ & 1 & & t & & & & t \\ & & 1 & & & & 1 & \\ & & & 1 & & & & \\ & & & & 1 & & a & \\ & & & & & 1 & a & 1 \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

does not commute with γ. □

Thus, hypothesis (H3) of Theorem 2.29 is too much to hope for in bad characteristic. Nevertheless, the hypothesis holds frequently enough in practice to still be of value.

Chapter 3

Optimal Cocharacters

Recall that Theorem 2.24 ensures the existence of a cocharacter λ pushing γ all the way to the unique cocharacter-closed orbit inside the cocharacter-closure of its orbit. But without additional care as to the choice of λ, the resulting parabolic subgroup $P(\lambda)$ will typically fail to satisfy even the mild hypothesis (H1) of Theorem 2.29.

The theory of optimal cocharacters, first introduced by Kempf [19] and Rousseau [28] and further developed by Hesselink [17] and Bate-Martin-Röhrle-Tange [6] will provide a better source of cocharacters λ. In this chapter we will present the theory of optimal cocharacters in the special case of unipotent elements limiting to the identity.

3.1 Optimal Cocharacters

Let $G \times V \to V$ be an action of a (connected) reductive group G on an affine algebraic variety V, with G, V, and the action all defined over a fixed non-archimedean local field F. To simplify the theory, we will assume that there exists a point $* \in V(F)$ fixed by the action of $G(F)$. Let $Y(G)$ denote the set of "virtual cocharacters" of the form $\frac{1}{n}\lambda$ for n a positive integer and $\lambda\colon \mathbb{G}_m \to G$ a cocharacter of G defined over F. We can make this precise by defining $Y(G)$ to be the quotient of the set of pairs (λ, n) by the relation $(\lambda, n) \sim (\mu, m)$ for $m\lambda = n\mu$. Note that if T is a torus, then $Y(T) = X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a vector space over $\mathbb{Q}$.

The theory of optimal cocharacters will produce a cocharacter or virtual cocharacter λ that moves a given point $v \in V(F)$ as "efficiently" as possible. This efficiency will be computed in terms of the norm (also called length) of λ and the speed (also called instability measure) of v under the action of λ. First we must construct a norm function on $Y(G)$.

Proposition 3.1. *There exists a norm function $\|\cdot\| : Y(G) \to \mathbb{Q}$ satisfying the following properties:*

1. *$\|g \cdot \lambda\| = \|\lambda\|$ for all $g \in G(F)$ and $\lambda \in Y(G)$.*
2. *For every F-split torus T of G, the restriction of the norm to $Y(T)$ is a positive definite quadratic form on $Y(T)$.*

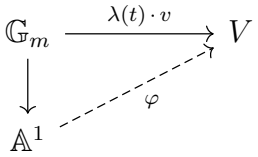
Proof. Fix a maximal split torus T of G and a positive definite quadratic form $q\colon Y(T) \to \mathbb{Q}$. By summing over the Weyl group of T, we may assume that q is invariant under the action of the Weyl group. By conjugacy of maximal split tori, every virtual cocharacter of G can be written as $g \cdot \lambda$ for $g \in G(F)$ and $\lambda \in Y(T)$. Now set $\|g \cdot \lambda\| = q(\lambda)$ for $g \in G(F)$ and $\lambda \in Y(T)$. To see that this norm is well defined, it is enough to show that if $g \cdot \lambda = \lambda'$ for $g \in G(F)$ and cocharacters $\lambda, \lambda' \in Y(T)$, then $q(\lambda) = q(\lambda')$. The Levi subgroup $M(\lambda)$ is reductive and contains both T and $g^{-1} \cdot T$. By conjugacy of maximal split tori, we can write $g^{-1} \cdot T = z \cdot T$ for some $z \in M(\lambda)(F)$. Then gz normalizes T, so we have

$$q(\lambda') = q(g \cdot \lambda) = q(g \cdot z \cdot \lambda) = q(gz \cdot \lambda) = q(\lambda)$$

by Weyl-invariance. □

We will fix a norm function $\|\cdot\| : Y(G) \to \mathbb{Q}$ satisfying the properties listed in Proposition 3.1. Next we must define the speed $s(v, \lambda)$ of v under the action of λ.

Definition 3.2. Let $v \in V(F)$ and let $\lambda\colon \mathbb{G}_m \to G$ be a cocharacter. We say that the limit $\lim_{t\to 0} \lambda(t) \cdot v$ exists if the morphism $\mathbb{G}_m \to V$ defined by $t \mapsto \lambda(t) \cdot v$ factors through $\mathbb{A}^1$.

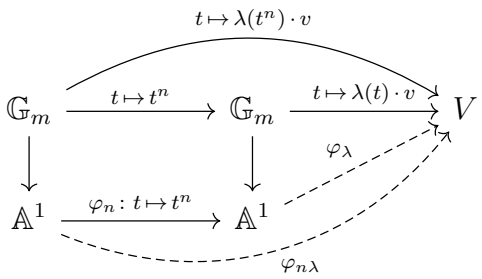


We then define the speed $s(v, \lambda)$ in terms of the limit $\overline{v} = \lim_{t\to 0} \lambda(t) \cdot v$ as follows:

- If the limit $\overline{v}$ fails to exist, then we set $s(v, \lambda) = -\infty$.
- If the limit $\overline{v}$ exists but is unequal to $*$, then we set $s(v, \lambda) = 0$.
- If the limit $\overline{v}$ exists and is equal to $*$, then we set $s(v, \lambda) = n$ where $n \in \mathbb{N} \cup \{\infty\}$ is such that the local ring homomorphism of stalks $\varphi^\sharp\colon \mathcal{O}_{V,*} \to F[t]_{(t)}$ maps the maximal ideal $\mathfrak{m}_* \subseteq \mathcal{O}_{V,*}$ to the power $(t)^n \subseteq F[t]_{(t)}$.

Lemma 3.3. *If $\lambda, \mu\colon \mathbb{G}_m \to G$ are cocharacters with $m\lambda = n\mu$, then $\frac{1}{n}s(v, \lambda) = \frac{1}{m}s(v, \mu)$.*

Proof. It is enough to show that $s(v, n\lambda) = ns(v, \lambda)$. Now consider the commutative diagram



In particular, observe that if the limit $\lim_{t\to 0} \lambda(t)\cdot v$ exists, then the limit $\lim_{t\to 0}\lambda(t^n)\cdot v$ also exists with $\lim_{t\to 0}\lambda(t^n)\cdot v = \lim_{t\to 0}\lambda(t)\cdot v$, and we further have

$$(t)^{s(v,n\lambda)} = \varphi^\sharp_{n\lambda}(\mathfrak{m}_*) = \varphi^\sharp_n(\varphi^\sharp_\lambda(\mathfrak{m}_*)) = \varphi^\sharp_n((t)^{s(v,\lambda)}) = (t)^{ns(v,\lambda)}.$$

And conversely, if the limit $\lim_{t\to 0}\lambda(t^n)\cdot v$ exists, then the ring homomorphism $\mathcal{O}_V \to F[t,\frac{1}{t}]$ has integral image, so the limit $\lim_{t\to 0}\lambda(t)\cdot v$ also exists. □

Definition 3.4. We extend the notion of speed to virtual cocharacters by setting $s(v,\frac{1}{n}\lambda) = \frac{1}{n}s(v,\lambda)$ for $v\in V(F)$ and $\lambda\colon \mathbb{G}_m\to G$ a cocharacter. This is well defined by Lemma 3.3.

We can now define optimal cocharacters as those with small norm relative to $s(v,\lambda)$. We would like optimal cocharacters to form a conjugacy class, so we will need to pick out a specific scaling of λ as canonical. There are multiple ways to achieve this, but we will impose $s(v,\lambda)\geq 1$ following Hesselink [17].

Definition 3.5. For $v\in V(F)$, we define the set of optimal cocharacters $\Lambda(v)\subseteq Y(G)$ as the set of λ with minimal norm $\|\lambda\|$ among those $\lambda\in Y(G)$ satisfying the constraint $s(v,\lambda)\geq 1$.

Before stating the main theorem, we first give a few basic properties of the set $\Lambda(v)$.

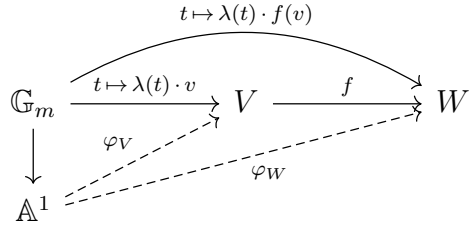
Lemma 3.6. *The set $\Lambda(v)$ contains 0 if and only if $v=*$.*

Proof. If $0\in\Lambda(v)$, then $s(v,0)\geq 1$. In general, if $s(v,\lambda)>0$, then the limit $\lim_{t\to 0}\lambda(t)\cdot v$ exists and is equal to $*$. Here $\lambda=0$, so the limit $\lim_{t\to 0}\lambda(t)\cdot v$ is just v, so v is equal to $*$. Conversely, if $v=*$, then $s(v,0)=\infty$. In particular, $\lambda=0\in Y(G)$ satisfies the constraint $s(v,\lambda)\geq 1$ and trivially has minimal norm $\|\lambda\|=0$, so $0\in\Lambda(v)$. □

Lemma 3.7. *Let $f\colon V\to W$ be a G-equivariant closed-immersion to another affine algebraic variety W. Then the following properties hold:*

1. *A limit $\lim_{t\to 0}\lambda(t)\cdot v$ exists if and only if the limit $\lim_{t\to 0}\lambda(t)\cdot f(v)$ exists, in which case we have $\lim_{t\to 0}\lambda(t)\cdot f(v) = f(\lim_{t\to 0}\lambda(t)\cdot v)$.*
2. *The speed satisfies $s(f(v),\lambda) = s(v,\lambda)$.*
3. *The optimal cocharacters satisfy $\Lambda(f(v)) = \Lambda(v)$.*

Proof. Consider the commutative diagram



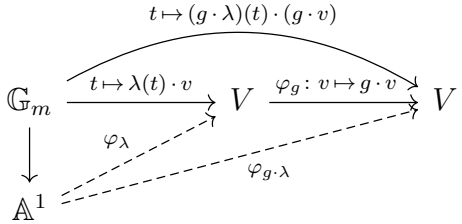
In particular, if the limit $\lim_{t\to 0}\lambda(t)\cdot v$ exists, then the limit $\lim_{t\to 0}\lambda(t)\cdot f(v)$ also exists with $\lim_{t\to 0}\lambda(t)\cdot f(v) = f(\lim_{t\to 0}\lambda(t)\cdot v)$. Conversely, if the limit $\lim_{t\to 0}\lambda(t)\cdot f(v)$ exists, then the ring homomorphism $\mathcal{O}_W \to F[t,\frac{1}{t}]$ has integral image. By surjectivity of $\mathcal{O}_W \to \mathcal{O}_V$, the ring homomorphism $\mathcal{O}_V \to F[t,\frac{1}{t}]$ must also have integral image, so the limit $\lim_{t\to 0}\lambda(t)\cdot v$ exists. This proves (1). For (2), if the limits exist and are equal to $*$, then we have

$$(t)^{s(f(v),\lambda)} = \varphi_W^\sharp(\mathfrak{m}_*) = \varphi_V^\sharp(f^\sharp(\mathfrak{m}_*)) = \varphi_V^\sharp(\mathfrak{m}_*) = (t)^{s(v,\lambda)}.$$

And then (3) follows directly from the definition of $\Lambda(v)$. □

Lemma 3.8. *For $g \in G(F)$, we have $s(g\cdot v, g\cdot\lambda) = s(v,\lambda)$ and $\Lambda(g\cdot v) = g\cdot\Lambda(v)$.*

Proof. The proof is similar to that of Lemma 3.7, except with the commutative diagram

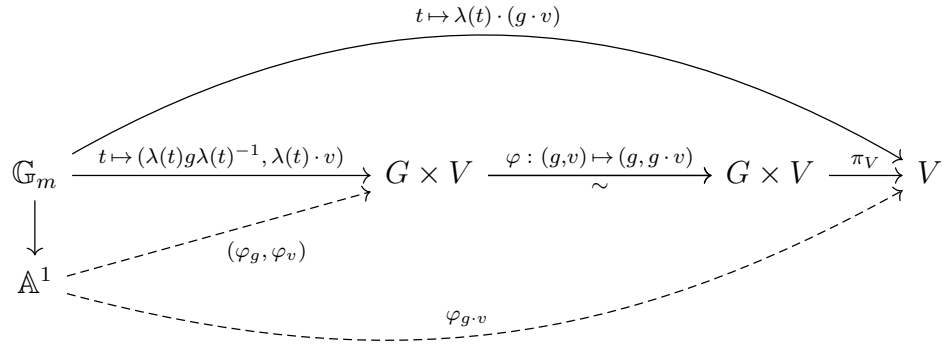


where we have

$$(t)^{s(g\cdot v,g\cdot\lambda)} = \varphi_{g\cdot\lambda}^\sharp(\mathfrak{m}_*) = \varphi_\lambda^\sharp(\varphi_g^\sharp(\mathfrak{m}_*)) = \varphi_\lambda^\sharp(\mathfrak{m}_*) = (t)^{s(v,\lambda)}.$$ □

Lemma 3.9. *For $g \in P(\lambda)(F)$, we have $s(g\cdot v,\lambda) = s(v,\lambda)$. If $\lambda\in\Lambda(v)$, then $g\cdot\lambda\in\Lambda(v)$.*

Proof. The proof is similar to that of Lemma 3.7, except with the commutative diagram



where the central map φ is an automorphism, so we have

$$(t)^{s(g\cdot v,\lambda)} = \varphi_{g\cdot v}^\sharp(\mathfrak{m}_*) = (\varphi_g,\varphi_v)^\sharp(\varphi^\sharp(\pi_V^\sharp(\mathfrak{m}_*))) = (\varphi_g,\varphi_v)^\sharp(\pi_V^\sharp(\mathfrak{m}_*)) = \varphi_v^\sharp(\mathfrak{m}_*) = (t)^{s(v,\lambda)}.$$ □

We can now state the main theorem in the theory of optimal cocharacters, corresponding to Theorem 2.2 of [19], Theorem 5.2 of [17], and Theorem 4.5 of [6].

Theorem 3.10. *Let $v \in V(F)$ with $*$ in the cocharacter-closure of v.*

1. *The set $\Lambda(v)$ is nonempty.*
2. *There is a unique parabolic subgroup $P(v)$ such that $P(\lambda) = P(v)$ for all $\lambda \in \Lambda(v)$.*
3. *For every maximal split torus T of $P(v)$, $Y(T)$ contains a unique element of $\Lambda(v)$.*

Proof. By Lemma 2.16 and Lemma 3.7, we may assume that V is a finite-dimensional representation of G over F. By Theorem 2.24, there exist virtual cocharacters $\lambda \in Y(G)$ satisfying $s(v, \lambda) \geq 1$. To prove (1), we must show that some $\lambda \in Y(G)$ attains the minimal norm among all $\lambda \in Y(G)$ satisfying $s(v, \lambda) \geq 1$. Fix a maximal split torus T of G. By conjugacy of maximal split tori, it suffices to prove that some $\lambda \in Y(T)$ attains the minimal norm among all $\lambda \in Y(T)$ satisfying $s(v, g \cdot \lambda) \geq 1$ for some $g \in G(F)$. By Lemma 3.8, it is equivalent to prove that some $\lambda \in Y(T)$ attains the minimal norm among all $\lambda \in Y(T)$ satisfying $s(g \cdot v, \lambda) \geq 1$ for some $g \in G(F)$.

Now consider a fixed $w \in G(F) \cdot v$. We can decompose $V = \bigoplus_\alpha V_\alpha$ where T acts on V_α via $\alpha \colon T \to \mathbb{G}_m$. Then write $w = \sum_\alpha w_\alpha$ and $* = \sum_\alpha *_\alpha$. Let S_w denote the set of all virtual cocharacters $\lambda \in Y(T)$ satisfying $\lim_{t \to 0} \lambda(t) \cdot w = *$ and $s(w, \lambda) \geq 1$. Then the condition $\lambda \in S_w$ boils down to the following constraints:

1. If $w_\alpha = 0$, then $*_\alpha = 0$.
2. If $w_\alpha \neq 0$ and $*_\alpha = 0$, then $\langle \alpha, \lambda \rangle \geq 1$.
3. If $w_\alpha \neq 0$ and $*_\alpha \neq 0$, then $w_\alpha = *_\alpha$ and $\langle \alpha, \lambda \rangle = 0$.

In particular, each set S_w is closed and convex, so each nonempty S_w contains a unique cocharacter of minimal norm. Furthermore, notice that the set S_w is determined by which equalities hold between w_α, $*_\alpha$, and 0. Then there are only finitely many possibilities for S_w as w varies (this is the essence of Kempf's "state formalism" [19]). Taking the minimal norm as w varies gives a virtual cocharacter $\lambda \in Y(T)$ with minimal norm subject to $s(g \cdot v, \lambda) \geq 1$ for some $g \in G(F)$. This proves (1).

To prove that there is at most one optimal cocharacter within a fixed maximal split torus T of G, we must prove that there is at most one $\lambda \in Y(T)$ attaining the minimal norm among all $\lambda \in Y(T)$ satisfying $s(v, \lambda) \geq 1$. But this follows from S_v being closed and convex.

To prove (2), let $\lambda, \mu \in \Lambda(v)$. The parabolic subgroups $P(\lambda)$ and $P(\mu)$ share a maximal split torus T. We can conjugate $g_\lambda \cdot \lambda \in Y(T)$ and $g_\mu \cdot \mu \in Y(T)$ for $g_\lambda \in P(\lambda)(F)$ and $g_\mu \in P(\mu)(F)$. By Lemma 3.9, $g_\lambda \cdot \lambda, g_\mu \cdot \mu \in \Lambda(v)$. Then $g_\lambda \cdot \lambda = g_\mu \cdot \mu$ and

$$P(\lambda) = P(g_\lambda \cdot \lambda) = P(g_\mu \cdot \mu) = P(\mu).$$

Finally, to prove (3), all maximal split tori of $P(v)$ are conjugate under $P(v)(F)$, so all maximal split tori of $P(v)$ contain an element of $\Lambda(v)$ by Lemma 3.9. □

Definition 3.11. The optimal parabolic subgroup associated to $v \in V(F)$ is the subgroup $P(v)$ from Theorem 3.10.

Lemma 3.12. *The optimal parabolic subgroup satisfies* $P(g \cdot v) = g \cdot P(v)$.

Proof. If $\lambda \in \Lambda(v)$, then $g \cdot \lambda \in \Lambda(g \cdot v)$ by Lemma 3.8. Then

$$P(g \cdot v) = P(g \cdot \lambda) = g \cdot P(\lambda) = g \cdot P(v). \qquad \square$$

Proposition 3.13. *The optimal parabolic subgroup satisfies* $G_v(F) \leq P(v)(F)$.

Proof. Let $g \in G_v(F)$. Then $g \cdot P(v) = P(g \cdot v) = P(v)$ by Lemma 3.12. But parabolic subgroups are self-normalizing, so $g \in P(v)(F)$. $\square$

Proposition 3.13 shows that these optimal parabolic subgroups satisfy hypothesis (H1) of Theorem 2.29. We now turn our attention to hypothesis (H3).

3.2 The Dimension Conjecture

Let G be a (connected) reductive group over a non-archimedean local field F. Let $\gamma \in G(F)$ be a unipotent element containing 1 in its cocharacter-closure. Then Theorem 3.10 gives an optimal parabolic subgroup P and an optimal cocharacter λ in each maximal split torus of P. Now observe that we can characterize the F-points of P and its unipotent radical N in terms of speed as

$$P(F) = \{g \in G(F) : s(g, \lambda) \geq 0\},$$
$$N(F) = \{g \in G(F) : s(g, \lambda) > 0\}.$$

Further constraints on the speed give rise to smaller algebraic subgroups.

Proposition 3.14 ([17, Lemma 2.5])**.** *For each* $r > 0$*, there is a connected unipotent normal subgroup* N_r *of* P *whose* F*-points are characterized in terms of speed as*

$$N_r(F) = \{g \in G(F) : s(g, \lambda) \geq r\}$$

for any optimal cocharacter $\lambda \in \Lambda(\gamma)$.

Since N_1 is a normal subgroup of P, the map $\iota\colon P/P_\gamma \hookrightarrow N$ defined by $\iota(p) = p\gamma p^{-1}$ will have image lying in N_1, so N_1 is a natural candidate for the subgroup U in Theorem 2.29. By Remark 2.27, it will be enough to check equality of dimension.

Conjecture 3.15. *Let* G *be a (connected) reductive group over a non-archimedean local field* F*, and let* $\gamma \in G(F)$ *be unipotent containing 1 in its cocharacter-closure. Let* P *be the optimal parabolic subgroup for* γ*. Then* $\dim N_1 = \dim P - \dim P_\gamma$.

Remark 3.16. Conjecture 3.15 is not true for all reductive groups in general. Proposition 2.38 is an example of an orthogonal group in bad characteristic where Conjecture 3.15 fails.

Now following Proposition 47 of [25], we will need to modify the Haar measure on $N_1(F)$ by a suitable function. We will need a couple additional technical assumptions on λ and P.

Lemma 3.17. *Let G be a reductive group over a non-archimedean local field F. Let $\gamma \in G(F)$ be a unipotent element containing 1 in its cocharacter-closure. Assume that Conjecture 3.15 holds for γ. Let λ be an optimal cocharacter for γ, and let P, N, and N_1 be the associated subgroups. Assume that 2λ is integral and that $G_\gamma \leq P$. Assume that G_γ is smooth. Then there exists a nonzero continuous function $\phi\colon N_1(F) \to \mathbb{R}$ such that $\phi(pup^{-1}) = \left|\det \mathrm{Ad}_p|_{\mathrm{Lie}\, N/\mathrm{Lie}\, N_1}\right| \phi(u)$ for all $p \in P(F)$ and $u \in N_1(F)$.*

Proof. Since 2λ is integral, we can decompose

$$\mathrm{Lie}\, G = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_i.$$

The map $\mathrm{Ad}_\gamma - \mathrm{id}\colon \bigoplus_{i \geq -1/2} \mathfrak{g}_i \to \bigoplus_{i \geq 1/2} \mathfrak{g}_i$ is surjective since the kernel $\mathrm{Lie}\, G_\gamma$ has

$$\dim \mathrm{Lie}\, G_\gamma = \dim G_\gamma = \dim(P/N_1) = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{1/2} = \dim \mathfrak{g}_{-1/2} + \dim \mathfrak{g}_0$$

where $\dim \mathfrak{g}_{1/2} = \dim \mathfrak{g}_{-1/2}$ since opposite weight spaces have the same dimension. Then the projection $T_\gamma\colon \mathfrak{g}_{-1/2} \to \mathfrak{g}_{1/2}$ must be surjective, hence invertible since $\dim \mathfrak{g}_{-1/2} = \dim \mathfrak{g}_{1/2}$. More generally, let $T_u\colon \mathfrak{g}_{-1/2} \to \mathfrak{g}_{1/2}$ be the projection of $\mathrm{Ad}_u - \mathrm{id}\colon \bigoplus_{i \geq -1/2} \mathfrak{g}_i \to \bigoplus_{i \geq 1/2} \mathfrak{g}_i$ and let $\phi(u) = |\det(T_\gamma^{-1} T_u)|^{1/2}$. Note that $\phi(\gamma) = 1$, so ϕ is nonzero. Now let $p \in P(F)$ with Levi quotient $m \in M(F)$. Then we can compute

$$\begin{aligned}
\phi(pup^{-1}) &= \left|\det T_\gamma^{-1} T_{pup^{-1}}\right|^{1/2} \\
&= \left|\det T_\gamma^{-1} \mathrm{Ad}_m T_u \mathrm{Ad}_{m^{-1}}\right|^{1/2} \\
&= \left|\det T_\gamma^{-1} \mathrm{Ad}_m T_\gamma T_\gamma^{-1} T_u \mathrm{Ad}_m^{-1}\right|^{1/2} \\
&= \left|\det T_\gamma^{-1} \mathrm{Ad}_m T_\gamma\right|^{1/2} \left|\det T_\gamma^{-1} T_u\right|^{1/2} \left|\det \mathrm{Ad}_m^{-1}\colon \mathfrak{g}_{-1/2} \to \mathfrak{g}_{-1/2}\right|^{1/2} \\
&= \left|\det \mathrm{Ad}_m\colon \mathfrak{g}_{1/2} \to \mathfrak{g}_{1/2}\right|^{1/2} \left|\det \mathrm{Ad}_m\colon \mathfrak{g}_{-1/2} \to \mathfrak{g}_{-1/2}\right|^{-1/2} \phi(u) \\
&= \left|\det \mathrm{Ad}_m\colon \mathfrak{g}_{1/2} \to \mathfrak{g}_{1/2}\right| \phi(u) \\
&= \left|\det \mathrm{Ad}_p|_{\mathrm{Lie}\, N/\mathrm{Lie}\, N_1}\right| \phi(u).
\end{aligned}$$

□

Proposition 3.18. *Let G be a reductive group over a non-archimedean local field F. Let $\gamma \in G(F)$ be a unipotent element containing 1 in its cocharacter-closure. Assume that Conjecture 3.15 holds for γ. let λ be an optimal cocharacter for γ, and let P and N_1 be the associated subgroups. Assume that 2λ is integral and that $G_\gamma \leq P$. Assume that G_γ is smooth. Then there exists a stricly positive Radon measure du on $N_1(F)$ satisfying*

$$\int_{N_1(F)} f(p_0 u p_0^{-1})\, du = \delta_P^{-1}(p_0) \int_{N_1(F)} f(u)\, du$$

for all $f \in C_c(N_1(F))$ and all $p_0 \in P(F)$.

Proof. We can take $du = \phi(n)\, dn_1$ for the function $\phi\colon N_1(F) \to \mathbb{R}$ from Lemma 3.17. □

Now we can specialize Theorem 2.29 to unipotent elements.

Theorem 3.19. *Let G be a reductive group over a non-archimedean local field F. Let $\gamma \in G(F)$ be a unipotent element containing 1 in its cocharacter-closure. Assume that Conjecture 3.15 holds for γ. Let λ be an optimal cocharacter for γ, and let P be the associated optimal parabolic subgroup for γ. Assume that 2λ is integral and that $G_\gamma \leq P$. Assume that G_γ is smooth and that the orbit space $(P/P_\gamma)(F)/P(F)$ is finite. Then $G_\gamma(F)$ is unimodular and the orbital integrals*

$$\int_{G(F)/G_\gamma(F)} f(g\gamma g^{-1})\, \frac{dg}{dg_\gamma}$$

converge for all $f \in C_c(G(F))$.

Proof. Conjecture 2.21 for $\overline{\gamma} \in M(F)$ follows from $\overline{\gamma} = 1$ and unimodularity of reductive groups. Hypothesis (H1) of Theorem 2.29 is exactly Proposition 3.13. Hypothesis (H2) of Theorem 2.29 follows from $\overline{\gamma} = 1$. Hypothesis (H3) of Theorem 2.29 follows from Conjecture 3.15, Remark 2.27, and the strictly positive Radon measure du on $N_1(F)$ from Proposition 3.18. It remains to verify Conjecture 2.26 for the embedding $\iota\colon P/P_\gamma \hookrightarrow N$. Since G_γ is smooth, the orbit space $(P/P_\gamma)(F)/P(F)$ is parameterized by the profinite kernel of Galois cohomology $\ker(H^1(F, P_\gamma) \to H^1(F, P))$. Then the orbit space is Hausdorff, hence discrete by finiteness. In particular, the rational orbit $P(F)/P_\gamma(F)$ is open in the stable orbit $(P/P_\gamma)(F)$, so Conjecture 2.26 becomes trivial. □

Remark 3.20. Our statement of Theorem 3.19 unfortunately has many technical assumptions. In particular, we must assume finiteness of $(P/P_\gamma)(F)/P(F)$ due to our inability to prove finiteness of the descended measure in Conjecture 2.26 in the case of infinitely many rational orbits within a stable orbit. However, if one cares only about stable orbital integrals, then smoothness of G_γ and finiteness of $(P/P_\gamma)(F)/P(F)$ can be dropped.

Chapter 4

Unipotent Orbital Integrals

We conclude by applying Theorem 3.19 to orbital integrals in GL_n, SL_n, and Sp_{2n}. This requires verifying Conjecture 3.15 in these cases.

4.1 Orbital Integrals in GL_n

Let F be a non-archimedean local field, and let γ be a unipotent element of $\mathrm{GL}_n(F)$. Then γ is conjugate in $\mathrm{GL}_n(F)$ to its Jordan normal form $\sum_i J_{n_i}$ with $\sum_i n_i = n$. Here J_{n_i} denotes the $n_i \times n_i$ Jordan block

$$J_{n_i} = \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix}.$$

More generally, let $J^*_{n_i}$ denote an $n_i \times n_i$ unipotent block

$$J^*_{n_i} = \begin{bmatrix} 1 & * & \cdots & * \\ & 1 & \ddots & \vdots \\ & & \ddots & * \\ & & & 1 \end{bmatrix}$$

such that the superdiagonal entries $(J^*_{n_i})_{k,k+1}$ are all nonzero. To each block $J^*_{n_i}$, we can associate the virtual cocharacter

$$\lambda_{n_i}(t) = \begin{bmatrix} t^{(n_i-1)/2} & & & \\ & t^{(n_i-3)/2} & & \\ & & \ddots & \\ & & & t^{-(n_i-1)/2} \end{bmatrix}.$$

For example, the first few of these virtual cocharacters are

$$\lambda_1(t) = \begin{bmatrix} 1 \end{bmatrix}, \qquad \lambda_2(t) = \begin{bmatrix} t^{1/2} & \\ & t^{-1/2} \end{bmatrix}, \qquad \lambda_3(t) = \begin{bmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{bmatrix}.$$

Let T_0 be the standard split torus of GL_n. The Euclidean norm on $Y(T_0)$ is Weyl-invariant and thus provides a norm function on $Y(\mathrm{GL}_n)$ following Proposition 3.1. We can then determine the optimal cocharacters with respect to this norm.

Proposition 4.1. *The virtual cocharacter $\sum_i \lambda_{n_i}$ is an optimal cocharacter for $\gamma = \sum_i J^*_{n_i}$ in $\mathrm{GL}_n(F)$.*

Proof. Let P_0 be the standard Borel of GL_n with Levi decomposition $P_0 = T_0 N_0$. The optimal parabolic P for γ shares a maximal split torus T with P_0. Then there exists an optimal cocharacter $\lambda \in Y(T)$ for γ. By conjugating by $g \in N_0(F)$, there exists an optimal cocharacter $\lambda_0 = g \cdot \lambda \in Y(T_0)$ for $g \cdot \gamma$. But observe that $g \cdot \gamma$ retains all of the nonzero superdiagonal entries of γ, so the restriction

$$\begin{bmatrix} t^{a_1} & & & & \\ & t^{a_2} & & & \\ & & \ddots & & \\ & & & t^{a_{n_i}} & \end{bmatrix}$$

of λ_0 to the block

$$\begin{bmatrix} 1 & * & \cdots & * \\ & 1 & \ddots & \vdots \\ & & \ddots & * \\ & & & 1 \end{bmatrix}$$

of $g \cdot \gamma$ must satisfy $a_k - a_{k+1} \geq 1$. Let $b_k = (n_i + 1 - 2k)/2$ be the exponents appearing in

λ_{n_i}, and let $c_k = a_k - b_k$. Then we can bound

$$\begin{aligned}
\sum_{k=1}^{n_i} a_k^2 &= \sum_{k=1}^{n_i} (b_k + c_k)^2 \\
&= \sum_{k=1}^{n_i} b_k^2 + 2\sum_{k=1}^{n_i} b_k c_k + \sum_{k=1}^{n_i} c_k^2 \\
&= \sum_{k=1}^{n_i} b_k^2 + \sum_{k=1}^{n_i} (n_i + 1 - 2k)(a_k - b_k) + \sum_{k=1}^{n_i} c_k^2 \\
&= \sum_{k=1}^{n_i} b_k^2 + \sum_{k=1}^{n_i - 1} k(n_i - k)((a_k - b_k) - (a_{k+1} - b_{k+1})) + \sum_{k=1}^{n_i} c_k^2 \\
&= \sum_{k=1}^{n_i} b_k^2 + \sum_{k=1}^{n_i - 1} k(n_i - k)(a_k - a_{k+1} - 1) + \sum_{k=1}^{n_i} c_k^2 \\
&\geq \sum_{k=1}^{n_i} b_k^2.
\end{aligned}$$

Summing over i gives $\|\lambda_0\| \geq \|\sum_i \lambda_{n_i}\|$. In particular, $\sum_i \lambda_{n_i}$ must also be an optimal cocharacter for γ. □

Theorem 4.2. *Conjecture 3.15 holds for unipotent elements of* $\mathrm{GL}_n(F)$.

Proof. Every unipotent element of $\mathrm{GL}_n(F)$ is conjugate over F to its Jordan normal form $\sum_i J_{n_i}$ so it is enough to consider $\gamma = \sum_i J_{n_i}$. By Proposition 4.1, the virtual cocharacter $\sum_i \lambda_{n_i}$ is an optimal cocharacter for γ in $\mathrm{GL}_n(F)$. Let d_k denote the number of times that $k/2$ appears as an exponent in the virtual cocharacter $\sum_i \lambda_{n_i}$. Then

$$\dim P = \frac{1}{2}\left(n^2 + \sum_k d_k^2\right)$$

and

$$\dim N_1 = \frac{1}{2}\left(n^2 - \sum_k d_k^2\right) - \sum_k d_k d_{k+1}.$$

But

$$\dim G_\gamma = \sum_{i,j} \dim_F \mathrm{Hom}_{F[X]}\left(\frac{F[X]}{X^{n_i}}, \frac{F[X]}{X^{n_j}}\right) = \sum_{i,j} \min(n_i, n_j).$$

So it suffices to prove that

$$\sum_{i,j} \min(n_i, n_j) = \sum_k d_k^2 + \sum_k d_k d_{k+1}. \tag{4.1}$$

We will give a combinatorial proof of (4.1). First, form a "layercake" with rows n_i and columns d_k. For example, if $(n_i) = [1, 1, 2, 4, 4, 5]$ then this would be:

				X				
				X				
			X		X			
	X		X		X		X	
	X		X		X		X	
X		X		X		X		X

Then the left hand side of (4.1) counts the number of ways to select an ordered pair of rows (possibly the same row twice) along with an element in the highest of the two rows. The sum $\sum_k d_k^2$ counts the number of ways to select an ordered pair of elements in the same column. The sum $\sum_k d_k d_{k+1}$ counts the number of ways to select an ordered pair of elements in adjacent columns with the first element in the left column and the second element in the right column. Then we can establish the following bijections:

$$\left\{\begin{array}{c}\text{ordered pair of rows of}\\ \text{the same parity along}\\ \text{with an element in the}\\ \text{highest of the two rows}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{an ordered pair}\\ \text{of elements in}\\ \text{the same column}\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{ordered pair of rows of}\\ \text{opposite parity along}\\ \text{with an element in the}\\ \text{highest of the two rows}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{an ordered pair of elements}\\ \text{in adjacent columns with}\\ \text{the first element in the left}\\ \text{column and the second}\\ \text{element in the right column}\end{array}\right\}$$

For the forward direction of the first bijection, we can select an element in the lowest of the two rows by taking the element directly below the element in the highest of the two rows. For the backward direction of the first bijection, we can simply read off the two rows and the element in the highest of the two rows. For the backward direction of the second bijection, we can simply read off the two rows and the element in the highest of the two rows. For the forward direction of the second bijection, if the first row is highest then we should select the element in the second row in the column to the right. And if the second row is highest then we should select the element in the first row in the column to the left. □

Corollary 4.3. *Let F be a non-archimedean local field, and let $\gamma \in \mathrm{GL}_n(F)$ be unipotent. Then the centralizer of γ in $\mathrm{GL}_n(F)$ is unimodular and orbital integrals of γ in $\mathrm{GL}_n(F)$ converge.*

Proof. This follows from Theorem 3.19 and Theorem 4.2 (including the computation of λ, P, and G_γ). For smoothness of G_γ, see [16, Theorem 1.1]. For finiteness of the orbit space, see [25, Remark 57]. □

4.2 Orbital Integrals in SL_n

Let F be a non-archimedean local field, and let γ be a unipotent element of $\mathrm{SL}_n(F)$. Then γ is conjugate in $\mathrm{SL}_n(F)$ to a sum $\sum_i J^*_{n_i}$ of blocks

$$J^*_{n_i} = \begin{bmatrix} 1 & * & & \\ & 1 & \ddots & \\ & & \ddots & * \\ & & & 1 \end{bmatrix}$$

with each $* \neq 0$ and $\sum_i n_i = n$. As with GL_n, if T_0 denotes the standard maximal split torus of SL_n, then the Euclidean norm on $Y(T_0)$ gives rise to a norm function on $Y(\mathrm{SL}_n)$. This norm function is compatible with the embedding $\mathrm{SL}_n \hookrightarrow \mathrm{GL}_n$.

Proposition 4.4. *The virtual cocharacter $\sum_i \lambda_{n_i}$ is an optimal cocharacter for $\gamma = \sum_i J^*_{n_i}$ in $\mathrm{SL}_n(F)$.*

Proof. By Proposition 4.1, $\sum_i \lambda_{n_i}$ is an optimal cocharacter for $\gamma = \sum_i J^*_{n_i}$ in $\mathrm{GL}_n(F)$. Then $\sum_i \lambda_{n_i}$ is also an optimal cocharacter for γ in $\mathrm{SL}_n(F)$. □

Theorem 4.5. *Conjecture 3.15 holds for unipotent elements of $\mathrm{SL}_n(F)$.*

Proof. When passing from GL_n to SL_n, the dimensions of P and G_γ both drop by one. Then the equality $\dim N_1 = \dim P - \dim G_\gamma$ from Theorem 4.2 remains true in SL_n. □

Corollary 4.6. *Let F be a non-archimedean local field, and let $\gamma \in \mathrm{SL}_n(F)$ be unipotent. Assume that the characteristic of F does not divide n. Then the centralizer of γ in $\mathrm{SL}_n(F)$ is unimodular and orbital integrals of γ in $\mathrm{SL}_n(F)$ converge.*

Proof. This follows from Theorem 3.19 and Theorem 4.5 (including the computation of λ, P, and G_γ). For smoothness of G_γ, see [16, Theorem 1.1]. For finiteness of the orbit space, see [25, Remark 57]. □

4.3 Orbital Integrals in Sp_{2n}

Finally, we turn to the case of unipotent orbital integrals in Sp_{2n}. This will require conjugacy class representatives and centralizer dimensions of unipotent elements in $\mathrm{Sp}_{2n}(F)$. These are provided by the references [15] and [12] which concern finite classical groups in odd characteristic but work equally well over arbitrary fields of characteristic not equal to 2.

As with SL_n, if T_0 denotes the standard maximal split torus of Sp_{2n}, then the Euclidean norm on $Y(T_0)$ gives rise to a norm function on $Y(\mathrm{Sp}_{2n})$ compatible with the embedding $\mathrm{Sp}_{2n} \hookrightarrow \mathrm{GL}_{2n}$.

Theorem 4.7. *Assume that the characteristic of F is not equal to 2. Conjecture 3.15 holds for unipotent elements of* $\mathrm{Sp}_{2n}(F)$.

Proof. Let $n_1 \geq \cdots \geq n_r$ be the sizes of the Jordan blocks. By Proposition 2.3 of [15], every unipotent element of $\mathrm{Sp}_{2n}(F)$ is conjugate over F to a matrix of the form $\sum_i J^*_{n_i}$, so it is enough to consider $\gamma = \sum_i J^*_{n_i}$. By Proposition 4.1, the virtual cocharacter $\sum_i \lambda_{n_i}$ is an optimal cocharacter for γ in $\mathrm{GL}_{2n}(F)$. Then $\sum_i \lambda_{n_i}$ is also an optimal cocharacter for γ in $\mathrm{Sp}_{2n}(F)$. By Lemma 4.1 of [12] applied to the Cayley transform $X = (\gamma - I)(\gamma + I)^{-1}$ with the same Jordan blocks and centralizer [21], we have

$$\dim G_\gamma = \sum_{j<i} n_i + \sum_i \left\lceil \frac{n_i}{2} \right\rceil.$$

Let d_k denote the number of times that $k/2$ appears as an exponent in the virtual cocharacter $\sum_i \lambda_{n_i}$. Then the Levi subgroup $M \cong \mathrm{Sp}_{d_0} \times \prod_{k>0} \mathrm{GL}_{d_k}$ has dimension

$$\dim M = \frac{d_0(d_0+1)}{2} + \sum_{k\geq 1} d_k^2$$

so the optimal parabolic subgroup P has dimension

$$\dim P = \frac{1}{2}\left(n(2n+1) + \frac{d_0(d_0+1)}{2} + \sum_{k\geq 1} d_k^2 \right)$$

and the unipotent subgroup N_1 has dimension

$$\dim N_1 = \frac{1}{2}\left(n(2n+1) - \frac{d_0(d_0+1)}{2} - \sum_{k\geq 1} d_k^2 \right) - \sum_{k\geq 0} d_k d_{k+1}.$$

Then we can directly compute

$$\begin{aligned}
\dim G_\gamma &= \sum_{j<i} n_i + \sum_i \left\lceil \frac{n_i}{2} \right\rceil \\
&= \frac{1}{2}\sum_{i\neq j} \min(n_i, n_j) + \sum_i \left\lceil \frac{n_i}{2} \right\rceil \\
&= \frac{1}{2}\sum_{i,j} \min(n_i, n_j) - \frac{1}{2}\sum_i n_i + \sum_i \left\lceil \frac{n_i}{2} \right\rceil \\
&= \frac{1}{2}\sum_k d_k^2 + \frac{1}{2}\sum_k d_k d_{k+1} + \sum_i \left(\left\lceil \frac{n_i}{2} \right\rceil - \frac{n_i}{2} \right) \\
&= \frac{1}{2} d_0^2 + \sum_{k\geq 1} d_k^2 + \sum_{k\geq 0} d_k d_{k+1} + \frac{d_0}{2} \\
&= \dim P - \dim N_1.
\end{aligned}$$

□

Corollary 4.8. *Let F be a non-archimedean local field of characteristic not equal to 2, and let $\gamma \in \mathrm{Sp}_{2n}(F)$ be unipotent. Then the centralizer of γ in $\mathrm{Sp}_{2n}(F)$ is unimodular and orbital integrals of γ in $\mathrm{Sp}_{2n}(F)$ converge.*

Proof. This follows from Theorem 3.19 and Theorem 4.7 (including the computation of λ, P, and G_γ). For smoothness of G_γ, see [16, Theorem 1.1]. For finiteness of the orbit space, see [25, Remark 57]. □

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