## Title
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**Authors**
Ding, Yifeng
Holliday, Wesley Halcrow
Icard, Thomas Frederick, III

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# Logics of Imprecise Comparative Probability

Yifeng Ding[†], Wesley H. Holliday[†], and Thomas F. Icard, III[‡]
† University of California, Berkeley and ‡ Stanford University

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**Abstract**

This paper studies connections between two alternatives to the standard probability calculus for representing and reasoning about uncertainty: *imprecise probability* and *comparative probability*. The goal is to identify complete logics for reasoning about uncertainty in a comparative probabilistic language whose semantics is given in terms of imprecise probability. Comparative probability operators are interpreted as quantifying over a set of probability measures. Modal and dynamic operators are added for reasoning about epistemic possibility and updating sets of probability measures.

**Keywords:** imprecise probability, comparative probability, logic and probability.

## 1 Introduction

While the standard probability calculus remains the dominant formal framework for representing uncertainty across numerous disciplines, a small but significant tradition in philosophy, economics, computer science, and statistics has contended that the precision inherent in assigning "sharp" probabilities to uncertain events is often inappropriate. The reasons are several. One obvious concern is the psychological reality of arbitrarily precise real-valued judgments (Boole 1854; Keynes 1921; Koopman 1940; Good 1962; Suppes 1974). As Suppes (1974) expresses the concern, "Almost everyone who has thought about the problems of measuring beliefs in the tradition of subjective probability or Bayesian statistical procedures concedes some uneasiness with the problem of always asking for the next decimal of accuracy in the prior estimation of a probability" (p. 160). Another quite distinct concern is that even for a certain kind of idealized agent free of computational or representational limitations, in many important cases the available evidence somehow underdetermines the "right" probability function to have, and it would be epistemically unfitting to opt for any one of them (Carnap 1936; Levi 1974; Joyce 2005; Konek Forthcoming).

A number of alternative formal frameworks have been advanced (see, e.g., Halpern 2003). Our focus here is on two especially prominent alternatives. Some authors favor a sort of generalization of the probability calculus, allowing uncertainty to be measured by *sets* of probability functions (Good 1962; Levi 1974; Walley 1991; Seidenfeld et al. 2012; see Bradley 2019 for a philosophical overview). This *imprecise probability* framework retains many of the benefits of standard Bayesian representation and reasoning—indeed allowing the standard picture to emerge as a special case—while also affording a wider range of epistemic attitudes. Philosophical questions about imprecise probability have generated a great deal of discussion in recent years (see, e.g., Joyce 2005; Schoenfield 2012; Rinard 2013; Bradley and Steele 2014; Moss 2020). A second line of work renounces the demand for explicit numerical judgments altogether, arguing that qualitative, especially *comparative*, judgments should be the primitive building blocks for the theory of uncertainty (Keynes 1921; Koopman 1940; Fine 1973; Hawthorne 2016; see Konek 2019 for a philosophical overview). Aside from being intuitively simpler and arguably closer to "ordinary" expressions of uncertainty, some authors have argued that this setting of *comparative probability* is perhaps uniquely suited to solving notable epistemic puzzles (Fine 1977; DiBella 2018; Eva 2019). Others have sought more ameliorative reconciliations between the quantitative and qualitative approaches so as to capitalize on the advantages of each (see, e.g., Suppes and Zanotti 1976 and Elliott 2020).

Our aim in this paper is neither to weigh in on the debate between precise and imprecise versions of probabilism, nor to adjudicate between the quantitative and the qualitative alternatives, but rather to shed light on the connections between them. Only quite recently have even the most basic questions about such connections been clarified (Ríos Insua 1992; Alon and Lehrer 2014; Alon and Heifetz 2014; Harrison-Trainor et al. 2016). This is of interest from all perspectives. If one takes sets of probability measures as primitive, it would nevertheless be desirable to understand some of the core qualitative commitments implicit in this representation, including how such commitments relate to those of precise probability and other frameworks. Most conspicuously, the generalization to sets of measures brings with it a rejection of the infamous comparability principle (also sometimes called *opinionation* or *totality*), according to which every two events ought to be compared in probability. Indeed, rejection of this principle has served as one of the primary arguments against precise probabilism. As Keynes (1921) expressed it a century ago:

> Is our expectation of rain, when we start out for a walk, always more likely than not, or less likely than not, or as likely as not? I am prepared to argue that on some occasions none of these alternatives hold, and that it will be an arbitrary matter to decide for or against the umbrella. If the barometer is high, but the clouds are black, it is not always rational that one should prevail over the other in our minds, or even that we should balance them. (p. 30)

Aside from the rejection of comparability, are there other differences between the precise and imprecise probabilistic frameworks that surface in this qualitative setting? Likewise, we can ask about various additional qualitative notions aside from the usual “weak” comparison ‘at least as likely as’. For example, whereas the strict version of this judgment, ‘more likely than’, is easily definable in the precise setting in terms of weak comparison, this is no longer the case in the imprecise setting (see Section 2 below), raising new questions about the qualitative principles characterizing this distinctive kind of *unanimity* operator.

If, on the other hand, one takes qualitative judgments as primitive, this has the potential advantage of discarding principles forced upon us by (even imprecise) probabilistic representations. This may be desirable, e.g., if one is solely concerned with certain epistemic virtues such as maximizing accuracy (Fitelson and McCarthy 2014). At the same time, there are also arguments that purport to show why an agent who maintains only comparative judgments would not want to violate qualitative probabilistic principles (Fishburn 1986; Fitelson and McCarthy 2014; Icard 2016). For example, suppose that we operationalize a judgment of the form ‘$A$ is more likely than $B$’ in terms of a disposition to opt for a prospect that pays some positive dividend conditional on $A$ over one that pays the same amount conditional on $B$. Moreover, suppose that satisfying this preference is worth some cost, while judgments of the form ‘$A$ and $B$ are equally likely’ engender no such disposition. Then one can show that an agent will be forced into choosing strictly dominated actions (worse than some other available option no matter how the world turns out) if and only if the agent’s judgments fail to comport with any set of probability measures (Icard 2016). Arguments like these highlight the importance of gaining a better understanding of what compatibility of comparative judgments with imprecise probability means.

In the present paper we take a logical approach, studying a sequence of increasingly expressive qualitative formal systems, all interpreted over sets of probability measures. To illustrate the type of reasoning we would like to systemize, consider the following examples.

**Example 1.1.** A patient learns from her doctor of the existence of a gland in the human body and of a disease previously unknown to her.[1] The doctor informs her that if her gland

[1] This example is inspired by van Benthem’s (2011, p. 164, p. 166) example of the hypochondriac.

is swollen, then it is more likely than not that she has the disease. Subsequently the patient's gland is examined, and she learns that it is swollen. As a result, she comes to think it is more likely than not that she has the disease.

How should we model the patient's evolving uncertainty? A natural approach is to represent her relevant uncertainty using the following set of four possible states:

$$\{\langle \text{swollen, disease}\rangle, \langle \text{swollen, no disease}\rangle, \langle \text{not swollen, disease}\rangle, \langle \text{not swollen, no disease}\rangle\}.$$

Initially, the patient knows nothing about the gland or the disease. We represent this ignorance using the set $\mathcal{P}$ of all probability measures on the state space above. Next, when her doctor informs her that if her gland is swollen, then it is more likely than not that she has the disease, we eliminate from her set of measures all measures except those for which the probability of disease conditional on a swollen gland is greater than the probability of no disease conditional on a swollen gland. This gives us a new set $\mathcal{P}'$ of measures. Finally, when she has the gland examined and learns that it is swollen, we condition each measure in $\mathcal{P}'$ on the information that the gland is swollen, giving us a final set $\mathcal{P}''$ of measures. All measures in $\mathcal{P}''$ give a higher probability to disease than no disease.

How should one model the example using the standard representation of an agent's uncertainty with a single probability measure? First, the standard representation forces the agent to have sharp probabilities that her gland is swollen and that she has the disease, even when she just learns of their existence and knows nothing else about them. It also forces her to have a sharp conditional probability for having the disease conditional on her gland being swollen, before the doctor tells her anything about the connection between the two. Suppose she thinks that disease and no disease are equally likely conditional on her gland being swollen. What do we then do with her probability measure when the doctor informs her that if her gland is swollen, then it is more likely than not that she has the disease? One idea would be to replace her probability measure with the "closest" measure for which the conditional probability of disease given a swollen gland is greater than that of no disease given a swollen gland; but the existence of a unique closest such measure is clearly problematic. Another idea is that we must give up the simple state space above. Instead, we must use a complicated state space involving possibilities for what her doctor might say to her. On this approach, the patient must start out with a sharp conditional probability for having the disease conditional on her doctor uttering at time $t$ the words "if your gland is swollen, then it is more likely than not that you have the disease." Assuming this conditional probability is greater than .5, it follows that conditional on the doctor *not* uttering those words at time $t$, the probability she assigns to having the disease will be less than .5. In order to allow that time $t$ may pass in silence without the patient changing her probability for disease, we must introduce still further distinctions in the state space, beyond the distinction that the doctor may or may not utter the indicated words at $t$.

Though we will not argue that the modeling approach with a single probability measure is unworkable, in this paper we wish to explore the multi-measure approach sketched above. We will fully formalize the swollen gland example in Section 6.2. There we will even model the patient's becoming aware of the distinction between having a swollen gland and not having a swollen gland and of the distinction between having the disease and not having the disease, creating the state space and set $\mathcal{P}$ of measures above.

The next example is one in which it is essential to consider the possibilities for what an informant may say. It was made famous by vos Savant (1991) in the Monty Hall version of the puzzle posed by Selvin (1975). We will present the earlier but mathematically equivalent Three Prisoners version of the puzzle from Gardner (1959a; 1959b).

**Example 1.2.** The following is Diaconis and Zabell's (1986, p. 30) description of the Three Prisoners puzzle (also see Diaconis 1978 and Halpern 2003):

> Of three prisoners $a$, $b$, and $c$, two are to be executed, but $a$ does not know which. He therefore says to the jailer, "Since either $b$ or $c$ is certainly going to

be executed, you will give me no information about my own chances if you give me the name of one man, either $b$ or $c$, who is going to be executed." Accepting this argument, the jailer truthfully replies, "$b$ will be executed." Thereupon $a$ feels happier because before the jailer replied, his own chance of execution was two-thirds, but afterward there are only two people, himself and $c$, who could be the one not executed, and so his chance of execution is one-half.

Under what conditions could $a$'s reasoning possibly be sound? Imagine there are four relevant ways the world could be: $w_{ab}, w_{ac}, w_{bc}$, and $w_{cb}$, where in $w_{ij}$ prisoner $i$ is the one who lives and prisoner $j$ is the one who the jailer says will be executed. Assuming that each prisoner is equally likely to be spared, we can assume $w_{bc}$ and $w_{cb}$ both have probability one-third, and the disjunction "$w_{ab}$ *or* $w_{ac}$" has probability one-third. Concerning the relative probability of $w_{ab}$ and $w_{ac}$, we could apply a principle of indifference and proclaim that the jailer is equally likely to announce $b$ or announce $c$, in case $a$ is the one to be spared. It is then easy to compute that the conditional probability of being spared after learning that $b$ will be executed (and thus $w_{ac}$ and $w_{bc}$ can be eliminated as possibilities) is still one-third. In this case $a$ learns nothing from the jailer's announcement.

By contrast, if for whatever reason $a$ thinks the jailer is certain to tell him it is $b$ who will be executed when $a$ is the one to be spared, then learning $b$ will be executed does rationally lead $a$ to conclude that he now has a one-half chance of survival.

There is an intuition in this scenario that the right way to respond to the evidence is to leave the relative likelihood of $w_{ab}$ and $w_{ac}$ open: to represent $a$'s uncertainty in terms of the set of all probability measures that assign one-third to each of $w_{bc}$, $w_{cb}$, and the disjunction "$w_{ab}$ *or* $w_{ac}$." In this case the probabilities of $w_{ab}$ and $w_{ac}$ each range from zero to one-third, under the constraint that their sum is one-third. Updating each such measure by eliminating $w_{ac}$ and $w_{bc}$ results in a range of posterior probability values for $a$ surviving, from zero to one-half. Thus, the probability that $a$ is spared (the disjunction "$w_{ab}$ *or* $w_{ac}$") has *dilated* (Walley 1991) from precisely one-third to the entire interval $[0, 1/2]$.

Examples 1.1 and 1.2 illustrate some important aspects of imprecise probabilistic reasoning, which surface already in a purely qualitative setting. By the end of this paper, we will be able to formalize Examples 1.1 and 1.2 in a dynamic logic of updating imprecise comparative probability (Examples 6.5 and 6.21).

The outline of the paper is as follows. In Section 2, we consider the pure order-theoretic setting of comparative probability and prove a representation theorem extending previous results in the literature. The theorem concerns both a weak and a strict comparative relation together represented by a set of probability measures (Theorem 2.7). In Section 3, we turn to the logical setting and review some completeness theorems for logics of precise and imprecise probability with a single weak comparative relation (Theorems 3.7, 3.9). In Section 4, we consider a logical language that includes both weak and strict comparative relations and, using the representation in Theorem 2.7, prove a corresponding completeness theorem (Theorem 4.4). Section 5 explores the addition of a primitive "possibility" operator asserting the existence of a probability measure with a given property, culminating again in a complete axiomatization (Theorem 5.5), plus an analysis of complexity (Theorem 5.12).

In Section 6, we turn to modeling the dynamics of learning. In Section 6.1, we add to our language an update operator whose semantics is given by a process of discarding from one's set of measures any measure assigning zero probability to the learned proposition and then conditioning the remaining measures on the learned proposition. With this we can model updating on pure comparative probability formulas (through the discarding part), as well as non-probabilistic (ontic) formulas (through the conditioning part) and mixed probabilistic-ontic formulas. The language also allows the formalization of basic comparative conditional probabilities. Yet we prove that the extended language is in fact no more expressive than the previous system from Section 5: the extended language can be completely axiomatized by a set of "reduction axioms" (Theorem 6.8). Finally, in Section 6.2, we add a second dynamic operator for *becoming aware* of a new proposition (recall how the patient becomes aware of the existence of the gland and disease in Example 1.1). When an agent becomes aware of a new proposition, we form a new state space by splitting each state in her old state space in two, one where the new proposition is true and the other where it is false, and we form a new set of probability measures by taking all measures on the new set of propositions that when restricted to just the old propositions coincide with some old measure. We show that this language is more expressive than our previous languages, allowing us to express any linear inequality with integer coefficients about the probability of formulas.

What emerges is a landscape of increasingly expressive logical systems, consistent with both precise and imprecise probabilistic representations, simple but sufficiently powerful to model sophisticated reasoning about uncertainty. Perhaps surprisingly, the computational complexity of reasoning (e.g., determining validity or consistency) in each of the "static" systems is no worse than for the classical propositional calculus. The complexity of reasoning in the dynamic logic of updating sets of probability measures is an open problem, as is the complexity and axiomatization of the dynamic logic of becoming aware.

# 2 Representation

Before introducing any explicit logical calculus, in this section we consider the pure order-theoretic setting of comparative probability. A comparative notion of probability is most naturally formalized as a binary relation on an algebra of events. However, not all binary relations can be intuitively interpreted as comparing how likely events are, just as not all functions from events to $[0, 1]$ can be interpreted as assigning quantitative probabilities. Taking the usual axiomatization of quantitative probability for granted, a natural question—posed early on by de Finetti (1949)—is what would be a set of axioms that are intuitive and in harmony with those quantitative axioms.

This question was first solved for finite event algebras by Kraft et al. (1959). Given a binary relation $\succsim$ on $\wp(W)$, where $W$ is a finite set, and a probability measure $\mu$ on $\wp(W)$, we say that $\succsim$ is *precisely represented by* $\mu$ if for all $X, Y \subseteq W$, $X \succsim Y$ iff $\mu(X) \geq \mu(Y)$.

**Theorem 2.1** (Kraft et al. 1959)**.** Let $W$ be a nonempty finite set and $\succsim$ a binary relation on $\wp(W)$. Then $\succsim$ is precisely represented by some probability measure on $\wp(W)$ if and only if:

- $\varnothing \not\succsim W$, $\{w\} \succsim \varnothing$ for all $w \in W$, and for all $A, B \in \wp(W)$, $A \succsim B$ or $B \succsim A$, and
- $\succsim$ satisfies the finite cancellation condition (FC): letting $\mathbf{1}_X$ denote the characteristic function of $X$, for any two finite sequences $\langle A_i \rangle_{i=1}^n, \langle B_i \rangle_{i=1}^n$ of events in $\wp(W)$ such that $\sum_{i=1}^n \mathbf{1}_{A_i} = \sum_{i=1}^n \mathbf{1}_{B_i}$ (additions are done in the vector space $\mathbb{R}^W$), if for all $i < n$, $A_i \succsim B_i$, then $B_n \succsim A_n$.

Following the same paradigm, we can consider a comparative notion of imprecise probability and ask the following question: which binary relations on a finite algebra of events can be naturally interpreted as an imprecise version of the at-least-as-likely-as relation? More precisely, given a binary relation $\succsim$ on $\wp(W)$, where $W$ is a finite set, and a set $\mathcal{P}$ of probability measures on $\wp(W)$, we say that $\succsim$ is *imprecisely represented as the weak relation by* $\mathcal{P}$ if for all $X, Y \subseteq W$, $X \succsim Y$ iff for all $\mu \in \mathcal{P}$, $\mu(X) \geq \mu(Y)$. The following analogue of Theorem 2.1 was proved by Ríos Insua (1992) (also see Alon and Lehrer 2014).

**Theorem 2.2** (Ríos Insua 1992)**.** Let $W$ be a nonempty finite set and $\succsim$ a binary relation on $\wp(W)$. Then $\succsim$ is imprecisely represented as the weak relation by some set $\mathcal{P}$ of probability measures on $\wp(W)$ if and only if:

- $\varnothing \not\succsim W$, $\{w\} \succsim \varnothing$ for all $w \in W$, and

- $\succsim$ satisfies the generalized finite cancellation condition (GFC): for any two finite sequences $\langle A_i \rangle_{i=1}^n, \langle B_i \rangle_{i=1}^n$ of events in $\wp(W)$ and $k \in \mathbb{N} \setminus \{0\}$ such that $\sum_{i=1}^{n-1} \mathbf{1}_{A_i} + k\mathbf{1}_{A_n} = \sum_{i=1}^{n-1} \mathbf{1}_{B_i} + k\mathbf{1}_{B_n}$, if for all $i < n$, $A_i \succsim B_i$, then $B_n \succsim A_n$.[2]

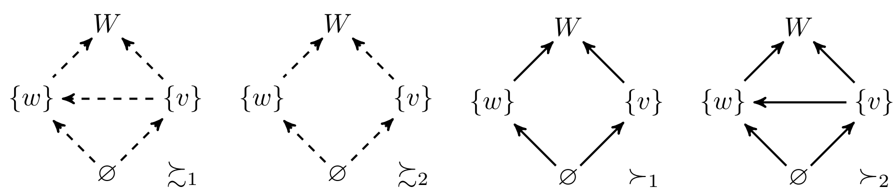
**Remark 2.3.** Harrison-Trainor et al. (2016) prove that there are relations $\succsim$ satisfying the conditions of Theorem 2.1 *except for the comparability principle* (that for all $A, B \in \wp(W)$, $A \succsim B$ or $B \succsim A$) and which fail to satisfy the GFC condition in Theorem 2.2. Thus, it is necessary to strengthen FC to GFC when dropping comparability to obtain Theorem 2.2.

A subtlety not covered by Theorem 2.2 is that given a set $\mathcal{P}$ of probability measures, there are two natural ways to generate a strict relation, corresponding to the strict and the weak dominance relation in game theory:

- $X$ *strictly dominates* $Y$ in $\mathcal{P}$ iff for all $\mu \in \mathcal{P}$, $\mu(X) > \mu(Y)$;
- $X$ *weakly dominates* $Y$ in $\mathcal{P}$ iff for all $\mu \in \mathcal{P}$, $\mu(X) \geq \mu(Y)$, and there is a $\mu \in \mathcal{P}$ such that $\mu(X) > \mu(Y)$.

When $\succsim$ is represented as the weak relation by $\mathcal{P}$, it is easy to see that $X$ weakly dominates $Y$ iff $X \succsim Y$ but $Y \not\succsim X$. However, we cannot pin down the strict dominance relation simply from the weak relation $\succsim$ or vice versa, as shown by the following example.

**Example 2.4.** Let $W = \{w, v\}$ and consider the four binary relations $\succsim_1, \succsim_2, \succ_1, \succ_2$ pictured below from left to right (for dashed arrows, reflexive and transitive arrows are omitted; for solid arrows, transitive arrows are omitted).




If all we know about a set $\mathcal{P}$ of probability measures on $\wp(W)$ is that its weak relation is $\succsim_1$, then both $\succ_1$ and $\succ_2$ may be $\mathcal{P}$'s strict dominance relation. For example, we can define a probability measure $\mu_{w<v}$ on $\wp(W)$ that favors $v$ so that $\mu_{w<v}(\{w\}) = 1/3$. Then let $\mu_{w=v}$ be the uniform distribution on $\wp(W)$: $\mu_{w=v}(\{w\}) = \mu_{w=v}(\{v\}) = 1/2$. Then for both $\{\mu_{w<v}, \mu_{w=v}\}$ and $\{\mu_{w<v}\}$, their weak relation is $\succsim_1$. Yet the strict dominance relation of the former is $\succ_1$ while the strict dominance relation of the latter is $\succ_2$.

Similarly, if all we know about $\mathcal{P}$ is that its strict dominance relation is $\succ_1$, then both $\succsim_1$ and $\succsim_2$ may be its weak relation. For this, define a probability measure $\mu_{w>v}$ that favors $w$ so that $\mu_{w>v}(\{w\}) = 2/3$. Then we see that the strict dominance relation of both $\{\mu_{w<v}, \mu_{w=v}\}$ and $\{\mu_{w<v}, \mu_{w>v}\}$ is $\succ_1$ while the weak relation of the former is $\succsim_1$ and the weak relation of the latter is $\succsim_2$.

In light of these considerations, we introduce the following definition that accounts for both relations; cf. Konek (2019, p. 275, footnote 4), who suggests that the study of comparative probability ought to start with pairs $\langle \succsim, \succ \rangle$ because an agent who judges that $X$ is at least as likely as $Y$ but withholds judgment about whether $Y$ is at least as likely as $X$ does not necessarily judge that $X$ is strictly more likely than $Y$.

**Definition 2.5.** Given a pair $\langle \succsim, \succ \rangle$ of binary relations on $\wp(W)$ and a set $\mathcal{P}$ of probability measures on $\wp(W)$, we say that $\langle \succsim, \succ \rangle$ is *represented by* $\mathcal{P}$ iff for all $X, Y \subseteq W$,

- $X \succsim Y$ iff for all $\mu \in \mathcal{P}$, $\mu(X) \geq \mu(Y)$, and
- $X \succ Y$ iff for all $\mu \in \mathcal{P}$, $\mu(X) > \mu(Y)$.

[2]Note that $n$ can be 1, in which case the condition simply expresses the reflexivity of $\succsim$.

**Remark 2.6.** Define $X \succcurlyeq Y$ as *not* $Y \succ X$, i.e., there is some $\mu \in \mathcal{P}$ such that $\mu(X) \geq \mu(Y)$ (cf. the notion of *justifiable preference* in Lehrer and Teper 2011). Then the pair $\langle \succsim, \succcurlyeq \rangle$ of weak relations is what Giarlotta and Greco (2013) call a *necessary and possible preference.*

The following theorem characterizes the representable relation pairs.

**Theorem 2.7.** Let $W$ be a nonempty finite set and $\succsim, \succ$ two binary relations on $\wp(W)$. Then $\langle \succsim, \succ \rangle$ is represented by a set $\mathcal{P}$ of probability measures on $\wp(W)$ if and only if:

- $\succ$ is irreflexive and $\succ \subseteq \succsim$;
- $W \succ \varnothing$, and $\{w\} \succsim \varnothing$ for all $w \in W$;
- $\succsim$ satisfies (GFC) and $\succ$ satisfies the strict generalized finite cancellation condition (SGFC): for any two finite sequences $\langle A_i \rangle_{i=1}^n, \langle B_i \rangle_{i=1}^n$ of events in $\wp(W)$ and $k \in \mathbb{N} \setminus \{0\}$ such that $\sum_{i=1}^{n-1} \mathbf{1}_{A_i} + k\mathbf{1}_{A_n} = \sum_{i=1}^{n-1} \mathbf{1}_{B_i} + k\mathbf{1}_{B_n}$, if for all $i < n$, $A_i \succsim B_i$ and there is $i < n$ with $A_i \succ B_i$, then $B_n \succ A_n$.

The rest of this section is devoted to the proof of Theorem 2.7. The proof is adapted from the proof of Theorem 2.2 above in Alon and Lehrer 2014, which also generalizes the proof in Scott 1964 for Theorem 2.1 (also see Mierzewski 2018, § 3.3 for a representation theorem concerning $\langle \succsim, \succ \rangle$ in the setting of precise probability). For this, pick a nonempty finite set $W$ and a pair $\langle \succsim, \succ \rangle$ satisfying the conditions (the necessity of the conditions is easy). The main strategy is to reframe the representability of $\langle \succsim, \succ \rangle$ in terms of the existence of solutions to some systems of homogeneous linear inequalities in the vector space $\mathbb{R}^W$. Hence we use vectors in $\Delta(W) = \{\mu \in \mathbb{R}^W \mid \mu \cdot \mathbf{1}_W = 1 \text{ and for all } w \in W, \mu(w) \geq 0\}$ as probability measures.

Define $D_{\succsim} = \{\mathbf{1}_A - \mathbf{1}_B \mid A, B \subseteq W, A \succsim B\}$ and $D_{\succ} = \{\mathbf{1}_A - \mathbf{1}_B \mid A, B \subseteq W, A \succ B\}$. Intuitively, $D_{\succsim}$ contains vectors that always receive non-negative measures and $D_{\succ}$ contains vectors that always receive positive measures. Given the conditions satisfied by $\succsim$ and $\succ$, we can prove the following lemmas.

**Lemma 2.8.** If $f \in \{-1, 0, 1\}^W$ is a non-negative linear combination of vectors in $D_{\succsim}$, then $f \in D_{\succsim}$.

*Proof.* Suppose $f \in \{-1, 0, 1\}^W$ is a non-negative linear combination of vectors in $D_{\succsim}$. Since all the vectors are in $\{-1, 0, 1\}^W$, we can assume that all coefficients are rational since a system of linear inequalities with rational coefficients has a solution if and only if it has a rational solution. Then we can clear the denominators and obtain a $k \in \mathbb{N} \setminus \{0\}$ such that $kf$ is simply a sum of vectors in $D_{\succsim}$ possibly with repetitions: $\sum_{i=1}^n g_i$. Since $f$ and the $g_i$'s are in $D_{\succsim}$, we can find subsets $A_i, B_i$ for $i = 1 \ldots n+1$ of $W$ such that

- $g_i = \mathbf{1}_{A_i} - \mathbf{1}_{B_i}$ for $i = 1 \ldots n$ and $f = \mathbf{1}_{B_{n+1}} - \mathbf{1}_{A_{n+1}}$ (take $B_{n+1} = f^{-1}(1)$ and $A_{n+1} = f^{-1}(-1)$), and
- $A_i \succsim B_i$ for $i = 1 \ldots n$.

Then given that $kf = \sum_{i=1}^n g_i$, we have $\sum_{n=1}^n \mathbf{1}_{A_i} + k\mathbf{1}_{A_{n+1}} = \sum_{i=1}^n \mathbf{1}_{B_i} + k\mathbf{1}_{B_{n+1}}$. Hence we can apply (GFC) to $\langle A_i \rangle_{i=1}^{n+1}$ and $\langle B_i \rangle_{i=1}^{n+1}$ and see that $B_{n+1} \succsim A_{n+1}$. Therefore, $f = \mathbf{1}_{B_{n+1}} - \mathbf{1}_{A_{n+1}} \in D_{\succsim}$. □

**Lemma 2.9.** If $f \in \{-1, 0, 1\}^W$ is a non-negative linear combination of vectors in $D_{\succsim} \cup D_{\succ}$ with a coefficient for a vector in $D_{\succ}$ being positive, then $f \in D_{\succ}$.

*Proof.* Similar to the proof of the previous lemma. The only change in this case is that when we find $k$ and express $kf$ as a sum of vectors in $D_{\succsim} \cup D_{\succ}$, at least one vector in $D_{\succ}$ must figure in the sum since initially the non-negative linear combination resulting in $f$ has a positive coefficient for a vector in $D_{\succ}$. Then we can find sets $A_i$'s and $B_i$'s similarly and apply (SGFC) to see that $f$ must be in $D_{\succ}$ already. □

Now define

$$\mathcal{P} = \{\mu \in \Delta(W) \mid \forall f \in D_{\succsim},\ \mu \cdot f \geq 0 \text{ and } \forall f \in D_{\succ},\ \mu \cdot f > 0\}.$$

Our goal is to show that $\langle \succsim, \succ \rangle$ is represented by this $\mathcal{P}$. Note that one direction is done already: for any $A, B \subseteq W$,

- if $A \succsim B$, then by the definition of $\mathcal{P}$, for all $\mu \in \mathcal{P}$, $\mu \cdot (\mathbf{1}_A - \mathbf{1}_B) \geq 0$, which means that $\mu \cdot \mathbf{1}_A \geq \mu \cdot \mathbf{1}_B$;
- similarly, if $A \succ B$, then for all $\mu \in \mathcal{P}$, $\mu \cdot (\mathbf{1}_A - \mathbf{1}_B) > 0$, which means that $\mu \cdot \mathbf{1}_A > \mu \cdot \mathbf{1}_B$.

Hence all that are left to prove are the following two claims:

(a) If $A \not\succsim B$, then there is a $\mu \in \mathcal{P}$ such that $\mu \cdot (\mathbf{1}_A - \mathbf{1}_B) < 0$;

(b) If $A \not\succ B$, then there is a $\mu \in \mathcal{P}$ such that $\mu \cdot (\mathbf{1}_A - \mathbf{1}_B) \leq 0$.

For (a), it is enough to prove that for all $f \in \{-1, 0, 1\}^W$, if $f \notin D_{\succsim}$, then there is $\mu \in \mathcal{P}$ such that $\mu \cdot -f > 0$, since for any $A, B \subseteq W$, we have $\mathbf{1}_A - \mathbf{1}_B \in \{-1, 0, 1\}^W$. Hence take such an $f \in \{-1, 0, 1\}^W \setminus D_{\succsim}$. We need to find a $\mu$ such that (i) $\mu \in \mathcal{P}$ and (ii) $\mu \cdot -f > 0$. Given the definition of $\mathcal{P}$, this amounts to the existence of a solution to the following system of homogeneous linear inequalities (where we write $[D]$ for the matrix containing as columns the vectors in a set $D$ of vectors):

$$[D_{\succsim}]^\top \vec{x} \geq \vec{0}, \quad [D_{\succ} \cup \{-f\}]^\top \vec{x} > \vec{0}. \tag{1}$$

The existence of a $\mu$ satisfiying (i) and (ii) is equivalent to the existence of a solution to the above system of inequalities because by assumption, $W \succ \varnothing$ and $\{w\} \succsim \varnothing$ for all $w \in W$, which means that $\mathbf{1}_W \in D_{\succ}$ and $\mathbf{1}_{\{w\}} \in D_{\succsim}$ for all $w \in W$, so any solution can be scaled to be an element in $\mathcal{P}$. The condition of the existence of a solution is given by a special case of Motzkin's Transposition Theorem (see Motzkin 1951).

**Theorem 2.10** (Motzkin's Transposition Theorem)**.** The linear inequality system $M_1\vec{x} \geq 0, M_2\vec{x} > \vec{0}$ has a solution if and only if there is no solution to the system $M_1^\top \vec{y}_1 + M_2^\top \vec{y}_2 = \vec{0}, \vec{y}_1 \geq \vec{0}, \vec{y}_2 \geq \vec{0}, \vec{y}_2 \neq \vec{0}$.

Suppose toward a contradiction that there is no solution to (1). Then by Motzkin's Transposition Theorem, there are non-negative $\vec{y}_1, \vec{y}_2$ with $\vec{y}_2$ non-trivial such that $[D_{\succsim}]^\top \vec{y}_1 + [D_{\succ} \cup \{-f\}]^\top \vec{y}_2 = \vec{0}$. In other words, $\vec{0}$ is a non-negative linear combination of vectors in $D_{\succsim} \cup D_{\succ} \cup \{-f\}$ with one of the vectors in $D_{\succ} \cup \{-f\}$ having a positive coefficient. Now there are two possibilities: either $-f$ has a positive coefficient or not. If not, then $\vec{0}$ is a non-negative linear combination of vectors in $D_{\succsim} \cup D_{\succ}$ with a vector in $D_{\succ}$ having a positive coefficient. Then, by Lemma 2.9, $\vec{0} \in D_{\succ}$. This contradicts the assumption that $\succ$ is irreflexive. If $-f$ has a positive coefficient, then $f$ is a linear combination of vectors in $D_{\succsim} \cup D_{\succ} = D_{\succsim}$ since $\succ \subseteq \succsim$. By Lemma 2.8, $f \in D_{\succsim}$, but we picked $f$ specifically from outside $D_{\succsim}$. Hence, either way, we have a contradiction. This completes the proof of (a).

The proof of (b) is almost identical. It is enough to show that for any $f \in \{-1, 0, 1\}^W \setminus D_{\succ}$, the following has a solution:

$$[D_{\succsim} \cup \{-f\}]^\top \vec{x} \geq \vec{0}, \quad [D_{\succ}]^\top \vec{x} > \vec{0}.$$

If there is no solution, then by Motzkin's Transposition Theorem, $\vec{0}$ is a non-negative linear combination of vectors in $D_{\succsim} \cup \{-f\} \cup D_{\succ}$ with at least one vector in $D_{\succ}$ having a positive coefficient. Again, we consider whether $-f$ has a positive coefficient or not. If not, then $\vec{0}$ should again be in $D_{\succ}$, which is not the case. If indeed $-f$ has a positive coefficient, then $f$ is a linear combination of vectors in $D_{\succsim} \cup D_{\succ}$ with at least one vector in $D_{\succ}$ having a positive coefficient. By Lemma 2.9, $f \in D_{\succ}$, contradicting the way we picked $f$. Hence (b) is also proved, which completes the proof of Theorem 2.7.

**Remark 2.11.** The sets $D_{\succsim}$ and $D_{\succ}$ used in the proof above are reminiscent of an alternative but also prominent way of modelling uncertainty in the imprecise probability literature: sets of desirable gambles (see Walley 2000 and chapters in Augustin et al. 2014 for introductions). For any event $A \subseteq W$, we may interpret it as a gamble that returns a unit of utility for the states in $A$ and returns nothing for states outside $A$. In other words, we can understand comparing the likelihoods of two events $A$ and $B$ as comparing the two corresponding gambles $\mathbf{1}_A$ and $\mathbf{1}_B$, which in turn reduces to the question of whether the gamble $\mathbf{1}_A - \mathbf{1}_B$ is acceptable/desirable. However, there are two important differences between our setting and the desirable gambles approach commonly presented in the literature.

First, since we are only comparing propositions, we do not need to appeal to the desirability of gambles not in $\{-1, 0, 1\}^W$. In the literature on desirable gambles, all gambles in $\mathbb{R}^W$ are considered, and that is partly the reason for succinct axioms for coherent sets of desirable gambles such as closure under positive scaling and pairwise addition. The same cannot be done when restricting to $\{-1, 0, 1\}^W$, since for example $\mathbf{1}_W + \mathbf{1}_W$ is no longer in $\{-1, 0, 1\}^W$. Also, it is not hard to see that different coherent sets of desirable gambles have the same intersection with $\{-1, 0, 1\}^W$. This means that using sets of desirable gambles in $\mathbb{R}^W$, we may encode more information than needed for comparing propositions.

Second, we model an agent's uncertainty with a pair of binary relations, and hence when translated to sets of desirable gambles, we use a pair of sets of desirable gambles instead of a single one. This can be easily seen from the proof above: we constructed a pair $\langle D_{\succsim}, D_{\succ} \rangle$ from $\langle \succsim, \succ \rangle$. If we disregard the previous difference and consider all gambles in $\mathbb{R}^W$, our approach can be understood as generalizing representation by a single set of *almost desirable* gambles (using the terminology in Couso and Moral 2011) by pairing it with another set of gambles that can be interpreted as *strictly desirable* gambles. However, the axiomatic requirement for this set is weaker than the requirement for "sets of strictly desirable gambles" in Couso and Moral 2011. More importantly, our axiomatic requirement concerns two sets jointly, as can be seen from Lemma 2.9. In this way, we achieve greater generality (expressivity) than merely using a set of almost desirable gambles. We leave further comparison between these two approaches to imprecise probability for future work.

# 3 The Logic $\mathsf{IP}(\succsim)$

In this section and the following sections, we turn to the formalization of imprecise comparative probabilistic reasoning in logical systems. The representation theorems of Section 2 lead to completeness theorems for these logical systems.

The logics we consider form a hierarchy of increasing expressive power of their languages. The least expressive language we will consider is the following.

**Definition 3.1.** The language $\mathcal{L}(\succsim)$, generated from a nonempty set $\mathsf{Prop}$ of propositional variables, is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \succsim \varphi)$$

where $p \in \mathsf{Prop}$. A *propositional* (or *Boolean*) *formula* is a formula generated from $\mathsf{Prop}$ using only $\neg$ and $\wedge$. We define the other propositional connectives $\vee$, $\rightarrow$, $\leftrightarrow$, $\top$, and $\bot$ as usual. Finally, we define $\varphi \succnsim \psi$ as $(\varphi \succsim \psi) \wedge \neg(\psi \succsim \varphi)$ and $\varphi \approx \psi$ as $(\varphi \succsim \psi) \wedge (\psi \succsim \varphi)$.

We will consider several semantics for this language, each of which builds on the standard possible world models for propositional logics.

**Definition 3.2.** A *propositional model* is a pair $\mathcal{M} = \langle W, V \rangle$ where $W$ is a nonempty set and $V : \mathsf{Prop} \to \wp(W)$. We may abuse notation and write '$w \in \mathcal{M}$' to mean $w \in W$.

The first semantics for $\mathcal{L}(\succsim)$ that we will consider, which may be considered its "intended semantics," equips a propositional model with one or more probability measures, as follows.

**Definition 3.3.** An *imprecise probabilistic model* (IP model) is a pair $\langle \mathcal{M}, \mathcal{P} \rangle$ where $\mathcal{M} = \langle W, V \rangle$ is a propositional model and $\mathcal{P}$ is a set of finitely additive probability measures on a field $\mathcal{F}$ of subsets of $W$ such that $V(p) \in \mathcal{F}$ for each $p \in \mathsf{Prop}$. A *precise probabilistic model* is an imprecise probabilistic model $\langle \mathcal{M}, \mathcal{P} \rangle$ such that $|\mathcal{P}| = 1$.

The key part of the truth definition of formulas of $\mathcal{L}(\succsim)$ in IP models matches the notion of imprecise representation from Section 2: $\varphi \succsim \psi$ is true just in case *according to all the probability measures in* $\mathcal{P}$, the probability of the set of worlds where $\varphi$ is true is at least as great as the probability of the set of worlds where $\psi$ is true.

**Definition 3.4.** Given an IP model $\langle \mathcal{M}, \mathcal{P} \rangle$, $w \in \mathcal{M}$, and $\varphi \in \mathcal{L}(\succsim)$, we define $\mathcal{M}, \mathcal{P}, w \vDash \varphi$ and $[\![\varphi]\!]^{\mathcal{M},\mathcal{P}} = \{w \in W \mid \mathcal{M}, \mathcal{P}, w \vDash \varphi\}$ as follows:

1. $\mathcal{M}, \mathcal{P}, w \vDash p$ iff $w \in V(p)$;
2. $\mathcal{M}, \mathcal{P}, w \vDash \neg\varphi$ iff $\mathcal{M}, \mathcal{P}, w \nvDash \varphi$;
3. $\mathcal{M}, \mathcal{P}, w \vDash (\varphi \wedge \psi)$ iff $\mathcal{M}, \mathcal{P}, w \vDash \varphi$ and $\mathcal{M}, \mathcal{P}, w \vDash \psi$;
4. $\mathcal{M}, \mathcal{P}, w \vDash \varphi \succsim \psi$ iff for all $\mu \in \mathcal{P}$, $\mu([\![\varphi]\!]^{\mathcal{M},\mathcal{P}}) \geq \mu([\![\psi]\!]^{\mathcal{M},\mathcal{P}})$.

If $\alpha$ is a propositional formula, we may write '$V(\alpha)$' for $[\![\alpha]\!]^{\mathcal{M},\mathcal{P}}$ to emphasize that the set of worlds where $\alpha$ is true does not depend on the set $\mathcal{P}$ of probability measures.

Finally, given a class $\mathsf{K}$ of IP models, $\varphi$ is *valid with respect to* $\mathsf{K}$ iff for all $\langle \mathcal{M}, \mathcal{P} \rangle \in \mathsf{K}$ and $w \in \mathcal{M}$, we have $\mathcal{M}, \mathcal{P}, w \vDash \varphi$.

An easy induction shows that for any formula $\varphi$, the set of worlds where $\varphi$ is true belongs to the algebra $\mathcal{F}$ of measurable sets.

**Lemma 3.5.** For every IP model $\langle \mathcal{M}, \mathcal{P} \rangle$ and $\varphi \in \mathcal{L}(\succsim)$, we have $[\![\varphi]\!]^{\mathcal{M},\mathcal{P}} \in \mathcal{F}$.

Below we define logics that are sound and complete with respect to the classes of imprecise probabilistic models and precise probabilistic models, respectively. To do so, we first need to define a syntactic abbreviation that allows us to express the finite cancellation condition of Theorem 2.1 using formulas of our language. Given formulas $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n \in \mathcal{L}(\succsim)$, and $1 \leq k \leq n$, define $C_k$ to be the disjunction of all conjunctions

$$f_1\varphi_1 \wedge \cdots \wedge f_n\varphi_n \wedge g_1\psi_1 \wedge \cdots \wedge g_n\psi_n$$

where exactly $k$ of the $f$'s and $k$ of the $g$'s are the empty string, and the rest are $\neg$. Thus, $C_k$ is true at a state $w \in W$ iff exactly $k$ of the $\varphi$'s and $k$ of the $\psi$'s are true at $w$. Then let

$$(\varphi_1, \ldots, \varphi_n) \equiv (\psi_1, \ldots, \psi_n) := C_1 \vee \cdots \vee C_n,$$

which is true at a state $w \in W$ iff the number of $\varphi$'s true at $w$ is exactly the same as the number of $\psi$'s true at $w$. Using these abbreviations, we can express the finite cancellation condition with the axiom schema (**A4**) below.

**Definition 3.6.** The set of theorems of $\mathsf{SP}(\succsim)$ (the *logic of sharp probability*) is the smallest subset of $\mathcal{L}(\succsim)$ that contains all tautologies of propositional logic, is closed under *modus ponens* (if $\varphi \in \mathsf{SP}(\succsim)$ and $\varphi \to \psi \in \mathsf{SP}(\succsim)$, then $\psi \in \mathsf{SP}(\succsim)$) and *necessitation* (if $\varphi \in \mathsf{SP}(\succsim)$, then $\varphi \succsim \top \in \mathsf{SP}(\succsim)$), and contains all instances of the following axiom schemas for all $n \in \mathbb{N}$:[3]

(**A0**) $(\varphi \succsim \psi) \vee (\psi \succsim \psi)$;

(**A1**) $\varphi \succsim \bot$;

[3]The labeling of axioms here follows Alon and Heifetz 2014.

(**A2**) $\varphi \succsim \varphi$;[4]

(**A3**) $\neg(\bot \succsim \top)$;

(**A4**) $\big((\varphi_1 \succsim \psi_1) \wedge \cdots \wedge (\varphi_n \succsim \psi_n) \wedge (\varphi_1, \ldots, \varphi_n, \varphi') \equiv (\psi_1, \ldots, \psi_n, \psi') \succsim \top\big) \to (\psi' \succsim \varphi')$;

(**A5**) $(\varphi \succsim \psi) \to \big((\varphi \succsim \psi) \succsim \top\big)$;

(**A6**) $\neg(\varphi \succsim \psi) \to \big(\neg(\varphi \succsim \psi) \succsim \top\big)$.

The representation result in Theorem 2.1 may be used to prove the following completeness theorem for $\mathsf{SP}(\succsim)$.

**Theorem 3.7** (Segerberg 1971; Gärdenfors 1975)**.** For all $\varphi \in \mathcal{L}(\succsim)$: $\varphi$ is a theorem of $\mathsf{SP}(\succsim)$ if and only if $\varphi$ is valid with respect to the class of all precise probabilistic models.

To obtain a complete logic for *imprecise* probabilistic models, we express the generalized finite cancellation conditions of Theorem 2.2 using formulas of our language as follows.

**Definition 3.8.** The logic $\mathsf{IP}(\succsim)$ (the *logic of imprecise probability*) is defined in the same way as $\mathsf{SP}(\succsim)$ except without axiom (**A0**) and with (**A4**) replaced by:

(**A4′**) $$\big((\varphi_1 \succsim \psi_1) \wedge \cdots \wedge (\varphi_n \succsim \psi_n) \wedge (\varphi_1, \ldots, \varphi_n, \underbrace{\varphi', \ldots, \varphi'}_{k\text{ times}}) \equiv (\psi_1, \ldots, \psi_n, \underbrace{\psi', \ldots, \psi'}_{k\text{ times}}) \succsim \top\big)$$
$$\to (\psi' \succsim \varphi').$$

The representation result in Theorem 2.2 may be used to prove the following completeness theorem for $\mathsf{IP}(\succsim)$.

**Theorem 3.9** (Alon and Heifetz 2014)**.** For all $\varphi \in \mathcal{L}(\succsim)$: $\varphi$ is a theorem of $\mathsf{IP}(\succsim)$ if and only if $\varphi$ is valid with respect to the class of all imprecise probabilistic models.

In Section 4 we will give a completeness proof that shows how the proof of Theorem 3.9 goes as well.

# 4 The Logic $\mathsf{IP}(\succsim, \succ)$

Our first step beyond the existing literature on logics of imprecise comparative probability is to add to our formal language the primitive strict operator $\succ$ from Section 2.

**Definition 4.1.** The language $\mathcal{L}(\succsim, \succ)$ is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \succsim \varphi) \mid (\varphi \succ \varphi)$$

where $p \in \mathsf{Prop}$. As before, we define $\varphi \gnsim \psi$ as $(\varphi \succsim \psi) \wedge \neg(\psi \succsim \varphi)$. Let $\mathcal{L}(\succ)$ be the fragment of $\mathcal{L}(\succsim, \succ)$ in which $\succsim$ does not occur.

**Definition 4.2.** We extend the semantics of Definition 3.4 to $\mathcal{L}(\succsim, \succ)$ as follows:

- $\mathcal{M}, \mathcal{P}, w \vDash \varphi \succ \psi$ iff for all $\mu \in \mathcal{P}$, $\mu([\![\varphi]\!]^{\mathcal{M},\mathcal{P}}) > \mu([\![\psi]\!]^{\mathcal{M},\mathcal{P}})$.

It follows from Example 2.4 that the formula $p \succ q$ is not equivalent to any formula of $\mathcal{L}(\succsim)$, including $p \gnsim q$, while the formula $p \succsim q$ is not equivalent to any formula of $\mathcal{L}(\succ)$.

In the following, we first present a sound and complete logic for $\mathcal{L}(\succsim, \succ)$ whose axioms match the conditions of the representation result in Theorem 2.7. Then we discuss the expressivity of this language, including how it is more expressive than $\mathcal{L}(\succsim)$.

[4]Axiom (**A2**) is redundant given (**A0**), but below we consider a logic that drops (**A0**). In fact, (**A2**) is also derivable from the $n = 0$ case of (**A4**) and (**A4′**), but we include (**A2**) to match Alon and Heifetz 2014.

## 4.1 Logic

**Definition 4.3.** The logic $\mathsf{IP}(\succsim,\succ)$ is the smallest subset of $\mathcal{L}(\succsim,\succ)$ that contains all tautologies of propositional logic, is closed under *modus ponens* (if $\varphi \in \mathsf{IP}(\succsim,\succ)$ and $\varphi \to \psi \in \mathsf{IP}(\succsim,\succ)$, then $\psi \in \mathsf{IP}(\succsim,\succ)$) and *necessitation* (if $\varphi \in \mathsf{IP}(\succsim,\succ)$, then $\varphi \succsim \top \in \mathsf{IP}(\succsim,\succ)$), and contains all instances of the following axiom schemas for all $n \in \mathbb{N}$:

(**B1**) $\varphi \succsim \bot$;

(**B2**) $\top \succ \bot$;

(**B3**) $(\varphi \succ \psi) \to (\varphi \succsim \psi)$;

(**B4**) $\neg(\varphi \succ \varphi)$;

(**B5**) $$\big(\varphi_1, \ldots, \varphi_n, \underbrace{\varphi', \ldots, \varphi'}_{k \text{ times}}) \equiv (\psi_1, \ldots, \psi_n, \underbrace{\psi', \ldots, \psi'}_{k \text{ times}}) \succsim \top\big) \to \big((\textstyle\bigwedge_{i=1}^n (\varphi_i \succsim \psi_i)) \to (\psi' \succsim \varphi')\big);$$

(**B6**) $$\big((\varphi_1, \ldots, \varphi_n, \underbrace{\varphi', \ldots, \varphi'}_{k \text{ times}}) \equiv (\psi_1, \ldots, \psi_n, \underbrace{\psi', \ldots, \psi'}_{k \text{ times}}) \succsim \top\big) \to \big((\textstyle\bigwedge_{i=1}^n (\varphi_i \succsim \psi_i) \wedge \bigvee_{i=1}^n (\varphi_i \succ \psi_i)) \to (\psi' \succ \varphi')\big);$$

(**B7**) $(\varphi \succsim \psi) \to ((\varphi \succsim \psi) \succsim \top)$;

(**B8**) $\neg(\varphi \succsim \psi) \to (\neg(\varphi \succsim \psi) \succsim \top)$;

(**B9**) $(\varphi \succ \psi) \to ((\varphi \succ \psi) \succsim \top)$;

(**B10**) $\neg(\varphi \succ \psi) \to (\neg(\varphi \succ \psi) \succsim \top)$.

The rest of this section is devoted to the proof of the following theorem.

**Theorem 4.4** (Soundness and Completeness)**.** For all $\varphi \in \mathcal{L}(\succsim,\succ)$: $\varphi$ is a theorem of $\mathsf{IP}(\succsim,\succ)$ if and only if $\varphi$ is valid with respect to the class of all imprecise probabilistic models.

The soundness direction is easy to check. For completeness, as usual, pick an arbitrary formula $\gamma$ consistent in $\mathsf{IP}(\succsim,\succ)$ and let $\mathsf{p}$ be the set of propositional variables appearing in $\gamma$ and $\mathcal{L}_0$ the restriction of $\mathcal{L}(\succsim,\succ)$ to $\mathsf{p}$. Then extend $\{\gamma\}$ to a set $\Gamma$ that is maximally consistent in $\mathsf{IP}(\succsim,\succ)$ with respect to $\mathcal{L}_0$. Now our goal is to build an IP model of $\gamma$ by extracting information from $\Gamma$. To this end, we view $\mathcal{L}_0$ as a term algebra of the type of Boolean algebras expanded with two binary operations. Then define $\blacksquare\varphi$ by $\varphi \wedge (\varphi \succsim \top)$, $F = \{\varphi \in \mathcal{L}_0 \mid \blacksquare\varphi \in \Gamma\}$, and define a binary relation $\sim$ on $\mathcal{L}_0$ by $\varphi \sim \psi$ iff $(\varphi \leftrightarrow \psi) \in F$.

**Lemma 4.5.** *$F$ contains $\top$ and is closed under deduction in $\mathcal{L}_0$: whenever $\varphi \to \psi \in \mathcal{L}_0$ is a theorem of $\mathsf{IP}(\succsim,\succ)$ and $\varphi \in F$, then $\psi \in F$ too. Also, $\sim$ is an equivalence relation extending the provable equivalence relation on $\mathcal{L}_0$ and is congruential over $\neg, \wedge, \succsim$, and $\succ$: for all $\varphi, \psi, \chi \in \mathcal{L}_0$, if $\varphi \sim \psi$, then $\neg\varphi \sim \neg\psi$, $(\varphi \wedge \chi) \sim (\psi \wedge \chi)$, $(\chi \wedge \varphi) \sim (\chi \wedge \psi)$, $(\varphi \succsim \chi) \sim (\psi \succsim \chi)$, $(\chi \succsim \varphi) \sim (\chi \succsim \psi)$, $(\varphi \succ \chi) \sim (\psi \succ \chi)$, and $(\chi \succ \varphi) \sim (\chi \succ \psi)$.*

*Proof.* When $n = 0$, (**B5**) together with necessitation shows that for every $\varphi$, $\varphi \succsim \varphi$ is a theorem. Then clearly $\top \in F$. To show that $F$ is closed under deduction in $\mathcal{L}_0$, noting that $\Gamma$ is clearly closed under deduction in $\mathcal{L}_0$ due to its being a maximally consistent set, it is enough to show that whenever $\varphi \to \psi \in \mathsf{IP}(\succsim,\succ)$, we have $(\varphi \succsim \top) \to (\psi \succsim \top) \in \mathsf{IP}(\succsim,\succ)$ too. For this, apply (**B5**) to $\langle \varphi, \psi \wedge \neg\varphi, \top \rangle$ and $\langle \top, \bot, \psi \rangle$.

Since $F$ is closed under deduction in $\mathcal{L}_0$ and contains $\top$, $F$ also contains all theorems of $\mathsf{IP}(\succsim,\succ)$ in $\mathcal{L}_0$. Hence it is easy to show that $\sim$ is an equivalence relation extending the provable equivalence relation on $\mathcal{L}_0$ that is congruential over $\neg$ and $\wedge$. To show that $\sim$ is congruential over $\succsim$ and $\succ$, using again that $\Gamma$ is closed under deduction in $\mathcal{L}_0$, we only need to show that the following are derivable:

- $((\varphi \leftrightarrow \psi) \succsim \top) \to ((\varphi \succsim \chi) \leftrightarrow (\psi \succsim \chi))$;
- $((\varphi \leftrightarrow \psi) \succsim \top) \to (((\varphi \succsim \chi) \leftrightarrow (\psi \succsim \chi)) \succsim \top)$;
- $((\varphi \leftrightarrow \psi) \succsim \top) \to ((\varphi \succ \chi) \leftrightarrow (\psi \succ \chi))$;
- $((\varphi \leftrightarrow \psi) \succsim \top) \to (((\varphi \succ \chi) \leftrightarrow (\psi \succ \chi)) \succsim \top)$.

In fact, the second and the fourth follow from the first and the third using (**B7**) to (**B10**), the closure of $(\cdot \succsim \top)$ under deduction, and Boolean reasoning. The first and the third are again simple exercises using (**B5**) and (**B6**), respectively. ☐

**Lemma 4.6.** $\mathcal{B} = \mathcal{L}_0/{\sim}$ is a Boolean algebra expanded with two binary operations which we denote again by $\succsim$ and $\succ$. Moreover, by axioms (**B7**) to (**B10**), for any $a, b \in \mathcal{B}$, $a \succsim b$ is either the top element or the bottom element, and so is $a \succ b$. In addition, $\mathcal{B}$ is finite.

*Proof.* Since $\sim$ is a congruence extending the provable equivalence relation and $\mathsf{IP}(\succsim, \succ)$ has all Boolean reasoning principles, $\mathcal{B}$ is a Boolean algebra. To see that $a \succsim b$ is either the top element or the bottom element, pick any $\varphi, \psi \in \mathcal{L}_0$ such that $[\varphi]_\sim = a$ and $[\psi]_\sim = b$. Then note that either $\varphi \succsim \psi \in \Gamma$ or $\neg(\varphi \succsim \psi) \in \Gamma$. In the former case, given (**B7**), we have $(\varphi \succsim \psi) \in F$ and hence $a \succsim b = [\varphi \succsim \psi]_\sim$ is the top element. In the latter case, using (**B8**), $\neg(a \succsim b)$ is the top element, which means that $a \succsim b$ is the bottom element. The same reasoning goes for $a \succ b$, using (**B9**) and (**B10**). Finally, to see that $\mathcal{B}$ is finite, note first that it has a finite set of generators: $[\mathsf{p}]_\sim = \{[p]_\sim \mid p \in \mathsf{p}\}$. Since we have just shown that $\succsim$ and $\succ$ only bring elements to either the top element or the bottom element, in generating $\mathcal{B}$ from $[\mathsf{p}]_\sim$ we can use only the Boolean operations. Hence the Boolean reduct of $\mathcal{B}$ is a finitely generated Boolean algebra, which must be finite. ☐

Since (the Boolean reduct of) $\mathcal{B}$ is a finite Boolean algebra, it is isomorphic to the powerset algebra of its set of atoms. However, to facilitate the proof of the completeness theorem of the next section, we take the set that includes all possible truth-assignments of propositional variables in $\mathsf{p}$.

**Definition 4.7.** Let $W_\mathsf{p} = \{0, 1\}^\mathsf{p}$ and $V_\mathsf{p} : \mathsf{Prop} \to \wp(W)$ be the natural valuation function defined by $V_\mathsf{p}(p) = \{f \in W_\mathsf{p} \mid f(p) = 1\}$ when $p \in \mathsf{p}$ and $V_\mathsf{p}(p) = \varnothing$ when $p \notin \mathsf{p}$. Finally, let $\mathcal{M}_\mathsf{p} = \langle W_\mathsf{p}, V_\mathsf{p} \rangle$.

In this way, $\wp(W_\mathsf{p})$ is essentially the free Boolean algebra generated by the images of $\mathsf{p}$ under $V_\mathsf{p}$. The difference between $\wp(W_\mathsf{p})$ and the Boolean reduct of $\mathcal{B}$ is that $\mathcal{B}$ might be missing some of the atoms in the sense that some truth-assignments to $\mathsf{p}$ may be inconsistent in $\mathcal{B}$. However, from the probabilistic point of view, it is enough to make them impossible probabilistically by assigning them 0 probability. This gives us the advantage of always using the same $\mathcal{M}_\mathsf{p}$ when satisfying any consistent subset of $\mathcal{L}_0$.

To connect $\mathcal{M}_\mathsf{p}$ to $\mathcal{B}$, first let $\pi$ be the natural Boolean quotient map $\pi$ from $\wp(W_\mathsf{p})$ to $\mathcal{B}_0$ such that $\pi(V_\mathsf{p}(p)) = [p]_\sim$. This map is uniquely given since $\wp(W_\mathsf{p})$ is the free Boolean algebra generated by $\{V(p) \mid p \in \mathsf{p}\}$ and $\mathcal{B}$ is generated by $\{[p]_\sim \mid p \in \mathsf{p}\}$ using Boolean operations. Then, on $\wp(W_\mathsf{p})$, we define two binary relations:

- $X \succsim_\Gamma Y$ iff $\pi(X) \succsim \pi(Y)$ is the top element of $\mathcal{B}$;
- $X \succ_\Gamma Y$ iff $\pi(X) \succ \pi(Y)$ is the top element of $\mathcal{B}$.

Then it is not hard to show the following using the axioms (**B1**) to (**B6**).

**Lemma 4.8.** $\langle \succsim_\Gamma, \succ_\Gamma \rangle$ satisfies all the conditions required in Theorem 2.7.

*Proof.* Note that for every $a \in \mathcal{B}$, $a = [\varphi]_\sim$ for some $\varphi \in \mathcal{L}_0$. Hence any quantification over $\mathcal{B}$, and by the quotient map $\pi$, any quantification over $\wp(W_\mathsf{p})$ as well, can be simulated by quantification over $\mathcal{L}_0$. Since the axioms are schematic, (**B1**) to (**B4**) directly translate the first two bullet points of Theorem 2.7.

For (GFC) and (SGFC), it is enough to note that for any two finite sequences $\langle A_i\rangle_{i=1}^n$ and $\langle B_i\rangle_{i=1}^n$ of sets in $\wp(W_{\mathsf{p}})$ such that $\sum_{i=1}^n \mathbf{1}_{A_i} = \sum_{i=1}^n \mathbf{1}_{B_i}$, we can find two sequences $\langle \varphi_i\rangle_{i=1}^n$ and $\langle \psi_i\rangle_{i=1}^n$ of formulas in $\mathcal{L}_0$ such that:

- for all $i = 1 \dots n$, we have $[\varphi_i]_\sim = \pi(A_i)$ and $[\psi_i]_\sim = \pi(B_i)$, which implies that $A_i \succsim_\Gamma B_i$ iff $\varphi_i \succsim \psi_i \in \Gamma$ and that $A_i \succ_\Gamma B_i$ iff $\varphi_i \succ \psi_i \in \Gamma$;
- $[(\varphi_1, \dots, \varphi_n) \equiv (\psi_i, \dots, \psi_n)]_\sim = [\top]_\sim$ and hence $(\varphi_1, \dots, \varphi_n) \equiv (\psi_i, \dots, \psi_n) \in F$, which in turn implies that $((\varphi_1, \dots, \varphi_n) \equiv (\psi_i, \dots, \psi_n) \succsim \top) \in \Gamma$.

The existence of these formulas means that we can use (**B5**) and (**B6**) to show (GFC) and (SGFC), respectively. □

Hence, by Theorem 2.7, we obtain a set $\mathcal{P}_\Gamma$ of probability measures on $\wp(W_{\mathsf{p}})$ such that

- $X \succsim_\Gamma Y$ iff for all $\mu \in \mathcal{P}_\Gamma$, $\mu(X) \geq \mu(Y)$, and
- $X \succ_\Gamma Y$ iff for all $\mu \in \mathcal{P}_\Gamma$, $\mu(X) > \mu(Y)$.

From this, we can show the following truth lemma.

**Lemma 4.9.** For all $\varphi \in \mathcal{L}_0$, $\pi([\![\varphi]\!]^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}) = [\varphi]_\sim$.

*Proof.* By a simple induction on $\mathcal{L}_0$. The only cases of interest are the inductive steps for $\succsim$ and $\succ$. Note that $[\![\varphi \succsim \psi]\!]^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}$ is either $W_{\mathsf{p}}$ or $\varnothing$. Similarly, we have shown that $[\varphi \succsim \psi]_\sim$ is either $[\top]_\sim$ or $[\bot]_\sim$. Then the only missing connection is the following:

$$\begin{aligned}
[\![\varphi \succsim \psi]\!]^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle} = W_{\mathsf{p}} &\iff \forall \mu \in \mathcal{P}_\Gamma, \mu([\![\varphi]\!]^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}) \geq \mu([\![\psi]\!]^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}) \\
&\iff [\![\varphi]\!]^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle} \succsim_\Gamma [\![\psi]\!]^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle} \\
&\iff (\pi([\![\varphi]\!]^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}) \succsim \pi([\![\psi]\!]^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle})) = [\top]_\sim \\
&\iff ([\varphi]_\sim \succsim [\psi]_\sim) = [\top]_\sim \\
&\iff [\varphi \succsim \psi]_\sim = [\top]_\sim .
\end{aligned}$$

The proof for the case with $\varphi \succ \psi$ is almost identical. □

Now note that $[\gamma]_\sim$ is not the bottom element in $\mathcal{B}$, since otherwise $[\neg\gamma]_\sim$ would be the top element, and then $\blacksquare\neg\gamma \in \Gamma$, which means $\neg\gamma \in \Gamma$ too, rendering $\Gamma$ inconsistent. Hence $[\![\gamma]\!]^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}$ is nonempty because $\pi(\varnothing)$ must be $[\bot]_\sim$, which is not $[\gamma]_\sim$. Take a $w \in [\![\gamma]\!]^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}$. Then $\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle, w \vDash \gamma$, and we are done.

To sum up, we now have the following strengthening of the completeness theorem, noting that there are only finitely many logically inequivalent formulas all using only a finite set $\mathsf{p}$ of propositional variables (see Lemma 6.11).

**Proposition 4.10.** For any finite subset $\mathsf{p}$ of $\mathsf{Prop}$ with $\mathcal{L}_0$ being the set of formulas in $\mathcal{L}(\succsim, \succ)$ using only the propositional variables in $\mathsf{p}$, and for any $\Gamma \subseteq \mathcal{L}_0$ that is consistent relative to $\mathsf{IP}(\succsim, \succ)$, there is a set $\mathcal{P}_\Gamma$ of probability measures on $\wp(W_{\mathsf{p}})$ and a $w \in W_{\mathsf{p}}$ such that $\mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma, w \vDash \gamma$ for all $\gamma \in \Gamma$.

Before we discuss the expressivity of $\mathcal{L}(\succsim, \succ)$, we comment on the logic of *precise* probabilistic models. While $\succ$ is not definable in $\mathcal{L}(\succsim, \succ)$ with respect to all IP models, with respect to precise probabilistic models, $\varphi \succ \psi$ can be defined simply by $\neg(\psi \succsim \psi)$. Hence we can define the logic $\mathsf{SP}(\succsim, \succ)$ as follows.

**Definition 4.11.** The logic $\mathsf{SP}(\succsim, \succ)$ is the smallest subset of $\mathcal{L}(\succsim, \succ)$ that is closed under *modus ponens* (if $\varphi \in \mathsf{SP}(\succsim, \succ)$ and $\varphi \to \psi \in \mathsf{SP}(\succsim, \succ)$, then $\psi \in \mathsf{SP}(\succsim, \succ)$) and *necessitation* (if $\varphi \in \mathsf{SP}(\succsim, \succ)$ then $\varphi \succsim \top \in \mathsf{SP}(\succsim, \succ)$), contains all instances of tautologies of propositional logic, all instances of the axiom schemas (**A1**) to (**A6**) for $\mathsf{SP}(\succsim)$, and all instances of the axiom schema (**A7**) $(\varphi \succ \psi) \leftrightarrow \neg(\psi \succsim \varphi)$.

Then the following completeness theorem for $\mathsf{SP}(\succsim, \succ)$ can be shown in the same way that we just showed the completeness of $\mathsf{IP}(\succsim, \succ)$ using instead the representation result in Theorem 2.1. It will be used in the completeness proof for $\mathsf{IP}(\succsim, \succ, \lozenge)$ in the next section.

**Proposition 4.12.** For any finite subset $\mathsf{p}$ of $\mathsf{Prop}$ with $\mathcal{L}_0$ being the set of formulas in $\mathcal{L}(\succsim, \succ)$ using only the propositional variables in $\mathsf{p}$, and for any $\Gamma \subseteq \mathcal{L}_0$ that is consistent relative to $\mathsf{SP}(\succsim, \succ)$, there is a probability measure $\mu_\Gamma$ on $\wp(W_{\mathsf{p}})$ and a $w \in W_{\mathsf{p}}$ such that $\mathcal{M}_{\mathsf{p}}, \{\mu_\Gamma\}, w \vDash \gamma$ for all $\gamma \in \Gamma$.

## 4.2 Expressivity

In this subsection we discuss the expressivity of $\mathcal{L}(\succsim)$ and $\mathcal{L}(\succsim, \succ)$ in distinguishing IP models. Given Example 2.4, it should not be surprising that $\mathcal{L}(\succsim, \succ)$ is more expressive than $\mathcal{L}(\succ)$. But here we precisely characterize the expressivity of these languages.

**Definition 4.13.** For any probability measure $\mu$ defined on a field $F$ of sets, let $\succsim_\mu$ and $\succ_\mu$ be binary relations on $F$ such that for any $X, Y \in F$, $X \succsim_\mu Y$ iff $\mu(X) \geq \mu(Y)$, and $X \succ_\mu Y$ iff $\mu(X) > \mu(Y)$. In addition, for any set $\mathcal{P}$ of probability measures defined on $F$, let $\succsim_{\mathcal{P}} = \bigcap\{\succsim_\mu \mid \mu \in \mathcal{P}\}$ and $\succ_{\mathcal{P}} = \bigcap\{\succ_\mu \mid \mu \in \mathcal{P}\}$.

For IP models $\langle W, V, \mathcal{P}\rangle$ and $\langle W', V', \mathcal{P}'\rangle$, we say that they are $\succsim$*-order-similar in* $\mathsf{p} \subseteq \mathsf{Prop}$ if for any Boolean formulas $\alpha, \beta$ using only letters in $\mathsf{p}$,

- $[\![\alpha]\!]^{\langle W,V\rangle} \succsim_{\mathcal{P}} [\![\beta]\!]^{\langle W,V\rangle}$ iff $[\![\alpha]\!]^{\langle W',V'\rangle} \succsim_{\mathcal{P}'} [\![\beta]\!]^{\langle W',V'\rangle}$.

We say that they are *order-similar in* $\mathsf{p} \subseteq \mathsf{Prop}$ if in addition to the above biconditional for $\succsim$, it is also true that for any Boolean formulas $\alpha, \beta$ using only letters in $\mathsf{p}$,

- $[\![\alpha]\!]^{\langle W,V\rangle} \succ_{\mathcal{P}} [\![\beta]\!]^{\langle W,V\rangle}$ iff $[\![\alpha]\!]^{\langle W',V'\rangle} \succ_{\mathcal{P}'} [\![\beta]\!]^{\langle W',V'\rangle}$.

A special case for ($\succsim$-)order-similarity is worth mentioning.

**Proposition 4.14.** Let $\langle W, V, \mathcal{P}\rangle$ and $\langle W, V, \mathcal{P}'\rangle$ be IP models and $\mathsf{p}$ a subset of $\mathsf{Prop}$. Let $F$ be the field of sets generated by the image of $\mathsf{p}$ under $V$. Then $\langle W, V, \mathcal{P}\rangle$ and $\langle W, V, \mathcal{P}'\rangle$ are $\succsim$-order-similar (resp. order-similar) in $\mathsf{p}$ iff $\succsim_{\mathcal{P}}|_F = \succsim_{\mathcal{P}'}|_F$ (resp. $\succsim_{\mathcal{P}}|_F = \succsim_{\mathcal{P}'}|_F$ and $\succ_{\mathcal{P}}|_F = \succ_{\mathcal{P}'}|_F$).

**Proposition 4.15.** Let $\langle W, V, \mathcal{P}\rangle$ and $\langle W', V', \mathcal{P}'\rangle$ be IP models and $w, w'$ worlds in $W$ and $W'$, respectively. Then $w$ and $w'$ satisfy the same formulas in $\mathcal{L}(\succsim, \succ)$ (resp. $\mathcal{L}(\succsim)$) using only propositional variables in $\mathsf{p} \subseteq \mathsf{Prop}$ iff

1. $w$ and $w'$ satisfy the same propositional variables in $\mathsf{p}$, and
2. $\langle W, V, \mathcal{P}\rangle$ and $\langle W', V', \mathcal{P}'\rangle$ are order-similar (resp. $\succsim$-order-similar) in $\mathsf{p}$.

*Proof.* The left-to-right direction is trivial since failure of either 1 or 2 directly translates to a formula in the appropriate language with respect to which $w$ and $w'$ disagrees. For the right-to-left direction, note first that any comparative formula $\chi$ of the form $\varphi \succsim \psi$ or $\varphi \succ \psi$ is true at one world iff it is true at all worlds. This means that a formula $\varphi$ with $\chi$ occurring is equivalent to $(\chi \wedge \varphi[\chi/\top]) \vee (\neg\chi \wedge \varphi[\chi/\bot])$ where $\varphi[\chi/\top]$ is the result of replacing $\chi$ by $\top$ in $\varphi$, and similarly for $\varphi[\chi/\bot]$. By repeated use of this method, it is not hard to see that every formula in $\mathcal{L}(\succsim, \succ)$ using only letters in $\mathsf{p}$ is semantically equivalent to a Boolean combination of formulas of one of the following types:

- a propositional variables in $\mathsf{p}$,
- $\alpha \succsim \beta$ where $\alpha$, $\beta$ are Boolean formulas using only letters in $\mathsf{p}$, and
- $\alpha \succ \beta$ where $\alpha$, $\beta$ are Boolean formulas using only letters in $\mathsf{p}$.

The case with $\mathcal{L}(\succsim)$ is similar (without the last kind of formula in the above list). The proposition then follows easily. $\Box$

Now we can translate Example 2.4 to a pair of pointed IP models that $\mathcal{L}(\succsim, \succ)$ can distinguish but $\mathcal{L}(\succsim)$ cannot. Let $W = \{w, v\}$ and $V$ be the valuation such that $V(p) = \{w\}$ and $V(q) = \varnothing$ for all $q \in \mathsf{Prop} \setminus \{p\}$. Let $\mu_{w<v}$ and $\mu_{w=v}$ be defined as in Example 2.4. Then by Propositions 4.15 and 4.14, $\mathcal{L}(\succsim)$ cannot distinguish $\langle W, V, \{\mu_{w<v}, \mu_{w=v}\}\rangle, w$ from $\langle W, V, \{\mu_{w<v}\}\rangle, w$, since $\succsim_{\{\mu_{w<v}, \mu_{w=v}\}}$ and $\succsim_{\{\mu_{w<v}\}}$ are the same on $\wp(W)$. However, $\neg p \succ p$ distinguishes the pointed models.

# 5 The Logic $\mathsf{IP}(\succsim, \succ, \Diamond)$

In this section, we further extend our language with a *possibility modal* $\Diamond$. In the context of natural language semantics, one proposal for the meaning of "possibly $\varphi$" in *precise* probabilistic models is that $\varphi$ has non-zero probability (Lassiter, 2010, §4.4). In imprecise probabilistic models, we could require either (a) that all measures in $\mathcal{P}$ give $\varphi$ non-zero probability or (b) that at least some measure in $\mathcal{P}$ gives $\varphi$ non-zero probability. We adopt the weaker interpretation (b) of "possibly $\varphi$" (not as a proposal in natural language semantics, but because it suits our technical purposes in the next section). In addition to making claims about the possibility of factual states of affairs, e.g., "It is possible that it is raining," we would like to be able to make claims about the possibility of likelihood relations, e.g., "It is possible that hail is more likely than lightning tonight." According to the formal semantics given below, the latter will be true when there exists a probability measure in $\mathcal{P}$ such that *according to that measure* hail is more likely than lightning.

**Definition 5.1.** The language $\mathcal{L}(\succsim, \succ, \Diamond)$ is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \succsim \varphi) \mid (\varphi \succ \varphi) \mid \Diamond\varphi$$

where $p \in \mathsf{Prop}$. We define $\Box\varphi := \neg\Diamond\neg\varphi$.

**Definition 5.2.** We extend the semantics of Definition 4.2 to $\mathcal{L}(\succsim, \succ, \Diamond)$ as follows:

- $\mathcal{M}, \mathcal{P}, w \vDash \Diamond\varphi$ iff there is a $\mu \in \mathcal{P}$ such that $\mu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\mu\}}) \neq 0$.

Note that with $\Diamond$ added, we no longer need $\succ$ as a primitive in the language, since $\varphi \succ \psi$ is definable as $\neg\Diamond(\psi \succsim \varphi)$, but we keep $\succ$ as a primitive for convenience.

In the following, we first present a sound and complete logic for the valid formulas in $\mathcal{L}(\succsim, \succ, \Diamond)$. Then we briefly comment on the logic's complexity. Finally, we show how $\mathcal{L}(\succsim, \succ, \Diamond)$ is more expressive than $\mathcal{L}(\succsim, \succ)$ and characterize the expressivity of $\mathcal{L}(\succsim, \succ, \Diamond)$.

## 5.1 Logic

An important logical fact about the set of valid formulas of $\mathcal{L}(\succsim, \succ, \Diamond)$ is that it is not closed under *uniform substitution* of arbitrary formulas for propositional variables.

**Example 5.3.** The formula $(p \succ \bot) \to \Diamond(p \succ \bot)$ is valid but

$$(\neg((p \succsim q) \vee (q \succsim p)) \succ \bot) \to \Diamond(\neg((p \succsim q) \vee (q \succsim p)) \succ \bot)$$

is not valid. The reason is that there is no single probability measure that can make true the non-comparability formula $\neg((p \succsim q) \vee (q \succsim p))$.

While the failure of uniform substitution can complicate efforts to axiomatize a set of validities (cf. Holliday et al. 2012, 2013), we will completely axiomatize the validities of $\mathcal{L}(\succsim, \succ, \Diamond)$ with the logic $\mathsf{IP}(\succsim, \succ, \Diamond)$ defined below.

**Definition 5.4.** The logic $\mathsf{SP}(\succsim, \succ, \Diamond)$ is the smallest subset of $\mathcal{L}(\succsim, \succ, \Diamond)$ that is (i) closed under modus ponens, uniform substitution, and the rule of replacement of provable equivalents, and (ii) contains all theorems of $\mathsf{SP}(\succsim, \succ)$ and $\Diamond p \leftrightarrow (p \succ \bot)$.

The logic $\mathsf{IP}(\succsim, \succ, \Diamond)$ is the smallest subset of $\mathcal{L}(\succsim, \succ, \Diamond)$ that is (i) closed under modus ponens, the rule of replacement of provable equivalents, and the rule that if $\varphi \in \mathsf{SP}(\succsim, \succ, \Diamond)$, then $\Box\varphi \in \mathsf{IP}(\succsim, \succ, \Diamond)$, and (ii) contains all substitution instances in $\mathcal{L}(\succsim, \succ, \Diamond)$ of the theorems in $\mathsf{IP}(\succsim, \succ)$ and also all instances of the following axiom schemas, where $\alpha$ and $\beta$ are propositional:

(**C1**) $(\Box\varphi \wedge \Box(\varphi \to \psi)) \to \Box\psi$;

(**C2**) $\Diamond\top$;

(**C3**) $\Box\varphi \to (\Box\varphi \succsim \top)$;

(**C4**) $\Diamond\varphi \to (\Diamond\varphi \succsim \top)$;

(**C5**) $\Box\varphi \leftrightarrow \Box(\varphi \succsim \top)$;

(**C6**) $(\alpha \succsim \beta) \leftrightarrow \Box(\alpha \succsim \beta)$;

(**C7**) $(\alpha \succ \beta) \leftrightarrow \Box(\alpha \succ \beta)$.

The rest of this section is devoted to the proof of the following theorem.

**Theorem 5.5** (Soundness and Completeness)**.** *For all $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)$: $\varphi$ is a theorem of $\mathsf{IP}(\succsim, \succ, \Diamond)$ if and only if $\varphi$ is valid with respect to the class of all imprecise probabilistic models.*

To prove Theorem 5.5, we first show that (1) there is no need for a $\Diamond$ to scope over a $\Diamond$ and (2) there is no need for a $\succsim$ or $\succ$ to scope over a $\Diamond$. In other words, we will find a significantly simpler fragment of $\mathcal{L}(\succsim, \succ, \Diamond)$, which we call $\mathcal{L}_{\text{Simp}}$, such that every formula in $\mathcal{L}(\succsim, \succ, \Diamond)$ is provably equivalent to a formula in $\mathcal{L}_{\text{Simp}}$ in $\mathsf{IP}(\succsim, \succ, \Diamond)$.

**Definition 5.6.** Define $T_{-\Diamond} : \mathcal{L}(\succsim, \succ, \Diamond) \to \mathcal{L}(\succsim, \succ)$ by:

- $T_{-\Diamond}(p) = p$ for all $p \in \mathsf{Prop}$;
- $T_{-\Diamond}(\neg\varphi) = \neg T_{-\Diamond}(\varphi)$;
- $T_{-\Diamond}(\varphi \wedge \psi) = T_{-\Diamond}(\varphi) \wedge T_{-\Diamond}(\psi)$;
- $T_{-\Diamond}(\varphi \succsim \psi) = T_{-\Diamond}(\varphi) \succsim T_{-\Diamond}(\psi)$;
- $T_{-\Diamond}(\varphi \succ \psi) = T_{-\Diamond}(\varphi) \succ T_{-\Diamond}(\psi)$;
- $T_{-\Diamond}(\Diamond\varphi) = \neg(\bot \succsim T_{-\Diamond}(\varphi))$.

**Lemma 5.7.** *For every $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)$, $\varphi \leftrightarrow T_{-\Diamond}(\varphi)$ is in $\mathsf{SP}(\succsim, \succ, \Diamond)$. Moreover, $T_{-\Diamond}(\varphi)$ uses the same propositional variables as $\varphi$ does.*

*Proof.* A simple induction with repeated use of replacement of equivalents suffices. □

**Lemma 5.8.** *In $\mathsf{IP}(\succsim, \succ, \Diamond)$, formulas of the form $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ are theorems. In addition, $\Box$ is a normal operator: for any $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)$, $(\Box\varphi \wedge \Box(\varphi \to \psi)) \to \Box\psi$ is in $\mathsf{IP}(\succsim, \succ, \Diamond)$, and whenever $\varphi$ is in $\mathsf{IP}(\succsim, \succ, \Diamond)$, so is $\Box\varphi$.*

*Proof.* To derive $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$, it is enough to derive $\Diamond\varphi \leftrightarrow \Diamond\neg\neg\varphi$. But this is clearly derivable by replacement of equivalents since $\Diamond\varphi \leftrightarrow \Diamond\varphi$ and $\varphi \leftrightarrow \neg\neg\varphi$ are theorems. □

**Definition 5.9.** Let $\mathcal{L}_{\text{Simp}}$ be the fragment of $\mathcal{L}(\succsim, \succ, \Diamond)$ generated from $\mathsf{Prop}$ and $\{\Diamond\varphi \mid \varphi \in \mathcal{L}(\succsim, \succ)\}$ by $\neg$ and $\wedge$.

In the following, for any $\mathsf{p} \subseteq \mathsf{Prop}$, we append $[\mathsf{p}]$ to the name of a language to denote the set of formulas in that language using only variables in $\mathsf{p}$.

**Lemma 5.10.** For every $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)$, there is a $T(\varphi) \in \mathcal{L}_{\mathrm{Simp}}$ such that $\varphi \leftrightarrow T(\varphi) \in \mathsf{IP}(\succsim, \succ, \Diamond)$. Moreover, $T(\varphi)$ and $\varphi$ use the same propositional variables.

*Proof.* By induction on $\mathcal{L}(\succsim, \succ, \Diamond)$. The base case is trivial: we can simply define $T(p) = p$. The Boolean cases are also trivial: we can define $T(\neg\varphi) = \neg T(\varphi)$ and $T(\varphi \wedge \psi) = T(\varphi) \wedge T(\psi)$. For the $\Diamond$ case, define $T(\Diamond\varphi) = \Diamond T_{-\Diamond}(\varphi)$. To see that $\Diamond\varphi$ is provably equivalent to $\Diamond T_{-\Diamond}(\varphi)$, first note that by Lemma 5.7, $\varphi \leftrightarrow T_{-\Diamond}(\varphi) \in \mathsf{SP}(\succsim, \succ, \Diamond)$. But then $\Box(\varphi \leftrightarrow T_{-\Diamond}(\varphi)) \in \mathsf{IP}(\succsim, \succ, \Diamond)$. By the normality of $\Box$, we have $\Diamond\varphi \leftrightarrow \Diamond T_{-\Diamond}(\varphi) \in \mathsf{IP}(\succsim, \succ, \Diamond)$.

To find the appropriate $T(\varphi \succsim \psi)$, given that the required $T(\varphi)$ and $T(\psi)$ in $\mathcal{L}_{\mathrm{Simp}}$ have been found, we need to extract all $\Diamond$'ed formulas in $T(\varphi) \succsim T(\psi)$ so that they are no longer in the scope of the main connective $\succsim$ in $T(\varphi) \succsim T(\psi)$. Clearly this can be done by iteratively using the following claim:

(*) for any $\chi \in \mathcal{L}(\succsim, \succ)$ and $\varphi, \psi \in \mathcal{L}_{\mathrm{Simp}}$,

$$(\varphi \succsim \psi) \leftrightarrow (\Diamond\chi \wedge (\varphi[\Diamond\chi/\top] \succsim \psi[\Diamond\chi/\top])) \vee (\neg\Diamond\chi \wedge (\varphi[\Diamond\chi/\bot] \succsim \psi[\Diamond\chi/\bot]))$$

is in $\mathsf{IP}(\succsim, \succ, \Diamond)$.

The claim is easily proved using (**C3**) and (**C4**). Note that since $\varphi, \psi$ are in $\mathcal{L}_{\mathrm{Simp}}$, they are Boolean combinations of propositional variables and formulas of the form $\Diamond\chi$ where $\chi \in \mathcal{L}(\succsim, \succ)$. List all the $\Diamond$'ed formulas appearing in $\varphi$ or $\psi$ as $\delta_1, \delta_2, \ldots, \delta_n$. Then for any $f \in \{0,1\}^n$, let $\delta_f$ be $\bigwedge_{i=1}^{n} \neg^{f(i)}\delta_i$ where $\neg^0\delta_i$ is $\neg\delta_i$ and $\neg^1\delta_i$ is simply $\delta_i$. Moreover, let $\varphi[f]$ be $\varphi[\delta_1/\top^{f(1)}, \cdots, \delta_n/\top^{f(n)}]$ and similarly for $\psi[f]$, where $\top^{f(i)} = \top$ if $f(i) = 1$ and $\top^{f(i)} = \bot$ if $f(i) = 0$. With this notation, it is not hard to see that by repeatedly applying (*), $\varphi \succsim \psi$ is provably equivalent to $\bigvee_{f \in \{0,1\}^n}(\delta_f \wedge (\varphi[f] \succsim \psi[f]))$ and then also to $\bigvee_{f \in \{0,1\}^n}(\delta_f \wedge \Box(\varphi[f] \succsim \psi[f]))$ since for any $f$, $\varphi[f]$ and $\psi[f]$ are propositional since we have replaced all the $\Diamond$'ed formulas by either $\top$ or $\bot$ and by axiom (**C6**) we can add a $\Box$ there. The formula $\bigvee_{f \in \{0,1\}^n}(\delta_f \wedge \Box(\varphi[f] \succsim \psi[f]))$ is the desired $T(\varphi \succsim \psi)$ since it is clearly in $\mathcal{L}_{\mathrm{Simp}}$ now. The definition of $T(\varphi \succ \psi)$ is almost identical: we can simply replace $\varphi[f] \succsim \psi[f]$ by $\varphi[f] \succ \psi[f]$. In this case, we use (**C7**) instead. $\Box$

Now we are ready to prove the soundness and completeness of $\mathsf{IP}(\succsim, \succ, \Diamond)$. Soundness is clear as usual. For completeness, pick an arbitrary $\gamma$ that is consistent relative to $\mathsf{IP}(\succsim, \succ, \Diamond)$, and let $\mathsf{p}$ be the set of propositional variables used in $\gamma$. Then take an arbitrary $\Gamma$ that is maximally consistent containing $\gamma$. Following the standard strategy, let $\Sigma = \{(\varphi \succsim \top) \mid \Box\varphi \in \Gamma, \varphi \in \mathcal{L}(\succsim, \succ)[\mathsf{p}]\}$. Note that $\Sigma \subseteq \mathcal{L}(\succsim, \succ)[\mathsf{p}]$. Also, $\Sigma$ must be consistent relative to $\mathsf{SP}(\succsim, \succ)$ since otherwise there are formulas $(\varphi_1 \succsim \top), (\varphi_2 \succsim \top), \ldots, (\varphi_n \succsim \top)$ in $\Sigma$ such that $((\varphi_1 \succsim \top) \wedge \cdots \wedge (\varphi_n \succsim \top)) \to \bot$ is in $\mathsf{SP}(\succsim, \succ)$. But then by the rules of $\mathsf{IP}(\succsim, \succ, \Diamond)$ and the normality of $\Box$, we have that $(\Box(\varphi_1 \succsim \top) \wedge \cdots \wedge \Box(\varphi_n \succsim \top)) \to \Box\bot$ is in $\mathsf{IP}(\succsim, \succ, \Diamond)$. Since $\Box\varphi$ is provably equivalent to $\Box(\varphi \succsim \top)$ by (**C5**), we have that $\Box\bot$ is in $\Gamma$ according to the maximality of $\Gamma$, rendering $\Gamma$ inconsistent since we have (**C2**).

Now let $\mathcal{D} = \{\Sigma \cup \{\neg(\varphi \succsim \top)\} \mid \neg\Box\varphi \in \Gamma, \varphi \in \mathcal{L}(\succsim, \succ)[\mathsf{p}]\}$. Note that for each $\Delta = \Sigma \cup \{\neg(\varphi \succsim \top)\} \in \mathcal{D}$, $\Delta$ is also a set of formulas in $\mathcal{L}(\succsim, \succ)[\mathsf{p}]$. Moreover, $\Delta$ must be consistent relative to $\mathsf{SP}(\succsim, \succ)$ as well. If not, then since $\Sigma$ is consistent, we must have formulas $(\varphi_1 \succsim \top), \ldots, (\varphi_n \succsim \top)$ in $\Sigma$ such that $((\varphi_1 \succsim \top) \wedge \cdots \wedge (\varphi_n \succsim \top)) \to (\varphi \succsim \top) \in \mathsf{SP}(\succsim, \succ)$. Then by reasoning similar to that above, $\Box(\neg\varphi \succsim \top)$ and hence $\Box\neg\varphi$ are in $\Gamma$ using (**C5**), rendering $\Gamma$ inconsistent.

Thus, for each $\Delta \in \mathcal{D}$, according to Proposition 4.12, there is a probability measure $\mu_\Delta$ on $\wp(W_{\mathsf{p}})$ and a $w \in W_{\mathsf{p}}$ such that $\mathcal{M}_{\mathsf{p}}, \{\mu_\Delta\}, w \vDash \Delta$. Note that since all formulas in $\Delta$ are comparison formulas of the form $\varphi \succsim \top$ or its negation, it does not matter what $w$ is. Hence we have that $\mathcal{M}_{\mathsf{p}}, \{\mu_\Delta\} \vDash \Delta$. Take $\mathcal{P}$ to be the set $\{\mu_\Delta \mid \Delta \in \mathcal{D}\}$. Then we are left only to show that there is a $w \in W_{\mathsf{p}}$ such that $\mathcal{M}_{\mathsf{p}}, \mathcal{P}, w \vDash \varphi$ for all $\varphi \in \Gamma \cap \mathcal{L}(\succsim, \succ, \Diamond)[\mathsf{p}]$.

Let $w_0$ be the element in $W_{\mathsf{p}} = \{0,1\}^{\mathsf{p}}$ defined by $w_0(p) = 1$ iff $p \in \Gamma$ for all $p \in \mathsf{p}$. Then we are ready to show the following truth lemma.

**Lemma 5.11.** For all $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)[\mathsf{p}]$, $\mathcal{M}_{\mathsf{p}}, \mathcal{P}, w_0 \vDash \varphi$ iff $\varphi \in \Gamma$.

*Proof.* It is enough to show that for all $\varphi \in \mathcal{L}_{\text{Simp}}[\mathsf{p}]$, $\mathcal{M}_{\mathsf{p}}, \mathcal{P}, w_0 \vDash \varphi$ iff $\varphi \in \Gamma$. This is because for any $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)[\mathsf{p}]$, according to Lemma 5.10, $\varphi \in \Gamma$ iff $T(\varphi) \in \Gamma$ with $T(\varphi) \in \mathcal{L}_{\text{Simp}}[\mathsf{p}]$. But then

$$T(\varphi) \in \Gamma \iff \mathcal{M}_{\mathsf{p}}, \mathcal{P}, w_0 \vDash T(\varphi) \iff \mathcal{M}_{\mathsf{p}}, \mathcal{P}, w_0 \vDash \varphi.$$

The first equivalence holds by the fact that $T(\varphi) \in \mathcal{L}_{\text{Simp}}[\mathsf{p}]$ and the truth lemma we will show below in this fragment. The second is by soundness.

We now focus on the fragment $\mathcal{L}_{\text{Simp}}[\mathsf{p}]$. Since the generating operations of this fragment are Boolean, the inductive cases are trivial. The atomic case for propositional variables in $\mathsf{p}$ is also trivial by the definition of $w_0$. Hence we are left to show that for any $\varphi \in \{\Diamond\psi \mid \psi \in \mathcal{L}(\succsim, \succ)[\mathsf{p}]\}$, we have $\varphi \in \Gamma$ iff $\mathcal{M}_{\mathsf{p}}, \mathcal{P}, w_0 \vDash \varphi$. In other words, we only need to show that for all $\varphi \in \mathcal{L}(\succsim, \succ)[\mathsf{p}]$, we have $\Diamond\varphi \in \Gamma$ iff $\mathcal{M}_{\mathsf{p}}, \mathcal{P}, w_0 \vDash \Diamond\varphi$.

- Suppose $\Diamond\varphi \notin \Gamma$, so $\Box\neg\varphi \in \Gamma$. Then $(\neg\varphi \succsim \top) \in \Sigma$ since $\neg\varphi \in \mathcal{L}(\succsim, \succ)[\mathsf{p}]$, which means $(\neg\varphi \succsim \top) \in \Delta$ for all $\Delta \in \mathcal{D}$. Then, for any $\mu_\Delta \in \mathcal{P}$, $\mathcal{M}_{\mathsf{p}}, \{\mu_\Delta\} \vDash \neg\varphi \succsim \top$ since $(\neg\varphi \succsim \top) \in \Delta$, which in turn means that $\mu_\Delta([\![\varphi]\!]^{\mathcal{M}_{\mathsf{p}}, \{\mu_\Delta\}}) = 0$. This is precisely the condition for $\Diamond\varphi$ to be false at $\mathcal{M}_{\mathsf{p}}, \mathcal{P}, w_0$.
- Suppose $\Diamond\varphi \in \Gamma$, so $\neg\Box\neg\varphi \in \Gamma$. Then there is a $\Delta$ such that $\neg(\neg\varphi \succsim \top) \in \Delta$ again because $\neg\varphi \in \mathcal{L}(\succsim, \succ)[\mathsf{p}]$. For this $\mu_\Delta$ then, $\mathcal{M}_{\mathsf{p}}, \{\mu_\Delta\} \nvDash \neg\varphi \succsim \top$. In other words, $\mu_\Delta([\![\varphi]\!]^{\mathcal{M}_{\mathsf{p}}, \{\mu_\Delta\}}) \neq 0$. The existence of this $\mu_\Delta \in \mathcal{P}$ shows that $\Diamond\varphi$ is true at $\mathcal{M}_{\mathsf{p}}, \mathcal{P}, w_0$. □

Given the above truth lemma, $\mathcal{M}_{\mathsf{p}}, \mathcal{P}, w_0 \vDash \gamma$ since $\gamma \in \Gamma$ and $\gamma \in \mathcal{L}(\succsim, \succ, \Diamond)[\mathsf{p}]$. Hence we have successfully found a model for the arbitrarily chosen consistent $\gamma$, completing the proof of the completeness of $\mathsf{IP}(\succsim, \succ, \Diamond)$.

## 5.2 Complexity

In this section, we briefly comment on the complexity of the consistency problem of the logic $\mathsf{IP}(\succsim, \succ, \Diamond)$ or equivalently the satisfiability problem of $\mathcal{L}(\succsim, \succ, \Diamond)$. First, adapting the proof of Theorem 9 in Harrison-Trainor et al. 2017, it is not hard to see that the satisfiability problem for a conjunction of literals where we take formulas in both $\mathsf{Prop}$ and $\{\Diamond\varphi \mid \varphi \in \mathcal{L}(\succsim, \succ)\}$ as atomic formulas is in $\mathsf{NP}$ (note that Theorem 2.6 in Fagin et al. 1990, used in the proof of Harrison-Trainor et al. 2017, allows strict inequalities). Hence the satisfiability problem for $\mathcal{L}_{\text{Simp}}$ is also in $\mathsf{NP}$. Then to see that the satisfiability problem for $\mathcal{L}(\succsim, \succ, \Diamond)$ is in $\mathsf{NP}$, it is enough to show that every $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)$ is equivalent to a disjunction of formulas in $\mathcal{L}_{\text{Simp}}$ where each disjunct's length is bounded by $O(|\varphi|)$. In our proof of Lemma 5.10 above, this is done by extracting $\Diamond$ from the scope of $\succsim$ and $\succ$ and eliminating $\Diamond$ in the scope of $\Diamond$. Note that the elimination of $\Diamond$ in the scope of $\Diamond$ can be done before the extraction: given an input formula $\varphi$, replace each subformula $\Diamond\chi$ not in the scope of any $\Diamond$ by $\Diamond T_{-\Diamond}(\chi)$. The resulting formula, which we call $\varphi'$, is clearly at most four times longer than $\varphi$. Then we only need to run the process of (1) extracting $\Diamond$'ed formulas in the scope of $\succsim$ or $\succ$ and (2) adding a $\Box$ to a $\succsim$ formula or a $\succ$ formula when both arguments to the $\succsim$ or $\succ$ no longer contain modal operators. This process, while introducing disjunctions exponentially, only grows the length of the disjuncts by at most a constant for each extracting operation. The number of total extracting operations is clearly at most the length of the input formula $\varphi'$. Thus, we obtain the following.

**Theorem 5.12.** The complexity of the satisfiability problem for $\mathcal{L}(\succsim, \succ, \Diamond)$ is $\mathsf{NP}$-complete.

## 5.3 Expressivity

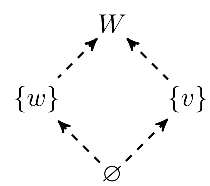
Reflecting the failure of uniform substitution, for any purely propositional formula $\alpha$, $\Diamond\alpha$ is already expressible in $\mathcal{L}(\succsim)$.

**Lemma 5.13.** Let $\alpha, \beta$ be propositional formulas. Then:

1. $\Diamond\alpha$ is equivalent to $\neg(\bot \succsim \alpha)$;
2. $\Diamond(\alpha \succsim \beta)$ and $\Diamond\neg(\beta \succ \alpha)$ are both equivalent to $\neg(\beta \succ \alpha)$;
3. $\Diamond(\beta \succ \alpha)$ and $\Diamond\neg(\alpha \succsim \beta)$ are both equivalent to $\neg(\alpha \succsim \beta)$.

However, $\Diamond\varphi$ is not in general expressible without $\Diamond$.

**Example 5.14.** The formula $\Diamond(p \approx \neg p)$ is not equivalent to any formula of $\mathcal{L}(\succsim, \succ)$. Consider again the propositional model $\mathcal{M} = \langle W, V\rangle$ where $W = \{w, v\}$ and $V(p) = \{w\}$ while $V(q) = \varnothing$ for all $q \in \mathsf{Prop} \setminus \{p\}$. Then let $\mathcal{P}$ be the set of all probability measures on $\wp(W)$ and $\mathcal{P}'$ the set of all probability measure $\mu$ on $\wp(W)$ except the ones that give equal probability to $\{w\}$ and $\{v\}$. Then $\succsim_{\mathcal{P}}$ and $\succsim_{\mathcal{P}'}$ (resp. $\succ_{\mathcal{P}}$ and $\succ_{\mathcal{P}'}$) are the same on $\wp(W)$ and are pictured below:




Thus, using Propositions 4.15 and 4.14, for any $\varphi \in \mathcal{L}(\succsim, \succ)$, $\mathcal{M}, \mathcal{P}, w \vDash \varphi$ iff $\mathcal{M}, \mathcal{P}', w \vDash \varphi$. Yet $\mathcal{M}, \mathcal{P}, w \vDash \Diamond(p \approx \neg p)$ while $\mathcal{M}, \mathcal{P}', w \nvDash \Diamond(p \approx \neg p)$.

Now we characterize the expressivity of $\mathcal{L}(\succsim, \succ, \Diamond)$ precisely.

**Proposition 5.15.** Let $\langle W, V, \mathcal{P}\rangle$ and $\langle W', V', \mathcal{P}'\rangle$ be IP models and $w, w'$ worlds in $W$ and $W'$, respectively. Let $\mathsf{p}$ be a subset of $\mathsf{Prop}$. Then $w$ and $w'$ satisfy the same formulas in $\mathcal{L}(\succsim, \succ, \Diamond)$ using only propositional variables in $\mathsf{p}$ if

1. $w$ and $w'$ satisfy the same propositional variables in $\mathsf{p}$,
2. for any $\mu \in \mathcal{P}$, there is $\mu' \in \mathcal{P}'$ such that $\langle W, V, \{\mu\}\rangle$ and $\langle W', V', \{\mu'\}\rangle$ are order-similar in $\mathsf{p}$, and
3. for any $\mu' \in \mathcal{P}'$, there is $\mu \in \mathcal{P}$ such that $\langle W, V, \{\mu\}\rangle$ and $\langle W', V', \{\mu'\}\rangle$ are order-similar in $\mathsf{p}$.

The converse also holds if in addition $\mathsf{p}$ is finite.

*Proof.* The left-to-right direction is again easy. For the only non-obvious case, suppose for example that the second clause fails: there is a $\mu \in \mathcal{P}$ such that for any $\mu' \in \mathcal{P}'$, $\langle W, V, \{\mu\}\rangle$ and $\langle W', V', \{\mu'\}\rangle$ are not order-similar in $\mathsf{p}$. Then let $\{\alpha_i\}_{1\leq i\leq n}$ be a finite set of Boolean formulas such that every Boolean formula using only letters in $\mathsf{p}$ is logically equivalent to some $\alpha_i$ (such a set can be found using disjunctive normal forms). We can now describe $\mu$ in full relative to $\mathsf{p}$ by the conjunction $\chi = \bigwedge_{1\leq i,j\leq n} s_i(\alpha_i \succsim \alpha_j)$ where $s_i$ is empty if $\mu([\![\alpha_i]\!]^{\langle W,V\rangle}) \succsim \mu([\![\alpha_j]\!]^{\langle W,V\rangle})$ and is $\neg$ otherwise. Indeed, by the definition of order-similarity, whenever $\langle W, V, \{\mu\}\rangle$ and $\langle W', V', \{\mu'\}\rangle$ are not order-similar in $\mathsf{p}$, at any world in $W'$, $\chi$ is false. This means that $w'$ would falsify $\Diamond\mu$, but $w$ satisfies $\Diamond\mu$, showing that the two worlds disagree on a formula in $\mathcal{L}(\succsim, \succ, \Diamond)$.

The right-to-left direction follows from the normal form lemma, Lemma 5.10. If the last two clauses hold, then for any formula of the form $\Diamond\varphi$ where $\varphi \in \mathcal{L}(\succsim, \succ)[\mathsf{p}]$, $\Diamond\varphi$ is true at $\mathcal{M}, \mathcal{P}, w$ iff it is true at $\mathcal{M}, \mathcal{P}', w'$. By the first clause, the two pointed IP models also satisfy the same propositional variables in $\mathsf{p}$. Then by a simple induction, they satisfy the same formulas in $\mathcal{L}_{\text{Simp}}[\mathsf{p}]$. But by Lemma 5.10, this is enough for them to satisfy the same formulas in $\mathcal{L}(\succsim, \succ, \Diamond)[\mathsf{p}]$. $\square$

The special case where the two IP models share the same propositional model is again worth spelling out.

**Proposition 5.16.** Let $\mathcal{M} = \langle W, V \rangle$ be a propositional model, $w$ and $w'$ two worlds in $W$, and $\mathcal{P}$ and $\mathcal{P}'$ nonempty sets of probability measures defined on fields of sets extending $V[\mathsf{Prop}]$. Let $\mathsf{p}$ be a subset of $\mathsf{Prop}$ and $F$ the field of sets on $W$ generated by $V[\mathsf{p}]$. Then $\mathcal{M}, \mathcal{P}, w$ and $\mathcal{M}, \mathcal{P}', w'$ satisfy the same formulas in $\mathcal{L}(\succsim, \succ, \Diamond)[\mathsf{p}]$ if

- $w$ and $w'$ satisfy the same propositional variables in $\mathsf{p}$,
- for any $\mu \in \mathcal{P}$, there is $\mu' \in \mathcal{P}'$ such that $\succsim_\mu|_F = \succsim_{\mu'}|_F$, and
- for any $\mu' \in \mathcal{P}'$, there is $\mu \in \mathcal{P}$ such that $\succsim_\mu|_F = \succsim_{\mu'}|_F$.

The converse also holds if in addition $\mathsf{p}$ is finite.

# 6 Dynamics

In this section, we consider two kinds of information dynamics in the context of imprecise probability. The first is a standard notion of *updating* a set of probability measure on new evidence (see, e.g., Halpern 2003, p. 81) where we can eliminate both possible worlds (keeping only the worlds compatible with the evidence) and probability measures (keeping only the probability measures that give the evidence a positive probability measure). Usually, especially in a Bayesian framework, such updates are all we need for information dynamics, since we can always model agents with a universal and all-inclusive state space, anticipating all distinctions that could be made among states. However, there are numerous examples where an agent is not initially aware of a distinction. In Example 1.1, the agent is not initially aware of the gland and hence the distinction between a swollen and normal gland. When the doctor tells the agent about the gland, we can model the agent as first learning the mere existence of a new proposition—the swollen gland proposition—and then learning how this proposition relates probabilistically to her having the disease. Without imprecise probability, we face the perennial question of how to assign a probability for such a new proposition. Given imprecise probability, however, we can simply choose the set of all probability measures that are compatible with one of the old probability measures. This models how an agent can "initialize" her uncertainty toward a newly introduced proposition.

In the next two subsections, we discuss the two dynamic operators in more detail. For the update operator, we show how it does not add expressivity to the language $\mathcal{L}(\succsim, \succ, \Diamond)$, and we present a sound and complete logic following the standard "reduction axiom" strategy in dynamic epistemic logic. For the operators modeling the introduction of new propositions, however, we show that they significantly increase expressivity, and we leave the axiomatization of the valid formulas as an open question.

## 6.1 Updating Probabilities and the Logic $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$

In this subsection, we introduce the update operator $\langle\,\rangle$ that models learning the truth of a proposition. Given an initial set $\mathcal{P}$ of probability measures, after learning some proposition $U \subseteq W$ with certainty, we update the set $\mathcal{P}$ to the set

$$\mathcal{P}_U = \{\mu(\cdot \mid U) : \mu \in \mathcal{P}, \mu(U) > 0\},$$

where $\mu(\cdot \mid U)$ is defined by conditionalization as usual: for any $V \subseteq W$, $\mu(V \mid U) = \frac{\mu(V \cap U)}{\mu(U)}$.

Since we have a formal language with comparative probability operators, we can model updating on sentences containing not only factual formulas but also comparative probability formulas (cf. Weatherson 2007; Yalcin 2011; Moss 2018), as in "it is raining, and it is more likely that there will be hail than it is that there will be lightning" $(r \wedge (h \succ \ell))$. Intuitively, if Ann tells Bob that "hail is more likely than lightning," she is not telling Bob something about *his own* epistemic state (which he already knows, in the models of this paper) but is rather recommending that he update his epistemic state to one according to which hail is more likely than lightning—which he can do by discarding from his set of measures any measure according to which hail is not more likely than lightning.[5] Our semantics below, developed in the style of *dynamic epistemic logic* (see, e.g., van Ditmarsch et al. 2008; van Benthem 2011), will allow such updates in response to comparative probability claims.

**Definition 6.1.** The language $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \succsim \varphi) \mid (\varphi \succ \varphi) \mid \Diamond\varphi \mid \langle\varphi\rangle\varphi$$

where $p \in \mathsf{Prop}$. We read $\langle\alpha\rangle\varphi$ as "(update with $\alpha$ is possible and) after update with $\alpha$, $\varphi$ is the case." As usual, $[\alpha]\varphi$ abbreviates $\neg\langle\alpha\rangle\neg\varphi$.

**Definition 6.2.** We extend the semantics of Definition 5.2 to $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ as follows:

- $\mathcal{M}, \mathcal{P}, w \vDash \langle\varphi\rangle\psi$ iff there is a $\mu \in \mathcal{P}$ such that $\mu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\mu\}}) \neq 0$ and $\mathcal{M}, \mathcal{P}_\varphi, w \vDash \psi$,

where

$$\mathcal{P}_\varphi = \{\nu(\cdot \mid \llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) : \nu \in \mathcal{P} \text{ and } \nu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) \neq 0\}.$$

**Lemma 6.3.** The semantics for $[\varphi]\psi$ is as follows:

- $\mathcal{M}, \mathcal{P}, w \vDash [\varphi]\psi$ iff if there is a $\mu \in \mathcal{P}$ such that $\mu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\mu\}}) \neq 0$, then $\mathcal{M}, \mathcal{P}_\varphi, w \vDash \psi$.

The following lemma states how updating with a formula $\varphi \succsim \psi$, if possible, results in restricting one's set of measures to just those that individually satisfy $\varphi \succsim \psi$.

**Lemma 6.4.** For any IP model $\langle\mathcal{M}, \mathcal{P}\rangle$ and $\varphi, \psi \in \mathcal{L}(\succsim, \succ, \Diamond)$, $\mathcal{P}_{\varphi \succsim \psi} = \varnothing$ or

$$\mathcal{P}_{\varphi \succsim \psi} = \{\nu \in \mathcal{P} : \mathcal{M}, \{\nu\} \vDash \varphi \succsim \psi\}.$$

Let us see how this framework can be used to formalize the three prisoners scenario from Example 1.2.

**Example 6.5.** Let $e_i$ and $s_i$ stand for 'prisoner $i$ will be *executed*' and 'the jailer *says* that prisoner $i$ will be executed', respectively. Define a propositional model $\mathcal{M} = \langle W, V\rangle$ with

$$W = \{w_{ab}, w_{ac}, w_{bc}, w_{cb}\}$$

where at $w_{ij}$, prisoner $i$ is the only prisoner who lives and prisoner $j$ is the prisoner who the jailer says will be executed, so

$$V(e_a) = \{w_{bc}, w_{cb}\}, V(e_b) = \{w_{ab}, w_{ac}, w_{cb}\}, V(e_c) = \{w_{ab}, w_{ac}, w_{bc}\},$$

$$V(s_b) = \{w_{ab}, w_{cb}\}, V(s_c) = \{w_{ac}, w_{bc}\}.$$

Since prisoner $a$ knows that each prisoner is equally likely to be executed but has no idea about how the jailer is likely to answer his question about which of $b$ or $c$ will be executed (except that the jailer is certain to give a true answer), prisoner $a$'s epistemic state may be modelled by the following set of probability measures:

$$\mathcal{P} = \{\mu : \mu(\{w_{ab}, w_{ac}\}) = \mu(\{w_{bc}\}) = \mu(\{w_{cb}\}) = 1/3\}.$$

[5] Another possible interpretation is that there is some objectively correct probability measure, and Ann is telling Bob a fact about that measure, which he wants his probabilities to ultimately match.

Then the following formulas together capture what is distinctive about the puzzle, all coming out true in this model. First, we can state that each prisoner is equally likely to be spared—indeed that each has one-third chance:

$$\alpha := \big(\bot \gtrsim (e_a \wedge e_b \wedge e_c)\big) \wedge \big(((e_a \wedge e_b) \vee (e_a \wedge e_c) \vee (e_b \wedge e_c)) \gtrsim \top\big) \wedge (e_a \approx e_b) \wedge (e_b \approx e_c).$$

Second, we can state that the jailer only announces truthfully one of $s_b$ and $s_c$:

$$\beta := ((s_b \to e_b) \gtrsim \top) \wedge ((s_c \to e_c) \gtrsim \top) \wedge (\bot \gtrsim (s_b \wedge s_c)).$$

Given the dynamic operator, we can also express a fact about how $a$'s uncertainty is affected upon learning that $b$ is to be executed. After this announcement, $a$'s credences *dilate* from a sharp two-thirds probability to including the possibilities that he is sure to be executed and that he has merely one-half probability of being executed:

$$\langle s_b \rangle \big(\lozenge(e_a \gtrsim \top) \wedge \lozenge(e_a \approx \neg e_a)\big).$$

If, however, $a$ first updates with the information that the jailer is following a protocol of reporting $b$ or reporting $c$ with equal probability in the case that $a$ is to be spared, then dilation no longer occurs. In fact, the probability of $e_a$ remains at two-thirds, and for instance the following formula is true:

$$\langle (\neg e_a \wedge s_b) \approx (\neg e_a \wedge s_c) \rangle \langle s_b \rangle \big((e_a \succ e_c) \wedge (e_a \succ \neg e_a) \wedge (\top \succ e_a)\big).$$

Finally, were $a$ to update with the information that the jailer would certainly announce $e_b$ in case $e_a$ were false, then the probabilities of $e_a$, $e_b$, and $e_c$ would all remain equally likely:

$$\langle \bot \gtrsim (\neg e_a \wedge s_c) \rangle \alpha.$$

But after learning that $b$ will be executed, the probability of $e_a$ decreases to one-half:

$$\langle \bot \gtrsim (\neg e_a \wedge s_c) \rangle \langle s_b \rangle (e_a \approx \neg e_a).$$

It is important to note that we do not have to resort to the particular model above to model the prisoner case. Indeed, the following formulas are true at any pointed IP model and hence also provable in the complete logic to be presented:

$$(\alpha \wedge \beta) \to [(\neg e_a \wedge s_b) \approx (\neg e_a \wedge s_c)] \langle s_b \rangle \big((e_a \succ e_c) \wedge (e_a \succ \neg e_a) \wedge (\top \succ e_a)\big) \tag{2}$$

$$(\alpha \wedge \beta) \to [\bot \gtrsim (\neg e_a \wedge s_c)](\alpha \wedge \langle s_b \rangle (e_a \approx \neg e_a)) \tag{3}$$

$$\begin{aligned}&(\alpha \wedge \beta) \to \\ &\quad ((\lozenge(\bot \gtrsim (\neg e_a \wedge s_b)) \wedge \lozenge(\bot \gtrsim (\neg e_a \wedge s_c))) \to \langle s_b \rangle (\lozenge(e_a \gtrsim \top) \wedge \lozenge(e_a \approx \neg e_a)))\,.\end{aligned} \tag{4}$$

In (2) and (3), we have to use $[\,]$ instead of $\langle\,\rangle$ since there are models that satisfy $\alpha \wedge \beta$ but do not contain probability measures satisfying either $(\neg e_a \wedge s_b) \approx (\neg e_a \wedge s_c)$ or $\bot \gtrsim (\neg e_a \wedge s_c)$, unlike the particular model above using the all-inclusive $\mathcal{P}$. To cope with this, we need to use the box version of the update operator. In formula (4), the extra premise $\lozenge(\bot \gtrsim (\neg e_a \wedge s_b)) \wedge \lozenge(\bot \gtrsim (\neg e_a \wedge s_c))$ is again required since dilation crucially relies on $\mathcal{P}$ containing both a measure assigning 0 to $\neg e_a \wedge s_b$ and a measure assigning 0 to $\neg e_a \wedge s_c$. In our current language, using the $\lozenge$ operator is the most straightforward way to express this. An equivalent way is to use $\neg((\neg e_a \wedge s_b) \succ \bot) \wedge \neg((\neg e_a \wedge s_b) \succ \bot)$. However, the $\lozenge$ in $\lozenge(e_a \approx \neg e_a)$ is necessary: there is no formula in $\mathcal{L}(\gtrsim, \succ)$ that is equivalent to $\lozenge(e_a \approx \neg e_a)$.

To obtain a complete logic for reasoning about updating sets of probability measures, we follow the standard "reduction axiom" strategy used in dynamic epistemic logic: identify a set of valid biconditionals that allow us to reduce any formula containing the dynamic operators $\langle \varphi \rangle$ to an equivalent formula of $\mathcal{L}(\gtrsim, \succ, \lozenge)$ without dynamic operators, which can then be handled by the complete logic for $\mathcal{L}(\gtrsim, \succ, \lozenge)$.

**Definition 6.6.** The logic $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$ is the smallest set of $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ formulas that is (i) closed under modus ponens and the rule of replacement of equivalents, and (ii) contains all theorems of $\mathsf{IP}(\succsim, \succ, \Diamond)$ as well as all instances of the following axiom schemas where $p \in \mathsf{Prop}$ and $\alpha$ and $\beta$ are propositional:

(**R0**) $\langle\varphi\rangle p \leftrightarrow (\Diamond\varphi \wedge p)$;

(**R1**) $\langle\varphi\rangle\Diamond\psi \leftrightarrow \Diamond\langle\varphi\rangle\psi$;

(**R2**) $\langle\varphi\rangle\neg\psi \leftrightarrow (\Diamond\varphi \wedge \neg\langle\varphi\rangle\psi)$;

(**R3**) $\langle\varphi\rangle(\psi \wedge \chi) \leftrightarrow (\langle\varphi\rangle\psi \wedge \langle\varphi\rangle\chi)$;

(**R4**) $\langle\varphi\rangle(\alpha \succsim \beta) \leftrightarrow (\Diamond\varphi \wedge \Box((\varphi \wedge \alpha) \succsim (\varphi \wedge \beta)))$;

(**R5**) $\langle\varphi\rangle(\alpha \succ \beta) \leftrightarrow (\Diamond\varphi \wedge \Box((\varphi \succ \bot) \to ((\varphi \wedge \alpha) \succ (\varphi \wedge \beta))))$.

**Example 6.7.** In a given model, we may ask if *after the agent updates with the information that it is raining and that hail is more likely than lightning tonight* the agent judges that it is at least as likely that a window will break as it is that the power will go out:

$$\langle r \wedge (h \succ l)\rangle(w \succsim p).$$

This is equivalent, in light of the reduction axiom (**R4**), to

$$\Diamond(r \wedge (h \succ l)) \wedge \Box\big(\big((r \wedge (h \succ l)) \wedge w\big) \succsim \big((r \wedge (h \succ l)) \wedge p\big)\big),$$

which is in turn equivalent to

$$\Diamond(r \wedge (h \succ l)) \wedge \Box\big((h \succ l) \to \big((r \wedge w) \succsim (r \wedge p)\big)\big),$$

i.e., there is some measure that gives $r$ non-zero probability and gives $h$ greater probability than $l$, and every measure that gives $h$ greater probability than $l$ also makes the probability of $w$ conditional on $r$ at least as great as the probability of $p$ conditional on $r$.

The rest of this section is devoted to the proof of the following theorem.

**Theorem 6.8** (Soundness and Completeness)**.** For all $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$: $\varphi$ is a theorem of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$ if and only if $\varphi$ is valid with respect to the class of all imprecise probabilistic models.

The soundness of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$ is less trivial than the soundness of the previous systems. More importantly, we will use its soundness to prove its completeness, similar to the proof of completeness of other dynamic epistemic logics axiomatized by reduction axioms.

**Proposition 6.9.** For all $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$: if $\varphi$ is a theorem of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$, then $\varphi$ is valid with respect to the class of all imprecise probabilistic models.

*Proof.* Clearly it is enough to check the validity of (**R0**) to (**R5**).

- For (**R0**), note that the valuation of $p$ is invariant under the updating.
- For (**R1**), the key is to treat $\langle\varphi\rangle\Diamond$ as a whole, whence the semantics of $\langle\varphi\rangle\Diamond\psi$ at $\mathcal{M}, \mathcal{P}, w$ is that there is a $\mu \in \mathcal{P}_\varphi$ such that $\mu([\![\psi]\!]^{\mathcal{M},\{\mu\}}) > 0$. But given the construction of $\mathcal{P}_\varphi$, this is precisely saying that there is a $\nu \in \mathcal{P}$ such that $\nu([\![\varphi]\!]^{\mathcal{M},\{\nu\}}) > 0$ and that, letting $\mu = \nu(\cdot \mid [\![\varphi]\!]^{\mathcal{M},\{\nu\}})$, we have $\mu([\![\psi]\!]^{\mathcal{M},\{\mu\}}) > 0$. Now note that for any $\nu \in \mathcal{P}$ such that $[\![\varphi]\!]^{\mathcal{M},\{\nu\}} > 0$, letting $\mu = \nu(\cdot \mid [\![\varphi]\!]^{\mathcal{M},\{\nu\}})$, we have $[\![\langle\varphi\rangle\psi]\!]^{\mathcal{M},\{\nu\}} = [\![\psi]\!]^{\mathcal{M},\{\mu\}}$ since $\{\nu\}_\varphi = \{\mu\}$. Hence the truth condition of $\langle\varphi\rangle\Diamond\psi$ is transformed into the existence of $\nu \in \mathcal{P}$ such that $\nu([\![\varphi]\!]^{\mathcal{M},\{\nu\}}) > 0$ and that $[\![\langle\varphi\rangle\psi]\!]^{\mathcal{M},\{\nu\}} > 0$. But this is precisely the truth condition of $\Diamond\langle\varphi\rangle\psi$.

- For (**R2**), the key insight is that at $\mathcal{M}, \mathcal{P}, w$, assuming that there is a $\nu \in \mathcal{P}$ such that $\nu([\![\varphi]\!]^{\mathcal{M},\{\nu\}}) > 0$, we have:

$$\begin{aligned}
\mathcal{M}, \mathcal{P}, w \vDash \langle\varphi\rangle\neg\psi &\iff \mathcal{M}, \mathcal{P}_\varphi, w \vDash \neg\psi \\
&\iff \mathcal{M}, \mathcal{P}_\varphi, w \nvDash \psi \\
&\iff \mathcal{M}, \mathcal{P}, w \vDash \neg\langle\varphi\rangle\psi.
\end{aligned}$$

- For (**R3**), the idea is similar to the above.

- For (**R4**), it is enough to observe the following chain of equivalences assuming that there is a $\nu \in \mathcal{P}$ such that $\nu([\![\varphi]\!]^{\mathcal{M},\{\nu\}}) > 0$:

$$\begin{aligned}
\mathcal{M}, \mathcal{P}, w \vDash \langle\varphi\rangle(\alpha \succsim \beta) &\iff \mathcal{M}, \mathcal{P}_\varphi, w \vDash \alpha \succsim \beta \\
&\iff \forall\mu \in \mathcal{P}_\varphi, \mu([\![\alpha]\!]^{\mathcal{M},\mathcal{P}_\varphi}) \geq \mu([\![\beta]\!]^{\mathcal{M},\mathcal{P}_\varphi}) \\
&\iff \forall\mu \in \mathcal{P}_\varphi, \mu(V(\alpha)) \geq \mu(V(\beta)) \\
&\iff \forall\nu \in \mathcal{P} \text{ such that } \nu([\![\varphi]\!]^{\mathcal{M},\{\nu\}}) > 0, \\
&\qquad \nu(V(\alpha) \mid [\![\varphi]\!]^{\mathcal{M},\{\nu\}}) \geq \nu(V(\beta) \mid [\![\varphi]\!]^{\mathcal{M},\{\nu\}}) \\
&\iff \forall\nu \in \mathcal{P} \text{ such that } \nu([\![\varphi]\!]^{\mathcal{M},\{\nu\}}) > 0, \\
&\qquad \nu(V(\alpha) \cap [\![\varphi]\!]^{\mathcal{M},\{\nu\}}) \geq \nu(V(\beta) \cap [\![\varphi]\!]^{\mathcal{M},\{\nu\}}) \\
&\iff \forall\nu \in \mathcal{P}, \nu(V(\alpha) \cap [\![\varphi]\!]^{\mathcal{M},\{\nu\}}) \geq \nu(V(\beta) \cap [\![\varphi]\!]^{\mathcal{M},\{\nu\}}) \\
&\iff \forall\nu \in \mathcal{P}, \mathcal{M}, \{\nu\} \vDash (\varphi \land \alpha) \succsim (\varphi \land \beta) \\
&\iff \forall\nu \in \mathcal{P}, \nu([\![(\varphi \land \alpha) \succsim (\varphi \land \beta)]\!]^{\mathcal{M},\{\nu\}}) = 1 \\
&\iff \mathcal{M}, \mathcal{P}, w \vDash \Box((\varphi \land \alpha) \succsim (\varphi \land \beta)).
\end{aligned}$$

  Note that the last three equivalences extensively use the fact that a Boolean combination of comparison formulas is true at a world if and only if it is true at all worlds. The sixth equivalence is true because when $\nu([\![\varphi]\!]^{\mathcal{M},\{\nu\}}) = 0$, it trivially holds that $\nu(V(\alpha) \cap [\![\varphi]\!]^{\mathcal{M},\{\nu\}}) \geq \nu(V(\beta) \cap [\![\varphi]\!]^{\mathcal{M},\{\nu\}})$.

- For (**R5**), the strategy is the same—it is enough to observe the following chain of equivalences assuming that there is a $\nu \in \mathcal{P}$ such that $\nu([\![\varphi]\!]^{\mathcal{M},\{\nu\}}) > 0$:

$$\begin{aligned}
\mathcal{M}, \mathcal{P}, w \vDash \langle\varphi\rangle(\alpha \succ \beta) &\iff \mathcal{M}, \mathcal{P}_\varphi, w \vDash \alpha \succ \beta \\
&\iff \forall\mu \in \mathcal{P}_\varphi, \mu([\![\alpha]\!]^{\mathcal{M},\mathcal{P}_\varphi}) > \mu([\![\beta]\!]^{\mathcal{M},\mathcal{P}_\varphi}) \\
&\iff \forall\mu \in \mathcal{P}_\varphi, \mu(V(\alpha)) > \mu(V(\beta)) \\
&\iff \forall\nu \in \mathcal{P} \text{ such that } \nu([\![\varphi]\!]^{\mathcal{M},\{\nu\}}) > 0, \\
&\qquad \nu(V(\alpha) \mid [\![\varphi]\!]^{\mathcal{M},\{\nu\}}) > \nu(V(\beta) \mid [\![\varphi]\!]^{\mathcal{M},\{\nu\}}) \\
&\iff \forall\nu \in \mathcal{P} \text{ such that } \nu([\![\varphi]\!]^{\mathcal{M},\{\nu\}}) > 0, \\
&\qquad \nu(V(\alpha) \cap [\![\varphi]\!]^{\mathcal{M},\{\nu\}}) > \nu(V(\beta) \cap [\![\varphi]\!]^{\mathcal{M},\{\nu\}}) \\
&\iff \forall\nu \in \mathcal{P}, \text{ if } \mathcal{M}, \{\nu\} \vDash \varphi \succ \bot \text{ then } \mathcal{M}, \{\nu\} \vDash (\varphi \land \alpha) \succ (\varphi \land \beta) \\
&\iff \forall\nu \in \mathcal{P}, \mathcal{M}, \{\nu\} \vDash (\varphi \succ \bot) \to ((\varphi \land \alpha) \succ (\varphi \land \beta)) \\
&\iff \forall\nu \in \mathcal{P}, \nu([\![(\varphi \succ \bot) \to ((\varphi \land \alpha) \succ (\varphi \land \beta))]\!]^{\mathcal{M},\{\nu\}}) = 1 \\
&\iff \mathcal{M}, \mathcal{P}, w \vDash \Box((\varphi \succ \bot) \to ((\varphi \land \alpha) \succ (\varphi \land \beta))).
\end{aligned}$$

  Again, the last four equivalences extensively use the fact that a Boolean combination of comparison formulas is true at a world if and only if it is true at all worlds. □

For completeness, we first show that the axioms allow us to provably-equivalently reduce any formula in $\mathcal{L}(\succsim, \succ, \Diamond, \langle\rangle)$ to a fragment $\mathcal{L}_{\text{Simpd1}}$ that is even simpler than the fragment $\mathcal{L}_{\text{Simp}}$: the comparison formulas in the scope of any $\Diamond$ must not have nested comparison.

**Definition 6.10.** Let $\mathcal{L}_{\text{Bool}}$ be the set of propositional formulas. In other words, this is the fragment generated from $\mathsf{Prop}$ by $\neg$ and $\wedge$.

Let $\mathcal{L}_{\text{Compd1}}$ be the fragment of $\mathcal{L}(\succsim, \succ)$ with no nesting of $\succsim$ and $\succ$. In other words, this is the fragment generated from $\mathsf{Prop}$ and $\{(\alpha \succsim \beta), (\alpha \succ \beta) \mid \alpha, \beta \in \mathcal{L}_{\text{Bool}}\}$ by $\neg$ and $\wedge$.

Finally, let $\mathcal{L}_{\text{Simpd1}}$ be the fragment of $\mathcal{L}(\succsim, \succ, \Diamond)$ generated from $\mathsf{Prop}$ and $\{\Diamond\varphi \mid \varphi \in \mathcal{L}_{\text{Compd1}}\}$ by $\neg$ and $\wedge$.

**Lemma 6.11.** For every $\varphi \in \mathcal{L}(\succsim, \succ)$, there is a $T_{\text{Compd1}}(\varphi) \in \mathcal{L}_{\text{Compd1}}$ such that $\varphi \leftrightarrow T_{\text{Compd1}}(\varphi) \in \mathsf{IP}(\succsim, \succ)$. Moreover, $\varphi$ and $T_{\text{Compd1}}(\varphi)$ use the same propositional variables.

*Proof.* We use a standard argument for extracting comparisons embedded in comparisons. Formally, an induction over $\mathcal{L}(\succsim, \succ)$ is needed. The base case and the inductive cases for $\neg$ and $\wedge$ are trivial as we can simply define $T_{\text{Compd1}}(p) = p$, $T_{\text{Compd1}}(\neg\varphi) = \neg T_{\text{Compd1}}(\varphi)$, and $T_{\text{Compd1}}(\varphi \wedge \psi) = T_{\text{Compd1}}(\varphi) \wedge T_{\text{Compd1}}(\psi)$.

For the non-trivial cases for $\succsim$ and $\succ$, we only need the following: for any $\alpha, \beta \in \mathcal{L}_{\text{Bool}}$ and $\varphi, \psi \in \mathcal{L}_{\text{Compd1}}$, the following are in $\mathsf{IP}(\succsim, \succ)$:

$$(\varphi \succsim \psi) \leftrightarrow (((\alpha \succsim \beta) \wedge (\varphi[\alpha \succsim \beta/\top] \succsim \psi[\alpha \succsim \beta/\top])) \vee (\neg(\alpha \succsim \beta) \wedge (\varphi[\alpha \succsim \beta/\bot] \succsim \psi[\alpha \succsim \beta/\bot])));$$
$$(\varphi \succsim \psi) \leftrightarrow (((\alpha \succ \beta) \wedge (\varphi[\alpha \succsim \beta/\top] \succsim \psi[\alpha \succsim \beta/\top])) \vee (\neg(\alpha \succ \beta) \wedge (\varphi[\alpha \succsim \beta/\bot] \succsim \psi[\alpha \succsim \beta/\bot])));$$
$$(\varphi \succ \psi) \leftrightarrow (((\alpha \succsim \beta) \wedge (\varphi[\alpha \succsim \beta/\top] \succ \psi[\alpha \succsim \beta/\top])) \vee (\neg(\alpha \succsim \beta) \wedge (\varphi[\alpha \succsim \beta/\bot] \succ \psi[\alpha \succsim \beta/\bot])));$$
$$(\varphi \succ \psi) \leftrightarrow (((\alpha \succ \beta) \wedge (\varphi[\alpha \succsim \beta/\top] \succ \psi[\alpha \succsim \beta/\top])) \vee (\neg(\alpha \succ \beta) \wedge (\varphi[\alpha \succsim \beta/\bot] \succ \psi[\alpha \succsim \beta/\bot]))).$$

They are proven mainly by (**B7**) to (**B10**). The key idea is to first derive the following:

$$\begin{aligned}
(\alpha \succsim \beta) &\to ((\varphi \leftrightarrow \varphi[\alpha \succsim \beta/\top]) \succsim \top);\\
\neg(\alpha \succsim \beta) &\to ((\varphi \leftrightarrow \varphi[\alpha \succsim \beta/\bot]) \succsim \top);\\
(\alpha \succ \beta) &\to ((\varphi \leftrightarrow \varphi[\alpha \succ \beta/\top]) \succsim \top);\\
\neg(\alpha \succ \beta) &\to ((\varphi \leftrightarrow \varphi[\alpha \succ \beta/\bot]) \succsim \top).
\end{aligned}$$

Together with $((\varphi \leftrightarrow \psi) \succsim \top) \to ((\varphi \succsim \chi) \leftrightarrow (\psi \succsim \chi))$ and $((\varphi \leftrightarrow \psi) \succsim \top) \to ((\varphi \succ \chi) \leftrightarrow (\psi \succ \chi))$, the required equivalences can easily be derived. □

**Proposition 6.12.** For every $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)$ there is a $T_{\text{Simpd1}}(\varphi) \in \mathcal{L}_{\text{Simpd1}}$ such that $\varphi \leftrightarrow T_{\text{Simpd1}}(\varphi)$ is in $\mathsf{IP}(\succsim, \succ, \Diamond)$.

*Proof.* The result of replacing all $\Diamond\chi$ in $T_{\text{Simp}}(\varphi)$ by $\Diamond T_{\text{Compd1}}(\chi)$ is the desired $T_{\text{Simpd1}}(\varphi)$. □

**Proposition 6.13.** For every $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ there is a $T_{\text{Simpd1}}(\varphi) \in \mathcal{L}_{\text{Simpd1}}$ such that $\varphi \leftrightarrow T_{\text{Simpd1}}(\varphi)$ is in $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$.

*Proof.* We proceed by induction. Given Proposition 6.12 and the rule of replacement of equivalents, the only non-trivial case is to show that there is a $T_{\text{Simpd1}}(\langle\varphi\rangle\psi)$ that is provably equivalent to $\langle\varphi\rangle\psi$ in $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$ where $\varphi, \psi$ are in $\mathcal{L}_{\text{Simpd1}}$. By repeated use of (**R1**) to (**R3**) and the rule of replacement of equivalents, obviously we can push the $\langle\varphi\rangle$ into $\psi$ over Boolean connectives and $\Diamond$ and obtain a Boolean combination of formulas of the form $\langle\varphi\rangle p$ or of the form $\langle\varphi\rangle(\alpha \succsim \beta)$ or $\langle\varphi\rangle(\alpha \succ \beta)$ since in $\mathcal{L}_{\text{Simpd1}}$, $\succsim$ and $\succ$ only scope over propositional formulas. All three kinds of formulas can be replaced by formulas in $\mathcal{L}(\succsim, \succ, \Diamond)$ provably equivalently. Then we apply $T_{\text{Simpd1}}$ again to finish off (to eliminate any $\Diamond$'s appearing inside $\Diamond$'s). □

With the above reduction method, the completeness of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$ follows.

**Proposition 6.14.** For all $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$: if $\varphi$ is valid with respect to the class of all imprecise probabilistic models, then $\varphi$ is a theorem of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$.

*Proof.* Let $\varphi$ be any valid formula in $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$. Then by the soundness of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$ and the fact that $\varphi \leftrightarrow T_{\mathrm{Simpd1}}(\varphi) \in \mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$, $T_{\mathrm{Simpd1}}(\varphi)$ is also valid. But $T_{\mathrm{Simpd1}}(\varphi) \in \mathcal{L}_{\mathrm{Simpd1}} \subseteq \mathcal{L}(\succ, \succ, \Diamond)$. By the completeness of $\mathsf{IP}(\succsim, \succ, \Diamond)$, $T_{\mathrm{Simpd1}}(\varphi) \in \mathsf{IP}(\succsim, \succ, \Diamond)$. By the definition of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$, it contains all theorems of $\mathsf{IP}(\succsim, \succ, \Diamond)$. Hence $T_{\mathrm{Simpd1}}(\varphi)$ is in $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$. Then by Boolean reasoning, $\varphi$ is in $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$. $\square$

Although the reduction axioms for $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ allow us to reduce the satisfiability problem for $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ to that for $\mathcal{L}(\succsim, \succ, \Diamond)$, which is in $\mathsf{NP}$ (Theorem 5.12), it does not immediately follow that the satisfiability problem for $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ is in $\mathsf{NP}$, due to the blowup in the length of formulas during the reduction process. A similar obstacle occurs in the case of the simplest dynamic epistemic logic (public announcement logic), in which case a solution is to use a satisfiability-preserving reduction with only polynomial blowup instead of the standard validity-preserving reduction with exponential blowup (Lutz 2006). Whether this or other techniques apply to $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ we leave as an open problem.

**Problem 6.15.** Determine the complexity of the satisfiability problem for $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$.

## 6.2 Introducing a New Proposition

In the previous subsection, we considered the dynamic update operator that concerns learning the truth of a proposition. In this subsection, we consider the complementary dynamics of learning the mere existence of a proposition and then being maximally uncertain about it in the way of imprecise probability (cf. Joyce 2005). Our goal is to show that this kind of information dynamics is expressively helpful, especially in formalizing examples in a natural way, and we leave the complete axiomatization of its logic as an open question.

**Definition 6.16.** The language $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$ is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \succsim \varphi) \mid (\varphi \succ \varphi) \mid \Diamond\varphi \mid \langle\varphi\rangle\varphi \mid I_p^+\varphi \mid I_p^-\varphi$$

where $p \in \mathsf{Prop}$. We read $I_p^+\varphi$ as "letting $p$ be a true proposition that is newly introduced to the agent, $\varphi$"; similarly, $I_p^-\varphi$ reads "letting $p$ be a false proposition that is newly introduced to the agent, $\varphi$". We also take $I_p\varphi$ as an abbreviation of $(I_p^+\varphi \wedge I_p^-\varphi)$.

We treat both $I_p^+$ and $I_p^-$ as a kind of propositional quantifier, since they change the meaning (denotation) of $p$, and we define free and bound propositional variables in the obvious way. For any $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$, let $\mathsf{Prop}(\varphi)$ be the set of freely occurring propositional variables in $\varphi$.

Now we specify the semantics for $I^+$ and $I^-$. First, we define how a model changes when we introduce a new proposition.

**Definition 6.17.** Given a non-empty set $W$, a field of sets $\mathcal{F}$ on $W$, a valuation $V$ such that $V(p) \in \mathcal{F}$ for all $p \in \mathsf{Prop}$, and a set of finitely additive probability measure $\mathcal{P}$ on $\mathcal{F}$, we interpret $\mathcal{F}$ as the collection of the "old" propositions. Our goal is to define the result of adding a "new" proposition $P$. Intuitively, we first split each $w \in W$ into $\langle w, 1\rangle$ and $\langle w, 0\rangle$ corresponding to $P$ being true and false, respectively, while keeping the truth value of the old propositions. For the probability measures, we take all probability measures defined on both the old and new propositions that, when restricted to just the old propositions, coincide with some old probability measure. The following gives the formal details.

- Let $\mathcal{F} \times 2 = \{X \times \{0,1\} \mid X \in \mathcal{F}\}$, which is a field of sets on $W \times \{0,1\}$.
- Let $\mathsf{Split}(\mathcal{F})$ be the smallest field of sets on $W \times \{0,1\}$ extending $\mathcal{F} \cup \{W \times \{0\}\}$.
- Let $V \times 2$ be defined such that $V \times 2(p) = V(p) \times \{0,1\}$ for all $p \in \mathsf{Prop}$; note that $V \times 2(p) \in \mathcal{F} \times 2$ for all $p \in \mathsf{Prop}$.

- For any $p \in \mathsf{Prop}$, let $V^{+p}$ be defined such that

$$V^{+p}(q) = \begin{cases} V(q) \times \{0,1\} & \text{if } q \neq p \\ W \times \{1\} & \text{if } q = p\,; \end{cases}$$

  note that $V^{+p}(q) \in \mathsf{Split}(\mathcal{F})$ for all $q \in \mathsf{Prop}$.

- For any finitely additive measure $\mu$ on $\mathcal{F}$, define $\mu \times 2$, a finitely additive measure on $\mathcal{F} \times 2$, by $\mu \times 2(X \times \{0,1\}) = \mu(X)$ for all $X \in \mathcal{F}$.

- Let $\mathcal{P} \times 2 = \{\mu \times 2 \mid \mu \in \mathcal{P}\}$.

- Let $\mathsf{Split}(\mathcal{P})$ be the set of all finitely additive measures $\mu$ on $\mathsf{Split}(\mathcal{F})$ such that $\mu|_{\mathcal{F}\times 2} \in \mathcal{P} \times 2$.

Using the above definition, given a propositional variable $p \in \mathsf{Prop}$ and a propositional model $\mathcal{M} = \langle W, V\rangle$, let $\mathcal{M} \times 2 = \langle W \times \{0,1\}, V \times 2\rangle$ and $\mathcal{M}^{+p} = \langle W \times \{0,1\}, V^{+p}\rangle$. Then if $\langle \mathcal{M}, \mathcal{P}\rangle$ is an IP model, so is $\langle \mathcal{M}^{+p}, \mathsf{Split}(\mathcal{P})\rangle$, and $\langle \mathcal{M}^{+p}, \mathsf{Split}(\mathcal{P})\rangle$ represents the result of adding a new proposition, now denoted by $p$, to $\langle \mathcal{M}, \mathcal{P}\rangle$.

**Remark 6.18.** In the algebraic theory of Boolean algebras, there is a standard operation of freely adjoining a new element to a Boolean algebra: for any Boolean algebra $\mathcal{B}$ and any $a \notin \mathcal{B}$, there is a unique up to isomorphism Boolean algebra $\mathcal{B}^{+a}$ such that

- $\mathcal{B}$ is a subalgebra of $\mathcal{B}^{+a}$, and every element in $\mathcal{B}^{+a}$ is generated from $\mathcal{B} \cup \{a\}$;
- for any $b \in \mathcal{B}$, $b \wedge a$ and $b \wedge \neg a$ are not the bottom element in $\mathcal{B}^{+a}$.

The operation of $\mathsf{Split}(\mathcal{F})$ is precisely the dual of this algebraic operation.

Hence, if we use an algebraic model $\langle \mathcal{B}, V, \mathcal{P}\rangle$ where $\mathcal{B}$ is a Boolean algebra (of propositions of which the agent is currently aware), $V$ a valuation function from $\mathsf{Prop}$ to $\mathcal{B}$, and $\mathcal{P}$ a set of finitely additive functions from $\mathcal{B}$ to $[0,1]$, we can easily define the result of adding a new proposition $a \notin \mathcal{B}$ to be denoted by $p$ as $\langle \mathcal{B}^{+a}, V', \mathcal{P}'\rangle$ where $V'$ coincides with $V$ on $\mathsf{Prop}$ except that $V'(p) = a$ and $\mathcal{P}' = \{\mu : \mathcal{B}^{+a} \to [0,1] \mid \mu \text{ is finitely additive and } \mu|_{\mathcal{B}} \in \mathcal{P}\}$.

**Remark 6.19.** The model construction from $\langle \mathcal{M}, \mathcal{P}\rangle$ to $\langle \mathcal{M}^{p}, \mathsf{Split}(\mathcal{P})\rangle$ can also be viewed as an event-model update from (probabilistic) dynamic epistemic logic (van Benthem et al. 2009). The event model contains two events $\{1, 0\}$ corresponding to whether the new proposition is true or not with no preconditions, and the agent is maximally ignorant about these two events: at any of the old worlds, she cannot distinguish between these two events, is completely ignorant about the relative likelihood of these two events, and does not observe which event happens. Using the terminology from van Benthem et al. (2009), the agent is maximally and imprecisely ignorant about the occurrence probability of these two events and makes no observation about these two events.

**Definition 6.20.** The semantics of $I_p^+$ and $I_p^-$ are given by

$$\mathcal{M}, \mathcal{P}, w \models I_p^-\varphi \text{ iff } \mathcal{M}^{+p}, \mathsf{Split}(\mathcal{P}), \langle w, 0\rangle \models \varphi\,,$$
$$\mathcal{M}, \mathcal{P}, w \models I_p^+\varphi \text{ iff } \mathcal{M}^{+p}, \mathsf{Split}(\mathcal{P}), \langle w, 1\rangle \models \varphi\,.$$

Now let us put the new operators to work. We first use them to formalize the medical example (Example 1.1).

**Example 6.21.** The following sentence is valid and represents the medical example if we take $p$ to mean that the agent has the disease (that is, the proposition introduced by $I_p$ is that the agent has the disease) and $q$ to mean that the gland is swollen (that is, the proposition introduced by $I_q$ is that the gland is swollen).

$$I_p\langle \neg p \succ p\rangle I_q\langle (q \wedge p) \succ (q \wedge \neg p)\rangle\langle q\rangle(p \succ \neg p). \tag{5}$$

We interpret the first update by $\neg p \succ p$ as the result of the agent observing that she is not feeling uncomfortable and hence believing that her not having the disease is more likely than her having it. The second update represents what the agent learns from the doctor, and the third update represents a medical examination revealing that her gland is swollen.

The above simple sentence does not capture more nuanced probabilistic relationships between $p$ and $q$ such as that conditioning on $q$, $p$ is twice as likely as $\neg p$ or that the medical examination does not reveal $q$ but only a signal that is probabilistically related to $q$. But with the new operator $I$, we can easily say these things. For example, to express that $p$ is twice as likely as $\neg p$ conditioning on $q$, we may introduce two new propositions (like two coin flips) by $I_r$ and $I_s$ at the beginning of the formula (note that our syntax forbids embedding $I$ in updates) and later add after $I_q$ the update $\langle(((q \wedge r \wedge s) \approx (q \wedge r \wedge \neg s)) \wedge ((q \wedge \neg r \wedge s) \approx (q \wedge \neg r \wedge \neg s)) \wedge ((q \wedge r \wedge s) \approx (q \wedge \neg r \wedge s)) \wedge (\bot \succsim (q \wedge \neg r \wedge \neg s))\rangle$, which says that conditioning on $q$ the two coin flips are fair and independent but the two tail situation is impossible (perhaps because the two coins will be retossed if they are both tails up). Then, using $\langle(q \wedge p) \approx (q \wedge s)\rangle$, we essentially say that $p$'s probability conditioning on $q$ is $2/3$ and thus twice as likely as $\neg p$. To express that the medical examination only provides an informative signal related to $q$, we may again introduce a new proposition $t$ representing that signal and then let the agent learn the probabilistic relationship between $t$ and $q$.

**Example 6.22.** For the prisoner example, recall that $\alpha$ is the formula

$$\big(\bot \succsim (e_a \wedge e_b \wedge e_c)\big) \wedge \big(\big((e_a \wedge e_b) \vee (e_a \wedge e_c) \vee (e_b \wedge e_c)\big) \succsim \top\big) \wedge (e_a \approx e_b) \wedge (e_b \approx e_c),$$

saying that two of the prisoners will be executed and the probabilities for the three situations are equal. Recall also that $\beta$ is the following formula

$$((s_b \to e_b) \succsim \top) \wedge ((s_c \to e_c) \succsim \top) \wedge (\bot \succsim (s_b \wedge s_c)),$$

saying that the jailer will announce one and only one prisoner to be executed truthfully. Then the following formula is valid and represents the dilation when $a$ hears that the jailer announces that $b$ will be executed:

$$I_{e_a} I_{e_b} I_{e_c} \langle \alpha \rangle I_{s_b} I_{s_c} \langle \beta \rangle \langle s_b \rangle (\Diamond(e_a \succsim \top) \wedge \Diamond(e_a \approx \neg e_a)).$$

As we have seen in Example 6.21, $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$ is capable of expressing numerical relationships. Leveraging this capability, it is easy to observe that $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$ is more expressive than $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$.

**Example 6.23.** Consider a propositional model $\mathcal{M} = \langle W, V \rangle$ where $W = \{w, u\}$ has two worlds, $V(p) = \{w\}$, and $V(q) = \varnothing$ for all $q \in \mathsf{Prop} \setminus \{p\}$. Then let $\mu_1$ be a probability measure on $\wp(W)$ such that $\mu_1(\{w\}) = 0.6$, and let $\mu_2$ also be a probability measure on $\wp(W)$ such that $\mu_2(\{w\}) = 0.9$. Then it is easy to see that $\mathcal{M}, \{\mu_1\}, w$ and $\mathcal{M}, \{\mu_2\}, w$ satisfy the same formulas in $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$. However, the following formula

$$I_q I_r \langle ((q \wedge r) \approx (q \wedge \neg r)) \wedge ((q \wedge r) \approx (\neg q \wedge r)) \wedge ((q \wedge r) \approx (\neg q \wedge \neg r)) \rangle (p \succ \neg(q \wedge r)),$$

which intuitively says that $p$ is more likely than not getting two heads up from two randomly and independently flipped fair coins, is true at $\mathcal{M}, \{\mu_2\}, w$, but false at $\mathcal{M}, \{\mu_1\}, w$.

Indeed, we will show that $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$ can express any linear inequality with integer coefficients about the probability of formulas. For this, we first introduce some notation.

**Definition 6.24.** Let $\Gamma$ be a finite set of formulas, $C(\Gamma)$ the set of all clauses (conjunctions of the form $\bigwedge_{\varphi \in \Gamma} \pm\varphi$ where $\pm$ is either the empty string or $\neg$), and $p$ a propositional variable. Then define $(p|\Gamma)$ to be the formula

$$\bigwedge_{\psi \in C(\Gamma)} ((\psi \wedge p) \approx (\psi \wedge \neg p)).$$

Intuitively, $(p|\Gamma)$ says that $p$ represents a fair coin flip independent of all events expressible using formulas in $\Gamma$.

**Proposition 6.25.** For any sequences $\langle\varphi_i\rangle_{i=1\ldots n}$ and $\langle\psi_i\rangle_{i=1\ldots m}$ of formulas in $\mathcal{L}(\succsim,\succ,\Diamond,\langle\,\rangle,I)$ and any sequences $\langle a_i\rangle_{i=1\ldots n}$ and $\langle b_i\rangle_{i=1\ldots m}$ of natural numbers, there is a formula $\chi\in\mathcal{L}(\succsim,\succ,\Diamond,\langle\,\rangle,I)$ such that for any IP model $\mathcal{M},\mathcal{P},w$,

$$\mathcal{M},\mathcal{P},w\models\chi \text{ iff } \forall\mu\in\mathcal{P},\sum_{i=1}^{n}a_i\mu([\![\varphi_i]\!]^{\mathcal{M},\mathcal{P}})\geq\sum_{i=1}^{m}b_i\mu([\![\psi_i]\!]^{\mathcal{M},\mathcal{P}}).$$

*Proof.* The central idea is already in Kraft et al. (1959) and is also described in Section 2 of Ding et al. Forthcoming: we use $I$ operators to introduce new propositions that evenly partition the logical space spanned by $\varphi_i$'s so that we can take the union of multiple copies of the partitioned $\varphi_i$'s to simulate addition.

Let $l$ be the smallest natural number such that $2^l$ is larger than the sum of all $a_i$'s and $b_i$'s and pick propositional variables $\langle p_i\rangle_{i=1\ldots l}$ not occurring in any of the $\varphi_i$'s and $\psi_i$'s. Then let $C$ list all logically inequivalent clauses made from $p_i$'s. Since $|C|=2^l$ and $2^l$ is larger than the sum of all coefficients, let $f$ be a function from $\{1,\ldots n\}\times\{0\}\cup\{1,\ldots m\}\times\{1\}$ to $\wp(C)$ such that $f(x)\cap f(y)=\varnothing$ whenever $x\neq y$ and $|f(i,0)|=a_i$ and $|f(i,1)|=b_i$. Let $\Gamma$ be set of all $\varphi_i$'s and $\psi_i$'s. Then consider the following formula:

$$I^+_{p_1}I^+_{p_2}\cdots I^+_{p_l}\langle(p_1|\Gamma)\wedge(p_2|\Gamma\cup\{p_1\})\cdots(p_l|\Gamma\cup\{p_1,p_2,\cdots,p_{l-1}\})\rangle$$
$$(\bigvee_{i=1}^{n}\bigvee_{c\in f(i,0)}(\varphi_i\wedge c))\succsim(\bigvee_{i=1}^{n}\bigvee_{c\in f(i,1)}(\psi_i\wedge c)). \quad (6)$$

This is the required formula since after the introduction of new propositions and the announcement, the probability of $\bigvee_{c\in f(i,0)}(\varphi_i\wedge c)$ (resp. $\bigvee_{c\in f(i,1)}(\psi_i\wedge c)$) is precisely $a_i/2^l$ (resp. $b_i/2^l$) times the probability of $\varphi_i$ (resp. $\psi_i$). Cancelling out the common denominator $2^l$, we see that the inequality expressed by formula (6) is the required one. □

Therefore, we see that with the new operators $I^+_p$ and $I^-_p$, $\mathcal{L}(\succsim,\succ,\Diamond,\langle\,\rangle,I)$ is capable of expressing quantitative (and in particular arbitrary additive) information. This also means that we cannot use the same reduction strategy we used for $\mathcal{L}(\succsim,\succ,\Diamond,\langle\,\rangle)$ to axiomatize the logic in $\mathcal{L}(\succsim,\succ,\Diamond,\langle\,\rangle,I)$. However, we conjecture that there is a computable translation from $\mathcal{L}(\succsim,\succ,\Diamond,\langle\,\rangle,I)$ to $\mathcal{L}(\succsim,\succ,\Diamond,\langle\,\rangle)$ that preserves satisfiability. Such a translation can then be coded as rules instead of axioms that completely axiomatize the logic.

**Problem 6.26.** Find an axiomatization of the set of valid formulas in $\mathcal{L}(\succsim,\succ,\Diamond,\langle\,\rangle,I)$.

**Problem 6.27.** Determine the complexity of the satisfiability problem for $\mathcal{L}(\succsim,\succ,\Diamond,\langle\,\rangle,I)$.

# 7 Conclusion

In this paper, we have investigated a hierarchy of languages

$$\mathcal{L}(\succsim)\subseteq\mathcal{L}(\succsim,\succ)\subseteq\mathcal{L}(\succsim,\succ,\Diamond)\subseteq\mathcal{L}(\succsim,\succ,\Diamond,\langle\,\rangle)\subseteq\mathcal{L}(\succsim,\succ,\Diamond,\langle\,\rangle,I)$$

and matching complete logics for imprecise comparative probabilistic reasoning in the first four languages:

$$\mathsf{IP}(\succsim)\subseteq\mathsf{IP}(\succsim,\succ)\subseteq\mathsf{IP}(\succsim,\succ,\Diamond)\subseteq\mathsf{IP}(\succsim,\succ,\Diamond,\langle\,\rangle).$$

The first four languages have straightforward extensions to the multi-agent setting, in which each agent $i$ has their own comparative probability relations $\succsim_i$ and $\succ_i$, allowing us to formalize statements such as "Ann judges it more likely than not that Bob thinks hail is more likely than lightning": $(h \succ_b l) \succ_a \neg(h \succ_b l)$. A multi-agent version of the language $\mathcal{L}(\succsim)$ was already studied in Alon and Heifetz (2014). Generalizing the other languages in this paper to the multi-agent setting presents no major challenges, although the complexity of the resulting multi-agent logics goes beyond that of the single-agent versions, just as the complexity of the basic epistemic logic $\mathsf{S5}$ jumps from $\mathsf{NP}$ to $\mathsf{PSPACE}$ when moving from the single-agent to multi-agent setting (see Halpern and Moses 1992). When generalizing the language $\mathcal{L}(\succsim, \succ, \Diamond, \langle\, \rangle, I)$ to the multi-agent setting, there is a distinction between introducing a new proposition to every agent publicly and introducing a new proposition for only one agent so that she becomes privately aware of it. Our semantics naturally generalizes to model all agents publicly becoming aware of a new proposition, but the modeling of some agent's privately becoming aware of a new proposition requires a different treatment.

Further extensions to the language are natural to consider, such as adding comparative conditional probability formulas $(\varphi \mid \psi) \succsim (\alpha \mid \beta)$ (resp. $(\varphi \mid \psi) \succ (\alpha \mid \beta)$) expressing that the conditional probability of $\varphi$ given $\psi$ is at least as great as (resp. greater than) the conditional probability of $\alpha$ given $\beta$ for every measure in one's set of measures, which is not expressible in the languages of this paper (see Luce 1968). For precise probabilistic models, such a quarternary operator is investigated in, e.g., Domotor 1969, § 2.6 and Suppes and Zanotti 1982 (and recently in Hawthorne 2016 using so-called Popper functions), but the interpretation in imprecise probabilistic models seems yet to be explored. Allowing inequalities of probabilistic products $(\varphi \times \psi) \succsim (\alpha \times \beta)$ would allow even greater expressivity (such an extension in the precise case is also considered in Domotor 1969, §2.4).

More generally, the systems in this paper are part of a much broader hierarchy of probabilistic languages, ranging from the very simple $\mathcal{L}(\succsim)$ all the way to highly expressive probabilistic languages encompassing full quantified real number arithmetic (Halpern, 1990). In addition to their inherent theoretical interest, probabilistic logics have emerged as a foundational tool for many central computational tasks, from core knowledge representation (Russell, 2015), to reasoning about strategic interaction (Dekel and Siniscalchi, 2015; van Benthem and Klein, 2019), to causal inference (witness *do-calculus*, which is built on top of a probability calculus; see, e.g., Pearl 2009; Bareinboim et al. 2020; Ibeling and Icard 2020). Furthermore, applications in these contexts have motivated some of the very systems presented here (e.g., Alon and Heifetz 2014). Understanding the capacities and limitations of such systems may well be an important step toward further integration of explicit probabilistic tools in these and other domains.

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