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Asymptotic Behavior of Nonlinear PDE: Dynamic Stability of a Droplet Model and Boundary Data Homogenization

A dissertation submitted in partial satisfaction
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by

William Myers Feldman

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ABSTRACT OF THE DISSERTATION

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Professor Christina Kim, Chair

This dissertation comes in two parts. The problems considered are all related to asymptotic behavior of solutions to nonlinear PDE in various scaling limits. Part I considers long time behavior for a dynamic model and Part II considers the macroscopic averaging behavior of stationary models with multiple length scales. These kinds of problems, which have been well studied in other settings, face significant additional difficulties when the interior equations are coupled with a moving boundary or an irregular boundary data on a fixed boundary. This dissertation addresses several of these questions and develops new techniques to deal with PDE problems involving irregularity on a lower dimensional interface.

Part I, Chapter 2, considers a model for the contact angle motion of quasi-static capillary drops resting on a flat surface. The evolution is formally the gradient flow of a free energy involving surface tension and adhesion to the hydrophilic surface. We prove the nonlinear dynamic stability of the round drop stationary solution under a strong star-shapedness assumption on the initial data. The key tool is a new geometric a priori estimate for the regularity of the contact line. Based on this estimate we are able to show the existence of a global in time uniformly regular weak solution via a discrete gradient flow type scheme. The convergence to equilibrium then follows from a classical symmetry result of Serrin on overdetermined boundary value problems.

Part II of this dissertation is split into three chapters in which we study several problems related to

the homogenization of oscillating boundary conditions for second order nonlinear non-divergence form elliptic operators in both the random and periodic settings.

Chapter 3 examines the case of a nonlinear elliptic equation with a periodic oscillating Dirichlet boundary condition in a general bounded domain. We prove a qualitative homogenization result under quite general conditions. Here the major difficulty is the lack of continuity of the homogenized boundary condition caused by singular behavior near boundary points with rational normal direction. For fully nonlinear equations the well-posedness of the Dirichlet problem with possibly discontinuous boundary condition is proved when the Hausdorff dimension of the discontinuity set is sufficiently small. This is the key estimate leading to the qualitative homogenization result.

Chapter 4 is concerned with the same periodic oscillating Dirichlet data problem, but now the focus is on the fine properties of the homogenized boundary condition near the rational directions. We show that the behavior depends essentially on the form of the operator. Linear operators and rotationally invariant nonlinear operators lead to a homogenized boundary condition which extends continuously to the rational directions. An explicit modulus of continuity can then be established, namely Hölder - $\frac{1}{d}$ up to logarithmic factors in the linear case. On the other hand, for a typical, in the Baire category sense, nonlinear non-rotation invariant operator the homogenized boundary condition does not extend continuously at any rational direction.

Finally, in Chapter 5, we discuss both the Dirichlet and Neumann boundary homogenization problems for nonlinear operators when the boundary data is a random field. The main result is the homogenization in half-spaces and general domains when the boundary data has the form of an i.i.d. random checkerboard and the interior operator is non-random. In this case we establish almost sure homogenization with an algebraic rate of convergence in probability. The result extends as well to the strongly mixing setting with sufficient decay.

The dissertation of William Myers Feldman is approved.

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To whom it may concern.

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Parts of this dissertation are based on published joint works. Chapter 2 is based on a joint work with Inwon Kim, "Dynamic Stability of Equilibrium Capillary Drops," which appeared in *Archive for Rational Mechanics and Analysis*, **211**(3):819-878, 2014. Chapter 3 is based on my paper "Homogenization of the Oscillating Dirichlet Boundary Condition in General Domains" which appeared in *Journal de Mathématiques Pures et Appliquées*, **101**(5):599-622, 2014. Part II Chapter 6 is based on a joint work with Inwon Kim and Takis Souganidis "Quantitative Homogenization of Elliptic PDE with Random Oscillatory Boundary Data" which appeared in *Journal de Mathématiques Pures et Appliquées*, **103**(4):958-1002, 2015. Chapter 4 is based on a joint work with Inwon Kim which is currently in submission. I would like to acknowledge my collaborators and the journals listed above. I would also like to acknowledge my collaborators Norbert Pozar, Stan Osher, Hayden Schaeffer and Giang Tran.

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PUBLICATIONS

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CHAPTER 1

Introduction

In broad terms this dissertation is concerned with limiting regimes for nonlinear partial differential equations, long time behavior for dynamical problems and the averaging behavior of equations with multiple scales. The common theme is a focus on problems where the structure of a lower dimensional interface plays an important role. We begin with a high level overview of the topics covered before providing a more detailed introduction to the results of the dissertation.

Part I, Chapter 2, of the dissertation considers a model for the contact angle motion of quasi-static capillary drops resting on a flat surface. The evolution is formally the gradient flow of a free energy involving surface tension and adhesion to the surface. The goal of this research is to prove nonlinear stability of the global energy minimizer, a round drop.

Part II of the dissertation is divided into three chapters, each focused on separate but inter-related problems in homogenization theory for oscillating boundary conditions. Chapter 3 examines the case of a nonlinear elliptic equation on the interior of a smooth domain in $\mathbb{R}^d$ with a $\mathbb{Z}^d$-periodic oscillating Dirichlet boundary condition on its the $(d-1)$-dimensional boundary. The main difficulty is to control the singular nature of the homogenization procedure near the boundary points with rational normal directions – namely those aligned with the $\mathbb{Z}^d$ periodicity lattice of the boundary condition. Chapter 4 is concerned with the same periodic oscillating Dirichlet data problem, but now the focus is on the fine properties of the homogenized boundary condition near the rational directions. Finally in Chapter 5 we discuss both the Dirichlet and Neumann boundary homogenization problems for nonlinear operators when the boundary data is a random field.

There remain many important open questions related to homogenization and/or long time behavior for oscillatory boundary value problems and moving interfaces. This dissertation resolves some questions in this area, and develops techniques which hopefully can prove fruitful for addressing similar problems. For example, we expect that the techniques of Chapter 2 may be useful to address the convergence to equilibrium of the volume preserving mean curvature flow. In another direction, we hope that the techniques developed to analyze the continuity properties of the homogenized boundary condition in Chapter 4 will have application in the divergence form systems setting as well and in optimizing the rates of convergence for these problems.

1.1 Part I: A Model for Capillary Drops

The first part of this dissertation is based largely on a joint work with Inwon Kim [FK14] on a model for the contact line dynamics of capillary drops. We will consider $u : \mathbb{R}^2 \to [0,+\infty)$, compactly supported, to be the height profile of the droplet above the solid surface. In terms of u the droplet occupies the region $\{(x,z) : 0 < z < u(x,t)\}$ in $\mathbb{R}^2 \times [0,\infty)$. The volume of this region, which can be expressed as $\int_{\mathbb{R}^2} u(x,t)\,dx$, is assumed to be invariant under the flow, this is a consequence of the incompressibility of the underlying fluid flow and the conservation of mass. In particular we are ignoring mechanisms, such as evaporation, that allow for a change in the total mass present in the drop. The height profile will satisfy the following free boundary problem in $\mathbb{R}^2$,

$$\begin{cases} -\Delta u(x,t) = \lambda(t) & \text{in} \quad \Omega_t := \{u(\cdot,t) > 0\}, \\ \frac{\partial_t u}{|Du|} = |Du|^2 - 1 & \text{on} \quad \partial\Omega_t. \end{cases} \tag{1.1.1}$$

Here $\lambda(t)$ is a Lagrange multiplier that enforces the volume constraint,

$$\int u(x,t)\,dx = \text{Vol} \ \text{ for all } \ t > 0.$$

The positivity set of $u(\cdot,t)$, Ω_t, is called the wetted set, $\partial\Omega_t$ is called the contact line and $\frac{\partial_t u}{|Du|}$ is the normal velocity of the contact line. One thing to note immediately is that the evolution can be described entirely in terms of the wetted set Ω_t once Vol is fixed. In particular, the height function u and the Lagrange multiplier $\lambda(t)$ are uniquely determined by Ω_t and Vol. This is a reflection of the model being "quasi-static", the physical derivation of the model has assumed a separation of time scales between local equilibration of the shape of the water droplet above its wetted set Ω_t and the movement of the contact line. In terms of the mathematical model a quasi-static assumption typically results in a coupling between an elliptic equation and a time evolution as is the case here.

Formally one can check that the evolution has an associated Lyapunov functional,

$$J(u) := \int_{u>0} |Du|^2 \, dx + |\{u > 0\}|,$$

which is decreasing along the flow. Moreover, it turns out, the evolution is a gradient flow of J. Thus one expects convergence to a local or global energy minimizer in the long time limit. One can check using rearrangement techniques that the global minimizer of J in H^1 with the volume constraint is of the form,

$$u_{\min}(x) = \frac{\lambda_*}{4}(r_*^2 - |x|^2)_+ \text{ with } \lambda_*, r_* \text{ uniquely determined by Vol.}$$

In fact the energy also has many local minima corresponding to finite or possibly countable unions of disjoint round drops. Global in time weak energy solutions of the problem have been shown to exist [GK11], but these lack sufficient regularity to prove any kind of convergence to equilibrium. As a result, especially when the volume is small so that the contact line is receding, there could be many changes of topology.

Topology changes pose a barrier to proving existence and uniqueness of a sufficiently strong notion of solution. For this reason, at least as a first step in attacking the general problem, we are interested in finding conditions on the initial data that will rule out any topological changes of the wetted set. One would hope to use viscosity solution based methods since, at least when $\lambda(t)$ is a pre-selected function of time, the problem does formally satisfy a comparison principle. This allows us to define viscosity solutions, which are much easier to work with than energy solutions when it comes to considering geometric properties of the free boundary. Unfortunately once the volume constraint is imposed the standard comparison principle no longer holds.

In order to show global existence of a sufficiently regular solution we need some kind of a priori estimate controlling the geometry of the evolving domain. Although, as mentioned, the standard comparison does not hold for the volume preserving flow, there are still some forms of comparison that do work. The main idea is to compare the solution with its push-forward under some symmetry of the flow. A scaling symmetry is often useful for this purpose, and we make use of an approximate scaling symmetry for the droplet problem to prove a short time existence and regularity result. This is insufficient to study long time behavior, instead we were able to make use of the reflection symmetry of the flow.

Consider the orthogonal reflection $\phi_H : \mathbb{R}^2 \to \mathbb{R}^2$ through a hyperplane $H(\sigma, x_0) = \{(x - x_0) \cdot \sigma = 0\}$, $x_0 \in \mathbb{R}^2$ and a unit vector σ. Note that ϕ_H is a symmetry of the evolution, that is $u(\phi_H(\cdot), t)$ is a solution of the volume preserving evolution when u is, and moreover it has the same Lagrange multiplier $\lambda(t)$. We say that Ω_0 has reflection comparison with respect to H if,

$$\Omega_0 \cap \{(x - x_0) \cdot \sigma > 0\} \subseteq \phi_H(\Omega_0) \cap \{(x - x_0) \cdot \sigma > 0\} \text{ or the reverse.}$$

The property "has reflection comparison with respect to H" is preserved by the flow. This follows from the comparison principle in the domain $\{(x - x_0) \cdot \sigma > 0\}$ using that $u(\cdot, t)$ and $u(\phi_H(\cdot), t)$ have the same Lagrange multiplier and they are equal along the boundary $H(\sigma, x_0)$ since the map ϕ_H restricted to H is the identity. As a result, any property of initial domains Ω_0 defined entirely in terms of reflection comparisons with respect to a fixed class of hyperplanes in $\mathbb{R}^2$ will be preserved under the flow.

Based partially on this observation we make the following definition.

Definition 1.1.1. We say $\Omega_0 \subset \mathbb{R}^2$ has ρ-reflection for some $\rho \geq 0$ if there exists $x_0 \in \Omega_0$ such that $\overline{B_\rho(x_0)} \subset \Omega_0$ and Ω_0 has reflection comparison with respect to every hyperplane $H \subset \mathbb{R}^2$ which does not intersect $B_\rho(x_0)$.

Based on the comments above ρ-reflection is preserved by the flow. We will see in Chapter 2 that the ρ-reflection property is a measure of the symmetry of Ω_0. If $\rho = 0$ then Ω_0 is a ball centered at x_0. When $\rho > 0$, ρ-reflection implies a strong star-shapedness property of Ω_0 which depends only on lower bound for the distance between $\partial B_\rho(x_0)$ and the complement of Ω_0. In particular, if the initial data for the droplet problem has ρ-reflection, then the wetted set will remain connected and its boundary will have a uniform Lipschitz property as long as the contact line does not recede too much and cross into $B_\rho(x_0)$. This reduces the control of regularity of the free boundary to control of the location. When the volume is sufficiently large one can show that the free boundary does not recede too much and therefore the wetted set remains uniformly regular. This construction provides the a priori estimate we desired.

Our approach to existence of solutions is based on a discrete time approximation scheme for the gradient flow. This kind of approximation procedure is often called a JKO scheme after the paper of Jordan, Kinderlehrer and Otto [JKO98]. There are other possible schemes to show existence, but the discrete gradient flow has the convenient property that the energy decrease holds by the definition. This avoids some difficulty with proving that the formal energy decay calculation is rigorous, but of course it introduces other difficulties. We will now attempt to give an informal description of the issues that arise.

First, it is useful to formulate the problem entirely in terms of the wetted set. For each $h > 0$ we define iteratively $\omega_k \subset \mathbb{R}^2$, the solution of our discretized gradient flow scheme, by the variational problem,

$$\omega_k := \operatorname*{argmin}_X \left[\min\left\{ \int |Du|^2 : \int u = \text{Vol and } u = 0 \text{ on } \partial\omega \right\} + |\omega| + \frac{1}{2h}\operatorname{dist}(\omega, \omega_{k-1})^2 \right].$$

Here "dist" is a metric on an appropriate class of subsets of $\mathbb{R}^2$ chosen so that the resulting gradient flow leads to the "correct" evolution. The difficulty we focus on explaining here is about the class of sets being minimized over. One approach is to add a small perimeter term as a regularization and then minimize over all sets of finite perimeter (namely, the Cacciopoli sets), see [GK11]. The problem with this approach is that it is very difficult to control what topological changes may occur in this formulation. Since we expect ρ-reflection to be preserved by the limiting continuous flow, we force this to hold for the discrete gradient flow by constraining the allowed class of sets X in the minimization to have ρ'-reflection for some ρ' slightly larger than ρ. At a very formal level, because we expect the unconstrained minimizer to be "in the interior" of the constraint set, the constrained and unconstrained minimizer should be the same. Using the preservation of ρ-reflection for viscosity solutions and a comparision principle property of the discrete gradient flow proved by [GK11] we are able to make a rigorous argument out of these formal ideas and prove that the continuum limit of the JKO schemes defined above is in fact a viscosity solution.

Combining the above results we obtained the following theorem:

Theorem 1.1.2. *Suppose that Ω_0 has ρ-reflection with some $0 < \rho \leq \frac{1}{10}\mathrm{Vol}^{\frac{1}{N+1}}$. Then there exists a global viscosity solution of* (1.1.1) *such that the energy $J(u(\cdot,t))$ is decreasing and, modulo translation, u converges uniformly to $u_{\min}$.*

We will show that ρ-reflection for very small ρ is equivalent to being close in Lipschitz norm to a ball. This is one way to interpret the condition on ρ of the above Theorem. We remark however that the restriction on ρ is more naturally considered as a condition that Vol is sufficiently large rather than that ρ is sufficiently small (i.e. the domain is close in Lipschitz norm to a ball). Also we remark that when one assumes an a-priori uniform $C^{1,\alpha}$ bound for the free boundary the modulo translation can be removed in the convergence.

These results will be discussed in full detail in Chapter 2.

1.2 Part II: Homogenization of Oscillating Boundary Value Problems

The second part of this dissertation is divided into several chapters, each focused on one specific aspect of homogenization of oscillating boundary conditions. To introduce this topic we should first describe some background from the theory of homogenization of second order elliptic equations with non-oscillating boundary conditions. In doing so we will naturally arive at a problem involving oscillatory boundary data. We will focus on non-divergence form operators. All of the problems we discuss can also be posed for divergence form equations, both scalar and system cases, but we will not treat these problems here. For simplicity we will begin by describing the linear case but eventually we will be more interested in general fully nonlinear equations.

In the simplest case one considers the following problem set in a domain $U \subset \mathbb{R}^d$, $d \geq 1$, with a parameter $\varepsilon \ll 1$,

$$\begin{cases} -\mathrm{Tr}(A(\frac{x}{\varepsilon})D^2u^\varepsilon) = f & \text{in} \quad U \\ u^\varepsilon = g & \text{on} \quad \partial U. \end{cases} \tag{1.2.1}$$

We take A to be a $\mathbb{Z}^d$ periodic, smooth, $d \times d$ matrix valued function on $\mathbb{R}^d$ satisfying the uniform ellipticity assumption,

$$\lambda \mathrm{Id} \leq A(x) \leq \Lambda \mathrm{Id} \ \text{ for all } \ x \in \mathbb{R}^d.$$

Here, as is usual, we interpret $M \geq N$ for $d \times d$ matrices M,N to mean $M - N$ is non-negative definite. The question is to determine an effective limiting operator $\overline{A}$ so that,

$$u^{\varepsilon} \to \overline{u} \ \text{ as } \ \varepsilon \to 0.$$

where $\overline{u}$ solves the homogenized problem,

$$\begin{cases} -\mathrm{Tr}(\overline{A}D^2\overline{u}) = f & \text{in} \quad U \\ \overline{u} = g & \text{on} \quad \partial U. \end{cases} \tag{1.2.2}$$

After computing $\overline{A}$ one is interested in the rate of convergence and more amibitiously a full asymptotic expansion of $u^{\varepsilon} - \overline{u}$ in powers of ε. The following formal asymptotic expansion is expected,

$$u^{\varepsilon}(x) = \overline{u} + \varepsilon(v_1(x, \tfrac{x}{\varepsilon}) + v_1^{\varepsilon,bl}(x)) + \varepsilon^2(v_2(x, \tfrac{x}{\varepsilon}) + v_2^{\varepsilon,bl}(x)) + \cdots .$$

Here $v_k(x,y)$ are called the interior correctors, they are defined in $U \times \mathbb{R}^d$ and are $\mathbb{Z}^d$ periodic in the y variable. The definitions of the interior correctors are somewhat complicated so we will not go into full detail here. See Kim and Lee [KL14] for a derivation and rigorous justification of this expansion. We just mention that v_j are defined in terms of $\overline{u}, v_1, \ldots, v_{j-1}$ and the solution of a "cell problem" on $\mathbb{R}^d/\mathbb{Z}^d$ which depends only on A and not on U. In this sense they are simple to compute, one just needs to solve several pde problems with smooth coefficients. The problem is that v_j are not zero on ∂U. To correct for this the so-called boundary layer correctors $v_k^{\varepsilon,bl}$ must be included in the asymptotic expansion. These solve,

$$\begin{cases} -\mathrm{Tr}(A(\tfrac{x}{\varepsilon})D^2 v_k^{\varepsilon,bl}) = 0 & \text{in} \quad U \\ v_k^{\varepsilon,bl} = -v_k(x, \tfrac{x}{\varepsilon}) & \text{on} \quad \partial U. \end{cases} \tag{1.2.3}$$

Here we have come upon yet another homogenization problem in a general domain, now even more difficult than the original problem since the boundary condition is also oscillating at the ε scale. This type of problem we call boundary data homogenization. We would like to show that this problem has some averaging as $\varepsilon \to 0$, i.e. that we have the convergence,

$$v_k^{\varepsilon,bl} \to \overline{v}_k^{bl} \ \text{ as } \varepsilon \to 0$$

for $\overline{v}_k^{bl}$ solving a Dirichlet problem in U for the homogenized operator $\overline{A}$ and some *homogenized boundary condition* determined by A, v_k and U. This would allow for an improved understanding of the asymptotic expansion above since $v_2^{\varepsilon,bl}$ (the first non-trivial boundary layer corrector in this case), to a first approximation, could be replaced with $\bar{v}_2^{bl}$ incurring an error of order $o(\varepsilon^2)$. In the following subsections we will discuss boundary data homogenization problems which come in an abstract form which includes the boundary layer corrector problem (1.2.3) as a particular case.

1.2.1 Homogenization of Oscillating Dirichlet Data: Periodic Case

As described above we are interested to understand the asymptotic behavior as the micro-scale $\varepsilon \to 0$ of the solution u^ε to

$$\begin{cases} F(D^2u^\varepsilon, \frac{x}{\varepsilon}) = 0 & \text{in} \quad U \\ u^\varepsilon = g(x, \frac{x}{\varepsilon}) & \text{on} \quad \partial U. \end{cases}$$

Here F is a uniformly elliptic operator, possibly nonlinear, and $F(M,y)$ and $g(x,y)$ are assumed to be $\mathbb{Z}^d$ periodic in y. We restrict to $d \geq 2$, when $d = 1$ the problem is exactly solvable and there will be no convergence of u^ε as $\varepsilon \to 0$ except in trivial cases.

Even in $d \geq 2$ there may be no convergence as $\varepsilon \to 0$, we illustrate this with a simple example. Let $U = (-1,1)^2$ and $g(x,y) = \sin(2\pi y_1)$. Then $u^\varepsilon(x) \equiv \sin\frac{2\pi}{\varepsilon}$ on $\{y_1 = 1\} \cap \partial U$. Since this boundary condition is uniformly smooth on $\{y_1 = 1\} \cap \partial U$ continuity up to the boundary implies that

$$|u^\varepsilon(1-\delta, 0) - \sin\frac{2\pi}{\varepsilon}| \leq \frac{1}{2}$$

for a δ sufficiently small and independent of ε. In particular $u^\varepsilon(1-\delta,0)$ cannot converge as $\varepsilon \to 0$. The problem in this particular example is that ∂U contains a flat part with normal direction aligned with the periodicity lattice of the boundary data. This is a central issue in the periodic boundary homogenization problem, and we will explain it in greater detail below.

As in the case of interior homogenization we first attempt to formally derive a cell problem determining the homogenized operator, or in this case the homogenized boundary condition. One may immediately note that u^ε, the solution of (4.1.1), is highly oscillatory near the boundary of U when ε is small. There is no uniform continuity in the whole of U as $\varepsilon \to 0$. It is reasonable to ask why any kind of averaging should even be expected in the limit $\varepsilon \to 0$. To begin to answer this question we consider a rescaling of u^ε near a boundary point $x_0 \in \partial U$,

$$v^\varepsilon(y) = u^\varepsilon(x_0 + \varepsilon y).$$

Let us denote σ_{x_0} the inward unit normal to U at x_0. The behavior of u^ε near ∂U at the unit scale, but far at the ε scale, can be understood through $v^\varepsilon(R\sigma_{x_0})$ for $1 \ll R \ll \varepsilon^{-1}$. For example, if we could identify a unique constant μ such that $v^\varepsilon(R_\varepsilon \sigma_{x_0}) \to \mu$ for any sequence R_ε with $R_\varepsilon \to \infty$ and $\varepsilon R_\varepsilon \to 0$, then this would be a good candidate for the homogenized boundary condition at x_0. In this case the solution u^ε would be oscillatory in a boundary layer of size $\sim \varepsilon$, and outside of this boundary layer averaging would occur.

Now that we have explained the usefulness of considering v^ε, we proceed to explain why $v^\varepsilon(R\sigma_{x_0})$ should be converging to a unique constant. Furthermore we define a "cell problem" which identifies the correct constant in a systematic way. We consider the equation solved by the v^ε,

$$\begin{cases} F(D^2 v^\varepsilon, y + \varepsilon^{-1}x_0) = 0 & \text{in} \quad \varepsilon^{-1}(U - x_0) \\ v^\varepsilon = g(y + \varepsilon^{-1}x_0) & \text{on} \quad \varepsilon^{-1}(\partial U - x_0). \end{cases}$$

Since F and g are assumed to be $\mathbb{Z}^d$ periodic in y, $\varepsilon^{-1}x_0$ can be replaced by $\tau_\varepsilon = \varepsilon^{-1}x_0 \mod \mathbb{Z}^d$. Since U is a smooth domain, as $\varepsilon \to 0$ the domain $\varepsilon^{-1}(\partial U - x_0)$ converges on compact sets to the half space $\{y \cdot \sigma_{x_0} > 0\}$. This motivates the definition of the "cell problem" solution $v_{\sigma,\tau}(\cdot;(\psi,F))$ for a direction $\sigma \in S^{d-1}$, a $\tau \in [0,1)^d$ and a continuous $\mathbb{Z}^d$-periodic ψ by,

$$\begin{cases} F(D^2 v_{\sigma,\tau}, y + \tau) = 0 & \text{in} \quad P_\sigma := \{y \cdot \sigma > 0\} \\ v_{\sigma,\tau} = \psi(y + \tau) & \text{on} \quad \partial P_\sigma. \end{cases}$$

We expect at a formal level that as $\varepsilon \to 0$,

$$|v^\varepsilon(y) - v_{\sigma_{x_0},\tau}(y; g(x_0,\cdot))| \to 0 \ \text{ along subsequences with } \ \tau_\varepsilon \to \tau.$$

From this identification we can replace the problem of understanding $v^\varepsilon(R\sigma_{x_0})$ for R large and εR small with the easier problem of understanding the limit $v_{\sigma,\tau}(R\sigma)$ as $R \to \infty$ for every $\tau \in [0,1)^d$.

The fundamental problem in this periodic setting is that this limit of $v_{\sigma,\tau}(R\sigma)$ may depend on τ. In fact this is generically case at the rational directions $\sigma \in \mathbb{R}\mathbb{Z}^d$. In that case the hyperplane orthogonal to σ is not dense in the unit square modulo $\mathbb{Z}^d$, and as a result the $v_{\sigma,\tau}(R\sigma)$ may have different limits at infinity for different values of $\tau \in [0,1)^d$. This is exactly the issue that appeared in the example of non-homogenization we gave at the beginning of this section. We return to that example to see how the non-homogenization manifests itself at the level of the cell problem. In that case the inner normal of the problematic boundary component was $\sigma = e_1$ and the boundary condition was $\psi(y) = \sin(2\pi y_1)$. Here $v_{\sigma,\tau}(y) \equiv \sin(2\pi\tau \cdot e_1)$ a constant in y but varying in τ. In terms of v^ε this would mean that along different subsequences of $\varepsilon \to 0$ where $\tau_\varepsilon = \varepsilon^{-1}x_0 \bmod \mathbb{Z}^d$ converges to different limits, $v^\varepsilon(R\sigma_{x_0})$ would approach different limits as well.

For irrational directions σ the distribution of g on $P_\sigma + \tau\sigma$ is, in an appropriate sense, invariant with respect to τ. We will show in Chapter 3 that there exists a limit $\mu(\sigma,\psi,F)$, the so-called *boundary layer tail* of $v_{\sigma,\tau}$, such that

$$\sup_{\tau\in[0,1)^d} \sup_{y\in\partial P_\sigma} |v_{\sigma,\tau}(y+R\sigma;\psi) - \mu(\sigma,\psi,F)| \to 0 \ \text{ as } \ R \to \infty. \tag{1.2.4}$$

Without too much additional difficulty we will also be able to show that $\mu(\cdot,\psi,F)$ is continuous at irrational directions.

In the linear case when the interior operator F has no y dependence one can also check that $\mu(\sigma,\psi,F) = \fint_{[0,1)^d} \psi \, dx$, just the average of ψ over the torus, at every irrational direction. In particular one may wonder whether there is some similar regularity in general? Or even could μ be independent of σ at the irrational directions for y-dependent and/or nonlinear operators? Choi and Kim [CK14] showed that, when F is rotation invariant, $\mu(\cdot,\psi,F)$ has a continuous extension from the irrational directions to the entire unit sphere for the Neumann version of this problem. They used this result to prove homogenization of the Neumann problem for rotation invariant operators. Actually, as we will show later, their methods also permit to show that for linear operators μ admits a continuous extension. Note that this is truly a separate case, the only rotation invariant linear operator is the Laplacian. We continue along this line of investigation in Chapter 4.

The methods of [CK14] suggested to us that, in contrast to the rotation invariant case, it might be possible to show that μ can have true discontinuities at rational directions. In fact we showed that this phenomenon is generic in the topological sense at least for homogeneous operators. We expect that the result should also hold for inhomogeneous operators since morally there is no reason the situation should be better there, however our current methods are not capable of handling this case. Call X the space of pairs (ψ, F) of continuous boundary data and spatially homogeneous uniformly elliptic operators. We avoid the technical details at this point and simply say that under a very natural metric on this space the following result holds:

Theorem 1.2.1. *The set of (ψ, F) such that $\mu(\cdot, \psi, F)$ is discontinuous at every rational direction is residual, i.e. it is an intersection of countably many open dense sets in X.*

The same framework that we use to prove this result also facilitates an improved, quantitative, result about the continuity of μ when the operator is rotation invariant or linear. In the linear case we optimized many of the steps and were able to obtain Hölder-$\frac{1}{d}$ continuity up to logarithmic factors, here $d \geq 2$ is the spatial dimension.

Theorem 1.2.2. *When F is rotation invariant μ has a uniform Hölder-α modulus of continuity at every irrational direction with α depending on the d and the ellipticity ratio. When F is linear there is a uniform Hölder-$\frac{1}{d}$ modulus continuity up to logarithmic factors.*

Both of these results will be discussed in full detail in Chapter 4.

Now, taking into account the difficulties outlined above, we return to the problem of showing homogenization in general domains. From the solution of the cell problem we expect that, along subsequences, the u^ε will converge to a solution of,

$$\begin{cases} \overline{F}(D^2\overline{u}) = 0 & \text{in} \quad U \\ u = \overline{g}(x) := \mu(\sigma_x, g(x,\cdot), F) & \text{on} \quad \partial U \setminus \Gamma(U). \end{cases} \tag{1.2.5}$$

Here we have called,

$$\Gamma(U) := \{x \in \partial U : \sigma_x \text{ is a rational direction}\}.$$

The analysis at irrational directions will allow us to show that $\overline{g}$ is continuous on $\partial U \setminus \Gamma(U)$. The problem is that such a solution may not be unique since the boundary condition is not defined uniquely on the entire boundary. For example when ∂U contains a flat part with rational normal direction then, generally speaking, there will be multiple subsequential limits. Although these limits can be described quite nicely in terms of the cell problem this kind of result is unstable with respect to a small change of the normal direction and is therefore not fully satisfactory. In the linear case if $\Gamma(U)$ has zero harmonic measure then there will be a uniquely defined solution of the homogenized problem. Under that case we would be able to prove $u^\varepsilon \to u$. Unfortunately things are not so simple in the nonlinear setting.

When the interior operator is nonlinear, the dependence of the interior values on variations of the boundary data on small sets of the boundary is still not well understood. It seems quite possible that in fact measure zero sets of the boundary could influence the interior values of the solutions. We describe briefly by heuristics why the problem is difficult. Without a convexity/concavity assumption the regularity of solutions to fully nonlinear elliptic equations, at least in high dimensions, is not better than $C^{1,\varepsilon}$ or $W^{2,\varepsilon}$ for a small $\varepsilon > 0$. By the $W^{2,\varepsilon}$ estimate the Hessian exists almost everywhere and at formal level solutions of fully nonlinear elliptic equations can be considered to solve their own linearization, a linear uniformly elliptic non-divergence form pde with bounded measurable coefficients. It is known that the harmonic meausure for such linear operators can be completely singular with respect to the surface measure on the boundary, even supported on a set of lower Hausdorff dimension [Wu96].

For our homogenization problem however we do not need to completely resolve this well-posedness question. For example, when the domain U is uniformly convex the set $\Gamma(U)$ is countable and therefore has Hausdorff dimension 0. This is a significantly stronger condition than having zero surface measure. We obtained a well-posedness result partially explaining when there exists a unique solution of the homogenized problem. This is sufficient for most cases of interest in the periodic homogenization problem, but it does not fully resolve the underlying question about the influence of small sets of ∂U on the solutions of the Dirichlet problem.

Theorem 1.2.3. *There exists $0 < \beta < d-1$ depending on the ellipticity ratio of $\overline{F}$ and the dimension such that when the Hausdorff dimension of $\Gamma(U) < \beta$ there exists a unique solution of the homogenized problem* (1.2.5).

Based on this result, which is of course not specific to the setting of homogenization, we obtained the following Theorem.

Theorem 1.2.4. *Under the conditions of Theorem 1.2.3,*

$$u^\varepsilon \to \overline{u} \text{ locally uniformly in } U \text{ as } \varepsilon \to 0.$$

Both of these results will be discussed in detail in Chapter 3.

1.2.2 Homogenization of Oscillating Dirichlet Data: Random Case

Finally in Chapter 5 we will discuss the boundary data homogenization problem with random data. We will consider both Dirichlet and Neumann problems, but for now we just describe the Dirichlet case. Ideally we would like to understand the behavior of

$$\begin{cases} F(D^2u^\varepsilon, \frac{x}{\varepsilon}, \omega) = 0 & \text{in} \quad U \\ u^\varepsilon = g(x, \frac{x}{\varepsilon}, \omega) & \text{on} \quad \partial U, \end{cases} \tag{1.2.6}$$

with $\omega \in \Omega$ some probability space and g, F stationary ergodic random fields on $\mathbb{R}^d$. We expect homogenization to hold for random interior operators but, as will be explained in more detail in Chapter 5, we are currently only able to handle spatially homogeneous (i.e. non-random) interior operators F. The issue is that we do not yet know how to estimate the dependence of the interior values of the solution on the interior randomness. For this reason we assume from here on that F has no dependence on (y, ω).

Since the periodic setting embeds into the stationary ergodic setting we know that trying to treat the general stationary ergodic case will result in some serious difficulties. The problem, as before, is that ergodicity is not necessarily preserved under restricting the permitted translations to lie in a hyperplane. A property more amenable to these restrictions is strong mixing.

In order to avoid this difficulty with the ergodicity and to give as simple as possible an explanation of our results we will also consider here a more specialized setting. As far as we are aware this is not directly embeddable as special case of (1.2.6). Instead of restricting a stationary ergodic random field on $\mathbb{R}^d$ to ∂U, we will instead locally lift random fields on $\mathbb{R}^{d-1}$ up to ∂U. Given a smooth partition of unity of ∂U, $(\varphi_j)_{j=1}^N$, points $x_j \in \operatorname{supp}(\varphi_j) \cap \partial U$, diffeomorphisms ζ_j mapping an open subset of $\mathbb{R}^{d-1}$ onto the interior of supp(φ_j), and a collection of stationary ergodic random fields on $\mathbb{R}^{d-1}$, $g_j, (x', \omega)$, we define a random field on ∂U,

$$g^{\varepsilon}(x, \omega) = \sum_{j=1}^{N} \varphi_j(x) g_j(\varepsilon^{-1} \zeta_j^{-1}(x), \omega).$$

This g^{ε} will be our Dirichlet boundary data for another random homogenization problem analogous to (1.2.6). Informally we refer to the general problem as "restricting from the whole space" and the specialized problem we introduce here as "lifting from hyperplanes". The lifting from hyperplanes problem is no longer associated with the boundary layer correctors from interior homogenization described earlier, but it could equally well have interest for physical problems.

A formal derivation similar to the one carried out in the previous section allows us to guess the form of the cell problem. For the restricting from the whole space problem the cell problem has the familiar form,

$$\begin{cases} F(D^2 v) = 0 & \text{in} \quad \{y \cdot \sigma > 0\} \\ v = \psi(y, \omega) & \text{on} \quad \partial\{y \cdot \sigma > 0\}, \end{cases} \tag{1.2.7}$$

where ψ will be at least a strongly mixing random field on $\mathbb{R}^d$ which is stationary with respect to $\mathbb{R}^d$ translations. This $\mathbb{R}^d$ translation stationarity is important, if for example we only had $\mathbb{Z}^d$ translation stationarity we would again run into problems with rational and irrational directions. For the lifting from hyperplanes case the cell problem takes the slightly different form,

$$\begin{cases} F(D^2 v) = 0 & \text{in} \quad \{y \cdot \sigma > 0\} \\ v = \sum_{j=1}^{N} a_j \psi_j(T_j y, \omega) & \text{on} \quad \partial\{y \cdot \sigma > 0\}, \end{cases} \tag{1.2.8}$$

where $\sum a_j = 1$ and $T_j : \mathbb{R}^{d-1} \to \partial\{y \cdot \sigma > 0\}$ are invertible linear transformations. In terms of the original boundary data g^{ε}, the cell problem at a point $x \in \partial U$ would have $a_j = \varphi_j(x)$, $\psi_j = g_j$, and $T_j = D\zeta_j\big|_{\zeta_j^{-1}(x)}$. We emphasize that the important difference from the previous cell problem (1.2.7) is that ψ_j may be permitted to only have stationarity with respect to $\mathbb{Z}^{d-1}$ translations. The problem is to show that there exists a limit at infinity,

$$\lim_{R \to \infty} v(R\sigma, \omega) = \mu(\psi, T, a, F) \text{ almost surely.} \tag{1.2.9}$$

Here ψ, T, a refer to the collections $(\psi_j)_{j=1}^N$, $(T_j)_{j=1}^N$ and $(a_j)_{j=1}^N$ respectively. In fact for the general domain problem, unlike in the periodic case, it will also be necessary for us to show a rate of convergence even to achieve qualitative homogenization.

In the case of boundary data given by lifting from hyperplanes the homogenized problem is formally,

$$\begin{cases} F(D^2\overline{u}) = 0 & \text{in} \quad U \\ u^\varepsilon = \overline{g}(x) & \text{on} \quad \partial U, \end{cases} \tag{1.2.10}$$

where

$$\overline{g}(x) = \mu((g_j)_{j=1}^N, (D\zeta_j\big|_{\zeta_j^{-1}(x)})_{j=1}^N, (\varphi_j(x))_{j=1}^N, F).$$

The goal is to show u^ε converge almost surely to $\overline{u}$ solving (1.2.10).

To state our results we will focus on the simplest setting for the random medium, an i.i.d. random checkerboard of uniform $[-1,1]$ random variables. The restricting from the whole space problem has some difficulties which we wish to avoid in this introduction. For this reason we now restrict to discussing the case of lifting from the hyperplane. We take ξ_k^j for $1 \leq j \leq N$ and $k \in \mathbb{Z}^{d-1}$ to be i.i.d. uniform random variables in $[-1,1]$. Let ρ be some smooth compactly supported bump function. We define,

$$g_j(x,\omega) = \sum_{k \in \mathbb{Z}^n} \xi_k^j(\omega)\rho(x-k).$$

This random field will only be stationary with respect to $\mathbb{Z}^n$.

We now give a brief description the techniques we employ to show that the cell problem (1.2.8) homogenizes when F is non-random. Our method relies on concentration inequalities for Lipschitz functions of many independent random variables. In the case of the i.i.d. uniform random checkerboard described above the necessary concentration result is a consequence of the log-Sobolev inequality satisfied by the random field. We show that the solution of the cell problem (1.2.8), $v(R\sigma;\xi)$, is a Lipschitz function of the boundary data ξ with Lipschitz constant decaying to zero as $R \to \infty$. This allows us to show that $v(R\sigma;\xi)$ concentrates about its mean. We combine this with PDE arguments to show that the sequence of means $\mathbb{E}v(R\sigma;\xi)$ has a limit. This will imply, in particular, that $v(R\sigma;\xi)$ converges almost surely to a non-random constant as $R \to \infty$.

In the general domain case we are able to use our result for the cell problem in local neighborhoods to control the difference $u^\varepsilon - \overline{u}$ on an event of high probability. Our results are summarized by the following Theorem.

Theorem 1.2.5. *Let the random media g^ε be of "random checkboard"-type as described above. The cell problem* (1.2.8) *homogenizes at each $x \in \partial U$, i.e.* (1.2.9) *holds and furthermore there is an algebraic rate of convergence in probability. The general domain problem homogenizes, that is u^ε converge for all $x \in U$ almost surely to $\overline{u}$ solving* (1.2.10)*. Moreover there is an algebraic rate of convergence in probability in any compact subset of U.*

There are many details left out of this description see Chapter 5 for the full results and proofs.

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Part I

Droplet Model

CHAPTER 2

Dynamic Stability of Equilibrium Capillary Droplets

We consider the function $u(x,t) : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ solving the following free boundary problem

$$\begin{cases} -\Delta u(x,t) = \lambda(t) & \text{in} \quad \{u > 0\}, \\ \mathscr{V} = F(|Du|) & \text{on} \quad \partial\{u > 0\}, \end{cases} \tag{P-V}$$

where the Lagrange multiplier $\lambda(t)$ is associated to the volume constraint

$$\int u(\cdot,t)dx \equiv \text{Vol} > 0. \tag{2.0.1}$$

(see Figure 1.)

$\mathscr{V} = \mathscr{V}_{x,t}$ given above denotes the (outward) normal velocity of the free boundary $\partial\{u > 0\}$ at (x,t). Indeed we denote our problem as (P-V) to emphasize the volume-preserving feature of the problem (2.0.1). In contrast, later on we will introduce a corresponding problem called (P-λ) where $\lambda(t)$ is a priori given function of time, for which the volume is not necessarily preserved but comparison principle holds.

The prescribed normal velocity $F : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function with $F(1) = 0$, where 1 denotes the equilibrium contact angle of the profile u with the surface. See Section 2 for a rigorous formulation of weak solutions for (P-V).

In two dimensions, (P-V) is a simplified model to describe a liquid droplet resting on a flat surface. The model is quasi-static in the sense that the speed of the contact line is much slower than the capillary relaxation time: see below for more discussion on the derivation of the model (also see [Gre78]). In this context, u denotes the height of the drop. We call $\Omega_t := \{u(\cdot,t) > 0\}$ the *wetted set* and the free boundary $\Gamma_t := \partial\{u(\cdot,t) > 0\}$ the *contact line* between the liquid and the flat surface. The function

$$F : [0,+\infty) \to \mathbb{R}$$

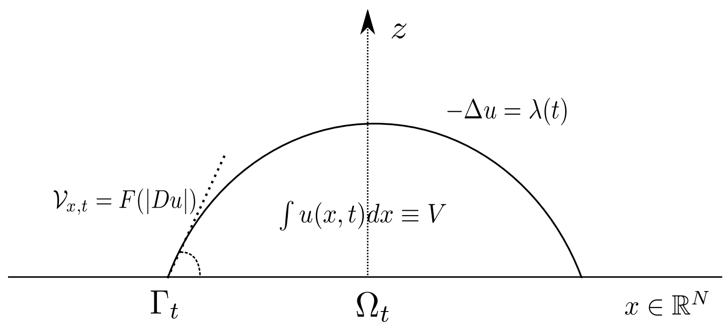

Figure 2.1: Description of the problem

represents the dependence of the normal velocity of the contact line Γ_t on the contact angle. The deviation of the contact angle $|\nabla u|$ on $\Gamma_t(u)$ from the equilibrium value 1 is responsible for the motion of the contact line. The relation between the velocity of the contact line and the contact angle $|\nabla u|$ is not a settled issue, and various velocity laws have been proposed and studied (e.g. [BR97], [Gre78], [HM93]) in the fluids literature. This motivates us to study our problem (P-V) with the most general possible F, at least for the study of geometric properties of solutions.

In section 4 and 5, we will focus on $F(|Du|) = |Du|^2 - 1$ for simplicity of presentation, since the associated energy structure is simplest for this choice of F. The approach we present nevertheless will apply to the general F, as we will discuss in section 4.

Our aim is to address the long time behavior of the solution of (P-V). Note that the equilibrium solution $v(x)$ for (P-V) solves

$$\begin{cases} -\Delta v(x) = \lambda & \text{in} \quad \{v > 0\}, \\ |Dv| = 1 & \text{on} \quad \partial\{v > 0\}. \end{cases} \tag{EQ}$$

Assuming that v is $C^{2,\alpha}$, the classical result of Serrin [Ser71] based on Alexandrov's moving planes method [Ale62] (also see for example [BN91], [GNN79]) yields that v is radial. We would like to show the dynamic stability of the above result, in the context of our model. More precisely, we would like to show that the solutions of (P-V) uniformly converges to the round drop given by (EQ). Such a stability result is, to the best of our knowledge, new in any model describing evolution of volume-preserving drops. (see [ABR99] and [BNS08] for stability results on stationary problems).

In general, the behavior of evolving drops is highly difficult to observe due to the diverse possibility of topological changes the drop may go through, which generates instabilities in the evolution. In the event of the topology changes it seems like a highly challenging task to track the long-time behavior of solutions, which would most likely end up looking like a multiple (or countable) collection of round drops. On the other hand, it is reasonable to expect that the drops stay simply connected if its initial shape is star-shaped with respect to some ball inside of the drop. This is what we confirm in terms of ρ-*reflection* (see Definition 2.2.10) which measures star-shapedness of a drop with respect to the reflection comparison. Numerical simulations suggest that initially convex drops may develop an wedge while it contracts (see e.g. [Gla05]), and thus our definition, which allows wedges on the free boundary, seems to be appropriate to address the global-time behavior of the drop.

Theorem 2.0.1 (Theorem 5.1)**.** *Suppose u_0 satisfies ρ-reflection and $\int u_0 = \mathrm{Vol}$, with $0 < \rho \leq \frac{1}{10}\mathrm{Vol}^{\frac{1}{N+1}}$. Then the following holds:*

(a) there exists an "energy" solution (see the definition in section 5) $u(x,t)$ of (P-V) *with initial data u_0 which stays star-shaped for all times and is Hölder continuous in space and time variable.*

(b) Any energy solution of (P-V) *with initial data u_0 converges uniformly modulo translation to the unique equilibrium solution v of* (EQ) *with volume V. Moreover the set $\{u(\cdot,t) > 0\}$ converges, modulo translation, to a ball in Hausdorff topology.*

Remark 2.0.6. It should be pointed out that the result is not a perturbative one. In fact for any initial base of the drop $\{u_0 > 0\}$ which is star-shaped with respect to some ball inside it (e.g. a star or a triangle), our theorem implies given that the drop has sufficiently large volume. Indeed the proof is based on geometric, moving-plane method-type arguments as mentioned above (see section 3). As a consequence we obtain explicit estimates on the size of the parameter ρ.

Remark 2.0.7. It may be possible to obtain the rate of the convergence by further investigation of the formal energy dissipation inequality carried out in the beginning of section 4. Unfortunately in our setting u is not regular enough for the calculation to go through.

Remark 2.0.8. Corresponding results were proved for convex solutions of the volume preserving, anisotropic mean curvature flow by Belletini et al. [BCC09], but their approach strongly depends on the level set formulation of the problem as well as the convexity preserving and regularizing feature of the mean curvature flow.

The use of maximum principle and moving plane-type arguments is motivated by the study of stationary problem by Serrin [Ser71]. It is an open question whether there exist other geometric properties besides ρ-reflection that are preserved throughout the evolution (P-V). In fact, finding such a geometric property and thus ensuring that the drop stays simply connected is one of the main novel features in our result. Let us point out that, in particular, it is unknown whether the convexity of the drop is preserved in the system (P-V).

2.0.3 The Model

The energy of a static droplet which occupies a subset of $\mathbb{R}^{N+1}$ resting on the plane $\{x_{N+1} = 0\}$ is given by

$$J(E) = P(E \cap \{x_{N+1} > 0\}) + \int_{E\cap\{x_{N+1}=0\}} \sigma d\mathscr{H}^N + \int_E \rho g x_{N+1} dx. \tag{2.0.2}$$

Here $P(F)$ is the perimeter of a subset F of $\mathbb{R}^{N+1}$ defined at least for bounded domains with Lipschitz boundaries. Stationary droplets correspond to minimizers of the energy J when the volume $|E| = \text{Vol}$ is fixed. The first term is the energy due to surface tension of the free surface of the droplet. The second term is the energy due to the adhesion between the droplet and the surface it rests on, σ is called the relative adhesion coefficient and in general it can be spatially dependent. In this dissertation however we will assume σ is constant. The third is a gravitational potential energy, ρ is the mass per unit volume of water and g the acceleration due to gravity. We will neglect the effects of gravity and assume that

$$g = 0$$

one expects this to be a reasonable approximation when the total volume is small.

We will make the further assumption that our droplets are the region under the graph of a function

$u : \mathbb{R}^N \to [0, +\infty)$,

$$E = \{(x', x_{N+1}) \in \mathbb{R}^N \times [0, +\infty) : x_{N+1} < u(x')\}.$$

Now, thinking of the droplet in terms of the function u, its height profile, the energy simplifies to become,

$$J(u) = \int_{\{u>0\}} \sqrt{1 + |Du|^2} dx + \sigma |\{u > 0\}|. \tag{2.0.3}$$

The Euler-Lagrange equation corresponding to fixed volume energy minimizers involves the mean curvature of the free surface. Finally, to simplify our analysis, we linearize (2.0.3) and taking $\sigma = -1/2$ for concreteness we get

$$\mathscr{J}(u) = \int_{\mathbb{R}^N} |Du|^2 dx + |\{u > 0\}|. \tag{2.0.4}$$

Some remarks are due on the account of the linearized energy. First of all let us mention that such linearization is most legitimate in the setting of small volume or with small gradients, even though our analysis is not restricted to such settings. We consider the linearized energy mainly because it is easier to fit into the setting of viscosity solutions theory. With the original energy, the resulting Euler-Lagrange equation for the height of the drop is that of prescribed mean curvature equation, which does not always have classical solutions if the support of the drop is less than C^2. While this may be only a technical issue and could be overcome with the variational techniques, we have made a deep investigation of the original energy as of yet.

Formally our problem (P-V) can be written as the gradient flow associated with the energy $\mathscr{J}(u)$ (see the heuristic argument in section 4). In this dissertation we show that, in the graph setting with the energy $\mathscr{J}(u)$, for a large class of initial data (see Remark 2.0.6) we are able to give a point-wise description on the movement of the contact line $\partial\{u > 0\}$ using the notion of viscosity solutions. We mention that in the general non-graph setting, and in a different time scale where evolution occurs only by evaporation and gravity, a minimizing movement scheme in the spirit of [Mie07] was carried out for the general energy $J(E)$ in [AD11] to derive a weak solution in the continuum limit, where the solution is given as an evolution of entire drop (sets of finite perimeters in $\mathbb{R}^{N+1}$).

2.0.4 Main challenges and strategy

The major difficulty of this problem is the lack of strong compactness (either in the associated energy given above or in the problem itself) which hinders most approximation arguments without geometric restrictions. In particular the standard comparison principle fails due to $\lambda(t)$, or, more fundamentally, the volume preserving nature of the problem. We emphasize that one cannot simplify the problem by replacing λ fixed in the problem, especially when one is interested in the long-time behavior of the solutions: in fact, one can show that, if we fix λ then most drops either vanish in finite time or grow to infinity. In comparison to other volume-preserving problems such as the volume-preserving mean curvature flow or the Hele-Shaw flow with surface tension, the evolution of our problem (P-V) is driven by the contact angle, not by higher-order regularizing terms such as the mean curvature. This aspect of the problem challenges, for example, the diffusive-interface approach which are present in many other problems.

To get around aforemented difficulties and establish the existence of solutions for (P-V), we utilize the energy structure of the problem. Formally, solutions of the problem (P-V) are gradient flows of an energy associated to the capillary drop problem. This formulation of the problem is best understood from the perspective of the wetted set. The energy of a capillary drop of volume V resting above a domain $\Omega \subset \mathbb{R}^N$ is taken to be

$$\mathscr{J}(\Omega) := \int_\Omega |Du|^2 + |\Omega| \tag{2.0.5}$$

where, for now, u is the height profile above the wetted set Ω and is defined by

$$u := \operatorname{argmin}\{\int |Dv|^2 : v \in H_0^1(\Omega),\ \int_{\mathbb{R}^N} v = \mathrm{Vol}\}. \tag{2.0.6}$$

Then one models the motion of the capillary droplet as a gradient flow of the functional $\mathscr{J}$ in an appropriate space of subsets of $\mathbb{R}^N$.

A rigorous justification of this approach is carried out using a regularized discrete gradient flow scheme in the space of Cacciopoli sets (sets of finite perimeter) in [GK11]. Since the analysis has been performed in the general setting which allows pinching and merging of droplet components, the resulting continuum solution is rather weak and is unstable under the variation of initial data. In this dissertation we show that much stronger results can be obtained when the initial data satisfies the ρ-reflection (see Theorem 2.0.1). Our analysis will be based on the aforementioned energy structure of the problem with geometric arguments (reflection maximum principle) as well as a modified viscosity solution theory where one views $\lambda(t)$ as a prescribed parameter.

Let us explain the geometric argument in more detail. For *a priori* given positive and continuous $\lambda(t)$, let us consider the problem

$$\begin{cases} -\Delta u(x,t) = \lambda(t) & \text{in} \quad \{u>0\}, \\ u_t = F(|Du|)|Du| & \text{on} \quad \partial\{u>0\}. \end{cases} \tag{P-λ}$$

For a given function $\lambda(t)$ as above and under some assumption on the niceness of the initial data Ω_0 and boundedness and ellipticity assumptions on F, a comparison principle holds for (P-λ) and, by a standard application of Perron's method, there exist global-in-time viscosity solutions.

We show, under an additional assumption on the star-shapedness of the initial data Ω_0 (ρ-reflection), the existence of such a function $\lambda(t)$ defined for all times such that a viscosity solutions u of (P-λ) satisfies

$$\int u(x,t)\,dx = \int u(x,0)\,dx =: \mathrm{Vol}$$

for all $t>0$. The proof is carried out by showing that, when Ω_0 is strongly star-shaped, then the "energy solution", constructed with the discrete time scheme associated with gradient flow structure mentioned above, stays star-shaped and coincides with the viscosity solution of (P-λ) with the corresponding λ. We point out that it is not a priori clear whether the discrete-time energy solutions preserve the star-shaped condition, so we incorporate the restriction directly into the approximate scheme (See section 4) to obtain the continuum limit. The introduction of geometric restriction to the gradient flow scheme seems to be new and of independent interest.

2.0.5 Outline of the Chapter

In section 2 we recall known results about the solutions of (EQ). It turns out that viscosity solutions of (EQ) with connected and Lipschitz positive phase are radial and minimize the associated energy.

In section 3 we investigate the geometric properties of the viscosity solutions of (P-λ). In particular we show that ρ-reflection property is preserved over the time (see Corollary 2.2.24), based on the reflection maximum principle. We point out that the reflection maximum principle holds only when the solution of (P-λ) is stable under perturbations, which is the case when the solutions are star-shaped (see section 3.5 as well as the Appendix).

In section 4 we discuss the energy structure of our original problem (P-V). Motivated by the formal gradient flow structure, we construct a solution of (P-V) as a continuum limit of a discrete-time "minimizing movement" scheme following the approach of [Cha04], [Mie07]. By putting the ρ-reflection property as a constraint for the minimizing movements, we obtain uniform convergence of solutions in the continuum limit. An important result proved here is Proposition 4.7, which states that the limiting "energy" solution is indeed a viscosity solution of (P-λ) with volume-preserving property.

In section 5 we make use of the energy structure of the problem to show that any energy solution obtained in section 4 uniformly converges to the radial solution as $t \to \infty$, modulo translation (Theorem 2.4.1) We point out that our result does not imply that our solutions converge to a unique radial solution centered at a given point, since the drops may slowly move around a range of round profiles and may not converge to a single one. The physical uniqueness of the limiting profile and the characterization of its center remain open at the moment (Though see Proposition 2.4.1 for the discussion on the uniqueness).

2.1 The Equilibrium Problem

Now consider the volume constrained minimization problem

$$\inf\{J(u) : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u = \text{Vol}\}. \tag{2.1.1}$$

One can show immediately using symmetric decreasing rearrangements that any infimizer of

$$\inf\{J(u) : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u = \text{Vol and } u \text{ radial }\}$$

is also an infimizer of (2.1.1). That the only minimizers are radial is more delicate. One can show this using a theorem from [BZ88]. We will not discuss the specifics here since this fact will not be needed for our arguments.

To find the unique (up to translation) minimizers of (2.1.1) among radial functions first fix a particular radius r for the radially symmetric support set of the droplet

$$\{u > 0\} = B_r(0).$$

Now the minimization becomes

$$\inf\{J(u) : u \in H_0^1(B_r(0)) \text{ and } \int_{\mathbb{R}^N} u = \text{Vol and } u \text{ radial }\}$$

the unique infimizer is just the solution of the Dirichlet problem for the Laplace operator with lagrange multiplier $\lambda = \lambda(r) \in (0, +\infty)$ chosen so that the height profile

$$u_r(x) = \max\{\frac{\lambda}{2N}(r^2 - |x|^2), 0\}$$

has the correct volume. Then explicitly calculating $\lambda(r)$ and minimizing J over $r > 0$ one can easily check that for

$$r_* = (\frac{N}{N+2})^{1/(N+1)} V^{1/(N+1)}$$

u_{r_*} is the strict minimizer of J among radial H^1 functions and therefore – by the rearrangement argument that was mentioned above – is also a minimizer among all H^1 functions.

Alternatively, one can consider the Euler-Lagrange equation for (2.1.1) which is given by,

$$\begin{cases} -\Delta u = \lambda & \text{in} \quad \{u > 0\}, \\ \lambda > 0 & \text{s.t.} \quad \int u = \text{Vol}, \\ |Du| = 1 & \text{on} \quad \partial\{u > 0\}. \end{cases} \tag{EQ}$$

Then there is a classical theorem of Serrin [Ser71] regarding the uniqueness of solutions of (EQ).

Theorem 2.1.1. (Serrin) *Let $u : \mathbb{R}^N \to [0, +\infty)$ be compactly supported such that $\partial\{u > 0\}$ a C^2 hypersurface in $\mathbb{R}^N$ and u is a classical solution of (EQ). Then $\{u > 0\} = B_{r^*}(x_0)$ for some $x_0 \in \mathbb{R}^N$ and*

$$r^* = (\frac{N}{N+2})^{1/(N+1)} \text{Vol}^{1/(N+1)}. \tag{2.1.2}$$

Serrin's proof of this theorem used the method of moving planes and a variant of the Hopf Lemma for domains with corners. The best possible spatial regularity we are able to show for the evolving contact line in (P-λ) is Lipschitz. In order to apply Serrin's result to show the convergence to equilibrium, we need the same result to hold for viscosity solutions of (EQ) with positivity sets which are Lipschitz domains. It is not clear whether it is possible to use a variant of the moving planes method to show this. Instead we use some regularity results for free boundary problems. First a theorem of De Silva from [De] shows that viscosity solutions of (EQ) with Lipschitz free boundaries are classical solutions.

Theorem 2.1.2. (De Silva) (Lipschitz implies $C^{1,\alpha}$) *Let $\Omega \subseteq \mathbb{R}^N$ a domain and $u : \mathbb{R}^N \to [0,+\infty)$ be a viscosity solution of the free boundary problem in Ω,*

$$\begin{cases} -\sum_{i,j=1}^{N} a_{ij}\partial_{ij}u = f & in \quad \{x \in \Omega : u(x) > 0\} \\ |Du| = g & on \quad \Gamma(u) := \partial\{u > 0\} \cap \Omega \end{cases} \tag{2.1.3}$$

where a_{ij}, $g \in C^{0,\beta}(\Omega)$ for some $0 < \beta \leq 1$ and $f \in C(\Omega) \cap L^\infty(\Omega)$ and $g \geq 0$. Moreover, we have the ellipticity condition: there exists $0 < \lambda < \Lambda$ such that for all $\xi \in \partial B_1(0) \subset \mathbb{R}^N$ and all $x \in \Omega$,

$$\lambda \leq \sum a_{ij}(x)\xi_i\xi_j \leq \Lambda.$$

Then if $x_0 \in \Gamma(u)$, $g(x_0) > 0$ and $\Gamma(u)$ is a Lipschitz graph in a neighborhood of x_0 then $\Gamma(u)$ is $C^{1,\alpha}$ in a smaller neighborhood of x_0 for $\alpha > 0$ depending only on N, β, and the ellipticity constants λ and Λ.

Using above result, higher regularity of u can be derived from the Hodograph method [KNS78] when the coefficients are smooth and $\Gamma(u)$ is locally Lipschitz. See Appendix C for more details .

Corollary 2.1.1. *Let u solve (2.1.3) with (a_{ij}) as identity matrix and $f = g = 1$. In addition suppose that $\Gamma(u)$ is locally Lipschitz and is bounded. Then u and $\Gamma(u)$ is C^∞.*

2.2 Viscosity Solutions

2.2.1 Basic Definitions and Assumptions

We will recall the basic viscosity solution theory for (P-λ). A more detailed exposition for free boundary problems of a similar form can be found in [Kim03]. First we restate the problem under consideration,

$$\begin{cases} -\Delta u(x,t) = \lambda(t) & \text{in} \quad \Omega_t, \\ u_t = F(|Du|)|Du| & \text{on} \quad \Gamma_t. \end{cases} \tag{P-λ}$$

$\lambda : [0,+\infty) \to [0,+\infty)$ will be bounded and continuous. We will work in a space-time *parabolic domain* defined as any $Q = U \times (a,b]$ where $0 \leq a < b \leq +\infty$ and U is a domain (possibly unbounded) in $\mathbb{R}^N$, with *parabolic boundary*,

$$\partial_p Q := \overline{U} \times \{t = 0\} \cup \partial U \times [a,b].$$

The positive phase of a height profile $u : Q \to [0,+\infty)$ is defined as

$$\Omega(u) := \{u > 0\} \quad \text{and} \quad \Gamma(u) := \partial\Omega(u) \tag{2.2.1}$$

and at particular times,

$$\Omega_t(u) = \Omega(u) \cap \{(y,s) \in Q : s = t\} \quad \text{and} \quad \Gamma_t(u) = \partial\Omega_t(u).$$

The dependence on u will be omitted when it is unambiguous which droplet profile we are referring to. The free boundary velocity F will satisfy the boundedness and ellipticity assumption,

Assumption A. $F : [0,+\infty) \to \mathbb{R}$ is strictly monotone increasing and continuous, and $F(1) = 0$.

Assumption B. There exists $c > 0$ such that for ε sufficiently small,

$$(1+\varepsilon)F((1+\varepsilon)^{-1}s) + c\varepsilon \geq F(s). \tag{2.2.2}$$

Notice if Assumption 2 holds then necessarily F has sublinear growth at ∞. For any $s > 0$ using (2.2.2) $\lfloor s/\varepsilon \rfloor$ times with $\varepsilon \to 0$ yields:

$$F(s) \leq cs + \max\{(1+s)F(0), 0\}.$$

The monotonicity assumption on F implies, at least formally, that the problem (P-λ) has a comparison principle. This is the underlying reason why viscosity solutions are the natural definition of weak solution for this PDE. The second assumption is important for proving the strong comparison results and thereby the regularity of the viscosity solutions of (P-λ). It is not clear to us whether this assumption is only technical. Some examples which satisfy the Assumptions A and B are

Example 2.2.1. If $F(s) := s^p - 1$ for $p \leq 1$ then by a simple calculation (2.2.2) is satisfied for all $\varepsilon > 0$ with $c = 1$.

The following example in the case $p = 2$ will be important to us later,

Example 2.2.2. Let $M > 0$ and $p > 1$, suppose $F(s) := \max\{s^p - 1, M\}$ then a calculation shows (2.2.2) is satisfied for $\varepsilon \leq 1 + \frac{1}{2}(p-1)^{-1}$ and $c \geq 2(M+2)$.

Now we turn to defining a notion of solution for (P-λ). For $E \subset \mathbb{R}^N \times [0, +\infty)$ we use the notation $C^{2,1}(E)$ for functions f with two continuous derivatives in the spatial variables and one continuous derivative in time. First we define a classical solution of the free boundary problem.

Theorem 2.2.1. *(Classical Solutions) A profile $u : Q \to [0, +\infty)$ is a classical subsolution (supersolution) for the free boundary problem* (P-λ) *on Q if $u \in C^{2,1}(\overline{\Omega(u)} \cap Q)$ and* (P-λ) *is satisfied in the pointwise classical sense with equality signs replaced by $\leq$ ($\geq$ in the supersolution case). Classical solutions are both super and subsolutions.*

For general initial data the contact line motion problem will not have a classical solution. To get existence of a weak solution we define the notion of viscosity solution of (P-λ). First though we define the following standard notion.

Theorem 2.2.2. *(Strictly separated) Let v, w be defined on a set $D \subseteq \mathbb{R}^N \times [0, \infty)$ then we say v and w are strictly separated on D and write $v \prec w$ if $\overline{\Omega(v)} \cap \overline{D}$ is compact and $v < w$ on $\overline{\Omega(v)} \cap D$.*

Next we define subsolutions and supersolutions, then a viscosity solution is defined to satisfy both the sub and supersolution properties. Informally, u is a subsolution of (P-λ) if u cannot be crossed from above by any strict classical supersolution. More precisely:

Theorem 2.2.3. *(Subsolution) A non-negative upper semi-continuous function $u : Q \to \mathbb{R}_+$ is a subsolution of* (P-λ) *if, for any parabolic domain $Q' \subseteq Q$, and any strict classical supersolution ϕ with $u \prec \phi$ on $\partial_p Q'$, then $u \prec \phi$ in Q'.*

Theorem 2.2.4. *(Supersolution) A non-negative lower semi-continuous function $u : Q \to \mathbb{R}_+$ is a supersolution of* (P-λ) *if, for any parabolic domain $Q' \subseteq Q$, and any strict classical subsolution ϕ with $\phi \prec u$ on $\partial_p Q'$, then $\phi \prec u$ in Q'.*

Theorem 2.2.5. *(Solution) A viscosity solution of* (P-λ) *is a non-negative continuous function u on Q which is both a supersolution and a subsolution.*

Viscosity solutions with sufficient regularity are classical solutions. For example see [BV05, Kim03]. Naturally, one can assign boundary data on the parabolic boundary. We will usually have $Q = \mathbb{R}^N \times (0,T]$ and in that case this will reduce to assigning initial data.

Theorem 2.2.6. *(Subsolution with boundary data g) Let $g : \partial_p Q \to [0,+\infty)$ bounded then u is a subsolution of* (P-λ) *on Q with boundary data g if u is a subsolution of* (P-λ) *on Q as defined above and*

$$\limsup_{(y,s)\to(x,t)\in\partial_p Q} u(y,s) \leq g(x,t)$$

Supersolutions and solutions are then defined analogously.

For the rest of the chapter, in the case $Q = \mathbb{R}^N \times (0,T]$, when we assign initial data we will specify the positivity set as some $\Omega_0 \subset \mathbb{R}^N$ an open domain and then u_0 the initial data will be taken to solve

$$-\Delta u_0(x) = \lambda(0) \quad \text{ and } \quad u_0(x) = 0 \text{ for } x \in \Gamma_0 = \partial\Omega_0.$$

As mentioned in the introduction, it is often more natural to think of the problem as an evolution on the positivity sets.

Remark 2.2.3. In general, the solution of (P-λ) is not expected to be continuous. First of all, the solutions can vanish or blow up in finite time. A simple calculation in the radially symmetric case demonstrates these behaviors. Even without blow up the solution may be discontinuous for certain initial data: one can construct an example in one dimension where two adjacent drops merge after a short time, causing a discontinuity. To address the long time behavior of our solutions without the complexity of multiple components, we will make restrictions on the initial data such that our solutions actually are continuous.

An important basic property of viscosity solutions theory is the stability of viscosity solutions under uniform convergence. We state this in the following lemma. For example see [CIL92].

Lemma 2.2.4. *Let $u_n : \mathbb{R}^N \times [0,T]$ be a sequence of viscosity solutions of the problems*

$$\begin{cases} -\Delta u_n(x,t) = \lambda_n(t) & \text{in} \quad \Omega_t(u_n), \\ \partial_t u_n = F_n(|Du_n|)|Du_n| & \text{on} \quad \Gamma_t(u_n). \end{cases}$$

where F_n are all monotone increasing and continuous. Suppose that $u_n \to u$, $F_n \to F$ and $\lambda_n \to \lambda$ uniformly on compact sets, then u is a viscosity solution of

$$\begin{cases} -\Delta u(x,t) = \lambda(t) & \text{in} \quad \Omega_t(u), \\ \partial_t u = F(|Du|)|Du| & \text{on} \quad \Gamma_t(u). \end{cases}$$

As mentioned in the introduction, it is useful sometimes to formulate the problem (P-λ) as an evolution on domains in $\mathbb{R}^N$. We quickly define what it means for an evolution

$$\Omega_t : [0,+\infty) \to \{ \text{ bounded subsets of } \mathbb{R}^N\}$$

to solve (P-λ) in the viscosity sense.

Theorem 2.2.7. *The evolution Ω_t is a supersolution (subsolution) of* (P-λ) *if and only if,*

$$u[\Omega_t] := \text{ the solution of } \begin{cases} -\Delta u(x,t) = \lambda(t) & \text{in} \quad \Omega_t \\ u = 0 & \text{on} \quad \Gamma_t. \end{cases} \tag{2.2.3}$$

is a supersolution (subsolution) in the sense of Definition 2.2.4 (Definition 2.2.3).

2.2.2 Sup and Inf Convolutions

An important property of subsolutions (supersolutions) is the closure under sup (inf) convolutions. These convolutions will be used to perturb the originial solutions and to overcome the lack of scaling invariance in the problem (P-λ).

Lemma 2.2.5. (Convolutions in space)

(a) *Let u be a subsolution of* (P-λ) *on* $\mathbb{R}^N \times [0,+\infty)$*, and* $r > 0$*,* $c \geq 0$ *and define the sup-convolution of* u

$$\widetilde{u}(x,t) := \sup_{y \in B_{r-ct}(x)} u(y,t). \tag{2.2.4}$$

Then $\widetilde{u}$ is a subsolution with free boundary speed $F(|D\widetilde{u}|) - c$ as long as $r - ct > 0$.

(b) *Let u be a supersolution of* (P-λ) *on* $\mathbb{R}^N \times [0,+\infty)$*, and $r, c > 0$ and define the inf-convolution of u*

$$\widetilde{u}(x,t) := \inf_{y \in B_{r-ct}(x)} u(y,t). \tag{2.2.5}$$

Then $\widetilde{u}$ is a supersolution with free boundary speed $F(|D\widetilde{u}|) + c$ as long as $r - ct > 0$.

We refer to [GK09] for the proof. For the intuition on why the free boundary speed is increased (decreased) by inf (sup) convolutions we recommend that the reader think of the one dimensional case first.

2.2.3 Comparison

We state the general strictly separated comparison principle for viscosity solutions of (P-λ),

Theorem 2.2.1. (Comparison for strictly separated data) *Suppose u is a supersolution and v a subsolution of* (P-λ) *on Q. Suppose u and v are strictly separated (defined above in Definition 2.2.2) on the parabolic boundary of Q,*

$$v \prec u \quad \text{on} \quad \partial_p Q$$

then $v < u$ in $\overline{\Omega(v)} \cap Q$, in particular $\Gamma(v)$ cannot touch $\Gamma(u)$ from the interior on any compact subset of Q.

Proof. The proof can be found in various places in the literature [Kim03, BV05, CV99, KP12]. We will go into more detail later in Section 2.2.3. □

As mentioned in Remark 2.2.3, we will need to make a geometric restriction on our initial data to expect the existence of a viscosity solution which is stable under a family of perturbations. First, we recall the definition of a set star-shaped with respect to a point:

Theorem 2.2.8. *A domain $\Omega \subseteq \mathbb{R}^N$ is called star-shaped with respect to a point x if for every $y \in \partial\Omega$ the line segment between x and y is contained in $\overline{\Omega}$.*

Then the assumption on our initial data Ω_0 will be called *strong star-shapedness*:

Theorem 2.2.9. *We call a domain Ω strongly star-shaped if there exists $r > 0$ such that Ω is star-shaped with respect to every point of $B_r(0)$. For each $R > 0$ we define the class of uniformly bounded strongly star-shaped sets:*

$$\mathscr{S}_{r,R} := \{\Omega \subset B_R(0) : \Omega \textit{ star-shaped with respect to } B_r(0)\}, \quad \mathscr{S}_r := \textstyle\bigcup_{R>0} \mathscr{S}_{r,R}, \tag{2.2.6}$$

and the class of strongly star-shaped sets

$$\mathscr{S}_0 := \bigcup_{r>0} \mathscr{S}_r. \tag{2.2.7}$$

That the ball $B_r(0)$ is centered at the origin is only for convenience, the problem is translation invariant. We will often refer to the strongly star-shaped property by saying that a set $\Omega_0 \in \mathscr{S}_{r,R}$ in order to clarify the role of r. We note a basic property of sets in $\mathscr{S}_{r,R}$ – they are Lipschitz domains with the Lipschitz constant depending only on r, R. First, we define a notation for cones with apex at the origin. For $x \in \mathbb{R}^N$ and $\theta \in (0, \pi)$ the cone in the direction x with opening angle θ is called,

$$C(x, \theta) := \{y : \langle x, y\rangle \geq (\cos\theta)|x||y|\}. \tag{2.2.8}$$

We show that $\Omega \in \mathscr{S}_r$ have interior and exterior cones at every boundary point:

Lemma 2.2.6. *The following are equivalent for a bounded domain $\Omega \supset B_r(0)$:*

(i) The domain $\Omega \in \mathscr{S}_r$.

(ii) There is an $\varepsilon_0 \in (0, \infty]$ such that for all $x \in \Omega^C$ there is an exterior cone to Ω,

$$x + C(x, \theta_x) \cap B_{\varepsilon_0}(x) \subset \Omega^C \quad \textit{where} \quad \sin\theta_x = \tfrac{r}{|x|}. \tag{2.2.9}$$

(iii) For all $x \in \partial\Omega$ there is an interior 'cone' to Ω,

$$(x + C(-x, \theta_x)) \cap C\left(x, \tfrac{\pi}{2} - \theta_x\right) \cup B_r(0) \subset \Omega \quad \textit{where} \quad \sin\theta_x = \tfrac{r}{|x|}. \tag{2.2.10}$$

(iv) There exists $\varepsilon_0 > 0$ so that

$$\Omega \subset\subset \bigcap_{|z| \le a\varepsilon} [(1+\varepsilon)\Omega + z] \;\; \textit{for all } \varepsilon_0 > \varepsilon > 0, \tag{2.2.11}$$

for every $0 < a < r$.

Proof. The proof for parts (i)-(iii) is essentially in the picture, see Figure 2.2. For part (iv) let us first note that (see figure 2.2),

$$x + C(x, \theta_x) \cap B_{\varepsilon_0 r}(x) \subset \bigcup_{\varepsilon_0 > \varepsilon > 0} \bigcup_{0 < a < r} B_{a\varepsilon}((1+\varepsilon)x).$$

Therefore by (ii) $\Omega \in \mathscr{S}_r$ if and only if for all $x \in \Omega^C$ all $\varepsilon_0 > \varepsilon > 0$ and all $0 < a < r$,

$$B_{a\varepsilon}((1+\varepsilon)x) \subset\subset \Omega^C,$$

or equivalently, for every $\varepsilon_0 > \varepsilon > 0$ and all $0 < a < r$,

$$\{x : d(x, (1+\varepsilon)\Omega^C) \le a\varepsilon\} \subset\subset \Omega^C.$$

Again, equivalently, for every $\varepsilon_0 > \varepsilon > 0$ and all $0 < a < r$,

$$\Omega \subset\subset \{x : d(x, (1+\varepsilon)\Omega^C) \le a\varepsilon\}^C = \mathbb{R}^N \setminus \bigcup_{|z| \le a\varepsilon} [(1+\varepsilon)\Omega^C + z] = \bigcap_{|z| \le a\varepsilon} [(1+\varepsilon)\Omega + z].$$

□

We prove a comparison principle for reflections through hyperplanes. This will be very useful to us later because the reflection ordering is a property which does not depend on the radius from the strong star-shapedness property. Let H be a hyperplane in $\mathbb{R}^N$ with $y \in H$ define the reflection through H by

$$\phi_H(x) = x - 2\langle x - y, \nu_H \rangle \nu_H.$$

The symmetry of the problem (P-λ) with respect to reflections at least formally implies that if a solution $u(\cdot, t)$ and its reflection $u(\phi_H(\cdot), t)$ through H are initially ordered in the half spaces of H then they will remain so.

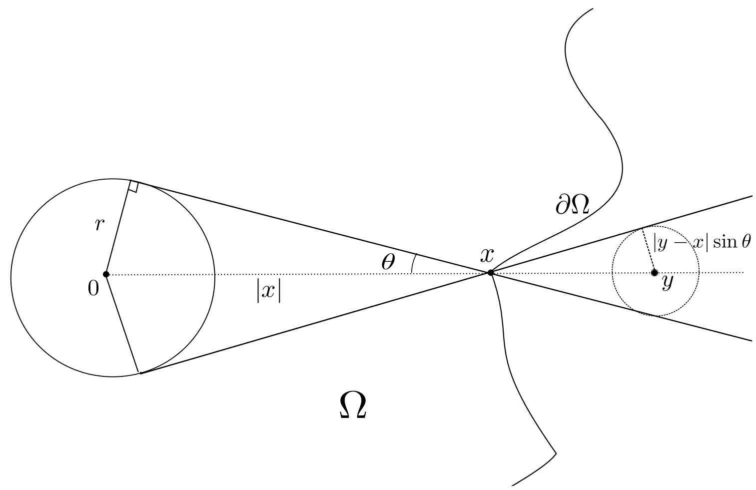

Figure 2.2: Interior and exterior cones for a strongly star-shaped domain.

Proposition 2.2.7. (Reflection Comparison) *Suppose* $u : \mathbb{R}^N \times [0,T] \to [0,+\infty)$ *is a solution of* (P-λ) *such that* $\Omega_t(u) \in \mathscr{S}_r$ *for some* $r > 0$ *and for all* $t \in [0,T]$*. Let* H *be a hyperplane in* $\mathbb{R}^N$ *with* $H \cap B_r(0) = \emptyset$ *and define* H_+ *and* H_- *the half spaces of* H *such that* $B_r(0) \subset H_+$*. Suppose that* $\phi_H(\Omega_0) \cap H_+ \subseteq \Omega_0 \cap H_+$ *i.e.*

$$u(\phi_H(x),0) \leq u(x,0) \text{ for } x \in H_+.$$

Then we have the ordering:

$$u(\phi_H(x),t) \leq u(x,t) \text{ for } (x,t) \in H_+ \times [0,T].$$

A similar argument will give the more standard strong comparison result,

Lemma 2.2.8. (Strong comparison) *Suppose* u *and* v *are respectively a sub- and supersolution of* (P-λ) *on* $\mathbb{R}^N \times [0,+\infty)$ *which are initially ordered,*

$$\Omega_0(u) \subseteq \Omega_0(v).$$

Suppose that that there exists $T \geq 0$ *such that for each* $t \in [0,T]$ *either* $\Omega_t(u)$ *or* $\Omega_t(v)$ *is in* $\mathscr{S}_{r,R}$ *for some* $r,R > 0$ *independent of* t*. Then let* c *from Assumption B and define* $t_0 := r^2/(Rc)$ *then*

$$\Omega_t(u) \subseteq \Omega_t(v) \text{ for } t \leq T + t_0.$$

Proof. The proof is a slightly easier version of the proof of Proposition 2.2.7 so we omit it. □

In order to get a strong comparison type result we want to slightly perturb the supersolution (or the subsolution) so that strict separation holds initially and we can use Theorem 2.2.1. To achieve this we use the inf and sup convolutions introduced above. This will be where the 'technical' assumption on F B comes in.

Proof of Proposition 2.2.7: Let ε_0 and c from Assumption B, $0 < a < r$, and define for $0 < \varepsilon \leq \varepsilon_0$,

$$v_\varepsilon(x,t) = (1+\varepsilon)^2 u((1+\varepsilon)^{-1}x,t); \text{ and}$$
$$u_\varepsilon(x,t) = \inf_{y \in B_{a\varepsilon - c\varepsilon t}(x)} v_\varepsilon(y,t) \quad \text{for} \quad 0 \leq t < \frac{a}{c}.$$

Then, from Lemma 2.2.6 part (iv) and the strong maximum principle for harmonic functions, we have

$$u(x,t) < u_\varepsilon(x,t) \ \text{ in } \ \mathbb{R}^N \times [0, \frac{a}{c}) \text{ for } \varepsilon > 0.$$

We check the super-solution property of u_ε. Note that $v_\varepsilon(x,t)$ is a supersolution of,

$$\begin{cases} -\Delta v_\varepsilon(x,t) \geq \lambda(t) & \text{in} \quad \Omega_t(v_\varepsilon) \\ \partial_t v_\varepsilon \geq (1+\varepsilon)F((1+\varepsilon)^{-1}|Dv_\varepsilon|)|Dv_\varepsilon| & \text{on} \quad \Gamma_t(v_\varepsilon). \end{cases}$$

Then, from Lemma 2.2.5, $u_\varepsilon(\cdot,t)$ is a supersolution of

$$\begin{cases} -\Delta u_\varepsilon(x,t) \geq \lambda(t) & \text{in} \quad \Omega_t(u_\varepsilon) \\ \partial_t u_\varepsilon \geq [(1+\varepsilon)F((1+\varepsilon)^{-1}|Du_\varepsilon|) + c\varepsilon]|Du_\varepsilon| & \text{on} \quad \Gamma_t(v_\varepsilon). \end{cases}$$

Using Assumption B we conclude that u_ε is a super-solution of (P-λ).

Since u_ε is a super-solution of (P-λ), by the strictly separated comparison principle Theorem 2.2.1,

$$u(\phi_H(x),t) \leq u_\varepsilon(x,t) \text{ in } \ H_+ \times [0, \frac{a}{c}).$$

Since $u_\varepsilon \searrow u$ as $\varepsilon \to 0$, it follows that

$$u(\phi_H(x),t) \leq u(x,t) \text{ in } \ H_+ \times [0, \frac{a}{c}).$$

We now iterate $\lfloor T_0 c/a \rfloor + 1$ times to conclude.

□

2.2.4 Short Time Existence

As usual in the viscosity solutions theory, the comparison theorem is the key ingredient needed to use Perron's method to show existence. In the existence proof we will use the following elementary facts about solutions of Poisson's equation at the boundary.

Lemma 2.2.9. (Boundary Hölder Estimates in Lipschitz Domains) *Let $\Omega \subset \mathbb{R}^N$ be a domain and $x_0 \in \partial\Omega$ and suppose that there exists $\theta \in (0, \pi/2)$ and $\nu \in S^{N-1}$ such that,*

$$x_0 + \{x : \langle x, \nu \rangle > \cos\theta |x|\} \subset \mathbb{R}^N \setminus \Omega.$$

In other words, Ω has an exterior cone of opening angle θ at x_0. Suppose $\Lambda > \lambda > 0$ and $u : \Omega \to [0, +\infty)$ satisfies

$$-\Delta u(x) \le \lambda \quad \text{for} \quad x \in \Omega$$

$$u(x) = 0 \qquad \text{on} \quad \partial\Omega$$

Then there exists $\alpha = \alpha(\theta, N) \in (0,1)$ and $C = C(\lambda, N) > 0$ such that for all $h > 0$,

$$\sup_{|z| \le h} u(x_0 + z) \le C h^{\alpha}.$$

Lemma 2.2.10. (Boundary Gradient Estimate in $C^{1,1}$ Domains) *Let $\Omega \subset \mathbb{R}^N$ be a domain $x_0 \in \partial\Omega$ and suppose that there exists $r > 0$, $y \in \mathbb{R} \setminus \overline{\Omega}$ such that*

$$B_r(y) \cap \overline{\Omega} = \{x\}.$$

In other words, Ω has an exterior ball at x_0 of radius r. Suppose $\Lambda > \lambda > 0$ and $u : \Omega \to [0, +\infty)$ satisfies $u(x_0) = 0$ and

$$-\Delta u \le \lambda \quad \text{for} \quad x \in \Omega.$$

Then

$$\lim_{s \to 0} \sup_{z \in \Omega \cap B_s(x)} \frac{u(z)}{s} \le C(\Lambda, N) \frac{\sup_{B_{2r}(x_0)} u}{r}.$$

We move on to the 'short time' existence theorem. Actually this shows the existence of a global in time discontinuous viscosity solution without any need for Assumption B. The uniqueness and continuity however rely on the strong comparison result which requires Assumption B and only holds for a short time. This is to be expected as noted in Remark 2.2.3. In particular the limiting factor will be the strong comparison principle Lemma 2.2.8 which only holds for a short time barring any a priori knowledge about strong star-shapedness.

Theorem 2.2.2. (Short time existence and uniqueness) *Let $r > 0$ and $\Omega_0 \in \mathscr{S}_r$, $\lambda(t)$ a given Lipschitz function of time, and let u_0 solve*

$$-\Delta u_0(x) = \lambda(0) \quad \textit{and} \quad u_0(x) = 0 \textit{ for } x \in \Gamma_0.$$

Then there is a $t_0 > 0$ depending only on r such that there exists a (unique) continuous viscosity solution u of (P-λ) *on $Q = \mathbb{R}^N \times (0,t_0]$ with initial data u_0.*

Proof. This proof, for $F(Du) = \max(|Du|^3 - 1, M)$, can be found in [GK09]. We construct a sub and supersolution for general F which take the initial data continuously then apply Perron's method.

1. First we construct a subsolution. Let $\gamma > 0$ to be chosen later and define,

$$U(x,t) := \begin{cases} \frac{\lambda(t)}{\lambda(0)}(1-\gamma t)^2 u_0(\frac{x}{1-\gamma t}) & \text{for} \quad t < \gamma^{-1} \\ 0 & \text{for} \quad t > \gamma^{-1} \end{cases} \tag{2.2.12}$$

Then, calculating formally, if $x_0 \in \Gamma_0$ then $(1-\gamma t)x_0$ is in Γ_t and

$$\frac{U_t}{|DU|}((1-\gamma t)x_0) = \gamma x_0 \cdot \frac{Du_0}{|Du_0|} \leq -\gamma |x_0| \frac{r}{|x_0|} \leq \min F$$

as long as $\gamma = \gamma(r)$ is chosen larger than $r^{-1} \min F$. Note that in the above calculation we have used the fact that for $\Omega_0 \in \mathscr{S}_r$ with smooth boundary, $\nu_x \cdot x \geq \frac{r}{|x|}$ on $\partial\Omega$, see Lemma 2.2.6. This calculation shows that U is a subsolution of (P-λ) when $\partial\Omega_0$ is smooth enough that Du_0 is defined on $\partial\Omega_0$. The calculation can be transferred to the test functions to show that U is a subsolution in the general case.

2. Next we construct a sequence of supersolutions which approximately take the initial data. Let $\alpha = \alpha(r/2, R) \in (0,1)$ from Lemma 2.2.9 combined with Lemma 2.2.6. Define, for any $h > 0$,

$$V_h(x,t) = \text{ the solution of } \begin{cases} -\Delta V_h(\cdot,t) = \lambda(t) & \text{in} \quad \bigcap_{|z| \leq \rho(t)} (1+h)(\Omega_0 + z) \\ V_h(\cdot,t) = 0 & \text{on} \quad \partial \bigcap_{|z| \leq \rho(t)} (1+h)(\Omega_0 + z) \end{cases} \tag{2.2.13}$$

where

$$\rho(t) = \frac{r}{2}\left(h - \frac{2}{r}\max\{F\left(Ch^{\alpha-1}\right),1\}t\right) \text{ for } 0 \le t \le \tfrac{1}{2}rh\max\{F\left(Ch^{\alpha-1}\right),1\}^{-1}.$$

Here $r/2 < r$ so that $u_0 \prec V_h$ as in Lemma 2.2.6. It follows from Lemma 2.2.5 that V_h satisfies

$$\frac{\partial_t V_h}{|DV_h|}(x,t) \ge \max\{F\left(Ch^{\alpha-1}\right),1\} \quad \text{on} \quad \Gamma_t(V_h)$$

in the viscosity sense. Let $x \in \Gamma_t(V_h)$, then there exists $y \in \partial B_{\rho(t)}(x) \cap (1+h)\Gamma_0$ and $\Omega_t(V_h)$ has the exterior ball at x

$$B_{\rho(t)}(y) \subset \mathbb{R}^N \setminus \Omega_t(V_h).$$

Then, since $\Omega_t(V_h)$ is an intersection of domains star-shaped with respect to $B_{r/2}(0)$, we apply Lemma 2.2.10 and then Lemma 2.2.9 to get,

$$|DV_h|(x,t) \le Ch^{-1} \sup_{|z|\le 2h} V_h(x+z,t) \le Ch^{\alpha-1}$$

and it follows from the monotonicity of F that V_h is a supersolution.

3. Now we apply Perron's method. Let us define

$$u(x,t) := \sup\{v : \mathbb{R}^N \times [0,+\infty) \to [0,+\infty) : v \text{ is a subsolution of (P-}\lambda\text{) with initial data } u_0\}.$$

From its definition and Theorem 2.2.1 we get the ordering for all $h > 0$

$$U(x,t) \le u(x,t) \le V_h(x,t) \quad \text{for} \quad 0 \le t \le \tfrac{1}{2}rh\max\{F\left(Ch^{\alpha-1}\right),1\}^{-1}. \tag{2.2.14}$$

Let us now define the upper and lower semi-continuous envelopes of u, respectively

$$u^*(x,t) = \limsup_{(y,s)\to(x,t)} u(y,s) \quad \text{and} \quad u_*(x,t) = \liminf_{(y,s)\to(x,t)} u(y,s).$$

It is standard in viscosity solution theory (see [CIL92]) to show that u^* is a subsolution and u_* is a supersolution. Due to (2.2.14) one can check that $u(\cdot,t) \to u_0(\cdot)$ uniformly as $t \to 0$, i.e. $u^* = u_*$ at $t = 0$. Thus, letting $t_0 = t_0(r,R) > 0$ as in Lemma 2.2.8, we have $u^* \le u_*$ on $[0,t_0]$. Combining this with the fact $u_* \le u^*$, we obtain the existence of a unique continuous viscosity solution of (P-λ) on $[0,t_0]$.

□

We point out that the above proof yields a modulus of continuity in time of the contact line at $t=0$. For solutions which are uniformly strongly star-shaped on a time interval $[0,T]$ this extends to give a uniform modulus of continuity in time for the contact line on $[0,T]$. We make this explicit in the case when F has polynomial growth below. For the rest of this section we will make the following assumption:

Assumption C. There exists $p \geq 0$ such that:

$$\limsup_{s \to \infty} \frac{F(s)}{s^p} < +\infty \tag{2.2.15}$$

We prove a modulus continuity for strongly star-shaped solutions of (P-λ) depending only on r, R, p, $\sup \lambda$ and $\sup F(s)/s^p$:

Corollary 2.2.11. *Suppose $u : \mathbb{R}^N \times [0,T] \to [0,+\infty)$ is a viscosity solution of* (P-λ) *with $0 < \lambda(t) \leq \Lambda$ bounded and F satisfying Assumptions A, B and C. Moreover suppose that $\Omega_t(v) \in \mathscr{S}_{r,R}$ for some $r, R > 0$ and all $t \in [0,T]$. Let α from Lemma 2.2.10 and*

$$\beta := \tfrac{1}{1+p(1-\alpha)},$$

then $\Gamma_t(u)$ is C^β in time in the Hausdorff metric, i.e.,

$$d_H(\Gamma_{t_1}(u), \Gamma_{t_2}(u)) \leq CRr^{-\beta}|t_1 - t_2|^\beta \text{ for all } t_1, t_2 \in [0,T].$$

Here C depends only on $\sup F(s)/s^p$*, p, Λ and N.*

Remark 2.2.12. An analogous equicontinuity result is true for general F satisfying Assumptions A and B. In this case the Hölder modulus of continuity will have to be replaced by a more general modulus of continuity depending on the growth of F at infinity. We restrict to the polynomial growth case for simplicity of presentation.

Remark 2.2.13. In particular, when F is bounded, Γ_t is Lipschitz in time with respect to d_H metric.

Proof. Let $T > t_2 > t_1 > 0$ and γ from the proof of Theorem 2.2.2. Using the barriers from the previous theorem,

$$u(x,t) \geq U(x,t) := \begin{cases} \frac{\lambda(t)}{\lambda(t_1)}(1-\gamma t)^2 u(\frac{x}{1-\gamma t}, t_1) & \text{for} \quad t - t_1 < \gamma^{-1} \\ 0 & \text{for} \quad t - t_1 > \gamma^{-1}. \end{cases} \tag{2.2.16}$$

Also for $h = Cr^{-\beta}(t_2 - t_1)^{\beta}$ we have that

$$u(x,t_2) \le V_h(x,t_2) \le \tfrac{\lambda(t_2)}{\lambda(t_1)}(1+h)^2 u\left(\frac{y}{1+h}, t_1\right). \tag{2.2.17}$$

Then the following containments hold:

$$[(1-\gamma(t_2-t_1)) \vee 0]\Omega_{t_1}(u) \subseteq \Omega_{t_2}(u) \subseteq (1 + Cr^{-\beta}(t_2-t_1)^{\beta})\Omega_{t_1}(u).$$

From which we derive,

$$d_H(\Gamma_{t_1}(u), \Gamma_{t_2}(u)) \le CRr^{-\beta}|t_1 - t_2|^{\beta}. \tag{2.2.18}$$

□

Remark 2.2.14. See Lemma 2.A.5 for some implications of (2.2.18).

We will be able to use this Corollary to show existence of a continuous viscosity solution for (P-λ) when F has polynomial growth. The idea is to take the unique solution u_M for the problem with the free boundary velocity $\max\{F, M\}$ and let $M \to \infty$. Corollary 2.2.11 will give equicontinuity to take a convergent subsequence of the u_M as long as the $\Omega_t(u_M) \in \mathscr{S}_{r,R}$ with r, R uniform in M and t. In order to show this we will need some kind of preservation of the strongly star-shaped property which does not depend on M.

Lemma 2.2.15. *Let $T > 0$ and $u_n : \mathbb{R}^N \times [0,T] \to [0,+\infty)$ be viscosity solutions of:*

$$\begin{cases} -\Delta u_n(x,t) = \lambda_n(t) & \text{in} \quad \Omega_t(u_n), \\ \partial_t u_n = F_n(|Du_n|)|Du_n| & \text{on} \quad \Gamma_t(u_n). \\ \Omega_0(u_n) = \Omega_0. \end{cases} \tag{2.2.19}$$

Here $\lambda_n \to \lambda > 0$ uniformly so that in particular there exists $\Lambda > 0$ so that:

$$\sup_n \sup_t \lambda_n(t) \le \Lambda < +\infty.$$

We suppose that $F_n \le F$ for all n and $F_n \to F$ uniformly on compact sets. The limiting free boundary speed F has polynomial growth of order $p > 1$,

$$0 < \limsup_{s \to \infty} \frac{F(s)}{s^p} < +\infty.$$

The F_n are assumed to all satisfy Assumptions A, B and C, with Assumption C satisfied uniformly :

$$\sup_n \sup_s \tfrac{F_n(s)}{s^p} < +\infty. \tag{2.2.20}$$

Moreover, suppose that there exist $r, R > 0$ such that for all n and all $t \in [0,T]$ we have $\Omega_t(u_n) \in \mathscr{S}_{r,R}$. However note that we do NOT assume that the constants c_n from Assumption B are uniformly bounded. Then the u_n converge uniformly on $\mathbb{R}^N \times [0,T]$ to a viscosity solution of:

$$\begin{cases} -\Delta u(x,t) = \lambda(t) & \text{in} \quad \Omega_t(u), \\ \partial_t u = F(|Du|)|Du| & \text{on} \quad \Gamma_t(u). \\ \Omega_0(u) = \Omega_0. \end{cases} \tag{2.2.21}$$

If additionally $\lambda_n(t) \leq \lambda(t)$ for all n then u is the minimal viscosity solution (2.2.21). *In particular, in this case, the limit does not depend on the approximating sequence F_n.*

Proof. 1. First we show that $\Omega_t(u_n)$ is an equicontinuous sequence of paths in $\mathscr{S}_{r,R}$. From Lemma 2.A.5 and Corollary 2.2.11 we derive for $t, s > 0$:

$$\sup_n d_H(\Omega_t(u_n), \Omega_s(u_n)) \leq \sup_n d_H(\Gamma_t(u_n), \Gamma_s(u_n)) \leq C|t-s|^\beta.$$

Then from the compactness lemma 2.A.6 for paths in $\mathscr{S}_{r,R}$ up to a subsequence $\Omega_t(u_n)$ converge uniformly to a continuous path Ω_t in $\mathscr{S}_{r,R}$ with free boundary $\Gamma_t := \partial\Omega_t$.

Let $u(x,t)$ be the solution of

$$\begin{aligned} -\Delta u(x,t) &= \lambda(t) \quad \text{for} \quad x \in \Omega_t \\ u(x,t) &= 0 \qquad\; \text{for} \quad x \in \Gamma_t. \end{aligned} \tag{2.2.22}$$

We claim that u_n converges uniformly to u in $\mathbb{R}^N \times [0,T]$, we will demonstrate this by showing the following estimate:

$$|u_n - u|(x,t) \lesssim_{r,R} d_H(\Omega_t(u_n), \Omega_t(u))^\alpha + |\lambda_n(t) - \lambda(t)|.$$

Let $x \in \Omega_t(u)\Delta\Omega_t(u_n)$, then:

$$d(x, \partial(\Omega_t(u) \cup \Omega_t(u_n))) \leq d_H(\Gamma_t(u_n), \Gamma_t(u)) \lesssim_{r,R} d_H(\Omega_t(u_n), \Omega_t(u)).$$

Since $u_n - u = 0$ on $\partial(\Omega_t(u) \cup \Omega_t(u_n))$ we get from Lemma 2.2.9,

$$|u_n - u|(x,t) \leq Cd(x, \partial(\Omega_t(u) \cup \Omega_t(u_n)))^\alpha \lesssim_{r,R} d_H(\Omega_t(u_n), \Omega_t(u))^\alpha.$$

Meanwhile for $x \in \Omega_t(u) \cap \Omega_t(u_n)$ we combine the above inequality which holds on the boundary of $\Omega_t(u) \cap \Omega_t(u_n)$ with the fact that $\operatorname{diam}(\Omega_t(u) \cap \Omega_t(u_n)) \lesssim_R 1$ to get,

$$|u_n - u|(x,t) \lesssim d_H(\Omega_t(u_n), \Omega_t(u))^\alpha + |\lambda_n(t) - \lambda(t)|.$$

We apply the stability of viscosity solutions under uniform convergence, Lemma 2.2.4, to see that u is a viscosity solution of the PDE (2.2.21) as claimed.

2. Now we show that if $\lambda_n \nearrow \lambda$ then u must be the smallest viscosity solution of (2.2.21). Let v be another viscosity solution of (2.2.21). Note that due to the orderings $F_n \leq F$ and $\lambda_n \leq \lambda$ we have that v is a supersolution of each of the approximating problems (2.2.19). Using the strong comparison principle Lemma 2.2.8 which holds for each problem (2.2.19) we get

$$u_n \leq v \text{ on } \mathbb{R}^N \times [0,T].$$

□

2.2.5 Preservation of the Strongly Star-shaped Property

Here we describe some of the properties of the viscosity solutions of (P-λ) and (P-V). First, we will show that, if the droplet initially is in $\mathscr{S}_0$, then this property persists for a short time with the radius of the strong star-shaped property going to zero in some finite amount of time. This short-time regularity will not be sufficient to prove any kind of long-time behavior. However, in Proposition 2.2.7 we have shown that a reflection comparison principle holds as long as the positivity set is in $\mathscr{S}_{r,R}$ for any $r, R > 0$. The reflection comparison principle in turn will allow us to show, in some situations, that actually there was no loss of star-shapedness. In particular for (P-V) we will show that initial data which is in $\mathscr{S}_r$ for a sufficiently large r along with a condition on the smallness of

$$\sup_{x \in \Omega_0} |x| - \inf_{x \in \Omega_0} |x|$$

will maintain some regularity globally in time. In fact, it will be strongly star-shaped for all time with a possibly smaller radius.

We show that strong star-shapedness cannot disappear immediately. The below Lemma is essentially contained in [GK09], we use a different proof.

Lemma 2.2.16. (Short time strong star-shapedness) *Let* $u : \mathbb{R}^N \times [0, +\infty) \to [0, +\infty)$ *be the solution of* (P-λ) *with initial positive phase* $\Omega_0 \in \mathscr{S}_r$ *for some* $r > 0$. *Then*

$$\Omega_t \in \mathscr{S}_{r-ct} \text{ for } 0 \le t < \frac{r}{c}$$

where c is from Assumption B.

Proof. Let $\varepsilon_0 > \varepsilon > 0$ and $0 < a < r$, then from Lemma 2.2.6 and Assumption B we know that

$$u_\varepsilon(x,t) := \inf_{|z| \le a\varepsilon - c\varepsilon t} (1+\varepsilon)^2 u((1+\varepsilon)^{-1}(x+z), t) \quad \text{for} \quad t \le \tfrac{a}{c}$$

is a supersolution of (P-λ) which has $u(\cdot, 0) \prec u_\varepsilon(\cdot, 0)$. Therefore from the strict comparison result Theorem 2.2.1 $u \le u_\varepsilon$ for $t < a/c$ and in particular,

$$\Omega_t(u) \subset \Omega_t(u_\varepsilon) = \bigcap_{|z| \le (a-ct)\varepsilon} ((1+\varepsilon)\Omega_t(u) + z) \text{ for every } \varepsilon_0 > \varepsilon > 0.$$

Since ε and $0 < a < r$ were arbitrary the converse direction of Lemma 2.2.6 part (iv) implies that $\Omega_t(u)$ has the claimed star-shapedness.

$\square$

We would like to show $\Omega_t(u) \in \mathscr{S}_r$ as long as $B_r(0)$ is contained in the positive phase $\Omega_t(u)$. This kind of preservation of the star-shapedness property would be very useful because we could get regularity results by simply showing that the positivity set must always contain some small ball around the origin. However, the arguments of Lemma 2.2.16 are insufficient to prove such a result. This is where the reflection comparison principle Proposition 2.2.7 is useful. The key fact is that reflection comparison holds as long as $\Omega_t(u) \in \mathscr{S}_r$ for any $r > 0$.

We begin by defining a slightly stronger property than strong star-shapedness which is defined in terms of reflections. In order to do this we will need some notations. Let Ω be an open bounded domain in $\mathbb{R}^N$. For $\nu \in S^{N-1}$, let $H = H(\nu)$ be the hyperplane through the origin orthogonal to ν with the half spaces it defines:

$$H_+ = H_+(\nu) = \{x : x \cdot \nu > 0\}, \quad H_- = H_-(\nu) = \{x : x \cdot \nu < 0\}.$$

Now for $s > 0$ define the translates,

$$H(s) = H + s\nu, \quad H_+(s) = H_+ + s\nu, \quad H_-(s) = H_- + s\nu.$$

For s sufficiently large

$$\Omega \subset H_-(s)$$

and so trivially

$$\phi_{H(s)}(\Omega \cap H_+(s)) \subset \Omega \cap H_-(s). \tag{2.2.23}$$

See Figure 2.3 for a depiction of the situation. Now let us slide the hyperplane inwards towards the origin by decreasing s until (2.2.23) no longer holds. We will call $s_{\min}(\nu, \Omega)$ the closest to the origin that we can move this plane with (2.2.23) always holding. More precisely, we define

$$s_{\min}(\nu, \Omega) := \inf\{s > 0 : \phi_{H(t)}(\Omega \cap H_+(t)) \subset \Omega \cap H_-(t) \text{ for all } t > s\}. \tag{2.2.24}$$

In the following we omit the dependence of $s_{\min}$ on Ω when it will not present any confusion. If $s_{\min}(\nu) = 0$ for every direction ν then Ω must be a ball. The basic approach of Serrin in [Ser71] to showing the symmetry of solutions of (EQ) is to show that $s_{\min}(\nu) = 0$ for all ν. In our case it is useful to use this same idea of symmetry but in a weaker form.

Theorem 2.2.10. *We say that a bounded, open set Ω has ρ-reflection if $B_\rho(0) \subseteq \Omega$ and*

$$\sup_{\nu \in S^{n-1}} s_{min}(\nu) \leq \rho. \tag{2.2.25}$$

Remark 2.2.17. As with strong star-shapedness, given some initial data Ω_0, we will in general assume that

$$\sup_{\nu \in S^{N-1}} s_{\min}(\nu, \Omega_0) = \inf_{z \in \mathbb{R}^N} \sup_{\nu \in S^{N-1}} s_{\min}(\nu, \Omega_0 + z).$$

Of course if this is not the case it can be corrected by a spatial translation.

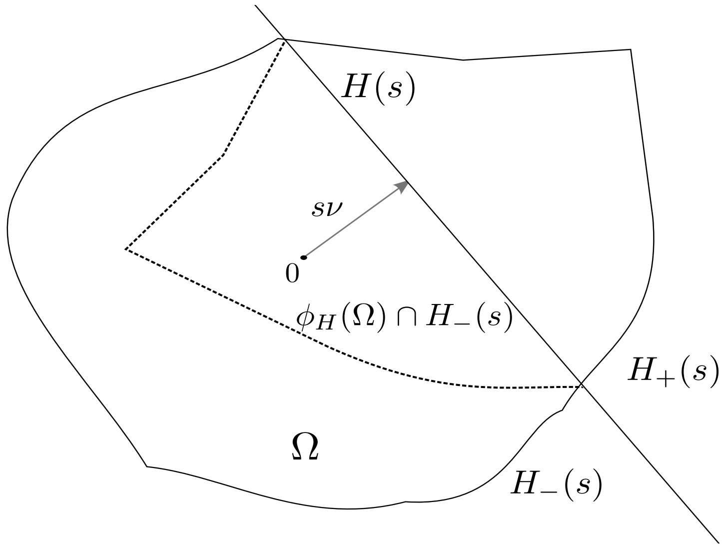

Figure 2.3: Reflection comparison will hold.

Remark 2.2.18. One can show that (see Lemma 2.A.3), upon fixing a maximal diameter R, a set has ρ-reflection if its boundary normals deviate from the radial direction by $O(\rho/R)$ and additionally the oscillation of the boundary is $O(\rho^2/R)$. Although we would like to know whether the flow preserves convexity, the ρ-reflection property does not seem to be very useful in this direction. In particular ρ-reflection does not imply convexity, and neither does the converse hold.

Note that the property (2.2.25) is preserved over time due to the reflection comparison, as long as the positivity set $\Omega_t(u) \in \mathscr{S}_r$ for any $r > 0$. We list some basic facts about sets which have ρ-reflection. The proofs are postponed to the appendix. The first says that the condition ρ-reflection imposes a condition on the spatial location of the boundary of the set.

Lemma 2.2.19. *Suppose Ω has ρ-reflection, then*

$$\sup_{x \in \partial\Omega} |x| - \inf_{x \in \partial\Omega} |x| \leq 4\rho. \tag{2.2.26}$$

We mention that (2.2.25) imposes a bound on what kind of normal vectors $\partial\Omega$ can have. In fact Ω will be strongly star-shaped with radius depending only on $\mathrm{dist}(\partial\Omega, B_\rho(0))$.

Lemma 2.2.20. *Suppose Ω has ρ-reflection. Then Ω satisfies the following:*

(a) for all $x \in \partial\Omega$ there is an exterior cone to Ω at x,

$$x + C(x, \phi_x) \subset \mathbb{R}^N \setminus \Omega \quad \textit{where} \quad \cos\phi_x = \tfrac{\rho}{|x|},\ \phi_x \in (0, \pi/2), \tag{2.2.27}$$

and $C(x, \phi_x)$ is the cone in direction x of opening angle ϕ_x as defined in (2.2.8)*;*

(b) $\Omega \in \mathscr{S}_r$ where

$$r = r(\rho, \inf_{x\in\partial\Omega} |x|) = (\inf_{x\in\partial\Omega} |x|^2 - \rho^2)^{1/2}.$$

Now, combining the strong star-shapedness from Lemma 2.2.20 with the reflection comparison principle Proposition 2.2.7, we obtain that ρ-reflection is preserved as long as the $B_\rho(0)$ is contained in evolving positive phase.

Lemma 2.2.21. *Suppose $u : \mathbb{R}^N \times [0, +\infty) \to [0, +\infty)$ is a solution of* (P-λ) *and Ω_0 has ρ-reflection for some $\rho > 0$. Let $I = [0, T)$ be the maximal time interval containing* 0 *on which $\overline{B_\rho(0)} \subseteq \Omega_t(u)$. Then $\Omega_t(u)$ has ρ-reflection and in particular $\Omega_t(u) \in \mathscr{S}_r$ for some $r > 0$ for every $t \in I$.*

Proof. Note that by comparing with a radial subsolution placed below $u(\cdot, 0)$ one can see that $T > 0$. Suppose towards contradiction that the lemma fails, that is

$$t_* := \inf\{t \in [0, T) : \Omega_t(u) \text{ does not have } \rho\text{-reflection }\} < T.$$

Since $\overline{B_\rho(0)} \subseteq \Omega_{t_*}(u)$ we have that

$$h := \operatorname{dist}(\Omega_{t_*}(u)^C, B_\rho(0)) > 0$$

so that by Lemma 2.2.20, $\Omega_{t_*}(u)$ is star-shaped with respect to $B_{(h^2+2\rho h)^{1/2}}(0)$. Then by Lemma 2.2.16

$$\Omega_t(u) \in \mathscr{S}_r \quad \text{for} \quad r = \tfrac{1}{2}(h^2 + 2\rho h)^{1/2} \tag{2.2.28}$$

on some slightly larger interval $[t_*, t_* + r/c)$. Now let H be a hyperplane which does not intersect $B_\rho(0)$. We can apply the reflection comparison Proposition 2.2.7 to $u(x,t)$ on $[t_*, t_* + r/c)$ to see that,

$$u(\phi_H(x), t) \le u(x, t) \text{ for } (x, t) \in H_+ \cap [t_*, t_* + r/c)$$

and therefore

$$\phi_H(\Omega_t(u)) \cap H_+ \subseteq \Omega_t(u) \cap H_+.$$

This holds for all admissible H, so $\Omega_t(u)$ has ρ-reflection on $[t_*, t_* + r/c)$, contradicting the definition of t_*. $\square$

Notice that an immediate consequence of Lemma 2.2.21 is that for any initial data Ω_0 which has ρ-reflection the evolution Ω_t has ρ-reflection for at least as long as,

$$t \leq \frac{\min_{x \in \Gamma_0} |x| - \rho}{\min F}.$$

The time on the right hand side above is independent of c from Assumption B and of $\lambda(t)$.

Now we present an application of this idea to the volume preserving problem, showing that for certain initial data any solution of the volume preserving problem must have ρ-reflection for all time. One can think of this as an a priori estimate for solutions of (P-V). If the initial data Ω_0 has ρ-reflection then Lemma 2.2.21 says that the solution Ω_t of the volume preserving flow also has ρ-reflection until such a time as Ω_t touches $B_\rho(0)$ from the exterior. For a domain $\Omega \subset \mathbb{R}^N$ let us define the associated Lagrange multiplier,

$$\lambda[\Omega] = \inf\{\int |Dv|^2 : v \in H_0^1(\Omega) \text{ and } \int v = \text{Vol}\}.$$

Then from Lemma 2.2.19 at the touching time t_* we know that $\Omega_{t_*} \subset B_{5\rho}(0)$. This allows us to get a lower bound on the Lagrange multiplier $\lambda[\Omega_{t_*}]$ and show that $B_\rho(0)$ is a strict subsolution near the touching time in the sense that

$$\frac{\lambda[\Omega(t)]}{2N}(\rho^2 - |x|^2)$$

is a strict classical subsolution for $|t - t_*|$ small. Of course, this will be a contradiction of the supersolution property of Ω_t. We make this precise in the following Lemma.

Lemma 2.2.22. *Suppose $u : \mathbb{R}^N \times [0,T] \to [0,+\infty)$ is a solution of* (P-V) *with initial data Ω_0 that has ρ-reflection with $B_\rho \subset\subset \Omega_0$. Then there is a dimensional constant C_N such that if*

$$0 < \rho < C_N V^{\frac{1}{N+1}} \tag{2.2.29}$$

then there exists $a > 0$ such that $B_{(1+a)\rho}(0) \subseteq \Omega_t(u)$ for all $t > 0$. In particular, we establish that the above holds with

$$C_N = \frac{1}{5^{\frac{N+2}{N+1}}} \left(\frac{N(N+2)}{|S^{N-1}|}\right)^{\frac{1}{N+1}}.$$

Remark 2.2.23. The scaling $\rho \lesssim \mathrm{Vol}^{\frac{1}{N+1}}$ is the natural one for such an inequality. In particular, if ρ is larger than the unique radial stationary solution,

$$\rho > r_* = \left(\frac{N}{N+2}\right)^{\frac{1}{N+1}} V^{\frac{1}{N+1}}$$

then we can construct a counter-example using radially symmetric solutions of (P-V). The solution of (P-V) with initial data $B_r(0)$ for any $r > \rho$ has ρ-reflection and converges as $t \to \infty$ to $B_{r_*}(0)$ so, in particular, there is some time after which $B_\rho(0)$ is no longer contained in the evolving positivity set. Of course in this case the initial data has ρ-reflection for every $\rho > 0$.

Proof. From the compact containment $B_\rho \subset\subset \Omega_0$ along with the strict inequality (2.2.29) let $a > 0$ small so that:

$$B_{(1+a)\rho}(0) \subset\subset \Omega_0 \quad \text{and} \quad \rho < \left(\tfrac{N(N+2)}{(1+a+4)^{N+2}|S^{N-1}|}\right)^{\frac{1}{N+1}}.$$

Towards a contradiction assume that $B_{(1+a)\rho}(0)$ touches $\Omega_t(u)$ from the inside for the first time at $0 < t_* < T$. More precisely let

$$t_* = \inf\{T > t > 0 : B_{(1+a)\rho(0)} \cap \Omega_t(u)^C \neq \emptyset\}$$

where we have assumed that the set being infimized over is non-empty. Note that $t_* > 0$ by comparing u with a radial subsolution starting on some ball slightly larger than $B_{(1+a)\rho}$ but still contained in Ω_0 and by continuity of the free boundary (Corollary 2.2.11 for example applies)

$$B_{(1+a)\rho}(0) \subseteq \Omega_{t_*}(u) \text{ and } \exists\, x_* \in \partial B_{(1+a)\rho}(0) \cap \Gamma_{t_*}(u).$$

Now by Lemma 2.2.21 and

$$B_\rho(0) \subset\subset \Omega_t(u) \text{ for } t \in [0, t_*+]$$

we have that $\Omega_t(u)$ has ρ-reflection and therefore by Lemma 2.2.20

$$B_{(1+a)\rho}(0) \subseteq \Omega_t(u) \subseteq B_{(1+a+4)\rho}(0).$$

Therefore we have

$$\lambda(t_*) > \lambda[B_{(1+a+4)\rho}] = \frac{N^2(N+2)V}{|S^{N-1}|((1+a+4)\rho)^{N+2}}.$$

Now, define

$$h(x) := \frac{N(N+2)V}{2|S^{N-1}|((1+a+4)\rho)^{N+2}}(\rho^2 - |x|^2),$$

which satisfies $-\Delta h(x) < \lambda(t_*)$ where $h > 0$ and h touches $u(x,t)$ from below at (x_0, t_*). On its free boundary, due to the assumption on ρ and a, h satisfies

$$F(|Dh|(x)) = F\left(\frac{N(N+2)V(1+a)\rho}{|S^{N-1}|((1+a+4)\rho)^{N+2}}\right) > 0.$$

The barrier h is a strict classical subsolution which touches $u(x,t)$ from below at (x_*, t_*), a contradiction of u being a viscosity supersolution. □

The following holds now due to Lemma 2.2.21 and Lemma 2.2.22.

Corollary 2.2.24. *Let u and Ω_0 be given as above. Then $\Omega_t(u) \in \mathscr{S}_{r,R}$ for some $r, R > 0$ for all $t > 0$.*

2.3 Gradient Flow

Now we consider the capillary droplet problem, contact line motion with volume preservation. We recall the PDE here for convenience,

$$\begin{cases} -\Delta u(x,t) = \lambda(t) & \text{in} \quad \Omega(u) \\ u_t = F(|Du|)|Du| & \text{on} \quad \Gamma(u). \end{cases} \tag{P-V}$$

We will consider the problem when

$$F(|Du|) = |Du|^2 - 1. \tag{2.3.1}$$

We restrict ourselves to the above choice of F because it gives the problem a simple gradient flow structure, although we expect similar results to hold for general F which satisfy Assumption A. The viscosity solution theory developed in the previous section does not apply directly to this choice of free boundary velocity. Therefore we will instead deal with

$$F(|Du|) = \max\{|Du|^2 - 1, M\}. \tag{2.3.2}$$

for $M > 0$ large and send $M \to \infty$ to get results for (2.3.2). As mentioned in the introduction, the problem (P-V) with free boundary velocity (2.3.1) is formally a gradient flow of the energy,

$$\mathscr{J}(\Omega) := \int_\Omega |Du|^2 + |\Omega|. \tag{2.0.5}$$

in the space of compact subsets of $\mathbb{R}^N$ (we leave the metric unspecified for now). Above we call

$$u = u[\Omega] := \operatorname{argmin}\{\int |Dv|^2 : v \in H_0^1(\Omega),\ \int_{\mathbb{R}^N} v = \mathrm{Vol}\},$$

Recall that u_0 solves,

$$\begin{cases} -\Delta u(x) = \lambda[\Omega] & \text{in} \quad \Omega \\ u(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.3.3}$$

where the Lagrange multiplier $\lambda[\Omega]$ is given by

$$\lambda[\Omega] := \min\{\int |Dv|^2 : v \in H_0^1(\Omega),\ \int_{\mathbb{R}^N} v = \mathrm{Vol}\}.$$

Now we give the formal derivation of the gradient flow structure as motivation. One can think of the Cacciopoli subsets of $\mathbb{R}^N$ as an infinite dimensional manifold such that the tangent space at Ω is

$$T_\Omega = L^\infty(\partial\Omega, |D\chi_\Omega|),$$

and the metric is specified as

$$g_\Omega(f,g) := \int_{\partial\Omega} fg|D\chi_\Omega|.$$

Then suppose that Ω_t is a smooth set valued solution of (P-V), i.e. $u[\Omega_t]$ is a smooth viscosity solution. In the geometric framework Ω_t is a path with velocity

$$T_{\Omega_t} \ni v(t) = F(|Du|(t)) = |Du|(t)^2 - 1.$$

Then we calculate,

$$\begin{aligned} \frac{d}{dt}\mathscr{J}(\Omega_t) &= \int_{\Omega_t} 2Du \cdot Du_t + \int_{\Gamma_t} (|Du|^2+1)F(|Du|) \\ &= -2\lambda(t)\int_{\Omega_t} u_t + \int_{\Gamma_t} 2u_t Du \cdot n + (|Du|^2+1)F(|Du|) \\ &= -2\lambda(t)\frac{d}{dt}\left(\int_{\Omega_t} u\right) + \int_{\Gamma_t} -2F(|Du|)|Du|^2 + (|Du|^2+1)F(|Du|) \\ &= \int_{\Gamma_t} (1-|Du|^2)F(|Du|) = -g_{\Omega_t}(v(t),v(t)). \end{aligned}$$

Of course, viscosity solutions of (P-V) lack the required smoothness to make the above framework anything but formal. Additionally, as far as we are aware, the distance associated with the metric g has no nice form even in the smooth case. Instead we use the pseudo-distance

$$\widetilde{\operatorname{dist}}^2(\Omega_0,\Omega_1) = \int_{\Omega_0 \Delta \Omega_1} d(x,\partial\Omega_0)dx. \tag{2.3.4}$$

This pseudo-distance was used previously for the same problem in [GK11] and for the mean curvature flow in [Cha04] and is motivated by the L^2-based Riemannian structure formally given to the space of sets with finite perimeters (see [GK11]). Unfortunately, $\widetilde{\operatorname{dist}}$ is not a distance. For example, it does not satisfy the triangle inequality. We will address this deficiency in Lemma 2.3.2.

Our plan is to show the existence of viscosity solutions for (P-V) by constructing a gradient flow of the energy $\mathscr{J}$. There is a standard approach to defining curves of maximal slope for energies in metric spaces through a discrete time approximation. Some early works on this subject are [JKO98, LS95] and the recent book [AGS08] contains a quite general treatment of the method. As noted previously our problem does not live in a nice metric space, however the essential idea of the construction is the same. In [GK11] it is shown that the curves of maximal slope $\mathscr{J}$ with $\widetilde{\operatorname{dist}}$ satisfy a barrier property with respect to strict smooth sub and supersolutions of (P-λ). From the a priori estimate Lemma (2.2.21) any solution with sufficiently round initial data of (P-V) will be in $\mathscr{S}_r$ for all time. With this in mind, it is reasonable to restrict the class of admissible sets in the gradient flow to only include sets which are uniformly strongly star-shaped with respect to the origin. With this additional regularity it becomes possible to show that the solutions of the discrete gradient flow scheme will converge to a viscosity solution of (P-V).

2.3.1 Discrete Gradient Flow Scheme

We will now define the discrete JKO type scheme to construct approximate solutions of the gradient flow (P-V). We denote these solutions with ω. This is to distinguish between solutions of the gradient flow and $\Omega(\cdot)$ which we use to denote the positive phase of a height profile. Let ω_0 be the initial positive phase and let $u_0 = u[\omega_0]$ be the corresponding initial height profile. Under the assumption that $\omega_0 \in \mathscr{S}_{r_0}$ for some $r_0 > 0$, let us define a discrete scheme approximating the gradient flow of the droplet energy $\mathscr{J}$ as follows:

Theorem 2.3.1. *Let $M, r, h > 0$ we define $\omega_{0,h} = \omega_0$, and define iteratively,*

$$\omega_{k+1,h} = \operatorname{argmin}\left\{ \mathscr{J}(\omega) + h^{-1}\widetilde{dist}^2(\omega_{k,h}, \omega) : \omega \in \mathscr{A}^h(\omega_{k,h}) \right\}. \tag{2.3.5}$$

Here $\mathscr{A}^h(\cdot)$ is the class of admissible sets which we will take to be

$$\mathscr{A}^h(\omega) := \{U \in \mathscr{S}_{r_0} \text{ and } d_H(U, \omega) \leq Mh\}. \tag{2.3.6}$$

We suppress the dependence M, r_0 and h in the above notations when there will be no confusion. In order to show that to show that the iteration defined in (2.3.5) is well defined Now we apply the Compactness Lemma 2.A.6:

Lemma 2.3.1. *The iteration* (2.3.5) *is well defined.*

Proof. We show that for any $\omega_0 \in \mathscr{S}_{r,R}$ the infimum from (2.3.5),

$$\inf\left\{ \mathscr{J}(\omega) + h^{-1}\widetilde{\mathrm{dist}}^2(\omega_0, \omega) : \omega \in \mathscr{A}^h(\omega_0) \right\}, \tag{2.3.7}$$

is achieved by some $\omega \in \mathscr{A}^h(\omega_0)$. Let $\omega_j \in \mathscr{A}^h(\omega_0)$ be an infimizing sequence for (2.3.7). In particular from the admissibility conditions (2.3.6),

$$\omega_j \in \mathscr{S}_{r_0, R+Mh}.$$

Applying Lemma 2.A.6 up to a subsequence the ω_j converge in d_H metric to some $\omega \in \mathscr{S}_{r_0,R+Mh}$ and we easily derive,

$$d_H(\omega, \omega_0) \leq Mh,$$

so that $\omega \in \mathscr{A}^h(\omega_0)$. Then, from Lemma 2.A.5, $\mathscr{J}(\cdot)$ and $\widetilde{\mathrm{dist}}^2(\omega_0, \cdot)$ are continuous with respect to convergence in $(\mathscr{S}_{r_0,R+Mh}, d_H)$ so ω achieves the infimum in (2.3.7).

□

Note that since ω_k is admissible at each stage we automatically get the discrete energy decay estimate,

$$\mathscr{J}(\omega_{k+1}) + \frac{1}{h}\widetilde{\mathrm{dist}}^2(\omega_{k+1}, \omega_k) \leq \mathscr{J}(\omega_k). \tag{2.3.8}$$

We would like to use the triangle inequality (which we don't have) to get the energy decay inequality over time of the form:

$$\frac{1}{nh}\widetilde{\text{dist}}^2(\omega_{k+n},\omega_k) \leq \mathscr{J}(\omega_k) - \mathscr{J}(\omega_{n+k}).$$

It turns out that this is possible as long as we are restricting to strongly star-shaped sets.

Lemma 2.3.2. *Let $\omega_0,\ldots,\omega_n \in \mathscr{S}_{r,R}$, then we have the inequality,*

$$\frac{C(r,R,N)}{n}\widetilde{\text{dist}}^2(\omega_n,\omega_1) \leq \sum_{j=1}^{n}\widetilde{\text{dist}}^2(\omega_{j+1},\omega_j) \tag{2.3.9}$$

In the following it will be useful to introduce some notation. We say that $A \lesssim B$ if there exists a constant $C > 0$ such that $A \leq CB$. We say $A \sim B$ or A is equivalent up to constants to B if $A \lesssim B \lesssim A$. The dependence of the constant C on various parameters q_i is expressed by writing $\lesssim_{q_1,\ldots,q_n}$ or $\sim_{q_1,\ldots,q_n}$. If no dependence is expressed, then the constant is assumed to depend at most on the dimension N.

Proof. Let $\omega_a,\omega_b \in \mathscr{S}_{r,R}$, then we define the distance between the sets ω_a and ω_b in the direction $\theta \in S^{N-1}$,

$$d_\theta(\omega_a,\omega_b) = |X_a(\theta)-X_b(\theta)| \quad \text{where} \quad \{X_i(\theta)\} = \partial\omega_i \cap \{x : \tfrac{x}{|x|} = \theta\},\ i \in \{a,b\}.$$

Notice, due to the strict star-shapedness of the ω_i, there is only one point $X_i(\theta)$ along the direction θ in $\partial\omega_i$. It is easy to check that d_θ has the triangle inequality:

$$d_\theta(\omega_a,\omega_b) \leq d_\theta(\omega_a,\omega_c) + d_\theta(\omega_c,\omega_b),$$

for any bounded strictly star-shaped sets ω_i, $i \in \{a,b,c\}$.

Claim. The $\widetilde{\text{dist}}$ is equivalent up to constants to a true distance:

$$\widetilde{\text{dist}}(\omega_a,\omega_b) \sim_{r,R,N} \left(\int_{S^{N-1}} d_\theta(\omega_a,\omega_b)^2 d\theta\right)^{1/2}. \tag{2.3.10}$$

Using the claim along with Cauchy-Schwarz and the triangle inequality for d_θ we will get,

$$\sum_{j=1}^{n}\widetilde{\text{dist}}^2(\omega_{j+1},\omega_j) \gtrsim_{r,R,N} \sum_{j=1}^{n}\int_{S^{N-1}} d_\theta(\omega_{j+1},\omega_j)^2 d\theta \geq \frac{1}{n}\int_{S^{N-1}} d_\theta(\omega_n,\omega_1)^2 d\theta \gtrsim_{r,R,N} \widetilde{\text{dist}}^2(\omega_n,\omega_1).$$

So it suffices to show the claimed equivalence (2.3.10). We calculate:

$$\begin{aligned}
\widetilde{\operatorname{dist}}^2(\omega_a,\omega_b) &= \int_{\omega_a\Delta\omega_b} d(x,\partial\omega_a)dx = \int_{S^{N-1}}\int_0^R d(\rho\theta,\partial\omega_b)\chi_{\omega_a\Delta\omega_b}(\rho\theta)\rho^{N-1}\,d\rho d\theta \\
&\sim_{r,R,N} \int_{S^{N-1}}\int_0^R d(\rho\theta,\partial\omega_b)\chi_{\omega_a\Delta\omega_b}(\rho\theta)\,d\rho d\theta \\
&= \int_{S^{N-1}}\int_{\min\{|X_a(\theta)|,|X_b(\theta)|\}}^{\max\{|X_a(\theta)|,|X_b(\theta)|\}} d(\rho\theta,\partial\omega_b)\,d\rho d\theta \\
&\sim_{r,R} \int_{S^{N-1}}\int_{\min\{|X_a(\theta)|,|X_b(\theta)|\}}^{\max\{|X_a(\theta)|,|X_b(\theta)|\}} |\rho-|X_a(\theta)||\,d\rho d\theta \\
&= \int_{S^{N-1}}\int_0^{d_\theta(\omega_a,\omega_b)} \rho\,d\rho d\theta = \frac{1}{2}\int_{S^{N-1}} d_\theta(\omega_a,\omega_b)^2 d\theta.
\end{aligned}$$

Here we have used an easy consequence of Lemma 2.2.6, for $x\in\mathbb{R}^N$ the radial distance to $\partial\omega_i$ (i.e. $|X_i(\hat{x})-x|$) is equivalent to $d(x,\partial\omega)$ up to constants depending on r and R. In particular for $x\in\mathbb{R}^N\setminus\omega_i$ from Figure 2.2,

$$B_{\frac{r}{R}|x-X_i(\hat{x})|}\subset\mathbb{R}^N\setminus\Omega$$

so that

$$|x-X_i(\hat{x})|\geq d(x,\partial\omega_i)\geq\frac{r}{R}|x-X_i(\hat{x})|.$$

□

As in the proof of Lemma 2.3.1, the restrictions on the admissible sets will allow us to take a continuum limit of the discrete time scheme. In particular, the assumption of uniform strong star-shapedness gives equicontinuity of the contact lines $\partial\omega_k$ in space, and the Hausdorff distance movement restriction gives equicontinuity in time.

Lemma 2.3.3. (Continuum Limit) *Let $\omega_{k,h}$ be given as in* (2.3.5). *Then the following holds for the interpolants*

$$\omega_h(t)=\omega_{k,h}\text{ for }t\in[kh,(k+1)h). \tag{2.3.11}$$

(i) There exists $R=R(r_0,N)>0$ such that $\omega_h(t)\subset B_R(0)$ for all $h,t>0$ and therefore

$$Lip(\partial\omega_h(t))\leq C(r_0,N).$$

(ii) $\omega_h(t)$ (up to a subsequence) locally uniformly converges in $t\in[0,+\infty)$ to $\omega(t)$, which satisfies

$$d_H(\omega(t_1),\omega(t_2))\leq M|t_1-t_2|\text{ for all }t_1>t_2\geq 0.$$

(iii) For $C = C(r_0, R, N)$ given as in Lemma 2.3.2, $\omega_h(t)$ satisfies the energy decay estimate

$$\frac{C}{t_2 - t_1} \widetilde{dist}^2(\omega_h(t_1), \omega_h(t_2)) \leq \mathscr{J}(\omega_h(t_1)) - \mathscr{J}(\omega_h(t_2)). \tag{2.3.12}$$

Proof. We give a sketch of the proof for (i). Let $x \in \partial\omega_h(t)$ and recall that

$$|\omega_h(t)| \leq \mathscr{J}(\omega_h(t)) \leq \mathscr{J}(\omega_0),$$

and $\omega_h(t)$ has the interior cone at x,

$$\{y : y \cdot x \geq 0 \text{ and } (x-y)\cdot x \geq |x-y|\sqrt{|x|^2 - r_0^2}\} \cup B_r(0) \subset \omega_h(t).$$

This cone contains $\sim |x|/r_0$ balls of radius $r_0/2$ centered along it's axis, so that

$$|x| \lesssim R(r_0, N) := r_0^{N-1}\mathscr{J}(\omega_0).$$

Now we prove (ii) and (iii). For simplicity we will take $h = 2^{-j}$ and then call the interpolant (2.3.11) $\omega_j(t)$. From (i) and Lemma 2.A.6 the sequence $\omega_j(t)$ for $t \in h\mathbb{Z}^+$ has a subsequence in j which converges in Hausdorff distance. Diagonalizing, we get a subsequence which we continue to call j along which $\omega_j(t)$ converges in Hausdorff distance to a limit we call $\omega(t)$. From (2.3.6) and the triangle inequality we get:

$$d_H(\omega_j(t_1), \omega_j(t_2)) \leq M|t_1 - t_2| \text{ for } t_1, t_2 \in h\mathbb{Z}^+ \text{ and for sufficiently large } j,$$

and $\omega(t)$ inherits the same estimate. $\omega(t)$ is densely defined and uniformly continuous and thus has a unique M-Lipschitz extension to $[0, +\infty)$. The local uniform convergence then directly follows. Estimate (2.3.12) follows from the continuity of $\mathscr{J}(\cdot)$ and $\widetilde{\text{dist}}^2(\cdot,\cdot)$ with respect to convergence in Hausdorff distance: see Lemma 2.A.5. □

2.3.2 Barrier Properties for the Gradient Flow

In [GK11] a discrete gradient flow is constructed for (P-V) without restrictions on the star-shapedness of the domain or the maximum speed of the free boundary. There it is proven that the discrete solutions satisfy a barrier property with respect to smooth strict sub and super-solutions of (P-λ). An analogous result can be proven for the solutions of the restricted iteration scheme defined in

(2.3.5). The cost of the restriction to nicer sets in (2.3.5) is that the class of barriers for which the sub and super-solution properties hold is reduced. The necessary conditions on admissible barriers can be deduced by inspecting the proofs in Section 3 of [GK11]. We give modified statement of the super-solution barrier property applicable to our situation below.

Theorem 2.3.1. (Grunewald, Kim) (Super-solution barrier property) *Let $u_h = u[\omega_h]$ be the profile after one iteration of the scheme with time step size $h > 0$, and let*

$$\lambda = \min\{\lambda[\omega_0], \lambda[\omega_h]\}.$$

Let $\rho > 0$, $x_0 \in \mathbb{R}^N$ and suppose there exists ϕ smooth with $|D\phi| \neq 0$ in $B_\rho(x_0) \times [0,h]$ with a $\delta > 0$ such that

$$\begin{aligned} -\Delta\phi(\cdot,t) &< \lambda - \delta && \text{in} \quad B_\rho(x_0) \times [0,h] \\ \frac{\phi_t}{|D\phi|} - (|D\phi|^2 - 1) &< -\delta && \text{on} \quad \Gamma(\phi) \cap (B_\rho(x_0) \times [0,h]). \end{aligned}$$

Moreover, we require that there exists $C > 0$ which does not depend on h such that the sets

$$\omega_h \cup (\{\phi(\cdot,h) > \eta\} \cap B_\rho(x_0)), \ -h^2 \leq \eta \leq Ch^{1/2} \tag{2.3.13}$$

are in the admissible class $\mathscr{A}^h_{\omega_0}$. Then, for $0 < h < h_0 = h_0(\rho,\phi,\delta)$, the following holds:

$$\textit{If } \phi(\cdot,0) \leq u_0(\cdot), \textit{ then } \phi(\cdot,h) \leq u_h(\cdot).$$

Essentially, (2.3.13) requires that the perturbed sets of ω_h by the barrier ϕ which crosses u_h from below stays in the admissible class. An analogous barrier property with respect to supersolutions holds as well.

Now we show that the barrier property carries over to the continuum limit $\Omega(t)$.

Lemma 2.3.4. (Restricted super-solution property for the gradient flow) *Let $\omega(t)$ be a continuum limit of the discrete gradient flow described in Lemma 2.3.3. Let $\phi \in C^{2,1}(\overline{\Omega(\phi)} \cap \{0 \leq t \leq T\})$ such that $|D\phi| \neq 0$ on $\Gamma(\phi) \cap \{0 \leq t \leq T\}$, the initial strict separation $\Omega_0(\phi) \subset\subset \omega(0)$ holds and ϕ is a strict subsolution,*

$$\begin{aligned} -\Delta\phi(\cdot,t) &< \lambda[\omega(t)] && \text{in} \quad \Omega(\phi) \cap \{0 \leq t \leq T\} \\ \phi_t - \max\{|D\phi|^2 - 1, M\}|D\phi| &< 0 && \text{on} \quad \Gamma(\phi) \cap \{0 \leq t \leq T\}. \end{aligned} \tag{2.3.14}$$

Then ϕ cannot touch $u[\omega(t)]$ from below at any free boundary point $(x_0,t_0) \in \Gamma(u)$ such that

$$D\phi(x_0,t_0)\cdot x_0 < -r_0|D\phi|(x_0,t_0). \tag{2.3.15}$$

Remark 2.3.5. Test functions $\phi \in C^{2,1}(\overline{\Omega(\phi)} \cap \{0 \le t \le T\})$ with $|D\phi| \neq 0$ on $\Gamma(\phi)$ can be extended to be $C^{2,1}(\mathbb{R}^N \times [0,T])$.

Proof. Suppose not. Then there exists $\phi \in C^{2,1}(\mathbb{R}^N \times [0,T])$ which touches $u = u[\omega(t)]$ from below in $\mathbb{R}^N \times (0,T)$ without satisfying (2.3.15). From the continuity of the free boundaries of ϕ and u, this occurs for the first time at some $(x_0,t_0) \in \Gamma(u) \cap \Gamma(\phi)$, with $0 < t_0 \le T$. That is, there exists $\rho > 0$ such that

$$\phi(x,t) \le u(x,t) \quad \text{for} \quad (x,t) \in Q := B_\rho(x_0) \times [t_0-\rho, t_0].$$

We may assume that ϕ touches u strictly from below by making the replacement,

$$\phi \to (\phi - \varepsilon|x-x_0|^2 - \varepsilon(t_0-t))_+.$$

In particular, we will have,

$$\phi(x,t) < u(x,t) \quad \text{for} \quad (x,t) \in Q \cap \overline{\Omega(\phi)} \setminus \{(x_0,t_0)\},$$

and for $\varepsilon > 0$ sufficiently small all the strict subsolution conditions for ϕ along with (2.3.15) still hold. Furthermore, we can make a subsolution which crosses u from below in Q by making a small translation in the normal direction to $\Gamma_{t_0}(\phi)$ at x_0,

$$\varphi(x,t) := \phi(x+\varepsilon\nu, t) \quad \text{where} \quad \nu = \tfrac{D\phi}{|D\phi|}(x_0,t_0).$$

Here $\varepsilon > 0$ is chosen sufficiently small based on the C^2 (in space) norm of ϕ so that (2.3.16) holds for φ, $\varphi(x_0,t_0) > 0 = u(x_0,t_0)$ and also

$$\varphi(x,t) \prec u(x,t) \quad \text{for} \quad (x,t) \in \partial_p Q.$$

Then we take ρ small depending on the modulus of continuity of $D\varphi$ such that there exists $\delta > 0$ with:

$$|D\varphi| > \delta, \quad \varphi_t < (M-\delta)|D\varphi|, \quad u - \varphi > \delta \text{ on } \partial_p Q \cap \Omega(\varphi),$$

and (2.3.15) hold on all of Q, that is

$$D\varphi(x,t)\cdot x < -r|D\varphi|(x,t) \quad \text{for} \quad (x,t)\in Q\cap\overline{\Omega(\varphi)}. \tag{2.3.16}$$

In order to make use of Lemma 2.3.1 we need to take this information back to the approximating sequence of discrete solutions. Let $h_j \to 0$ be the sequence along which the discrete gradient flow scheme

$$d_H(\omega_{h_j}(t),\omega(t)) \to 0$$

uniformly for $t_0-\rho \le t \le t_0$. From Lemma 2.A.5 $u_j(\cdot,t) := u[\omega_{h_j}(t)]$ converge uniformly in (x,t) to u. Let j be sufficiently large (h_j small) so that Lemma 2.3.1 will apply and also $h_j \le \delta^2$, $\varphi(x_0,t_0) > u_j(x_0,t_0)$ and

$$\varphi(x,t) \prec u_j(x,t) \quad \text{for} \quad (x,t)\in\partial_p Q. \tag{2.3.17}$$

Now there exists $t_0-\rho-h_j < t_k := kh_j < t_0$ such that

$$\begin{aligned} &\varphi(x,t_k) \le u_j(x,t_k) && \text{for all} && x\in B_\rho(x_0)\\ &\varphi(x_1,t_{k+1}) > 0 = u_j(x_1,t_{k+1}) && \text{for some} && x_1\in B_\rho(x_0). \end{aligned}$$

To apply Lemma 2.3.1 and get a contradiction it is sufficient to show the following:

Claim. Let $\eta \ge -h_j^2$ then, calling $\omega_{h_j,k} = \omega_k$:

$$U_\eta := (\{\phi(\cdot,t_{k+1}) > \eta\}\cap B_\rho(x_0))\cup\omega_{k+1} \in \mathscr{A}^{h_j}(\omega_k).$$

Let $x\in U_\eta$, we claim that:

$$d(x,\omega_k) \le Mh_j.$$

If $x\in\omega_{k+1}$ then $d(x,\omega_k)\le Mh_j$ from the admissibility condition (2.3.6) for ω_{k+1}. If $x\in\Omega_{t_{k+1}}(\varphi-\eta)\cap B_\rho(x_0)$ then in the case $\eta\ge 0$ from the containment $\Omega_{t_k}(\varphi)\cap B_\rho(x_0)\subset\omega_k$ and $\phi_t < M|D\phi|$ we get,

$$d(x,\omega_k)\le d(x,\Omega_{t_k}(\varphi)) < Mh_j.$$

In the case $-\delta^2 h_j \le -h_j^2 \le \eta < 0$:

$$\begin{aligned} d(x,\omega_k) \le d(x,\Omega_{t_k}(\varphi)\cap B_\rho(x_0)) &\le d_H(\Omega_{t_{k+1}}(\varphi-\eta),\Omega_{t_{k+1}}(\varphi)) + (M-\delta)h_j\\ &\le \frac{-\eta}{\delta} + (M-\delta)h_j\\ &\le Mh_j. \end{aligned}$$

Let $x \in \omega_k$ then $d(x, U_\eta) \leq d(x, \omega_{k+1}) \leq Mh_j$. Combining with the above arguments,

$$d_H(U_\eta, \omega_{k+1}) \leq Mh_j.$$

That $U_\eta \in \mathscr{S}_{r_0}$ is more subtle and this is where the condition (2.3.15) comes in. For a given $x \in \partial U_\eta$, we need to show that for every $y \in B_{r_0}(0)$,

$$L_{x,y} := \{z \in \mathbb{R}^N : z = y + s(x-y),\ s \in [0,1]\} \subset \overline{U_\eta}. \tag{2.3.18}$$

This is trivially true for all $x \in \partial U_\eta \cap \partial \omega_{k+1}$ from the strong star-shapedness of ω_{k+1}. So we consider

$$x \in \partial U_\eta \cap \partial(\Omega_{t_{k+1}}(\varphi - \eta) \cap B_\rho(x_0)) \subset B_\rho(x_0) \cap \partial \Omega_{t_{k+1}}(\varphi - \eta).$$

Let $y \in B_{r_0}(0)$ and we consider the line segment from x to y parametrized by

$$z(s) = x + s(y - x).$$

First we show that – for $s > 0$ small depending on the C^2 norm of ϕ – $z(s) \in U_\eta$. Using $|y| < r$ and (2.3.16),

$$\phi(z(s)) = \eta + sD\varphi(x) \cdot (y - x) + o(s) > s(-|y| + r_0)|D\varphi|(x) + o(s) > 0.$$

Suppose towards a contradiction that $z(s)$ exits $\overline{U_\eta}$ before $s = 1$, then by continuity it must pass through ∂U_η for the first time at $0 < s_0 < 1$ defined by,

$$1 > s_0 = \inf\{s > 0 : z(s) \in \partial U_\eta\} > 0.$$

If $z(s_0) \in \partial \omega_{k+1}$ then we are done due to $\omega_{k+1} \in \mathscr{S}_r$:

$$L_{z(s_0),y} \subset \overline{\omega_{k+1}} \subset \overline{U_\eta}.$$

So it remains to consider the case $z(s_0) \in B_\rho(x_0) \cap \partial \Omega_{t_{k+1}}(\varphi)$ and in this case:

$$0 \geq \frac{d}{ds}\varphi(z(s))\bigg|_{s=s_0^-} = D\varphi(z(s_0)) \cdot (y - x) = (1 - s_0)D\varphi(z(s_0)) \cdot (y - z(s_0)),$$

rearranging this and using $|y| < r_0$ we get

$$D\varphi(z(s_0)) \cdot z(s_0) \geq D\varphi(z(s_0)) \cdot y > -r_0|D\varphi(z(s_0))|$$

which contradicts (2.3.16). □

An analogous statement about touching from above by strict classical super-solutions holds as well. Lemma 2.3.4 almost says that $\omega(t)$ is a viscosity super-solution of (P-V) except for the restriction (2.3.15) on the barriers. In fact we can make the following statement:

Proposition 2.3.6. *If there exists $r' > r_0$ such that $\omega(t) \in \mathscr{S}_{r'}$ for all $t \in [0,T]$ then $u[\omega(t)]$ cannot be touched from below (above) by any strict classical sub-solution (super-solution) of (P-$\lambda[\omega(t)]$) and is therefore a viscosity solution of* (P-V).

Proof. This is simply because any $\phi \in C^{2,1}(\mathbb{R}^N \times [0,T])$ which touches $u[\omega(t)]$ from below (above) at a free boundary point must necessarily satisfy (2.3.15). To see this fact, let $x \in \partial\omega(t)$, now from the strong star-shapedness of $\omega(t)$,

$$\{x + s(x-y) : s \geq 0 \text{ and } y \in B_{r'}(0)\} \subset \mathbb{R}^N \setminus \omega(t) \subset \mathbb{R}^N \setminus \Omega_t(\phi)$$

and so for any $y \in B_{r'}(0)$

$$0 \geq \frac{d}{ds}\phi(x+s(x-y))\bigg|_{s=0} = D\phi(x)\cdot(x-y).$$

Rearranging above inequality, we obtain (2.3.15):

$$D\phi(x)\cdot x \leq \inf_{y\in B_{r'}(0)} D\phi(x)\cdot y = -r'|D\phi|(x) < -r_0|D\phi|(x).$$

$\square$

Unfortunately this statement is not that useful to us because we would still need to show that the gradient flow preserves strong star-shapedness. That is if we had a Lemma for $\omega(t)$ like Lemma 2.2.16 then from the above Proposition we would know that $\omega(t)$ was a viscosity solution of (P-V) at least for a short time, and then we could apply a result about preservation of strong star-shapedness for viscosity solutions like Lemma 2.2.22 to show that $\omega(t)$ is a viscosity solution globally in time. Instead of attempting to prove Lemma 2.2.16 for solutions of the gradient flow we will show directly that $\omega(t)$ and the viscosity solution of (P-$\lambda[\omega(t)]$) are the same. The idea comes from the following corollary of the proof of Proposition 2.3.6. As usual the corresponding result for supersolutions is also true.

Corollary 2.3.7. *(Comparison with star-shaped classical subsolutions) Suppose* $\phi \in C^{2,1}(\mathbb{R}^N \times [0,T])$ *is a strict classical sub-solution of (P-*$\lambda[\omega(t)]$*) and the initial strict separation*

$$\phi(\cdot,0) \prec u[\omega(0)]$$

holds. Moreover for some $r' > r_0$ *the positive phase of* ϕ*,* $\Omega_t(\phi)$*, is in* $\mathscr{S}_{r'}$ *for all* $t \in [0,T]$*. Then*

$$\phi(\cdot,T) \leq u[\omega(T)].$$

2.3.3 Comparison

Let $\omega(t)$ be the continuum limit of the discrete gradient flow scheme defined above. We want to show that $u = u[\omega(t)]$ is a viscosity solution of the problem,

$$\begin{cases} -\Delta v(x,t) = \lambda[\omega(t)] & \text{in} \quad \omega(t) \\ v_t = \min\{|Dv|^2 - 1, M\}|Dv| & \text{on} \quad \partial\omega(t), \end{cases} \tag{2.3.19}$$

at least as long as the unique viscosity solution of (P-$\lambda[\omega(t)]$) satisfies the appropriate strong star-shaped condition.

In fact u satisfies a comparison principle with respect to sub and super-solutions of (2.3.19) which are star-shaped with respect to a slightly larger ball than $B_{r_0}(0)$. This is the non-smooth version of the idea of Corollary 2.3.7. Various iterations of the proof can be found in the literature [CV99, Kim03, BV05], we will only sketch most of the details of the proof the essential idea is, as in Corollary 2.3.7, to demonstrate that the restricted barrier property of the gradient flow solution $\omega(t)$ is sufficient to make the argument work.

Proposition 2.3.8. (Comparison with sub-solutions) *Let* $u(\cdot,t) = u[\omega(t)]$ *as above from the gradient flow and* v *a continuous sub-solution of*

$$\begin{cases} -\Delta v(x,t) \leq \lambda[\omega(t)] & \textit{in} \quad \Omega_t(v) \\ v_t \leq \min\{|Dv|^2 - 1, M\}|Dv| & \textit{on} \quad \Gamma_t(v), \end{cases}$$

with initial data ordered,

$$\Omega_0(v) \subseteq \omega(0).$$

Let $r > 0$ from the restriction on admissible sets in the discrete scheme (2.3.5)*. Suppose there exists $r' > r_0$ such that $\Omega_t(v) \in \mathscr{S}_{r'}$ for all $t \in [0,T]$. Then*

$$v(\cdot,t) \le u(\cdot,t) \;\; on \;\; [0,T].$$

Proof. 1. First we show it suffices to treat the case when u and v are initially strictly separated,

$$\Omega_0(v) \subset\subset \omega(0), \tag{2.3.20}$$

and there exists $\varepsilon > 0$ so that v satisfies,

$$-\Delta v(x,t) \le (1-\varepsilon)\lambda(t) \quad \text{for} \quad (x,t) \in \Omega(v). \tag{2.3.21}$$

Let us assume temporarily that result holds in this case and we show that it also holds under the non-strict conditions given in the statement of the proposition. This is accomplished by a similar device to the one used in the proof of the strong comparison type results Lemma 2.2.8 and Proposition 2.2.7. Let $\varepsilon > 0$ small enough that $(1+\varepsilon)^{-1}r' > r_0$, c from Assumption B, and $0 < a < r_0$ then define the perturbation,

$$\widetilde{v}_\varepsilon(x,t) = \sup\nolimits_{|z|\le a\varepsilon - c\varepsilon t}(1+\varepsilon)^{-3}v((1+\varepsilon)(x+z),t) \quad \text{for} \quad 0 \le t \le \tfrac{a}{c}.$$

Now the perturbed subsolution $\widetilde{v}_\varepsilon$ satisfies (2.3.20) and (2.3.21) and so we get,

$$v(x,t) = \sup\nolimits_{\varepsilon>0} \widetilde{v}_\varepsilon(x,t) \le u(x,t) \quad \text{for} \quad 0 \le t \le a/c$$

Iterating this $\lfloor Tc/a \rfloor$ times we get the desired result.

2. Now we work in the case when (2.3.20) and (2.3.21) hold. Suppose towards a contradiction that v crosses u from below on $\mathbb{R}^N \times [0,T]$. In order to have some regularity at the touching point we use the space-time sup and inf convolutions, for some basic properties see [Kim03, BV05]. Let $\varepsilon, \delta > 0$ with $\delta < \varepsilon/T$ define the sup-convolution of v

$$V(x,t) = \sup_{|(y,s)-(x,t)|\le\varepsilon} v(y,s) \tag{2.3.22}$$

and the inf-convolution of u

$$U(x,t) = \inf_{|(y,s)-(x,t)|\le\varepsilon-\delta t} Z(y,s), \quad Z(y,s) := \inf_{|w-y|\le\varepsilon} u(w,s). \tag{2.3.23}$$

defined in the domain

$$\Sigma_\varepsilon := \mathbb{R}^N \times [\varepsilon, T-\varepsilon].$$

We will show that V still has the following properties:

(a) $\Omega_t(V)$ strongly star-shaped with respect to a ball of radius larger than r,

(b) V is a subsolution of (P-$\lambda[\omega(t)]$) and is initially strictly separated from U,

$$\Omega_\varepsilon(V) \subset\subset \Omega_\varepsilon(U).$$

First we show (a), let $\varepsilon \leq (r'-r)/2$, then

$$\Omega_t(V) = \bigcup_{|(y,s)|\leq\varepsilon} y + \Omega_{t+s}(v). \tag{2.3.24}$$

Noting that all of the sets in the above union are star-shaped with respect to $B_{(r+r')/2}(0)$ implies that $\Omega_t(V)$ is as well. This proves (a).

Now we show (b). Choosing ε smaller if necessary based on η and the modulus of continuity of $\lambda(t)$, the subsolution property for V is a standard result about sup-convolutions see for example [Kim03, BV05]. Then simply from the initial strict separation of u and v along with the Hausdorff distance continuity of the free boundaries of u and v we also have the strict separation of U and V at time ε

$$\Omega_\varepsilon(V) \subset\subset \Omega_\varepsilon(U).$$

Now by our assumption we know U crosses V from below for the first time at some point $P_0 = (x_0,t_0) \in \Sigma_\varepsilon$. By the strong maximum principle for subharmonic functions

$$x_0 \in \Gamma_{t_0}(U) \cap \Gamma_{t_0}(V).$$

At the touching point P_0 the positivity set of V has an interior space-time ball of radius ε centered at some point $P_1 = (x_1,t_1) \in \Gamma(v)$, i.e. $|P_1 - P_0| = \varepsilon$ and

$$B_1 := \{(y,s) \in \mathbb{R}^N \times [0,T] : |(y,s) - P_1| < \varepsilon\} \subset \Omega_+(V).$$

This ball has the tangent hyperplane through P_0

$$H_1 = \{(y,s) \in \mathbb{R}^N \times [0,T] : ((y,s) - P_0) \cdot (P_1 - P_0) = 0\}.$$

Let (ν, m) be normal to H_1 in the direction inward to $\Omega(V)$ with $|\nu| = 1$, m is the advance speed of the free boundary $\Gamma(V)$ at P_0. We will show below that m is finite, but at this point let us include the possibilities that $m = \infty$ or $m = -\infty$.

Similarly, there exists $P_2 = (x_2, t_2) \in \Gamma(Z)$ with $|P_2 - P_0| = \varepsilon - \delta t_0$ and $\Omega(U)$ has an exterior space-time ellipse at P_0 of the form

$$\widetilde{B}_2 := \{(y,s) \in \mathbb{R}^N \times [0,T] : |(y,s) - P_2| \leq \varepsilon - \delta s\}$$

while there exists $(x_3, t_2) \in \Gamma(u)$ with $|x_2 - x_3| = \varepsilon$ and $\Omega(u)$ contains the following set:

$$E = \{(y,s) \in \mathbb{R}^N \times [0,T] : |y - w| \leq \varepsilon, |(w,s) - P_2| < \varepsilon - \delta t_0\} \subset \Omega(u), \quad (x_3, t_2) \in \bar{E} \cap \Gamma(u). \tag{2.3.25}$$

Let $\widetilde{H}_2$ be the tangent hyperplane to $\widetilde{B}_2$ at P_0 and H_2 be the tangent hyperplane to B_2 at P_2.

Lemma 2.3.9. *H_1 is not horizontal.*

Proof. We refer to the proof of Lemma 2.5 in [Kim03], which rules out the possibility that $m = +\infty$.

Next let us rule out the possibility that $m = -\infty$. Let x_3 and E be the set given in (2.3.25). Due to the star-shapedness and Hölder continuity of $\Omega(u)$, it follows that (x_3, t_2) lies in the spatial boundary of $E \cap \{t = t_2\}$. We consider the classical subsolution

Take $\tau > 0$ sufficiently small that $D_0 := E \cap \{t = t_2 - \tau\}$ is nonempty. Let D_1 be the top portion of the closure of E, i.e., $D_1 := \bar{E} \cap \{t = t_2\}$, and let us define the interpolation of D_0 and D_1, i.e.,

$$D_t := (1 - s(t))D_0 + s(t)D_1, \text{ where } s(t) \text{ is linear and satisfies } s(t_2 - \tau) = 0, s(t_2) = 1.$$

We choose τ small that $s'(t) < -1$. Now let us consider the space-time domain

$$\Sigma := \bigcup_{\tau \leq t - t_2 \leq 0} [D(t) - \frac{1}{2} D(t)]$$

Then, due to the fact that $\Sigma \subset E$, the positive set $\Omega(u)$ crosses Σ for the first time at (x_3, t_2). Let us consider the classical subsolution ψ in the domain satisfying

$$\begin{cases} -\Delta_x \phi(x,t) = \lambda[w(t)] & \text{for} \quad (x,t) \in \Sigma \\ \phi = 0 & \text{for} \quad (x,t) \in \partial D(t) \\ \phi = \min_{\frac{1}{2}D(t)} u(x,t) > 0 & \text{for} \quad (x,t) \in \frac{1}{2}\partial D(t) \end{cases} \tag{2.3.26}$$

Then ψ crosses u from below at (x_3, t_2). One can check from the definition of u and the strong star-shapedness of $\Omega(V)$ that the outer normal ν of D_1 at x_3 satisfies $\nu \cdot x_2 < -r_0$ and thus it contradicts Lemma 2.3.4. $\square$

Let (ν', m') be the normal to the hyperplane H_2 rescaled as before. Then the normal to $\widetilde{H}_2$ is $(\nu', m' + \delta)$. As a consequence of the ordering $V(x,t) \leq U(x,t)$ for all $t \leq t_0$ we also get the ordering of the advance speeds,

$$m' + \delta \leq m.$$

Since $U \leq V$ for $t \leq t_0$ we get the following inclusions

$$B_1 \cap \{t \leq t_0\} \subseteq \{U > 0\} \cap \{V > 0\} \quad \text{and} \quad \widetilde{B}_2 \cap \{t \leq t_0\} \subseteq \{U = 0\} \cap \{V = 0\}.$$

In particular $\Omega_{t_0}(U)$ and $\Omega_{t_0}(V)$ both have interior and exterior spatial balls at x_0,

$$B_1 \cap \{t = t_0\} \text{ and } \widetilde{B}_2 \cap \{t = t_0\}$$

which must both have centers lying along the same axis, or equivalently both free boundaries have spatial inward normal ν at x_0. From the strong star-shapedness of $\Omega_{t_0}(V)$, and this is a key point as we have seen in the proof of above lemma: we get that

$$\nu \cdot x_0 \leq -r'' < -r_0. \tag{2.3.27}$$

Now the idea is to construct a smooth strict sub-solution ϕ which touches $\Gamma(V)$ from below at (x_0, t_0). Then a translation of ϕ will touch u from below at P_2 from the inadmissible direction,

$$\nu = \frac{\phi_t}{|D\phi|}(x_0, t_0)$$

leading to a contradiction of Lemma 2.3.4.

Lemma 2.3.10.

$$-1 < m \leq M.$$

Proof. The fact that $m \leq M$ follows from a relatively simple barrier argument, based on the condition $v_t \leq M|Dv|$ on $\Gamma_t(v)$ as well as the fact that $\Omega(v)$ has an exterior space-time ball at (x_1, t_1).

It remains to show that $m \leq -1$. If it is, then

$$m' \leq -1 - \delta. \tag{2.3.28}$$

Take $\tau > 0$ sufficiently small that $D_0 := B_2 \cap \{t = t_2 - \tau\}$ is nonempty. We can now construct a classical subsolution ψ in the domain

$$\tilde{\Sigma} := \cup_{t_2 - \tau \leq t \leq s_0} (B_2 - \frac{1}{2} B_2),$$

similarly as in the proof of $m > -\infty$ in the lemma above, and use (2.3.28) to derive a contradiction.

□

Lemma 2.3.11. *Near the point P_0 we have the nontangential estimate*

$$\liminf_{d \to 0^+} \frac{V(x_0 - d\nu, t_0)}{d} \geq \sqrt{m+1}. \tag{2.3.29}$$

Proof. The proof is based on construction of the barrier for v to yield a contradiction, in the event that the lemma holds false, and it is parallel to the proof of Lemma 2.6 in [Kim03].

□

Now as in [Kim03] for any $\eta > 0$ we construct a smooth test function $\phi(x,t) = \phi(|x - x_0|, t)$ with the following properties:

$$\begin{cases} -\Delta_x \phi(x,t) < 0 & \text{for} \quad (x,t) \notin \frac{1}{4} B_2 \\ \phi > 0 & \text{for} \quad (x,t) \in B_2 \\ \phi < 0 & \text{for} \quad (x,t) \notin \overline{B_2} \\ |D\phi|(x,t) = \sqrt{m+1}(1-\eta) & \text{on} \quad \partial B_2 \cap \{t = t_2\}. \end{cases} \tag{2.3.30}$$

Now from the definition of m' we have for $x \in \partial B_2 \cap \{t = t_2\}$,

$$\tfrac{\phi_t}{|D\phi|}(x, t_2) = m' \leq m - \delta$$

and choosing τ sufficiently small depending on the C^2 norm of ϕ and η small depending on δ and m we get for $x \in \partial B_2 \cap \{t_2 - \tau \leq t \leq t_2\}$,

$$\tfrac{\phi_t}{|D\phi|}(x, t_2) < m - \delta/2 < |D\phi|^2 - 1.$$

Therefore ϕ is a strict subsolution in the region

$$(B_2 \setminus \tfrac{1}{4}B_2) \cap \{t_2 - \tau \leq t \leq t_2\}.$$

Next we show that ϕ touches u from below at $P_2 = (x_2, t_2)$. For $d > 0$ sufficiently small depending on the C^2 norm of ϕ we have

$$\phi(x,t) \leq \sqrt{m+1}(1-2\eta)d \quad \text{on} \quad \partial(1-d)B_2 \cap \{t_2 - \tau \leq t \leq t_2\}. \tag{2.3.31}$$

Let $(x,s) \in B_2$ and let

$$d := d(x, \partial B_2 \cap \{t = s\})$$

then since P_0 is the center of B_2 we have that $|P_0 - (x,s)| = \varepsilon - \delta t_0 - d$ and

$$|(x_0 + d\nu, t_0) - (x,s)| \leq |P_0 - (x,s)| + d = \varepsilon - \delta t_0.$$

Therefore from the definition of U as an infimum and from Lemma 2.3.11 for any $\eta > 0$ there exists d_0 such that $d < d_0$ implies

$$\sqrt{m+1}(1-2\eta)d \leq U(x_0 + d\nu, t_0) \leq u(x,s). \tag{2.3.32}$$

Now combining (2.3.31) and (2.3.32) with the fact that $B_2 \subset \Omega(u)$ and $\phi = 0$ on ∂B_2 we get that

$$\phi(x,t) \leq u(x,t) \quad \text{for} \quad (x,t) \in (\partial(1-d)B_2 \cup \partial B_2) \cap \{t_2 - \tau \leq t \leq t_2\}. \tag{2.3.33}$$

Then since $u - \phi$ is superharmonic by the strong minimum principle we get,

$$\phi(x,t) < u(x,t) \quad \text{for} \quad (x,t) \in B_2 \setminus (1-d)B_2 \cap \{t_2 - \tau \leq t \leq t_2\}, \tag{2.3.34}$$

and $u(P_2) = \phi(P_2) = 0$ so ϕ touches u from below at P_2. This is a contradiction of Lemma 2.3.4 since from (2.3.27)

$$\frac{D\phi}{|D\phi|}(x_2, t_2) \cdot x_2 = \nu \cdot x_2 < \nu \cdot x_0 < -r_0.$$

$\square$

2.4 Convergence for Solutions with Global in Time Star-Shapedness

Now we can combine the results of the previous sections to get our main result. Under the assumption of sufficient roundness of the initial data Ω_0 phrased in terms of ρ-reflection any continuum limit of the restricted gradient flow of the functional $\mathscr{J}$ is also a global in time viscosity solution of the problem (P-V). To make this precise let us define the class of weak solutions arising from the gradient flow scheme described in section 2.3.

Theorem 2.4.1. *An evolution $\omega(t) : [0,\infty) \to \mathscr{S}_r$ is an energy solution of* (P-V) *if there exist $\omega_{M_k}(t)$, $M_k \to \infty$ as $k \to \infty$, which are minimizing movements of the restricted gradient flow scheme from Definition 2.3.1 with initial data $\omega(0)$ such that,*

$$d_H(\omega_{M_k}(t), \omega(t)) \to 0 \ \textit{ as } \ k \to \infty \ \textit{ locally uniformly in } [0,\infty).$$

A droplet profile $u : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ is an energy solution if $u = u[\omega(t)]$ for an energy solution $\omega(t)$.

Now we can say the following about any energy solution arising from an initial data with ρ-reflection.

Theorem 2.4.1. *Let $V > 0$ and Ω_0 a domain in $\mathbb{R}^N$ such that Ω_0 has ρ-reflection with ρ satisfying,*

$$\rho < C_N \mathrm{Vol}^{\frac{1}{N+1}},$$

where C_N is a dimensional constant. Then there exists an energy solution $u : \mathbb{R}^N \times [0,+\infty) \to [0,+\infty)$. Moreover any energy solution u, the following holds:

(a) u is also a viscosity solution of the free boundary problem,

$$\begin{cases} -\Delta u(x,t) = \lambda(t) & \textit{in} \quad \Omega_t(u) \\ u_t = (|Du|^2 - 1)|Du| & \textit{on} \quad \Gamma_t(u) \\ \{u(\cdot,0) > 0\} = \Omega_0, \end{cases} \tag{P-V}$$

where $\lambda(t)$ is chosen to enforce the volume constraint for all $t > 0$,

$$\int u(x,t)dx = \mathrm{Vol}.$$

(b) The positivity set $\Omega_t(u)$ has ρ-reflection for all $t > 0$ with $\inf_{x\in\Gamma_t(u)}(|x|-\rho)$ bounded below. The energy $\mathscr{J}$ is decreasing along the flow and additionally u satisfies the energy decay estimate for all $t > s \geq 0$,

$$\frac{1}{t-s}\widetilde{\operatorname{dist}}(\Omega_s(u),\Omega_t(u)) \lesssim_{\rho,N} \mathscr{J}(\Omega_s(u)) - \mathscr{J}(\Omega_t(u)).$$

(c) The flow of the sets $\Omega_t(u)$ converges uniformly modulo translation to the radially symmetric equilibrium solution,

$$\frac{\lambda_*}{2N}(r_*^2 - |x|^2)_+,$$

where r_* is given in (2.1.2) and λ_* can be calculated from the volume constraint. More precisely we mean that,

$$\inf\{d_H(\Omega_t(u),B_{r*}(x)) : x \in \overline{B_\rho(0)}\} \to 0 \text{ as } t \to \infty.$$

Proof. 1. The existence proof as well as the energy estimate follow from a fairly straightforward combination of the results we have already proven. We give an outline of the proof. Given the assumption on Ω_0 having ρ-reflection we know due to the apriori estimate Lemma 2.2.22 that any solution of (P-V) will be in $\mathscr{S}_r$ for some $r > 0$ as long as it exists. Then we construct the restricted gradient flow solution with initial data Ω_0 as in Section 2.3 where the restriction on the radius of the strong star-shapedness is strictly weaker than that from the apriori estimate. Given the Lagrange multplier $\lambda(t)$ associated with the restricted gradient flow we then solve (P-λ). Then the idea is that the viscosity solution of (P-λ) is strongly star-shaped with a larger radius than the restriction on the gradient flow and so we will be able to use Proposition 2.3.8 to show that the two solutions agree for all time.

Let $b > 0$ so that

$$B_{(1+b)\rho}(0) \subset \Omega \quad \text{and} \quad (1+b)\rho < C_N V^{\frac{1}{N+1}}.$$

Then we expect thanks to the apriori estimate Lemma 2.2.22 that $(1+b)\rho$ will be contained in Ω_t for all time. In particular we expect Ω_t will be in $\mathscr{S}_r$ for $r = b\rho$. Let $R = R(r) > 0$ be so large that any set in $\mathscr{S}_r$ which touches B_R from the inside must have larger energy than Ω_0 and then define $\gamma = \frac{r}{2R}$. That it is possible to choose R in this way is described in Lemma 2.3.3. Let $M > 0$ and let

$\omega_M(t) = \omega_{M,\gamma r}(t)$ be the restricted gradient flow solution with initial data Ω_0 as constructed in the last section. The notation indicates that $\omega_{M,\gamma r}(t)$ is restricted to remain in $\mathscr{S}_r$ and its free boundary can move no faster than M. We suppress the dependence on γr. Define the putative Lagrange multiplier λ_M to be

$$\lambda_M(t) := \lambda[\omega_M(t)],$$

and let u_M be the possibly discontinuous viscosity solution of $(P - \lambda_M)$ constructed by Perron's method in Theorem 2.2.2. We will show that u_M and $u[\omega_M]$ are the same.

Let I be the largest interval containing the origin on which u_M and $u[\omega_M]$ agree. We will show that I is open and closed in $[0,+\infty)$. Since u_M agrees with $u[\omega_M]$ on I it is continuous and has constant volume and thus it is a volume preserving viscosity solution. Therefore Lemma 2.2.22 implies that:

$$\Omega_t(u_M) \text{ has } \rho\text{-reflection and } B_{(1+b)\rho} \subset \Omega_t(u_M) \text{ for } t \in I.$$

In particular $\Omega_t(u_M)$ is in $\mathscr{S}_r$ on I. Suppose $I = [0,T)$ for some $T > 0$, then the short time existence theorem implies that, for some small $t_0(r,R)$, u_M is continuous on $[0, T + t_0)$. Since the set where two continuous functions agree is closed $I = [0,T]$. Suppose $I = [0,T]$ where T may be equal to zero. Then thanks to Lemma 2.2.16 there exists $t_M > 0$ such that:

$$\Omega_t(u_M) \in \mathscr{S}_{r'} \text{ for some } r' > \gamma r \text{ on } [0, T + t_M).$$

Then from Proposition 2.3.8 $u[\omega_M]$ satisfies a strong comparison principle with respect to viscosity solutions which are strongly star-shaped with a larger radius than γr so $u_M = u[\omega_M]$ on $[0, T+t_M)$. Therefore $I = [0,+\infty)$ and there exists a global in time continuous viscosity solution of (P-V)$_M$ which has ρ-reflection and is in $\mathscr{S}_{r,R}$ for all $t > 0$.

2. Now we show the existence for (P-V) without the restriction on a maximum speed. The key point in this case is the equicontinuity afforded by the fact that the $\Omega_t(u_M) \in \mathscr{S}_{r,R}$ with r,R independent of M. In particular from Corollary 2.2.11 we get for some $C > 0$ and $\alpha \in (0,1)$ independent of M,

$$d_H(\Gamma_t(u_M), \Gamma_s(u_M)) \leq C|t-s|^{\alpha}. \tag{2.4.1}$$

Then we also derive thanks to Lemma 2.A.5 the equicontinuity of the Lagrange multipliers $\lambda_M(t)$. Taking a subsequence such that the $\lambda_M(t)$ converge uniformly on compact subsets of $[0,+\infty)$ to some $\lambda(t)$ we can apply Lemma 2.2.15. We get that along this subsequence the u_M converge uniformly to a viscosity solution u of (P-V) with free boundary velocity $F(|Du|) = |Du|^2 - 1$. Due to the uniform convergence of $\Omega_t(u_M)$ to $\Omega_t(u_M)$ in hausdorff distance sense the energy estimates for the u_M and the Hölder regularity in time (2.4.1) carry over to u. Then u is an energy solution by the definition. Unfortunately by the compactness method we do not know whether there is any uniqueness of the limiting Lagrange multiplier $\lambda(t)$.

3. Now we show that any subsequential limit of the $u(\cdot, t_n)$ must be a viscosity solution of the equilibrium problem (EQ). Note that the same result is true for all the u_M.

Claim. Let $t_n \to \infty$ such that $\Omega_{t_n}(u)$ converges in Hausdorff topology to Ω_∞. Then $u[\Omega_\infty]$ is a stationary solution of (P-V), that is it solves the equilibrium problem (EQ) in the viscosity sense.

Define $U_n : [0, +\infty) \to \mathscr{S}_{r,R}$ by

$$U_n(t) := \Omega_{t+t_n}, \tag{2.4.2}$$

and we also consider the viscosity solutions of (P-V) which lie above the $U_n(t)$,

$$v_n(x,t) := u[U_n(t)](x) \quad \text{with Lagrange Multipliers} \quad \lambda_n(t) := \lambda[U_n(t)]. \tag{2.4.3}$$

First we show that, uniformly in $t > 0$,

$$\mathscr{J}(U_n(t)) \to \mathscr{J}(\Omega_\infty).$$

Since is $\mathscr{J}(U_n(t))$ is monotone decreasing for all $n > 0$ we have for all $t > 0$

$$\mathscr{J}(\Omega_\infty) = \inf_{s>0} \mathscr{J}(\Omega_s) \leq \mathscr{J}(U_n(t)) \leq \mathscr{J}(U_n(0)) = \mathscr{J}(\Omega_{t_n}),$$

but due to Lemma 2.A.5 along with the convergence $\Omega_n \to \Omega_\infty$ in Hausdorff distance,

$$\mathscr{J}(\Omega_{t_n}) \searrow \mathscr{J}(\Omega_\infty).$$

Now we will show that up to a subseqeunce the v_n converge uniformly on compact time intervals. Recalling the uniform Hölder estimates from (2.4.1) for $\Omega_t(u_M)$ which carry over in the limit to $\Omega_t(u)$,

$$d_H(U_n(t), U_n(s)) \leq C|t-s|^\alpha, \tag{2.4.4}$$

and from Lemma 2.3.3 the energy estimates,

$$\frac{C}{t-s}\widetilde{\operatorname{dist}}(U_n(s),U_n(t)) \le \mathscr{J}(U_n(s)) - \mathscr{J}(U_n(t)). \tag{2.4.5}$$

The paths U_n are a sequence of equicontinuous maps into $(\mathscr{S}_{r,R}, d_h)$ and so from the Compactness Lemma 2.A.6 there exists $U_\infty : [0,+\infty) \to \mathscr{S}_{r,R}$ with $U_\infty(0) = \Omega_\infty$ such that up to taking a subsequence,

$$U_n \to U_\infty \text{ uniformly on compact subsets of } [0,+\infty).$$

Now we show that v_∞ is a stationary viscosity solution of (P-V). From Lemma 2.A.5 we get the following:

(i) $v_n \to v_\infty$ uniformly in (x,t) on compact time intervals and therefore from the stability of viscosity solutions under uniform convergence – Lemma 2.2.4 – v is a viscosity solution of (P-V).

(ii) $\widetilde{\operatorname{dist}}(U_n(s),U_n(t)) \to \widetilde{\operatorname{dist}}(U_\infty(s),U_\infty(t))$ uniformly on compact subsets of $[0,+\infty)\times[0,+\infty)$.

Combining (ii) with (2.4.5) we derive the energy estimate for $U_\infty(\cdot)$ for all $t > s > 0$:

$$\frac{C}{t-s}\widetilde{\operatorname{dist}}(U_\infty(s),U_\infty(t)) \le \mathscr{J}(U_\infty(s)) - \mathscr{J}(U_\infty(t)) = 0. \tag{2.4.6}$$

So $v_\infty = u[\Omega_\infty]$ is a stationary viscosity solution of (P-V). Then due to Theorem 2.1.1, Theorem 2.1.2 and Corollary 2.1.1 it follows that $\Omega_\infty = B_{r*}(x_0)$ for some point $x_0 \in \mathbb{R}^N$. Actually x_0 is not completely arbitrary since we know that Ω_∞ must have ρ-reflection. One can easily check that this implies $x_0 \in \overline{B_\rho(0)}$.

4. Finally we show that the convergence is uniform modulo translation. Suppose that there exists a sequence of times $t_n \to \infty$ and a $\delta > 0$ such that

$$\inf_{x\,\in\overline{B_\rho(0)}} d_H(\Omega_{t_n}(u), B_{r*}(x)) > \delta.$$

By taking a subsequence of the t_n we may assume that $\Omega_{t_n}(u)$ converges in Hausdorff distance to some Ω_∞. By part 3 of the argument $u[\Omega_\infty]$ must be equal to $B_{r*}(x_0)$ for some $x_0 \in B_\rho(0)$. Choosing n sufficiently large so that

$$d(\Omega_{t_n}(u), B_{r*}(x_0)) < \delta$$

we derive a contradiction.

□

Note that above theorem leaves the possibility that the drop oscillates between a family of round drops with its speed going to zero but not fast enough to converge to a single profile. Below we show that, if the drops are sufficiently regular at large times, then this does not happen. Such regularity, when the time is sufficiently large so that the profile of u is sufficiently close to a round one is suspected to be true in the light of existing results introduced by Caffarelli et. al. (see e.g. the book [CS05]), but verifying this for our setting would be highly nontrivial and thus we do not pursue this question here.

Proposition 2.4.1. (Conditional uniqueness of the limit) *Suppose additionally that $u(\cdot,t)$ are uniformly $C^{1,\alpha}$, then $\Omega_t \to B_{r*}(x_0)$ for some x_0.*

Proof. Let $t_n \to \infty$ be a sequence of times along which $\Gamma_{t_n}(u)$ converges in Hausdorff distance. The limit is a ball $\partial B_{r*}(x_0)$ for some x_0 which is compatible with the reflection symmetry of Ω. By translating we may assume that $x_0 = 0$. Note under this translation $\Omega_t(u)$ no longer need have ρ-reflection or be strongly star shaped with respect to the origin, this will not affect the proof. From Lemma 2.A.5 we get that $u(\cdot,t_n)$ converge uniformly to

$$u_{EQ} = \frac{\lambda_*}{2N}(r_*^2 - |x|^2)_+.$$

Due to the assumption, we have

$$\sup_n ||u(\cdot,t_n)||_{C^{1,\alpha}} < C_1 < +\infty.$$

In particular the $Du(\cdot,t_n)$ are uniformly bounded and equicontinuous so they must converge uniformly to Du_{EQ}. Let $\delta > 0$ small enough that $C_1\delta^\alpha \leq 1/2$ and choose n sufficiently large that

$$d_H(\Gamma_{t_n}(u), \partial B_{r*}) \leq \delta \quad \text{and} \quad ||Du(\cdot,t_n) - Du_{EQ}||_\infty \leq \delta.$$

Now let $x \in \Gamma_{t_n}(u)$ and let $y = y(x) \in \partial B_{r_*}$ such that $|x-y| = d(x,\partial B_{r_*})$. We calculate,

$$|Du(x,t_n) - Du_{EQ}(y)| \leq ||Du(\cdot,t_n) - Du_{EQ}||_\infty + C_1 d(x,\partial B_{r_*})^\alpha \leq C\delta^\alpha$$

so since $|Du_{EQ}| = 1$ on ∂B_{r*} we have that $|Du(x,t_n)| \geq 1/2$ and

$$|\tfrac{Du}{|Du|}(x,t_n) - \tfrac{y}{|y|}| \leq C\delta^{\alpha}.$$

Rephrasing the above in terms of the interior normal field ν_n to $\Gamma_{t_n}(u)$ we get that

$$\langle \nu_n(x), x \rangle \geq 1 - o(1). \tag{2.4.7}$$

In particular this means, thanks to Lemma 2.A.3, that for any $\rho > 0$ there exists n sufficiently large so that $\Omega_{t_n}(u)$ has ρ-reflection. Since ρ-reflection is preserved under the flow for ρ small this means that for every $\rho > 0$ there is a T such that for $t \geq T$, $\Omega_t(u)$ has ρ-reflection. Thus any subsequential limit of the $\Omega_t(u)$ has ρ-reflection for every $\rho > 0$ and must be a ball centered at the origin.

□

There is one nontrivial case where we can say that the limiting equilibrium solution is unique. When the initial data is symmetric with respect to N orthogonal hyperplanes through the origin in addition to all the conditions given in the statement of Theorem 2.4.1 then the limit is unique. In this case the reflection symmetries are preserved by the equation and so any Hausdorff distance limit of the $\Omega_t(u)$ will share these symmetries. Then it is basic to check that the only ball of radius r^* which is symmetric with respect to N orthogonal axes through the origin is in fact centered at the origin. We record this fact in the following corollary:

Corollary 2.4.2. *If Ω_0 satisfies all the conditions of Theorem 2.4.1 and additionally is symmetric with respect to N mutually orthogonal hyperplanes through the origin then any solution u of* (P-V) *with initial data Ω_0 constructed from the discrete gradient flow scheme of Section 2.3 satisfies,*

$$d_H(\Omega_t(u), B_{r*}(0)) \to 0 \ \ as \ \ t \to \infty.$$

2.A Geometric Properties

Let Ω be an open bounded domain in $\mathbb{R}^N$ which is strictly star-shaped with respect to 0. Let $\rho > 0$ such that $B_\rho(0) \subseteq \Omega$ and let H be a hyperplane in $\mathbb{R}^N$ such that $H \cap B_\rho(0) = \emptyset$. Let H_+ be the open half-space of H which contains $B_\rho(0)$ and H_- be the interior of its complement. Then define

$$\Omega_+ = \Omega \cap H_+ \text{ and } \Omega_- = \Omega \cap H_-$$

and define the reflection through H by

$$\phi_H(x) = x - 2\langle x - y, \nu_H \rangle \nu_H$$

where ν_H is the unit normal to H (pointing inward towards H_- for concreteness) and $y \in H$.

Lemma 2.A.1. *Suppose Ω has ρ-reflection, then*

$$\sup_{x\in\partial\Omega} |x| - \inf_{x\in\partial\Omega} |x| \leq 4\rho.$$

Proof. Take $x_0 \in \partial\Omega$ such that

$$|x_0| = \inf_{x\in\partial\Omega} |x|.$$

Let H be any hyperplane tangent to $B_\rho(0)$ such that $x_0 \in H_+$.

1. Claim: The reflections $\phi_H(x_0)$ with respect to the hyperplanes described above cover all directions in the sphere, more precisely

$$\{\frac{\phi_H(x_0)}{|\phi_H(x_0)|} : H \text{ tangent to } B_\rho(0) \text{ and } x_0 \in \overline{H}_+\} = S^{N-1}.$$

We index H by its normal vector, so for $\nu \in S^{N-1}$ let H_ν be the hyperplane through 0 normal to ν and then define:

$$\phi_\nu(x) = \phi_{H_\nu+\rho\nu}(x).$$

Let $\omega \in S^{N-1}$. Without loss, by changing coordinate names, we may assume that $\widehat{x_0} = e_1$ and $\omega \in \operatorname{span}\{e_1, e_2\}$. We restrict ourselves to hyperplanes with normal direction $\nu \in \operatorname{span}\{e_1, e_2\}$ and thereby reduce to the case $N = 2$. Let φ such that $\cos\varphi = \rho/|x_0|$. Then for any $\theta \geq \varphi$ the plane through

$$\rho((\cos\theta)e_1 + (\sin\theta)e_2)$$

has x_0 in the same half-space as B_ρ. Let us consider the continuous map $f : [0,\pi] \to [0,\pi]$ defined by,

$$f(\theta) = \cos^{-1}\left\langle \tfrac{\phi_{\nu(\theta)}(x_0)}{|\phi_{\nu(\theta)}(x_0)|}, e_1 \right\rangle \text{ where } \nu(\theta) = \cos\left(\varphi + \left(\tfrac{\pi-\varphi}{\pi}\right)\theta\right) e_1 + \sin\left(\varphi + \left(\tfrac{\pi-\varphi}{\pi}\right)\theta\right) e_2.$$

We show that $f(0) = 0$ and $f(\pi) = \pi$ and thus f maps $[0,\pi]$ onto itself. We chose φ so that $x_0 \in H_{\nu(0)} + \rho\nu(0)$,

$$\langle x_0 - \rho\nu(0), \nu(0)\rangle = \langle x_0, \nu(0)\rangle - \rho = |x_0|\cos\varphi - \rho = 0.$$

Therefore $\phi_{\nu(0)}(x_0) = x_0$ and $f(0) = 0$. Meanwhile since $\nu(\pi) = \pi$ we compute directly that

$$\phi_{\nu(\pi)}(x_0) = (\rho - |x_0|)e_1$$

and thus $f(\pi) = \pi$. Now in order to show that ω is hit we just choose θ so that:

$$f(\theta) = \cos^{-1}\langle\omega, e_1\rangle$$

or in other words,

$$\frac{\phi_{\nu(\theta)}(x_0)}{|\phi_{\nu(\theta)}(x_0)|} = \omega.$$

2. Let $\omega \in S^{N-1}$, by Ω strictly star-shaped with respect to 0 there exists $t > 0$ such that $t\omega \in \partial\Omega$ and $s \leq t$ implies $s\omega \in \Omega$, $s > t$ implies $s\omega \in \Omega^C$. We want to show that

$$t \leq |x_0| + 4\rho.$$

Let H be such that $x_0 \in H_+$ and $\frac{\phi_H(x_0)}{|\phi_H(x_0)|} = \omega$, which is possible by part 1 of the proof above. Then by star-shapedness $\lambda x_0 \in \overline{\Omega}^C$ for $\lambda \in (1, +\infty)$ and the analagous statement for the reflected domain,

$$\lambda\phi_H(x_0) \in \overline{\phi_H(\Omega_+)}^C \text{ for } \lambda \in (1, +\infty).$$

Moreover since Ω has ρ-reflection $\Omega_- \subseteq \phi_H(\Omega_+)$ so in particular

$$\frac{t}{|\phi_H(x_0)|}\phi_H(x_0) = t\omega \ \in \ \overline{\Omega_-} \subseteq \overline{\phi_H(\Omega_+)}$$

so $t \leq |\phi_H(x_0)|$. Now, noting that $-\rho\nu_H \in H$ due to H being tangent to $B_\rho(0)$,

$$\phi_H(x_0) = x - 2\langle x_0 + \rho\nu_H, \nu_H\rangle\nu_H = (x_0 - \langle x_0, \nu_H\rangle\nu_H) - (2r + \langle x_0, \nu_H\rangle)\nu_H$$

so by pythagoras

$$\begin{aligned}
|\phi_H(x_0)|^2 &= |x_0 - \langle x_0, \nu_H\rangle\nu_H|^2 + (2r + \langle x_0, \nu_H\rangle)^2 \\
&= |x_0 - \langle x_0, \nu_H\rangle\nu_H|^2 + |\langle x_0, \nu_H\rangle|^2 + 4\rho^2 + 4\rho\langle x_0, \nu_H\rangle \\
&= |x_0|^2 + 4\rho^2 + 4\rho\langle x_0, \nu_H\rangle
\end{aligned}$$

rearranging (and noticing that $\langle x_0, \nu_H \rangle > -\rho$ due to our assumption that $x_0 \in H_+$),

$$\begin{aligned} t \leq |\phi_H(x_0)| &\leq |x_0| + 4\frac{\rho^2 + \rho\langle x_0, \nu_H\rangle}{|\phi_H(x_0)| + |x_0|} \\ &\leq |x_0| + 4\frac{\rho^2 + \rho|x_0|}{|\phi_H(x_0)| + |x_0|} \\ &\leq |x_0| + 4\frac{2r}{1 + |\phi_H(x_0)|/|x_0|} \\ &\leq |x_0| + 4\rho, \end{aligned}$$

where we have used $|x_0| \geq \rho$ and $|\phi_H(x_0)| \geq |x_0|$ in the last two lines. This completes the proof. $\square$

Lemma 2.A.2. *Suppose Ω has ρ-reflection. Then Ω satisfies the following:*

(a) for all $x \in \partial\Omega$ there is an exterior cone to Ω at x,

$$x + C(x, \phi_x) \subset \mathbb{R}^N \setminus \Omega \quad \textit{where} \quad \cos\phi_x = \tfrac{\rho}{|x|},\ \phi_x \in (0, \pi/2), \tag{2.A.1}$$

and $C(x, \phi_x)$ is the cone in direction x of opening angle ϕ_x as defined in (2.2.8)*;*

(b) $\Omega \in \mathscr{S}_r$ where

$$r = r(\rho, \inf_{x\in\partial\Omega} |x|) = (\inf_{x\in\partial\Omega} |x|^2 - \rho^2)^{1/2}.$$

Proof. Let $x_0 \in \partial\Omega$ and let $\mathscr{R}$ be the collection of planes which pass through x_0 and are admissible for reflection i.e.

$$\mathscr{R} = \{H \text{ hyperplane in } \mathbb{R}^N :\ x_0 \in H \text{ and } \ H \cap \overline{B_\rho(0)} = \emptyset\}.$$

Notice that if $H \in S$ then for any $a > 0$ the plane $H + a\nu_H \subset H_-$ so $H + a\nu_H$ is also admissible for reflection. We can also characterize $\mathscr{R}$ as hyperplanes $H \subset \mathbb{R}^N$ such that $\inf_{y\in H} |y| \geq \rho$ and $x_0 \in H$,

$$\rho \leq \inf_{y\in H} |y| = |\langle x_0, \nu_H\rangle|.$$

So if we think of these planes as being indexed by their normal vectors,

$$\mathscr{R} = \{H : \langle \nu_H, x_0\rangle \geq \rho \text{ and } x_0 \in H\}.$$

Now let $y \in x_0 + C(x_0, \phi_{x_0})$ as above, then from the definition of ϕ_{x_0},

$$\langle \frac{y - x_0}{|y - x_0|}, x_0\rangle > \rho$$

so the plane H_y with normal $\nu_{H_y} = \frac{y-x_0}{|y-x_0|}$ through x_0 is in $\mathscr{R}$. Now we will use the reflection comparison with respect to the plane

$$\widehat{H} = H_y + \frac{1}{2}(y - x_0)$$

which is admissible for reflection by the remark above that $\widehat{H} \subset H_{y-}$. Then from the ρ-reflection property of Ω,

$$y = \phi_{\widehat{H}}(x_0) \in \mathbb{R}^N \setminus \phi_{\widehat{H}}(\Omega \cap \widehat{H}_+) \subseteq \mathbb{R}^N \setminus (\Omega \cap \widehat{H}_-)$$

and therefore $y \in \mathbb{R}^N \setminus \Omega$. Since y was arbitrary,

$$C(x_0, \phi_{x_0}) \subset \mathbb{R}^N \setminus \Omega.$$

This proves (a), (b) follows from (a) and Lemma 2.2.6. □

Conversely we can show that given a domain Ω which is uniformly close to a ball around zero and has some uniform condition on the directions of its normal vectors (i.e. star-shapedness with respect to a large ball) has ρ-reflection. So ρ-reflection is a natural condition to describe closeness to being round.

Lemma 2.A.3. *Let $\Omega \in \mathscr{S}_0$, $n(x)$ be normal to $\partial\Omega$, and let $\rho > 0$ such that $B_\rho(0) \subset \Omega$. Suppose that for $\mathscr{H}^{N-1}$ almost every $x \in \partial\Omega$*

$$|\langle n(x), x\rangle|^2 \geq |x|^2 - \rho^2/5 \tag{2.A.2}$$

and also,

$$\sup_{x,y\in\partial\Omega} |x|^2 - |y|^2 \leq \rho^2 \tag{2.A.3}$$

then Ω has ρ-reflection.

Remark 2.A.4. The conditions (2.A.2) and (2.A.3) should be interpreted together as a smallness requirement on the Lipschitz norm distance between $\partial\Omega$ and the nearest sphere or alternatively that $\Omega \in \mathscr{S}_{r,R}$ for r sufficiently large depending on R and ρ.

Proof. We first prove the result when $\partial\Omega$ is C^1. Then derive the result for nonsmooth Ω by density. Let $n(x) \in C(\mathbb{R}^N;\mathbb{R}^N)$ be a vector field normal to $\partial\Omega$. Let $\nu \in S^{N-1}$ and define

$$H = \{x : x \cdot \nu = 0\}, \quad H_+ = \{x : x \cdot \nu > 0\}, \quad H_- = \{x : x \cdot \nu < 0\}$$

the hyperplane normal to ν through the origin and corresponding half spaces. For $s > 0$ define the translates of H,

$$H(s) = H + s\nu, \quad H_+(s) = H_+ + s\nu, \quad H_-(s) = H_- + s\nu.$$

We want to show that for $s \geq \rho$

$$\phi_{H(s)}(\Omega) \cap H_-(s) \subseteq \Omega \cap H_-(s). \tag{2.A.4}$$

We make the following notations for simplicity,

$$\Omega_s := \Omega \cap H_-(s), \quad \text{and} \quad \widetilde{\Omega}_s := \phi_{H(s)}(\Omega) \cap H_-(s).$$

From $\Omega \subset B_R(0)$ we know that $H(R)$ does not intersect Ω and therefore (2.A.4) holds for $s \geq R$. Now move the plane H inward towards the origin until it touchs Ω for the first time at

$$s_0 := \inf\{s > 0 : H(s) \cap \Omega = \emptyset\}.$$

Then the containment (2.A.4) holds until $s_{\min}(\nu)$ which we call t for convenience. We want to show that $t \leq \rho$. Note that $t \leq s_0$ since for $s \geq s_0$ the intersection $\Omega \cap H_-(s)$ is empty and so (2.A.4) holds trivially. Moreover $t < s_0$ because $\partial\Omega$ is C^1. At t there are two possibilities. The first is that $\widetilde{\Omega}_t$ touches Ω_t from the inside at some point off of $H(t)$, that is there exists $x \in \partial\Omega \cap H_+(t)$ such that:

$$\phi_{H(t)}(x) \in \partial\Omega_t \cap \partial\widetilde{\Omega}_t \setminus H(t).$$

The second is that $H(t)$ intersects $\partial\Omega$ at a point where ν is tangent to $\partial\Omega$.

In the first case, call $y = \phi_{H(t)}(x)$ to be the point in $\partial\Omega_t \cap \partial\widetilde{\Omega}_t \setminus H(t)$. Note that $\phi_H(n(x))$ is the normal to $\partial\Omega$ at y. Initially we suppose that

$$\langle y, \nu \rangle \leq \frac{t}{4}.$$

Then a simple calculation, or some geometry, shows that:

$$t = \tfrac{1}{2}(\langle x, \nu \rangle + \langle y, \nu \rangle).$$

Now we derive a bound for t from above in this situation,

$$\begin{aligned}|x|^2 &= |x-y|^2+|y|^2+2\langle x-y,y\rangle = \langle x-y,\nu\rangle^2+|y|^2+2|x-y|\langle \nu,y\rangle \\ &= (2t-2\langle y,\nu\rangle)^2+|y|^2-4(t-\langle y,\nu\rangle)\langle \nu,y\rangle \\ &= 4t^2-12t\langle \nu,y\rangle+8\langle \nu,y\rangle^2+|y|^2 \\ &\geq t^2+|y|^2.\end{aligned}$$

Keeping this in mind we now work in the case when

$$\langle y,\nu\rangle > \frac{t}{4}.$$

Because x lies to the in $H_+(\nu)$ and we have assumed a bound on the angle between x and $n(x)$ we also get a bound on the angle between $n(x)$ and ν,

$$\langle n(x),\nu\rangle = \langle n(x)-\widehat{x},\nu\rangle+\frac{t}{|x|} \geq -2\left(1-\sqrt{1-\frac{\rho^2}{5|x|^2}}\right)+\frac{t}{|x|} \geq \frac{t}{|x|}-\frac{\rho^2}{5|x|^2}.$$

Meanwhile, $\phi_H(n(x))$ inherits the opposite bound,

$$\langle \phi_H(n(x)),\nu\rangle = -\langle n(x),\nu\rangle \leq -\frac{t}{|x|}+\frac{\rho^2}{5|x|^2}.$$

Due to our assumption on the angle between y and ν we also get a bound in the other direction,

$$\langle \phi_H(n(x)),\nu\rangle = \langle \phi_H(n(x))-\widehat{y},\nu\rangle+\langle \widehat{y},\nu\rangle > -\frac{\rho^2}{5|y|^2}+\frac{t}{4|y|}.$$

Combining the above two bounds we get that,

$$t \leq \frac{\rho^2|x|}{5|y|^2}+\frac{\rho^2}{5|x|}-\frac{t|x|}{4|y|},$$

then calling $\gamma = |x|/|y|$ and using $\rho \leq \min\{|x|,|y|\}$ we rearrange to get,

$$t \leq \frac{1}{5}\left(\rho\frac{\rho}{|y|}\frac{\gamma}{1+\frac{1}{4}\gamma}+\rho\frac{\rho}{|x|}\right) \leq \rho.$$

In the second case, let $x \in \partial\Omega \cap H(t)$ such that,

$$\langle n(x),\nu\rangle = 0, \quad \text{and therefore} \quad |\langle x,\nu\rangle|^2+|\langle x,n(x)\rangle|^2 \leq |x|^2,$$

Rearranging and noting that $x \in H(t)$ implies $|\langle x, \nu\rangle| = t$,

$$t \leq \left(|x|^2 - (|x|^2 - \rho^2))\right)^{1/2} = \rho.$$

Finally in the case of general $\Omega \in \mathscr{S}_0$ one can approximate $\partial\Omega$ by boundaries of C^1 domains which converge in Lipschitz norm (in the appropriately interpreted sense) and use the fact the ρ-reflection is preserved by convergence in Hausdorff distance.

□

Lemma 2.A.5. *Let Ω_j for $j \in \{1,2\}$ and U be in $\mathscr{S}_{r,R}$, and let $\alpha = \alpha(r,R) \in (0,1)$ from the Hölder estimates for harmonic functions in Lipschitz domains Lemma 2.2.9. Then the following estimates hold:*

$$d_H(\Omega_1, \Omega_2) \leq d_H(\partial\Omega_1, \partial\Omega_2) \tag{2.A.5}$$

$$|\lambda[\Omega_1] - \lambda[\Omega_2]| \lesssim_r d_H(\Omega_1, \Omega_2) \tag{2.A.6}$$

$$||u[\Omega_1] - u[\Omega_2]||_\infty \lesssim_{r,R} d_H(\Omega_1, \Omega_2)^\alpha \tag{2.A.7}$$

$$|\Omega_1 \Delta \Omega_2| \lesssim_{r,R} d_H(\Omega_1, \Omega_2) \tag{2.A.8}$$

$$|\mathscr{J}(\Omega_1) - \mathscr{J}(\Omega_2)| \lesssim_{r,R} d_H(\Omega_1, \Omega_2)^\alpha \tag{2.A.9}$$

$$|\widetilde{\text{dist}}^2(\Omega_1, U) - \widetilde{\text{dist}}^2(\Omega_2, U)| \quad \textit{and} \quad |\widetilde{\text{dist}}^2(U, \Omega_1) - \widetilde{\text{dist}}^2(U, \Omega_2)| \quad \lesssim_{r,R} d_H(\Omega_1, \Omega_2) \tag{2.A.10}$$

Proof. We will start by showing (2.A.8), the only necessary assumption on the Ω_j is that one of their boundaries be rectifiable with $\mathscr{H}^{N-1}$ Hausdorff measure bounded in terms of r and R,

$$|\Omega_1 \Delta \Omega_2| \leq d_H(\Omega_1, \Omega_2)\mathscr{H}^{N-1}(\partial\Omega_1) \lesssim_{r,R} d_H(\Omega_1, \Omega_2).$$

Then (2.A.10) follows easily from (2.A.8),

$$\begin{aligned} |\widetilde{\text{dist}}^2(U, \Omega_1) - \widetilde{\text{dist}}^2(U, \Omega_2)| &= \left| \int_{\Omega_1 \Delta U} d(x, \partial U)dx - \int_{\Omega_2 \Delta U} d(x, \partial U)dx \right| \\ &\lesssim_R |\Omega_1 \Delta \Omega_2| \end{aligned}$$

and because the $\widetilde{\text{dist}}^2$ is asymmetric we also check

$$\begin{aligned}|\widetilde{\text{dist}}^2(\Omega_1,U)-\widetilde{\text{dist}}^2(\Omega_2,U)| &= \left|\int_{\Omega_1\Delta U} d(x,\partial\Omega_1)dx-\int_{\Omega_2\Delta U} d(x,\partial\Omega_2)dx\right| \\ &\lesssim_R |\Omega_1\Delta\Omega_2|+\int_{(\Omega_1\cap\Omega_2)\Delta U}|d(x,\partial\Omega_1)-d(x,\partial\Omega_2)|dx \\ &\lesssim_R d_H(\Omega_1,\Omega_2)+\int_{(\Omega_1\cap\Omega_2)\Delta U} d_H(\Omega_1,\Omega_2)dx \\ &\lesssim_R d_H(\Omega_1,\Omega_2)\end{aligned}$$

where in the third line we have used that if $y_1\in\partial\Omega_1$ there exists $y_2\in\partial\Omega_2$ with $|y_1-y_2|\le d_H(\Omega_1,\Omega_2)$ and therefore $d(x,\partial\Omega_2)\le d(x,\partial\Omega_1)+d_H(\Omega_1,\Omega_2)$ and vice versa. Note that (2.A.9) is an easy consequence of (2.A.7) and (2.A.6),

$$|\mathscr{J}(\Omega_1)-\mathscr{J}(\Omega_2)|\le ||u[\Omega_1]-u[\Omega_2]||_\infty \max\{|\Omega_1|\,,\,|\Omega_2|\}+|\Omega_1\Delta\Omega_2|,$$

so we are left to prove the first three estimates.

First recall the behavior of λ under spatial dilations,

$$\lambda[a\Omega]=a^{-(N+2)}\lambda[\Omega]. \tag{2.A.11}$$

We will show that taking,

$$a:=\frac{r}{d_H(\partial\Omega_1,\partial\Omega_2)+r} \tag{2.A.12}$$

one gets the following containments,

$$a\Omega_1\subseteq\Omega_2\subseteq\frac{1}{a}\Omega_1. \tag{2.A.13}$$

This will imply by (2.A.11) that

$$a^{-(N+2)}\lambda[\Omega_1]\ge\lambda[\Omega_2]\ge a^{N+2}\lambda[\Omega_1]$$

and by rearranging,

$$|\lambda[\Omega_1]-\lambda[\Omega_2]|\le\max\{\lambda[\Omega_1],\lambda[\Omega_2]\}(1-a^{N+2}).$$

Now to get an estimate of the form (2.A.6), for $t\ge 0$ note that

$$1-(1+t)^{-(N+2)}\le(N+2)t$$

to get

$$|\lambda[\Omega_1]-\lambda[\Omega_2]| \leq r^{-1}(N+2)\lambda[B_r]d_H(\partial\Omega_1,\partial\Omega_2).$$

Let us prove (2.A.13) for a chosen as in (2.A.12). If $x\in\Omega_1\setminus\Omega_2$ let $t\in(0,1)$ such that $tx\in\partial\Omega_2$. Then from the star-shaped property there is an exterior cone to Ω_2 at tx,

$$tx+C(x,\theta_{tx})\subset\mathbb{R}^N\setminus\Omega_2 \quad \text{where} \quad \sin\theta_{tx}=\tfrac{r}{t|x|},\ \theta_{tx}\in(0,\pi/2)$$

so that

$$B_{\frac{(1-t)}{t}r}(x)\subset tx+C(x,\theta_{tx})\subset\mathbb{R}^N\setminus\Omega_2.$$

Then $d(x,\partial\Omega_2)\geq\left(\frac{1}{t}-1\right)r$ and moreover,

$$\left(\frac{1}{t}-1\right)r\leq d_H(\partial\Omega_1,\partial\Omega_2).$$

or by rearranging,

$$a\leq t.$$

Then by making the same argument for $x\in\Omega_2\setminus\Omega_1$ we get

$$a\Omega_1\subseteq\Omega_2\subseteq\frac{1}{a}\Omega_1.$$

Now we show (2.A.7). Without loss of generality suppose that $\lambda[\Omega_1]\geq\lambda[\Omega_2]$. Consider the difference of the droplet profiles on the two domains,

$$w(x)=u[\Omega_1](x)-u[\Omega_2](x).$$

First we estimate the size of $w(x)$ on $\Omega_1\Delta\Omega_2$. If $x\in\Omega_1\Delta\Omega_2$ let $h=d(x,\partial(\Omega_1\cup\Omega_2))$ then

$$h\leq d_H(\Omega_1,\Omega_2) \ \text{ and } \ B_h(x)\subset\Omega_1\cup\Omega_2 \ \text{ with } \ \partial B_h(x)\cap\partial(\Omega_1\cup\Omega_2)\neq\emptyset.$$

By the standard construction of barriers for domains with the exterior cone property there exist $0<\alpha<1$ and $C>0$ depending on the uniform lower bound on the opening angle of the exterior cones such that

$$|w(x)|\leq Ch^{\alpha}+\frac{\lambda[\Omega_1]}{2N}h^2.$$

So on $\Omega_1 \Delta \Omega_2$ and in particular on $\partial(\Omega_1 \cap \Omega_2)$ we have that

$$|w(x)| \lesssim_{r,R} \max\{d_H(\Omega_1,\Omega_2)^{\alpha}, d_H(\Omega_1,\Omega_2)^2\} \lesssim_{r,R} d_H(\Omega_1,\Omega_2)^{\alpha},$$

where the last inequality is due to the fact that $d_H(\Omega_1,\Omega_2) \leq 2R$. Finally on intersection $x \in \Omega_1 \cap \Omega_2$,

$$|w(x)| \lesssim_{r,R} d_H(\Omega_1,\Omega_2)^{\alpha} + |\lambda[\Omega_1] - \lambda[\Omega_2]| \lesssim_{r,R} d_H(\Omega_1,\Omega_2)^{\alpha}.$$

Finally we show (2.A.5). Let $x_0 \in \Omega_1 \setminus \Omega_2$, then $tx \in \mathbb{R}^N \setminus \Omega_2$ for all $t > 1$. In particular let $s > 1$ such that $x_1 = sx_0 \in \partial\Omega_1$. We claim that:

$$h := d(x_0, \Omega_2) \leq d(x_1, \Omega_2). \tag{2.A.14}$$

Let $x \in B_h(x_1)$, then consider the point $s^{-1}x$,

$$|s^{-1}x - x_0| = s^{-1}|x - x_1| \leq |x - x_1| = h.$$

Therefore $s^{-1}x \in B_h(x_0) \subset \mathbb{R}^N \setminus \Omega_2$ and so from the star-shapedness of Ω_2 the point $x = ss^{-1}x$ is as well, this proves the claimed inequality (2.A.14). In particular:

$$\sup_{x\in\Omega_1} d(x,\Omega_2) = \sup_{x\in\Omega_1\setminus\Omega_2} d(x,\partial\Omega_2) \leq \sup_{x\in\partial\Omega_1} d(x,\partial\Omega_2),$$

and by noting that Ω_1 and Ω_2 play symmetric roles:

$$d_H(\Omega_1,\Omega_2) \leq d_H(\partial\Omega_1,\partial\Omega_2).$$

$\square$

Define the metric space of boundaries of strongly star-shaped sets,

$$\partial\mathscr{S}_{r,R} := \{\partial\Omega : \Omega \in \mathscr{S}_{r,R}\}$$

with metric d_H. This space embeds continuously into

$$\{f \in C^{0,1}(S^{N-1}) : r \leq f \leq R \text{ and } ||Df||_{L^\infty} \leq C(r,R)\}$$

with the L^∞ distance. As a direct consequence of this we get the following compactness for $\partial\mathscr{S}_{r,R}$:

Lemma 2.A.6. (Compactness) *The metric space $(\partial\mathscr{S}_{r,R}, d_H)$ is compact:*

(i) Suppose that $\Gamma_j \in (\partial\mathscr{S}_{r,R}, d_H)$ for some $r, R > 0$ and all $j \in \mathbb{N}$. Then $\{\Gamma_j\}_{j\in\mathbb{N}}$ has a subsequence that converges and any subsequential limit is also in $\partial\mathscr{S}_{r,R}$.

(ii) Let I be a compact interval in $\mathbb{R}$ and $\Gamma_j : I \to (\partial\mathscr{S}_{r,R}, d_h)$ for $j \in \mathbb{N}$ are an equicontinuous sequence of paths in $(\partial\mathscr{S}_{r,R}, d_h)$. Then there is a subsequence of the $\Gamma_j(\cdot)$ that converges uniformly on I to a path $\Gamma : I \to (\partial\mathscr{S}_{r,R}, d_h)$.

Proof. Arzela-Ascoli. The only interesting point is that $\partial\mathscr{S}_{r,R}$ is closed in $(\mathscr{K}, d_H)$ where $\mathscr{K}$ is the class of compact subsets of $\mathbb{R}^N$. It suffices to establish the estimate:

$$d_H(\partial\Omega_1, \partial\Omega_2) \lesssim_{r,R} d_H(\Omega_1, \Omega_2) \tag{2.A.15}$$

where $\Omega_j \in \mathscr{S}_{r,R}$. Let $x \in \partial\Omega_1$ and $y \in \partial\Omega_2$ such that:

$$|x-y| = d(x, \partial\Omega_2) = d_H(\partial\Omega_1, \partial\Omega_2).$$

If $x \in \mathbb{R}^N \setminus \Omega_2$ then,

$$d_H(\partial\Omega_1, \partial\Omega_2) = |x-y| = d(x, \Omega_2) \le d_H(\Omega_1, \Omega_2).$$

Otherwise $x \in \Omega_2$, then let $t > 0$ such that $(1+t)x \in \partial\Omega_2$. From the strong star-shapedness of Ω_1 we know that

$$d((1+t)x, \Omega_1) \ge t|x|\frac{r}{|x|} \ge |(1+t)x - x|\frac{r}{R} \ge |x-y|\frac{r}{R}.$$

Rearranging we get that

$$d_H(\partial\Omega_1, \partial\Omega_2) = |x-y| \le \frac{R}{r} d((1+t)x, \Omega_1) \le \frac{R}{r} d_H(\Omega_1, \Omega_2).$$

$\square$

2.B Higher Regularity for the Equilibrium Problem

In this appendix we describe the method in [De] and [KNS78] to show that if u is a solution of the equilibrium problem

$$\begin{cases} -\Delta u(x) = \lambda & \text{in} \quad \Omega = \Omega(u), \\ |Du| = 1 & \text{on} \quad \Gamma = \Gamma(u). \end{cases} \tag{EQ}$$

with $\Omega \in \mathscr{S}_0$ then Γ is $C^{2,\alpha}$. The first step is Theorem 2.1.2 from [De] which says that Γ is $C^{1,\alpha}$. As a consequence of this $u \in C^{1,\alpha}(\overline{\Omega})$. Then one uses a transformation of the problem first used for this purpose by [KNS78] that has the effect of flattening the free boundary while making the PDE and the boundary condition nonlinear. Below we describe the transformation in detail.

We may assume that $0 \in \Gamma$ and $1 = |Du|(0) = \partial_n u(0)$. Then near 0 we make the change of variables:

$$y_j = x_j \quad \text{for} \quad 1 \leq j \leq n-1, \quad y_n = u, \quad v = x_n,$$

Then we can write the y derivatives of v in terms of the x derivatives of u so that $v(y)$ solves the problem:

$$\begin{cases} \sum_{j=1}^{n} \partial_j (F_j(Dv)) = \lambda \quad \text{for} \;\; y_n > 0 \\ (\partial_n v)^2 - \sum_{j=1}^{n-1} (\partial_j v)^2 = 1 \quad \text{on} \quad y_n = 0 \end{cases} \tag{2.B.1}$$

where

$$\begin{cases} F_j(p) = \dfrac{p_j}{p_n} \quad \text{for} \quad 1 \leq j \leq n-1, \\ F_n(p) = -\frac{1}{2} \dfrac{1 + \sum_{j=1}^{n-1} (p_j)^2}{(p_n)^2}. \end{cases} \tag{2.B.2}$$

It is straightforward to check that this operator is uniformly elliptic when $\partial_n v$ is bounded from below. This fact as well as the nondegeneracy of $\partial_n v$ from the boundary condition enables the argument in [KNS78] to go through for our problem and yield the desired regularity result in Corollary 2.1.1.

Let us show that $(C.1)$ is the correct equation. Based on the definitions of y_j and v, we have

$$u(y_1, ..., y_{n-1}, v(y_1, ..., y_n)) = y_n. \tag{2.B.3}$$

Taking the n-th partials in (2.B.3) we get

$$\partial_n u = \frac{1}{\partial_n v}, \quad -\partial_{nn} u = \frac{\partial_{nn} v}{(\partial_n v)^3} \quad \text{and} \quad \partial_{nj} u = \frac{\partial_{nn} v \, \partial_j v}{(\partial_n v)^3} - \frac{\partial_{nj} v}{(\partial_n v)^2}$$

where any derivative of u is assumed to be evaluated at $(y_1, ..., y_{n-1}, v(y_1, ..., y_n))$ and any derivative of v at $(y_1, ..., y_n)$. Similarly taking derivatives in (2.B.3) for $1 \leq j \leq n-1$, we have $\partial_j u + \partial_n u \, \partial_j v =$

0 and thus

$$
\begin{aligned}
-\partial_{jj}u &= 2\partial_{nj}u\,\partial_j v + \partial_{nn}u(\partial_j v)^2 + \partial_n u\,\partial_{jj}v \\
&= \frac{\partial_{nn}v\,(\partial_j v)^2}{(\partial_n v)^3} - \frac{\partial_{nj}v\,\partial_j v}{(\partial_n v)^2} + \frac{\partial_{jj}v}{\partial_n v}.
\end{aligned}
$$

Then we substitute into (EQ) to get

$$
\begin{aligned}
\lambda &= -\sum_{j=1}^{n-1}\partial_{jj}u - \partial_{nn}u = \sum_{j=1}^{n-1}\left[\frac{\partial_{nn}v\,(\partial_j v)^2}{(\partial_n v)^3} - 2\frac{\partial_{nj}v\,\partial_j v}{(\partial_n v)^2} + \frac{\partial_{jj}v}{\partial_n v}\right] + \frac{\partial_{nn}v}{(\partial_n v)^3} \\
&= \sum_{j=1}^{n-1}\frac{\partial_{jj}v}{\partial_n v} + \left(\frac{1+\sum_{j=1}^{n-1}(\partial_j v)^2}{(\partial_n v)^3}\right)\partial_{nn}v - 2\sum_{j=1}^{n-1}\frac{\partial_{nj}v\,\partial_j v}{(\partial_n v)^2} \\
&= \sum_{j=1}^{n-1}\partial_j\left(\frac{\partial_j v}{\partial_n v}\right) - \partial_n\left(\frac{1}{2}\frac{1+\sum_{j=1}^{n-1}(\partial_j v)^2}{(\partial_n v)^2}\right).
\end{aligned}
$$

The calculation of the boundary condition is easier

$$
1 = |Du|^2 = \sum_{j=1}^{n-1}\left(\frac{\partial_j v}{\partial_n v}\right)^2 + \frac{1}{(\partial_n v)^2}.
$$

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Part II

Boundary Homogenization

CHAPTER 3

Homogenization of Oscillating Dirichlet Data: Periodic Case

3.1 Introduction

We consider the homogenization of the following Dirichlet problem set in a bounded domain in $\Omega \subset \mathbb{R}^n$ with smooth boundary,

$$\begin{cases} F_\varepsilon(D^2 u^\varepsilon, x, \frac{x}{\varepsilon}) = 0 & \text{for} \quad x \in \Omega \\ \\ u^\varepsilon(x) = g(x, \frac{x}{\varepsilon}) & \text{for} \quad x \in \partial\Omega, \end{cases} \tag{3.1.1}$$

with Hölder continuous boundary data $g \in C^\alpha(\mathbb{R}^n \times \mathbb{R}^n)$ for some $\alpha \in (0,1)$. The operator $(M,x,y) \to F_\varepsilon(M,x,y)$ is assumed to be uniformly elliptic, and both the operator and the boundary condition $g(x,y)$ are assumed to be $\mathbb{Z}^n$-periodic in the y variable. See Section 3.2.3 for the precise assumptions we make on the operators.

The homogenization of the problem (3.1.1) when $g(x,y) = g(x)$ is independent of the fast variable is somewhat classical at this point and was done by Evans [Eva92, Eva89], the ideas originated in the unpublished paper of Lions, Papanicolaou and Varadhan [LPV88] on the periodic homogenization of Hamilton-Jacobi equations. There are too many works investigating the homogenization of the interior operators in various settings for us to list all of them here, we simply mention that more recently the interior homogenization has been shown in the very general setting of stationary ergodic random media by Caffarelli, Souganidis and Wang in [CSW05].

As far as oscillating boundary data is concerned there is less literature. For divergence form operators the case of oscillating co-normal Neumann data is in the classical book of Bensoussan, Papanicolaou and Lions [BLP78]. For the case of non co-normal Neumann data and fully nonlinear operators there are a number of recent works. The papers of Arisawa [Ari03] and Barles, Da Lio, Lions and Souganidis [BDL08] identify the cell problem and show the homogenization in half-space type domains. The general domain case for the Neumann problem was solved by Choi and Kim in [CK14] (see also [CKL13]). They require that the operator F_ε homogenizes to a rotationally invariant operator, in this situation the homogenized boundary data is continuous which guarantees the uniqueness of the limiting problem and thus stability of the ε-problem as $\varepsilon \to 0$. This issue is explained in more detail in [CK14]. Our work is based on the ideas in [CK14,CKL13]. Indeed, one of the novelties of the article this chapter is based on was to show the uniqueness and stability result for a general class of operators even though the homogenized boundary data may be discontinuous, refer to the following chapter of this dissertation for further results related to this issue. In particular we do not need to put any special assumptions on the operators besides the uniform ellipticity. We believe that similar ideas should apply to the case of oscillatory Neumann data.

For the oscillating Dirichlet data case there are even less references. For linear divergence form equations the homogenization in general domains was shown recently by Gerard-Varet and Masmoudi in [GM11,GM12] where they are able to consider linear systems, see also [LS12]. The work of Barles and Mironescu [BM13] deals with fully nonlinear non-divergence form elliptic operators when Ω is a half-space. Barles and Mironescu identify the cell problem and the attendant difficulties in solving it under quite general assumptions on the operators. We will be concerned with extending this result to general domains. In order to clarify the main issues that arise from the general domain rather than from the cell problem we will avoid giving the most general assumptions on the operators. We expect that our results should also hold in the generality at which [BM13] show the cell problem can be solved, but this will be addressed in future work.

At least on the surface a difficulty of this problem, especially in contrast to the corresponding oscillating Neumann data problem, is a lack of equicontinuity for the u^ε. The optimal estimate for the continuity of the u^ε up to the boundary cannot be any better than the continuity of $g(x, \frac{x}{\varepsilon})$ which has oscillations of size 1 at arbitrarily small scales in the limit $\varepsilon \to 0$. In particular the homogenization of (3.1.1), if it occurs, must be happening outside of a 'boundary layer' of width $o(1)$. Keeping this in mind, let us call $\overline{F}$ to be the homogenized operator on the interior given by the results of [Eva92, Eva89, CSW05]. Our goal is to show that the solutions u^ε of the ε-problem (3.1.1) converge locally uniformly in Ω to $\overline{u}$ solving an equation of the form,

$$\begin{cases} \overline{F}(D^2\overline{u},x) = 0 & \text{for} \quad x \in \Omega \\ \\ \overline{u}(x) = \overline{g}(x) & \text{for} \quad x \in \partial\Omega. \end{cases} \tag{3.1.2}$$

As we will see, the homogenized boundary condition $\overline{g}$ will depend not only on the function g, but also on σ_x, the inner normal to Ω at x and on the F_ε.

Let us describe heuristically the approach taken in [BM13] to identifying the homogenized boundary condition. The analysis of the problem (3.1.1) proceeds by blowing up about points $x \in \partial\Omega$. As in [BM13], the introduction of the localizations,

$$v^\varepsilon(y) = u^\varepsilon(x_0 + \varepsilon y),$$

at points $x_0 \in \partial\Omega$ in the limit as ε is taken to 0 leads us formally to the following "cell problem" set in the half space $P(x_0,\sigma) := \{y : (y-x_0)\cdot\sigma > 0\}$ where $\sigma \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$,

$$\begin{cases} F(D^2v,y) = 0 & \text{in} \quad P(x_0,\sigma) \\ v(y) = \psi(y) & \text{on} \quad \partial P(x_0,\sigma). \end{cases} \tag{3.1.3}$$

Some appropriate assumptions on the F_ε and g will yield that F is uniformly elliptic and that both F and ψ are $\mathbb{Z}^n$-periodic in y. Then by a Liouville type result the limit $v(y+R\sigma)$ as $R \to \infty$ will be a constant $\mu(\sigma,\psi,F)$, we will identify this as the homogenized boundary condition. Then, at least formally, one expects that $|u^\varepsilon(x+\varepsilon R\sigma) - v(R\sigma)| \to 0$ as $\varepsilon \to 0$. In particular, if we pull away from the boundary sufficiently far near x_0, we can recover the interior averaging. This suggests that our homogenized boundary condition in (3.1.2) should be $\overline{g}(x) = \mu(\sigma_x, g(x,\cdot), F^x)$ where σ_x is the inner unit normal to Ω at x and F^x is an appropriate blow up limit of F_ε.

The main difficulty that one has to deal with in extending to general domains is the *rational boundary points* $x \in \partial\Omega$ such that the interior normal σ_x to Ω at x is in $\mathbb{R}\mathbb{Z}^n$. Exactly when $\sigma \in \mathbb{R}\mathbb{Z}^n$ the cell problem (3.1.3) does not yield a unique constant μ. This corresponds to an a-priori lack of uniform control in σ of the rate of convergence of $v(y+R\sigma) \to \mu(\sigma)$. Some stability of this rate of convergence in σ is necessary in order to show that the behavior of the cell problem governs the homogenization near irrational boundary points.

It turns out that $v(x+R\sigma) \to \mu(\sigma)$ uniformly in $\sigma \in S^{n-1}$ is equivalent to $\mu(\cdot,\psi,F)$ having a unique continuous extension from the irrational directions $S^{n-1} \setminus \mathbb{R}\mathbb{Z}^n$ to all of S^{n-1}. When $\overline{F}$ is a rotationally invariant operator one can show using the methods of [CKL13, CK14] that such a continuous extension exists. For general F the homogenized boundary condition μ is expected to have no such continuous extension [KS13], so we are left to consider weaker continuity conditions which still suffice to prove the homogenization.

One of the main contributions of this work is showing that the weak stability which one has 'intrinsically' is sufficient for the homogenization to hold. We find this stability by carefully analyzing the rates of convergence for the cell problem in terms of the discrepancy function, see Sections 3.2.2 and 3.3.

Let us call Γ the set of rational boundary points,

$$\Gamma(\Omega) := \{x \in \partial\Omega : \sigma_x \in \mathbb{R}\mathbb{Z}^n\}.$$

In light of the above discussion, the best continuity result we expect for general F_ε is,

$$\overline{g} : \partial\Omega \to \mathbb{R} \text{ is continuous at } x \in \partial\Omega \setminus \Gamma, \tag{3.1.4}$$

and it turns out that $\partial\Omega \setminus \Gamma$ is also exactly where one is able to exploit the connection between the cell problem and the ε-problem to show that any subsequential locally uniform limit of the u^ε satisfies the homogenized boundary data. In order to show that all subsequential limits are the same the fundamental question becomes, when is there uniqueness for the problem:

$$\begin{cases} \overline{F}(D^2\overline{u},x) = 0 & \text{for} \quad x \in \Omega \\ \\ \overline{u}(x) = \overline{g}(x) & \text{for} \quad x \in \partial\Omega \setminus \Gamma(\Omega). \end{cases} \tag{3.1.5}$$

We are not aware of any existing results in this direction, it is worth noting that this problem is related to the open question of whether there exists a boundary version of the Alexandroff-Bakelman-Pucci estimate. We provide a resolution of this question when $\Gamma(\Omega)$ has sufficiently small Hausdorff dimension in the following Theorem, see Section 3.4 for a precise explication of all the assumptions,

Theorem 3.1.1. [Theorem 3.4.3] *Let $\overline{F}(M,y)$ be uniformly elliptic with constants (λ,Λ) and satisfy appropriate continuity assumptions in (M,y). Suppose that Ω has the strict γ exterior cone condition for some $0<\gamma\leq 1$. There exists $\beta_0(n,\lambda,\Lambda,\gamma)>\max\{0,\frac{\lambda}{\Lambda}(n-1)-1\}$ such that if*

$$\beta_0 > \dim_{\mathscr{H}} \Gamma(\Omega) \geq 0,$$

then (3.1.5) *has a unique bounded solution $\overline{u}$.*

Remark 3.1.2. The constraint on the Hausdorff dimension of $\Gamma(\Omega)$ allows us to construct a sequence of super solution barriers ϕ_j which are large on $\Gamma(\Omega)$, but converge to zero on compact subsets of Ω. Then for any bounded solution v of (3.1.5) one can use the classical comparison principle to show $v\leq \overline{u}+\phi_j$, and letting $j\to\infty$ will yield the result.

Now we will give a statement of our main result. The results are not stated in their full generality in order that it be understandable at this point in the exposition. For the exact statements one can refer to the main text, in particular the main result is in Theorem 3.5.3.

Theorem 3.1.3. *(Solution of the Cell Problem and Homogenization)*

(i) [Lemmas 3.3.1, 3.3.3 and 3.3.4] *Let $v_{x_0,\sigma}$ solve* (3.1.3) *in $P(\sigma,x_0)$, there exists a constant $\mu(\sigma,\psi,F)$ such that for all $\sigma\in S^{n-1}\setminus\mathbb{R}\mathbb{Z}^n$,*

$$\sup_{x_0\in\mathbb{R}^n}\sup_{y\in P(x_0,\sigma)}|v_{x_0,\sigma}(y+R\sigma)-\mu(\sigma,\psi,F)|\to 0 \ \text{ as } \ R\to\infty.$$

Moreover μ is continuous in all of its arguments at irrational directions $\sigma\in S^{n-1}\setminus\mathbb{R}\mathbb{Z}^n$ (for the precise meaning of this refer to the referenced Lemmas).

(ii) [Theorem 3.5.3] *Let u^ε be the solutions of the ε-problem* (3.1.1) *and Ω a bounded domain with C^2 boundary satisfying the condition of Theorem 3.1.1. Then the u^ε converge locally uniformly in Ω to the unique solution $\overline{u}$ of* (3.1.5) *with,*

$$\overline{g}(x)=\mu(\sigma_x,g(x,\cdot),F^x).$$

Remark 3.1.4. In particular the homogenization result holds whenever the set of rational boundary points Γ is countable for general uniformly elliptic operators F_ε satisfying the assumptions of Section 3.2.3. For example, by the inverse function theorem, this includes the classical assumptions of [BLP78] for the homogenization of co-normal Neumann data, that $\partial\Omega$ has no flat parts in the sense that $D_\tau \sigma_x$ (the gradient in the tangential directions at x) has rank $n-1$.

Remark 3.1.5. Here F^x is a rescaling of the F_ε at $x \in \partial\Omega$. Under the assumptions we will put on the F_ε, the operators F_ε themselves will not actually depend on ε and one can simply take $F^x(M,y) = F_\varepsilon(M,x,y)$. For a precise characterization of the F^x see Lemma 3.5.1.

Remark 3.1.6. It turns out that $\mu(\sigma,\psi,F)$ is just the average of ψ over a unit cell when F is linear and spatially homogeneous and σ is an irrational direction, see Section 3.3.1 for the proof. Since there is no dependence on the operator in the linear case one may wonder whether μ is always just the average even when F is nonlinear. We show that in fact this is not the case in Section 3.3.1, generically when F is nonlinear $\mu(\sigma,\psi,F)$ depends on F and is not linear in the ψ argument.

3.1.1 Outline of the Chapter

In Section 3.2 we discuss various notations and previous results which will be used throughout this chapter. For readers well versed in the field of homogenization of uniformly elliptic equations most of the material in Section 3.2 except for Section 3.2.2 should be familiar. In particular we will discuss in detail the properties of $\mathbb{Z}^n$ periodic functions restricted to hyperplanes with rational and irrational normal directions. These results are essentially refinements of the classical equidistribution theorem of Weyl [Wey10] on irrational rotations of the torus. Then we describe precisely the assumptions on the differential operators we will consider and the associated comparison, regularity and interior homogenization results.

In Section 3.3 we analyze the cell problem (3.1.3). We prove Theorem 3.1.3 part (i) and show properties of the homogenized boundary condition μ in particular the continuity properties discussed above. We show that the convergence of the solutions of the cell problem $v_{x_0,\sigma}(y+R\sigma)$ to μ is sufficiently uniform at irrational directions to show the claimed continuity. We also give an example which demonstrates the nonlinearity of the homogenized boundary condition when F is nonlinear.

In Section 3.5 we use the results of the previous section to show the homogenization. In particular we use the stability of the uniform convergence $v_{x_0,\sigma}(y+R\sigma) \to \mu$ at irrational directions to show that the upper and lower half-relaxed limits of the u^ε are respectively sub and supersolutions of (3.1.5). Then we use the result of Theorem 3.1.1 to conclude that the upper and lower half-relaxed limits are the same and so the homogenization holds.

3.2 Set Up

3.2.1 Notations

Let us make a note of the notational conventions we will follow in this work. The numbers $0 < \lambda < \Lambda$ will always refer to the ellipticity constants. The meaning of the term ellipticity constants will be explained in Section 3.2.3. The number $\alpha \in (0,1)$ will almost always refer to the Hölder continuity of whatever Dirichlet boundary data is under consideration. Let $\Omega \subset \mathbb{R}^n$ the associated Hölder spaces are,

$$C^\alpha(\Omega) := \left\{ \psi : \Omega \to \mathbb{R} : \|\psi\|_{C^\alpha(\Omega)} := \sup_{x\in\Omega} |\psi(x)| + |\psi|_{C^\alpha(\Omega)} < +\infty \right\}$$

where $|\psi|_{C^\alpha}$ is the Hölder semi-norm defined by,

$$|\psi|_{C^\alpha} := \sup_{x\neq y\in\Omega} \frac{|\psi(x)-\psi(y)|}{|x-y|^\alpha}.$$

We also define the oscillation of a real valued function ψ on $\mathbb{R}^n$ over a set $E \subset \mathbb{R}^n$,

$$\operatorname*{osc}_E \psi := \sup_E \psi - \inf_E \psi.$$

Constants denoted by C and c will denote universal constants, they depend only on λ, Λ and the dimension n, they may change value from line to line and sometimes even within the same line. Let A and B be two quantities, we write

$$A \lesssim B \ \text{ to mean } \ A \leq CB \ \text{ where } C \text{ is a universal constant.}$$

If we want to emphasize the dependency of a constant on a quantity A we will write $C(A)$.

3.2.2 Rational and Irrational Directions

We define the following classes of normal directions with respect to the $\mathbb{Z}^n$ lattice,

Definition 3.2.1. (i) $\sigma \in S^{n-1}$ is called a rational direction if $\sigma \in \mathbb{R}\mathbb{Z}^n$.

(ii) $\sigma \in S^{n-1}$ is called an irrational direction if it is not a rational direction.

Given a $\mathbb{Z}^n$ periodic function ψ we we will be interested in the properties of $\psi|_H$, the restriction of ψ to the hyperplane $H = \{x \in \mathbb{R}^n : (x - x_0) \cdot \sigma = 0\}$. When σ is an irrational direction the distribution of the values taken by $\psi|_H$ does not change too much under translation of H (changing x_0). This 'fact' will be very important to understanding the homogenization of the boundary data in half spaces.

We will first give a heuristic description of the issue at hand. Let us think of ψ as a function on the unit periodicity cell $[0,1)^n$. Then we may consider the parts of the unit periodicity cell which are cut by the hyperplane H,

$$H_{per} = \{\tau \in [0,1)^n : \tau + z \in H \text{ for some } z \in \mathbb{Z}^n\} = \bigcup_{z \in \mathbb{Z}^n} (H + z) \cap [0,1)^n.$$

For hyperplanes H whose normal direction is rational the union defining H_{per} will be equivalent to a finite union. As a result of this the restriction $\psi|_H([0,1)^n)$ may be only a non-dense subset of all the values taken by ψ. Moreover, this subset will be highly variable under changing x_0. For example this issue is immediately evident in $\mathbb{R}^2$ when we consider the periodic function,

$$\psi(x) = \psi(x_1) = \sin 2\pi x_1,$$

and the hyperplanes $H = \{x_1 = a\}$.

When the direction is irrational H_{per} will be 'uniformly dense' (in a sense to be made precise) in $[0,1)^n$. This is exactly equivalent to the well known irrational rotation of the torus example in the case $n = 2$. In this case changing x_0 will not change the overall distribution of the values taken by $\psi|_H$. To be slightly more precise, this is the situation in which homogenization of the oscillating boundary condition $\psi(x/\varepsilon)$ will hold.

Now we proceed to give a precise description of the above heuristics in a way that will be useful to us in the future. Parts of the following discussion are borrowed from [CKL13]. The reason we repeat it here is we will need to make use of a more refined version of the results outlined there.

Definition 3.2.2. A bounded sequence of real numbers $(x_j)_{j=1}^{\infty}$ is said to be *equidistributed* in an interval $[a,b]$ if for any $[c,d] \subseteq [a,b]$ we have,

$$\lim_{n\to\infty} \frac{|\{x_1,\ldots,x_n\}\cap[c,d]|}{n} = \frac{d-c}{b-a}.$$

For $x \in \mathbb{R}$ let $[x]$ be the largest integer less than or equal to x.

Definition 3.2.3. A sequence is *equidistributed mod* 1 if the sequence $(x_j - [x_j])_{j\in\mathbb{N}}$ is equidistributed in $[0,1]$.

Theorem 3.2.4. (Weyl's equidistribution theorem [Wey10]) *If x is irrational then $(jx)_{j\in\mathbb{N}}$ is equidistributed mod* 1.

In order to make quantitative estimates we introduce the notion of the discrepancy. The following definition is from the book [KN74] via [CKL13]. For a subset $E \subset [0,1]$, a natural number N and a sequence $(x_j)_{j\in\mathbb{N}} \subset [0,1]$ call $A(E;N)$ the number of elements of the sequence x_j for $1 \le j \le N$ contained in E.

Definition 3.2.5. For any $x \in [0,1]$ let $x_j = jx - [jx]$ and define the discrepancy function,

$$D_N(x) = \sup_{E=[a,b)\subseteq[0,1]} \left| \frac{A(E;N)}{N} - |E| \right|.$$

Then by Weyl's equidistribution theorem $D_N(x) \to 0$ as $N \to \infty$ for each x irrational. Now let us replace $D_N(x)$ by a function equivalent up to constants which is continuous at every irrational. We define the modified discrepancy function as in [KN74],

$$D_N^* = D_N^*(x_1,\ldots,x_N) = \sup_{0<a\le 1} \left| \frac{A([0,a);N)}{N} - a \right|.$$

Then we have the following properties for $D_N^*(x)$,

Lemma 3.2.6. *(Properties of D_N^*)*

(i) (Theorem 1.3 in [KN74]) The discrepancies D_N and D_N^* are equivalent up to constants,

$$D_N^* \le D_N \le 2D_N^*.$$

(ii) (Theorem 1.4 in [KN74]) Let $x_1 \le x_2 \le \cdots \le x_N$ be in $[0,1)$. Then their discrepancy D_N^* is given by,

$$D_N^* = \frac{1}{2N} + \max_{i=1,\ldots,N} \left| x_i - \frac{2i-1}{2N} \right|.$$

(iii) Define the modified discrepancy function for all $x \in \mathbb{R}$

$$D_N^*(x) := D_N^*(x-[x], 2x-[2x], \ldots, Nx-[Nx]),$$

then this function is continuous in a neighborhood of every irrational $x \in \mathbb{R} \setminus \mathbb{Q}$.

Proof. The proofs of the first two parts can be found in the book [KN74]. We prove part (iii). Given an irrational x and an integer $N > 0$ let us call

$$\delta = \min_{1 \le j < k \le N} |jx - kx| \wedge \min_{1 \le j \le N} [(jx - [jx]) \wedge ([jx] + 1 - jx)] > 0.$$

Let $s : \{1, \ldots, N\} \to \{1, \ldots, N\}$ be the permutation which orders the sequence $x - [x], \ldots, Nx - [Nx]$, that is,

$$s(1)x - [s(1)x] < s(2)x - [s(2)x] < \cdots < s(N)x - [s(N)x].$$

Then for $|y - x| < \frac{\delta}{2N}$ we have that the above ordering is preserved for y, in fact for each $1 \le j < N$,

$$\begin{aligned} s(j)y &\le s(j)x + s(j)|y-x| < s(j)x + \delta/2 \\ &\le s(j+1)x - \delta/2 < s(j+1)x - s(j+1)|y-x| \\ &\le s(j+1)y. \end{aligned}$$

Furthermore from the first line of the above inequality and the definition of δ, $[s(j)y] = [s(j)x]$. Then from part (ii) we have that for $|y - x| < \frac{\delta}{2N}$ the discrepancy is has the form,

$$D_N^*(y) = \frac{1}{2N} + \max_{i=1,\ldots,N} \left| s(i)y - [s(i)y] - \frac{2i-1}{2N} \right|,$$

which, as a maximum of finitely many continuous functions which are continuous in a neighborhood of x, is also continuous in a neighborhood of x. $\square$

Now let $\sigma = (\sigma_1, ..., \sigma_n) \in S^{n-1}$ be a direction, and let $1 \leq i \leq n$ be the largest index corresponding to the largest component of σ, that is,

$$i = \max\{1 \leq j \leq n : |\sigma_j| = |\sigma|_{\ell^\infty}\}.$$

Let $H_\sigma = \{x \in \mathbb{R}^n : x \cdot \sigma = 0\}$ be the hyperplane through the origin orthogonal to σ. For $1 \leq j \leq n$ define $m_j(\sigma)$ to be the slope of H_σ in the plane $span\{e_i, e_j\}$,

$$0 = -m_j(\sigma)|\sigma_i| + |\sigma_j| = -m_j(\sigma)|\sigma|_{\ell^\infty} + |\sigma_j|. \tag{3.2.1}$$

The first equality indicates that one of the vectors $m_j(\sigma)e_i \pm e_j$ is in the hyperplane H_σ. The second equality makes manifest that $m_j(\sigma)$ thus defined are continuous in σ and $m_j(\sigma) \in [0,1]$. Now for every $\varepsilon > 0$ and some $\gamma \in (0,1)$ we define,

$$\omega_\sigma(N) := 2 \min_{1 \leq j \leq n} D^*_{[N]}(m_j(\sigma)) \text{ for } N > 1. \tag{3.2.2}$$

For irrational directions $\sigma \in S^{n-1} \setminus \mathbb{Z}^n$ at least one of the $m_j(\sigma)$ will be irrational. As a result, $\omega_\sigma(N) \to 0$ as $N \to \infty$ when σ is an irrational direction. Furthermore, for N sufficiently large, the min in (3.2.2) above will actually be the same as the min over j such that $m_j(\sigma) \in \mathbb{R} \setminus \mathbb{Q}$. Then for each such $N > 1$ fixed $\omega_\cdot(N)$ will be continuous in a S^{n-1} neighborhood of σ with size depending on N (it is a minimum of finitely many compositions of continuous functions).

Now we state a lemma which quantifies our heuristic argument that hyperplanes with irrational normals are uniformly dense modulo $\mathbb{Z}^n$ in $[0,1)^n$. The lemma is from [CKL13], but we state it in a modified form which will be useful to us. Due to the modifications we will also present the proof.

Lemma 3.2.7. (Lemma 2.7 in [CKL13]) *For $\sigma \in S^{n-1}$ and $x_0 \in \mathbb{R}^n$ we define $H(x_0) = \{x \in \mathbb{R}^n : (x - x_0) \cdot \sigma = 0\}$. Let $\omega_\sigma : \mathbb{Z}_{>1} \to \mathbb{R}_+$ be as defined in* (3.2.2)*, then the following hold:*

(i) There exists a dimensional constant $C = C(n) > 0$ such that the following is true: for any $x \in H(x_0)$ and any $N > 1$ there exists $y \in \mathbb{R}^n$ such that

$$|x - y| \leq C(n)N \text{ with } y - x_0 \in \mathbb{Z}^n,$$

and

$$\operatorname{dist}(y, H(x_0)) < \omega_\sigma(N).$$

(ii) For any $\delta > 0$, there exists $z \in \mathbb{Z}^n$ such that,

$$\operatorname{dist}(z, H(x_0)) \leq \inf_{N>0} \omega_\sigma(N) + \delta.$$

Proof. Let σ be any direction in S^{n-1} and $N > 1$. Without loss we may translate so that $x = 0$ and assume that $\sigma_n = |\sigma|_{\ell^\infty} \geq 1/\sqrt{n}$. Let j such that $\omega_\sigma(N) = D^*_{[N]}(m_j(\sigma))$. Recall that one of the vectors $m_j(\sigma)e_n \pm e_j$ is in the hyperplane $H = H(0)$, let us assume it is $m_j(\sigma)e_n + e_j$, the other case will work out the same. Now from the Lemma 3.2.6 we know that the the discrepancy $D_{[N]}(m_j(\sigma)) \leq \omega_\sigma(N)$, and therefore every interval $[a,b) \subset [0,1)$ with length at least $\omega_\sigma(N)$ contains at least one of the $km_j(\sigma) - [km_j(\sigma)]$ with $k \leq N$.

Let $s = s_n e_n$, with $s_n \in [0,1)$, then by the above argument there exists $1 \leq k \leq N$ such that $|km_j(\sigma) - [km_j(\sigma)] - s_n| \leq \omega_\sigma(N)$, and therefore,

$$|k(m_j(\sigma)e_n + e_j) - s| \bmod \mathbb{Z}^n \leq \omega_\sigma(N).$$

In particular there exists $y \in H_\sigma$ such that $|y - s| \bmod \mathbb{Z}^n \leq \omega_\sigma(N)$ and $|y| \leq \sqrt{2}N$.

Now let us consider $s \in [0,1)^n$ arbitrary. Then let us take

$$s' = \frac{s \cdot \sigma}{\sigma_n} e_n \text{ so that } s - s' \in H.$$

Now, by the above arguments, there exists $y' \in H$ such that $|y' - s'| \bmod \mathbb{Z}^n \leq \omega_\sigma(N)$ and $|y'| \leq \sqrt{2}N$. Then let $y = y' + (s - s')$ which is still in H and $|y| \leq |y'| + |s - s'| \leq 2\sqrt{n}N$ and, moreover,

$$|y - s| \bmod \mathbb{Z}^n = |y' - s'| \bmod \mathbb{Z}^n \leq \omega_\sigma(N).$$

Now let $s = x_0 \bmod \mathbb{Z}^n$, then there exists $\tilde{y} \in H$ as above with

$$|\tilde{y}| \leq 2\sqrt{n}N \text{ and } |\tilde{y} - x_0| \bmod \mathbb{Z}^n = |\tilde{y} - s| \bmod \mathbb{Z}^n \leq \omega_\sigma(N).$$

Now let $y = \tilde{y} + (s - \tilde{y} \bmod \mathbb{Z}^n)$, then we have $|y| \leq (2\sqrt{n} + 1)N$ and

$$y \bmod \mathbb{Z}^n = s = x_0 \bmod \mathbb{Z}^n \text{ and } \operatorname{dist}(y, H) \leq \omega_\sigma(N).$$

This completes the proof of part (i).

The proof of part (ii) is very similar. Let $\delta > 0$ and N sufficiently large so that $\omega_\sigma(N) \le \inf_{N>0} \omega_\sigma(N) + \delta$. Let $s = -x_0$ and by the arguments above there exists $1 \le k \le N$ such that

$$|k(m_j(\sigma)e_n + e_j) + x_0| \bmod \mathbb{Z}^n \le \omega_\sigma(N) \le \inf_{N>0} \omega_\sigma(N) + \delta.$$

Since $x_0 + k(m_j(\sigma)e_n + e_j)$ is in $H(x_0)$ this completes the proof of (ii). $\square$

3.2.3 Assumptions on the Operators

Now we give the technical assumptions on the differential operators under which we can solve the cell problem. The operator F which arises in the cell problem is obtained as a scaling limit of the general operators F_ε being homogenized over. This connection will be made more explicit later in Lemma 3.5.1. We will work in the class of fully nonlinear uniformly elliptic operators without gradient dependence. This is mostly for convenience and we believe that our framework can be extended to the gradient dependent case by using the arguments of [BM13].

Let $\mathcal{M}^n$ be the class of $n \times n$ symmetric matrices with real entries. Then the differential operator $F : \mathcal{M}^n \times \mathbb{R}^n \to \mathbb{R}$ will be assumed to satisfy the following properties:

(F1) (Lipschitz continuity) F is locally Lipschitz continuous in $\mathcal{M}^n \times \mathbb{R}^n$, and moreover there exists $C > 0$ so that for all $z, y \in \mathbb{R}^n$ and $M, N \in \mathcal{M}^n$,

$$|F(M,z) - F(N,y)| \le C(|z-y|(1 + \|M\| + \|N\|) + \|M - N\|).$$

(F2) (Uniform Ellipticity) There exists $\Lambda > \lambda > 0$ such that for all $y \in \mathbb{R}^n$,

$$\lambda \operatorname{Tr}(N) \le F(M,y) - F(M+N,y) \le \Lambda \operatorname{Tr}(N) \text{ for all } M, N \in \mathcal{M}^n \text{ such that } N \ge 0.$$

(F3) (Periodicity) For every $z \in \mathbb{Z}^n$, $y \in \mathbb{R}^n$ and $M \in \mathcal{M}^n$ we have,

$$F(M, y+z) = F(M,y).$$

(F4) (Homogeneity) For all $t > 0$, $M \in \mathcal{M}^n$ and $y \in \mathbb{R}^n$ we have $F(tM,y) = tF(M,y)$ and, in particular $F(0,y) = 0$.

A wide class of operators of this form come from the theory of optimal control of diffusion processes, for example the Hamilton-Jacobi-Bellman operators,

$$F(M,y) = -\sup_{\alpha\in\mathscr{A}} \mathrm{Tr}(A^{\alpha}(y)M) \text{ where } \lambda \le A^{\alpha}(y) \le \Lambda\, \forall\, \alpha \in \mathscr{A}.$$

In much greater generality there are Isaacs operators coming from differential games which have the form,

$$F(M,y) = -\inf_{\beta\in\mathscr{B}} \sup_{\alpha\in\mathscr{A}} \mathrm{Tr}(A^{\alpha\beta}(y)M) \text{ where } \lambda \le A^{\alpha\beta}(y) \le \Lambda\, \forall\, (\alpha,\beta) \in \mathscr{A}\times\mathscr{B}.$$

In fact, all operators satisfying (F1)-(F4) can be written as an Isaacs operator, the proof is a basic exercise.

The operator F_ε is assumed to satisfy (F1) with the local Lipschitz continuity in both the x and y variables, the uniform ellipticity (F2) and (F4) the F_ε are $\mathbb{Z}^n$ periodic in y. The most general form of such operators is,

$$F_\varepsilon(M,x,y) = f(x,y) - \inf_{\beta\in\mathscr{B}} \sup_{\alpha\in\mathscr{A}} \mathrm{Tr}(A^{\alpha\beta}(x,y)M) \text{ where } \lambda \le A^{\alpha\beta} \le \Lambda\, \forall\, (\alpha,\beta) \in \mathscr{A}\times\mathscr{B}. \quad (3.2.3)$$

Here the $A^{\alpha\beta}$ have uniformly bounded Lipschitz norm, are periodic in their y variable, and satisfy the uniform ellipticity condition

$$\lambda \le A^{\alpha\beta} \le \Lambda\, \forall\, (\alpha,\beta) \in \mathscr{A}\times\mathscr{B}.$$

The function f is in $C^{0,1}(\mathbb{R}^n\times\mathbb{R}^n)$ and is $\mathbb{Z}^n$ periodic in its second variable.

The standard notion of weak solution for the equation $F(D^2u,y) = 0$ in a domain $\Omega \subseteq \mathbb{R}^n$ is the viscosity solution. The usual reference on the theory of viscosity solutions is [CIL92].

Definition 3.2.8. (Viscosity Solutions)

(i) $u : \Omega \to \mathbb{R}$ upper semi-continuous is a subsolution of $F(D^2u,y) \le 0$ if for any $\phi \in C^2(\Omega)$ such that $u-\phi$ has a local max at $y_0 \in \Omega$,

$$F(D^2\phi(y_0), y_0) \le 0.$$

(ii) $u : \Omega \to \mathbb{R}$ lower semi-continuous is a supersolution of $F(D^2u, y) \geq 0$ if for any $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local min at $y_0 \in \Omega$,

$$F(D^2\phi(y_0), y_0) \geq 0.$$

(iii) $u : \Omega \to \mathbb{R}$ continuous is a solution of $F(D^2u, y) = 0$ if it is both a subsolution and a supersolution.

The Dirichlet boundary data can be defined in the classical sense for viscosity solutions. We say that u is a subsolution of $F(D^2u, y) = 0$ in Ω with Dirichlet data $\psi : \partial\Omega \to \mathbb{R}$ if u is a subsolution and

$$\limsup_{z \to y} u(z) \leq \psi(y) \text{ for all } y \in \partial\Omega.$$

Supersolutions are defined analogously and solutions satisfy both the sub solution and the super solution property.

One very useful property of viscosity solutions is their stability under uniform convergence. Let $u_j : \overline{\Omega} \to \mathbb{R}$ with $j \in \mathbb{N}$ for some domain (possible unbounded) $\Omega \subset \mathbb{R}^n$. Let us defined the upper and lower half-relaxed limit operations,

$$\limsup{}^* u_j(x) = \lim_{\varepsilon \to 0} \sup\{u_j(y) : y \in \overline{\Omega} \text{ and } |y - x| + j^{-1} \leq \varepsilon\}$$

and

$$\liminf{}^* u_j(x) = \lim_{\varepsilon \to 0} \inf\{u_j(y) : y \in \overline{\Omega} \text{ and } |y - x| + j^{-1} \leq \varepsilon\}.$$

One can also have $u_j : \Omega_j \to \mathbb{R}$ with the $\Omega_j \to \Omega$ in the Hausdorff topology on compact subsets of $\mathbb{R}^n$. One can easily check that if the upper and lower half-relaxed limits agree then the u^j converge locally uniformly to their common value. In any case we have the following stability result for viscosity sub and supersolutions under the half-relaxed limit operations, see [CIL92] for more details,

Lemma 3.2.9. (Stability of viscosity solutions) *Let $F_j(M, y) : \mathscr{M}^n \times \mathbb{R}^n \to \mathbb{R}$ be a sequence of operators satisfying (F1) and (F2) which converge to an operator $F(M, y)$ locally uniformly in M*

and uniformly in $y \in \mathbb{R}^n$. Let Ω be a domain in $\mathbb{R}^n$ and let $u_j : \Omega \to \mathbb{R}$ be a bounded above (below) sequence of subsolutions (supersolutions) of,

$$F_j(D^2 u_j, y) \leq 0 \ \ in \ \ \Omega \quad (\ F_j(D^2 u_j, y) \geq 0 \ \ in \ \ \Omega \ ).$$

Define u^* the upper half-relaxed limit of the u^j (resp. u_* the lower half-relaxed limit), then we have,

$$F(D^2 u^*, y) \leq 0 \ \ in \ \ \Omega \quad (\ F(D^2 u_*, y) \geq 0 \ \ in \ \ \Omega \ ).$$

Then we state the interior homogenization result in terms of the half-relaxed limits:

Theorem 3.2.10. *(Interior homogenization [Eva92, Eva89, CSW05]) Let $F_\varepsilon(M, x, y)$ satisfying (F1) with the local Lipschitz continuity in both the x and y variables, (F2) and F_ε are $\mathbb{Z}^n$ periodic in y. Let Ω a bounded domain in $\mathbb{R}^n$ and u^ε satisfying,*

$$\begin{cases} F_\varepsilon(D^2 u^\varepsilon, x, \frac{x}{\varepsilon}) \leq 0 & in \quad \Omega \\ \\ u^\varepsilon(x) \leq M & on \quad \partial\Omega \end{cases} \tag{3.2.4}$$

for some $M > 0$. Then there exists an operator $\overline{F}(M, x)$ such that $u^* = \limsup^* u^\varepsilon$ is a subsolution of,

$$\begin{cases} \overline{F}(D^2 u^*, x) \leq 0 & in \quad \Omega \\ \\ u^*(x) \leq M & on \quad \partial\Omega. \end{cases} \tag{3.2.5}$$

The analogous result holds for supersolutions as well.

3.2.4 Maximal Operators and Comparison

Now we recall the Pucci maximal operators associated with the class of uniformly elliptic operators with constants λ, Λ. The basic results about the Pucci maximal operators can be found in the book [CC95]. For $M \in \mathscr{M}^n$ we can always decompose $M = M_+ - M_-$ with $M_\pm \geq 0$ and $M_+ M_- = 0$. We then define,

$$\mathscr{P}^+(M) = \Lambda \mathrm{Tr} M_+ - \lambda \mathrm{Tr} M_- \ \text{ and } \ \mathscr{P}^-(M) = \lambda \mathrm{Tr} M_+ - \Lambda \mathrm{Tr} M_-. \tag{3.2.6}$$

Then from (F2) we may derive for $M, N \in \mathscr{M}^n$ and $x \in \mathbb{R}^n$,

$$-\mathscr{P}^+(M-N) \leq F(M,y) - F(N,y) \leq -\mathscr{P}^-(M-N). \tag{3.2.7}$$

Note that by the inequality (3.2.7) any viscosity solution of $F(D^2u, x) = 0$ will satisfy, in the viscosity sense,

$$-\mathscr{P}^+(D^2u) \leq 0 \ \text{ and } \ -\mathscr{P}^-(D^2u) \geq 0.$$

Moreover, (3.2.7) implies that the difference $w = u - v$ of a classical supersolution u and a classical subsolution v is itself a supersolution of the equation,

$$-\mathscr{P}^-(D^2w) \geq 0.$$

This is also true of only semicontinuous viscosity sub and supersolutions using the method of sup and inf-convolutions originally used for this purpose by Jensen [Jen88]. We state the Theorem in a slightly different form closer to what appears in [CC95],

Theorem 3.2.11. (Comparison) *Let Ω be a domain in $\mathbb{R}^n$ not necessarily bounded. Let F satisfy (F1) and (F2). Suppose that u and v satisfy, in the viscosity sense,*

$$F(D^2u, y) \leq F(D^2v, y) \ \text{ in } \ \Omega.$$

Then $w = u - v$ is a subsolution of,

$$-\mathscr{P}^+(D^2w) \leq 0 \ \text{ in } \ \Omega.$$

In bounded domains the comparison principle for the Dirichlet problem follows from Theorem 3.2.11 in a standard way. A comparison principle for solutions of uniformly elliptic equations in half spaces with sublinear growth at infinity then follows from the following localization lemma which will be useful to us later,

Lemma 3.2.12. (Localization) *Let $\delta, \varepsilon > 0$ and $L > 1$ and suppose that v is a viscosity solution of*

$$\begin{cases} -\mathscr{P}^+(D^2v) \leq 0 & \text{in} \quad P(0,e_n) \cap Q_L \\ \\ v(y) \leq \delta & \text{on} \quad \partial P(0,e_n) \cap Q_L \\ \\ v(y) \leq L^{1-\varepsilon} & \text{on} \quad \partial Q_L \cap \overline{P}, \end{cases} \tag{3.2.8}$$

where Q_L is the cylindrical region,

$$Q_L = \{|x'| \leq L\} \times \{0 \leq x_n \leq L\} \ \ with \ \ x' = x - x_n e_n.$$

Then we have,

$$v(y) \leq \delta + 2\tfrac{\Lambda}{\lambda} n L^{-\varepsilon} \ \ in \ \ Q_1.$$

Remark 3.2.13. Since the Pucci maximal operators are translation and rotation invariant the same result holds for the analogous problem in the half space $P(x_0, \sigma)$ for any $\sigma \in S^{n-1}$ and $x_0 \in \mathbb{R}^n$.

Proof. Consider the following barrier ϕ,

$$\phi(x) = L^{-1-\varepsilon}(|x'|^2 - 2\tfrac{\Lambda}{\lambda}(n-1)x_n^2) + \delta + (2\tfrac{\Lambda}{\lambda}(n-1)+1)L^{-\varepsilon} x_n.$$

It is straightforward to check that $\phi(x) \geq v(x)$ on ∂Q_L and that ϕ is a smooth supersolution of $-\mathscr{P}^+(D^2\phi) \geq 2\Lambda(n-1)L^{-1-\varepsilon}$ in $x_n > 0$. From the definition of viscosity solution $v \leq \phi$ in Q_L. Therefore since

$$\phi(x) \leq \delta + 2(\tfrac{\Lambda}{\lambda}(n-1)+1)L^{-\varepsilon} \ \ \text{in} \ \ Q_1,$$

we get the result.

□

A straightforward application of the localization lemma is the following comparison principle for bounded solutions of the half-space problem (3.1.3):

Lemma 3.2.14. *Let $\psi_1, \psi_2 \in C^\alpha(\mathbb{R}^n)$. Let v_1 be a bounded upper semi-continuous subsolution of* (3.1.3) *with Dirichlet data ψ_1, and v_2 be a bounded lower semi-continuous supersolution of* (3.1.3) *with Dirichlet data ψ_2. Then,*

$$\sup_{y \in P} (v_1(y) - v_2(y))_+ \leq \sup_{y \in \partial P} (\psi_1(y) - \psi_2(y))_+.$$

3.2.5 Regularity Results

The Pucci maximal operators govern the worst possible behavior for solutions of equations satisfying the ellipticity condition (F2). In addition to the role they play in the comparison result, Theorem 3.2.11, they are useful because regularity results hold uniformly in the ellipticity class (λ,Λ). In particular we have the following result, for more details see the book [CC95]:

Lemma 3.2.15. (Interior Hölder estimates) *Assume (F2) and (F4). Let $x_0 \in \mathbb{R}^n$ and v be a continuous viscosity solution of*

$$-\mathscr{P}^+(D^2 v) \le 0 \ \ and \ \ -\mathscr{P}^-(D^2 v) \ge 0.$$

in $B_r(x_0)$. For every $\alpha \in (0,1)$ there exists $C = C(\lambda,\Lambda,n,\alpha) > 0$ such that,

$$\sup_{x,y\in B_{r/2}(x_0)} \frac{|v(x)-v(y)|}{|x-y|^\alpha} \le Cr^{-\alpha} \sup_{x\in B_r(x_0)} v(x).$$

By using barriers one can prove the following Hölder estimate up to the boundary,

Lemma 3.2.16. (Boundary Hölder estimates) *Let $g \in C^\alpha(\mathbb{R}^n)$ for some $\alpha \in (0,1)$ and suppose that u is a viscosity solution of*

$$\begin{cases} F(D^2u) = 0 & in \quad B_1^+ = \{|x| < 1,\ x_n > 0\} \\ \\ u = g(x) & on \quad T = \partial B_1^+ \cap \{x_n = 0\}. \end{cases} \tag{3.2.9}$$

Then we have the following estimate up to the boundary,

$$\|u\|_{C^\alpha(\overline{B_{1/2}^+})} \le C(n,\lambda,\Lambda)(\|g\|_{C^\alpha(T)} + \sup_{B_1^+}|u|).$$

3.3 The Cell Problem

Now we begin discussing the actual homogenization problem. First we will analyze the cell problem and identify the homogenized boundary condition. We will work in the following half-space type domains in $\mathbb{R}^N$,

$$P(\sigma,x_0) = \{x \in \mathbb{R}^n : (x-x_0)\cdot\sigma > 0\}.$$

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be $C^\alpha(\mathbb{R}^n)$ for some $\alpha \in (0,1)$ and periodic with respect to the $\mathbb{Z}^n$ lattice. For convenience in the future we will sometimes refer to this class of functions as $C^\alpha_{per}(\mathbb{R}^n)$. From here on α will always refer to the exponent of Hölder continuity of the Dirichlet boundary data for the problem under consideration. Now we consider solving the following Dirichlet problem in $P(\sigma, x_0)$, which we will call the cell problem,

$$\begin{cases} F(D^2 v_{x_0,\sigma}, y) = 0 & \text{in} \quad P(\sigma, x_0) \\ \\ v_{x_0,\sigma}(y) = \psi(y) & \text{on} \quad \partial P(\sigma, x_0). \end{cases} \tag{3.3.1}$$

A standard application of Perron's method in combination with the Localization Lemma 3.2.12 and the Comparison Principle 3.2.14 give a unique bounded solution to the cell problem. Standard barrier arguments imply that the solution given by Perron's method achieves the boundary data continuously.

Now let us define the upper and lower homogenized boundary conditions arising from the cell problems (3.3.1),

$$\begin{aligned} \mu^*(\sigma, F, \psi) &:= \sup\{\limsup_{R\to\infty} v_{x_0,\sigma}(y + R\sigma) : y \in \partial P(\sigma, x_0),\ x_0 \in \mathbb{R}^n\} \\ \mu_*(\sigma, F, \psi) &:= \inf\{\liminf_{R\to\infty} v_{x_0,\sigma}(y + R\sigma) : y \in \partial P(\sigma, x_0),\ x_0 \in \mathbb{R}^n\}. \end{aligned} \tag{3.3.2}$$

Then we call $\mu = \frac{1}{2}(\mu^* + \mu_*)$ to be the average of μ^* and μ_*. Notice that μ^*, μ_* and μ are invariant under translations of ψ. In fact we could have equivalently defined them as envelopes over $\psi(\cdot + \tau)$ for $\tau \in [0,1)^n$ rather than over the location of the hyperplane $\partial P(\sigma, x_0)$. From now on we will work mostly with the quantity $\mu(\sigma, \psi, F)$ but the upper and lower homogenized boundary conditions will be mentioned occasionally as well since we will see that they are relevant at the rational directions.

Now we will show that $\mu = \mu^* = \mu_*$ when σ is an irrational direction and that the convergence of $v_{x_0,\sigma}(y + R\sigma)$ to μ is uniform in x_0 and y as $R \to \infty$. Furthermore we provide a precise estimate on the difference $\mu^* - \mu_*$ when σ is a rational direction in terms of the discrepancy in the form of ω_σ. This lemma is analogous to Theorem 2.2 in [BM13].

Lemma 3.3.1. (Identification of the homogenized boundary condition) *Suppose that $v_{x_0} = v_{x_0,\sigma}$ is the unique bounded solution of* (3.3.1) *set in $P(\sigma, x_0)$ where σ is any direction in S^{n-1} and $x_0 \in \mathbb{R}^n$. Let $\mu(\sigma, \psi, F)$ defined as above then we have for all $N > 1$,*

$$\sup_{x_0 \in \mathbb{R}^n} \sup_{y \in \partial P} |v_{x_0}(y + R\sigma) - \mu| \le C(n, \lambda, \Lambda)(|\psi|_{C^\alpha} + \operatorname{osc} \psi)((N/R)^\alpha + \omega_\sigma(N)^\alpha) \ \textit{for}\ \ R \ge CN. \tag{3.3.3}$$

In particular, when σ is an irrational direction $\sigma \in S^{n-1} \setminus \mathbb{R}\mathbb{Z}^n$ we have that $\mu^* = \mu_* = \mu$ and for all directions $\sigma \in S^{n-1}$,

$$\mu^*(\sigma, \psi, F) - \mu_*(\sigma, \psi, F) \leq C(n, \lambda, \Lambda)(|\psi|_{C^\alpha} + \operatorname{osc} \psi) \inf_{N>1} \omega_\sigma(N)^\alpha. \tag{3.3.4}$$

Remark 3.3.2. The fact that the limit of $v_{x_0,\sigma}(y + R\sigma)$ as $R \to \infty$ does not depend on the location of the hyperplane ∂P (i.e. on x_0) for irrational directions is essential for homogenization to hold. As mentioned before this is heuristically because restrictions of ψ to irrational hyperplanes see all the values of ψ. The Hölder regularity of ψ is not essential, but in general continuity seems to be necessary.

Proof. We may assume without loss of generality due to invariance of the equation under adding a constant and scalar multiplication that $|\psi|_{C^\alpha} \leq 1$ and that $\|\psi\|_\infty = \frac{1}{2} \operatorname{osc} \psi \leq 1$.

1. First we show that $v_{0,\sigma}(y + R\sigma)$ converges along subsequences locally uniformly in $\mathbb{R}^n$ to constants. For now let us call $v = v_{0,\sigma}$, $P = P(\sigma, 0)$. By the comparison principle we have the uniform bounds,

$$-1 \leq v \leq 1.$$

Let $R_k \to \infty$ be a subsequence such that $v(R_k\sigma)$ converges as $R_k \to \infty$ to some constant $\tilde{\mu}$.

2. Due to Lemma 3.2.7 for any $p = p' + R\sigma \in \partial P + R\sigma$ there exists $z \in \mathbb{Z}^n$ such that,

$$|z - p'| \leq CN \text{ and } h := \operatorname{dist}(z, \partial P) \leq \omega_\sigma(N).$$

Note that we have $z \cdot \sigma = \pm h$, we assume the $+$ sign, the proof will be very similar in the other case. We define the translation of v,

$$\widetilde{v}(y) = v(y - z),$$

which is a viscosity solution of,

$$\begin{cases} F(D^2\widetilde{v}, y) = F(D^2\widetilde{v}, y - z) = 0 & \text{in} \quad P + h\sigma \\ \widetilde{v}(y) = \psi(y - z) = \psi(y) & \text{on} \quad \partial P + h\sigma. \end{cases}$$

Now for $y \in \partial P + h\sigma \subset \overline{P}$ we have,

$$|v(y) - \psi(y)| \leq |v(y) - v(y - h\sigma)| + |\psi(y - h\sigma) - \psi(y)| \leq Ch^\alpha.$$

Therefore by comparison principle,

$$\sup_{y\in P+h\sigma}|v(y)-v(y-z)|\leq Ch^{\alpha},$$

and then in particular, by the interior C^{α} estimates,

$$\begin{aligned}|v(p+R\sigma)-v(R\sigma)| &\leq |v(p+R\sigma)-v(z+R\sigma)|+|v(z+R\sigma)-v(R\sigma)|\\ &\leq \operatorname*{osc}_{|y-p|\leq CN} v(y)+C\omega_{\sigma}(N)^{\alpha}\leq C(\omega_{\sigma}(N)^{\alpha}+(N/R)^{\alpha}),\end{aligned}$$

where C is of course independent of the point p. Then using comparison,

$$\operatorname*{osc}_{y\in P} v(y+R\sigma)\leq \operatorname*{osc}_{y\in\partial P} v(y+R\sigma)\leq C(\omega_{\sigma}(N)^{\alpha}+(N/R)^{\alpha}).$$

Now for $R_k>R$, since $R_k\sigma\in P+R\sigma$,

$$\sup_{y\in P}|v(y+R\sigma)-v(y+R_k\sigma)|\leq C(\omega_{\sigma}(N)^{\alpha}+(N/R)^{\alpha}).$$

Letting $R_k\to\infty$ in the above inequality we derive,

$$\sup_{y\in P(0,\sigma)}|v_{0,\sigma}(y+R\sigma)-\tilde{\mu}|\leq C(\omega_{\sigma}(N)^{\alpha}+(N/R)^{\alpha}). \tag{3.3.5}$$

3. Now we show that the estimate (3.3.5) is independent of the spatial location of ∂P. Let $x_0\in\mathbb{R}^n$ and v_{x_0} be the corresponding solution of (3.3.1) in $P(\sigma,x_0)$, and v as before be the solution of (3.3.1) in $P(\sigma,0)$. Let $N>1$, due to Lemma 3.2.7 there exists $z\in\mathbb{Z}^n$ such that

$$\operatorname{dist}(\partial P(\sigma,x_0)+z,\partial P(\sigma,0))=h\leq\omega_{\sigma}(N).$$

Then $\widetilde{v}(y)=v(y-z)$ solves the following problem in $P(\sigma,z)$,

$$\begin{cases} F(D^2\widetilde{v},y)=F(D^2\widetilde{v},y-z)=0 & \text{in} \quad P(\sigma,z)\\ \widetilde{v}(y)=\psi(y-z)=\psi(y) & \text{on} \quad \partial P(\sigma,z).\end{cases}$$

Note that $\widetilde{v}$ trivially satisfies the estimate (3.3.5). Let us assume that $z\in P(\sigma,x_0)$, the other case will follow by a similar argument. For each $y\in\partial P(\sigma,z)$, we have that $y-h\sigma\in\partial P(\sigma,x_0)$, so by the C^{α} regularity up to the boundary for v_{x_0} we get,

$$|v_{x_0}(y)-\psi(y)|\leq|v_{x_0}(y)-v_{x_0}(y-h\sigma)|+|\psi(y-h\sigma)-\psi(y)|\leq Ch^{\alpha}.$$

Therefore by the comparison principle Lemma 3.2.14,

$$\sup_{y \in P(\sigma,z)} |\widetilde{v}(y) - v_{x_0}(y)| \le C\omega_\sigma(N)^\alpha,$$

Combining the above estimate with estimate (3.3.5) for $\widetilde{v}$, we get

$$\sup_{y \in P(0,\sigma)} |v_{x_0,\sigma}(y + R\sigma) - \tilde{\mu}| \le C(\omega_\sigma(N)^\alpha + (N/R)^\alpha), \tag{3.3.6}$$

where the constant C depends only on the uniform ellipticity of F by way of the interior C^α estimates, and in particular not on the arbitrary point x_0 or N. Taking the supremum over $x_0 \in \mathbb{R}^n$ and we get the desired estimate except with $\tilde{\mu}$. From here we simply note that $\tilde{\mu} \in [\mu_*, \mu^*]$ so estimate (3.3.3) with $\tilde{\mu}$ replacing μ still implies (3.3.4) which in turn implies (3.3.3) with μ (possibly at the cost of increasing the universal constant $C(n, \lambda, \Lambda)$).

□

We now list some properties of the homogenized boundary condition which can be derived in a straightforward way from the identification Lemma 3.3.1.

Lemma 3.3.3. (Properties of μ) *The following properties all hold for μ^* and μ_* as well unless otherwise specified. Recall that the distinction between the envelopes is only relevant at rational directions.*

(i) (Continuity with respect to F) Fix $\psi \in C^\alpha_{per}(\mathbb{R}^n)$, $\sigma \in S^{n-1}$ and let $F_j(M,y)$ be a sequence of operators satisfying (F1)-(F4) which converge to an operator $F_\infty(M,y)$ locally uniformly in M and uniformly in y, then,

$$\limsup_{j \to \infty} |\mu(\sigma, \psi, F_j) - \mu(\sigma, \psi, F_\infty)| \le C(n, \lambda, \Lambda)(|\psi|_{C^\alpha} + \operatorname{osc} \psi) \inf_{N > 0} \omega_\sigma(N)^\alpha.$$

In particular this implies the continuity of μ with respect to locally uniform convergence of its F argument at all irrational directions $\sigma \in S^{n-1} \setminus \mathbb{R}\mathbb{Z}^n$.

(ii) (Monotonicity) For any $\psi_1, \psi_2 \in C^\alpha_{per}(\mathbb{R}^n)$ and F_1, F_2 satisfying (F1)-(F4) we have,

$$\psi_1 \le \psi_2 \ \textit{ and } \ F_2 \le F_1 \ \textit{ implies } \ \mu(\sigma, \psi_1, F_1) \le \mu(\sigma, \psi_2, F_2).$$

For the following we fix $\sigma \in S^{n-1}$ and an operator F satisfying (F1)-(F4), and we call $\mu(\sigma,\cdot,F) = \mu(\cdot)$ for simplicity of notation.

(iii) (Continuity with respect to ψ) For $\psi_1, \psi_2 \in C^{\alpha}_{per}(\mathbb{R}^n)$,

$$|\mu(\psi_1) - \mu(\psi_1)| \leq \|\psi_1 - \psi_2\|_{L^\infty}.$$

(iv) (Homogeneity) For any $t > 0$ and $\psi \in C^{\alpha}_{per}(\mathbb{R}^n)$,

$$\mu(t\psi) = t\mu(\psi).$$

(v) (Constants) For $\psi \in C^{\alpha}_{per}(\mathbb{R}^n)$ and $c \in \mathbb{R}$,

$$\mu(\psi + c) = \mu(\psi) + c.$$

(vi) (Translation invariance) For any $\psi \in C^{\alpha}_{per}(\mathbb{R}^n)$ and any $\tau \in [0,1)^n$,

$$\mu(\psi(\cdot + \tau)) = \mu(\psi).$$

(vii) (Sub/Super-additivity) If the operator F is convex (concave) then, for any $\psi_1, \psi_2 \in C^{\alpha}_{per}(\mathbb{R}^n)$,

$$\mu_*(\psi_1 + \psi_2) \geq \mu_*(\psi_1) + \mu_*(\psi_2) \ \ (\text{ or } \ \mu^*(\psi_1 + \psi_2) \leq \mu^*(\psi_1) + \mu^*(\psi_2)).$$

In particular, if F is linear then μ is linear as well at irrational directions.

Proof. All except (i) are easy applications of Lemma 3.3.1 and the comparison principle. Let us therefore consider the situation in (i). As before we assume that $|\psi|_{C^\alpha} \leq 1$ and that $\|\psi\|_\infty = \frac{1}{2}\operatorname{osc}\psi$. Let $\mu_j = \mu_j(\psi, F_j, \sigma)$ and let v_j for $j \in \mathbb{N} \cup \{\infty\}$ respectively solve the cell problems (3.3.1) for F_j in the domain $P = P(0,\sigma)$. By the method of half-relaxed limits [BP88] the v_j converge uniformly on compact subsets of $P \cup \partial P$ to v_∞. Let $\delta > 0$ and let N sufficiently large so that $\omega_\sigma(N)^\alpha \leq c\delta + \inf_{N>0}\omega_\sigma(N)^\alpha$ and then taking $R \geq \delta^{-1/\alpha}CN$, we get from (3.3.3),

$$\sup_{j \in \mathbb{N} \cup \{\infty\}} |v_j(R\sigma) - \mu_j| \leq \delta + C\omega_\sigma(N)^\alpha \leq 2\delta + C\inf_{N>0}\omega_\sigma(N)^\alpha.$$

Then fixing such an R, since $v_j(R\sigma) \to v_\infty(R\sigma)$ as $j \to \infty$, for $j > J(\delta, R)$ sufficiently large,

$$|\mu_j - \mu_\infty| \leq |v_j(R\sigma) - \mu_j| + |v_j(R\sigma) - v_\infty(R\sigma)| + |v_\infty(R\sigma) - \mu_\infty| \leq 5\delta + C\inf_{N>0}\omega_\sigma(N)^\alpha.$$

Since δ was arbitrary we conclude.

$\square$

Now let us fix our boundary data $\psi \in C^{\alpha}(\mathbb{R}^n)$ which is $\mathbb{Z}^n$ periodic and $F(M,y)$ our operator satisfying (F1)-(F4). We will call $\mu(\sigma) = \mu(\sigma,\psi,F)$ which is defined in Lemma 3.3.1 for all directions $\sigma \in S^{n-1} \setminus \mathbb{R}\mathbb{Z}^n$. We now consider the continuity properties of μ with respect to the normal direction σ. The goal is to show that μ is continuous at every irrational direction.

Lemma 3.3.4. *Let $\sigma \in S^{n-1} \setminus \mathbb{R}\mathbb{Z}^n$. Then μ satisfies the following continuity estimate for any $N > 1$ and $|\sigma' - \sigma| \leq \eta(\sigma,N)$ sufficiently small,*

$$|\mu(\sigma') - \mu(\sigma)| \leq C(n,\lambda,\Lambda)(|\psi|_{C^\alpha} + \operatorname{osc}\psi)(N^{\frac{\alpha}{1+\alpha}}|\sigma - \sigma'|^{\frac{\alpha}{2+\alpha}} + \omega_\sigma(N)^\alpha).$$

Remark 3.3.5. In particular this estimate implies, since $\omega_\sigma(N) \to 0$ as $N \to \infty$ for irrational directions, that $\mu(\sigma)$ is continuous at every irrational direction. At every rational direction we have an upper bound on the limiting oscillation of μ in small neighborhoods of σ. In fact using the methods of [CKL13,CK14] when the operator $F(M,y)$ homogenizes to a rotationally invariant operator one can show that μ has a unique continuous extension from $S^{n-1} \setminus \mathbb{R}\mathbb{Z}^n$ to all of S^{n-1}. In general though, numerical experiments suggest that there are non rotationally invariant operators for which μ will not extend continuously to the rational directions [KS13]. In that case the above continuity estimate may be the best possible.

Proof. As before, without loss of generality (due to Lemma 3.3.3) we may assume that $|\psi|_{C^\alpha} \leq 1$ and that $\|\psi\|_\infty = \frac{1}{2}\operatorname{osc}\psi$.

Fix $\sigma \in S^{n-1} \setminus \mathbb{R}\mathbb{Z}^n$ and let $\sigma' \in S^{n-1}$ be any direction with $|\sigma' - \sigma| \leq \eta < 1/2$, let L,R with $L >> R$ and $\eta L << 1$. The choices of η, L and R will be made more explicit shortly. The idea is to explicitly compare the solutions of (3.3.1) in $P(0,\sigma)$ and $P(0,\sigma')$ and use the identification of the homogenized boundary condition Lemma 3.3.1.

Let us call $x' = x - (x\cdot\sigma)\sigma$, then we have,

$$\{x\cdot\sigma > |\sigma' - \sigma|L\} \subset P(0,\sigma') \cap \{|x'| \leq L\}.$$

Moreover, on $\{x\cdot\sigma = |\sigma'-\sigma|L\}$ letting $a(x') = -\frac{x'\cdot\sigma'}{\sigma\cdot\sigma'}$ so that $x'+a(x')\sigma \in \partial P(0,\sigma')$ we have that $|a(x')| \leq \frac{1}{2}L|\sigma'-\sigma|$ since $\eta < 1/2$ and by using the C^α estimates up to the boundary,

$$\begin{aligned}|v_\sigma(x)-v_{\sigma'}(x)| &\leq |v_\sigma(x)-v_\sigma(x')| + |\psi(x')-\psi(x'+a(x')\sigma)| + |v_{\sigma'}(x'+a(x')\sigma)-v_{\sigma'}(x)|\\ &\leq C(n,\lambda,\Lambda)L^\alpha|\sigma'-\sigma|^\alpha.\end{aligned}$$

Then we apply the localized comparison estimate of Lemma 3.2.12 in the domain $(P(0,\sigma)+|\sigma'-\sigma|L\sigma)\cap P(0,\sigma')$ to get,

$$|v_{\sigma'}-v_\sigma| \leq C(L^\alpha|\sigma'-\sigma|^\alpha + \tfrac{R}{L}) \text{ in } \{|x'|\leq R\}\times\{0\leq x\cdot\sigma\leq R\}.$$

Then by Lemma 3.3.1 for every $1<N<c(n)R$,

$$\begin{aligned}|\mu(\sigma)-\mu(\sigma')| &\leq |v_\sigma(R\sigma)-\mu(\sigma)| + C(L^\alpha|\sigma'-\sigma|^\alpha+\tfrac{R}{L}) + |v_{\sigma'}(R\sigma)-\mu(\sigma')|\\ &\leq C(L^\alpha|\sigma'-\sigma|^\alpha + \tfrac{R}{L} + (\tfrac{N}{R})^\alpha + \omega_\sigma(N)^\alpha + \omega_{\sigma'}(N)^\alpha).\end{aligned} \tag{3.3.7}$$

Now in order to make the first three terms on the right hand side above of comparable size we choose,

$$R = N|\sigma-\sigma'|^{-\frac{1}{2+\alpha}} \text{ and } L = N^{\frac{1}{1+\alpha}}|\sigma-\sigma'|^{-\frac{1+\alpha(2+\alpha)}{(1+\alpha)(2+\alpha)}}.$$

So for any $N>1$ choose η sufficiently small so that $R\geq C(n)N$, that is,

$$0<\eta\leq C(n)^{-(2+\alpha)}.$$

Then based on the continuity properties of $\omega_\cdot(N)$ with fixed N from Section 3.2.2 choose η smaller if necessary so that

$$\omega_{\sigma'}(N) \leq 2\omega_\sigma(N) \text{ when } |\sigma-\sigma'|\leq\eta(\sigma,N).$$

Then we get, by plugging in our choice of R, L for $|\sigma-\sigma'|\leq\eta(\sigma,N)$ into (3.3.7),

$$|\mu(\sigma)-\mu(\sigma')| \leq C(n,\lambda,\Lambda)(N^{\frac{\alpha}{1+\alpha}}|\sigma-\sigma'|^{\frac{\alpha}{2+\alpha}} + \omega_\sigma(N)^\alpha),$$

which was the desired result. □

3.3.1 The Nonlinearity of the Homogenized Boundary Condition

In the case when the operator $F(M,y) = \mathrm{Tr}(A(y)M)$ is linear the homogenized boundary condition $\mu(\psi)$ can be identified explicitly as the average of ψ,

$$\mu(\sigma,\psi,F) = \int_{[0,1)^n} \psi(y)dy. \tag{3.3.8}$$

Note that in particular this value is independent of $\sigma \in S^{n-1} \setminus \mathbb{R}\mathbb{Z}^n$ and of the operator F. The proof is a straightforward consequence of the Riesz Representation Theorem in combination with the properties given in Lemma 3.3.3. We state this in slightly greater generality in the following Lemma,

Lemma 3.3.6. *If F satisfying (F1)-(F4) has no y-dependence and is concave, then for all irrational directions σ and all $\psi \in C^{\alpha}_{per}(\mathbb{R}^n)$,*

$$\mu(\sigma,\psi,F) \geq \int_{[0,1)^n} \psi(y)dy.$$

The opposite inequality holds in case F is convex. More precisely we argue that if μ is linear in ψ then it is just the average of ψ over the unit periodicity cell. In particular this is the case when F is a linear operator.

Proof. We will just show (3.3.8). Then the result of the Lemma can be derived by noting that F concave (convex) with $F(0,x) = 0$ can be written as a supremum (infimum) of linear homogenous operators and using the monotonicity of μ with respect to F. By Lemma 3.3.3 we see that $\mu : C^{\alpha}(\mathbb{T}^n) \to \mathbb{R}$ is linear and satisfies,

$$|\mu(\psi)| \leq \|\psi\|_{\infty}.$$

Therefore μ extends to a bounded linear operator (with norm 1) on the space $C(\mathbb{T}^n)$, and so, by the Riesz Representation Theorem, there is a Radon measure σ on $\mathbb{T}^n$ so that,

$$\mu(\psi) = \int_{\mathbb{T}^n} \psi(y)d\sigma(y).$$

Then, from the translation invariance along with the fact that $\sigma(1) = 1$, we derive that σ must be the uniform measure. $\square$

Clearly this identification of μ, (3.3.8), relies strongly on the linearity of the operator F. Nonetheless one may imagine, since μ is independent of the interior operator in the linear case, that this may carry over to the nonlinear case as well. This is not the case. We give a simple example which shows the effect of the nonlinearity:

Example 3.3.7. In dimension $n = 2$ let us define, for $\Lambda > 1$ and σ, η orthogonal irrational directions, the following fully nonlinear and concave operator,

$$F(D^2u) = \min\{-\Delta u, -u_{\eta\eta} - \Lambda u_{\sigma\sigma}\}.$$

We will take our boundary data to be $\psi(x) = \cos(2\pi x_1)$ and consider the solution u of the following cell problem,

$$\begin{cases} F(D^2u) = 0 & \text{in} \quad P(\sigma) \\ \\ u(x) = \cos(2\pi x_1) & \text{on} \quad \partial P(\sigma). \end{cases} \tag{3.3.9}$$

Note that $\int_{[0,1)^n} \psi(x)dx = 0$ would be the homogenized boundary condition associated with ψ if F were linear. In this case we can construct explicit solutions for both of the operators in the definition of F using separation of variables. In particular we consider,

$$v_1(x) = e^{-x\cdot\sigma}\cos(2\pi\eta_1(x\cdot\eta)) \text{ and } v_2(x) = e^{-x\cdot\sigma/\Lambda^{1/2}}\cos(2\pi\eta_1(x\cdot\eta)),$$

which solve (3.3.9) with the interior operator replaced by Δu and $u_{\eta\eta} + \Lambda u_{\sigma\sigma}$ respectively. In particular v_1 and v_2 are both subsolutions of (3.3.9). Therefore, by comparison, we see that

$$u(x) \geq \max\{v_1(x), v_2(x)\}. \tag{3.3.10}$$

Now let us consider the values of v_1 and v_2 for example on the hyperplane $\{x \cdot \sigma = \Lambda^{1/4}\}$, here we have

$$\max\{v_1(x), v_2(x)\} = e^{-\Lambda^{-1/4}}[\cos(2\pi\eta_1(x\cdot\eta))]_+ + e^{-\Lambda^{1/4}}[\cos(2\pi\eta_1(x\cdot\eta))]_-.$$

Notice that this is the restriction to the hyperplane $\{x \cdot \sigma = \Lambda^{1/4}\}$ of the $\mathbb{Z}^n$ periodic function on $\mathbb{R}^n$,

$$\psi'(x) = e^{-\Lambda^{-1/4}}[\cos(2\pi(x_1 - \Lambda^{1/4}\sigma_1))]_+ + e^{-\Lambda^{1/4}}[\cos(2\pi(x_1 - \Lambda^{1/4}\sigma_1))]_-.$$

Or in other words,

$$\psi'(x) = \max\{v_1(x), v_2(x)\} \le u(x) \;\text{ on }\; \partial P(\sigma, \Lambda^{1/4}\sigma).$$

Therefore using Lemma 3.3.6 identification of μ for linear operators and the monotonicity of μ,

$$\mu(\cos(2\pi\cdot), F, \sigma) \ge \mu(\sigma, \psi', -\Delta) = \int_{[0,1)^n} \psi'(x)dx \ge \tfrac{1}{\pi}\left(e^{-\Lambda^{-1/4}} - e^{-\Lambda^{1/4}}\right) > 0.$$

As a result μ generically will not be just the average of the boundary data when F is nonlinear. Also note that in particular, since $\overline{\mu}$ must be the average if it is linear in ψ, it cannot be linear in ψ.

3.4 Comparison with Partial Boundary Data

Recalling that we do not expect the homogenization to hold for (3.1.1) at points on boundary with rational normal direction we are led to consider whether the comparison principle holds when the ordering on the boundary only holds on a 'large' subset. In particular one is led to consider the following situation, suppose that we have v and u bounded and respectively upper and lower semicontinuous in the closure of a bounded domain Ω and a subset $\Gamma \subset \partial\Omega$ such that,

$$\begin{cases} F(D^2 v, x) \le F(D^2 u, x) & \text{in} \quad \Omega \\ \\ \limsup_{y \to x} v(y) \le \liminf_{y \to x} u(y) & \text{on} \quad \partial\Omega \setminus \Gamma. \end{cases} \tag{3.4.1}$$

Then can we deduce that $v \le u$ in Ω, or more generally is there an estimate of the form,

$$\sup_{x \in K \subset\subset \Omega} (v - u)_+ \lesssim_K |\Gamma|,$$

where $|\cdot|$ is some measure of the size of Γ. Such an estimate, a form of the Alexandroff-Bakelman-Pucci estimate on the boundary, appears to be unknown even for linear F at least when $|\cdot| = \mathscr{H}^{n-1}(\cdot)$ the $n-1$ dimensional Hausdorff measure. We are able to prove such an estimate only for $|\cdot| = \mathscr{H}^{\beta_0}(\cdot)$ with some $\beta_0 < n-1$, possibly quite small depending on the ratio Λ/λ.

In order to prove an estimate of the above form we will use the singular solution that was constructed in [ASS12]. The following Theorem was proven there for even more general operators,

Theorem 3.4.1. (Theorem 1 from [ASS12]) *For any* $1 > \gamma > 0$ *let* $K_\gamma = \{x : x_n > (-1+\gamma)|x|\}$, *there exists a unique constant* $\beta_0(n,\lambda,\Lambda,\gamma) > \max\{\frac{\lambda}{\Lambda}(n-1)-1, 0\}$ *such that the problem,*

$$\begin{cases} -\mathscr{P}^+(D^2\Phi) = 0 & in \quad K_\gamma \\ \Phi = 0 & on \quad \partial K_\gamma \setminus \{0\} \end{cases} \tag{3.4.2}$$

has a positive solution $\Phi \in C(\overline{K}_\gamma \setminus \{0\})$ *which is* $-\beta_0$ *homogeneous, that is,*

$$\Phi(x) = |x|^{-\beta_0}\Phi(x/|x|).$$

Note that by the Evans-Krylov Theorem Φ is $C^{2,\alpha}$ on the interior of K_γ for some $\alpha \in (0,1)$, see [CC95]. For $\eta \in S^{n-1}$ let $\mathscr{O}_\eta$ be any rotation sending η to e_n and then we call,

$$\Phi_\eta(x) = \Phi(\mathscr{O}_\eta x). \tag{3.4.3}$$

In order to use the above barriers we will make the assumption that the domain Ω has the following exterior cone condition for some $0 < \gamma \leq 1$,

Definition 3.4.2. Let $0 < \gamma < 1$, then a bounded domain $\Omega \subset \mathbb{R}^n$ has the *strict* γ *exterior cone condition* if there exists $\rho > 0$ and $\gamma' > \gamma$ such that for every $x \in \partial\Omega$ there is a direction η_x with,

$$\{y : (y-x)\cdot\eta_x \leq (-1+\gamma')|y-x|\} \cap B_\rho(x) \subset \mathbb{R}^n \setminus \Omega.$$

Of course convex domains have the γ exterior cone condition for every $\gamma < 1$ since they have a supporting hyperplane at every boundary point. More generally any Lipschitz domain also has the strict γ exterior cone condition for some γ depending on the Lipschitz constant of $\partial\Omega$.

We will also use the notion Hausdorff dimension below, so we will recall the definition. The d-dimensional Hausdorff measure is defined in the following way for a subset $E \subset \mathbb{R}^n$,

$$\mathscr{H}^d(E) := \sup_{\delta>0} \inf\left\{\sum_{j=1}^{\infty} r_j^d : \exists x_j \in \mathbb{R}^n \text{ s.t. } E \subseteq \bigcup_{j\geq 1} B(x_j,r_j) \text{ and } r_j \leq \delta\right\}.$$

It is standard to check that $\mathscr{H}^d(E)$ is decreasing in d and $\mathscr{H}^d(E) = 0$ for $d > n$, this motivates us to define the Hausdorff dimension as,

$$\dim_{\mathscr{H}} E := \inf\{d \geq 0 : \mathscr{H}^d(E) = 0\}.$$

Theorem 3.4.3. *Let F satisfying (F1) and (F2). Suppose that Ω has the strict γ exterior cone condition for some $0 < \gamma < 1$ and let $\beta_0(n, \lambda, \Lambda, \gamma)$ as above from Theorem 1 of [ASS12]. Let $\Gamma \subset \partial\Omega$ such that $\mathscr{H}^{\beta_0}(\Gamma) = 0$, in particular this is the case if $\dim_{\mathscr{H}} \Gamma < \beta_0$. Then if u and v bounded and respectively upper and lower semicontinuous on the closure of Ω satisfy,*

$$\begin{cases} F(D^2 v, x) \leq F(D^2 u, x) & \text{in} \quad \Omega \\ \\ \limsup_{y \to x} v(y) \leq \liminf_{y \to x} u(y) & \text{on} \quad \partial\Omega \setminus \Gamma \end{cases} \tag{3.4.4}$$

then $u \leq v$ in Ω.

Remark 3.4.4. This Lemma is also true in the gradient dependent case as long as the operator is globally Lipschitz in the gradient argument. The only alteration is that the best estimate of β_0. In fact Theorem 1 of [ASS12] gives the existence of a singular solution in the gradient dependent case as well.

Proof. For simplicity we assume that the γ exterior cone condition holds globally, that is ρ from the definition of the γ exterior cone condition is equal to $+\infty$. A localization procedure using Lemma 3.2.12 will cover the general case. Let $w = u - v$ by the Comparison Theorem 3.2.11 w is a bounded subsolution of

$$\begin{cases} -\mathscr{P}^+(D^2 w) \leq 0 & \text{in} \quad \Omega \\ \\ \limsup_{y \to x} w(y) \leq 0 & \text{on} \quad \partial\Omega \setminus \Gamma. \end{cases} \tag{3.4.5}$$

Let Φ be the singular solution from Theorem 1 of [ASS12] corresponding to γ, then call $m = \min\{\Phi(\theta) : \theta \in K_{\gamma'} \cap S^{n-1}\} > 0$ and take $M > m^{-1}\|w\|_\infty$. Then from the definition of Hausdorff measure, $\mathscr{H}^{\beta_0}(\Gamma) = 0$ implies that for each $\delta > 0$ there exists $B(x_j, r_j)$ for $j \in \mathbb{N}$, an open covering of Γ by balls, so that $\sum_{j>0} r_j^{\beta_0} < \delta^2/M$. For each x_j let η_j be from the γ exterior cone condition for Ω and consider the following barrier, smooth in Ω,

$$\phi_\delta(x) = \delta + \sum_{j>0} M r_j^{\beta_0} \Phi_{\eta_j}(x - x_j).$$

We have the following convergence of the $\phi_\delta \searrow 0$,

$$0 \leq \phi_\delta(x) \leq 2\delta \ \text{ for } \ d(x, \partial\Omega) \gtrsim \delta^{1/\beta_0}.$$

Let us show that ϕ_δ is a strict smooth supersolution on the interior of Ω, since $\{y : (y-x_j)\cdot\eta_j \geq (-1+\gamma')|y-x_j|\} \subset \Omega^C$ for each j along with (3.4.2),

$$-\mathscr{P}^+(\phi_\delta) \geq \sum_{j>0} -Mr_j^{\beta_0}\mathscr{P}^+(D^2\Phi_{\eta_j}(x-x_j)) \geq 0.$$

Let us show that for each $x \in \partial\Omega$,

$$\limsup_{y\to x}\,(w-\phi_\delta)(y) \leq -\delta.$$

This is true by the assumption for $x \in \partial\Omega\setminus\Gamma$ since $\phi_\delta \geq \delta$ on $\partial\Omega$. For $x \in \Gamma$ we have that $x \in B(x_j, r_j)$ for some $j > 0$, so that, as above, for $y \in \Omega$ with $|y-x| < d(x, \partial B(x_j, r_j))$,

$$\phi_\delta(y) \geq \delta + Mr_j^{\beta_0}\Phi_{\eta_j}(y-x_j) \geq \delta + Mr_j^{\beta_0}mr_j^{-\beta_0} \geq \delta + \|w\|_\infty.$$

This proves the ordering of the Dirichlet data. Then since ϕ_δ is smooth in the interior of Ω simply by the definition of viscosity solution we have that $w \leq \phi_\delta$ in Ω. Letting $\delta \to 0$ gives the result.

$\square$

3.5 Homogenization in General Domains

Now we consider the homogenization of (3.1.1) in general domains Ω. We show that (3.1.1) homogenizes when $\partial\Omega$ does not have too many rational boundary points in the sense of Theorem 3.4.3. Unlike in the situation when Ω is a half space which was considered in [BM13] it is now necessary to make use of the continuity properties of the homogenized boundary condition μ with respect to $\sigma \in S^{n-1}$. The first reason is in order to guarantee that the homogenized problem,

$$\begin{cases} \overline{F}(D^2\overline{u}, x) = 0 & \text{in} \quad \Omega \\ \\ \overline{u}(x) = \overline{g}(x) & \text{on} \quad \partial\Omega. \end{cases} \tag{3.5.1}$$

has comparison. Here $\overline{F}$ is given by Theorem 3.2.10 and we will identify $\overline{g}(x)$ by the results of the previous section. This is where we will use the result of the previous section on comparison with partial boundary data.

In order to blow up at boundary points and use the results for the cell problem in a locally uniform way we will again need the stability of the rate of averaging at irrational directions. We make this precise in Lemma 3.5.2 which is based on Lemma 3.1 in [BM13], but first we make the following note on the behavior of F_ε under the blow up rescaling:

Lemma 3.5.1. *The following convergence holds:*

$$\varepsilon^2 F_\varepsilon(\varepsilon^{-2}M,x,y) \to F^x(M,y) \ \ as \ \ \varepsilon \to 0,$$

locally uniformly in M and uniformly in (x,y), with $F^x(M,y)$ satisfying assumptions (F1)-(F4). In particular we also have that if $x_j \to x$ then $F^x_j \to F^x$ locally uniformly in M and uniformly in y.

Proof. By our assumptions on F_ε in Section 3.2.3, there exists a collection of matrices, $\lambda I \le A^{\alpha\beta}(x,y) \le \Lambda I$, $A^{\alpha,\beta}$ have uniformly bounded Lipschitz norm and are periodic in their y variable, also there exists a function $f \in C^{0,1}(\mathbb{R}^n \times \mathbb{R}^n)$ periodic in it's second variable so that,

$$F_\varepsilon(M,x,y) = \inf_{\beta\in\mathscr{B}} \sup_{\alpha\in\mathscr{A}} \left[-\mathrm{Tr}(A^{\alpha\beta}(x,y)M)\right] + f(x,y).$$

This form of the operators makes the claim clear, and in this situation,

$$F^x(M,y) = \inf_{\beta\in\mathscr{B}} \sup_{\alpha\in\mathscr{A}} \left[-\mathrm{Tr}(A^{\alpha\beta}(x,y)M)\right]. \tag{3.5.2}$$

□

Lemma 3.5.2. (Behavior of u^ε away from the boundary layer) *Suppose that Ω is a bounded domain with $\partial\Omega$ being C^2. Let $x_0 \in \partial\Omega$ such that the interior normal σ_{x_0} is an irrational direction. For any $\delta > 0$ there exists R_0 and $\eta > 0$ such that for $R \ge R_0$ and for any sequence $x_\varepsilon \to x$ with $|x - x_0| \le \eta$,*

$$\limsup_{\varepsilon\to 0} |u^\varepsilon(x_\varepsilon + \varepsilon R\sigma_{x_0}) - \overline{g}(x)| \le \delta,$$

(in particular the quantity in the lim sup *is defined for ε sufficiently small) where the homogenized boundary data $\overline{g}$ is given by,*

$$\overline{g}(x) := \mu(\sigma_x, g(x,\cdot), F^x).$$

Proof. First we explain how to choose η. Given $\delta > 0$, and σ_{x_0} an irrational direction, there exists $N_0 > 1$ sufficiently large so that $\omega_{\sigma_{x_0}}(N_0) \lesssim \delta^{1/\alpha}$. Recalling that $\omega_\cdot$ is continuous at irrational directions and that σ_x is continuous in x since Ω is assumed to be a C^2 domain, let η be sufficiently small so that

$$|x - x_0| \leq \eta \;\text{ implies }\; \omega_{\sigma_x}(N) \lesssim \delta^{1/\alpha}.$$

Now fix $x \in \partial\Omega$ be fixed as above with $|x - x_0| \leq \eta$ as above and for every $\varepsilon > 0$ define

$$v^{\varepsilon}(y) := u^{\varepsilon}(x + \varepsilon y).$$

Let us call $\Omega^{\varepsilon} = \varepsilon^{-1}(\Omega - x)$. The v^{ε} satisfy the following equation in the viscosity sense,

$$\begin{cases} F_{\varepsilon}(\varepsilon^{-2}D^2 v^{\varepsilon}, x + \varepsilon y, \varepsilon^{-1}x + y) = 0 & \text{for} \quad y \in \Omega^{\varepsilon} \\ \\ v^{\varepsilon}(y) = g(x + \varepsilon y, \varepsilon^{-1}x + y) & \text{for} \quad y \in \partial\Omega^{\varepsilon}. \end{cases} \tag{3.5.3}$$

The $\Omega^{\varepsilon} \to P_x = P(0, \sigma_x)$ in Hausdorff distance when restricted to compact sets. Now we use the compactness afforded by the periodic setting. Let us call

$$[0,1)^n \ni \tau_{\varepsilon} = \varepsilon^{-1}x \mod \mathbb{Z}^n,$$

i.e. $\varepsilon^{-1}x = \tau_{\varepsilon} + z_{\varepsilon}$ with $z_{\varepsilon} \in \mathbb{Z}^n$. We may replace $\varepsilon^{-1}x$ in (3.5.3) above by τ_{ε} using the $\mathbb{Z}^n$ periodicity of F_{ε} and g. Then taking a subsequence of the ε the $\tau_{\varepsilon} \to \tau \in [0,1]^n$. Noting that $-\|g\|_{L^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)} \leq v^{\varepsilon} \leq \|g\|_{L^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)}$ we may take the upper and lower relaxed limits of v^{ε} along the above subsequence,

$$v^{*,\tau}(y) = \limsup{}^{*} v^{\varepsilon}(z) \quad \text{and} \quad v_{*,\tau}(y) = \liminf{}_{*} v_{\varepsilon}(z).$$

The upper and lower half-relaxed limits are defined for each $y \in \overline{P_x}$. Let us show that these are respectively sub and supersolutions of the limiting scale one equation,

$$\begin{cases} F^x(D^2 v_{\tau}, y + \tau) = 0 & \text{for} \quad y \in P_x \\ \\ v_{\tau}(y) = g(x, \tau + y) & \text{for} \quad y \in \partial P_x. \end{cases} \tag{3.5.4}$$

The interior sub/supersolution property follows from the stability of viscosity solutions under uniform convergence, Lemma 3.2.9. The condition on the boundary can be shown in the following way. Fix $y \in \partial P_x$ and $\delta > 0$. For ε sufficiently small, from the local Hausdorff distance convergence of $\partial\Omega^\varepsilon \to \partial P_x$, there exists $y' \in \partial\Omega_\varepsilon \cap B(y,\delta)$. Now for any $z \in \Omega^\varepsilon \cap B(y,\delta)$, by the estimates up to the boundary Lemma 3.2.16,

$$\begin{aligned}|v^\varepsilon(z) - g(x,\tau+y)| &\leq C\|g(x+\varepsilon\cdot,\tau_\varepsilon+\cdot)\|_{C^\alpha}|z-y'|^\alpha + |g(x+\varepsilon y',\tau_\varepsilon+y') - g(x,\tau+y)| \\ &\leq C(\|g\|_{C^\alpha})(\delta^\alpha + \varepsilon^\alpha|y'|^\alpha + |\tau_\varepsilon - \tau|^\alpha).\end{aligned}$$

Then taking the supremum over such z and letting $\varepsilon \to 0$ along any subsequence where $\tau_\varepsilon \to \tau$,

$$|v^{*,\tau}(y) - g(x,\tau+y)| \leq \lim_{\varepsilon\to 0} \sup_{z\in\Omega^\varepsilon\cap B(y,\delta)} |v^\varepsilon(z) - g(x,\tau+y)| \leq C\|g\|_{C^\alpha}\delta^\alpha.$$

Letting $\delta \to 0$ proves that $v^{*,\tau}$ achieves the boundary data in (3.5.4). The same argument works for $v_{*,\tau}$.

Since the equation (3.5.4) has comparison we derive $v^{*,\tau} \leq v_{*,\tau}$, of course the opposite inequality is always true so we get the local uniform convergence in P_x along subsequences of $v^\varepsilon \to v_\tau$ for some $\tau \in [0,1]^n$ depending on the subsequence. The point is that the limiting behavior of $v_\tau(y+R\sigma)$ will not depend on $\tau \in [0,1]^n$.

Let us consider $\widetilde{v}_\tau = v_\tau(y-\tau)$, these satisfy,

$$\begin{cases} F^x(D^2\widetilde{v}_\tau, y) = 0 & \text{for} \quad y \in P_x + (\tau\cdot\sigma_x)\sigma_x \\ \\ \widetilde{v}_\tau(y) = g(x,y) & \text{for} \quad y \in \partial P_x + (\tau\cdot\sigma_x)\sigma_x. \end{cases} \tag{3.5.5}$$

Then by Lemma 3.3.1, for any $N > 1$,

$$\sup_{\tau\in[0,1]^n} \sup_{y\in P_x+(\tau\cdot\sigma_x)\sigma_x} |\widetilde{v}_\tau(y+R\sigma_x) - \mu(\sigma_x, g(x,\cdot), F^x)| \lesssim \omega_{\sigma_x}(N)^\alpha + (N/R)^\alpha.$$

Given $\delta > 0$ fixing $N = N_0$ as in our choice of η above and letting $R_0 > 0$ sufficiently large we get,

$$\sup_{\tau\in[0,1]^n} \sup_{y\in\partial P_x+(\tau\cdot\sigma_x)\sigma_x} |\widetilde{v}_\tau(y+R\sigma_x) - \overline{g}(x)| \leq \delta.$$

Then for $R \geq 2R_0$, $\eta > 0$ sufficiently small by the continuity of σ_x so that $|\sigma_x - \sigma_{x_0}| \leq 1/2$, and each $\tau \in [0,1]^n$,

$$|u^\varepsilon(x+\varepsilon R\sigma_{x_0}) - \overline{g}(x)| \leq |u^\varepsilon(x+\varepsilon R\sigma_{x_0}) - v_\tau(R\sigma_{x_0})| + |v_\tau(R\sigma_{x_0}) - \overline{g}(x)| \leq |u^\varepsilon(x+\varepsilon R\sigma_{x_0}) - v_\tau(R\sigma)| + \delta,$$

and infimizing over τ and then taking $\limsup$ on both sides,

$$\limsup_{\varepsilon\to 0} |u^{\varepsilon}(x+\varepsilon R\sigma_{x_0}) - \overline{g}(x)| \leq \delta + \limsup_{\varepsilon\to 0} \inf_{\tau\in[0,1]^n} |u^{\varepsilon}(x+\varepsilon R\sigma_{x_0}) - v_{\tau}(R\sigma_{x_0})| = \delta.$$

Since the convergence of the u^{ε} is locally uniform along subsequences we may replace x by any sequence $x_{\varepsilon} \to x$ above. □

From Lemma 3.5.2 we will be able to deduce that homogenization of the boundary condition holds near boundary points with irrational directions. In general one cannot blow up to get an estimate of the form in Lemma 3.5.2 near boundary points with rational normal directions. However we have the comparison principle with partial boundary data from Theorem 3.4.3. This Theorem gives a condition under which satisfying the Dirichlet data only at boundary points with irrational normal directions is enough to deduce uniqueness.

Now let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary which has a strict γ exterior cone condition for some $0 < \gamma < 1$. Recalling that $\sigma_{\cdot} : \partial\Omega \to S^{n-1}$ is the inward unit normal vector field to $\partial\Omega$ we call the set of rational boundary points,

$$\Gamma = \Gamma(\Omega) := \{x \in \partial\Omega : \sigma_x \in \mathbb{R}\mathbb{Z}^n\}.$$

Let us suppose that Ω has no flat boundary pieces in any rational direction of too large Hausdorff dimension, precisely we assume,

$$\mathscr{H}^{\beta_0}(\Gamma) = 0,$$

where $n-1 > \beta_0(n,\lambda,\Lambda,\gamma) > \max\{0, \frac{\lambda}{\Lambda}(n-1)-1\}$ is the singular solution exponent from Theorem 1 in [ASS12]. In particular this is always true of uniformly convex domains which have an exterior half space at every boundary point and for which Γ will be countable. More generally any bounded C^2 domain such that the set of rational boundary points where $D_{\tau}\sigma_x$ (the tangential gradient) is degenerate has Hausdorff dimension smaller than β_0 will satisfy this condition.

Theorem 3.5.3. *Suppose that Ω satisfies the hypothesis given above. Let $\overline{g}(x) = \mu(\sigma_x, g(x,\cdot), F^x)$, then $\overline{g} : \partial\Omega \to \mathbb{R}$ is continuous at every point of $\partial\Omega \setminus \Gamma$, and the solutions u^{ε} of* (3.1.1) *converge locally uniformly in Ω to the unique, in the sense of Theorem 3.4.3, solution given by Perron's* *method $\overline{u}$ of*

$$\begin{cases} \overline{F}(D^2\overline{u},x) = 0 & in \quad \Omega \\ \overline{u}(x) = \overline{g}(x) & on \quad \partial\Omega \setminus \Gamma. \end{cases} \tag{3.5.6}$$

Proof. Let us first note the continuity of $\overline{g}$ at points of $\partial\Omega \setminus \Gamma$. Let $x_j \in \partial\Omega$ converging to $x \in \partial\Omega \setminus \Gamma$, then $\|g(x_j,\cdot) - g(x,\cdot)\|_{L^\infty} \leq C|x_n - x|^\alpha$, $\sigma_{x_n} \to \sigma_x$ since $\partial\Omega$ is C^2 and $F^{x_j}(M,y)$ converges to $F^x(M,y)$ locally uniformly in M and uniformly in y as $j \to \infty$. Then from the continuity of μ noted in lemma 3.3.3 (i) and (iii) and in lemma 3.3.4 we have the convergence,

$$\overline{g}(x_j) = \mu(g(x_j,\cdot), F^{x_j}, \sigma_{x_j}) \to \mu(\sigma_x, g(x,\cdot), F^x) = \overline{g}(x) \;\text{ as }\; j \to \infty. \tag{3.5.7}$$

This shows the claimed continuity of $\overline{g}$.

The main tool to show the homogenization is Lemma 3.5.2. Fixing $x_0 \in \partial\Omega \setminus S$, σ_{x_0} the interior normal to $\partial\Omega$ at x_0, $\delta > 0$ and $R \geq R_0(\delta)$ and $\eta = \eta(\delta) > 0$ from Lemma 3.5.2 we consider,

$$\widetilde{u}^\varepsilon(x) := u^\varepsilon(x + \varepsilon R\sigma_0) \;\text{ defined for }\; x \in \Omega - \varepsilon R\sigma_0.$$

Since the $\widetilde{u}^\varepsilon$ are uniformly bounded (by comparison with $\pm\|g\|_\infty$) the upper and lower relaxed limits are defined and finite in $\overline{\Omega}$,

$$\widetilde{u}^* := \limsup{}^* \widetilde{u}^\varepsilon \quad \text{and} \quad \widetilde{u}_* := \liminf{}_* \widetilde{u}^\varepsilon.$$

Note that for $x \in \Omega$ the upper (and lower) relaxed limits of $\widetilde{u}^\varepsilon$ and u^ε (which we call u^* and u_* respectively) agree. The only dependence on R, the manifestation of the boundary layer for u^ε, is on $\partial\Omega$. By our choice of R and η from Lemma 3.5.2 and the continuity shown above in (3.5.7) there exists $0 < \eta' \leq \eta$ sufficiently small so that for $x \in \partial\Omega$ with $|x - x_0| \leq \eta'$,

$$\widetilde{u}^*(x) \leq \overline{g}(x_0) + \delta \quad \text{and} \quad \widetilde{u}_*(x) \geq \overline{g}(x_0) - \delta,$$

and by the interior homogenization result for $x \in \Omega$ we get the inequalities,

$$\overline{F}(D^2\widetilde{u}^*,x) \leq 0 \quad \text{and} \quad \overline{F}(D^2\widetilde{u}_*,x) \geq 0.$$

Let $\phi_\pm$ be respectively super and subsolution barriers for the domain Ω at x_0. Then, as in the standard barrier argument, we consider

$$\widetilde{\phi}_+ = \frac{\|g\|_\infty}{\inf_{|x-x_0|\geq\eta} \phi_+}\phi_+ + (g(x_0) + \delta),$$

which satisfies $\widetilde{\phi}_+ \geq \widetilde{u}^*$ on $\partial\Omega$. By comparison $\widetilde{\phi}_+ \geq \widetilde{u}^*$ in Ω. In fact since $\phi_\pm$ can be chosen smooth this is even just by the definition of subsolution. In any case, since $u^* = \widetilde{u}^*$ in Ω we get,

$$\limsup_{z\to x_0} u^*(z) \leq \limsup_{z\to x_0} \widetilde{u}^*(z) \leq g(x_0) + \delta.$$

Making the same argument with ϕ_- and $\widetilde{u}_*$ we get,

$$\liminf_{z\to x_0} u_*(z) \geq g(x_0) - \delta.$$

Of course now the left hand side above is independent of δ, so we may take $\delta \to 0$ to get,

$$g(x_0) \leq \liminf_{z\to x_0} u_*(z) \leq \limsup_{z\to x_0} u^*(z) \leq g(x_0).$$

This argument works for every $x_0 \in \partial\Omega \setminus \Gamma$. Applying the comparison principle with partial boundary data Theorem 3.4.3 for $\overline{F}$ to u^* and u_* we get for $x \in \Omega$,

$$u^*(x) = u_*(x),$$

so that the u^ε converge locally uniformly in Ω to $\overline{u} = u^* = u_*$.

□

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CHAPTER 4

Continuity Properties of the Homogenized Boundary Condition

4.1 Introduction

We investigate the homogenization of oscillating Dirichlet boundary data problems,

$$\begin{cases} F(D^2u^\varepsilon, \frac{x}{\varepsilon}) = 0 & \text{in} \quad U \\ u^\varepsilon = g(\frac{x}{\varepsilon}) & \text{on} \quad \partial U. \end{cases} \tag{4.1.1}$$

Here $F(M,y)$ is uniformly elliptic, $g(y)$ is continuous, and both are $\mathbb{Z}^d$-periodic in y. The domain $U \subset \mathbb{R}^d$ is bounded and uniformly convex, although more general assumptions which rule out large flat portions of ∂U will also be sufficient for the results discussed below, see the previous chapter. In such domains we know from the previous chapter (or [Fel14]) that there exists $\overline{g} : S^{d-1} \to \mathbb{R}$, which may or may not be continuous, so that $u^\varepsilon \to \overline{u}$ locally uniformly in U where $\overline{u}$ is the unique solution of,

$$\begin{cases} \overline{F}(D^2\overline{u}) = 0 & \text{in} \quad U \\ \overline{u} = \overline{g}(x) & \text{on} \quad \partial U, \end{cases} \tag{4.1.2}$$

with $\overline{g}$ depending only on σ_x, the inward normal of U at $x \in \partial U$. There is no problem to include a large scale x dependence in g but we omit it here for clarity.

In this chapter we explore the continuity properties of the homogenized boundary data $\overline{g}$. In the Neumann case the continuity of the corresponding $\overline{g}$ has been studied by Choi-Kim-Lee and Choi-Kim [CKL13, CK14]. There it was shown that when the averaged operator $\bar{F}$ is rotation invariant, homogenization holds and the homogenized boundary data is continuous. Following these works [Fel14] showed homogenization for general F in the Dirichlet setting, due to the new observation that (4.1.2) has a unique solution if the discontinuity set of $\overline{g}(x) = \overline{g}(\sigma_x)$ on ∂U has sufficiently small Hausdorff dimension. This brings up the natural question of whether the homogenized boundary condition could in fact be discontinuous when $\bar{F}$ is not rotation-invariant. Our main results are (i) an explicit estimate of the continuity of $\overline{g}$ when $\overline{F}$ is rotation invariant or linear, (ii) when $\overline{F}$ is not rotation invariant or linear, $\overline{g}$ is 'generically' discontinuous at every boundary point with rational normal direction (see Theorem 4.1.2 and Corollary 4.1.3). These results seem to be new even in the linear case.

We expect our main results in this chapter to hold with parallel proofs in both the Dirichlet and Neumann case. On the other hand we hope to keep our illustration simple so that our main ideas are presented clearly. For this reason we will only discuss the Dirichlet problem, even though our arguments build on the framework introduced for the Neumann problem in [CK14]. We leave the task of proving parallel results for the Neumann problem, including the general homogenization results in [Fel14], for the future work.

We proceed to give a more precise, but still informal, description of our methods. We begin by reminding the reader of the derivation of the cell problem determining $\overline{g}$. We consider a rescaling of the solution u^ε of (4.1.1) near a boundary point $x_0 \in \partial U$ with unit inner normal σ_{x_0},

$$v^\varepsilon(y) = u^\varepsilon(x_0 + \varepsilon y).$$

The limit of $v^\varepsilon(R\sigma_{x_0})$ as $R \to \infty$ and $\varepsilon \to 0$, if it exists, will be the homogenized boundary data $\bar{g}(\sigma_{x_0})$ as long as $\varepsilon R \to 0$. The behavior of v^ε outside of the oscillating boundary layer is the quantity of interest. To proceed with the analysis we inspect the equation solved by the v^ε,

$$\begin{cases} F(D^2 v^\varepsilon, y + \varepsilon^{-1} x_0) = 0 & \text{in} \quad \varepsilon^{-1}(U - x_0) \\ v^\varepsilon = g(y + \varepsilon^{-1} x_0) & \text{on} \quad \varepsilon^{-1}(\partial U - x_0). \end{cases} \tag{4.1.3}$$

Since F and g are assumed to be $\mathbb{Z}^d$ periodic in y, $\varepsilon^{-1}x_0$ can be replaced by $\tau_\varepsilon = \varepsilon^{-1}x_0 \mod \mathbb{Z}^d$. Note that along various subsequences τ_ε could converge to any $\tau \in [0,1)^d$. This motivates the definition of the cell problem solution $v_{\sigma,\tau}(\cdot;(\psi,F))$ for a direction $\sigma \in S^{d-1}$, a $\tau \in [0,1)^d$ and a continuous $\mathbb{Z}^d$-periodic ψ by,

$$\begin{cases} F(D^2 v_{\sigma,\tau}, y + \tau) = 0 & \text{in} \quad P_\sigma := \{y \cdot \sigma > 0\} \\ v_{\sigma,\tau} = \psi(y + \tau) & \text{on} \quad \partial P_\sigma. \end{cases} \tag{4.1.4}$$

It is not too difficult to see, at least formally, that,

$$|v^{\varepsilon}(y) - v_{\sigma_{x_0},\tau_\varepsilon}(y; g(x_0,\cdot))| \to 0 \ \text{ as } \ \varepsilon \to 0.$$

From this identification we can replace understanding $\bar{g}(\sigma)$ with the easier problem of understanding the limit $v_{\sigma,\tau}(R\sigma)$ as $R \to \infty$ for every $\tau \in [0,1)^d$. For irrational directions σ the distribution of g on $P_\sigma + \tau\sigma$ is, in an appropriate sense, invariant with respect to τ. For this reason it was possible to show, in [CKL13, CK14, Fel14], that, for irrational directions σ, there exists a limit $\mu(\sigma,\psi,F)$, the so-called *boundary layer tail* of $v_{\sigma,\tau}$, such that

$$\sup_{\tau\in[0,1)^d} \sup_{y\in\partial P_\sigma} |v_{\sigma,\tau}(y+R\sigma;\psi) - \mu(\sigma,\psi,F)| \to 0 \ \text{ as } \ R \to \infty. \tag{4.1.5}$$

Note that in the context of (4.1.1) and (4.1.2) we should define $\bar{g}(\sigma) = \mu(g,F,\sigma)$. It was further shown in [CKL13, CK14, Fel14] that $\bar{g}(\sigma) : S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d \to \mathbb{R}$ is continuous, see Theorem 4.2.2. Thus the remaining question is to understand the limiting behavior of $\bar{g}(\sigma)$ as σ converges to a rational direction $\sigma_0 \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$.

The rate of convergence in (4.1.5) degenerates at rational directions where, in fact, the boundary layer tail does depend on τ. Indeed the rational directions σ are the possible discontinuity points of $\bar{g}$. It turns out that the asymptotic behavior near the rational directions is actually quite structured and a more careful analysis is warranted. We will show that there is a multi-scale homogenization occurring, near-boundary in the micro-scale and then further away from the boundary in an intermediate scale, as irrational directions approach a rational direction. This phenomenon, partially described previously in [CK14], leads to a secondary homogenization problem with its own 'cell problem' and 'effective operator'. This is far from obvious, and it will indeed be the main observation leading to the results of this chapter. Let us attempt to give a heuristic derivation of the secondary homogenization problem. The reader may wish to skip to the statement of the main results as the following description is unavoidably somewhat technical.

We begin with a lattice point $\xi \in \mathbb{Z}^d \setminus \{0\}$ and its associated unit direction $\hat{\xi}$. We may assume that ξ is irreducible in the sense that the greatest common divisor of its entries $\gcd(\xi_1,\dots,\xi_d)$ is 1. A $\mathbb{Z}^d$-periodic ψ on $\mathbb{R}^d$ restricts to ∂P_ξ to be periodic with respect to a lattice on ∂P_ξ with unit cell of size comparable to $|\xi|$ (by the irreducibility). The boundary layer tail exists by the periodicity of the boundary data, but the same argument applies to $\psi(\cdot+\tau)$ and, generically, the limit of $v_{\xi,\tau}$ is not the same unless $(\partial P_\xi+\tau) \bmod \mathbb{Z}^d = \partial P_\xi \bmod \mathbb{Z}^d$. We can concisely write down the set of limit points as,

$$m_\xi(t;(\psi,F)) := \lim_{R\to\infty} v_{\xi,t\hat{\xi}}(R\hat{\xi};(\psi,F)) \text{ which is a } \tfrac{1}{|\xi|}\text{-periodic function on } \mathbb{R}. \tag{4.1.6}$$

Now consider an irrational direction σ which is very close to $\hat{\xi}$. For a fixed $y_0 \in \partial P_\sigma$ the boundary ∂P_σ is close to $\partial P_\sigma + (y_0\cdot\xi)\hat{\xi}$ on a very large region of size $\sim |\sigma-\hat{\xi}|^{-1}$ and so the respective cell problem solutions are also close in a smaller region (See Figure 1 in section 4.1). This observation leads to the conclusion that, far from the boundary, v_σ averages similarly to the following *two-dimensional* problem:

$$\begin{cases} \overline{F}(D^2\overline{w}_{\xi,\eta}) = 0 & \text{in} \quad P_\xi \\ \overline{w}_{\xi,\eta} = m_\xi(y\cdot\eta;(\psi,F)) & \text{on} \quad \partial P_\xi, \end{cases} \tag{4.1.7}$$

where η is the “approaching direction” of σ to ξ,

$$\eta := -\exp_{\hat{\xi}}^{-1}(\sigma) \approx \hat{\xi}-\sigma. \tag{4.1.8}$$

The limit of the homogenized profile $\mu(\sigma,\psi,F)$ as $\sigma\to\hat{\xi}$ can then be identified in terms of the approaching direction η. In other words we can show that for given ψ, F, ξ and σ approaching ξ, there is a directional limit $L = L_\xi(\hat{\eta})$ with η as given in (4.1.8) such that

$$\mu(\sigma,\psi,F) - L_\xi(\hat{\eta}) \to 0 \text{ as } \sigma\to\hat{\xi}. \tag{4.1.9}$$

We refer to Section 4 for more precise statements.

The first half of the chapter is spent to rigorously justify this derivation of the secondary cell problem and to obtain a quantitative estimate on the asymptotics of v_σ near rational directions. When the effective operators L_ξ are constant for every rational direction the quantitative estimates allow us to derive an explicit modulus of continuity for the homogenized boundary condition. From the previous arguments in [CK14] and [Fel14] we know that L_ξ are constant, for instance, when $\bar{F}$ is either rotation invariant or linear.

The characterization of the asymptotic behavior near rational directions described in (4.1.7) and (4.1.9) also opens the possibility of proving discontinuity of μ. One would just need to show that L_ξ can be non-constant for some operator F, boundary condition ψ and lattice vector $\xi \in \mathbb{Z}^d \setminus \{0\}$. This simplicity turns out to be somewhat deceptive, as the situations where we can actually compute the boundary layer tail in (4.1.7) is when either the boundary data is trivial or the operator is linear, and L_ξ is constant in those cases. Another natural case to consider is when the operators are extremal, but then they are rotation invariant and thus L_ξ is continuous. While it is difficult to come up with any specific example, it turns out to be much more tractable to show that a *generic* operator and boundary data will result in non-constant L_ξ. In fact we are able to argue that *if* L_ξ were to be constant, then we would be able to find many nearby F', ψ' with L'_ξ non-constant. The perturbation of the operator is monotone and hence intrinsically nonlinear, and is designed to affect $\overline{F}$ in one direction $\eta \perp \xi$ while leaving another direction $\eta' \perp \eta, \xi$ unaffected. The existence of η, η' mutually orthogonal and orthogonal to ξ requires $d \geq 3$, and we are only able to achieve the desired perturbation of $\overline{F}$ when $F = \overline{F}$ is homogeneous.

4.1.1 Main Results

The operators $F(M,y)$ discussed below will be positively 1-homogeneous, uniformly elliptic with ellipticity ratio Λ, and $\mathbb{Z}^d$ periodic in y. When F is linear we will write $F(M,y) = -\mathrm{Tr}(A(y)M)$. If we say that F is spatially homogeneous we mean that $F = \overline{F}$ has no y dependence. For more details on these assumptions see Section 4.2.2.

First we state our result about continuity. More details can be found in Section 4.5, for the improved estimate in the linear case see Section 4.7.

Theorem 4.1.1. *Let $d \geq 2$ and F such that (i) the homogenized operator $\overline{F}$ is rotation invariant or (ii) F is linear. Then there exists $\alpha = \alpha(\Lambda) \in (0,1)$ such that for any $\beta \in (0,1)$, $\psi \in C^{0,\beta}(\mathbb{T}^d)$ and $\sigma, \sigma' \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ we have*

$$|\mu(\sigma,\psi,F) - \mu(\sigma',\psi,F)| \leq C(d,\Lambda,\beta)\|\psi\|_{C^{0,\beta}(\mathbb{T}^d)}|\sigma - \sigma'|^{\alpha\beta/d}.$$

In case (ii) *we have additionally,*

$$|\mu(\sigma,\psi,F)-\mu(\sigma',\psi,F)| \leq C(d,\Lambda,\|A\|_{C^5(\mathbb{T}^d)})\|\psi\|_{C^7(\mathbb{T}^d)}|\sigma-\sigma'|^{1/d}[1+(\log\tfrac{1}{|\sigma-\sigma'|})^3].$$

For linear, divergence form systems a mode of continuity for $\mu(\cdot,\psi,F)$ is obtained by Gérard-Varet and Masmoudi [GM12] on the set of Diophantine irrational directions. Our result on the other hand is based on the mode of continuity near rational directions, and the modulus of continuity we obtain is uniform on the entire sphere. In the linear case it may be possible to combine these two results, but we do not pursue this here.

Next we state our result about discontinuity. The statement is not completely precise, see Section 4.6 for the full details.

Theorem 4.1.2. *For* $d \geq 3$*, there is a residual set (in the Baire category sense) of continuous boundary conditions and spatially homogeneous nonlinear operators* (ψ,F) *such that* $\mu(\cdot,\psi,F)$ *does not extend continuously at any rational direction.*

The following question is left open.

Open Problem. *Does Theorem 4.1.2 hold when* (i) F *is taken to be inhomogeneous or* (ii) $d=2$*?*

The argument used in the proof of Theorem 4.1.2 appears to be insufficient to address the above questions, however we do believe that the theorem holds in both cases (i) and (ii).

The above results can be easily translated in terms of the original problem (4.1.2), since $\overline{g}(x) = \mu(\sigma_x,g,F)$ for $x \in \partial U$ with $\sigma_x \in S^{d-1}\setminus\mathbb{R}\mathbb{Z}^d$. The details can be found in [Fel14].

Corollary 4.1.3. *Let* $U \subset \mathbb{R}^d$ *be bounded uniformly convex domain, and let* $\bar{g}$ *be as given in* (4.1.2)*. Then the following holds:*

(a) Suppose $\overline{F}$ *is rotation invariant or linear, then the homogenized boundary data* $\bar{g}(x)$ *extends to be Hölder continuous on* ∂U*, with mode of continuity in* σ_x *as given in Theorem 4.1.1.*

(b) Let $d \geq 3$*. Then there is a residual set of* g *and* F *in* (4.1.1)*, in the sense of Theorem 4.1.2 such that* $\bar{g}$*, and hence* $\overline{u}$ *as well, are discontinuous at every* $x \in \partial U$ *with* σ_x *a rational direction.*

4.1.2 Literature

There has been a surge of recent interest in the homogenization of oscillating boundary conditions, both in the linear divergence form and nonlinear non-divergence form settings. These works have much in common but there are some key differences which necessitate differing approaches.

The problem is first addressed in the book of Benssoussan, Papanicolaou and Lions [BLP78], which considers linear divergence form operators with co-normal oscillating Neumann boundary condition in general domains with no flat sides. The case of an oscillating Dirichlet boundary condition remained mostly open for quite a long time. For linear, divergence form systems recent progress began with the works of Gérard-Varet and Masmoudi [GM11, GM12] where they show homogenization of the oscillating Dirichlet boundary condition problem with an explicit rate of convergence in $L^2(U)$. In that setting they show that the cell problem homogenizes at normal directions satisfying a Diophantine condition and that the rate of convergence to the boundary layer tail is better than polynomial. Continuing this investigation, in the direction of improved rates of convergence, are the works of Aleksanyan, Sjölin and Shahgholian [ASS13, ASS14b, ASS14a]. They identify the expected optimal L^p convergence rate in general domains and obtain this rate under certain assumptions on the inhomogeneity of the operator. In a slightly different direction is the work of Prange [Pra13] which extends the results of [GM11, GM12] to include all irrational directions. He shows that the convergence to the boundary layer tail can occur at an arbitrarily slow polynomial rate without the Diophantine assumption. Perhaps the most relevant work to this chapter of the dissertation is a recent result of Aleksanyan [Ale14] on the continuity of the homogenized boundary condition. He shows for layered media, where the operator is independent of translations in the e_d direction, that the homogenized boundary condition is as regular as the boundary data ψ away from a possible singular set on $x_d = 0$. Compared to his result, we do not rely on any structure assumption on the operator, but on the other hand we obtain only Hölder-$\frac{1}{d}$ continuity in the linear case. It should be remarked that our result is in the non-divergence setting, nonetheless it may be possible for our approach to carry over to the setting of linear systems.

Next we discuss the nonlinear, non-divergence form operators. For nonlinear operators there are several significant differences from the linear case. Firstly, due to the blow up procedure leading to the cell problem, the operators in the cell problem will always be positively homogeneous and therefore non-smooth at 0 (or linear). This makes the cell problem inherently impossible to linearize and so no regularity estimates better than $C^{1,\beta}$ (or $C^{2,\beta}$ in the convex case) should be expected. On the other hand, higher regularity seems to be essential to obtaining arbitrary polynomial rate of convergence to the boundary layer tail at irrational directions as was done in the linear case by [GM12]. For these reasons obtaining arbitrary polynomial rates of convergence for the cell problem seems quite difficult if not impossible in the nonlinear case. The second problem, explicated for the first time in the paper that this chapter of the dissertation is based on [FK15], is that the homogenized boundary condition can be discontinuous. For linear operators a discontinuous boundary condition does not pose such a serious issue because, by the Green's function representation the interior values of the homogenized solution can be estimated by measure theoretic norms of the homogenized boundary condition. In the nonlinear case no such "boundary ABP" estimate is known and so the uniqueness and stability of the homogenized problem is at issue (see [Fel14] for a partial resolution to this problem). In regards to the literature, most earlier works address the Neumann problem: some special cases were discussed in Arisawa [Ari03] in the half-space setting with periodic boundary data, and also by Tanaka [Tan84] using probabilistic methods. More general results were proved later by Barles, Da Lio, Lions and Souganidis [BDL08]. Only just recently the full problem in general domains was considered by Choi, Kim [CK14] and Choi, Kim and Lee [CKL13], wherein they show continuity of the homogenized Neumann boundary condition for rotationally invariant operators. For the Dirichlet problem Barles and Mironescu [BM13] obtained homogenization in half-spaces for a general class of nonlinear operators. The full problem in general domains was then considered by Feldman in [Fel14]. The random case was considered in Feldman, Kim and Souganidis [FKS14]. We also mention the recent work by Guillen and Schwab [GS14], where the half-space Neumann problem has been formulated as an interior homogenization problem for nonlinear non-local operators.

4.1.3 Outline of the Chapter

In Section 2 we start with notations and preliminary results to be used later in this chapter. In Section 3 we prove the exponential rate of convergence for the half-space cell problem, when the boundary data is periodic on the boundary. While the proof is relatively straightforward, our result appears to be new for nonlinear operators. In Section 4 we investigate the behavior of the homogenized boundary condition $\mu(\sigma,\psi,F)$ as σ approaches a rational direction $\hat{\xi}$ with $\xi \in \mathbb{Z}^d \setminus \{0\}$. We derive a second boundary homogenization problem that governs the directional limits as σ approaches $\hat{\xi}$. In Section 5 we prove Theorem 4.1.1 as a consequence of estimates in Section 4 and the Dirichlet's Theorem (Theorem 4.2.7). In Section 6 we show that, when F is nonlinear, μ is generically discontinuous (Theorem 4.1.2). Finally in Section 7 we show that when F is linear and ψ is sufficiently regular $\mu(\cdot,\psi,F)$ is Hölder-$\frac{1}{d}$ continuous up to logarithmic factors. In the Appendix we prove an extension of the result in section 3, which we make use of in section 7.

4.2 Preliminaries

This section contains notational conventions, fixing of the assumptions on the pde operators, statements of previously known results, and proofs of several technical Lemmas. The material here will be used throughout this chapter and we suggest that the reader refer back as needed to this section rather than begin a careful reading here.

4.2.1 Notation

We denote the half space with inner normal σ by $P_\sigma = \{y : y \cdot \sigma > 0\}$. For a vector $e \in \mathbb{R}^d$, $\hat{e}$ is the unit vector in the same direction $\hat{e} = |e|^{-1}e$. We will occasionally need to project a vector e onto the orthogonal complement of another vector $f \in \mathbb{R}^d$, this we denote,

$$\Pi_{f^\perp} e = e - (e \cdot \hat{f})\hat{f}.$$

We say that a constant $C > 0$ is universal if it depends only on the ellipticity ratio Λ and the dimension d. These constants may change from line to line without comment. If we need to refer to a specific universal constant which is not changing between lines we may call it C_0 or C_1. For two quantities A,B we write $A \lesssim B$ if $A \leq CB$ for a universal constant C. If C additionally depends on a parameter b which is not universal then we will write $A \lesssim_b B$.

We will work with the function spaces of Hölder continuous functions $C^{k,\beta}(X)$ for $k \in \mathbb{N} \cup \{0\}$ and $\beta \in (0,1]$ with (X,d) a complete separable metric space. Most often $X = \mathbb{T}^n = \mathbb{R}^n \mod \mathbb{Z}^n$ with metric inherited from Euclidean distance on $\mathbb{R}^n$. We will repeatedly use the Hölder semi-norm and norm for $\beta \in (0,1]$, for a $\phi : X \to \mathbb{R}$,

$$|\phi|_{C^{0,\beta}(X)} := \sup_{x \neq y \in X} \frac{|\phi(x)-\phi(y)|}{d(x,y)^\beta} \text{ and } \|\phi\|_{C^{0,\beta}(X)} = \sup_{x \in X} |\phi(x)| + |\phi|_{C^{0,\beta}(X)}.$$

On $\mathbb{R}^n$ (or $\mathbb{T}^n$) the norms for the higher order Hölder spaces are defined inductively for $k \geq 1$ and $\beta \in (0,1]$ by,

$$\|\phi\|_{C^{k,\beta}(\mathbb{R}^n)} = \|\phi\|_{C^{k-1,1}(\mathbb{R}^n)} + |D^k \phi|_{C^{0,\beta}(\mathbb{R}^n)}.$$

4.2.2 Operators

We will work in the class of fully nonlinear uniformly elliptic equations. Let $\mathscr{M}_{d\times d}$ be the class of $d \times d$ real symmetric matrices. For $F : \mathscr{M}_{d\times d} \to \mathbb{R}$ we say F is uniformly elliptic if there exist $0 < \lambda < \Lambda$ so that,

$$\lambda \mathrm{Tr}(N) \leq F(M) - F(M+N) \leq \Lambda \mathrm{Tr}(N) \text{ for all } M,N \in \mathscr{M}_{d\times d} \text{ with } N \geq 0. \tag{4.2.1}$$

Then we define the class of uniformly elliptic operators,

$$\mathscr{S}_{\lambda,\Lambda} = \{F : \mathscr{M}_{d\times d} \to \mathbb{R} : (4.2.1) \text{ holds } \}$$

We will assume the following:

(i) There is some $\Lambda > 0$ so that $F(\cdot,y) \in \mathscr{S}_{1,\Lambda}$ for all $y \in \mathbb{T}^d$.

(ii) F is Lipschitz continuous in y,

$$|F(M,y) - F(M,z)| \leq C(1+\|M\|)|y-z|.$$

(iii) F is positively 1-homogeneous,

$$F(tM,y) = tF(M,y) \text{ for all } t > 0.$$

We note that under the above assumptions $F(M,y)$ is in fact an Isaacs operator,

$$F(M,y) = \inf_{a \in \mathscr{A}} \sup_{b \in \mathscr{B}} -\mathrm{Tr}(A^{ab}(y)M) \;\text{ with }\; 1 \leq A^{ab}(y) \leq \Lambda.$$

4.2.3 Regularity in Two Dimensions

In $d \geq 3$ it is not known in general whether the solutions of fully nonlinear uniformly elliptic equations are smooth, examples of non-classical viscosity solutions in high dimensions ($d \geq 12$) have been given by Nadirashvili and Vlăduţ [NV07, NV08] . On the other hand in $d = 2$ it is a classical result of Nirenberg [Nir53] that solutions are $C^{2,\alpha}$ for a small α. We will be able to use this result because the asymptotics near rational directions of the homogenized boundary condition in any dimension naturally turn out to be determined by a two-dimensional problem. We state the result using more modern terminology, but our statement follows easily from Nirenberg's theorem in [Nir53].

Theorem 4.2.1 (Nirenberg)**.** *There exists $\alpha(\Lambda) \in (0,1)$ and $C(\Lambda) > 0$ so that if $u : B_1 \to \mathbb{R}$ is a viscosity solution of $F(D^2u) = 0$ in B_1 for some $F \in S_{1,\Lambda}$ then*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(\Lambda) \operatorname*{osc}_{B_1} u.$$

4.2.4 Results from Homogenization Theory

First we describe the results obtained in [Fel14] regarding the cell problem, the Neumann counterpart is in [CK14, CKL13]. Let $v_{\sigma,\tau}(\cdot;(\psi,F))$ solve the cell problem

$$\begin{cases} F(D^2 v_{\sigma,\tau}, y) = 0 & \text{in} \quad P_\sigma \\ v_{\sigma,\tau} = \psi(y+\tau) & \text{on} \quad \partial P_\sigma. \end{cases} \tag{4.2.2}$$

The following result says that, when σ is irrational, $v_{\sigma,\tau}$ has a limit as $y \cdot \sigma \to \infty$ and the limit is independent of τ.

Theorem 4.2.2 (Theorem 1.2 of [Fel14])**.** *For $\sigma \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ there exists $\mu(\sigma, \psi, F)$, called the boundary layer tail or homogenized boundary condition, such that,*

$$\sup_{\tau \in [0,1)^d} \sup_{y \in \partial P_\sigma} |v_{\sigma,\tau}(y + R\sigma) - \mu| \to 0 \;\text{ as }\; R \to \infty.$$

Moreover $\mu(\cdot,\psi,F)$ is continuous on $S^{d-1}\setminus\mathbb{RZ}^d$.

We will also need a rate of interior homogenization. In general this can be derived by the same methods used by Caffarelli-Souganidis [CS10] (also see Armstrong-Smart [AS14]). However for the purposes of this chapter we will only require an interior homogenization rate in the special situation where the solution of the homogenized problem in consideration is $C^{2,\alpha}$ due to our two dimensional reduction and Theorem 4.2.1. In this case it is straightforward to obtain a rate of convergence, so we provide the proof.

For $\sigma\in S^{d-1}$ and $R>0$, we consider the homogenization problem,

$$\begin{cases} F(D^2u^\varepsilon,\frac{x}{\varepsilon})=0 & \text{in} \quad 0<x\cdot\sigma<R \\ u^\varepsilon=g(x) & \text{on} \quad x\cdot\sigma\in\{0,R\} \end{cases} \text{which homogenizes to} \begin{cases} \overline{F}(D^2\overline{u})=0 & \text{in} \quad 0<x\cdot\sigma<R \\ \overline{u}=g(x) & \text{on} \quad x\cdot\sigma\in\{0,R\}, \end{cases} \tag{4.2.3}$$

where we are considering $g:\mathbb{R}^d\to\mathbb{R}$ to be bounded and continuous. Suppose g satisfies

$$g(x)=g_0(x\cdot\eta,x\cdot\sigma)\ \text{ for some unit vector } \eta\perp\sigma \ \text{ and some } \ g_0:\mathbb{R}^2\to\mathbb{R}. \tag{4.2.4}$$

Then by uniqueness $\overline{u}(x+t\zeta)=\overline{u}(x)$ for any $\zeta\perp\operatorname{span}\{\sigma,\eta\}$. In particular $\overline{u}(t\sigma+s\eta)$ actually solves a fully nonlinear uniformly elliptic problem in $d=2$ and hence has interior C^{2,α_0} estimates by Theorem 4.2.1.

We recall, for example from Evans [Eva89], that for each $M\in\mathscr{M}_{d\times d}$ there is a unique constant $\overline{F}(M)$ and a unique (modulo constants) $\mathbb{Z}^d$-periodic bounded solution $v(y;M)$ of

$$F(M+D^2v,y)=\overline{F}(M)\ \text{ in }\ \mathbb{R}^d, \tag{4.2.5}$$

satisfying $\|v(\cdot;M)\|_{L^\infty}\leq C(\Lambda,d)\|M\|$. Again from [Eva89], $\overline{F}$ turns out to be uniformly elliptic with the same ellipticity ratio Λ as $F(M,y)$.

Theorem 4.2.3. *Let u^ε,u,g be as given in (4.2.3) and (4.2.4). There exists $0<\alpha(\Lambda)<1$ such that for any $\beta\in(0,1)$ and any $R>0$,*

$$\sup_{0<x\cdot\sigma<R}|u^\varepsilon(x)-\overline{u}(x)|\leq C(\Lambda,d)(\operatorname*{osc}_{\partial U}g+R^\beta|g|_{C^{0,\beta}})(R^{-1}\varepsilon)^{\alpha\beta}.$$

Proof. After rescaling we may assume that $R=1$ and $U=\{0<x\cdot\sigma<1\}$. Let $\delta\in(0,\operatorname{osc}_{\partial U} g]$ to be chosen later and $\overline{u}^\delta$ solve

$$\begin{cases} \overline{F}(D^2\overline{u}^\delta)=\delta & \text{in} \quad U \\ \overline{u}^\delta=g(x) & \text{on} \quad \partial U. \end{cases} \tag{4.2.6}$$

By comparing with a parabola we have

$$\sup_U |\overline{u}^\delta-\overline{u}|\leq C\delta.$$

We will construct a supersolution barrier function based on $\bar{u}^\delta$ to compare with u^ε away from the boundary. We begin by collecting uniform estimates on $\overline{u}^\delta$. Let $h>0$ to be chosen small and call $U_h=\{x:d(x,\partial U)>h\}$. By the $C^{0,\beta}$ estimates up to the boundary for both $\overline{u}$ and u^ε at unit scale,

$$\|\overline{u}^\delta\|_{C^{0,\beta}(U)}+\|u^\varepsilon\|_{C^{0,\beta}(U)}\leq C\|g\|_{C^{0,\beta}(\partial U)}.$$

Moreover due to Theorem 4.2.1 we have

$$|D^2\overline{u}^\delta(x)|\leq C(h^{-2}\operatorname*{osc}_{B_h(x)}\overline{u}^\delta+\delta h^2)\leq Ch^{\beta-2}\|g\|_{C^{0,\beta}} \ \text{ for } \ x\in U_h$$

and similarly,

$$|D^2\overline{u}^\delta(x)|_{C^{2,\alpha_0}(U_h)}\leq Ch^{\beta-(2+\alpha_0)}\|g\|_{C^{0,\beta}}.$$

Note that for $x\in U\setminus U_h$ there is $y\in\partial U$ with $|y-x|\leq h$ and thus

$$|\overline{u}^\delta(x)-u^\varepsilon(x)|\leq|\overline{u}^\delta(x)-g(y)|+|u^\varepsilon(x)-g(y)|\leq C\|g\|_{C^{0,\beta}(\partial U)}h^\beta.$$

We wish to show that, in fact, the maximum of $u^\varepsilon(x)-\overline{u}^\delta(x)$ in U is obtained in $U\setminus U_h$. Suppose otherwise, then there exists $x_0\in U\setminus U_h$ such that

$$u^\varepsilon(x_0)-\overline{u}^\delta(x_0)=\max_U(u^\varepsilon-\overline{u}^\delta). \tag{4.2.7}$$

In particular $\overline{u}^\delta(x)+u^\varepsilon(x_0)-\overline{u}^\delta(x_0)$ touches u^ε from above at x_0. Let us define the barrier function

$$\phi^\varepsilon(x):=u^\varepsilon(x_0)+D\overline{u}^\delta(x_0)\cdot(x-x_0)+\tfrac{1}{2}(x-x_0)\cdot D^2\overline{u}^\delta(x_0)(x-x_0)+\varepsilon^2v(\tfrac{x}{\varepsilon};D^2\overline{u}^\delta(x_0))+\tfrac{\delta}{2\Lambda}|x-x_0|^2,$$

where v is the corrector given in (4.2.5). Note that

$$|\phi^\varepsilon(x_0)-u^\varepsilon(x_0)|\leq C_0\|D^2\overline{u}^\delta\|_{L^\infty(U_h)}\varepsilon^2. \tag{4.2.8}$$

One can verify that, using the uniform ellipticity and the definition of the corrector,

$$F(D^2\phi^\varepsilon(x), \tfrac{x}{\varepsilon}) \geq \overline{F}(D^2\overline{u}^\delta(x_0)) - \tfrac{1}{2}\delta \geq \tfrac{1}{2}\delta.$$

Let us choose

$$r = \min[\varepsilon^{\frac{2}{2+\alpha_0}}(\|D^2\overline{u}^\delta\|_{L^\infty(U_h)}/|D^2\overline{u}^\delta|_{C^{0,\alpha_0}(U_h)})^{\frac{1}{2+\alpha_0}}, h].$$

We claim that, for δ sufficiently small and C_0 from (4.2.8),

$$\phi^\varepsilon(x) \geq u^\varepsilon(x) + 2C_0\|D^2\overline{u}^\delta\|_{L^\infty(U_h)}\varepsilon^2 \text{ on } \partial B_r(x_0). \tag{4.2.9}$$

The comparison principle then would yield that the same inequality holds in $B_r(x_0)$, yielding a contradiction to (4.2.8). We now verify that δ can be chosen so that (4.2.9) holds. Using (4.2.7) we have

$$\phi^\varepsilon(x) \geq u^\varepsilon(x) + \tfrac{\delta}{2\Lambda}|x-x_0|^2 - C\|D^2\overline{u}^\delta\|_{L^\infty(U_h)}\varepsilon^2 - C|D^2\overline{u}^\delta|_{C^{0,\alpha_0}(U_h)}|x-x_0|^{2+\alpha_0},$$

we have chosen r above so that the last two terms are of the same size on $\partial B_r(x_0)$. Thus, evaluating this on $\partial B_r(x_0)$ we have

$$\phi^\varepsilon(x) \geq u^\varepsilon(x) + \tfrac{\delta}{2\Lambda}\varepsilon^{\frac{4}{2+\alpha_0}}(\|D^2\overline{u}^\delta\|_{L^\infty(U_h)}/|D^2\overline{u}^\delta|_{C^{0,\alpha_0}(U_h)})^{\frac{2}{2+\alpha_0}} - C_1\|D^2\overline{u}^\delta\|_{L^\infty(U_h)}\varepsilon^2.$$

Now suppose $r < h$ and let us choose

$$\delta \leq C\|g\|_{C^{0,\beta}} \max\{\varepsilon^{\frac{2\alpha_0}{2+\alpha_0}}h^{\beta-(2+\alpha_0)}, h^{\beta-4}\varepsilon^2\}, \tag{4.2.10}$$

then due to the regularity estimates on $\bar{u}^\delta$ given above we arrive at (4.2.9). If $r = h$ then $\delta = Mh^{\beta-4}\varepsilon^2$ to get the same contradiction. By a parallel argument we can show that the same choice of δ will result in the minimum of $u^\varepsilon(x) - \overline{u}^\delta(x)$ occurring in $U \setminus U_h$.

Now we put together the bounds obtained above. Since the maximum and minimum of $\overline{u}^\delta - u^\varepsilon$ are achieved in $U \setminus U_h$ for δ as above,

$$\sup_U |\overline{u} - u^\varepsilon| \leq \sup_U |\overline{u} - \overline{u}^\delta| + \sup_{U\setminus U_h} |\overline{u}^\delta - u^\varepsilon| \leq C\delta + C\|g\|_{C^{0,\beta}}h^\beta.$$

Using δ as chosen in (4.2.10),

$$\sup_U |\overline{u} - u^\varepsilon| \leq C\|g\|_{C^{0,\beta}}(\varepsilon^{\frac{2\alpha_0}{2+\alpha_0}}h^{\beta-(2+\alpha_0)} + h^{\beta-4}\varepsilon^2 + h^\beta).$$

By choosing $h = \varepsilon^{\frac{2\alpha_0}{(2+\alpha_0)^2}}$ we arrive at

$$\sup_U |\overline{u} - u^\varepsilon| \le C\|g\|_{C^{0,\beta}} \varepsilon^{\beta \frac{2\alpha_0}{(2+\alpha_0)^2}}.$$

$\square$

4.2.5 Continuity up to the Boundary

We will use the following result repeatedly in what follows. It is a fundamental technical tool used in estimating the difference between cell problem solutions in nearby half-spaces. The result addresses the continuity-up-to-the boundary for solutions of the Dirichlet problem, but it can be also viewed as a localization result.

Lemma 4.2.4. *Suppose that $\omega : [0,\infty) \to [0,\infty)$ is a modulus of continuity and $u \le \omega(1)$ satisfies,*

$$\begin{cases} -\mathscr{P}^+_{1,\Lambda}(D^2 u) \le 0 & \text{in} \quad B_1 \cap K \\ u(x) \le \omega(|x|) & \text{on} \quad B_1 \cap \partial K \end{cases}$$

where $0 \in \partial K$ and $B_1 \cap K \subset B_1^+$. Then there is a modulus $\bar{\omega}$ depending on Λ, d and ω such that

$$u(x) \le C(\Lambda, d)\bar{\omega}(|x|) \text{ in } B_1 \cap K.$$

If $\omega(r) = r^\beta$ for some $\beta \in (0,1)$ then $\bar{\omega}(r) = C(\Lambda, d, \beta) r^\beta$.

Because it is not obvious how to calculate $\bar{\omega}$ for general ω we will work with Hölder continuous boundary conditions throughout so that we get explicit estimates. The generalizations to arbitrary ω present only notational difficulties.

Proof. By rescaling, without loss $\omega(1) = 1$. Let ϕ be a positive, smooth function in $\overline{B_1^+}$ satisfying

$$\begin{cases} -\mathscr{P}^+_{1,\Lambda}(D^2 \phi) \ge 0 & \text{in} \quad B_1 \cap K \\ \phi \ge 1 & \text{on} \quad \partial B_1 \cap K \end{cases} \quad \text{and } \phi(x) \le C(\Lambda, d)|x| \text{ in } B_1 \cap K.$$

For example one can choose ϕ to be a rescaled translation of the fundamental solution for the Pucci operator. For each $r > 0$ and an $M > 1$ to be chosen large consider the barrier,

$$\phi_r(x) = \omega(Mr) + (\sup_{B_{Mr} \cap K} u)_+ \phi((Mr)^{-1} x).$$

Then $\phi_r(x) \geq (\sup_{B_{Mr}\cap K} u) \geq u$ on $\partial B_{Mr} \cap K$ and

$$\phi_r(x) \geq \omega(Mr) \geq \omega(|x|) \geq u(x) \text{ on } \partial K \cap B_{Mr},$$

since ω is monotone. By comparison principle $u \leq \phi_r$ in B_{Mr} and therefore it follows that

$$\sup_{B_r\cap K} u(x) \leq \omega(Mr) + (\sup_{B_{Mr}\cap K} u)_+ \phi(\tfrac{1}{M}). \tag{4.2.11}$$

To get a modulus of continuity, let $\varepsilon > 0$, choose $M \geq 2\varepsilon^{-1}C(\Lambda,d)$ and then choose r sufficiently small to make the right hand side in (4.2.11) less than or equal to ε.

On the other hand, since the argument is valid for every $r > 0$, applying the estimate repeatedly up until $M^{n+1}r \geq 1$,

$$\sup_{B_r\cap K} u(x) \leq \sum_{j=1}^{n-1} \omega(M^j r)\phi(\tfrac{1}{M})^{j-1} + (\sup_{B_1\cap K} u)_+ \phi(\tfrac{1}{M})^{n-1}$$

In case $\omega(r) = r^\beta$ for some $\beta \in (0,1)$, choose $M^{1-\beta} = 2C(\Lambda,d)$ so that

$$\begin{aligned}
\sup_{B_r\cap K} u(x) &\leq Mr^\beta \sum_{j=1}^{n-1} C(\Lambda,d)^j M^{-(1-\beta)j} + (\sup_{B_1\cap K} u)_+ (M^{-(1-\beta)}C(\Lambda,d))^{n-1} M^{-(n-1)\beta} \\
&\leq 2Mr^\beta + (\sup_{B_1\cap K} u)_+ 2^{-(n-1)} M^{2\beta} r^\beta \\
&\leq C'(\Lambda,d,\beta)(1 + (\sup_{B_1\cap K} u)_+) r^\beta.
\end{aligned}$$

□

4.2.6 Some Number Theory

Lastly we present some elementary number theoretic results which we will make use of. When $\sigma \in \mathbb{R}\mathbb{Z}^d$ is a rational direction then there is some minimal $T = T(\sigma) > 0$ such that,

$$(\partial P_\sigma + T\sigma) \bmod \mathbb{Z}^d = \partial P_\sigma. \tag{4.2.12}$$

Lemma 4.2.5. *If* $\xi \in \mathbb{Z}^d \setminus \{0\}$ *is irreducible in the sense that* $\gcd(\xi_1, \ldots, \xi_d) = 1$ *then,*

$$T(\hat{\xi}) = |\xi|^{-1}.$$

Proof. We need to show that there exists $x \in \mathbb{Z}^d$ such that $\xi \cdot x = 1$. If there exists such an x then

$$0 \in [\partial P_{\hat{\xi}} \bmod \mathbb{Z}^d] \cap [(\partial P_{\hat{\xi}} + \frac{1}{|\xi|}\hat{\xi}) \bmod \mathbb{Z}^d],$$

and so the two sets are the same. This is essentially just using the Euclidean algorithm repeatedly since we are looking for an integer solution of

$$\xi_1 x_1 + \cdots \xi_d x_d = \gcd(\xi_1, \ldots, \xi_d) = 1.$$

$\square$

Lemma 4.2.6. *If $\xi \in \mathbb{Z}^d \setminus \{0\}$ then ∂P_ξ is spanned by $d-1$ vectors $f^j \in \mathbb{Z}^d$ with $|f^j| \leq |\xi|$.*

Proof. Without loss assume that $|\xi_d| = \arg\max_{1\leq i\leq d} |\xi_i| > 0$ since $\xi \neq 0$. Then call, for $1 \leq j \leq d-1$,

$$f^j = \xi_d e_j - \xi_j e_d \text{ and from the definition } f^j \cdot \xi = 0.$$

$\square$

Next we state a classical number theoretic result, the simultaneous version of Dirichlet's approximation Theorem. The proof is by pigeon-hole principle.

Theorem 4.2.7. *For given real numbers $\alpha_1, ..., \alpha_n$ and $N \in \mathbb{N}$, there are integers $p_1, ..., p_n, q \in \mathbb{Z}$ with $1 \leq q \leq N$ such that*

$$|q\alpha_i - p_i| \leq \frac{1}{N^{1/n}}.$$

4.3 Asymptotics of Half-space Solutions with Periodic Boundary Conditions

Here we consider the convergence rate for homogenization of half-space problems. First consider the solution v of the following problem in a half-space,

$$\begin{cases} F(D^2 v, y) = f(y) & \text{in} \quad P_{e_d} \\ v = \phi(y) & \text{on} \quad \partial P_{e_d}. \end{cases} \tag{4.3.1}$$

We assume that F, f and ϕ are periodic with respect to linearly independent translations $\ell_1, \ldots \ell_{d-1} \in \partial P_{e_d}$. We define the lattice of periodicity and its unit cell,

$$\mathscr{L} := \left\{ z : z = \sum_{j=1}^{d-1} k_j \ell_j,\ k_j \in \mathbb{Z} \right\}, \quad Q := \left\{ \sum \lambda_j \ell_j : \lambda_j \in [0,1) \right\}, \quad \text{and } L := \operatorname{diam}(Q).$$

In this section we will only consider the case $f \equiv 0$ for the simplicity of presentation. The proof of Lemma 4.3.1 for the general case $f \neq 0$, which is needed in Section 4.7, is presented in Appendix 4.A. The calculations are rather delicate, since for later usage it will be important for us to keep track of the dependence on the unit cell size L.

The following lemma states that the rate of convergence to the homogenized boundary condition will be exponentially fast depending on L and universal constants. This result is originally due to Tartar [Tar09] for linear divergence form operators. To the best of our knowledge the result is new for nonlinear operators. The proof is an iterative argument using the $\mathscr{L}$-periodicity of the solution and the interior oscillation decay from Harnack inequality.

Lemma 4.3.1. *There exist $\mu(\phi, F)$ and $c_0(\Lambda, d) > 0$ such that,*

$$\sup_{y \cdot e_d \geq R} |v(y) - \mu| \leq C(\Lambda, d)(\operatorname{osc} \phi) \exp(-c_0 L^{-1} R),$$

and this estimate gives the optimal rate up to the determination of c_0.

Proof. By rescaling we may assume without loss that $\operatorname{osc} \phi = 1$. Let $\alpha(d, \Lambda) \in (0,1)$ be the Hölder continuity exponent and $C_0(\Lambda, d)$ the constant in the interior Hölder estimate for the maximal class (see [CC95]). For $r > C_0^{1/\alpha} L$ we claim that

$$\operatorname*{osc}_{P_{e_d} + kr\sigma} v \leq C_0^k (L/r)^{\alpha k} \quad \text{for any } k \in \mathbb{N}. \tag{4.3.2}$$

Supposing that this result holds, let $k = [R/r]$ to obtain

$$\operatorname*{osc}_{P_{e_d} + R\sigma} v \leq \operatorname*{osc}_{P_{e_d} + kr\sigma} v \leq \exp\left(k \log \frac{C_0 L^\alpha}{r^\alpha} \right).$$

When we choose $r = e C_0^{1/\alpha} L$ the estimate becomes,

$$\operatorname*{osc}_{P_{e_d} + R\sigma} v \leq \exp\left(- \left[\frac{R}{e C_0^{1/\alpha} L} \right] \right) \leq C \exp\left(-cR/L\right)$$

with $C = e$ and $c = e^{-1}C_0^{-1/\alpha}$.

It remains to prove (4.3.2) by induction. For $k = 0$ (4.3.2) follows from the maximum principle. Assuming (4.3.2) for k, we prove it for $k+1$. Note that

$$v_k(y) = C_0^{-k} r^{\alpha k} L^{-\alpha k} v(y + kre_d)$$

satisfies $F(\cdot, y + kre_d) = 0$ in P_{e_d} with boundary data $\phi_k(y) = C_0^{-k} r^{\alpha k} L^{-\alpha k} v(y + ke_d)$. Both the operator and the boundary data are periodic with respect to $(\ell_j)_{j=1}^{d-1}$ translations by uniqueness, and $\operatorname{osc} \phi_k \leq 1$ by the inductive hypothesis. Then by the interior Hölder estimate in $B_r(re_d)$,

$$|v_k|_{C^\alpha(B_{r/2}(re_d))} \leq C_0 r^{-\alpha} \text{ and so } \operatorname*{osc}_{Q+re_d} v_k \leq C_0 (L/r)^\alpha,$$

where we have used $r > 2L$ so that $Q \subset B_{r/2}(0)$. On the other hand v_k is periodic on $\partial P_{e_d} + re_d$ with respect to the translations $(\ell_j)_{j=1}^{d-1}$ and periodicity cell Q. Therefore

$$\operatorname*{osc}_{P_\sigma + re_d} v_k \leq \operatorname*{osc}_{\partial P_\sigma + re_d} v_k \leq C_0 (L/r)^\alpha,$$

where we have again used maximum principle for the first inequality. Rewriting this in terms of v,

$$\operatorname*{osc}_{P_\sigma + (k+1)re_d} v = C_0^k (L/r)^{\alpha k} \operatorname*{osc}_{P_\sigma + re_d} v_k \leq C_0^{k+1} (L/r)^{(k+1)\alpha}.$$

This completes the inductive proof.

Lastly to show that the rate is optimal we take $F = -\Delta$ and $\phi = \cos \frac{2\pi y_1}{L}$. Then $v(y)$ can be explicitly computed using separation of variables as

$$v(y) = \cos(\tfrac{2\pi y_1}{L}) \exp(-\tfrac{2\pi}{L} y_2).$$

Plugging in $y_1 = 0$ and $y_2 = R$ completes the proof since evidently $\mu = 0$ in this case. □

We will need a slight variant of Lemma 4.3.1 when the operator does not share the periodicity cell of the boundary data but its oscillations are at a smaller scale (see the proof of Lemma 4.4.3).

Lemma 4.3.2. *Let ϕ, v and L as given in Lemma 4.3.1, and suppose that there is $0 < \varepsilon \leq L$ such that for every $y \in \partial P_{e_d}$ there is y' with $|y - y'| \leq \varepsilon$ and $F(M, \cdot + y') = F(M, \cdot)$ in P_{e_d}. Then there exists $C, c > 0$ depending only on Λ, d such that,*

$$\operatorname*{osc}_{y \in \partial P_{e_d}} v(Re_d + y) \leq C[(\operatorname{osc} \phi) \exp(-cL^{-1}R) + \omega_v(\varepsilon)].$$

Here ω_v is the modulus of continuity of v at points of ∂P_{e_d},

$$\omega_v(r) := \sup\{|v(y)-v(y')| : y \in \partial P_{e_d}, y' \in P_{e_d} \ \textit{and} \ |y-y'| \le r\}.$$

Note that by Lemma 4.2.4 we have, for any $\beta \in (0,1)$, $\omega_v(\varepsilon) \le C(\Lambda,d,\beta)\|\phi\|_{C^{0,\beta}}\varepsilon^{\beta}$.

Proof. The proof is a minor modification of the proof of Lemma 4.3.1 and so we mainly focus on the difference in the proof. We will prove that there is $\alpha(\Lambda,d) \in (0,1), C_0(\Lambda,d) > 0$ such that for $r \ge (2C_0)^{1/\alpha}L$

$$\operatorname*{osc}_{P_{e_d}+kre_d} v \le C_0^k(L/r)^{\alpha k}(\operatorname{osc}\phi) + C_0\omega_v(\varepsilon) \quad \text{ for any } k \in \mathbb{N}. \tag{4.3.3}$$

Following the proof of Lemma 4.3.1 we are done as long as we can show (4.3.3).

The proof of (4.3.3) is again by induction. We wish to show that (4.3.3) holds for all k. Assuming that (4.3.3) holds up to k we prove for $k+1$. Note that $v_k(y) = v(y+kre_d)$ solves the equation

$$F_k(D^2v_k, y) := F(D^2v_k, y+kre_d) = 0 \ \text{ in } \ P_{e_d}$$

with boundary data $\phi_k(y) = v_{k-1}(y+re_d)$. Note that F_k satisfies the same assumption as F. By the interior oscillation decay for the ellipticity class, there exists a universal constant $C_1 > 1$ such that

$$\operatorname*{osc}_{\ell+Q+re_d} v_k \le C_1(L/r)^{\alpha} \operatorname*{osc}_{\partial P_{e_d}} \phi_k \quad \text{ for any } \ell \in \mathscr{L}.$$

For an arbitrary $y \in \partial P_{e_d}$ let $\ell \in \mathscr{L}$ such that $y \in \ell+Q$,

$$\begin{aligned} |v_k(y+re_d) - v_k(re_d)| &\le 2\operatorname*{osc}_{\ell+Q+re_d} v_k + |v_k(\ell+re_d) - v_k(re_d)| \\ &\le 2C_0'(L/r)^{\alpha}\operatorname*{osc}_{\partial P_{e_d}} \phi_k + |v_k(\ell+re_d) - v_k(re_d)|. \end{aligned} \tag{4.3.4}$$

The second term on the right hand side above appears since v_k is no longer periodic. By the assumption on F there is $\ell' \in P_{e_d}$ with $|\ell'-\ell| \le \varepsilon$ and $F(M,\cdot+\ell') = F(M,\cdot)$ in P_{e_d} for all symmetric matrices M. Therefore we can estimate,

$$|v_k(\ell+re_d) - v_k(re_d)| \le |v_k(\ell+re_d) - v_k(\ell'+re_d)| + |v_k(\ell'+re_d) - v_k(re_d)|.$$

The first term can be estimated by,

$$|v_k(\ell+re_d) - v_k(\ell'+re_d)| \le C_0'(|\ell-\ell'|/r)^{\alpha}\operatorname*{osc}_{\partial P_{e_d}} \phi_k. \tag{4.3.5}$$

For the second term it suffices to bound $w(y) = v(y+\ell') - v(y)$ in P_{e_d}. Using the periodicity of ϕ we have $w(y) = v(y+\ell') - \phi(y+\ell)$ on ∂P_{e_d}. Then by maximum principle and the definition of ω_v,

$$\sup_{P_{e_d}} |w| \leq \sup_{\partial P_{e_d}} |w| = \sup_{\partial P_{e_d}} |v(y+\ell') - \phi(y+\ell)| \leq \omega_v(|\ell - \ell'|). \tag{4.3.6}$$

Plugging (4.3.5) and (4.3.6) into (4.3.4) and using $|\ell - \ell'| \leq \varepsilon \leq L$ we obtain,

$$|v_k(y+re_d) - v_k(re_d)| \leq 3C_0'(L/r)^\alpha \operatorname*{osc}_{\partial P_{e_d}} \phi_k + \omega_v(\varepsilon). \tag{4.3.7}$$

We can conclude now since,

$$\begin{aligned}
\operatorname*{osc}_{\partial P_{e_d}+(k+1)re_d} v &= \operatorname*{osc}_{\partial P_{e_d}+re_d} v_k \\
&\leq 3C_1(L/r)^\alpha \operatorname*{osc}_{\partial P_{e_d}} \phi_k + \bar{\omega}_\phi(\varepsilon) \\
&\leq 3C_1C_0^k(L/r)^{(k+1)\alpha} \operatorname{osc} \phi + (3C_1C_0^k(L/r)^{k\alpha} + 1)\omega_v(\varepsilon) \\
&\leq C_0^{k+1}(L/r)^{(k+1)\alpha} \operatorname{osc} \phi + C_0\omega_v(\varepsilon)
\end{aligned}$$

where the last step holds if we choose $C_0 = 3C_1$ and $r \geq 2^{1/\alpha}LC_0^{1/\alpha}$ since $1 < C_1$.

$\square$

4.4 Asymptotics of Half-Space Solutions Near Rational Directions

In this section we study asymptotic behavior of half-space solutions as the normal direction σ approaches a rational direction, $\hat{\xi}$ for some $\xi \in \mathbb{Z}^d \setminus \{0\}$. Let us recall the cell problem solution $v_{\sigma,\tau}$ defined for a direction $\sigma \in S^{d-1}$, a $\tau \in \mathbb{R}^d$ and a continuous $\mathbb{Z}^d$-periodic ψ by,

$$\begin{cases} F(D^2 v_{\sigma,\tau}, y) = 0 & \text{in} \quad P_\sigma \\ v_{\sigma,\tau} = \psi(y) & \text{on} \quad \partial P_\sigma. \end{cases} \tag{4.4.1}$$

Due to Theorem 4.2.2, for irrational directions there exists a limit $\mu(\sigma, \psi, F)$ such that,

$$\sup_{\tau \in \mathbb{R}^d} \sup_{y \in \partial P_\sigma} |v_{\sigma,\tau}(y+R\sigma; (\psi, F)) - \mu(\sigma, \psi, F)| \to 0 \text{ as } R \to \infty. \tag{4.4.2}$$

We are interested to understand the asymptotic behavior of $\mu(\sigma, \psi, F)$ as σ approaches a rational direction $\hat{\xi}$ for $\xi \in \mathbb{Z}^d \setminus \{0\}$. The limiting behavior, it will turn out, depends on the direction of tangential approach. The main result of this section (stated quantitatively in Proposition 4.4.10) is the following:

Theorem 4.4.1. *Let $\beta \in (0,1)$ and $\psi \in C^{0,\beta}(\mathbb{T}^d)$. For any $\xi \in \mathbb{Z}^d \setminus \{0\}$, irreducible, there exists a function $L_\xi(\cdot) = L_\xi(\cdot;\psi,F)$ on unit vectors tangent to S^{d-1} at $\hat{\xi}$ and a mode of continuity, $\omega_{\xi,\beta}$, such that the following holds:*

$$|\mu(\sigma(t),\psi,F) - L_\xi(\sigma'(0))| \leq \omega_{|\xi|,\beta}(|\sigma(t)-\sigma(0)|)$$

for any $\sigma : [0,1) \to S^{d-1}$ a unit speed geodesic with $\sigma(0) = \hat{\xi}$.

The basic idea behind these asymptotics already appeared in [CKL13, CK14] for the Neumann problem as a part of their proof that, when $\overline{F}$ is rotation invariant, $\mu(\cdot,\psi,F)$ has a continuous extension from the irrational directions to the entire unit sphere. The proof proceeds by a series of reductions which will be carried out by a multi-scale homogenization argument. Our analysis is a quantitative and improved version of the proof given in [CK14] in the following sense. First we have tried to obtain optimal estimates at each stage of the argument. We do this with the hope of clarifying the proof and of achieving improved quantitative results on the continuity of μ in the end. Secondly we introduce the directional limit L and observe that L depends on a two-dimensional projected version of the problem (see Section 4.4.3). It is for this reason that we are able to use Nirenberg's two dimensional regularity result and the corresponding interior homogenization result, Theorem 4.2.3. By this careful exposition we are able to obtain a precise characterization of the asymptotic behavior of μ at rational directions and its dependence on the operator F and boundary data ψ. With this characterization we are able to understand both continuity and discontinuity, Sections 5 and 6 respectively, in a unified way.

4.4.1 Step 1: Replacing the Boundary Condition at an Intermediate Scale

Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ be irreducible and recall the cell problem solutions, $\mathbb{Z}^d$-periodic in $\tau \in \mathbb{R}^d$,

$$\begin{cases} F(D^2 v_{\xi,\tau}, y+\tau) = 0 & \text{in} \quad P_\xi \\ v_{\xi,\tau} = \psi(y+\tau) & \text{on} \quad \partial P_\xi. \end{cases} \tag{4.4.3}$$

By the results of Section 4.2.6 the boundary data $\psi|_{\partial P_\xi}$ is periodic with respect to a lattice on ∂P_ξ with unit cell size $\leq C_d|\xi|$. The result of the previous section implies that for each $\tau \in \mathbb{R}^d$ there is a limit at infinity in the ξ direction. The limit for $\tau, \tau' \in \mathbb{R}^d$ is the same when τ, τ' are both in $\partial P_\xi + t\hat{\xi}$ modulo $\mathbb{Z}^d$ for some $t \in \mathbb{R}$. By Lemma 4.2.5 this is exactly when $\tau \cdot \hat{\xi} \mod \frac{1}{|\xi|}\mathbb{Z} = \tau' \cdot \hat{\xi} \mod \frac{1}{|\xi|}\mathbb{Z}$. We define

$$m_\xi(t;(\psi,F)) := \lim_{R\to\infty} v_{\hat{\xi},t\hat{\xi}}(R\hat{\xi})$$

which is continuous, $\frac{1}{|\xi|}$-periodic on $\mathbb{R}$. By definition we have

$$\lim_{R\to\infty} v_{\xi,\tau}(R\hat{\xi}) = m_\xi(\tau \cdot \hat{\xi};(\psi,F)).$$

Generally speaking m_ξ inherits the up to the boundary regularity of the cell problem solutions. In the general fully nonlinear case this is limited to $C^{0,1}$, but for linear operators the result would hold for arbitrary $C^{k,\beta}$, see Section 4.7.

Lemma 4.4.2. *For $\beta \in (0,1)$,*

$$\|m_\xi(\cdot;(\psi,F))\|_{C^{0,\beta}} \leq C(\Lambda,d,\beta)\|\psi\|_{C^{0,\beta}},$$

and

$$\|m_\xi(\cdot;(\psi,F))\|_{C^{0,1}} \leq C(\Lambda,d,\beta)\|\psi\|_{C^{1,\beta}}.$$

The proof is a straightforward application of the boundary $C^{0,\beta}$ estimates combined with the definition of m_ξ, and is postponed till the end of this section. We drop the dependence of m_ξ on (ψ,F) as long as there is no ambiguity. Due to Lemma 4.3.1 of the previous section,

$$\sup_{P_\xi + R\hat{\xi}} |v_{\xi,t\hat{\xi}}(\cdot) - m_\xi(t)| \leq C(\operatorname{osc}\psi)\exp(-cR/|\xi|).$$

Let $\sigma \in S^{d-1}$ be an irrational direction and $\mu(\sigma,\psi,F)$ the boundary layer tail of the cell problem solutions $v_{\sigma,\tau}$ (see Theorem 4.2.2 for the definition of μ). Since the limit is independent of τ (from Theorem 4.2.2) we simply refer to $v_\sigma = v_{\sigma,0}$ when σ is irrational. We consider the asymptotics of $\mu(\sigma,\psi,F)$ as σ approaches $\hat{\xi}$.

When $\sigma \neq -\hat{\xi}$ there is a unique vector $\eta \perp \xi$ (see Figure 1) so that,

$$\sigma = (\cos|\eta|)\hat{\xi} - (\sin|\eta|)\hat{\eta} \ \text{ with } \ |\eta| = |\sigma - \hat{\xi}| + O(|\sigma-\hat{\xi}|^2) \tag{4.4.4}$$

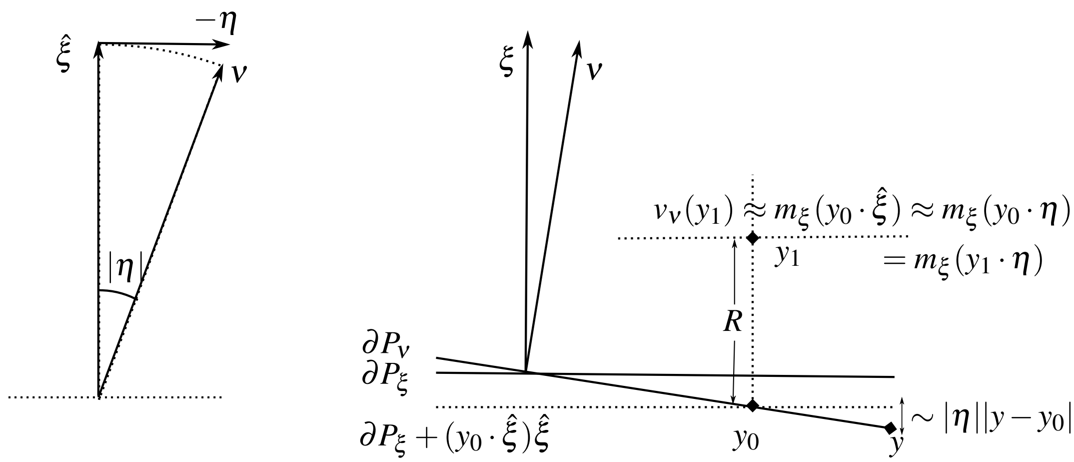

Figure 4.1: The appearance of m_ξ at an intermediate distance from ∂P_σ.

It may be helpful, although it is not essential, to note that this is just the minus of the inverse of the exponential map $\exp_{\hat{\xi}} : T_{\hat{\xi}} S^{d-1} \to S^{d-1} \setminus \{-\hat{\xi}\}$ where $T_{\hat{\xi}} S^{d-1}$ is the tangent space to S^{d-1} at $\hat{\xi}$. The goal of this section is to show that, after moving to the interior and rescaling, the cell problem solution v_σ is very close, in terms of $|\sigma - \hat{\xi}|$, to $w_{\xi,\eta}$ solving,

$$\begin{cases} F(D^2 w_{\xi,\eta}, |\eta|^{-1} y) = 0 & \text{in} \quad P_\xi \\ w_{\xi,\eta} = m_\xi(y \cdot \hat{\eta}) & \text{on} \quad \partial P_\xi, \end{cases} \tag{4.4.5}$$

in their common domain of definition. We do not claim that (4.4.5) has a boundary layer tail. Indeed the periodicity lattices of the boundary data and the operator may not be aligned. On the other hand, by Lemma 4.3.2, it will almost have a limit up to an error small in $|\eta|$ and this will be sufficient for our purposes. More precisely we aim to prove:

Proposition 4.4.3. *Let $\xi \in \mathbb{Z}^d \setminus \{0\}$, irreducible, and $\sigma \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ with $|\xi - |\xi|\sigma| \leq 1/2$ then, for any $\beta \in (0,1)$,*

$$\mu(\sigma, F, \psi) - \liminf_{R \to \infty} w_{\xi,\eta}(R\hat{\xi}; (\psi, F)) \leq C(\Lambda, d, \beta) |\psi|_{C^{0,\beta}} |\xi - |\xi|\sigma|^\beta \log \tfrac{1}{|\xi - |\xi|\sigma|}.$$

The parallel statement holds for the $\limsup$ *as well.*

We remark that the result does not depend on the Hölder continuity of ψ (any continuity modulus for ψ would yield an analogous result). Furthermore when $\psi \in C^{1,\beta}$ the estimate can be improved to

$$\mu(\sigma,F,\psi)-\liminf_{R\to\infty} w_{\xi,\eta}(R\hat{\xi};(\psi,F))\leq C(\Lambda,d,\beta)\|\psi\|_{C^{1,\beta}}|\xi-|\xi|\sigma|\log\tfrac{1}{|\xi-|\xi|\sigma|}.$$

Let us give a heuristic proof of Proposition 4.4.3, which is illustrated in Figure 1. Pick a point $y_0\in P_\sigma$. In a neighborhood of y_0 the boundary data ψ for v_σ is very close to that of ψ restricted to $\partial P_\xi+(y_0\cdot\hat{\xi})\hat{\xi}$. This causes v_σ to be close to $m_\xi(y_0\cdot\hat{\xi})$ at $y_1:=y_0+R\hat{\xi}$ for $R=o(|\hat{\xi}-\sigma|)$. Next, observe that $y_0\cdot\hat{\xi}\sim y_0\cdot\eta$, since $\hat{\xi}-\eta$ is almost σ and y_0 is perpendicular to σ. But now, since η is perpendicular to ξ, we have $y_0\cdot\eta=y_1\cdot\eta$. Consequently one can now say $v_\sigma(y)$ is now close to $m_\xi(y\cdot\eta)$ R-away from ∂P_σ, and this describes the near-boundary homogenization for v_σ. Now taking $m_\xi(y\cdot\eta)$ as the new boundary data for the interior homogenization, we arrive at the interior problem (4.4.5) and Proposition 4.4.3.

The actual proof is slightly more involved for technical reasons.

Lemma 4.4.4. *Let $\xi\in\mathbb{Z}^d\setminus\{0\}$ be irreducible. For $|\xi-|\xi|\sigma|\leq 1/2$ and $\beta\in(0,1)$ let us define $R_0:=c^{-1}|\xi|\log\frac{1}{|\xi||\sigma-\hat{\xi}|}$. Then we have*

$$\sup_{y\in\partial P_\sigma}|v_\sigma(R_0\sigma+y)-m_\xi(y\cdot\hat{\xi})|\leq C(\Lambda,d,\beta)|\psi|_{C^{0,\beta}}|\xi-|\xi|\sigma|^\beta\log\tfrac{1}{|\xi-|\xi|\sigma|}.$$

We remark that the log term in above esimate can be improved slightly as may be noticed from the proof.

Proof of Lemma 4.4.4. We first show that for all $R>1$,

$$\sup_{y\in\partial P_\sigma}|v_\sigma(R\sigma+y)-m_\xi(y\cdot\hat{\xi})|\lesssim(\operatorname{osc}\psi)\exp(-cR/|\xi|)+|\psi|_{C^{0,\beta}}R^\beta|\sigma-\hat{\xi}|^\beta.$$

This will imply the desired result by choosing $R_0=c^{-1}|\xi|\log\frac{1}{|\xi||\sigma-\hat{\xi}|}$ and using $(\operatorname{osc}\psi)\lesssim|\psi|_{C^{0,\beta}}$. Fix $y_0\in\partial P_\sigma$ and we consider comparing v_σ with the solution w of,

$$\begin{cases} F(D^2w,y)=0 & \text{in} \quad P_\xi+y_0\\ w=\psi(y) & \text{in} \quad \partial P_\xi+y_0.\end{cases}\tag{4.4.6}$$

Note that $y_0\in\partial P_\xi+(y_0\cdot\hat{\xi})\hat{\xi}$ and therefore, using that $\sigma\cdot\hat{\xi}\geq 1/2$,

$$|w(y_0+R\sigma)-m(y_0\cdot\hat{\xi})|\leq C(\operatorname{osc}\psi)\exp(-c|\xi|^{-1}R).\tag{4.4.7}$$

On the other hand for $y \in \partial[(P_\xi + y_0) \cap P_\sigma]$ there exists $y' \in \partial P_\xi + y_0$ such that

$$|y' - y| \le |\sigma - \hat{\xi}||y - y_0|,$$

and so by the Hölder continuity of w up to the boundary,

$$|w(y) - \psi(y)| \le |w(y) - w(y')| + |\psi(y') - \psi(y)| \le C(d,\Lambda)|\psi|_{C^{0,\beta}}|\sigma - \hat{\xi}|^\beta |y - y_0|^\beta.$$

The same argument holds for v_σ and by combining the two estimates we have

$$|v_\sigma(y) - w(y)| \le \min\{C|\psi|_{C^{0,\beta}}|\sigma - \hat{\xi}|^\beta |y - y_0|^\beta, \operatorname{osc}\psi\} \ \text{ for } \ y \in \partial[(P_\xi + y_0) \cap P_\sigma],$$

where the second term is from maximum principle, $\min\psi \le w, v_\sigma \le \max\psi$. Now we claim that,

$$|v_\sigma(y) - w(y)| \le C(\Lambda, d, \beta)\min\{|\psi|_{C^{0,\beta}}|\sigma - \hat{\xi}|^\beta |y - y_0|^\beta, \operatorname{osc}\psi\} \ \text{ for } \ y \in (P_\xi + y_0) \cap P_\sigma, \quad (4.4.8)$$

but this is just a rescaling of Lemma 4.2.4. In particular (4.4.8) combined with (4.4.7) implies,

$$\begin{aligned} |v_\sigma(R\sigma + y_0) - m(y_0 \cdot \hat{\xi})| &\le |v_\sigma(R\sigma + y_0) - w(R\sigma + y_0)| + C|\psi|_{C^{0,\beta}}R^\beta|\sigma - \hat{\xi}|^\beta \\ &\lesssim (\operatorname{osc}\psi)\exp(-cR/|\xi|) + |\psi|_{C^{0,\beta}}R^\beta|\sigma - \hat{\xi}|^\beta. \end{aligned}$$

This was the desired estimate. □

Next we return to the proof of Proposition 4.4.3 from Lemma 4.4.4.

Proof of Proposition 4.4.3. By maximum principle in the domain $P_\sigma + R_0\sigma$ Lemma 4.4.4 implies that,

$$|v_\sigma(y + R_0\sigma) - u(y)| \lesssim |\psi|_{C^{0,\beta}}|\xi|^\beta|\sigma - \hat{\xi}|^\beta \log\tfrac{1}{|\xi||\sigma - \hat{\xi}|} \ \text{ for } \ y \in P_\sigma + R_0\sigma \qquad (4.4.9)$$

where u solves,

$$\begin{cases} F(D^2u, y) = 0 & \text{in} \quad P_\sigma \\ u(y) = m_\xi(y \cdot \hat{\xi}) = m_\xi(y \cdot \Pi_{\sigma^\perp}\hat{\xi}) & \text{on} \quad \partial P_\sigma, \end{cases}$$

where we recall that $\Pi_{\sigma^\perp}\hat{\xi} := \hat{\xi} - (\hat{\xi} \cdot \sigma)\sigma$ is the orthogonal projection onto ∂P_σ. In particular an estimate of the same form as (4.4.9) holds for the respective boundary layer tails. Call $\eta_0 = \Pi_{\sigma^\perp}\hat{\xi}$ and recall that we had defined η as

$$\eta := -\exp_{\hat{\xi}}^{-1}(\sigma) \ \text{ defined so that } \ \sigma = (\cos|\eta|)\hat{\xi} - (\sin|\eta|)\hat{\eta}.$$

From the definition of η we calculate,

$$\eta_0 = \Pi_{\sigma^\perp}\hat{\xi} = \hat{\xi} - (\hat{\xi}\cdot\sigma)\sigma = (\sin|\eta|)^2\hat{\xi} + (\sin|\eta|)(\cos|\eta|)\hat{\eta},$$

and so,

$$\begin{aligned}|\eta_0 - \eta| &\leq |\sin|\eta| - |\eta|| + |\eta_0 - (\sin|\eta|)\hat{\eta}| \\ &\leq |\sin|\eta| - |\eta|| + (\sin|\eta|)\sqrt{2(1-\cos|\eta|)} \\ &\leq |\eta|^2\end{aligned}$$

Now we rescale to $\tilde{u}(z) = u(|\eta|^{-1}z)$ which solves

$$\begin{cases} F(D^2\tilde{u}, |\eta|^{-1}z) = 0 & \text{in} \quad P_\sigma \\ \tilde{u}(z) = m_\xi(z\cdot|\eta|^{-1}\eta_0) & \text{on} \quad \partial P_\sigma, \end{cases}$$

and estimate the difference of $\tilde{u}$ and $w_{\xi,\eta}$ in their common domain $P_\sigma \cap P_\xi$. The aim is to obtain an estimate on the difference of their respective boundary layer tails. From here the proof will follow a familiar argument. From the estimates above and Lemma 4.4.2,

$$|m_\xi(z\cdot|\eta|^{-1}\eta_0) - m_\xi(z\cdot\hat{\eta})| \leq |m_\xi|_{C^{0,\beta}}|z|^\beta|\eta|^\beta.$$

Using this we bound the difference $\tilde{u} - w_{\xi,\eta}$ for $z \in \partial(P_\sigma \cap P_\xi)$. First note that for $z \in \partial P_\sigma \cap P_\xi$ there is $z' \in \partial P_\xi$ with $|z' - z| = |z||\eta|$. Therefore we have

$$\begin{aligned}|\tilde{u}(z) - w_{\xi,\eta}(z)| &\leq |m_\xi(z\cdot|\eta|^{-1}\eta_0) - m_\xi(z\cdot\hat{\eta})| + |w_{\xi,\eta}(z) - m_\xi(z\cdot\hat{\eta})| \\ &\leq |m_\xi|_{C^{0,\beta}}|z|^\beta|\eta|^\beta + |w_{\xi,\eta}(z) - m_\xi(z'\cdot\hat{\eta})| + |m_\xi(z'\cdot\hat{\eta}) - m_\xi(z\cdot\hat{\eta})| \\ &\leq C|m_\xi|_{C^{0,\beta}}|z|^\beta|\eta|^\beta\end{aligned}$$

where the middle term in the second line is estimated using the continuity up to the boundary of $w_{\xi,\eta}$ from Lemma 4.2.4. Combining this with $\operatorname{osc} m_\xi \leq |m_\xi|_{C^{0,\beta}}|\xi|^{-\beta}$, and $|m_\xi|_{C^{0,\beta}} \leq C(\Lambda,d)|\psi|_{C^{0,\beta}}$ we have

$$\begin{cases} -\mathscr{P}^+_{1,\Lambda}(D^2(\tilde{u} - w_{\xi,\eta})) \leq 0 & \text{in} \quad P_\sigma \cap P_\xi \\ (\tilde{u} - w_{\xi,\eta})(z) \leq C(\Lambda,d)|\psi|_{C^{0,\beta}}\min\{|z|^\beta|\sigma - \hat{\xi}|^\beta, |\xi|^{-\beta}\} & \text{on} \quad \partial(P_\sigma \cap P_\xi). \end{cases}$$

Therefore by the rescaled version of Lemma 4.2.4,

$$\tilde{u}(R\hat{\xi}) - w_{\xi,\eta}(R\hat{\xi}) \leq C(d,\Lambda,\beta)|\psi|_{C^{0,\beta}} R^{\beta}|\sigma - \hat{\xi}|^{\beta} \text{ for any } R > 0. \tag{4.4.10}$$

At this stage we want to combine this estimate with the exponential convergence of $\tilde{u}, w_{\xi,\eta}$ to their respective boundary layer tails, but there is a minor technical issue that $F(M, |\eta|^{-1}z)$ does not share the same periodicity lattice as $m_{\xi}(z \cdot \eta)$. However, the conditions of Lemma 4.3.2 do hold and the rate of convergence established in Lemma 4.3.2 combined with (4.4.10) implies, for any $R > 0$,

$$\limsup_{R' \to \infty} \tilde{u}(R'\hat{\xi}) - \liminf_{R' \to \infty} w_{\xi,\eta}(R'\hat{\xi}) \lesssim (\operatorname{osc} m_{\xi}) \exp(-c|\xi|R) + |\psi|_{C^{0,\beta}} R^{\beta}|\sigma - \hat{\xi}|^{\beta} + |\eta|^{\beta}|\psi|_{C^{0,\beta}}.$$

Now we are free to minimize over $R > 0$, then plugging in $|\eta| \leq |\sigma - \hat{\xi}|$ to obtain

$$\limsup_{R \to \infty} \tilde{u}(R\hat{\xi}) - \liminf_{R \to \infty} w_{\xi,\eta}(R\hat{\xi}) \leq C|\psi|_{C^{0,\beta}}|\sigma - \hat{\xi}|^{\beta} \log \tfrac{1}{|\sigma - \hat{\xi}|}.$$

Finally combining with (4.4.9) and the remark below it that the same estimate holds for the boundary layer tails,

$$\mu(\sigma, \psi, F) - \liminf w_{\xi,\eta}(R\hat{\xi}) \leq C(\Lambda, d, \beta)|\psi|_{C^{0,\beta}}|\xi|^{\beta}|\sigma - \hat{\xi}|^{\beta} \log \tfrac{1}{|\xi||\sigma - \hat{\xi}|}.$$

A symmetric argument yields the same estimate for $\limsup_{R \to \infty} w_{\xi,\eta}(R\hat{\xi}) - \mu(\sigma, \psi, F)$.

□

Proof of Lemma 4.4.2. For the purposes of this proof it will be useful to work with a slightly different definition of the cell problem solution. We call $\tilde{v}_{\xi,\tau}(y) = v_{\xi,\tau}(y - \tau)$ which now solves,

$$\begin{cases} F(D^2\tilde{v}_{\xi,\tau}, y) = 0 & \text{in} \quad P_{\xi} + \tau \\ \tilde{v}_{\xi,\tau} = \psi(y) & \text{on} \quad \partial P_{\xi} + \tau. \end{cases} \tag{4.4.11}$$

Of course the boundary layer tail remains unchanged. The point is that the $\tilde{v}_{\xi,\tau}$ now solve the same interior equation for all $\tau \in \mathbb{R}^d$, but in different domains. When $\tau - \tau'$ is small the domains are close and we can combine the boundary continuity estimate of Lemma 4.2.4 with comparison principle to estimate the difference of the cell problem solutions, and hence of their boundary layer tails as well. It suffices to estimate the continuity of m_ξ at $t=0$. Let $\beta \in (0,1)$, $\varepsilon > 0$ and any $y \in \partial P_\xi$,

$$\begin{aligned}
\tilde{v}_{\xi,-\varepsilon\hat{\xi}}(y) - \tilde{v}_{\xi,0}(y) &= \tilde{v}_{\xi,-\varepsilon\hat{\xi}}(y) - \psi(y) \\
&= \tilde{v}_{\xi,-\varepsilon\hat{\xi}}(y) - v_{\xi,-\varepsilon\hat{\xi}}(y-\varepsilon\hat{\xi}) + \psi(y-\varepsilon\hat{\xi}) - \psi(y) \\
&\leq \varepsilon^\beta (\sup_\tau \|\tilde{v}_{\xi,\tau}\|_{C^{0,\beta}(P_\xi)} + \|\psi\|_{C^{0,\beta}(\mathbb{T}^d)}) \\
&\leq C(d,\Lambda,\beta)\varepsilon^\beta \|\psi\|_{C^{0,\beta}(\mathbb{T}^d)}.
\end{aligned}$$

Then, by maximum principle the same estimate holds in P_ξ and therefore,

$$\begin{aligned}
|m_\xi(-\varepsilon) - m_\xi(0)| &= \lim_{R\to\infty} |\tilde{v}_{\xi,-\varepsilon\hat{\xi}}(R\hat{\xi}) - \tilde{v}_{\xi,0}(R\hat{\xi})| \\
&\leq \sup_{P_\xi} |\tilde{v}_{\xi,-\varepsilon\hat{\xi}}(R\hat{\xi}) - \tilde{v}_{\xi,0}(R\hat{\xi})| \\
&\leq C(d,\Lambda,\beta)\varepsilon^\beta \|\psi\|_{C^{0,\beta}(\mathbb{T}^d)}.
\end{aligned}$$

Parallel arguments work for $\varepsilon < 0$. To get the Lipschitz estimate use the fact that

$$\sup_\tau \|\tilde{v}_{\xi,\tau}\|_{C^{0,1}(P_\xi)} \leq C(d,\Lambda,\beta)\|\psi\|_{C^{1,\beta}(\mathbb{T}^d)} \text{ for any } \beta > 0.$$

□

4.4.2 Step 2: Interior Homogenization at the Intermediate Scale

From the reduction performed in the first step we are left to consider the following problem. For an $\eta \perp \xi$ with $|\eta|$ small,

$$\begin{cases} F(D^2 w_{\xi,\eta}, |\eta|^{-1}y) = 0 & \text{in} \quad P_\xi \\ w_{\xi,\eta} = m_\xi(y\cdot\hat{\eta}) & \text{on} \quad \partial P_\xi, \end{cases} \quad \text{which homogenizes to} \quad \begin{cases} \overline{F}(D^2 \overline{w}_{\xi,\eta}) = 0 & \text{in} \quad P_\xi \\ \overline{w}_{\xi,\eta} = m_\xi(y\cdot\hat{\eta}) & \text{on} \quad \partial P_\xi. \end{cases} \tag{4.4.12}$$

Note that $\overline{w}_{\xi,\eta} = \overline{w}_{\xi,\hat{\eta}}$. We wish to make this convergence quantitative so that we can get an estimate of the difference between $\mu(\sigma, F, \psi)$ and boundary layer tail of the homogenized problem in (4.4.12).

At this stage it is useful to note that the homogenized solution $\overline{w}_{\xi,\eta}$ is actually two dimensional. The key observation here is that since the boundary data only varies in the η direction and the homogenized operator is translation invariant, the solution $\overline{w}_{\xi,\eta}$ only varies in the $\eta, \hat{\xi}$ directions. This is a simple consequence of uniqueness.

Claim. $\overline{w}_{\xi,\eta}(x)$ only depends on $x \cdot \xi$ and $x \cdot \eta$.

We prove the claim only to emphasize the importance of passing from $w_{\xi,\eta}$ to $\overline{w}_{\xi,\eta}$.

Proof. For any $\zeta \perp \eta, \xi$ and $t \in \mathbb{R}$ note that $\overline{w}' = \overline{w}_{\xi,\eta}(y + t\zeta)$ solves

$$\overline{F}(D^2 \overline{w}') = 0 \ \text{ in } \ P_\xi + \zeta = P_\xi \ \text{ with } \ \overline{w}'(y) = m((y + t\zeta) \cdot \eta) = m(y \cdot \eta) \ \text{ on } \ \partial P_\xi.$$

This is of course the same equation satisfied by $\overline{w}_{\xi,\eta}$ so by the uniqueness of bounded solutions $\overline{w}_{\xi,\eta}(y + t\zeta) = \overline{w}_{\xi,\eta}(y)$. □

In particular we have reduced to a situation where, by Nirenberg's Theorem, the homogenized solution is C^{2,α_0} on the interior. By using the exponential rate of convergence to the boundary layer tail established in Section 4.3 combined with Theorem 4.2.3 we are able to show, up to a logarithmic factor, that the same rate of convergence holds for (4.4.12).

Lemma 4.4.5. *Let $\eta \perp \xi$ with $|\eta||\xi| \leq 1/2$. Then there is $\alpha(\Lambda) \in (0,1)$ such that for any $\beta \in (0,1)$,*

$$|w_{\xi,\eta}(y) - \overline{w}_{\xi,\eta}(y)| \leq C(\Lambda, d) |\psi|_{C^{0,\beta}(\mathbb{T}^d)} |\xi|^{\beta(\alpha-1)} |\eta|^{\alpha\beta} (\log \tfrac{1}{|\xi||\eta|}),$$

and, in particular, their boundary layer tails have the same estimate.

Before we proceed with the proof we state a consequence of Proposition 4.4.3 and Lemma 4.4.5.

Lemma 4.4.6. *Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ irreducible and σ an irrational direction with $\eta = \eta(\sigma)$ as in* (4.4.4)*. Then there is $\alpha(\Lambda) \in (0,1)$ such that for any $\beta \in (0,1)$,*

$$|\mu(\sigma, \psi, F) - \lim_{R \to \infty} \overline{w}_{\xi,\eta}(R\hat{\xi})| \leq C(\Lambda, d, \beta) |\psi|_{C^{0,\beta}(\mathbb{T}^d)} |\xi|^{\alpha\beta} |\eta|^{\alpha\beta}.$$

Proof of Lemma 4.4.5. First recall that m_ξ is $\frac{1}{|\xi|}$-periodic on $\mathbb{R}$. Therefore $m_\xi(y \cdot \hat{\eta})$ is $\frac{1}{|\xi|}$-periodic on ∂P_ξ in the direction $\hat{\eta}$ and constant in the directions orthogonal to $\hat{\eta}$. Due to Lemma 4.4.2 we can estimate

$$\operatorname{osc} m_\xi \leq |\xi|^{-\beta} |m_\xi|_{C^{0,\beta}(\mathbb{R})} \leq C |\xi|^{-\beta} |\psi|_{C^{0,\beta}},$$

and

$$|m_\xi|_{C^{0,\alpha\beta}(\mathbb{R})} \leq |\xi|^{\beta(\alpha-1)}|m_\xi|_{C^{0,\beta}(\mathbb{R})}.$$

Next let μ and $\overline{\mu}$ respectively denote the limit of $w_{\xi,\eta}(R\xi)$ and $\overline{w}(R\xi)$ as $R \to \infty$, and let $\alpha(\Lambda)$ be as given in Theorem 4.2.3. Then from Lemma 4.3.2 we have

$$|w_{\xi,\eta}(y)-\mu|+|\overline{w}_{\xi,\eta}(y)-\overline{\mu}| \leq C(\operatorname{osc} m_\xi)\exp(-c|\xi|R)+C|m_\xi|_{C^{0,\alpha\beta}}|\eta|^{\alpha\beta} \;\text{ for }\; y \in \partial P_\xi + R\hat{\xi}. \tag{4.4.13}$$

We use (4.4.13) to restrict to a domain where we can use Theorem 4.2.3, then we simultaneously are able to estimate $\mu-\overline{\mu}$ and $w_{\xi,\eta}-\overline{w}_{\xi,\eta}$. Fix an R to be chosen and consider,

$$\tilde{w}_{\xi,\eta}(y) = w_{\xi,\eta}(y)+(\overline{\mu}-\mu)R^{-1}y\cdot\hat{\xi}+\sup_{\partial P_\xi+R\hat{\xi}}\left[|w_{\xi,\eta}(\cdot)-\mu|+|\overline{w}_{\xi,\eta}(\cdot)-\overline{\mu}|\right].$$

Note that with this modification $\tilde{w}_{\xi,\eta}$ still solves the same equation as $w_{\xi,\eta}$ in P_ξ with the same boundary condition on ∂P_ξ but also

$$\tilde{w}_{\xi,\eta}(y) \geq \overline{w}_{\xi,\eta}(y) \;\text{ on }\; \partial P_\xi + R\hat{\xi}.$$

Now Theorem 4.2.3 implies that

$$\overline{w}_{\xi,\eta}(y)-\tilde{w}_{\xi,\eta}(y) \leq C(|\xi|^{-\beta}+R^\beta)(R^{-1}|\eta|)^{\alpha\beta}|m_\xi|_{C^{0,\beta}(\mathbb{R})}.$$

Rewriting this in terms of $w_{\xi,\eta}$ using (4.4.13),

$$\overline{w}_{\xi,\eta}(y)-w_{\xi,\eta}(y) \leq (\overline{\mu}-\mu)R^{-1}y\cdot\hat{\xi}+C|\psi|_{C^{0,\beta}}|\xi|^{-\beta}[(R^{-1}|\eta|)^{\alpha\beta}(1+|\xi|^\beta R^\beta)+\exp(-c|\xi|R)+|\xi|^{\alpha\beta}|\eta|^{\alpha\beta}].$$

Let us choose $R = 2(c|\xi|)^{-1}\log\frac{1}{|\xi||\eta|}$ to obtain

$$\overline{w}_{\xi,\eta}(y)-w_{\xi,\eta}(y) \leq (\overline{\mu}-\mu)R^{-1}y\cdot\hat{\xi}+C|\psi|_{C^{0,\beta}}|\xi|^{\beta(\alpha-1)}|\eta|^{\alpha\beta}(\log\tfrac{1}{|\xi||\eta|}). \tag{4.4.14}$$

This implies an estimate for the limits as well by evaluating for $y \in \partial P_\xi + \frac{1}{2}R\hat{\xi}$:

$$\overline{\mu}-\mu \leq \tfrac{1}{2}(\overline{\mu}-\mu)+C|\psi|_{C^{0,\beta}}|\xi|^{\beta(\alpha-1)}|\eta|^{\alpha\beta}(\log\tfrac{1}{|\xi||\eta|}).$$

Here we have used (4.4.13) to estimate $\mu-w_{\xi,\eta}$ and $\overline{\mu}-\overline{w}_{\xi,\eta}$ on $\partial P_\xi + \frac{1}{2}R\hat{\xi}$, the error is of the same order as in (4.4.14) so we combined terms. Rearranging the last inequality and making a similar argument for the lower bound, we conclude that

$$|\overline{\mu}-\mu| \leq C|\psi|_{C^{0,\beta}}|\xi|^{\beta(\alpha-1)}|\eta|^{\alpha\beta}(\log\tfrac{1}{|\xi||\eta|}). \tag{4.4.15}$$

But now we can plug (4.4.15) back into (4.4.2) and obtain, for any $0 < y \cdot \hat{\xi} < R$ and hence for any $y \in P_\xi$ by maximum principle,

$$|\overline{w}_{\xi,\eta}(y) - w_{\xi,\eta}(y)| \leq C|\psi|_{C^{0,\beta}}|\xi|^{\beta(\alpha-1)}|\eta|^{\alpha\beta}(\log \tfrac{1}{|\xi||\eta|}). \tag{4.4.16}$$

□

4.4.3 Step 3: Reduction to a two-dimensional Problem

The third step of our reduction procedure is actually more of notation change. Let $\bar{F}$ be a homogeneous, uniformly elliptic operator. We are concerned with the solution of,

$$\begin{cases} \overline{F}(D^2\overline{w}_{\xi,\eta}) = 0 & \text{in} \quad P_\xi \\ \overline{w}_{\xi,\eta} = m_\xi(y\cdot\eta) & \text{on} \quad \partial P_\xi \end{cases} \tag{4.4.17}$$

for a fixed unit vector $\eta \in S^{d-1}$ with $\eta \cdot \xi = 0$.

In the previous section we have already observed that $\overline{w}_{\xi,\eta}$ varies only in the $\hat{\xi}, \eta$ directions. To emphasize the two-dimensionality of $\overline{w}_{\xi,\eta}$ let us define $W_{\xi,\eta} : \mathbb{R}^2_+ \to \mathbb{R}$ by,

$$W_{\xi,\eta}(z) = \overline{w}_{\xi,\eta}(z_1\eta + z_2\hat{\xi}). \tag{4.4.18}$$

Now $W_{\xi,\eta}$ will solve an equation in the upper half space with an operator $G_{\eta,\xi}$ which is essentially the projection of $\overline{F}$ onto the ξ-η plane. Let $M \in \mathscr{M}_{2\times 2}$ a symmetric 2×2 matrix, the definition of $G_{\xi,\eta}(M)$ is somewhat cumbersome in terms of notation but the idea is quite simple,

$$G_{\xi,\eta}(M) := \overline{F}(\sum_{e,f\in\{\eta,\hat{\xi}\}} M_{ij} e\otimes f). \tag{4.4.19}$$

It is quite important to note the dependence of G on the orientation of η; $G_{\eta,\xi}$ may not be the same operator as $G_{-\eta,\xi}$.

Lemma 4.4.7. *Let $W_{\xi,\eta}$ and $G_{\xi,\eta}$ be as given in* (4.4.18) *and* (4.4.19). *Then $W_{\xi,\eta}(z)$ is the unique solution of*

$$\begin{cases} G_{\xi,\eta}(D_z^2 W_{\xi,\eta}) = 0 & \textit{in} \quad \mathbb{R}^2_+ \\ W_{\xi,\eta} = m_\xi(z_1) & \textit{on} \quad \partial\mathbb{R}^2_+. \end{cases} \tag{4.4.20}$$

The key point of this reduction is that we realize $\overline{w}_{\xi,\eta}$ as the solutions of *different* pdes in the *same* domain with the *same* boundary conditions.

4.4.4 Step 4: The directional limits of μ at rational directions

We are now ready to precisely characterize the limiting behavior of $\mu(\cdot,\psi,F)$ near a rational vector $\xi \in \mathbb{Z}^d \setminus \{0\}$ of $\mu(\cdot,\psi,F)$ in terms of the boundary layer tails of the class of simpler two dimensional problems (4.4.20).

Definition 4.4.8. Let F be a uniformly elliptic operator as given in Section 2. For $\psi \in C^{0,\beta}(\mathbb{T}^d)$, a rational vector $\xi \in \mathbb{Z}^d$ and a unit vector $\eta \perp \xi$ define,

$$L_\xi(\eta;(\psi,F)) := \lim_{R\to\infty} W_{\xi,\eta}(0,R;(\psi,F)).$$

Similar arguments to those used in the previous section will show that L_ξ is continuous in η. For example see the proof of Proposition 4.4.3.

Lemma 4.4.9. *For $\xi \in \mathbb{Z}^d \setminus \{0\}$, $\beta \in (0,1)$ and any η,η' unit vectors orthogonal to ξ,*

$$|L_\xi(\eta;(\psi,F)) - L_\xi(\eta';(\psi,F))| \le C(\Lambda,d,\beta)\|\psi\|_{C^{0,\beta}}|\eta-\eta'|^\beta(1+\log\tfrac{1}{|\eta-\eta'|}).$$

A combination of the results of the previous sections yields the following classification of the asymptotic behavior of $\mu(\cdot,\psi,F)$ near ξ:

Proposition 4.4.10. *Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ be irreducible and let $\sigma : [0,1) \to S^{d-1}$ a geodesic path with unit speed and $\sigma(0) = \hat{\xi}$. Then there is $\alpha(\Lambda) \in (0,1)$ such that for any $\beta \in (0,1)$,*

$$|\mu(\sigma(t),\psi,F) - L_\xi(\sigma'(0);(\psi,F))| \le C(\Lambda,d)|\psi|_{C^{0,\beta}(\mathbb{T}^d)}|\xi|^{\alpha\beta}t^{\alpha\beta}.$$

4.5 Continuity of μ

One immediate consequence of Proposition 4.4.10 is a continuity result analogous to Theorem 4.1 of Choi and Kim [CK14] for operators $F(M,y)$ such that $\overline{F}$ is rotation invariant. Let us repeat that the proof we have given of this result is not new, rather we have made each of the steps [CK14] quantitative and elucidated the secondary two-dimensional cell problem underlying the limiting behavior near rational directions. This additional work will be essential to the results that follow but is not so important just to get a continuous extension of $\mu(\cdot,\psi,F)$ to the rational directions without an explicit modulus.

Theorem 4.5.1. *Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ be irreducible. If $\overline{F}$ is invariant with respect to the rotations/reflections that preserve ξ or $\overline{F}$ is linear, then $L_\xi(\cdot;(\psi,F)) \equiv L_\xi(\psi,F)$ independent of the approach direction. As a consequence, $\mu(\cdot,\psi,F)$ extends continuously to $\hat{\xi}$ with value $L_\xi(\psi,F)$ and,*

$$|\mu(\sigma,\psi,F) - L_\xi(\psi,F)| \leq C(\Lambda,d,\beta)|\psi|_{C^{0,\beta}(\mathbb{T}^d)}|\xi - |\xi|\sigma|^{\alpha\beta},$$

for some $\alpha(\Lambda) \in (0,1)$ and any $\beta \in (0,1)$. In particular, if $\overline{F}$ is rotation invariant or linear, $\mu(\cdot,\psi,F)$ extends from $S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$ to a continuous function on S^{d-1}.

Proof. It suffices to show L_ξ is constant in the cases claimed, the rest of the Theorem will then follow from Proposition 4.4.10. For any $\eta_1, \eta_2 \perp \xi$ let O be a rotation sending η_1 to η_2 and holding ξ fixed. Now $\overline{w}_{\xi,\eta_1}(O^t\cdot)$ has the same boundary data as $\overline{w}_{\xi,\eta_2}(\cdot)$ and by the rotation invariance of $\overline{F}$ they solve the same pde in P_ξ. Thus by uniqueness they are equal. In particular they have the same boundary layer tail so $L_\xi(\eta_1;(\psi,F)) = L_\xi(\eta_2;(\psi,F))$.

In the second case we refer to Lemma 3.6 of [Fel14] which shows, using Riesz Representation Theorem, that when F is linear and homogeneous $\mu(\sigma,\psi,F) = \langle\psi\rangle$ (the average over the torus). We apply this to $\overline{w}_{\xi,\eta}$ which satisfies the assumptions of the Lemma since it is a solution of $\overline{F}$ which is homogeneous and, by assumption, linear. We derive for every $\eta \perp \xi$,

$$L_\xi(\eta;(\psi,F)) = \lim_{R\to\infty} \overline{w}_{\xi,\eta}(R\hat{\xi}) = |\xi| \int_0^{1/|\xi|} m_\xi(t;(\psi,F))dt.$$

The right hand side is independent of η which was the desired result. □

As a corollary of Theorem 4.5.1 we will show an explicit modulus of Hölder continuity for the homogenized boundary condition when $\overline{F}$ is rotation invariant or linear. The argument is entirely number theoretic and relies on Dirichlet's Theorem, Theorem 4.2.7. A sharper estimate in the linear case can be found in Section 4.7. The improvement there is in the rate of convergence at a single rational direction. The argument using Dirichlet's Theorem stays the same.

Corollary 4.5.2. *Let F satisfying the assumptions of Theorem 4.5.1. There is $\alpha(\Lambda) \in (0,1)$ such that for all $\beta \in (0,1)$, $\psi \in C^{0,\beta}(\mathbb{T}^d)$, and σ_1 and σ_2 irrational vectors in S^{d-1} we have*

$$|\mu(\sigma_1,\psi,F) - \mu(\sigma_2,\psi,F)| \leq C(d,\Lambda,\beta)|\psi|_{C^{0,\beta}}|\sigma_1 - \sigma_2|^{\beta\alpha/d}.$$

Proof. Assume $|\psi|_{C^{0,\beta}} \leq 1$, the general case follows from scaling. Let $\varepsilon := |\sigma_1 - \sigma_2|$, and let $N = \varepsilon^{-(d-1)/d}$. Then due to Lemma 4.2.7 there exists $\xi \in \mathbb{Z}^d$ and $n \in \mathbb{Z}$ with $1 \leq n \leq N$ such that

$$\left| n\frac{\sigma_1}{|\sigma_1|_\infty} - \xi \right| \leq (d-1)^{1/2} N^{-1/(d-1)}.$$

Note that $n \gtrsim |\sigma_1|_\infty |\xi| \geq d^{-1/2}|\xi|$. Due to this and the choice of N we have

$$\begin{aligned}
|\sigma_2 - n^{-1}|\sigma_1|_\infty \xi| &= |\sigma_1 - \sigma_2| + \left|\sigma_1 - n^{-1}|\sigma_1|_\infty \xi\right| \\
&\leq \varepsilon + (d-1)^{1/2} n^{-1} |\sigma_1|_\infty N^{-1/(d-1)} \\
&\leq \varepsilon + C_d |\xi|^{-1} N^{-1/(d-1)}.
\end{aligned}$$

Now we apply Theorem 4.5.1 with $\sigma = \sigma_j$ at the rational direction ξ to conclude that

$$|\mu(\sigma_1) - \mu(\sigma_2)| \lesssim N^{-\frac{\alpha\beta}{(d-1)}} + |\xi|^{\alpha\beta} \varepsilon^{\alpha\beta}.$$

Using that $|\xi| \lesssim N$ we obtain

$$|\mu(\sigma_1) - \mu(\sigma_2)| \lesssim N^{-\frac{\alpha\beta}{(d-1)}} \log N + (N\varepsilon)^{\alpha\beta} \lesssim \varepsilon^{\alpha\beta/d}.$$

$\square$

4.6 Discontinuity of μ

Given the set up of the previous sections it may seem at least plausible to the reader that when F is nonlinear and not rotation invariant, for a given ξ the directional limit function L_ξ will typically be non-constant, resulting in the discontinuity of the homogenized boundary data. On the other hand it is not obvious, at least to us, how to prove that any specific pair (ψ, F) results in a non-constant L_ξ. Apart from explicitly computing the solutions the only way to differentiate the boundary layer tails of the $W_{\xi,\eta}$ would be to use maximum principle. However, except in some specially arranged cases, one cannot choose $\eta_1, \eta_2 \perp \xi$ so that $G_{\xi,\eta_1} \geq G_{\xi,\eta_2}$ and so there is no reason for W_{ξ,η_j} to be ordered in the whole of $\mathbb{R}^2_+$ for any such pair η_1, η_2. We instead find monotonicity by perturbing (ψ, F). We are then able to show that the class of (ψ, F) for which $L_\xi(\cdot;(\psi,F))$ is non-constant is open and dense in the appropriate topologies.

Let us give a heuristic description of how this monotonicity arises. The goal is to show that for any (ψ, F) and $\xi \in \mathbb{Z}^d \setminus \{0\}$ we can find a nearby (ψ', F') such that $L_\xi(\cdot;(\psi', F'))$ is non-constant. We are only able to show that a small perturbation of $\overline{F}$ would lead to L_ξ being non-constant, which directly corresponds to perturbation of the homogeneous operators since $F = \overline{F}$. In the general case of inhomogeneous F it is not clear to us whether it is possible to perturb F to correspond to the desired perturbation of $\bar{F}$; we leave this as an open question. Let us now describe the perturbation of homogeneous operators F. First note that we only need to perturb (ψ, F) when $L_\xi(\cdot;(\psi, F))$ is constant, otherwise we could take $(\psi', F') = (\psi, F)$. When $d \geq 3$ we can find two directions η_1, η_2 perpendicular both to each other and to ξ. We then perturb $\overline{F}$ in a monotone and hence intrinsically nonlinear way, heuristically affecting the choice of diffusions in the η_1 direction while leaving the η_2 direction unchanged. More concretely the perturbation will satisfy that $G'_{\xi,\eta_1} \lneq G_{\xi,\eta_1}$ while $G'_{\xi,\eta_2} = G_{\xi,\eta_2}$. Then, up to a small perturbation of ψ, strong maximum principle will imply that $W'_{\xi,\eta_1} < W_{\xi,\eta_1}$ and, since periodicity provides compactness in the lateral directions, also $L_\xi(\eta_1,(\psi', F')) < L_\xi(\eta_2,(\psi', F'))$, while $L_\xi(\eta_2)$ remains unchanged. Now, having assuming that $L_\xi(\cdot;(\psi, F))$ was originally constant, $L_\xi(\cdot;(\psi', F'))$ must be non-constant.

The only natural notion of genericity in this setting, to our knowledge, is topological. We make precise the topological setting. Our boundary data will be taken from the spaces

$$C^{k,\beta}(\mathbb{T}^d) = \{\psi : \mathbb{T}^d \to \mathbb{R} | \ \|\psi\|_{C^{k,\beta}(\mathbb{T}^d)} < +\infty\} \text{ with the norm } \|\cdot\|_{C^{k,\beta}} = \|\cdot\|_{C^{k-1,1}} + |D^k \cdot|_{C^{0,\beta}}.$$

Let us next define the space of uniformly elliptic operators,

$$\mathrm{UE}_d = \{F : \mathscr{M}_{d\times d} \to \mathbb{R} | \ F \in \cup_{\Lambda>1} \mathscr{S}_{1,\Lambda} \text{ uniformly elliptic and positively 1-homogeneous}\}.$$

For $F \in \mathrm{UE}_d$ we define the ellipticity ratio $\Lambda(F)$ to be the minimal $\Lambda > 1$ such that $F \in \mathscr{S}_{1,\Lambda}$. It is easy to check from this that $F \in \mathrm{UE}_d$ are Lipschitz continuous with respect to the operator norm metric on $\mathscr{M}_{d\times d}$ with Lipschitz constant $d\Lambda(F)$. Conversely consider an F which is Lipschitz continuous with respect to the operator norm metric on $\mathscr{M}_{d\times d} \simeq \mathbb{R}^{\frac{d(d+1)}{2}}$. For this F the gradient DF, from standard inner product $\mathrm{Tr}(AB)$ on $d \times d$ matrices A, B, is defined Lebesgue almost everywhere. The Lipschitz constant of F is $\|DF\|_{L^\infty(\mathscr{M}_{d\times d})}$ where we implicitly take the underlying matrix norm to be the dual of the operator norm. Based on this definition it is straightforward to check that $\Lambda(F) \le \|DF\|_\infty$. We take as the metric on UE_d,

$$d_{\mathrm{UE}_d}(F_1,F_2) := \sup_{\|M\|=1} |F_1(M) - F_2(M)| + \|DF_1 - DF_2\|_{L^\infty(\mathscr{M}_{d\times d})}.$$

Noting that Cauchy sequences have $\|DF_n\|_\infty$ bounded and hence $\Lambda(F_n)$ bounded we see that $(\mathrm{UE}_d, d_{\mathrm{UE}_d})$ is complete. We draw our operator and boundary data (ψ,F) from the spaces,

$$X = C^{k,\beta}(\mathbb{T}^d) \times \mathrm{UE}_d \;\text{ with distance }\; d_X((\psi_1,F_1),(\psi_2,F_2)) = \|\psi_1 - \psi_2\|_{C^{k,\beta}} + d_{\mathrm{UE}_d}(F_1,F_2),$$

which are complete metric spaces. The results below hold for all $k \in \mathbb{N}\cup\{0\}$ and $\beta \in (0,1)$.

Theorem 4.6.1. *Let $d \ge 3$ and $\xi \in \mathbb{Z}^d \setminus \{0\}$. Then the set*

$$E_\xi = \{(\psi,F) \in X \mid \mu(\cdot,\psi,F) \text{ is discontinuous at } \hat{\xi}\} \text{ is open and dense in } X.$$

In particular there is a residual set $E \subset X$, a countable intersection of open dense sets $E = \cap_{\xi\in\mathbb{Z}^d\setminus\{0\}} E_\xi$, such that for all $(\psi,F) \in E$,

$$\mu(\cdot,\psi,F) \text{ is discontinuous at } \textbf{every} \text{ rational direction.}$$

The proof of the theorem consists of the following two steps. First we prove that E_ξ is open. The proof of Lemma 4.6.2 is more or less standard, and is due to comparison principle and the stability of viscosity solutions with respect to uniform convergence.

Lemma 4.6.2. *For each $\xi \in \mathbb{Z}^d$, $(\psi,F) \in X$, $L_\xi : \{\eta \in S^d, \eta\cdot\xi = 0\} \times X \to \mathbb{R}$ is continuous with respect to d_X at (ψ,F),*

$$\sup_{\eta\in S^d,\eta\cdot\xi=0} |L_\xi(\eta;(\psi',F')) - L_\xi(\eta;(\psi,F))| \to 0 \;\text{ as }\; d_X((\psi',F'),(\psi,F)) \to 0.$$

In particular by Proposition 4.4.10 E_ξ is open.

Next we will show that E_ξ is dense, whose proof strongly depends on the conditions $d \ge 3$ and that F is homogeneous.

Proposition 4.6.3. *Let $d \ge 3$ and $\xi \in \mathbb{Z}^d$. Then for given $(\psi,F) \in X$ and $\varepsilon > 0$, there exists $(\psi_\varepsilon,F_\varepsilon)$ such that*

$$d_X((\psi_\varepsilon,F_\varepsilon),(\psi,F)) \le \varepsilon \;\text{ and }\; L_\xi(\cdot\,;(\psi_\varepsilon,F_\varepsilon)) \text{ is non-constant.}$$

In particular $\mu(\cdot,\psi_\varepsilon,F_\varepsilon)$ is discontinuous at ξ by Proposition 4.4.10.

Now we proceed with the proofs.

Proof of Lemma 4.6.2. It suffices to prove the result for $k=0$ and and $\beta>0$. Let (ψ_n, F_n) be a sequence in (X, d_X) converging to (ψ, F). Let us recall the definition of $L_\xi(\eta;(\psi_n, F_n))$ given in Definition 4.4.8:

$$\begin{cases} \overline{F}_n(D^2 w_n) = 0 & \text{in} \quad P_\xi \\ w_n(y) = m_\xi(y\cdot\eta;(\psi_n,F_n)) & \text{on} \quad \partial P_\xi \end{cases} \quad \text{and} \quad L_\xi(\eta;(\psi_n,F_n)) = \lim_{R\to\infty} w_n(R\xi). \tag{4.6.1}$$

Since $F_n's$ are homogeneous, $\overline{F}_n = F_n$ but we continue to write $\overline{F}_n$ to emphasize the correct definition of L_ξ. We begin by first investigating the continuity properties of m_ξ. The claim is

$$\sup_t |m_\xi(t;(\psi_n,F_n)) - m_\xi(t;(\psi,F))| \to 0 \text{ as } n\to\infty.$$

Observe that by maximum principle,

$$|m_\xi(\cdot;(\psi_n,F_n)) - m_\xi(\cdot;(\psi,F_n))| \le \|\psi_n - \psi\|_\infty.$$

Thus it remains to show that $\sup_t |m_\xi(t;(\psi,F_n)) - m_\xi(t;(\psi,F))| \to 0$. The pointwise convergence with fixed t is due to stability of viscosity solutions with respect to uniform convergence of F_n, but a little extra work is required to show that the convergence is uniform in t. Note that by Lemma 4.4.2, we have

$$|m_\xi(t;(\psi,F_n)) - m_\xi(t';(\psi,F_n))| \le C(\Lambda(F_n))|\psi|_{C^{0,\beta}}|t-t'|^\beta \;\text{ and }\; |m_\xi(t;(\psi,F_n))| \le \|\psi\|_\infty.$$

Since $F_n \to F$ is a convergent sequence in d_{UE_d}, $\|F_n\|_\infty$ and $\Lambda(F_n)$ are bounded. Since $m_\xi(\cdot;(\psi,F_n))$ are uniformly bounded and equicontinuous $\frac{1}{|\xi|}$-periodic functions on $\mathbb{R}$, every subsequence has a uniformly convergent subsequence. It follows that, since $m_\xi(\cdot;(\psi,F_n))$ converge pointwise to $m_\xi(\cdot;(\psi,F))$, they will also converge uniformly.

Now let us define $\tilde{w}_n$ to solve

$$\begin{cases} \overline{F}_n(D^2 \tilde{w}_n) = 0 & \text{in} \quad P_\xi; \\ \tilde{w}_n(y) = m_\xi(y\cdot\eta;(\psi,F)) & \text{on} \quad \partial P_\xi. \end{cases}$$

Since we have already proven that $m_\xi(y\cdot\eta;(\psi_n,F_n))\to m_\xi(y\cdot\eta;(\psi,F))$ uniformly on ∂P_ξ, by maximum principle,

$$\begin{aligned}|\lim_{R\to\infty}\tilde{w}_n(R\xi)-L_\xi(\eta;(\psi_n,F_n))| &\le \sup_{y\in P_\xi}|w_n(y)-\tilde{w}_n(y)|\\ &\le \sup_{t\in\mathbb{R}}|m_\xi(t;(\psi_n,F_n))-m_\xi(t;(\psi,F))|\to 0 \text{ as } n\to\infty.\end{aligned}$$

By a similar argument as above, since $\overline{F}_n\to\overline{F}$ uniformly on compact sets of $\mathscr{M}_{d\times d}$ when $d_{\mathrm{UE}_d}(F_n,F)\to 0$, we have that $\tilde{w}_n\to w$ locally uniformly in P_ξ and

$$|\lim_{R\to\infty}\tilde{w}_n(R\xi)-L_\xi(\eta;(\psi,F))|\to 0.$$

Combined with the previous estimate this yields that

$$|L_\xi(\eta;(\psi_n,F_n))-L_\xi(\eta;(\psi,F))|\to 0 \text{ as } n\to\infty.$$

This shows pointwise convergence of $L_\xi(\cdot;(\psi_n,F_n))$. Uniform convergence over all unit vectors η tangential to ξ will again follow from uniform boundedness and equicontinuity of L_ξ (see Lemma 4.4.9).

□

Finally we give the proof of Proposition 4.6.3.

Proof of Proposition 4.6.3. Let $\xi\in\mathbb{Z}^d$ and $(\psi,F)\in X$. If $L_\xi(\cdot;(\psi,F))$ is non-constant then we are done, so we suppose it is constant and construct $(\psi_\varepsilon,F_\varepsilon)$.

Let η_1,η_2 be unit vectors such that $\eta_j\perp\xi$ and $\eta_1\perp\eta_2$, which is possible since $d\ge 3$. We will aim to perturb F to construct $(F_\varepsilon,\psi_\varepsilon)$ so that

$$L_\xi(\eta_1;(\psi_\varepsilon,F_\varepsilon))<L_\xi(\eta_2;(\psi_\varepsilon,F_\varepsilon)). \tag{4.6.2}$$

To this end let us define,

$$F_\varepsilon(M):=\max\{F(M),F(M+\varepsilon(\eta_1^TM\eta_1)\eta_1\otimes\eta_1)\}.$$

It is not difficult to check the definition of ellipticity to see that $F_\varepsilon\in\mathrm{UE}_d$. Also,

$$\sup_{\|M\|=1}|F(M+\varepsilon(\eta_1^TM\eta_1)\eta_1\otimes\eta_1)-F(M)|\le\Lambda(F)\varepsilon,$$

and furthermore, since $D[F(M+\varepsilon(\eta_1^T M\eta_1)\eta_1\otimes\eta_1)] = DF+\varepsilon(\eta^T DF\eta)\eta\otimes\eta$,

$$|DF_\varepsilon - DF| \lesssim_d \varepsilon\Lambda(F) \text{ where it is defined.}$$

Combining these two estimates it follows that F_ε is close to F in d_{UE_d} metric,

$$d_{\mathrm{UE}_d}(F,F_\varepsilon) \lesssim_d \Lambda(F)\varepsilon.$$

Let us now show (4.6.2). From the definition of the 2-d operators $G_{\xi,e}$ for $N\in\mathscr{M}_{2\times 2}$,

$$G^\varepsilon_{\xi,\eta_2}(N) = \max\left\{F\left(\sum_{\alpha,\beta\in\{\xi,\eta_2\}} N_{ij}\alpha\otimes\beta\right), F\left(\sum_{\alpha,\beta\in\{\xi,\eta_2\}} N_{ij}(\alpha\otimes\beta+\varepsilon(\eta_1\cdot\alpha)(\eta_1\cdot\beta)\eta_1\otimes\eta_1)\right)\right\}$$

note that $\eta_1\cdot\alpha = \eta_1\cdot\beta = 0$ for any combination of $\alpha,\beta\in\{\xi,\eta_2\}$ since $\xi,\eta_2\perp\eta_1$ and so

$$G^\varepsilon_{\xi,\eta_2}(N) = G_{\xi,\eta_2}(N) \text{ for all } \varepsilon>0. \tag{4.6.3}$$

On the other hand, calling $e_1=(1,0)$,

$$G^\varepsilon_{\xi,\eta_1}(N) = \max\{G_{\xi,\eta_1}(N), G_{\xi,\eta_1}(N+\varepsilon N_{11}e_1\otimes e_1)\}.$$

Now for any N with $N_{11}\le 0$ by uniform ellipticity,

$$G_{\xi,\eta_1}(N+\varepsilon N_{11}e_1\otimes e_1) \ge G_{\xi,\eta_1}(N) - \varepsilon N_{11},$$

and thus

$$G^\varepsilon_{\xi,\eta_1}(N) > G_{\xi,\eta_1}(N) \text{ if } N_{11}<0. \tag{4.6.4}$$

Let us denote W_{ξ,η_i} and W_{ξ,η_i} as given in (4.4.18) for G_{ξ,η_i} and $G^\varepsilon_{\xi,\eta_i}$. From above discussions we know that, for any boundary data $\phi\in C(\mathbb{T}^d)$,

$$W_{\xi,\eta_1}(\cdot;(\phi,F_\varepsilon)) \le W_{\xi,\eta_1}(\cdot;(\phi,F)) \text{ and } W_{\xi,\eta_2}(\cdot;(\phi,F_\varepsilon)) \equiv W_{\xi,\eta_2}(\cdot;(\phi,F))$$

where the first inequality is due to the fact $G^\varepsilon_{\xi,\eta_1}\ge G_{\xi,\eta_1}$. On the other hand, since $G_{\xi,\eta_1}(D^2W_{\xi,\eta_1}(\cdot;(\phi,F_\varepsilon)))\le 0$, the dichotomy holds due to the strong maximum principle:

$$(i)\ W_{\xi,\eta_1}(\cdot;(\phi,F_\varepsilon)) < W_{\xi,\eta_1}(\cdot;(\phi,F)) \text{ or } (ii)\ W_{\xi,\eta_1}(\cdot;(\phi,F_\varepsilon)) \equiv W_{\xi,\eta_1}(\cdot;(\phi,F)). \tag{4.6.5}$$

In case (i), since W_{ξ,η_1} is $\frac{1}{|\xi|}$-periodic in the z_1 direction,

$$W_{\xi,\eta_1}(z_1,1;(\phi,F_\varepsilon)) \le W_{\xi,\eta_1}(z_1,1;(\phi,F)) - \delta \ \text{ for some } \ \delta > 0.$$

By maximum principle it follows

$$W_{\xi,\eta_1}(z;(\phi,F_\varepsilon)) \le W_{\xi,\eta_1}(z;(\phi,F)) - \delta \ \text{ for } \ z_2 > 1$$

and therefore

$$L_\xi(\eta_1;(\phi,F_\varepsilon)) = \lim_{z_2\to+\infty} W_{\xi,\eta_1}(z;(\phi,F_\varepsilon)) \le \lim_{z_2\to+\infty} W_{\xi,\eta_1}(z;(\phi,F)) - \delta < L_\xi(\eta_1;(\phi,F)). \quad (4.6.6)$$

We have just shown that (4.6.5) (i) implies (4.6.6) and therefore we actually have the dichotomy,

$$(i)\ L_\xi(\eta_1;(\phi,F_\varepsilon)) < L_\xi(\eta_1;(\phi,F)) \ \text{ or } \ (ii)\ W_{\xi,\eta_1}(\cdot;(\phi,F_\varepsilon)) \equiv W_{\xi,\eta_1}(\cdot;(\phi,F)). \quad (4.6.7)$$

Let us next prove that (i) holds whenever $m_\xi(\cdot;(\phi,F_\varepsilon))$ is non-constant. Suppose this is not the case, then $m_\xi(\cdot;(\phi,F_\varepsilon))$ is non-constant and (ii) holds in (4.6.7). By Nirenberg's Theorem (Theorem 4.2.1) $W_{\xi,\eta_1}, W^\varepsilon_{\xi,\eta_1}$ are C^{2,α_0} for a small $\alpha_0(\Lambda) > 0$ and are classical solutions of their respective equations. But then $D^2W^\varepsilon_{\xi,\eta_1} \equiv D^2W_{\xi,\eta_1}$ and

$$0 = G^\varepsilon_{\xi,\eta_1}(D^2W_{\xi,\eta_1}(z)) = G^\varepsilon_{\xi,\eta_1}(D^2W_{\xi,\eta_1}(z)) \ \text{ for every } z \in \mathbb{R}^2_+ \text{ and therefore } \ D^2_{11}W_{\xi,\eta_1} \ge 0.$$

On the other hand, by the $1/|\xi|$ periodicity of $W_{\xi,\eta}$ in the z_2 variable,

$$\int_a^{a+1/|\xi|} D^2_{11}W_{\xi,\eta_1}(z_1,z_2)dz_2 = 0 \ \text{ for all } \ a \in \mathbb{R}, z_2 > 0.$$

and therefore $D^2_{11}W_{\xi,\eta_1} = 0$ in the whole of $\mathbb{R}^2_+$. In particular for any $t > 0$, $W_{\xi,\eta_1}(t,\cdot)$ is constant, so W_{ξ,η_1} is also constant in $\{z_1 > 0\}$ by uniqueness of the bounded solution of $G_{\xi,\eta_1}(\cdot) = 0$. Since $t > 0$ was arbitrary W_{ξ,η_1} is constant in $\mathbb{R}^2_+$, but this contradicts the boundary data $m_\xi(\cdot;(\phi,F_\varepsilon))$ being non-constant.

Lastly we show that we can choose ψ_ε with $\|\psi_\varepsilon - \psi\|_{C^{k,\beta}} \le C(k,\beta)\varepsilon$ such that $m_\xi(\cdot;(\psi_\varepsilon,F_\varepsilon))$ is non-constant. By above arguments this would yield our claim (4.6.2). If $m_\xi(\cdot;(\psi,F_\varepsilon))$ is already non-constant then we don't need to do anything and can take $\psi_\varepsilon = \psi$. Otherwise let us take

$$\psi_\varepsilon(y) := \psi(y) + \varepsilon\cos(2\pi y\cdot\xi) \ \text{ which satisfies } \ \|\psi_\varepsilon - \psi\|_{C^{k,\beta}} \le C(k,\beta)\varepsilon.$$

Observe that for each fixed hyperplane $\partial P_\xi + t\hat{\xi}$ we have $\psi_\varepsilon(y) = \psi(y) + \varepsilon\cos(2\pi|\xi|t)$. Therefore

$$m_\xi(t;(\psi_\varepsilon, F_\varepsilon)) = m_\xi(t;(\psi, F_\varepsilon)) + \varepsilon\cos(2\pi|\xi|t).$$

This is evidently non-constant when $m_\xi(\cdot;(\psi, F_\varepsilon))$ is constant.

We have now proven that there is $(\psi_\varepsilon, F_\varepsilon)$ with $d_X((\psi_\varepsilon, F_\varepsilon),(\psi, F)) \leq C\varepsilon$ such that

$$L_\xi(\eta_1;(\psi_\varepsilon, F_\varepsilon)) < L_\xi(\eta_1;(\psi_\varepsilon, F)) \text{ and } L_\xi(\eta_2;(\psi_\varepsilon, F_\varepsilon)) = L_\xi(\eta_2;(\psi_\varepsilon, F)).$$

Now if $L_\xi(\eta_1;(\psi_\varepsilon, F)) \neq L_\xi(\eta_2;(\psi_\varepsilon, F))$ we would be done, and otherwise $L_\xi(\eta_1;(\psi_\varepsilon, F)) = L_\xi(\eta_2;(\psi_\varepsilon, F))$ but then by the above arguments

$$L_\xi(\eta_1;(\psi_\varepsilon, F_\varepsilon)) < L_\xi(\eta_1;(\psi_\varepsilon, F)) = L_\xi(\eta_2;(\psi_\varepsilon, F)) = L_\xi(\eta_2;(\psi_\varepsilon, F_\varepsilon))$$

and we are again done. $\square$

4.7 Improved Estimates in the Linear Case

In this section we show the best possible continuity estimates for $\mu(\sigma,\psi,F)$ by our current methods in the linear case. The main tool is the higher regularity estimates available for linear operators in $\mathbb{R}^d$ or in half-spaces with smooth boundary data. For our purpose $W^{3,d}$ estimates would be sufficient, but we do not pursue this minimal assumption since it would be too much to expect in the general nonlinear case anyway.

Consider u^ε solving, for $\sigma \in S^{d-1}$ and $R > 1$,

$$\begin{cases} F(D^2u^\varepsilon, \frac{x}{\varepsilon}) := -\mathrm{Tr}(A(\frac{x}{\varepsilon})D^2u^\varepsilon) = 0 & \text{in} \quad 0 < x\cdot\sigma < R \\ u^\varepsilon = g(x) & \text{on} \quad x\cdot\sigma \in \{0,R\}. \end{cases} \tag{4.7.1}$$

We assume that A is $\mathbb{Z}^d$-periodic and smooth and satisfies $(Id)_{d\times d} \leq A \leq \Lambda(Id)_{d\times d}$. Due to the linearity, the interior corrector can be written as $v(y,M) = \Sigma_{ij} v_{ij}(y) M_{ij}$, where $v_{ij}(y)$ solves

$$-\mathrm{Tr}\left[A(y)(e_i \otimes e_j + D_y^2 v_{ij})\right] = -\overline{A}_{ij} \text{ in } \mathbb{R}^d,$$

with the estimate

$$\|v_{ij}\|_\infty, \|Dv_{ij}\|_\infty \leq C(\Lambda, d). \tag{4.7.2}$$

The following result is not optimal in terms of the required regularity of g, but it is sufficient to improve the estimate of Section 4.4.2, to match the order of the estimate in Section 4.4.1.

Theorem 4.7.1. *Let g and u^ε as given above, and let $\bar{u}$ be as given in* (4.2.3) *solving the homogenized equation. Then u^ε converges to $\bar{u}$ with the convergence rate*

$$\sup_{0<x\cdot\sigma<R} |u^\varepsilon(x) - \overline{u}(x)| \leq C(\Lambda, d, \beta)\|g(R\cdot)\|_{C^{3,\beta}} R^{-1}\varepsilon \quad \textit{for any } R > 0.$$

In order to use the above theorem to obtain the desired interior homogenization rate, higher regularity of the boundary condition is needed. In our setting in Section 4.4.2, that boundary condition is $m_\xi(x\cdot\hat{\eta})$ for some $\eta \perp \xi$. In the following lemma we show that $m_\xi(\cdot;(\psi,F))$ has sufficient regularity to apply Theorem 4.7.1 when ψ is regular. Indeed the proof of this lemma is the most interesting and delicate part of this section.

Lemma 4.7.2. *Let F be as in* (4.7.1). *Then for all $k \in \mathbb{N}\cup\{0\}$, $\beta \in (0,1]$ and any $\delta > 0$,*

$$\|m_\xi(\cdot;(\psi,F))\|_{C^{k,\beta}(\mathbb{R})} \leq C(k,\beta,\delta,\Lambda,d,\|A\|_{C^{k+1}})|\xi|^\delta \|\psi\|_{C^{k+4}}.$$

Note that by Lemma 4.4.2 when $k = 0$, $\beta \in (0,1)$ an improved estimate holds with $\delta = 0$. It is important here that the power of $|\xi|$ does not get worse as k increases.

Now we can return to the proof of Lemma 4.4.5 and use Theorem 4.7.1 to achieve the following result:

Lemma 4.7.3. *Let $\xi \in \mathbb{Z}^d \setminus \{0\}$ and $\eta \perp \xi$. Define $w_{\xi,\eta}, \overline{w}_{\xi,\eta}$ as in* (4.4.12). *Then, when F is linear,*

$$\sup_{P_\xi} |w_{\xi,\eta} - \overline{w}_{\xi,\hat{\eta}}| \leq C(\beta,\Lambda,d,\|A\|_{C^5})\|m_\xi(|\xi|^{-1}\cdot)\|_{C^{3,\beta}(\mathbb{R})}|\xi||\eta|[1 + (\log \tfrac{1}{|\xi||\eta|})^3].$$

Note that by Lemma 4.7.2,

$$\|m_\xi(|\xi|^{-1}\cdot) - m_\xi(0)\|_{C^{3,\beta}(\mathbb{R})} \lesssim |\xi|^{-3-\beta}|D^3 m_\xi|_{C^{0,\beta}} \lesssim_A \|\psi\|_{C^{k+7}},$$

where we have used the $\frac{1}{|\xi|}$-periodicity to estimate the lower order terms by $|D^3 m_\xi|_{C^{3,\beta}}$. Combining the estimates of Theorem 4.7.1, Lemma 4.7.2 and Lemma 4.7.3, in the same way as we did before in the general case in Theorem 4.5.1 and Corollary 4.5.2, we now obtain the continuity estimate,

Theorem 4.7.4. *If F is linear and $\psi \in C^7(\mathbb{T}^d)$, then for every irreducible $\xi \in \mathbb{Z}^d \setminus \{0\}$ and $\sigma \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$,*

$$|\mu(\sigma,\psi,F) - L_\xi(\psi,F)| \le C(\Lambda,d,\|A\|_{C^5})\|\psi\|_{C^7}|\xi - |\xi|\sigma|[1+(\log\tfrac{1}{|\xi-|\xi|\sigma|})^3].$$

Furthermore, for every $\sigma,\sigma' \in S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d$,

$$|\mu(\sigma,\psi,F) - \mu(\sigma',\psi,F)| \le C(\Lambda,d,\|A\|_{C^5})\|\psi\|_{C^7}|\sigma-\sigma'|^{1/d}[1+(\log\tfrac{1}{|\sigma-\sigma'|})^3].$$

We omit the proof of Theorem 4.7.4 as it is a straightforward consequence of the improved estimates of Lemma 4.7.2 and Lemma 4.7.3 and its usage in the proof of Theorem 4.5.1 and Corollary 4.5.2. Before we proceed with the proofs of Theorem 4.7.1 and Lemmas 4.7.2 - 4.7.3, we make an interlude to explain some background results that we will make use of.

4.7.1 Background Results

For the proofs in the next subsection we will need to use the regularity results of Avellaneda-Lin [AL89] for solutions of non-divergence form linear homogenization problems. We state the result here in the form that we will use it. Suppose that $u^\varepsilon, v^\varepsilon$ solve respectively,

$$-\operatorname{Tr}(A(\tfrac{x}{\varepsilon})D^2u^\varepsilon) = f(x) \text{ in } B_1 \text{ and } \begin{cases} -\operatorname{Tr}(A(\frac{x}{\varepsilon})D^2v^\varepsilon) = f(x) & \text{in} \quad B_1 \cap P_\sigma \\ v^\varepsilon = g(x) & \text{on} \quad \partial P_\sigma \cap B_1. \end{cases} \tag{4.7.3}$$

The following results hold uniformly in $\sigma \in S^{d-1}$. We first state the classical results in unit scale.

Theorem 4.7.5. *For every $\beta \in (0,1)$ there exists a constant $C(d,\Lambda,\beta,\|DA\|_{L^\infty(\mathbb{T}^d)})$ such that,*

(1) $\|D^2u^1\|_{C^{0,\beta}(B_{1/2})} \le C(\operatorname{osc}_{B_1} u^1 + \|f\|_{C^{0,\beta}(B_1)})$

(2) $\|D^2v^1\|_{C^{0,\beta}(B_{1/2}\cap P_\sigma)} \le C(\operatorname{osc}_{B_1\cap P_\sigma} v^1 + \|f\|_{C^{0,\beta}(B_1\cap P_\sigma)} + \|g\|_{C^{2,\beta}(\partial P_\sigma \cap B_1)})$

Scaling arguments yields that the best possible uniform regularity estimate with respect to $\varepsilon > 0$ is $C^{1,1}$. [AL89] shows that this estimate indeed holds. Below is a slightly simplified version of the main Theorem in [AL89]:

Theorem 4.7.6 (Avellaneda-Lin)**.** *For every $\beta \in (0,1)$ there exist constants $C_j(d,\Lambda,\beta,\|DA\|_{L^\infty(\mathbb{T}^d)})$ such that for all $\varepsilon > 0$,*

$$\begin{aligned}
&(1) \qquad \|D^2u^\varepsilon\|_{L^\infty(B_{1/2})} &&\leq C_1(\operatorname{osc}_{B_1} u^\varepsilon + \|f\|_{C^{0,\beta}(B_1)})\\
&(2) \qquad \|Du^\varepsilon\|_{C^{0,\beta}(B_{1/2})} &&\leq C_2(\operatorname{osc}_{B_1} u^\varepsilon + \|f\|_{L^\infty(B_1)})\\
&(3) \quad \|D^2v^\varepsilon\|_{L^\infty(B_{1/2}\cap P_\sigma)} &&\leq C_3(\operatorname{osc}_{B_1\cap P_\sigma} v^\varepsilon + \|f\|_{C^{0,\beta}(B_1\cap P_\sigma)} + \|g\|_{C^{2,\beta}(\partial P_\sigma\cap B_1)})\\
&(4) \quad \|Dv^\varepsilon\|_{C^{0,\beta}(B_{1/2}\cap P_\sigma)} &&\leq C_4(\operatorname{osc}_{B_1\cap P_\sigma} v^\varepsilon + \|f\|_{L^\infty(B_1\cap P_\sigma)} + \|g\|_{C^{1,\beta}(\partial P_\sigma\cap B_1)})
\end{aligned}$$

We will also need a fairly standard interpolation result on Hölder spaces. Since we are unable to find a reference we provide the proof in Appendix 4.A.1.

Lemma 4.7.7. *For any $g : \mathbb{R}^d \to \mathbb{R}$ smooth and any $\beta,\gamma \in (0,1)$,*

$$\|Dg\|_{L^\infty(\mathbb{R}^d)} \leq C(d)|g|_{C^{0,1-\gamma}(\mathbb{R}^d)}^{\frac{\beta}{\beta+\gamma}}|Dg|_{C^{0,\beta}(\mathbb{R}^d)}^{\frac{\gamma}{\beta+\gamma}}.$$

4.7.2 Proofs of the results of Section 4.7

Finally we prove Theorem 4.7.1, and Lemmas 4.7.2 and 4.7.3.

Proof of Theorem 4.7.1. It suffices to consider the case $R = 1$, the case for general $R > 0$ follows from rescaling. Let $\overline{u}$ be as given in the theorem. Let $U := \{0 < x\cdot\sigma < 1\}$. By the classical $C^{2,1}$ estimate up to the boundary (for the Laplacian) at unit scale,

$$\|D^3\overline{u}\|_{L^\infty(U)} \leq C(\Lambda,d,\beta)(\operatorname*{osc}_U \overline{u} + \|D^3g\|_{C^{0,\beta}(\partial U)}) \leq C\|g\|_{C^{3,\beta}(\partial U)}.$$

Let $\overline{u}^\varepsilon(x) := \rho_\varepsilon \star \overline{u}(x)$ with a standard mollifier ρ_ε, well defined on $U_\varepsilon = \{\varepsilon < x\cdot\sigma < 1-\varepsilon\}$, and define

$$\phi^\varepsilon(x) := \overline{u}^\varepsilon(x) + \sum_{i,j}\varepsilon^2 v_{ij}(\tfrac{x}{\varepsilon})D^2_{ij}\overline{u}^\varepsilon(x).$$

Then we have

$$\begin{aligned}
|\phi^\varepsilon(x) - \overline{u}(x)| &\leq \varepsilon\|D\overline{u}\|_{L^\infty(U)} + C(d)\varepsilon^2\|D^2\overline{u}^\varepsilon\|_{L^\infty(U)}\sup_{ij}\|v_{ij}\|_{L^\infty(\mathbb{R}^d)}\\
&\leq C(\Lambda,d)\|g\|_{C^{2,1}(\partial U)}\varepsilon.
\end{aligned}$$

On $x\cdot\sigma=\varepsilon$ we have, by the regularity up to the boundary of Theorem 4.7.6,

$$|\phi^\varepsilon(x)-u^\varepsilon(x)|\le|\phi^\varepsilon(x)-\overline{u}(x)|+|\overline{u}(x)-g(x')|+|g(x')-u^\varepsilon(x)|\le C(\Lambda,d)\varepsilon\|g\|_{C^{2,1}}.$$

A similar estimate holds on $x\cdot\sigma=1-\varepsilon$. Thus we can estimate,

$$\sup_U|\overline{u}-u^\varepsilon|\le\sup_{U_\varepsilon}|\overline{u}-u^\varepsilon|+C(\Lambda,d)\varepsilon\|g\|_{C^{2,1}}\le\sup_{U_\varepsilon}|\phi^\varepsilon-u^\varepsilon|+C(\Lambda,d)\varepsilon\|g\|_{C^{2,1}}. \tag{4.7.4}$$

It remains to estimate $\sup_{U_\varepsilon}|\phi^\varepsilon-u^\varepsilon|$. In U_ε, using the uniform ellipticity and then the definition of the corrector,

$$\begin{aligned}-\mathrm{Tr}(A(\tfrac{x}{\varepsilon})D^2\phi^\varepsilon(x))\ge{}&-\mathrm{Tr}[A(\tfrac{x}{\varepsilon})(D^2\overline{u}^\varepsilon(x)+\sum_{i,j}D^2v_{ij}(\tfrac{x}{\varepsilon})D^2_{ij}\overline{u}^\varepsilon(x))]\cdots\\ &\cdots-\sum_{i,j}\Lambda\left[2\varepsilon|Dv_{ij}(\tfrac{x}{\varepsilon})||DD^2_{ij}\overline{u}^\varepsilon(x)|+\varepsilon^2|v_{ij}(\tfrac{x}{\varepsilon})||D^2D^2_{ij}\overline{u}^\varepsilon(x)|\right]\\ \ge{}&-\mathrm{Tr}(\overline{A}D^2\overline{u}(x))-C|D^2\overline{u}(x)-D^2\overline{u}^\varepsilon(x)|\cdots\\ &\cdots-C\varepsilon(\sup_{i,j}\|Dv_{ij}\|_\infty)|D^3\overline{u}^\varepsilon(x)|-C\varepsilon^2(\sup_{i,j}\|v_{ij}\|_\infty)]|D^4\overline{u}^\varepsilon(x)|.\end{aligned}$$

Recall that, from (4.7.2), $\|D^kv_{ij}\|_\infty\le C(\Lambda,d)$ for $k=0,1$. Since $-\mathrm{Tr}(\overline{A}D^2\overline{u}(x))=0$ we get the following supersolution/subsolution conditions,

$$f(x)\ge-\mathrm{Tr}(A(\tfrac{x}{\varepsilon})D^2\phi^\varepsilon(x))\ge-f(x)$$

where we have called $f(x)$ to be the error from the preceding calculation,

$$f(x):=C(\Lambda,d)[|D^2\overline{u}(x)-D^2\overline{u}^\varepsilon(x)|+\varepsilon|D^3\overline{u}^\varepsilon(x)|+\varepsilon^2|D^4\overline{u}^\varepsilon(x)|].$$

The first and third terms can be estimated in terms of $\bar{u}$ by standard mollification estimates:

$$|D^4\overline{u}^\varepsilon(x)|\le\varepsilon^{-1}\|D^3\overline{u}\|_{L^\infty}\ \text{ and }\ |D^2\overline{u}(x)-D^2\overline{u}^\varepsilon(x)|\le\varepsilon\|D^3\overline{u}\|_{L^\infty}\ \text{ for all }\ \varepsilon<x\cdot\sigma<R-\varepsilon.$$

Thus we have,

$$\|f\|_{L^\infty}\le C(\Lambda,d)\|D^3\overline{u}\|_{L^\infty}\varepsilon.$$

Finally, using $C(\Lambda,d)\|f\|_\infty(x\cdot\sigma)(1-x\cdot\sigma)$ as a barrier

$$\sup_{U_\varepsilon}|\phi^\varepsilon-u^\varepsilon|\le\sup_{\partial U_\varepsilon}|\phi^\varepsilon-u^\varepsilon|+C(\Lambda,d)\|f\|_{L^\infty(U_\varepsilon)},$$

which, when combined with the estimate near the boundary gives,

$$\sup_{U_\varepsilon}|\phi^\varepsilon - u^\varepsilon| \leq C(\Lambda,d,\beta)\|g\|_{C^{3,\beta}(\partial U)}\varepsilon.$$

Combining with (4.7.4) completes the proof. $\square$

The proof of Lemma 4.7.3 is almost exactly the same as the proof of Lemma 4.4.5 except we now use the improved interior homogenization rate given in Theorem 4.7.1.

Proof of Lemma 4.7.3. Recall that $m_\xi(y\cdot\hat{\eta})$ is $\frac{1}{|\xi|}$-periodic on ∂P_ξ in the direction $\hat{\eta}$ and constant in the directions orthogonal to $\hat{\eta}$. As usual we can restrict to the case $|\xi||\eta| \leq 1/2$. Let $w_{\xi,\eta}$, $\overline{w}$, μ and $\bar{\mu}$ as given in the proof of Lemma 4.7.3. Recall from Lemma 4.3.1 and Lemma 4.3.2 that,

$$|w_{\xi,\eta}(y)-\mu|+|\overline{w}_{\xi,\eta}(y)-\overline{\mu}| \leq C[(\operatorname{osc} m_\xi)\exp(-c|\xi|R)+\|Dw_{\xi,\eta}\|_{L^\infty(P_\xi)}|\eta|] \ \text{ for } \ y\in\partial P_\xi+R\hat{\xi}. \tag{4.7.5}$$

By Theorem 4.7.6, for any $r>0$,

$$\|Dw_{\xi,\eta}\|_{L^\infty(B_{|\xi|^{-1}}\cap P_\xi)} \leq C(|\xi|\operatorname{osc} m_\xi + |\xi|^{-1}\|D^2 m_\xi\|_\infty) \leq C\|m_\xi(|\xi|^{-1}\cdot)\|_{C^{3,\beta}}|\xi|,$$

Fix an $R \geq |\xi|^{-1}$ to be chosen and consider,

$$\tilde{w}_{\xi,\eta}(y) = w_{\xi,\eta}(y)+(\overline{\mu}-\mu)R^{-1}y\cdot\hat{\xi}+\sup_{\partial P_\xi+R\hat{\xi}}\left[|w_{\xi,\eta}(\cdot)-\mu|+|\overline{w}_{\xi,\eta}(\cdot)-\overline{\mu}|\right].$$

Note that with this modification $\tilde{w}_{\xi,\eta}$ still solves the same equation as $w_{\xi,\eta}$ in P_ξ with the same boundary condition on ∂P_ξ but now also,

$$\tilde{w}_{\xi,\eta}(y) \geq \overline{w}_{\xi,\eta}(y) \ \text{ on } \ \partial P_\xi + R\hat{\xi}.$$

Now Theorem 4.7.1 implies that,

$$\overline{w}_{\xi,\eta}(y)-\tilde{w}_{\xi,\eta}(y) \leq C\|m_\xi(R\cdot)\|_{C^{3,\beta}}R^{-1}|\eta| \leq C\|m_\xi(|\xi|^{-1}\cdot)\|_{C^{3,\beta}}(R|\xi|)^{3+\beta}R^{-1}|\eta|.$$

Rewriting this in terms of $w_{\xi,\eta}$,

$$\overline{w}_{\xi,\eta}(y)-w_{\xi,\eta}(y) \leq (\overline{\mu}-\mu)R^{-1}y\cdot\hat{\xi}+C\|m_\xi(|\xi|^{-1}\cdot)\|_{C^{3,\beta}}[R^{2+\beta}|\xi|^{3+\beta}|\eta|+C(\operatorname{osc} m_\xi)\exp(-c|\xi|R)+|\xi||\eta|].$$

Let us choose $R = 2(c|\xi|)^{-1}\log\frac{1}{|\eta||\xi|}$ to obtain

$$\overline{w}_{\xi,\eta}(y)-w_{\xi,\eta}(y) \leq (\overline{\mu}-\mu)R^{-1}y\cdot\hat{\xi}+C\|m_\xi(|\xi|^{-1}\cdot)\|_{C^{3,\beta}}|\xi||\eta|(\log\tfrac{1}{|\xi||\eta|})^{2+\beta}. \tag{4.7.6}$$

Due to (4.7.5), evaluating (4.7.6) for $y \in \partial P_\xi + \frac{1}{2}R\hat{\xi}$ yields

$$\overline{\mu} - \mu \leq \tfrac{1}{2}(\overline{\mu} - \mu) + C\|m_\xi(|\xi|^{-1}\cdot)\|_{C^{3,\beta}}|\xi||\eta|(\log \tfrac{1}{|\xi||\eta|})^{2+\beta}.$$

Rearranging the last inequality and making a similar argument for the lower bound,

$$|\overline{\mu} - \mu| \leq C\|m_\xi(|\xi|^{-1}\cdot)\|_{C^{3,\beta}}|\xi||\eta|(\log \tfrac{1}{|\xi||\eta|})^{2+\beta}. \tag{4.7.7}$$

But now we can plug (4.7.7) back into (4.7.6) to arrive at the desired result,

$$|\overline{w}_{\xi,\eta}(y) - w_{\xi,\eta}(y)| \leq C\|m_\xi(|\xi|^{-1}\cdot)\|_{C^{3,\beta}}|\xi||\eta|(\log \tfrac{1}{|\xi||\eta|})^{2+\beta}. \tag{4.7.8}$$

□

Proof of Lemma 4.7.2. Recall that $m_\xi(t)$ is defined as the boundary layer tail of $v_{\xi,t\hat{\xi}}$ solving the cell problem,

$$\begin{cases} -\sum_{ij} A_{ij}(y+\tau)D^2_{ij}v_{\xi,\tau} = 0 & \text{in} \quad P_\xi \\ v_{\xi,\tau} = \psi(y+\tau) & \text{on} \quad \partial P_\xi, \end{cases} \tag{4.7.9}$$

Let us denote the differential operator $\partial := \hat{\xi}\cdot D_\tau$. We will prove by induction that, for all $k \in \mathbb{N}\cup\{0\}$: if $\|\psi\|_{C^{k+3}} \leq 1$ then for every $\delta > 0$, the following hold for all $R > 0$ uniformly in $\tau \in \mathbb{R}^d$,

$$\sup_{y \in P_\xi} |\partial^k v_{\xi,\tau}(y)| \lesssim_{k,\delta,A} |\xi|^\delta \tag{i}$$

$$\operatorname*{osc}_{y\cdot\hat{\xi} \geq R} \partial^k v_{\xi,\tau}(y) \lesssim_{k,\delta,A} |\xi|^\delta \exp(-c_{k,\delta}R/|\xi|) \tag{ii}$$

$$\|D^2\partial^k v_{\xi,\tau}\|_{C^{0,1/2}(\{y\cdot\hat{\xi}\geq R\})} \lesssim_{k,\delta,A} |\xi|^\delta \exp(-c_{k,\delta}R/|\xi|) \tag{iii}$$

$$\|D^2\partial^k v_{\xi,\tau}\|_{L^\infty(\{y\cdot\hat{\xi}\geq R\})} \lesssim_{k,\delta,A} (2+R)^{\delta-2}|\xi|^\delta \exp(-c_{k,\delta}R/|\xi|) \tag{iv}$$

Once we have proven that (i) holds for all k we will be done since $\frac{d^k}{dt^k}m_\xi(t) = \lim_{R\to\infty}\partial^k v_{\xi,t\hat{\xi}}(R\hat{\xi})$. Let us point out that the choice of the $C^{0,1/2}$ norm in (iii) is arbitrary; any $C^{0,\beta}$ for $\beta \in (0,1)$ would work as well. The outline of the argument is as follows. For each k and $\delta > 0$ we prove (i)-(iii) using that (i)-(iv) hold for all $m < k$ and every $\delta' > 0$. Then we show (iv) by interpolating between (iii) for k and the large scale $C^{1,1-\gamma}$ estimates from Theorem 4.7.6 for $\gamma = \gamma(\delta)$ sufficiently small.

For $k = 0$, (i) is maximum principle, (ii) is a consequence of Lemma 4.3.1, and (iii) follows from the up to the boundary $C^{2,1/2}$ estimates in Theorem 4.7.5. Lastly to show (iv) note that Theorem 4.7.6

implies

$$\|D^2 v_{\xi,\tau}\|_{L^\infty(\{y\cdot\hat{\xi}\geq R\})} \lesssim_A R^{-2} \operatorname*{osc}_{\{y\cdot\hat{\xi}\geq R/2\}} v_{\xi,\tau}$$
$$\lesssim_A (R+2)^{-2}(\operatorname{osc}\psi)\exp(-\tfrac{1}{2}c_0 R/|\xi|) \ \text{ for } R>1,$$

where c_0 is the exponential rate of convergence to the boundary layer tail from Lemma 4.A.3. For $R<1$ the estimate is already contained in (*iii*).

Suppose that (*i*)-(*iv*) hold for all $\delta>0$ and all $m\leq k-1$. For a fixed $\delta_0>0$ we aim to prove (*i*)-(*iv*) for $\partial^k v_{\xi,\tau}$ under the assumption $\|\psi\|_{C^{k+3}}\leq 1$. Our induction is based on the fact that $\partial^k v_{\xi,\tau}$ solves

$$-\sum_{ij} A_{ij}(y+\tau)D^2_{ij}\partial^k v_{\xi,\tau}(y) = f_{\xi,\tau}(y) \quad \text{ in } P_\xi, \tag{4.7.10}$$

where

$$f_{\xi,\tau}(y) := \sum_{ij}\sum_{\ell+m=k,\ell\neq 0} [(\hat{\xi}\cdot D_y)^\ell A_{ij}(y+\tau)]D^2_{ij}\partial^m v_{\xi,\tau}(y)$$

with boundary data

$$\partial^k v_{\xi,\tau}(y) = [(\hat{\xi}\cdot D_y)^k\psi](y+\tau) \ \text{ on } \ \partial P_\xi.$$

(To be completely precise we should take difference quotients instead of $\partial^k v_{\xi,\tau}$; the reader can easily see how to make our formal argument rigorous.) Note that $f_{\xi,\tau}$, the boundary data $\partial^k v_{\xi,\tau}$, and the operator F are all periodic with respect to a lattice on ∂P_ξ with unit cell diameter of order $|\xi|$. This will allow us to use the results of the Appendix which extend Section 4.3 Lemma 4.3.1 to include equations with a right hand side. We use inductive hypothesis (*iv*) with $\delta=\delta_0/4$ to see that $f_{\xi,\tau}(y)$ satisfies

$$\|f_{\xi,\tau}\|_{L^\infty(\{y\cdot\hat{\xi}\geq R\})} \lesssim_{k,\delta_0,A} (2+R)^{\delta_0/4-2}|\xi|^{\delta_0/4}\exp(-c_{k-1,\delta_0}R/|\xi|). \tag{4.7.11}$$

In particular $f_{\xi,\tau}$ fits under the assumptions of the results of the Appendix. We can apply directly Lemma 4.A.2 in combination with Lemma 4.A.1 to get,

$$\sup_{\{y\cdot\hat{\xi}\geq R\}} \partial^k v_{\xi,\tau}(y) \lesssim_{\delta_0,k,A} \|\partial^k\psi\|_\infty + |\xi|^{\delta_0} \lesssim |\xi|^{\delta_0}.$$

We have used that, by assumption, $\|\psi\|_{C^{k+3}} \leq 1 \leq |\xi|^{\delta_0}$. Then we use Lemma 4.A.3 to get, with $c_{k,\delta_0} = \delta_0 c_{k-1,\delta_0}/8$

$$\operatorname*{osc}_{y\cdot\sigma\geq R} \partial^k v_{\xi,\tau} \lesssim_{k,\delta_0,A} (\|\partial^k \psi\|_\infty + |\xi|^{\delta_0}) \exp(-2c_{k,\delta_0} R/|\xi|) \lesssim |\xi|^{\delta_0} \exp(-2c_{k,\delta_0} R/|\xi|). \quad (4.7.12)$$

This establishes (*ii*). We proceed to prove (*iii*). For $R > 1$ the interior $C^{2,1/2}$ estimates at unit scale for the operator $-\mathrm{Tr}(A(y)M)$ imply,

$$\begin{aligned}\|D^2 \partial^k v_{\xi,\tau}\|_{C^{0,1/2}(\{y\cdot\hat{\xi}\geq R\})} &\lesssim (\operatorname*{osc}_{y\cdot\hat{\xi}\geq R-1} v_{\xi,\tau}(y) + \|f\|_{C^{0,1/2}(\{y\cdot\hat{\xi}\geq R-1\})}) \\ &\lesssim_{k,\delta_0,A} |\xi|^{\delta_0} \exp(-2c_{k,\delta_0} R/|\xi|),\end{aligned}$$

where we have used (4.7.12) inductive hypothesis (*iii*) to bound $\|f\|_{C^{0,1/2}}$. For $R < 1$ we instead use the up to the boundary $C^{2,1/2}$ estimate,

$$\begin{aligned}\|D^2 \partial^k v_{\xi,\tau}\|_{C^{0,1/2}(B_1\cap P_\xi)} &\lesssim (\operatorname*{osc}_{P_\xi} v_{\xi,\tau}(y) + \|f\|_{C^{0,1/2}(P_\xi)} + \|(\hat{\xi}\cdot D_y)^k \psi\|_{C^{2,1/2}(\mathbb{T}^d)}) \\ &\lesssim_{k,\delta_0,A} \|\psi\|_{C^{k+2,1/2}} |\xi|^{\delta_0},\end{aligned}$$

where again we have used inductive hypothesis (*iii*) for $m \leq k-1$ to bound $\|f\|_{C^{0,1/2}(P_\xi)}$. Combining these and using that, by assumption, $\|\psi\|_{C^{k+2,1/2}} \leq \|\psi\|_{C^{k+3}} \leq 1$ we get (*iii*),

$$\|D^2 \partial^k v_{\xi,\tau}\|_{C^{0,1/2}(\{y\cdot\hat{\xi}\geq R\})} \lesssim_{k,\delta_0,A} |\xi|^{\delta_0} \exp(-c_{k,\delta_0} R/|\xi|). \quad (4.7.13)$$

We have now proven that the inductive hypothesis (*i*), (*ii*) and (*iii*) hold for k.

It remains to prove (*iv*). Let $n+\beta < 2$, then (*iii*) and Theorem 4.7.6 yields that, for $R > 1$,

$$\begin{aligned}|\partial^k v_{\xi,\tau}|_{C^{n,\beta}(\{y\cdot\hat{\xi}\geq R\})} &\lesssim_A R^{-(n+\beta)} \operatorname*{osc}_{\{y\cdot\hat{\xi}\geq R/2\}} v_{\xi,\tau} + R^{2-(n+\beta)} \|f\|_{L^\infty(\{y\cdot\hat{\xi}\geq R/2\})} \\ &\lesssim_{k,\delta_0,A,n+\beta} (2+R)^{\delta_0/2-(n+\beta)} |\xi|^{\delta_0} \exp(-c_{k,\delta_0} R/|\xi|).\end{aligned}$$

For $R < 1$ we can use (4.7.13). Together we obtain for all $R > 0$,

$$|\partial^k v_{\xi,\tau}|_{C^{n,\beta}(\{y\cdot\hat{\xi}\geq R\})} \lesssim_{k,\delta_0,A,n+\beta} (2+R)^{\delta_0/2-(n+\beta)} |\xi|^{\delta_0} \exp(-c_{k,\delta_0} R/|\xi|). \quad (4.7.14)$$

We wish to interpolate using Lemma 4.7.7 between estimates (4.7.14) and (4.7.13) to get (*iv*). It suffices to prove the case $R > 1$, the case $R < 1$ is already contained in (4.7.14). In order to localize the estimate we will make use of a smooth spatial cutoff function . Let $\phi : \mathbb{R} \to [0,1]$, smooth, monotone increasing, with $\phi(x) \equiv 0$ for $x < 1/2$ and $\phi \equiv 1$ for $x > 1$. Then for $\phi_R(y) := \phi(y \cdot \hat{\xi}/R)$ and for any $\gamma > 0$,

$$\begin{aligned}
\|D_y^2 \partial^k v_{\xi,\tau}\|_{L^\infty(\{y\cdot\hat{\xi}\geq R\})} &= \|D_y^2 \partial^k v_{\xi,\tau}\|_{L^\infty(\{y\cdot\hat{\xi}\geq R\})} \\
&= \|D_y(\phi_R D_y \partial^k v_{\xi,\tau})\|_{L^\infty(\{y\cdot\hat{\xi}\geq R\})} \\
&\leq \|D_y(\phi_R D_y \partial^k v_{\xi,\tau})\|_{L^\infty(\mathbb{R}^d)} \\
&\lesssim |\phi_R D_y \partial^k v_{\xi,\tau}|_{C^{0,1-\gamma}(\mathbb{R}^d)}^{\frac{1/2}{\gamma+1/2}} |D_y(\phi_R D_y \partial^k v_{\xi,\tau})|_{C^{0,1/2}(\mathbb{R}^d)}^{\frac{\gamma}{\gamma+1/2}}
\end{aligned} \tag{4.7.15}$$

We continue to estimate the right hand side above using (4.7.14) for the first term and (4.7.13) for the second term. For the first term of (4.7.15),

$$\begin{aligned}
|\phi_R D_y \partial^k v_{\xi,\tau}|_{C^{0,1-\gamma}(\mathbb{R}^d)} &= |\phi_R D_y \partial^k v_{\xi,\tau}|_{C^{0,1-\gamma}(\{y\cdot\hat{\xi}\geq R/2\})} \\
&\leq \|\phi_R\|_{L^\infty} |D_y \partial^k v_{\xi,\tau}|_{C^{0,1-\gamma}} + \|D_y \partial^k v_{\xi,\tau}\|_{L^\infty} |\phi_R|_{C^{0,1-\gamma}} \\
&\lesssim_{k,\delta_0,A,\gamma} (2+R)^{\delta_0/2-(2-\gamma)} |\xi|^{\delta_0} \exp(-c_{k,\delta_0} R/|\xi|)
\end{aligned}$$

where the domains of the norms in the second line are all $\{y \cdot \hat{\xi} \geq R/2\}$ and we used (4.7.14) both for $(n,\beta) = (0,1)$ and $(1, 1-\gamma)$ in the last inequality. For the second term of (4.7.15), again omitting the domain $\{y \cdot \hat{\xi} \geq R/2\}$ in the norms,

$$\begin{aligned}
|D_y(\phi_R D_y \partial^k v_{\xi,\tau})|_{C^{0,1/2}(\{y\cdot\hat{\xi}\geq R/2\})} &\leq |\phi_R D_y^2 \partial^k v_{\xi,\tau}|_{C^{0,1/2}} + |D_y \partial^k v_{\xi,\tau} \otimes D_y \phi_R|_{C^{0,1/2}} \\
&\leq \|\phi_R\|_{L^\infty} |D_y^2 \partial^k v_{\xi,\tau}|_{C^{0,1/2}} + \|D_y^2 \partial^k v_{\xi,\tau}\|_{L^\infty} |\phi_R|_{C^{0,1/2}} + \cdots \\
&\quad \cdots + \|D_y \partial^k v_{\xi,\tau}\|_{L^\infty} |D_y \phi_R|_{C^{0,1/2}} + \|D_y \phi_R\|_{L^\infty} |D_y \partial^k v_{\xi,\tau}|_{C^{0,1/2}} \\
&\lesssim_{k,\delta_0,A} |\xi|^{\delta_0} \exp(-c_{k,\delta_0} R/|\xi|).
\end{aligned}$$

For the last inequality we have used (4.7.13) to estimate the first two terms and (4.7.14), with $(n,\beta) = (0,1)$ and $(1,1/2)$ respectively, to estimate the second two terms. Combining the two estimates via the interpolation (4.7.15) we obtain,

$$\|D_y^2 \partial^k v_{\xi,\tau}\|_{L^\infty(\{y\cdot\hat{\xi}\geq R\})} \lesssim_{k,\delta_0,A,\gamma} |\xi|^{\delta_0} (2+R)^{[\delta_0/2-(2-\gamma)]\frac{1}{1+2\gamma}} \exp(-c_{k,\delta_0} R/|\xi|).$$

Using this estimate with $\gamma = \gamma(\delta_0) = \frac{\delta_0}{1-2\delta_0}$ (we can assume $\delta_0 \leq 1/8$ since the result for smaller δ_0 is stronger) so that $\frac{\gamma}{1+2\gamma} = \delta_0/2$, we finally obtain (*iv*):

$$\|D_y^2 \partial^k v_{\xi,\tau}\|_{L^\infty(\{y\cdot\hat{\xi} \geq R\})} \lesssim_{k,\delta_0,A} |\xi|^{\delta_0}(2+R)^{\delta_0-2}\exp(-c_{k,\delta_0}R/|\xi|).$$

□

4.A Exponential Convergence to Boundary Layer Tail with Periodic Data and a Right Hand Side

Here we prove Lemma 4.3.1 but now including a right hand side in the equation. Let us consider v solving the following problem:

$$\begin{cases} F(D^2 v, y) = f(y) & \text{in} \quad P_{e_d} \\ v = \phi(y) & \text{on} \quad \partial P_{e_d}. \end{cases} \tag{4.A.1}$$

We assume that F, f and ϕ are periodic with respect to linearly independent translations $\ell_1, \ldots \ell_{d-1} \in \partial P_{e_d}$. Recall that we denote $\mathscr{L}$ by the periodicity lattice generated by $\{\ell_j\}$, Q its unit cell, and L the diameter of Q. Let us assume $L > 1$.

When f is continuous and has sufficient decay there is a unique bounded solution of (4.3.1). In order to investigate v we define some auxiliary functions relating to the decay of the right hand side. The fundamental quantities turn out to be the decay rate of f and it's second primitive,

$$M_f(R) = \sup_{y_d \geq R} |f(z)| \text{ and } I_f(R) = \int_R^\infty \int_s^\infty M_f(r)\, drds. \tag{4.A.2}$$

Since $M_f \geq 0$ the second primitive I_f is always well defined, although it may be infinite.

Lemma 4.A.1. *Suppose I_f is finite, then there exists a unique bounded solution of* (4.A.1) *with,*

$$\min\phi - I_f(0) \leq v(y) \leq \max\phi + I_f(0).$$

Proof. Once we have constructed barriers existence will follow from Perron's method. The uniqueness of bounded solutions can be found, for example, in Lemma 2.8 of [Fel14]. For a supersolution barrier we take $h(y) := \max\phi + I_f(0) - I_f(y_d)$. Due to the uniform ellipticity of F

$$F(D^2h(y), y) \geq \mathrm{Tr}(M_f(y_d)e_d \otimes e_d) = M_f(y_d) \geq f(y).$$

Thus h serves as a supersolution of (4.A.1) with the desired upper bound. $\square$

For notational simplicity we will only focus on the particular case to be used in this dissertation: let us assume

$$M_f(R) = K(1+R)^{\delta-2}e^{-bR/L} \text{ with } b > 0 \text{ and } 0 < \delta << 1. \tag{4.A.3}$$

With above assumption we estimate I_f in the following Lemma:

Lemma 4.A.2. *Let M_f as in* (4.A.3)*, then*

$$I_f(R) \lesssim_{b,\delta} KL^{2\delta}e^{-\delta bR/L}. \tag{4.A.4}$$

Proof. Without loss of generality we can set $K = 1$. Observe that

$$\begin{aligned} \tfrac{1}{1-\delta}(1+s)^{\delta-1}e^{-bs/L} &= \int_s^\infty -\frac{d}{ds}[\tfrac{1}{1-\delta}(1+s)^{\delta-1}e^{-bs/L}]\, ds \\ &\geq \int_s^\infty M_f(s)\, ds. \end{aligned}$$

Therefore

$$I_f(R) \leq \int_R^\infty \tfrac{1}{1-\delta}(1+s)^{\delta-1}e^{-bs/L}ds.$$

To estimate this integral we consider different ranges of R. When $R < R_0 := b^{-1}(1-\delta)L\log(1+L)$:

$$\begin{aligned} \int_R^\infty (1+s)^{\delta-1}e^{-bs/L}\, ds &\leq \int_R^{R_0}(1+s)^{\delta-1}\, ds + \int_{R_0}^\infty e^{-bs/L}\, ds \\ &= \tfrac{1}{\delta}(R_0^\delta - R^\delta) + \tfrac{L}{b}e^{-bR_0/L} \\ &\lesssim_{b,\delta} (L\log L)^\delta \\ &\lesssim_\delta L^{2\delta}e^{-\delta bR/L}. \end{aligned}$$

When $R \geq R_0$:

$$\begin{aligned}\int_R^\infty (1+s)^{\delta-1}e^{-bs/L}\,ds &\leq \int_R^\infty e^{-bs/L}\,ds\\&= b^{-1}Le^{-bR/L} = b^{-1}Le^{-(1-\delta)bR/L}e^{-\delta bR/L}\\&\leq b^{-1}LL^{-(1-\delta)^2}e^{-\delta bR/L}\\&\lesssim_b L^{2\delta}e^{-\delta bR/L}.\end{aligned}$$

□

Now we are able to prove the corresponding version of Lemma 4.3.1 when f is of the form in (4.A.3).

Lemma 4.A.3. *Suppose that f satisfies* (4.A.3) *for some $K,\delta,b>0$. Then there exists $\mu(\phi,F,f)$ and $c_0(\Lambda,d)>0$ such that,*

$$\sup_{y\cdot e_d\geq R}|v(y)-\mu| \leq C(\Lambda,d,\delta,b)[(\operatorname{osc}\phi)e^{-c_0R/L}+KL^{2\delta}e^{-\delta bR/L}].$$

Proof. After rescaling we may assume that $K,\operatorname{osc}\phi\leq 1$. Let $\alpha(d,\Lambda)\in(0,1)$ and C_0 as given in Lemma 4.3.1. Next define $A(k):=2I_f(jr)+r^2M_f((j+1/2)r)$ for $k\in\mathbb{N}$ and $r>2C_0^{1/\alpha}L$. We claim that

$$\operatorname*{osc}_{P_{e_d}+kr\sigma} v \leq C_0^k(2L/r)^{\alpha k}+\sum_{j=0}^{k-1}C_0^{k-j}(2L/r)^{\alpha(k-j)}A(j)+2I_f(kr). \tag{4.A.5}$$

Let us assume for now that (4.A.5) is correct and continue with the rest of the proof. Define

$$r:=(2eC_0)^{1/\alpha}L \ \text{ and } \ c_0:=(2eC_0)^{-1/\alpha}$$

so that $C_0^k(2L/r)^{\alpha k}=e^{-k}$ and

$$\operatorname*{osc}_{P_{e_d}+kr\sigma} v \leq e^{-k}+e^{-k}\sum_{j=0}^{k-1}e^jA(j)+2I_f(kr). \tag{4.A.6}$$

Due to our assumption (4.A.3) we have the bound

$$r^2M_f((j+1/2)r)\leq r^2(1+(j+1/2)r)^{-2+\delta}e^{-b(j+1/2)r/L}\lesssim r^\delta e^{-bjr/L}\lesssim L^\delta e^{-c_0^{-1}bj},$$

and using Lemma 4.A.2 we also have,

$$I_f(jr)\lesssim_{\delta,b} L^{2\delta}e^{-\delta bjr/L}=L^{2\delta}e^{-c_0^{-1}\delta bj}.$$

Plugging these estimates into (4.A.6) yields

$$\operatorname*{osc}_{P_{e_d}+kr\sigma} v \lesssim_{\delta,b} e^{-k} + L^{2\delta} e^{-k} \sum_{j=0}^{k-1} e^{(1-c_0^{-1}\delta b)j} + L^{2\delta} e^{-c_0^{-1}\delta bk} \lesssim e^{-k} + L^{2\delta} e^{-c_0^{-1}\delta bk}.$$

Now for any $R > 0$, let $k = [R/r]$ to get the desired result

$$\operatorname*{osc}_{P_{e_d}+R\sigma} v \leq \operatorname*{osc}_{P_{e_d}+kr\sigma} v \lesssim_{b,\delta} e^{-c_0 R/L} + L^{2\delta} e^{-\delta bR/L}.$$

It remains to prove (4.A.5) by induction. For $k = 0$ (4.A.5) follows from Lemma 4.A.1. Assuming (4.A.5) for k, we prove it for $k+1$. Note that

$$v_k(y) = v(y + kre_d)$$

satisfies $F(\cdot, y + kre_d) = f(y + kre_d)$ in P_{e_d} with boundary data $\phi_k(y) = v(y + ke_d)$. Both the operator and the boundary data are periodic with respect to $(\ell_j)_{j=1}^{d-1}$ translations by uniqueness, and

$$\operatorname*{osc}_{P_{e_d}} v_k \leq C_0^k (2L/r)^{\alpha k} + \sum_{j=0}^{k-1} C_0^{k-j} (2L/r)^{\alpha(k-j)} A(j) + 2I_f(kr)$$

by the inductive hypothesis. Then by the interior Hölder estimate in $B_{r/2}(re_d)$,

$$|v_k|_{C^\alpha(B_{r/4}(re_d))} \leq 2^\alpha C_0 r^{-\alpha} (\operatorname{osc} v_k + r^2 M_f((k+1/2)r)),$$

and so, since $r > 4L$, the oscillation on Q can be estimated by

$$\operatorname*{osc}_{Q+re_d} v_k \leq C_0 (2L/r)^\alpha (\operatorname{osc} v_k + r^2 M_f((k+1/2)r)),$$

On the other hand v_k is periodic on $\partial P_{e_d} + re_d$ with respect to the translations $(\ell_j)_{j=1}^{d-1}$ and periodicity cell Q so $\operatorname{osc}_{\partial P_{e_d}+re_d} v_k = \operatorname{osc}_{Q+re_d} v_k$. Using the inductive hypothesis the right hand side above is bounded by,

$$\begin{aligned}
\operatorname*{osc}_{\partial P_{e_d}+re_d} v_k &\leq C_0 (2L/r)^\alpha \Big(C_0^k (2L/r)^{\alpha k} + \sum_{j=0}^{k-1} C_0^{k-j} (2L/r)^{\alpha(k-j)} A(j) + 2I_f(kr) + r^2 M_f((k+1/2)r)\Big) \\
&= C_0^k (2L/r)^{\alpha(k+1)} + \sum_{j=0}^{k-1} C_0^{k+1-j} (2L/r)^{\alpha(k+1-j)} A(j) + C_0 (2L/r)^\alpha A(k) \\
&= C_0^k (2L/r)^{\alpha(k+1)} + \sum_{j=0}^{k} C_0^{k+1-j} (2L/r)^{\alpha(k+1-j)} A(j).
\end{aligned}$$

Finally using Lemma 4.A.1 in place of maximum principle to bound the oscillation in the entire half space,

$$
\begin{aligned}
\operatorname*{osc}_{P_\sigma+(k+1)re_d} v = \operatorname*{osc}_{P_\sigma+re_d} v_k \\
&\leq \operatorname*{osc}_{\partial P_\sigma+re_d} v_k + 2I_f((k+1)r) \\
&\leq C_0^k(2L/r)^{\alpha(k+1)} + \sum_{j=0}^{k} C_0^{k+1-j}(2L/r)^{\alpha(k+1-j)}A(j) + 2I_f((k+1)r),
\end{aligned}
$$

which is the inductive hypothesis for $k+1$. □

4.A.1 Proof of Lemma 4.7.7

The proof uses Littlewood-Paley theory. Fix a smooth, radially decreasing function $\varphi : \mathbb{R}^d \to \mathbb{R}$ satisfying $\varphi \equiv 1$ in $B_{9/10}$ and $\varphi \equiv 0$ outside B_1. Let us define $\varphi_N(\cdot) := \varphi(N^{-1}\cdot)$ and $\psi_N := \varphi_N - \varphi_{N/2}$ for dyadic integers $N \in 2^{\mathbb{Z}}$. Note that $\varphi_N = \sum_{M\leq N} \psi_M$. Let $\tilde{\psi}_N$ and $\tilde{\varphi}_N$ be respectively the inverse fourier transforms of ψ_N and φ_N. It is useful to note that $(\psi_{N/2} + \psi_N + \psi_{2N})\psi_N = \psi_N$. We call, for a $g : \mathbb{R}^d \to \mathbb{R}$ which is locally integrable, $g_{\leq N} = \tilde{\varphi}_N \star g$ and $g_N = \tilde{\psi}_N \star g$. One basic estimate is,

$$
\begin{aligned}
\|Dg_{N/2\leq\cdot\leq 2N}\|_\infty &= \|D(\tilde{\psi}_{N/2} + \tilde{\psi}_N + \tilde{\psi}_{2N}) \star g_N\|_\infty \\
&\leq (\|D\tilde{\psi}_{N/2}\|_1 + \|D\tilde{\psi}_N\|_1 + \|D\tilde{\psi}_{2N}\|_1)\|g_N\|_\infty \\
&\lesssim N\|g_N\|_\infty.
\end{aligned}
$$

Let us show

$$
\|g_N\|_\infty \leq N^{-\beta}|g|_{C^{0,\beta}}. \tag{4.A.7}
$$

To show (4.A.7) recall that since $\psi_N(0)=0$, $\int_{\mathbb{R}^d}\tilde{\psi}_N=0$, and that $\tilde{\psi}_N$ is Schwartz class,

$$\begin{aligned}
|g_N(x)| &= |\int_{\mathbb{R}^d} N^d\tilde{\psi}(Ny)(g(x-y)-g(x))\,dy| \\
&\le |g|_{C^{0,\beta}}\int_{\mathbb{R}^d} N^d\tilde{\psi}(Ny)|y|^\beta\,dy \\
&= |g|_{C^{0,\beta}}\int_{\mathbb{R}^d}\tilde{\psi}(z)|N^{-1}z|^\beta\,dz \\
&\lesssim N^{-\beta}|g|_{C^{0,\beta}}\int_{\mathbb{R}^d}(1+|z|)^{-(d+2)}|z|^\beta\,dz \\
&\lesssim N^{-\beta}|g|_{C^{0,\beta}}.
\end{aligned}$$

A parallel computation shows that $\|g-g_N\|_\infty\lesssim N^{-\beta}|g|_{C^{0,\beta}}$. Combining the established estimates,

$$\|Dg_{\le N}\|_\infty \lesssim \sum_{M\le N} M\|g_M\|_\infty \lesssim \sum_{M\le N} M^{1-\beta}|g|_{C^{0,\beta}} \lesssim N^{1-\beta}|g|_{C^{0,\beta}}. \tag{4.A.8}$$

This is the main useful estimate to proving the interpolation inequality.

Now let g as in the statement of the Lemma, and $N>0$ to be chosen later. From (4.A.8),

$$\|Dg_{\le N}\|_{L^\infty(\mathbb{R}^d)} \lesssim_d N^\gamma|g|_{C^{0,1-\gamma}(\mathbb{R}^d)}$$

and from the remark above applied now to Dg,

$$|Dg_{\le N}(x)-Dg(x)| \lesssim N^{-\beta}|Dg|_{C^{0,\beta}(\mathbb{R}^d)}.$$

Thus,

$$\|Dg\|_{L^\infty(\mathbb{R}^d)} \le \|Dg_{\le N}\|_{L^\infty(\mathbb{R}^d)} + \|Dg_{\le N}-Dg\|_{L^\infty(\mathbb{R}^d)} \lesssim N^\gamma|g|_{C^{0,1-\gamma}(\mathbb{R}^d)} + N^{-\beta}|Dg|_{C^{0,\beta}}.$$

Choosing now $N=(|Dg|_{C^{0,\beta}}/|g|_{C^{0,1-\gamma}(\mathbb{R}^d)})^{\frac{1}{\beta+\gamma}}$ and plugging in yields the desired result. $\square$

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CHAPTER 5

Homogenization of Random Oscillating Boundary Data

5.1 Introduction

In this chapter we investigate the averaging behavior of the solutions to nonlinear uniformly elliptic partial differential equations with random Dirichlet or Neumann boundary data oscillating on a small scale. Under conditions on the operator, the data and the random media leading to concentration of measure, we prove an almost sure and local uniform homogenization result with a rate of convergence in probability.

In particular, we consider the Dirichlet and Neumann boundary value problems

$$\begin{cases} F(D^2u^\varepsilon) = 0 & \text{in} \quad U, \\ u^\varepsilon = g(\cdot, \frac{\cdot}{\varepsilon}, \omega) & \text{on} \quad \partial U, \end{cases} \tag{5.1.1}$$

and

$$\begin{cases} F(D^2u^\varepsilon) = 0 & \text{in} \quad U \setminus K, \\ \partial_\sigma u^\varepsilon = g(\cdot, \frac{\cdot}{\varepsilon}, \omega) & \text{on} \quad \partial U, \\ u^\varepsilon = f & \text{on} \quad \partial K, \end{cases} \tag{5.1.2}$$

where U is a smooth bounded domain in $\mathbb{R}^d$ with $d \geq 2$, K is a compact subset of U, σ is the inward normal, F is positively homogeneous of degree one and uniformly elliptic, f is continuous on K and $g = g(x,y,\omega)$ is bounded and Lipshitz continuous in x, y uniformly in ω belonging to a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and, for each fixed $x \in U$, stationary with respect to the translation action of $\mathbb{R}^d$ on Ω and strongly mixing with respect to (y, ω) (the precise assumptions are given in Section 5.2.2).

We show that there exist a deterministic continuous functions $\overline{g}_D, \overline{g}_N : \partial U \to \mathbb{R}$ such that, as $\varepsilon \to 0$, the solutions $u^\varepsilon = u^\varepsilon(\cdot,\omega)$ of (5.1.1) and (5.1.2) converge almost surely and locally uniformly in U (with a rate in probability) to the unique solution $\overline{u}$ of respectively

$$\begin{cases} F(D^2\overline{u}) = 0 & \text{in} \quad U, \\ \overline{u} = \overline{g}_D & \text{on} \quad \partial U, \end{cases} \tag{5.1.3}$$

and

$$\begin{cases} F(D^2\overline{u}) = 0 & \text{in} \quad U \setminus K, \\ \partial_\sigma \overline{u} = \overline{g}_N & \text{on} \quad \partial U, \\ \overline{u} = f & \text{on} \quad \partial K. \end{cases} \tag{5.1.4}$$

The homogenized boundary condition $\overline{g}$ (here and in the rest of the chapter we omit the subscript and always denote the homogenized boundary condition by $\overline{g}$) depend on F, σ, d and the random field g. The rate of convergence depends on the regularity of U, the continuity and mixing properties of g, the dimension d, the ellipticity ratio of F and, in the case of the Neumann problem, the bounds of f.

We discuss next heuristically what happens as $\varepsilon \to 0$ in the Dirichlet problem (5.1.1). It follows from the up to the boundary continuity of the solutions to (5.1.1) that, close to the boundary, u^ε typically has unit size oscillations over distances of order ε. Therefore any convergence to a deterministic limit must be occurring outside of some shrinking boundary layer, where the solution remains random and highly oscillatory even as $\varepsilon \to 0$. In order to analyze the behavior of the u^ε near a point $x_0 \in \partial U$ with inner normal σ we "blow up" u^ε to scale ε, that is we consider

$$v^\varepsilon(y,\omega) = u^\varepsilon(x_0 + \varepsilon y, \omega).$$

If homogenization holds, then $v^\varepsilon(R\sigma,\omega)$ should converge to $\overline{g}(x_0)$ for $R > 0$ sufficiently large to escape the boundary layer. Noting that the random function $v^\varepsilon(\cdot,\omega)$ is uniformly continuous, as $\varepsilon \to 0$, we can approximate $v^\varepsilon(\cdot,\omega)$ by the solution of the half-space problem, obtained after "blowing up" in the tangent half-space at x_0,

$$\begin{cases} F(D^2 v) = 0 & \text{in} \quad \{y \in \mathbb{R}^d : y\cdot\sigma > 0\} \\ v(\cdot,\omega) = \psi(\cdot,\omega) & \text{on} \quad \{y \in \mathbb{R}^d : y\cdot\sigma = 0\}. \end{cases} \tag{5.1.5}$$

Formally, based on the problem satisfied by $v^\varepsilon(\cdot,\omega)$, we expect that,

$$|v^\varepsilon(y,\omega)-v(y,\omega)| \to 0 \text{ as } \varepsilon\to 0 \text{ when } \psi(y,\omega)=g(x_0,y+\varepsilon^{-1}x_0,\omega). \tag{5.1.6}$$

With the expectation that $u^\varepsilon(x+\varepsilon R\sigma,\omega)$ can be approximated, for small ε and large R, by $v(R\sigma,\omega)$, we are led to consider, for random fields ψ satisfying assumptions similar to g for fixed x, the existence of an almost sure limit, as $y\cdot\sigma\to\infty$, of $v(y,\omega)$. This is the analogue, in our setting, of the cell problem in classical homogenization theory.

We say that the cell problem (5.1.5) has a solution, if there exists a constant $\mu=\mu(\sigma,F,\psi)$, often referred to as the ergodic constant, such that

$$\lim_{R\to\infty} v(R\sigma,\omega)=\mu \text{ almost surely.} \tag{5.1.7}$$

As indicated above there are two main steps in the argument. The first is to show the existence of the limit (5.1.7) for the cell problem. This is where all the assumptions on F and the randomness come in (see Section 5.2.2). The second is to show that the approximation of u^ε near the boundary by the cell problem (5.1.5) holds with quantitative estimates so that the convergence results for the cell problem can be used to identify the boundary condition for the general domain problem.

Similar heuristic arguments lead to the Neumann cell problem, which is to show that there exists a deterministic (ergodic) constant $\mu=\mu(\sigma,F,\psi)$ such that, if $v_R(\cdot,\omega)=v_{R,\sigma}$ is the unique bounded solution of

$$\begin{cases} F(D^2v_R)=0 & \text{in} \quad \{y\in\mathbb{R}^d : 0<y\cdot\sigma<2R\},\\ \partial_\sigma v_R(\cdot,\omega)=\psi(\cdot,\omega) & \text{on} \quad \{y\in\mathbb{R}^d : y\cdot\sigma=0\},\\ v_R(\cdot,\omega)=0 & \text{on} \quad \{y\in\mathbb{R}^d : y\cdot\sigma=2R\}, \end{cases} \tag{5.1.8}$$

then

$$\lim_{R\to\infty}\frac{v_R(R\sigma,\omega)}{R}=\mu \text{ almost surely.} \tag{5.1.9}$$

In the Neumann problem, it is the gradient Du^ε of the solution to (5.1.2) that has an oscillatory boundary layer, outside of which it should be approaching a deterministic constant far from the Neumann part of the boundary.

In order to have any hope for the convergence described in (5.1.7) and (5.1.9), it is necessary to impose assumptions on the randomness. To motivate them, we consider the Dirichlet cell problem (5.1.5) in the linear case where we can represent $v(R\sigma,\omega)$, using the Poisson kernel, as

$$v(R\sigma,\omega) = \int_{\{y\in\mathbb{R}^d : y\cdot\sigma=0\}} P(Re_d,y)\psi(y,\omega)dy.$$

Since $P(R\sigma,y) \sim R^{1-d}$ on $B_R \cap \{y\in\mathbb{R}^d : y\cdot\sigma = 0\}$, $v(R\sigma,\omega)$ is essentially the average of the boundary values on $B_R \cap \{y\in\mathbb{R}^d : y\cdot\sigma = 0\}$. In this case the convergence, as $R\to\infty$, is a consequence of the ergodic theorem as long as the boundary data satisfies two important assumptions, namely stationarity and ergodicity, which we describe next.

Firstly the distribution of $\psi(y,\omega)$ should not depend on y, in other words $\psi(y,\omega)$ must be stationary with respect to translations parallel to the hyperplane $\{y\in\mathbb{R}^d : y\cdot\sigma = 0\}$. On the other hand, in view of (5.1.6), we actually need to consider the cell problem for translations of the boundary data parallel to σ as well and so we actually require stationarity of ψ with respect to all $\mathbb{R}^d$ translations. Without some assumption of this form one can easily construct examples for which the limit will not exist.

For reference it is useful to consider the periodic version of our problem. In this case, there is a $\mathbb{Z}^d$ translation action under which the underlying probability space, which is the torus $\mathbb{T}^d$, is stationary and ergodic. When the normal direction σ is rational, the boundary values are not stationary with respect to the translations parallel to σ.

One possible way to address this problem (see Section 5.2 for more discussion of this version of the problem) is to consider data defined on a hyperplane which is stationary with respect to an $\mathbb{R}^{d-1}$ or $\mathbb{Z}^{d-1}$ action, and then define data on pieces of the boundary of the general domain by "lifting up from the hyperplane". This is in contrast to the setting described already which we refer to as assigning boundary data by "restricting from the whole space". More specifically, one could take $\psi(y,\omega)$ on $\mathbb{R}^{d-1}$ and a diffeomorphism ζ from an open subset of $\mathbb{R}^{d-1}$ to an open subset of ∂U and then define the boundary data, for the general domain problem, by

$$g^\varepsilon(x,\omega) = \psi(\varepsilon^{-1}\zeta^{-1}(x),\omega). \tag{5.1.10}$$

Secondly, we need to assume that the action of the translations of $\mathbb{R}^d$ on Ω is ergodic. Indeed some assumption on the long range decorrolation of the values of $\psi(y,\omega)$, ergodicity being the weakest is always necessary in to prove a law of large numbers/ergodic theorem-type of result. The exact form of the ergodic behavior actually turns out to be quite a delicate issue for boundary data homogenization because it does not necessarily restrict to lower dimensional subspaces.

An instructive way to understand this difficulty is again to consider the periodic version of the problem. In this case, the translations parallel to $\{y \in \mathbb{R}^d : y \cdot \sigma = 0\}$ are not ergodic when the normal direction is rational; for more discussion see Choi and Kim [CK14] and Feldman [Fel14]. It turns out, however, that it is enough that most (in an appropriate sense) directions yield an ergodic action.

It is not clear to us what kind of generalization of the periodicity assumption would yield this kind of homogenization for almost every direction. Again if we take boundary data on the general domain by "lifting up from hyperplanes" as in (5.1.10), the concern above is not an issue. On the other hand, if one assumes a more quantitative decay of correlations like strong mixing, the translation action of the $d-1$ dimensional subspaces ∂P_σ on Ω will be ergodic and the resulting rate of homogenization is uniform in the normal direction.

We also remark that, as is always the case in random homogenization, we lack the compactness of the periodic setting. All the major assumptions are used to overcome the above difficulties, that is stationarity and ergodicity on hyperplanes and lack of compactness.

Qualitative homogenization results in the stochastic setting for elliptic equations with oscillations in the interior of the domain typically rely on identifying a quantity which controls, via Alexandrov-Bakelman-Pucci (ABP for short)-type inequalities, the asymptotic behavior of the solution and whose ergodic properties can be studied using the sub-additive ergodic theorem; see Caffarelli, Souganidis and Wang [CSW05] and, later, Armstrong and Smart [CS10].

It is not clear to to us whether such a quantity exists for boundary homogenization. This is related to the fact that it is not known whether there exists an estimate analogous to the ABP-inequality controlling solutions to boundary value problems for linear elliptic equations with bounded measurable coefficients in terms of a measure theoretic norm of the boundary data. Relatedly it is known that the harmonic measure for linear non-divergence form operators with only bounded measurable coefficients may be singular with respect to the surface measure on the boundary and, in fact, may be supported on a set of lower Hausdorff dimension (see Wu [Wu96] or Caffarelli, Fabes and Kenig [CFK81] for the divergence form case.)

On the other hand, an argument, which is more in the spirit of the linear problem, works well here since it turns out that the value of the solutions of the cell problems can still be thought of as Lipschitz "nonlinear averages" of the boundary values. This observation lends itself to using tools from the theory of concentration of measure, which, generally speaking, provide estimates in probability for the concentration of Lipschitz functions of many independent random variables about their mean. There is more detailed discussion about this later in the chapter.

To give an idea of the type of results we obtain, we state informally the main theorems without any technical assumptions. Exact statements are given in the next section. In the case that F is either convex or concave, there is a very powerful concentration inequality due to Talagrand which allows the homogenization result to hold without any additional assumptions on F. Without convexity/concavity, the available concentration inequalities either lead to a restriction on the ellipticity ratio of F or depend on stronger assumptions on the random media. We state the results separately depending on this property of F.

Theorem A. *Let F be positively homogeneous of degree one, uniformly elliptic and either convex or concave. Assume that the stationary random field ψ is bounded, uniformly Hölder continuous and satisfies a strong mixing condition with sufficient decay. Then the Dirichlet and Neumann cell problems* (5.1.5) *and* (5.1.8) *homogenize.*

Theorem B. *Let F be positively homogeneous of degree one and uniformly elliptic and ψ be a bounded and uniformly Hölder continuous stationary random field.*
(i) Under a restriction on the ellipticity ratio of F and if ψ satisfies a strong mixing condition with sufficient decay, the cell problems (5.1.5) *and* (5.1.8) *homogenize.*
(ii) If the random media is of "random checkboard"-type, stationary with respect to the $\mathbb{Z}^{d-1}$ translation action on $\{y \in \mathbb{R}^d : y \cdot \sigma = 0\}$, and has a log-Sobolev inequality, then the cell problems

(5.1.5) *and* (5.1.8) *homogenize without any assumption on the ellipticity ratio of F.*

Theorem C. *Under the assumptions of either Theorem A or Theorem B part (i), the general domain Dirichlet and Neumann boundary value problems* (5.1.1) *and* (5.1.2) *homogenize to* (5.1.3) *and* (5.1.4)*, with the homogenized boundary condition determined by the corresponding cell problems* (5.1.5) *and* (5.1.8)*.*

See Section 5.2 for more precise results. We expect that methods very similar to the ones presented here will also show that the homogenization of the cell problems (5.1.5) and (5.1.8) for boundary data as, for example in Theorem B part (ii), will yield a corresponding version of Theorem C for general domains with boundary data defined by "lifting up from the hyperplane" as in (5.1.10) (see Section 5.2 for more details).

Our approach can be extended to equations with nonhomogeneous of degree one nonlinearities and spatial oscillations inside the domain as long as the latter do not "interact" with the second-order derivatives, because the lower order terms scale out during the "blow up" procedure that leads to the cell problems.

Understanding the homogenization in the presence of random oscillations in both the second order term and the boundary data is still open. The main obstacle appears to be the fact that the methods applied to study the homogenization in the interior and on the boundary do not provide sufficient control to deal with the additional boundary layer created by the combined oscillations.

We present next a short review of the existing literature. Since there is a large body of work concerning the homogenization of elliptic pde, we organize this review around oscillatory and non-oscillatory boundary value problems. In each case we discuss general qualitative homogenization results and quantitative error estimates.

Classical references for homogenization of linear (divergence and non-divergence form) operators in periodic media are the books Benssousan, Lions, Papanicolaou [BLP78] and Jikov, Koslov, Oleĭnik [JKO94], while for fully nonlinear problems we refer to Evans [Eva89] and Caffarelli [Caf99]. The most general nonlinear result, without any convexity assumptions on F, is due to Caffarelli and Souganidis [CS10] using δ viscosity solutions.

The first results in random media for linear divergence and non-divergence form operators are due to Papanicolaou and Varadhan [PV81, PV82] and Kozlov [Koz78]. Nonlinear variational problems were studied by Dal Maso and Modica [DM86]. The first general nonlinear result was obtained by Caffarelli, Souganidis and Wang [CSW05], who introduced a sub-additive quantity, based on a family of auxiliary obstacle problems, which controls the behavior of the solutions using the ABP-inequality and, in view of the sub-additive ergodic theorem, has an ergodic behavior. Error estimates for linear divergence and non-divergence form equations under Cordes-type assumptions were obtained by Yurinskiĭ [Yur85, Yur88], while Gloria and Otto [GO11], Gloria, Neukamm and Otto [GNO13] and Marahrens and Otto [MO14] proved optimal rates of convergence in the discrete setting for linear divergence form elliptic pde. For nonlinear non-divergence form equations the first error estimate (logarithmic rate) was obtained by [CS10] in the strong mixing setting; under some assumptions this was upgraded recently to an algebraic error by Armstrong and Smart [AS13].

Much less is known about the homogenization of oscillatory boundary data. In the linear divergence form case with co-normal Neumann boundary data the homogenization is proved in the book [BLP78]. In the periodic setting, some special cases were discussed in Arisawa [Ari03], the Neumann problem in a special half-space setting with periodic boundary data was studied by Barles, Da Lio, Lions and Souganidis [BDL08] (some results were obtained earlier by Tanaka [Tan84] using probabilistic methods), and recently, for general domains and rotationally invariant equations, by Choi and Kim [CK14] and Choi, Kim and Lee [CKL13]. Qualitative results for the oscillatory Dirichlet problems in periodic media were obtained by Barles and Mironescu [BM13] in the half-space setting and, recently, by Feldman [Fel14] in general domains. In the periodic case, the analysis for general domains requires either a careful geometric analysis, such as in [CK14], to obtain a uniform modulus of continuity, or an argument as in [Fel14] that ignores discontinuities of the data in small parts of the boundary. We also point out the recent results of Garet-Varet and Masmoudi [GM11, GM12] about systems of divergence-form operators with oscillatory Dirichlet data in periodic media with error estimates. In a similar vein are the results of Kenig, Lin and Shen [KLS12] on the rate of convergence for interior homogenization with Dirichlet or Neumann boundary conditions in Lipschitz domains.

As far as we know, the results of this chapter are the first concerning homogenization of nonlinear, and, for general domains, even of linear Dirichlet and Neumann problems with random oscillatory data. Our approach is based on measuring, using appropriate barriers, the fluctuation of the interior values of the solutions to the cell problems in terms of the randomness on the boundary. Once such control has been established, we use tools from the theory of concentration inequalities to prove estimates on the deviation of the interior values of the solutions from their mean. Estimates from the elliptic theory and the maximum principle are then used to show that the means converge to a deterministic constant. The different versions of our results in the convex and non-convex cases are due to the nature of the available concentration estimates. Once the homogenization of the cell problem for half spaces in any direction has been established, we employ estimates from the elliptic theory to make rigorous the "blow up" argument described earlier to show that homogenization occurs for the general domain problem. We also discuss separately the cell problems of, what we called above, the "lifting up from the hyperplane" problem. The arguments in this case are similar but, since the probabilistic setting is simpler, we are able to obtain general results as far as F is concerned.

This chapter is organized as follows: In Section 5.2 we introduce the notation and some conventions, the assumptions on the equations, the probabilistic setting, the concentration inequalities we use in this chapter and some useful technical facts about uniformly elliptic pde. Then we present the precise statements of the results. In Section 5.3 we begin with an outline of the proof of the homogenization of the Dirichlet cell problem, then we continue with the details which consist of two steps. The first is the Lipschitz estimates of solutions of the cell problem in terms of the boundary data and the second is the probabilistic arguments based on the aforementioned concentration results. The presentation is divided into two parts depending on the assumptions on F and the randomness. We also prove the continuity of the homogenized boundary condition and the homogenization of the cell problem in the "lifting up from the hyperplane" setting. Section 5.4 contains the proof of homogenization for the Dirichlet problem in a general domain. The Neumann problem is discussed in Section 5.5. The structure of the proof mirrors that of the Dirichlet problem.

5.2 Preliminaries, Assumptions and Statements of Results

5.2.1 Notation and some terminology/conventions

We denote by $\mathscr{M}^d$, TrM and I_d the class of $d \times d$ symmetric matrices with real entries, the trace of $M \in \mathscr{M}^d$ and the the $d \times d$ identity matrix respectively. We write Q for the unit cube $[0,1)^{d-1}$. For each $\sigma \in S^{d-1}$, the sphere in $\mathbb{R}^d$, $P_\sigma := \{x \in \mathbb{R}^d : x \cdot \sigma > 0\}$, $\Pi_\sigma := \{x \in \mathbb{R}^d : 0 < x \cdot \sigma < 1\}$ and for $r > 0$ we call $\Pi_\sigma^r = r\Pi_\sigma$. We call $B_r(x_0) = \{x \in \mathbb{R}^d : |x - x_0| < r\}$ and B_r and B_r^+ refer to $B_r(0)$ and $B_r(0) \cap P_{e_d}$ respectively. Given $\sigma \in S^{d-1}$, we also write $x' = x - x \cdot \sigma$, and, for $L > 0$, we use the cylinders $\mathrm{Cyl}_{\sigma,L} = \{x \in \mathbb{R}^d : |x'| \leq L\} \times \{x \in \mathbb{R}^d : 0 \leq x \cdot \sigma \leq L\}$. Moreover, $2^{\mathbb{N}}$ is the set of dyadic numbers, $\mathbf{1}_A$ is the indicator function of the set A and $C^{0,\alpha}(D)$, $C^{0,1}(D)$, $C^{1,\alpha}(D)$ and $C^2(D)$ are the spaces of α-Hölder continuous, Lipschitz continuous, continuously differentiable with α-Hölder continuous derivatives and twice continuously differentiable functions $f : D \to \mathbb{R}$ with norms respectively $\|f\|_{C^{0,\beta}(D)}$, $\|f\|_{C^{0,1}(D)}$, $\|f\|_{C^{1,\alpha}(D)}$ and $\|f\|_{C^2(D)}$. We write $\mathrm{osc}_D f = \sup_D f - \inf_D f$ for the oscillation over D of the continuous function f and $\|f\|_{\infty,D}$ for the L^∞ norm of the bounded function $f : D \to \mathbb{R}$; if there is no ambiguity for the domain, for simplicity, we only write $\|f\|_\infty$. If U is an open subset of $\mathbb{R}^d$, then, for $r > 0$, $U_r := \{x \in U : \mathrm{dist}(x, \partial U) > r\}$, where dist is the usual Euclidean distance. For $a \in \mathbb{R}$, a_+ and a_- are respectively the positive and negative parts of a. Given a metric space X, $\mathscr{B}(X)$ is the σ-algebra of the Borel sets of X associated with the metric. On product spaces, for example $\Xi = X^{\mathbb{Z}^n}$, the notation $\mathscr{B}(\Xi)$ will refer to the cylinder σ-algebra of Borel sets. For bounded linear operators $L : \ell^2(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n)$ we write $|L|_{\ell^2}$ for the usual operator norm.

We say that constants are universal, if they only depend on the underlying parameters of the problem, which are the ellipticity ratio of F, the dimension d, the constants associated with the C^2-property of U and the mixing conditions. Also note that constants may change, without explicitly said, from line to line. Given two quantities A and B we write $A \lesssim B$ when $A \leq CB$ for some universal C. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$.

Throughout this chapter subsolutions, supersolutions and solutions should be interpreted in the Crandall-Lions viscosity sense. We refer to Crandall, Ishii and Lions [CIL92] for the facts we use throughout this chapter about viscosity solutions. In the special case of uniformly elliptic second order equations the book of Caffarelli and Cabré [CC95] is a useful reference.

We work on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. If $(\Xi, \mathscr{G})$ is a measurable space, $\xi : \Omega \to \Xi$ is a random variable, if it is a measurable mapping in the sense that $\xi^{-1}(E) \in \mathscr{F}$ for all $E \in \mathscr{G}$. When $\Xi = \mathbb{R}$ we write $\mathbb{E}\xi = \int \xi d\mathbb{P}$ when the integral is defined. If $f : D \times \Omega \to \mathbb{R}$, we write $\mathbb{E}f(x)$ for $\int_\Omega f(x,\omega) d\mathbb{P}(\omega)$. If Ξ has a metric space structure and $\mathscr{G} = \mathscr{B}(\Xi)$, then the composition of a continuous function $f : \Xi \to \mathbb{R}$ with a random variable ξ, $f \circ \xi : \Omega \to \mathbb{R}$, is still measurable mapping with respect to $\mathscr{B}(\mathbb{R})$. This justifies the fact that the various functions on Ω we consider throughout this chapter are indeed random variables. Finally, $\sigma(O_a : a \in A)$ is the smallest σ algebra containing the collection $(O_a)_{a \in A} \subset \mathscr{F}$.

5.2.2 The probabilistic setting and concentration inequalities

We are given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, which is endowed with the action of the group of $\mathbb{R}^d$-translations $(\tau_y)_{y \in \mathbb{R}^d}$ such that, for each $y \in \mathbb{R}^d$, $\tau_y : \Omega \to \Omega$ is measurable and measure preserving. More specifically, for every measurable set $A \in \mathscr{F}$ and every $y, z \in \mathbb{R}^d$,

$$\mathbb{P}(\tau_y(A)) = \mathbb{P}(A) \text{ and } \tau_z \tau_y = \tau_{y+z},\ \tau_0 = \text{id}.$$

We say that the action is ergodic if, for any $E \in \mathscr{F}$,

$$\text{if } \tau_x E = E \text{ for all } x \in \mathbb{R}^d, \text{ then } \mathbb{P}(E) \in \{0,1\}.$$

We will also use the above notions in the context of a $\mathbb{Z}^n$-action on a probability space and the definitions are analogous.

Given a bounded subset O of $\mathbb{R}^d$ and a random field $f : \mathbb{R}^d \times \Omega \to \mathbb{R}$, $\mathscr{G}(O)$ is the σ-algebra

$$\mathscr{G}(O) := \sigma(\{f(y,\cdot) : y \in O\}).$$

The ϕ-mixing rate function

$$\phi(r) := \sup\{|\mathbb{P}(E|F) - \mathbb{P}(E)| : \text{dist}(O,V) \geq r,\ E \in \mathscr{G}(O),\ F \in \mathscr{G}(V) \text{ with } \mathbb{P}(F) \neq 0\} \quad (5.2.1)$$

measures of the decorrelation of the values of the random field f. If we want to emphasize the dependence on the random field f, we write ϕ_f.

We say that

$$\text{the random field } f \text{ is } \phi\text{-mixing if } \ \phi(r) \to 0 \ \text{ as } \ r \to \infty. \tag{5.2.2}$$

We will also use the ϕ-mixing condition in the context of random fields on a $\mathbb{Z}^n$-lattice; the definition is completely analogous.

Let A be a complete separable metric space and $\mathscr{B}(A)$ the associated Borel σ-algebra. Fix $n \in \mathbb{N}$, let m be a probability measure on $\Xi := A^{\mathbb{Z}^n}$ with the cylinder σ-algebra and assume that the random field, $(X_j)_{j \in \mathbb{Z}^n}$, on $\mathbb{Z}^n$ given by the coordinate maps satisfies the $\mathbb{Z}^n$ -lattice version of the ϕ-mixing condition (5.2.2).

We say that $f : \Xi \to \mathbb{R}$ is Lipschitz with respect to the α-weighted Hamming distance with weight $\alpha \in \ell^2(\mathbb{Z}^n)$ if

$$|f(X) - f(Y)| \leq \sum_{i \in \mathbb{Z}^n} \alpha_i \mathbf{1}_{X_i \neq Y_i}. \tag{5.2.3}$$

We will be making use of the following concentration inequalities in the strong mixing setting due to Marton [Mar03] and Samson [Sam00]. This first is about functions which are Lipschitz with respect to the Hamming distance (5.2.3).

Theorem 5.2.1. *There exist positive constants c, C depending only on n such that, for any $f : \Xi \to \mathbb{R}$ which is 1-Lipschitz with respect to the α-weighted Hamming distance,*

$$m(\{|f - \int f dm| > t\}) \leq C \exp\left(- \frac{ct^2}{|\alpha|_{\ell^2} (\sum_{i \in \mathbb{Z}^n} \phi(|i|)^{1/2})^2} \right).$$

Suppose additionally that A is a closed convex subspace of some separable Banach space with norm $\|\cdot\|$ and, for $1 \leq p < \infty$, define the ℓ^p-distances on $\Xi^{\mathbb{Z}^n}$ by

$$|X - Y|_{\ell^p} := (\sum_{j \in \mathbb{Z}^n} \|X_j - Y_j\|^p)^{1/p} \ \text{ for } \ X, Y \in \Xi^{\mathbb{Z}^n}.$$

Concentration inequalities with respect to the Hamming, or ℓ^0, distance on a product space (which is just $d_{\ell^0}(X,Y) = \sum \mathbf{1}_{X_i \neq Y_i}$) are often suboptimal when the underlying space also has a Euclidean, that is ℓ^2, distance. This is essentially due to the fact that being 1-Lipschitz with respect to ℓ^0-distance on the Hamming cube $\{0,1\}^n$ is worse by a factor of $\sqrt{n}$ than being 1-Lipschitz with respect to ℓ^2-distance. For functions which are 1-Lipschitz with respect to the ℓ^2-distance and convex, there is the following result of Samson [Sam00], which is an extension of Talagrand's concentration inequality.

Theorem 5.2.2. *There exist positive constants c, C that depend only on n such that, for any convex and 1-Lipschitz with respect to ℓ^2-metric $f : \Xi \to \mathbb{R}$,*

$$m(\{|f - \int f dm| > t\}) \leq C \exp\left(-\frac{ct^2}{(\sum_{i\in\mathbb{Z}^n} \phi(|i|)^{1/2})^2}\right). \tag{5.2.4}$$

Talagrand's inequality is very powerful but it requires convexity. This assumption appears to be, in general, necessary. However, the known counterexamples without convexity do involve some irregularity of the underlying measure.

A concentration inequality like (5.2.4) does hold for Gaussian measures without the convexity assumption on the Lipschitz function f. A less stringent assumption, which also yields concentration for general Lipschitz functions, is the so-called log-Sobolev inequality that we describe next.

A probability measure m on $\mathscr{B}(\mathbb{R}^n)$ is said to have a log-Sobolev inequality (LSI for short), if there exists $\rho > 0$ so that, for any locally continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}$,

$$\int f^2 \log \frac{f^2}{\int f^2 dm} dm \leq \frac{1}{2\rho} \int \sum_{j=1}^{n} |\partial_j f|^2 dm. \tag{5.2.5}$$

The LSI can be defined on infinite product spaces as well, for example, $\mathbb{R}^{\mathbb{Z}^n}$. It turns out that measures which have log-Sobolev inequality satisfy the following dimension independent concentration property; for a proof we refer to Ledoux [Led01].

Theorem 5.2.3. *Suppose that m has LSI with constant ρ. For any 1-Lipschitz with respect to the ℓ^2-metric $f : \mathbb{R}^n \to \mathbb{R}$,*

$$m(\{|f - \int f dm| > t\}) \leq 2\exp(-2\rho t^2).$$

The same result holds on infinite product spaces like $\mathbb{R}^{\mathbb{Z}^n}$ as well.

We list next as lemmas two important properties of the log-Sobolev inequalities which are used to construct large classes of measures with LSI. The first is the so-called tensorization property, which says that LSI is preserved under taking product measures; we refer again to [Led01] for more explanations.

Lemma 5.2.4. *Fix $N \in \mathbb{N}$ and assume that, for $1 \leq j \leq N$, the measures m_j on $\mathbb{R}^{n_j}$ have LSI with constants ρ_j. Then the product measure $m = m_1 \otimes \cdots \otimes m_N$ on the product space $\mathbb{R}^{n_1 + \cdots n_N}$ also has LSI with constant $\rho = \min \rho_j$.*

The second is that the pushforward $\phi_\# m$ of a measure m with LSI under a Lipschitz transformation ϕ still has a log-Sobolev inequality; recall that $\phi_\# m(E) := m(\phi^{-1}E)$ for any $E \in \mathscr{B}(\mathbb{R}^n)$.

Lemma 5.2.5. *Let m be a measure on $\mathbb{R}^n$ with LSI and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ a (globally) Lipschitz and locally continuously differentiable function. Then the pushforward measure $\phi_\# m$ of m by ϕ also has log-Sobolev inequality.*

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally continuously differentiable. Writing, for notational convenience, $\tilde{f} := f \circ \phi$, we find

$$\begin{aligned}
\int f^2 \log \frac{f^2}{\int f^2 d\phi_\# m} d\phi_\# m = \int \tilde{f}^2 \log \frac{\tilde{f}^2}{\int \tilde{f}^2 dm} dm \leq \frac{1}{2\rho} \int |D\tilde{f}|^2 dm \\
\leq \frac{\sup |D\phi|^2_{\ell^2}}{2\rho} \int |Df \circ \phi|^2 dm = \frac{\sup |D\phi|^2_{\ell^2}}{2\rho} \int |Df|^2 d\phi_\# m.
\end{aligned}$$

□

Combining Lemma 5.2.4 and Lemma 5.2.5, it is possible to construct large classes of measures with LSI starting from some simple examples of such measures, like the uniform and Gaussian measures on the unit interval and $\mathbb{R}$ respectively, which are the prototypical examples of measures with log-Sobolev inequality; see [Led01].

For example, the distribution of $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$, with the X_i's i.i.d. Gaussians with mean 0 and variance 1, has LSI with the same constant ρ as the distribution of the X_i's. Moreover, given a $n \times n$ matrix σ, the random variable $Y = \sigma X$ is also Gaussian with covariance matrix

$\Sigma = (\mathbb{E}Y_iY_j)_{1\le i,j\le N}$ and its distribution m_σ on $\mathbb{R}^n$ has LSI, since

$$\int f^2 \log \frac{f^2}{\int f^2 dm_\sigma} dm_\sigma \le \frac{|\Sigma|^2_{\ell^2}}{2\rho} \int |Df|^2 dm_\sigma. \tag{5.2.6}$$

Based on the discussion above we can construct now some random fields on $\mathbb{Z}^{d-1}$ with LSI.

Let $X = (X_j)_{j\in\mathbb{Z}^{d-1}}$ be i.i.d. and such that law(X_0) has log-Sobolev inequality on $\mathbb{R}$. Then, by Lemma 5.2.4, the law, m, of X on $\mathbb{R}^{\mathbb{Z}^{d-1}}$ has LSI.

In the Gaussian case we can accommodate a more general mixing condition. Let $(Y_j)_{j\in\mathbb{Z}^{d-1}}$ be Gaussian random variables and $Y' = \phi(Y)$, where ϕ is Lipschitz with respect to $|\cdot|_{\ell^2}$ and $|\phi|_{\ell^\infty} \le 1$. If the covariance Σ of the Y_j's, with elements $\Sigma_{ij} = \mathbb{E}Y_iY_j$, has a finite operator norm as a map from ℓ^2 to ℓ^2, then the law, m, of Y' has LSI on $\mathbb{R}^{\mathbb{Z}^{d-1}}$. The finiteness of $|\Sigma|_{\ell^2}$ follows from the summability in $\mathbb{Z}^{d-1}$ of the covariances, since

$$|\Sigma|^2_{\ell^2} \le (\max_i \sum_j |\Sigma_{ij}|).$$

We note that from the above example we may infer, at least heuristically, that the assumption of a log-Sobolev inequality amounts to some regularity of the 1-dimensional distributions combined with a summable decay of correlations.

5.2.3 The assumptions

We present next our assumptions. For the convenience of the reader we divide them into separate groups.

The domain. We assume that

$$U \subset \mathbb{R}^d \ \text{ has } \ C^2\text{-boundary}, \tag{5.2.7}$$

and

$$K \ \text{ is a compact subset of } U. \tag{5.2.8}$$

The regularity of ∂U means, in particular, that, for each $x \in \partial\Omega$, there exists an inward normal direction σ_x, $r_0 > 0$ and $M > 0$ such that, for all $0 < r < r_0$ and all $x \in \partial U$,

$$d((x + \partial P_\sigma) \cap \overline{B_r}, \partial U \cap \overline{B_r}) \le Mr^2. \tag{5.2.9}$$

The nonlinearity. We assume that $F : \mathscr{M}^d \to \mathbb{R}$ is uniformly elliptic, that is there exist $\Lambda > \lambda > 0$ such that

$$\lambda \operatorname{tr}(N) \leq F(M) - F(M+N) \leq \Lambda \operatorname{tr}(N) \ \text{ for all } M, N \in \mathscr{M}^d \ \text{ such that } N \geq 0, \tag{5.2.10}$$

and homogeneous, that is

$$F(0) = 0. \tag{5.2.11}$$

Typical examples of F's satisfying (5.2.10) and (5.2.11) are the convex Hamilton-Jacobi-Bellman and the non convex Hamilton-Jacobi-Isaacs nonlinearities

$$F(M) = \sup_{\alpha \in \mathscr{A}} [-\operatorname{tr}(A^\alpha M)] \ \text{ with } \ \lambda I \leq A^\alpha \leq \Lambda I \text{ for all } \alpha \in \mathscr{A}$$

and

$$F(M) = \inf_{\beta \in \mathscr{B}} \sup_{\alpha \in \mathscr{A}} [-\operatorname{tr}(A^{\alpha\beta} M)] \ \text{ where } \ \lambda I \leq A^{\alpha\beta} \leq \Lambda I \ \text{ for all } (\alpha, \beta) \in \mathscr{A} \times \mathscr{B},$$

which arise respectively in the theories of stochastic control of diffusion processes and zero-sum stochastic differential games in both cases without running cost.

In fact all F's satisfying (5.2.10) and (5.2.11) have a max-min representation as above and, hence, are one-positively homogeneous, that is

$$F(tM) = tF(M) \ \text{ for all } t > 0 \ \text{ and } \ M \in \mathscr{M}^d. \tag{5.2.12}$$

The boundary data for the Neumann problem. As far as the deterministic (non-random) boundary condition on K is concerned we assume what is sufficient for the Neumann problems (5.1.2) and (5.1.4) to have unique solutions, that is

$$f : K \to \mathbb{R} \ \text{ is bounded and continuous.} \tag{5.2.13}$$

The random media and data. We make different assumptions depending on whether the boundary data is the restriction of a function defined on the whole space or is obtained by lifting from a hyperplane. To avoid repetition we state some properties for functions defined on $\mathbb{R}^n$ with $n = d$ in the former and $n = d-1$ in the latter cases.

We begin with the boundary data $\psi : \mathbb{R}^n \times \Omega \to \mathbb{R}$ arising in the cell problems. We assume that

$$\psi \text{ is measurable with respect to } \mathscr{B}(\mathbb{R}^n) \otimes \mathscr{F}, \tag{5.2.14}$$

stationary in (y, ω) with respect to the group action $(\tau_y)_{y \in \mathbb{R}^d}$, that is, for all $y \in \mathbb{R}^n$,

$$\psi(y, \omega) = \psi(0, \tau_y \omega), \tag{5.2.15}$$

and bounded uniformly in $\omega \in \Omega$, that is there exists $C > 0$ such that

$$|\psi(y, \omega)| \leq C \text{ for all } y \in \mathbb{R}^n \text{ and } \omega \in \Omega. \tag{5.2.16}$$

When $n = d$, which is the case of the cell problem (5.1.5), we also assume that ψ is Lipschitz continuous uniformly in ω, that is there exists $C > 0$ such that, for all $y, w \in \mathbb{R}^n$ and $\omega \in \Omega$,

$$|\psi(y, \omega) - \psi(w, \omega)| \leq C|y - w|; \tag{5.2.17}$$

we remark that, with only minor changes, we may assume that ψ is Hölder or even just uniformly continuous in y. We leave it up to the interested reader to fill in the details.

Furthermore, as mentioned in the introduction, we need some mixing assumption on the random field ψ. We assume that

$$\psi(\cdot, \cdot) \text{ is } \phi\text{-mixing with a rate } \phi \text{ such that } \rho = \int_0^\infty \phi(r)^{1/2} r^{d-2} dr < +\infty; \tag{5.2.18}$$

the value of this integral, ρ, will be considered a universal constant in the following sections.

In the case that the boundary data is assigned by lifting from a hyperplane, that is when $n = d - 1$, the condition on the boundary data is based on LSI.

We assume that there exists a collection of identically distributed $(X_j)_{j \in \mathbb{Z}^{d-1}}$ such that, if $X = (X_j)_{j \in \mathbb{Z}^{d-1}}$, then

$$\text{the measure law}(X) \text{ on } \mathbb{R}^{\mathbb{Z}^{d-1}} \text{ has LSI with constant } \rho \tag{5.2.19}$$

and

$$\psi(y, \omega) = X_j(\omega) \text{ when } y \in j + [0, 1)^{d-1}; \tag{5.2.20}$$

this is just the random checkerboard boundary data. Note that in this case we only have $\mathbb{Z}^{d-1}$ stationarity.

Next we state the conditions on the random boundary data $g : \partial U \times \mathbb{R}^d \times \Omega \to \mathbb{R}$. We assume that

$$g \text{ is measurable with respect to } \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{F}, \tag{5.2.21}$$

and bounded and Lipschitz continuous uniformly in $\omega \in \Omega$, that is there exists $C > 0$ such that, for all $(x,y),(z,w) \in \partial U \times \mathbb{R}^d$ and $\omega \in \Omega$,

$$|g(x,y,\omega)| \leq C \;\text{ and }\; |g(x,y,\omega) - g(z,w,\omega)| \leq C(|x-z| + |y-w|). \tag{5.2.22}$$

We remark again that, with only minor changes, we may assume that g is Hölder or even just uniformly continuous in (x,y); we leave it up to the interested reader to fill in the details.

Finally we assume that, for each fixed $x_0 \in \partial U$,

$$g(x_0,\cdot,\cdot) \;\text{ is stationary in } (y,\omega) \tag{5.2.23}$$

and

$$g(x_0,\cdot,\cdot) \text{ satisfies (5.2.18) with constants uniform in } x_0. \tag{5.2.24}$$

5.2.4 Some pde background

We recall and prove some background results from the theory of fully nonlinear uniformly elliptic equations that we will need for the proofs in the sequel. First we discuss (5.1.5) and then (5.1.8). Then we introduce a selection criterion to identify uniquely a particular solutions to (5.1.5) and then (5.1.8) when ψ is discontinuous.

In what follows it is convenient to use the extremal Pucci operators $\mathscr{P}^{\pm}_{\lambda,\Lambda}$ associated with F defined by

$$\mathscr{P}^{+}_{\lambda,\Lambda}(N) := \Lambda \mathrm{tr}(N_+) - \lambda \mathrm{tr}(N_-) \;\text{ and }\; \mathscr{P}^{-}_{\lambda,\Lambda}(N) := \lambda \mathrm{tr}(N_+) - \Lambda \mathrm{tr}(N_-); \tag{5.2.25}$$

it follows from (5.2.10) and (5.2.11) that, for all $M,N \in \mathscr{M}^d$,

$$-\mathscr{P}^{+}_{\lambda,\Lambda}(M-N) \leq F(M) - F(N) \leq -\mathscr{P}^{-}_{\lambda,\Lambda}(M-N). \tag{5.2.26}$$

The Dirichlet Problem (5.1.5). The first two results are an oscillation decay lemma, which is a consequence of the interior Lipschitz estimates for solutions to elliptic pdes, and a boundary Lipshitz estimate; for the proofs we refer to Caffarelli and Cabre [CC95].

Lemma 5.2.6. (Oscillation decay/Interior Lipschitz estimate) *Assume* (5.2.10), (5.2.11), (5.2.16) *and* (5.2.22) *and let* v *be the unique bounded solution of* (5.1.5). *Then, for* $R > 1$ *and* $y,z \in \partial P_\sigma + R\sigma$,

$$\sup_{|y-z|\leq 1} |v(y,\omega) - v(z,\omega)| \leq CR^{-1}\operatorname{osc}_{\partial P_\sigma}\psi.$$

Lemma 5.2.7. (Boundary Lipshitz) *Assume* (5.2.10) *and* (5.2.11). *If* $u \in C(\overline{B_1^+})$ *solves* $F(D^2u) = 0$ *in* B_1^+ *and* $u = h \in C^{0,1}(\partial P_{e_d} \cap B_1)$, *then* $u \in C^{0,1}(\overline{B_{1/2}^+})$ *and there exists* $C = C(n,\lambda,\Lambda) > 0$ *such that*

$$\|u\|_{C^{0,1}(\overline{B_{1/2}^+})} \leq C(\sup_{B_1^+}|u| + \|h\|_{C^{0,1}(\partial P_{e_d}\cap B_1)}).$$

The next lemma is an important tool for the proofs to come, because it quantifies the error introduced by solving the Dirichlet problem in a half space with a cut-off applied to the boundary data.

Lemma 5.2.8. (Localization) *Fix* $\sigma \in S^{d-1}$, *let* $M > 0$ *and* $L > 1$ *and suppose that* v *is a solution of*

$$\begin{cases} -\mathscr{P}^+_{\lambda,\Lambda}(D^2v) \leq 0 & \text{in} \quad P_\sigma \cap \mathrm{Cyl}_{\sigma,L}, \\ v \leq 0 & \text{on} \quad \partial P_\sigma \cap \mathrm{Cyl}_{\sigma,L}, \\ v \leq M & \text{on} \quad \partial \mathrm{Cyl}_{\sigma,L} \cap \overline{P_\sigma}. \end{cases} \tag{5.2.27}$$

Then

$$v \leq 2\tfrac{\Lambda}{\lambda} dML^{-1} \ \text{ in } \ \mathrm{Cyl}_{\sigma,1}.$$

Proof. Since the Pucci maximal operators are rotation invariant, without loss of generality, we assume that $\sigma = e_d$ and, for notational simplicity, write Cyl_L for $\mathrm{Cyl}_{e_d,L}$. Consider the barrier ψ

$$\psi(x) := ML^{-2}(|x'|^2 - 2\tfrac{\Lambda}{\lambda}(d-1)x_d^2) + (2\tfrac{\Lambda}{\lambda}(d-1)+1)ML^{-1}x_d.$$

It is straightforward to check that $\psi \geq v$ on $\partial\mathrm{Cyl}_L$ and that ψ is a smooth supersolution of $-\mathscr{P}^+_{\lambda,\Lambda}(D^2\psi) \geq 0$ in P_{e_d}. From the comparison of viscosity solution we have $v \leq \psi$ in Cyl_L

and the claim follows, since

$$\psi \leq 2(\tfrac{\Lambda}{\lambda}(d-1)+1)ML^{-1} \text{ in } \mathrm{Cyl}_1.$$

$\square$

Next we recall a particular case of Theorem 1 from Armstrong, Sirakov and Smart [ASS12] which yields the existence of singular solutions to nonlinear uniformly elliptic equations.

Theorem 5.2.9. *Assume* (5.2.10) *and* (5.2.11). *For any $\gamma \in (-1,1)$, there exists a unique constant $\beta = \beta(d,\lambda,\Lambda,\gamma) > \max\{\frac{\lambda}{\Lambda}(d-1)-1, 0\}$ such that the problem*

$$\begin{cases} -\mathscr{P}^+_{\lambda,\Lambda}(D^2\Phi) = 0 & in \quad \mathrm{Cone}_\gamma := \{x \in \mathbb{R}^d : x_d > \gamma|x|\}, \\ \Phi = 0 & on \quad \partial\mathrm{Cone}_\gamma \setminus \{0\}, \end{cases} \tag{5.2.28}$$

has a unique positive solution $\Phi \in C(\overline{\mathrm{Cone}_\gamma} \setminus \{0\})$ which is $-\beta$ homogeneous, that is $\Phi(x) = |x|^{-\beta}\Phi(x/|x|)$, and satisfies $\max_{\partial B_1 \cap K_\gamma} \Phi = 1$.

When $\gamma = 0$, that is in the half space case, Leoni [Leo12] shows that

$$\frac{\lambda}{\Lambda}(d-1) \geq \beta \geq \frac{\lambda}{\Lambda}d - 1; \tag{5.2.29}$$

from now on we will always use β to refer to the half-space singular solution exponent for the ellipticity ratio Λ/λ.

It will be useful for later to estimate, in the half-space case, the decay of Φ in the directions x' which are orthogonal to e_d.

We first note that Lemma 5.2.7 and a simple covering argument yield that Φ is Lipschitz with universal constant in $(B_2 \setminus B_{1/2}) \cap K_0$.

Next we show that $\Phi(x/|x|) \sim x_d/|x|$ and, therefore,

$$\Phi(x) = |x|^{-\beta}\Phi(x/|x|) \sim x_d/|x|^{\beta+1}. \tag{5.2.30}$$

The Lipschitz estimate and the fact that $\Phi = 0$ on ∂K_0 yield

$$\Phi(\frac{x}{|x|}) \leq C\mathrm{dist}\left(\frac{x}{|x|}, B_2 \setminus B_{1/2} \cap \partial K_0\right) \leq C\frac{x_d}{|x|}.$$

For the other direction we note that $A = \min_{B_2 \cap \{x_d > 1/2\}} \Phi > 0$ is universal due to the normalization $\max_{\partial B_1 \cap K_0} \Phi = 1$ and the Harnack inequality. Then, for each x, let $x' = x - x_d e_d$ and $y = \frac{3}{4} e_d + \frac{x'}{|x|}$, consider the radially symmetric barrier v satisfying

$$\begin{cases} -\mathscr{P}^{+}_{\lambda,\Lambda}(D^2 v) = 0 & \text{in} \quad B_{1/2}(y) \setminus B_{1/4}(y), \\ v = 0 & \text{on} \quad \partial B_{1/2}(y), \\ v = A & \text{on} \quad \partial B_{1/4}(y), \end{cases}$$

and observe that the maximum principle yields $\Phi \geq v$ in $B_{1/2}(y) \setminus B_{1/4}(y)$. Meanwhile, an explicit computation shows that

$$v(\frac{x}{|x|}) \geq cA \frac{x_d}{|x|},$$

and, hence,

$$\Phi(\frac{x}{|x|}) \geq v(\frac{x}{|x|}) \gtrsim \frac{x_d}{|x|},$$

which proves both directions of (5.2.30).

The Neumann Problem (5.1.8). The first result is a localization result analogous to Lemma 5.2.8.

Lemma 5.2.10. (Localization) *Assume* (5.2.10) *and* (5.2.11)*, fix* $\sigma \in S^{d-1}$ *and, for* $L > 0$*, let* $u : \Pi_\sigma \to \mathbb{R}$ *satisfy*

$$\begin{cases} -\mathscr{P}^{+}_{\lambda,\Lambda}(D^2 u) \leq 0, & \textit{in } \Pi_\sigma \\ \partial_\sigma u \leq 0 & \textit{on } \partial P_\sigma \cap B_L, \\ u \leq 0 & \textit{on } (\partial P_\sigma + \sigma) \cap B_L \\ u \leq M & \textit{on } \Pi_\sigma \cap \partial B_L. \end{cases}$$

Then

$$|u| \leq (1 + (d-1)\frac{\Lambda}{\lambda}) \frac{M}{L^2} \ \textit{ on } \ \Pi_\sigma \cap \{|x'| \leq 1\}$$

Proof. The claim follows from the comparison principle using the barrier

$$\varphi(x) = \frac{M}{L^2} \left(|x'|^2 + (d-1)\frac{\Lambda}{\lambda}(1 - (x \cdot \sigma)^2) \right).$$

□

The next result is about up to the boundary regularity of solutions of the Neumann problem. Its proof can be found in Milakis and Silvestre [MS06].

Lemma 5.2.11. *(Boundary $C^{1,\alpha}$-regularity) Assume* (5.2.10) *and* (5.2.11). *For every* $\alpha \in (0,1)$, *there exists* $C = C(d,\lambda,\Lambda,\alpha) > 0$ *such that, if v solves the Neumann problem*

$$\begin{cases} F(D^2 v) = 0 & \text{in } B_1^+, \\ \partial_{x_d} v = h & \text{on } \partial P_{e_d} \cap B_1, \end{cases}$$

then

$$\|v\|_{C^{0,\alpha}(B_{1/2}^+)} \le \sup_{B_1} |v| + C \max_{\partial P_{e_d} \cap B_1} |h|.$$

Furthermore, there is some $\alpha = \alpha(d,\lambda,\Lambda) \in (0,1)$ *such that,*

$$\|v\|_{C^{1,\alpha}(B_{1/2}^+)} \le \sup_{B_1} |v| + C \|h\|_{C^{0,\alpha}(\partial P_{e_d} \cap B_1)}.$$

As for the decay of oscillations, the estimate is slightly better than for the Dirichlet problem, since the oscillations decay up to the boundary.

Lemma 5.2.12. (Oscillation decay) *Assume* (5.2.10) *and* (5.2.11) *and fix* $\sigma \in S^{d-1}$. *If v_R is the solution to* (5.1.8)*, then,*

$$\sup_{|y-z|\le 1} R^{-1} |v_R(y,\omega) - v_R(z,\omega)| \le C R^{-1} \sup_{\partial P_\sigma} |\psi| \quad \text{if} \quad y \in \overline{\Pi}_\sigma^R + \frac{1}{2} R\sigma$$

and there exists $0 < \alpha = \alpha(d,\lambda,\Lambda)$ *such that*

$$\sup_{|y-z|\le 1} |\partial_\sigma (v_R(y,\omega) - v_R(z,\omega))| \le C R^{-\alpha} \operatorname{osc}_{\partial P_\sigma} \psi(y) \quad \text{if } y \in \overline{\Pi}_\sigma^R + \frac{1}{2} R\sigma.$$

Proof. To prove the first inequality we consider the rescaled function $\tilde v(y,\omega) := \frac{1}{R} v_R(Ry,\omega)$ and apply Lemma 5.2.11 to the Neumann problem in $B_2 \cap P_\sigma$ for $|y-z| \le R^{-1}$.

For the second inequality we apply the interior $C^{1,\alpha}$-regularity result for solutions to uniformly elliptic operators (see [CC95]) to $\tilde v$ and use the fact that $\operatorname{osc}_{2\Pi_\sigma} \tilde v \le 2 \operatorname{osc}_{\partial P_\sigma} \psi$. □

Finally we introduce the analogue of the singular solutions Theorem 5.2.9 in the Neumann case. Let

$$\Phi(x) = |x|^{-\beta} \text{ with } \beta = \frac{\lambda}{\Lambda}(d-1) - 1. \tag{5.2.31}$$

An explicit calculation shows that $\phi(x) = \Phi(x+e_d)$ satisfies

$$\begin{cases} -\mathscr{P}^{+}_{\lambda,\Lambda}(D^2\phi) = 0 & \text{in} \quad P_{e_d}, \\ -\partial_{e_d}\phi \geq c_d(1+|x|)^{\frac{\lambda}{\Lambda}(d-1)} & \text{on} \quad \partial P_{e_d}. \end{cases} \tag{5.2.32}$$

Discontinuous Boundary Data. We discuss the Dirichlet problem here, but similar arguments apply to the Neumann problem. In the course of the proof of the homogenization of the cell problem, we will have to consider (5.1.5) with bounded but discontinuous boundary data. Such boundary value problems may not have, in general, a unique solution unless F has some additional structure. It was shown, for example, in [Fel14] that uniqueness holds if the discontinuities are concentrated on a subset of ∂D of Hausdorff dimension less than $d-1$ and if $|\lambda/\Lambda - 1|$ is small depending on the dimension of the discontinuity set.

We describe next a selection mechanism we will be using throughout this chapter, which allows to talk about a unique solution to (5.1.5) satisfying the comparison principle

Since the argument is more general than (5.1.5), we consider the boundary value problem

$$\begin{cases} F(D^2u) = 0 \text{ in } D, \\ u = \phi \text{ on } \partial D; \end{cases} \tag{5.2.33}$$

here D is a general domain, which typically in this chapter will be P_σ for some $\sigma \in S^{d-1}$, and $\phi : \partial D \to \mathbb{R}$ is bounded but possibly discontinuous. In what follows we write $u(\cdot;\phi)$ to denote the solution of the boundary value problem with data ϕ.

A maximal/minimal solution to (5.2.33) can be constructed by the classical Perron's method (see, for example, [CIL92]). Here we describe an alternative method to select in a unique fashion a solution $u(\cdot;\phi)$, which is defined as the pointwise in $\overline{D}$ and local uniform in D limit of the (unique) solution $u(\cdot;\phi_\delta)$ of (5.2.33), for some appropriately defined regularization ϕ_δ of ϕ. Moreover, $u(\cdot;\phi)$ satisfies the contraction property, that is, for any two bounded possibly discontinuous ϕ, ϕ',

$$\sup_D (u(\cdot;\phi) - u(\cdot;\phi'))_\pm \leq \sup_{\partial D}(\phi - \phi')_\pm; \tag{5.2.34}$$

in what follows we say that the so defined $u(\cdot;\phi)$ satisfies the maximum principle.

To this end, given a bounded possibly discontinuous $\phi : \partial D \to \mathbb{R}$, for $\delta > 0$, we define

$$\phi_\delta(y) := \max_{z\in\partial D}[\phi(z) - \frac{1}{\delta}|y-z|].$$

It is immediate that ϕ_δ is Lipschitz continuous, $\phi_\delta \geq \phi$ and the ϕ_δ's decrease, as $\delta \to 0$, to ϕ at every point of continuity of ϕ. The standard comparison principle for bounded uniformly continuous solutions to (5.2.33) yields that the solutions $u(\cdot;\phi_\delta)$'s are decreasing in δ and, hence, for each $y \in \overline{D}$, we may define $u(y;\phi)$ by $u(y;\phi) := \lim_{\delta\to 0} u(y;\phi_\delta)$ pointwise in $\overline{D}$ and locally uniformly in D. Since the regularization of ϕ is contractive, that is, for any ϕ, ϕ' as above, $\sup_{\partial D}(\phi_\delta - \phi'_\delta)_\pm \leq \sup_{\partial D}(\phi - \phi')_\pm$, it follows, again from the comparison principle, that

$$\sup_D (u(\cdot;\phi_\delta) - u(\cdot;\phi'_\delta))_\pm \leq \sup_{\partial D}(\phi_\delta - \phi'_\delta)_\pm \leq \sup_{\partial D}(\phi - \phi')_\pm;$$

letting $\delta \to 0$ on the left hand side above yields (5.2.34).

5.2.5 The results

Now that we have set up the assumptions, we give the precise statements of the main results, Theorems A, B and C from the Introduction.

The Dirichlet cell problem (5.1.5). The first result assumes convexity/concavity and imposes no restrictions on the ellipticity ratio of F, while the second applies to general F's but requires a dimension dependent upper bound on Λ/λ.

In what follows $\beta(\lambda,\Lambda)$ is the homogeneity exponent of the singular solution for the ellipticity class of F in the half plane (see (5.2.29)).

Theorem 5.2.13. *Suppose that, in addition to* (5.2.10)*and* (5.2.11)*, F is either convex or concave and ψ satisfies* (5.2.14)*,* (5.2.15)*,* (5.2.16)*,* (5.2.17) *and* (5.2.18) *. Then, for each $\sigma \in S^{d-1}$, the cell problem* (5.1.5) *has a unique solution v_σ which concentrates about its mean with rate*

$$\mathbb{P}(\{\omega : |v_\sigma(y'+R\sigma,\omega) - \mathbb{E}v_\sigma(y'+R\sigma)| > t\}) \leq C\exp(-cR^{\hat\beta}t^2) \text{ for } y' \in \partial P_\sigma \text{ and } t > 0. \quad (5.2.35)$$

where $\hat\beta = \beta(\lambda,\Lambda)$. Moreover, the limit $\mu(\psi,F,\sigma) := \lim_{R\to\infty}\mathbb{E}v_\sigma(R\sigma)$ exists and equals the almost sure limit of $v_\sigma(R\sigma,\omega)$, as $R \to \infty$, and, furthermore, for some universal constant $C =$

$C(d,\lambda,\Lambda,\rho) = C(d,\lambda,\Lambda)\rho^2$ *and all* $y' \in \partial P_\sigma$,

$$|\mathbb{E}v_\sigma(y'+R\sigma) - \mu(\psi,F,\sigma)| \le C(\log R)^{1/2}R^{-\hat{\beta}/2}. \tag{5.2.36}$$

Theorem 5.2.14. *Assume that F and ψ satisfy* (5.2.10)*,* (5.2.11)*,* (5.2.14)*,* (5.2.15)*,* (5.2.16)*,* (5.2.17) *and* (5.2.18) *and assume that* $2\beta(\lambda,\Lambda)-(d-1)>0$*. Then the results of Theorem 5.2.13 hold with* $\hat{\beta} = 2\beta(\lambda,\Lambda)-(d-1)$*.*

Under the assumptions that lead to the homogenization of the cell problem, the ergodic constant (homogenized boundary condition) is continuous with respect to the normal direction and the boundary data. This a very important property to establish homogenization in general domains. Its proof relies in a critical way on having a rate of convergence for the cell problem.

Theorem 5.2.15. *Assume* (5.2.10)*,* (5.2.11) *and* $2\beta(\lambda,\Lambda)-(d-1)>0$ *if F is neither convex nor concave.*

(i) If ψ and ψ' satisfy (5.2.14)*,* (5.2.15)*,* (5.2.16)*,* (5.2.17) *and* (5.2.18) *and fix* $\sigma \in S^{d-1}$*, then*

$$|\mu(\psi,F,\sigma) - \mu(\psi',F,\sigma)| \le \|\psi-\psi'\|_{\infty,\mathbb{R}^d\times\Omega}.$$

(ii) There exists $C = C(d,\lambda,\Lambda) > 0$ *such that, for every* $\sigma,\sigma' \in S^{d-1}$ *and* $\alpha \in (0,\alpha')$ *with* $\alpha' = \alpha'(\hat{\beta}) := \frac{\hat{\beta}}{2(1+\hat{\beta})}$*, if ψ satisfies* (5.2.14)*,* (5.2.15)*,* (5.2.16)*,* (5.2.17) *and* (5.2.18)*, then*

$$|\mu(\psi,F,\sigma) - \mu(\psi,F,\sigma')| \le C(\operatorname{osc}\psi)(1+\|D\psi\|_\infty)^{\alpha'}|\sigma-\sigma'|^{\alpha}.$$

Lastly we have the following homogenization result for the Dirichlet problem in general domains with an algebraic rate in probability.

Theorem 5.2.16. *Assume* (5.2.7)*,* (5.2.10) *and* (5.2.11) *and define $\hat{\beta}$ as in either Theorem 5.2.13 or Theorem 5.2.14 depending on whether F is convex/concave or not. In addition suppose that the boundary data g satisfies* (5.2.21)*,* (5.2.22)*,* (5.2.23) *and* (5.2.24)*. Let u^ε and $\overline{u}$ be respectively the solutions to* (5.1.1) *and* (5.1.3) *with Dirichlet boundary data* $g(x,x/\varepsilon,\omega)$ *and* $\overline{g}(x) :=$

$\mu(g(x,\cdot,\cdot),F,\sigma_x)$ *respectively. The* u^ε *concentrates about its mean in the sense that, for* $\alpha_0 := \frac{\hat\beta}{4+3\hat\beta}$ *and every* $p>0$*, there exists a sufficiently large universal constant* M_p *such that*

$$\mathbb{P}(\{\omega : \sup_{\{x:d(x,\partial U)>\varepsilon^{1-2\alpha_0/\hat\beta}\}} |u^\varepsilon(x,\omega)-\mathbb{E}u^\varepsilon(x)| > M_p(\log\tfrac{1}{\varepsilon})^{1/2}\varepsilon^{\alpha_0}\}) \lesssim \varepsilon^p, \tag{5.2.37}$$

and the expected value $\mathbb{E}u^\varepsilon$ *converges, as* $\varepsilon\to 0$*, to* $\overline{u}$ *with the rate*

$$\sup_{\{x:d(x,\partial U)>\varepsilon^{1-2\alpha_0/\hat\beta}\}} |\mathbb{E}u^\varepsilon(x)-\overline{u}(x)| \lesssim (\log\tfrac{1}{\varepsilon})^{1/2}\varepsilon^{\alpha_0}. \tag{5.2.38}$$

The lifting from the hyperplane plane. We describe the general setting for the Dirichlet cell problem arising when assigning boundary data by “lifting from the hyperplane” as in (5.1.10) and then state the result which asserts that the cell problem homogenizes for general F without either a convexity/concavity assumption or a restriction on the ellipticity ratio.

We assume that

$$\zeta_0 \text{ is a diffeomorphism from an open subset of } \mathbb{R}^{d-1} \text{ to a relatively open subset of } \partial U, \tag{5.2.39}$$

and we extend ζ_0 to a local diffeomorphism $\zeta:\mathbb{R}^d\to\mathbb{R}^d$ by

$$\zeta(x) = (\zeta_0(x'), x_d\sigma_{\zeta(x')}). \tag{5.2.40}$$

Given a random field $\psi:\mathbb{R}^{d-1}\times\Omega\to\mathbb{R}$, we define the boundary data on the general domain U by

$$g^\varepsilon(x,\omega) := \psi(\varepsilon^{-1}\zeta^{-1}(x),\omega) \ \text{ for } \ x\in\partial U;$$

note that here we implicitly extend ψ to map $\psi:\partial U\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ which is constant in the y_d direction.

We study the asymptotic behavior of the rescaling $v^\varepsilon(y) := u^\varepsilon(x_0+\varepsilon y)$ of u^ε, the solution to (5.1.1) with g as above, at some x_0 in the image of ζ_0, which solves the boundary value problem which solves the boundary value problem

$$\begin{cases} F(D^2 v^\varepsilon) = 0 & \text{in} \quad \varepsilon^{-1}(U-x_0) \\ v^\varepsilon(y,\omega) = \psi(\varepsilon^{-1}\zeta^{-1}(x_0) + (D\zeta^{-1})(x_0)y + w_\varepsilon(y),\omega) & \text{on} \quad \varepsilon^{-1}\partial(U-x_0); \end{cases} \tag{5.2.41}$$

with $|w_\varepsilon(y)| \le C\varepsilon|y|^2$ for some $C = C(\|\zeta^{-1}\|_{C^2})$.

The corresponding cell problem is then to solve

$$\begin{cases} F(D^2 v_T) = 0 & \text{in} \quad TP_{e_d}, \\ v_T(y,\omega) = \psi(T^{-1}y,\omega) & \text{on} \quad T\partial P_{e_d}, \end{cases} \tag{5.2.42}$$

where $T : \mathbb{R}^d \to \mathbb{R}^d$ is an invertible linear map, and to show that there exists an ergodic constant $\tilde{\mu}(\psi,F,T)$ such that, almost surely,

$$\tilde{\mu}(\psi,F,T) = \lim_{R\to\infty} v_T(R\sigma,\omega).$$

We can further normalize T so that Te_d is a unit vector orthogonal to $T\partial P_{e_d}$ as this will not change the definition of v_T. We remark that, at the expense of changing F, we may reduce to the case of $T = I$ by changing variables to $z = T^{-1}y$. Notice that the ellipticity constants will remain bounded as long as we work with a class of maps with T and T^{-1} bounded. For example, if $T = D\zeta(\zeta^{-1}(x_0))$, the bound will depend only on the properties of the diffeomorphism ζ. We emphasize that such a transformation need not change β, since, in this case, we will use $\Phi(T\cdot)$, which has the same homogeneity as Φ, as a supersolution barrier for the transformed problem.

The result for (5.2.42) is the following theorem.

Theorem 5.2.17. *Assume that F and ψ satisfy* (5.2.10), (5.2.11), (5.2.14), (5.2.15), (5.2.16), *and the log-Sobolev assumptions* (5.2.19) *and* (5.2.20). *Then the conclusions of Theorem 5.2.13 holds for v_T with $\hat{\beta} = \min\{\beta(\lambda,\Lambda),2\}$ and constants that depend on λ,Λ and the bounds for T and T^{-1}*

.

The proof of Theorem 5.2.17 is based on the concentration inequalities for measures with log-Sobolev inequality and does not depend on the concentration inequalities for the strong-mixing setting Theorems 5.2.1 and 5.2.2. As a result no restrictions are imposed on the ellipticity ratio of F. We expect the a proof very similar to the one of Theorem 5.2.16 will also give a homogenization for general domains with data locally constructed by lifting from hyperplanes. Note that in order to prove the continuity of $\tilde{\mu}$ with respect to T, which is required for the general domain result, we actually need to take a regularized version of the random checkerboard boundary data (5.2.20) so that (5.2.17) holds. We leave the details to the reader.

The Neumann cell problem (5.1.8). The result when F either convex or concave is:

Theorem 5.2.18. *Assume that, in addition to* (5.2.10) *and* (5.2.11)*, F is either convex or concave and ψ satisfies* (5.2.14), (5.2.15), (5.2.16), (5.2.17) *and* (5.2.18) *. Then, for each $\sigma \in S^{d-1}$, the cell problem* (5.1.8) *has a unique solution $v_{\sigma,R}$ which concentrates about its mean with rate*

$$\mathbb{P}(\{\omega : R^{-1}|v_{\sigma,R}(y'+R\sigma,\omega) - \mathbb{E}v_{\sigma,R}(y'+R\sigma)| > t\}) \leq C\exp(-cR^{\hat{\beta}}t^2) \text{ on } \partial P_\sigma \text{ and} t > 0. \tag{5.2.43}$$

with $\hat{\beta} = \frac{\lambda}{\Lambda}(d-1)$. Moreover, the limit $\mu(\psi,F,\sigma) := \lim_{R\to\infty} R^{-1}\mathbb{E}v_{\sigma,R}(R\sigma)$ exists and equals the almost sure limit of $R^{-1}v_{\sigma,R}(R\sigma,\omega)$, as $R\to\infty$. Furthermore, for some universal constant $C = C(d,\lambda,\Lambda,\rho) = C(d,\lambda,\Lambda)\rho^2$ and all $y' \in \partial P_\sigma$,

$$|\mathbb{E}v_\sigma(y'+R\sigma) - \mu(\psi,F,\sigma)| \leq C(\log R)^{1/2}R^{-\hat{\beta}/2}. \tag{5.2.44}$$

In the nonconvex/nonconcave case the result is:

Theorem 5.2.19. *Assume that F and ψ satisfy* (5.2.10), (5.2.11), (5.2.14), (5.2.15), (5.2.16), (5.2.17) *and* (5.2.18) *and $\frac{\lambda}{\Lambda} > \frac{1}{2}$. Then the conclusions of Theorem 5.2.18 hold with $\hat{\beta} = (\frac{\lambda}{\Lambda} - \frac{1}{2})(d-1)$.*

As for the Dirichlet problem, under the assumptions that lead to the homogenization of the cell problem, the ergodic constant is continuous with respect to the normal direction and the boundary data. This an very important property to establish homogenization in general domains. Its proof relies in a critical way on the rate of convergence for the cell problem.

Theorem 5.2.20. *Assume* (5.2.10), (5.2.11) *and $\frac{\lambda}{\Lambda} > \frac{1}{2}$ if F is neither convex nor concave.*

(i) If ψ and ψ' satisfy (5.2.14), (5.2.15), (5.2.16), (5.2.17) *and* (5.2.18)*, then, for every $\sigma \in S^{d-1}$,*

$$|\mu(\psi,F,\sigma) - \mu(\psi',F,\sigma)| \leq \|\psi - \psi'\|_{\infty,\mathbb{R}^d\times\Omega}.$$

(ii) There exists $C = C(d,\lambda,\Lambda) > 0$ such that, for every $\sigma,\sigma' \in S^{d-1}$ and $\alpha \in (0,\alpha')$ with $\alpha'(\hat{\beta}) := \frac{\hat{\beta}}{2(1+\hat{\beta})}$, if ψ satisfies (5.2.14), (5.2.15), (5.2.16), (5.2.17) *and* (5.2.18)*, then*

$$|\mu(\psi,F,\sigma_1) - \mu(\psi,F,\sigma_2)| \leq C|\psi|_{C^{0,\alpha}})\|\psi\|_\infty|\sigma_1 - \sigma_2|^\alpha.$$

Finally we have the following homogenization result for the general domain problem whenever the cell problem homogenizes.

Theorem 5.2.21. *Assume* (5.2.8) *and* (5.2.7) *and suppose that the assumptions of either Theorem 5.2.18 or Theorem 5.2.19 hold and define $\hat{\beta}$ accordingly. Let u^ε and $\overline{u}$ be respectively the solutions to* (5.1.2) *and* (5.1.4)*, with Neumann data $g(x,x/\varepsilon,\omega)$ and $\overline{g}(x) := \mu(g(x,\cdot,\cdot),F,\sigma_x)$. The u^ε concentrates about its mean in the sense that, there exists $\alpha_0(\hat{\beta}) > 0$ and every $p > 0$, there exists a sufficiently large universal constant M_p such that*

$$\mathbb{P}(\{\omega : \sup_{x\in U\setminus K} |u^\varepsilon(x,\omega) - \mathbb{E}u^\varepsilon(x)| > M_p(\log\tfrac{1}{\varepsilon})^{1/2}\varepsilon^{\alpha_0}\}) \lesssim \varepsilon^p, \tag{5.2.45}$$

and the expected value $\mathbb{E}u^\varepsilon$ converges, as $\varepsilon \to 0$, to $\overline{u}$ with rate

$$\sup_{x\in U\setminus K} |\mathbb{E}u^\varepsilon(x) - \overline{u}(x)| \lesssim (\log\tfrac{1}{\varepsilon})^{1/2}\varepsilon^{\alpha_0}. \tag{5.2.46}$$

We remark that the analogue of Theorem 5.2.17 will hold in the Neumann case as well although we do not provide the proof as it is a natural adaptation of the other proofs presented. Again the extension to a result in general domains with Neumann data given by "lifting up from the hyperplane" should also follow with similar arguments.

5.3 The Dirichlet Cell Problem

Here we present the proofs of Theorem 5.2.13, Theorem 5.2.14, Theorem 5.2.15 and Theorem 5.2.17 which are about the existence and properties of the homogenized boundary condition or ergodic constant, that is the asymptotic behavior, as $R \to \infty$, of the solutions to Dirichlet cell problem (5.1.5) and the continuity with respect to the normal directions.

In many places throughout the section we will consider (5.1.5) with discontinuous data. In this case, when we talk about the solution satisfying the maximum/comparison principle we refer to the one constructed in the previous section.

Since the arguments are rather long, we have divided the section into three major parts. The first is about Theorem 5.2.13 and Theorem 5.2.14, the second deals with Theorem 5.2.15 and the third concerns Theorem 5.2.17.

5.3.1 The proofs of Theorem 5.2.13 and Theorem 5.2.14.

There are two main steps here. The first is to consider, for R large, $v_\sigma(R\sigma,\omega)$ as a function of the boundary data and prove a Lipschitz estimate in order to apply one of the concentration inequalities given in Theorem 5.2.1 and Theorem 5.2.2. The argument is essentially deterministic and will be the focus of one of the subsections below. In the second step we obtain a quantitative estimate for the concentration of $v_\sigma(y,\omega)$ and show the convergence of the means $\mathbb{E}v_\sigma(y)$, as $y\cdot\sigma\to\infty$, to a deterministic constant $\mu(\psi,F,\sigma)$. To motivate the arguments we begin with an outline of the proof.

The outline of the proof. We consider the cell problem (5.1.5). For simplicity we take $\sigma=e_d$ and boundary data $\psi(y,\omega)=\xi_{y \bmod \mathbb{Z}^{d-1}}(\omega)$, where $(\xi_k)_{k\in\mathbb{Z}^{d-1}}$ is a stationary ergodic field on $\mathbb{Z}^{d-1}$ with $\mu=\mathbb{E}\xi_0$ and $\sigma^2=\mathrm{var}(\xi_0)$. Let $v(\cdot;\psi)$ be the solution of the cell problem (5.1.5) (note that for notational simplicity we write v instead of v_{e_d}). Then, for large R, we consider the value $v_R(\psi)=v(Re_d;\psi)$ as a function $v_R:[-1,1]^{\mathbb{Z}^{d-1}}\to\mathbb{R}$.

To keep the ideas simple, we first describe the argument when the interior equation is just the Laplacian, $-\Delta$. Then, as we already said in the introduction, we write $v_R(\psi)$ as

$$v_R(\psi)=\int_{\partial P_{e_d}}P(Re_d,y)\psi(y)dy=\sum_{k\in\mathbb{Z}^{d-1}}a_k\xi_k,$$

where $P:P_{e_d}\times\partial P_{e_d}\to\mathbb{R}$ is the Poisson kernel for the upper half space and

$$a_k=a_k(R):=\int_{k+[0,1)^{d-1}}P(Re_d,y)\,dy,$$

and observe that

$$\mathbb{E}v_R(\psi) = (\mathbb{E}\xi_0)\int_{\partial P_{e_d}} P(Re_d, y)\, dy = \mu,$$

and

$$a_k \sim R(R^2 + |k|^2)^{-\frac{d}{2}}.$$

In particular we see that $v_R(\psi)$ has the form of an ergodic average and, hence, we can apply the ergodic theorem to get

$$v_R(\psi) - \mu \to 0 \quad \text{almost surely as } R \to \infty.$$

When the ξ_k's are i.i.d., a standard variance estimate yields

$$\operatorname{var}(v_R(\psi)) = \sigma^2 \sum_{k\in\mathbb{Z}^{d-1}} a_k^2 \le \int_{\partial P_{e_d}} P(Re_d, y)^2\, dy \le C_d R^{1-d},$$

which already implies, by Chebyshev's inequality, homogenization in probability.

This is not, however, optimal, since it is possible to obtain, using Hoeffding's inequality (see [Hoe63]), the following Gaussian-type concentration about the mean:

$$\mathbb{P}(\{\omega : |v_R(\psi) - \mu| > t\}) \le 2\exp\left(\frac{-t^2}{2\sum_{k\in\mathbb{Z}^d} a_k^2}\right) \le 2\exp\left(-c_d R^{d-1} t^2\right). \tag{5.3.1}$$

Next we describe a way to generalize the main components of the above argument to the nonlinear setting.

Arguing for the moment heuristically, we suppose that we can linearize v_R around $\psi \equiv 0$ and write

$$v_R(\psi) = \sum_{k\in\mathbb{Z}^{d-1}} a_k \xi_k + o(|\xi|_{\ell^\infty}),$$

where $a_k := \partial_{\xi_k} v_R(0)$ and $|\xi|_{\ell^\infty}$ is small.

Fix $j \in \mathbb{Z}^{d-1}$. Taking as boundary data $\xi_k = \chi_{\{j\}}$ and using the strong maximum principle, the positive homogeneity of F and Taylor's expansion, we find

$$0 < v_R(\psi) = \eta^{-1} v_R(\eta\psi) = a_j + \eta^{-1} o(\eta),$$

and, after sending $\eta \to 0$, $a_j > 0$.

Taking next $\psi \equiv 1$, in which case $v_R(\psi) = 1$, the above argument yields, after letting $\eta \to 0$,

$$|a|_{\ell^1} = \sum_{k \in \mathbb{Z}^{d-1}} a_k = 1.$$

To show homogenization then, it is enough to obtain a vanishing, as $R \to \infty$, bound on the ℓ^2-norm of a_k, and, since,

$$\sum_{k \in \mathbb{Z}^{d-1}} a_k^2 \leq |a|_{\ell^\infty} |a|_{\ell^1} \leq |a|_{\ell^\infty},$$

it suffices to show that, as $R \to \infty$, $|a|_{\ell^\infty} \to 0$.

Of course this depends on a concentration estimate of the form (5.3.1) holding in the nonlinear case. This is a delicate issue related to the difference between the various concentration results stated in Section 5.2.

This is where the structure of the pde is needed to complete the argument. It is, however, evident from the heuristics above, that, to obtain the result, it is enough to get bounds on $v_R(\psi)$ for boundary data of the form $\chi_{k+[0,1)^{d-1}}$.

For nonlinear equations we are not aware of any quantity other than $v_R(\psi)$ which controls the homogenization and for which we can prove satisfactory estimates. Here $v_R(\psi)$ is no longer necessarily a linear or even a subadditive average of the ξ_k and, hence, it is not possible, as far as we can tell, to apply any version of the ergodic/subadditive ergodic theorem.

We are, however, able to prove that $v_R(\psi)$ is a Lipschitz function of the variables ξ_k which, as we show, does not put too much weight on any of them in the sense that $|\partial_{\xi_k} v_R|_{\ell^\infty} \to 0$ as $R \to \infty$.

In this case, the heuristic argument above suggests that the problem will homogenize. We do need, however, to deal with the nonlinearity of the equation reflected in the fact that $v_R(\psi)$ is only a Lipschitz, instead of linear, average of the ξ_k. This is where we use the tools of concentration inequalities to show that the v_R's concentrate around their mean.

The discretized cell problem. We present here some estimates for the solution of the cell problem in terms of the boundary data, which can be seen as the analogue to the classical L^p-estimates for the Poisson kernel of the boundary data in the linear case.

We assume for simplify that $\sigma = e_d$, but the proof will apply in general.

Recall that $Q = [0,1)^{d-1}$ is the unit cube on ∂P_{e_d} and note that

$$\partial P_{e_d} = \bigcup_{i \in \mathbb{Z}^{d-1}} (i+Q).$$

Instead of using Q, it is possible to cover ∂P_{e_d} using rQ for some $r > 0$ which can be chosen later based on the mixing rate function to optimize the estimates. Since this would only change constants and not the rate of convergence, we just take $r = 1$ for simplicity.

For $X \in \Xi := C(\overline{Q})^{\mathbb{Z}^{d-1}}$, we consider the solution $u : P_{e_d} \times \Xi \to \mathbb{R}$ to

$$\begin{cases} F(D_y^2 u) = 0 & \text{in} \quad P_{e_d}, \\ u(y;X) = \sum_{i \in \mathbb{Z}^{d-1}} X_i(y-i)\chi_{i+Q}(y) & \text{on} \quad \partial P_{e_d}. \end{cases} \tag{5.3.2}$$

We remark that, if $-F$ is convex (an analogous claim is true if $-F$ is concave), then $u(\cdot;X)$ is convex with respect to X. This follows from the classical fact (see [CIL92]) that, when $-F$ is convex, linear combinations of supersolutions of (5.3.2) are also supersolutions.

In Lemma 5.3.1 below, we show that, for each fixed $y \in P_{e_d}$, by the maximum principle, $u(y;X)$ is continuous in the sup-norm with respect to each component X_j, and, hence $u(y;\cdot) : (\Xi, \mathscr{B}(\Xi)) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is a measurable mapping.

We think of the value of $u(Re_d;X)$ as a function of R and X and write

$$f_R(X) := u(Re_d;X). \tag{5.3.3}$$

We prove Lipshitz estimates for f_R with respect to the various norms ℓ^p-norms on $C(\overline{Q})^{\mathbb{Z}^{d-1}}$, which we define next. For $X \in C(\overline{Q})^{\mathbb{Z}^{d-1}}$, the $\ell^p(\mathbb{Z}^{d-1};C(\overline{Q}))$ norms are given for $1 \le p < \infty$ and $p = \infty$, by

$$|X|_{\ell^p} = \left(\sum_{i \in \mathbb{Z}^{d-1}} \sup_{y \in Q} |X_i(y)|^p \right)^{1/p}, \tag{5.3.4}$$

and

$$|X|_{\ell^\infty} = \sup_{i \in \mathbb{Z}^{d-1}} \sup_{y \in Q} |X_i(y)|.$$

Lemma 5.3.1. *Assume* (5.2.10) *and* (5.2.11). *The map* $f_R : C(\overline{Q})^{\mathbb{Z}^{d-1}} \to \mathbb{R}$ *has the following continuity properties for* $X,Y \in C(\overline{Q})^{\mathbb{Z}^{d-1}}$ *and some universal constant* $C =: C(d,\lambda,\Lambda) > 0$*:*

(i) $$|f_R(X) - f_R(Y)| \leq |X - Y|_{\ell^\infty}. \tag{5.3.5}$$

(ii) $$|f_R(X) - f_R(Y)| \leq CR^{-\beta}|X - Y|_{\ell^1}. \tag{5.3.6}$$

(iii) $$|f_R(X) - f_R(Y)| \leq CR^{-\beta/2}|X - Y|_{\ell^2}. \tag{5.3.7}$$

Proof. The first estimate is an immediate consequence of the definition of f_R and the comparison principle.

The proof of the ℓ^1-continuity consists of constructing a barrier which controls how much $u(Re_d;X)$ can change when the value of X is altered at a single site $k \in Z^{d-1}$.

The main tool in coming up with such barrier is the singular solution Φ of Theorem 5.2.9 with $\mathrm{Cone}_\gamma = P_{e_d}$, that is the solution Φ to

$$\begin{cases} \mathscr{P}^+_{\lambda,\Lambda}(D^2\Phi) = 0 & \text{in} \quad P_{e_d}, \\ \Phi = 0 & \text{on} \quad \partial P_{e_d} \setminus \{0\}, \end{cases} \tag{5.3.8}$$

with $\max_{\partial B_1 \cap P_{e_d}} \Phi = 1$, which is β homogeneous with $\beta = \beta(d,\lambda,\Lambda)$ as in (5.2.29). Let

$$\widetilde{\phi}(x) := 2m^{-1}\Phi(x + e_d),$$

with $m := \min\{\Phi(x) : x = (x',1) \text{ with } x' \in Q\} > 0$, and observe that, in view of the uniform ellipticity and convexity of $\mathscr{P}^+_{\lambda,\Lambda}$, $\widetilde{\phi} \in C^2(\overline{P}_{e_d})$ (see [CC95]), and moreover,

$$\widetilde{\phi} \geq 1 \text{ on } \partial P_{e_d} \cap Q.$$

Then, for $X,Y \in \ell^1(\mathbb{Z}^{d-1})$, let,

$$\phi(x) := \sum_{j \in \mathbb{Z}^{d-1}} (\sup_Q |X_j - Y_j|)\tilde{\phi}(x - j);$$

note that the sum, which is only over sites where $X_j \neq Y_j$, converges in $\overline{P}_{e_d}$ since $X,Y \in \ell^1(\mathbb{Z}^{d-1};C(\overline{Q}))$ and , in view of the regularity of $\widetilde{\phi}$, $\phi \in C^{2,\alpha}(\overline{P}_{e_d})$.

Using the definition and regularity of ϕ and the subadditivity of the maximal operator $\mathscr{P}^+_{\lambda,\Lambda}$ we get

$$F(D^2u(y;X)+D^2\phi,y)\geq 0 \text{ in } P_{e_d} \text{ and } u(y;X)+\phi(y)\geq u(y;Y) \text{ on } \partial P_{e_d}.$$

The comparison principle and the homogeneity of Φ yield, for some universal C,

$$u(Re_d;Y)\leq u(Re_d;X)+\phi(Re_d)\leq u(Re_d;X)+m^{-1}C\sum_{j\in\mathbb{Z}^{d-1}}R(|j|^2+R^2)^{-(\beta+1)/2}\sup_Q|X_j-Y_j|;$$

it then follows that

$$|f_R(X)-f_R(Y)|\leq C\sum_{j\in\mathbb{Z}^{d-1}}R(|j|^2+R^2)^{-(\beta+1)/2}\sup_Q|X_j-Y_j|.$$

Finally, employing Hölder's inequality, we find for any $p\in[1,\frac{1}{1-\frac{\beta+1}{d-1}})$ or $p\in[1,\infty]$, if $\beta+1>d-1$,

$$|f_R(X)-f_R(Y)|\leq R^{-\beta+(d-1)/p'}|X-Y|_{\ell^p(\mathbb{Z}^{d-1})},$$

where p' is the Hölder dual exponent of p; note that, using the Poisson kernel for the upper half space, it is possible to derive the same result as in the linear case with $\beta=d-1$.

For the ℓ^2-continuity, we consider $X\in C(\overline{Q})^{\mathbb{Z}^{d-1}}$ and $Z\in C(\overline{Q})^{\mathbb{Z}^{d-1}}$ with $|Z|_{\ell^2}<+\infty$ and, for some $t>0$ to be chosen later, we write

$$Z=Z_{>t}+Z_{\leq t} \text{ where } (Z_{>t})_j=Z_j\chi_{\{\sup_Q|Z_j|>t\}} \text{ and } Z_{\leq t}=Z-Z_{>t}.$$

Then

$$\begin{aligned}|f_R(X+Z)-f_R(X)|&\leq|f_R(X+Z)-f_R(X+Z_{>t})|+|f_R(X+Z_{>t})-f_R(X)|\\&\leq|Z_{\leq t}|_{\ell^\infty}+CR^{-\beta}|Z_{>t}|_{\ell^1}\\&\leq t+CR^{-\beta}t^{-1}|Z|^2_{\ell^2},\end{aligned}$$

and, after choosing $t=R^{-\beta/2}|Z|_{\ell^2}$,

$$|f_R(X+Z)-f_R(X)|\leq CR^{-\beta/2}|Z|^2_{\ell^2}.$$

□

The homogenization of the cell problem (5.1.5). We apply the estimates of Lemma 5.3.1 to the random problem and establish the homogenization of the cell problem (5.1.5)

As we did before, in what follows, for simplicity, we assume that $\sigma = e_d$, $|\psi| \leq 1$ almost surely, and we work with the solution v_{e_d} of the cell problem

$$\begin{cases} F(D^2 v_{e_d}) = 0 & \text{in} \quad P_{e_d}, \\ v_{e_d}(\cdot,\omega) = \psi(\cdot,\omega) & \text{on} \quad \partial P_{e_d}. \end{cases} \tag{5.3.9}$$

We now transform to the setting of the previous section to apply the concentration results Theorem 5.2.1 and Theorem 5.2.2. More precisely, we consider the Banach space $C(\overline{Q})$ with the sup-norm and the associated Borel σ-algebra $\mathscr{B}(C(\overline{Q}))$ and the measure space $(\Xi, \mathscr{B}(\Xi))$, where, as before $\Xi := C(\overline{Q})^{\mathbb{Z}^{d-1}}$, and we define the measurable map $\Psi : \Omega \to \Xi$ given, for each $j \in \mathbb{Z}^{d-1}$, by

$$\Psi_j(\omega)(\cdot) := \psi(j + \cdot, \omega),$$

which induces a probability measure P on $(\Xi, \mathscr{B}(\Xi))$, the pushforward of $\mathbb{P}$ by Ψ or, equivalently, the law of the random variable Ψ. Recall that, by the definition of the pushforward measure, for any measurable $f : \Xi \to \mathbb{R}$, $\int_\Xi f(X) dP(X) = \int_\Omega f(\Psi) d\mathbb{P}$.

It is immediate, in view of the properties of ψ and $\mathbb{P}$, that P is stationary with respect to the natural $\mathbb{Z}^{d-1}$ action on Ξ. Furthermore, the random field Ψ on $\mathbb{Z}^{d-1}$ has the ϕ-mixing property (5.2.2) adapted to the lattice with the same rate function as ψ up to a dimensional constant, and, in view of (5.2.18),

$$\sum_{i \in \mathbb{Z}^{d-1}} \phi(|i|)^{1/2} \lesssim \rho. \tag{5.3.10}$$

We also remark that, in view of the definition of Ψ, $u(\cdot;\Psi) = v_{e_d}(\cdot,\omega)$ on ∂P_{e_d}. Since $\psi(\cdot,\omega)$ is continuous for each ω, in view of the uniqueness of the solutions to the boundary value problem,

$$u(\cdot;\Psi) = v_{e_d}(\cdot,\omega) \text{ on } \overline{P}_{e_d}.$$

Furthermore, given the definition of P and in view of the relationship between $u(\cdot;\Psi)$ and $v_{e_d}(\cdot,\omega)$, the concentration of $u(R\sigma;\Psi)$ with respect to P is the same as concentration of $v_{e_d}(y,\omega)$ with respect $\mathbb{P}$. In other words any probability estimates and expectations on $u(\cdot;\Psi)$ with respect to P are the same as probability estimates and expectations on $v_{e_d}(\cdot,\cdot)$ with respect to $\mathbb{P}$.

We apply next Theorem 5.2.1 or Theorem 5.2.2 depending on whether F and, hence, $u(Re_d;\cdot)$ are convex or concave.

When F is neither convex nor concave, we need to assume that the ellipticity constants (λ,Λ) of F are such that

$$2\beta(\lambda,\Lambda) > d-1, \tag{5.3.11}$$

where again β is the exponent of the upward pointing half-space singular solution for the maximal operator. In view of (5.2.29), a sufficient condition for (5.3.11) is

$$\frac{\lambda}{\Lambda} > \frac{1}{2}\frac{d+1}{d}. \tag{5.3.12}$$

In this case, recalling that $|\psi|\le 1$ (and therefore $|\Psi|_{\ell^\infty}\le 1$) almost surely, we have, from the proof of Lemma 5.3.1, the following weighted Hamming distance continuity of $f_R = u(Re_d;\cdot)$:

$$|f_R(X)-f_R(Y)| \lesssim \sum_{k\in\mathbb{Z}^{d-1}} R(|k|^2+R^2)^{-(\beta+1)/2}\chi_{\{X_k\neq Y_k\}} \text{ for all } X,Y\in \operatorname{supp}(P).$$

Since we can bound the ℓ^2-norm of the coefficients in the above inequality as follows

$$\sum_{k\in\mathbb{Z}^{d-1}} R^2(|k|^2+R^2)^{-(\beta+1)} \lesssim R^{d-1-2\beta}\int_{\mathbb{R}^{d-1}}(1+|x|^2)^{-(\beta+1)}dx,$$

applying Theorem 5.2.1 and using (5.3.11) we find

$$\mathbb{P}(\{\omega : |v_{e_d}(Re_d,\omega)-\mathbb{E}v_{e_d}(Re_d,\cdot)|\ge t\}) = P(\{X\in\Xi : |f_R(X)-\mathbb{E}_P f_R|\ge t\}) \le C\exp\left(-cR^{2\beta-(d-1)}t^2\right),$$

with $c=c'/\rho^2$ and $c'=c'(d,\lambda,\lambda)>0$.

When $-F$ is convex, we use the convexity of the solutions with respect to the boundary data, the ℓ^2-continuity of Lemma 5.3.1 and Theorem 5.2.2 to obtain

$$\mathbb{P}(\{\omega : |v_{e_d}(Re_d,\omega)-\mathbb{E}v_{e_d}(Re_d,\cdot)|\ge t\}) = P(\{X\in\Xi : |f_R(X)-\mathbb{E}_P f_R|\ge t\}) \le C\exp\left(-cR^{\beta}t^2\right),$$

with, as before, $c=c'/\rho^2$ and $c'=c'(d,\lambda,\lambda)>0$. The same applies in the concave case.

For brevity we write from now on

$$\hat{\beta} := \beta(\lambda,\Lambda) \text{ when } F \text{ is either convex or concave and, otherwise, } \hat{\beta} := 2\beta(\lambda,\Lambda) - (d-1) \tag{5.3.13}$$

assuming, of course, for the latter case (5.3.12).

We proceed now with the proofs of Theorem 5.2.13 and Theorem 5.2.14.

Proof of Theorem 5.2.13 and Theorem 5.2.14. It suffices to consider the case $\sigma = e_d$ since the proof for general σ is analogous, and, for simplicity, we write $v = v_{e_d}$. Although v is a random function, in what follows, to keep the notation simple, we sometimes omit ω.

Let $N \in 2^{\mathbb{N}}$. The first step in the proof is to show that the expected average at height N

$$\mu_N := \mathbb{E}u(Ne_d;\Psi) = \mathbb{E}v(Ne_d) = \mathbb{E}v(Ne_d + y') \text{ for all } y' \in \partial P_{e_d}$$

is Cauchy.

For a large constant $A = A(d,\lambda,\Lambda) >> (1+\hat{\beta}/2)(d-1)$ and $k \in \mathbb{Z}^{d-1}$, we define the events

$$E_k^N = \{\omega \in \Omega : |v(Ne_d + N^{1-\hat{\beta}/2}k,\omega) - \mu_N| \geq A^{1/2}(\log N)^{1/2}N^{-\hat{\beta}/2}\}. \tag{5.3.14}$$

In view of the assumed stationarity and either Theorem 5.2.1 or 5.2.2 we have

$$\mathbb{P}(E_k^N) = \mathbb{P}(E_0^N) = \mathbb{P}(\{\omega : |u(Ne_d;\Psi) - \mu_N| \geq A^{1/2}(\log N)^{1/2}N^{-\hat{\beta}/2}\}) \leq C\exp(-cA\log N).$$

It is then immediate, from a simple union bound, that, if $E^N := \bigcup_{|k|\leq N^{\hat{\beta}}} E_k^N$, then

$$\mathbb{P}(E^N) \leq CN^{\hat{\beta}(d-1)}\exp(-cA\log N).$$

We fix some large $M \in 2^{\mathbb{N}}$ and observe that, as long as $A > \hat{\beta}(d-1)/c$, if $E^{\geq M} := \bigcup_{N\geq M} E^N$, then

$$\mathbb{P}(E^{\geq M}) \leq C\sum_{N\geq M} N^{\hat{\beta}(d-1)-cA} < +\infty.$$

It follows that, for some universal sufficiently large M, $P(E^{\geq M}) < 1$, which, of course, implies that $\mathbb{P}(\Omega \setminus E^{\geq M}) > 0$; note that on $\Omega \setminus E^{\geq M}$,

$$|v(Ne_d + N^{1-\hat{\beta}/2}k, \omega) - \mu_N| \lesssim (\log N)^{1/2} N^{-\hat{\beta}/2} \text{ for all } N \geq M \text{ and } |k| \leq N^{\hat{\beta}}.$$

Then, using the oscillation decay estimates (Lemma 5.2.6), for every $N \geq M$, we have

$$|v(y' + Ne_d, \omega) - \mu_N| \lesssim N^{1-\hat{\beta}/2} N^{-1} + (\log N)^{1/2} N^{-\hat{\beta}/2} \lesssim (\log N)^{1/2} N^{-\hat{\beta}/2} \text{ for } |y'| \leq N^{1+\hat{\beta}/2}.$$

Moreover the localization Lemma 5.2.8 gives

$$|v(2Ne_d, \omega) - \mu_N| \lesssim \sup_{|y'| \leq N^{1+\hat{\beta}/2}} |v(y' + Ne_d, \omega) - \mu_N| + N^{1-(1+\hat{\beta}/2)} \lesssim (\log N)^{1/2} N^{-\hat{\beta}/2}.$$

Combining the previous two estimates we get

$$|\mu_{2N} - \mu_N| \lesssim (\log 2N)^{1/2} (2N)^{-\hat{\beta}/2} + |v(2Ne_d, \omega) - \mu_N| \lesssim (\log N)^{1/2} N^{-\hat{\beta}/2},$$

and, for every $N, L \geq M$ in $2^{\mathbb{N}}$,

$$|\mu_N - \mu_L| \lesssim \sum_{K \geq M} (\log K)^{1/2} K^{-\hat{\beta}/2} \lesssim (\log M)^{1/2} M^{-\hat{\beta}/2}.$$

Thus the sequence $(\mu_N)_{N \in 2^{\mathbb{N}}}$ is Cauchy sequence and, therefore, has a limit μ.

Next we use the above facts to prove (5.2.36) for $R > 1$ not necessarily dyadic and $y' \in \partial P_{e_d}$.

Let $N \in 2^{\mathbb{N}}$ such that $N \leq R \leq 2N$. The above arguments yield that, if R and, hence, N are sufficiently large depending only universal constants, then for any $y' \in \partial P_{e_d}$

$$\mathbb{P}(\tau_{-y'} E^{\geq N}) = \mathbb{P}(E^{\geq N}) < 1/2.$$

The convergence rate of μ_N to μ and the same localization argument as above yield that for $\omega \notin \tau_{-y'} E^{\geq N}$,

$$\begin{aligned} |v(y'+Re_d,\omega)-\mu| &= |v(Re_d,\tau_{y'}\omega)-\mu| \\ &\leq |v(Re_d,\tau_{y'}\omega)-\mu_N| + C(\log R)^{1/2}R^{-\hat{\beta}/2} \leq C(\log R)^{1/2}R^{-\hat{\beta}/2}, \end{aligned} \tag{5.3.15}$$

since $\tau_{y'}\omega \notin E^{\geq N}$. On the other hand, since

$$\mathbb{P}(\{\omega : |v(y'+Re_d,\omega)-\mathbb{E}v(y'+Re_d)| > t\}) \leq C\exp(-cR^{\hat{\beta}}t^2),$$

taking $t = AR^{-\hat{\beta}/2}$, where A is sufficiently large universal, gives

$$\mathbb{P}(\{\omega : |v(y'+Re_d,\omega)-\mathbb{E}v(y'+Re_d)| > t\}) \leq 1/2,$$

and thus,

$$\{|v(y'+Re_d,\omega)-\mathbb{E}v(y'+Re_d)| \leq A(\log R)^{1/2}R^{-\hat{\beta}/2}\} \cap (\tau_{-y'}E^{\geq N})^C \neq \emptyset.$$

Evaluating (5.3.15) for an ω in this intersection, we get

$$|\mathbb{E}v(y'+Re_d)-\mu| \lesssim (\log R)^{1/2}R^{-\hat{\beta}/2}.$$

Arguments along the same lines also imply that the limit $\lim_{R\to\infty} v(Re_d,\omega) = \mu$ holds pointwise on the set $\Omega_0 = \Omega \setminus \cap_{M\geq 1} E^{\geq M}$ which has probability 1. We do not go into the details, since this fact will not be used in the general domain case.

$\square$

For the proof of the homogenization of the general domain Dirichlet problem we need an additional estimate which we state and prove next. Essentially, because the concentration has exponential rate, we can combine the oscillation decay with the concentration to get uniform estimates on a polynomially large (in R) subset of P_σ.

Lemma 5.3.2. *Under the assumptions of Theorem 5.2.13 and Theorem 5.2.14 and for $\sigma \in S^{d-1}$ and $R > 1$, the solution $v_\sigma(\cdot,\omega)$ of the cell problem* (5.1.5) *satisfies the spatially uniform concentration estimate*

$$\mathbb{P}(\{\omega : \sup_{y\in K(R,\sigma)} |v_\sigma(y,\omega)-\mathbb{E}v_\sigma(y,\cdot)| > t\}) \leq CR^{d\hat{\beta}/2}\exp(-cR^{\hat{\beta}}t^2), \tag{5.3.16}$$

where

$$K(R,\sigma) := \{y \in \mathbb{R}^d : R/2 \le y \cdot \sigma \le 2R,\ |y-(y\cdot\sigma)\sigma| \le 3R\}. \tag{5.3.17}$$

Proof. Again we take $\sigma = e_d$ and write $v = v_{e_d}$; the general case follows similarly.

We divide $K(R,e_d)$ into $O(\varepsilon^{-d})$ disjoint cubes $\tilde{Q}_j$ centered at $c(\tilde{Q}_j)$ of size $O(\varepsilon R)$. On each of the $\tilde{Q}_j$'s, we use the interior Lipschitz estimates along with the fact that $d(\tilde{Q}_j, \partial P_{e_d}) \sim R$ to derive that, almost surely,

$$\sup_{y\in\tilde{Q}_j} |v(y,\omega) - v(c(\tilde{Q}_j),\omega)| \lesssim \varepsilon,$$

and, hence,

$$\sup_{y\in K(R,e_d)} |v(y,\omega) - \mathbb{E}v(y)| \le \sup_j |v(c(\tilde{Q}_j),\omega) - \mathbb{E}v(c(\tilde{Q}_j,\cdot))| + C_0\varepsilon.$$

Then, for $t > 2C_0\varepsilon$, we have

$$\begin{aligned}
\mathbb{P}(\{\omega : \sup_{y\in K(R,e_d)} |v(y,\omega) - \mathbb{E}v(y)| > t\}) &\le \mathbb{P}(\{\omega : \sup_j |v(c(\tilde{Q}_j,\omega) - \mathbb{E}v(c(\tilde{Q}_j))| + C_0\varepsilon > t\}) \\
&\le \mathbb{P}(\{\omega : \sup_j |v(c(\tilde{Q}_j),\omega) - \mathbb{E}v(c(\tilde{Q}_j))| > t/2\}) \\
&\le C\varepsilon^{-d}\mathbb{P}(\{\omega : |v(c(\tilde{Q}_j),\omega) - \mathbb{E}v(c(\tilde{Q}_j))| > t/2\}) \\
&\le C\varepsilon^{-d}\exp(-cR^{\hat{\beta}}t^2).
\end{aligned}$$

On the other hand, when $t \le 2C_0\varepsilon$,

$$\mathbb{P}(\{\omega : \sup_{y\in K(R,e_d)} |v(y,\omega) - \mathbb{E}v(y)| > t\}) \le 1 \le [C^{-1}\varepsilon^d \exp(cC_0R^{\hat{\beta}}\varepsilon^2)][C\varepsilon^{-d}\exp(-cR^{\hat{\beta}}t^2)].$$

Choosing $\varepsilon^2 \sim R^{-\hat{\beta}}$ and combining the two cases yields the claim.

$\square$

5.3.2 The continuity properties of the homogenized boundary condition

We discuss the continuity properties of $\mu(\sigma,F,\psi)$ and, in particular, we prove Theorem 5.2.15. Both parts of the theorem are used to establish the continuity of the homogenized boundary condition and the homogenization in general domains. In particular, (i) yields the continuity of $\overline{g}(x) = \mu(g(x,\cdot,\cdot),F,\sigma)$ with respect to the large scale x-dependence of g, while (ii) implies continuity of $\overline{g}$ with respect to changing normal directions.

Proof of Theorem 5.2.15. Part (i) is a direct consequence of the comparison principle.

To show (ii), we fix $\sigma,\sigma' \in S^{d-1}$ and, without loss of generality, we assume that $\sigma\cdot\sigma' \geq 1/2$ and, after a rescaling, that $|\psi| \leq 1$.

We estimate $|\mu-\mu'|$, where $\mu = \mu(\sigma,F,\psi)$ and $\mu' = \mu(\sigma',F,\psi)$, by comparing the solutions $v = v_\sigma$ and $v' = v_{\sigma'}$ of the corresponding cell problems in the intersection of the half spaces P_σ and $P_{\sigma'}$. Here, for simplicity, we do not display in most places the dependence of v and v' on ω.

Let us now show that $v(\cdot,\omega)$ and $v'(\cdot,\omega)$ are close in B_R, if their respective domains are close in B_L for $L \gg R$. Note that,

$$\sup_{y\in B_L(0)\cap\partial P_{\sigma'}} y\cdot\sigma \leq L|\sigma'-\sigma|.$$

and, therefore,

$$E := \{y\in\mathbb{R}^d : L|\sigma'-\sigma| < y\cdot\sigma < L, \quad |y-(y\cdot\sigma)\sigma| \leq L\} \subset P_\sigma \cap P_{\sigma'}.$$

The up to the boundary Lipschitz continuity the v and v' (Lemma 5.2.7) and the (almost sure) Lipschitz continuity of ψ, imply that, almost surely,

$$\sup_{y\in\partial E\cap\{y\in\mathbb{R}^d : y\cdot\sigma = L|\sigma'-\sigma|\}} |v(y,\omega)-\psi(y,\omega)| + |v'(y,\omega)-\psi(y,\omega)| \lesssim L(1+\|D\psi\|_\infty)|\sigma'-\sigma|.$$

Since, in view of the comparison principle, $|v(\cdot,\omega)|, |v'(\cdot,\omega)| \leq 1$, we use the localization result Lemma 5.2.8 to get

$$|v'(\cdot,\omega)-v(\cdot,\omega)| \lesssim L(1+\|D\psi\|_\infty)|\sigma'-\sigma| + \frac{R}{L} \ \text{ in } \ E\cap B_{2R}.$$

Using the rate of convergence of the expectations from Theorem 5.2.13 or Theorem 5.2.14 and taking expectations on both sides of the above inequality we get,

$$\begin{aligned} |\mu-\mu'| &\leq |\mu-\mathbb{E}v(R\sigma)| + |\mu'-\mathbb{E}v'(R\sigma)| + \mathbb{E}|v(R\sigma)-v'(R\sigma)| \\ &\lesssim L(1+\|D\psi\|_\infty)|\sigma'-\sigma| + \frac{R}{L} + (\log R)^{1/2}R^{-\hat{\beta}/2}. \end{aligned}$$

Choosing $L=(\log R)^{-1/2}R^{1+\hat{\beta}/2}$ and $R^{-1}=(1+\|\psi\|_\infty)^{1/(1+\hat{\beta})}|\sigma-\sigma'|^{1/(1+\hat{\beta})}$ to make all the terms in the inequality above comparable size, we get, for $\alpha'=\frac{\hat{\beta}}{2(1+\hat{\beta})}$

$$|\mu-\mu'|\lesssim(1+\|\psi\|_\infty)^{\alpha'}\left(\log\frac{1}{|\sigma-\sigma'|}\right)^{1/2}|\sigma-\sigma'|^{\alpha'},$$

which is the claim.

$\square$

5.3.3 The "lifting from the hyperplane" cell problem.

We prove here Theorem 5.2.17. In view of the discussion before its statement in the previous section, for simplicity, we assume that $T=I$ and $\sigma=e_d$ and we consider the cell problem

$$\begin{cases} F(D^2v)=0 & \text{in} \quad P_{e_d}, \\ v(\cdot,\omega)=\psi(\cdot,\omega) & \text{on} \quad \partial P_{e_d}. \end{cases} \tag{5.3.18}$$

It follows from (5.2.20) and the above simplification that

$$\psi(\cdot,\omega):=\sum_{i\in\mathbb{Z}^{d-1}}X_i(\omega)\chi_{i+Q}(\cdot).$$

Let $u:P_{e_d}\times\mathbb{R}^{\mathbb{Z}^{d-1}}$ be defined as in (5.3.2). The comparison principle yields

$$v(\cdot,\omega)=u(\cdot;X(\omega)) \ \text{ in } \ P_{e_d}.$$

We note that Lemma 5.3.1 still applies and yields a universal constant C such that, for all $Y,Z\in\mathbb{R}^{\mathbb{Z}^{d-1}}$

$$|u(Re_d;Y)-u(Re_d;Z)|\leq CR^{-\beta/2}|Y-Z|_{\ell^2}.$$

Finally, we recall, again because of (5.2.20), that the law of $X=(X_j)_{j\in\mathbb{Z}^{d-1}}$ on $\mathbb{R}^{\mathbb{Z}^{d-1}}$, which we call P in analogy with the previous subsection, has LSI and, hence, from Theorem 5.2.3, we have the concentration estimate

$$\mathbb{P}(\{\omega:|u(Re_d;X(\omega))-\mathbb{E}u(Re_d;X)|>t\})=$$
$$P(\{Y\in\mathbb{R}^{\mathbb{Z}^{d-1}}:|u(Re_d;Y)-\int_{\mathbb{R}^{\mathbb{Z}^{d-1}}}u(Re_d;Z)dP(Z)|>t\})\leq C\exp(-cR^{\beta}t^2),$$

which is as good as the convex case.

Proof of Theorem 5.2.17. The proof is almost the same as Theorem 5.2.13 and Theorem 5.2.14 with one minor difference. Here, instead of $\mathbb{R}^{d-1}$, we only have $\mathbb{Z}^{d-1}$ translation invariance of the distribution of the boundary data. The only consequence of this is that the rate of convergence of $\mathbb{E}v(Re_d)$ is limited also by the interior oscillation decay.

5.4 Homogenization in General Domains

Since we can only expect homogenization to hold on every compact subset of the domain U, we will consider here the rate of convergence in a subdomain $U_{R\varepsilon}$and we will make use of the additional free parameter R.

We prove next a slightly more general version of Theorem 5.2.16 leaving R free and then explain the choice of R that leads to Theorem 5.2.16.

Theorem 5.4.1. *Suppose that all the assumptions of Theorem 5.2.16 hold. Then, as* $\varepsilon \to 0$, *almost surely and for every* $x \in U$, $u^\varepsilon(x,\omega) \to u(x)$. *Furthermore, for any* $1 < R \lesssim \varepsilon^{-\frac{2}{4+3\hat{\beta}}}$, u^ε *concentrates about its mean with the estimate*

$$\mathbb{P}(\{\omega : \sup_{x \in U_{R\varepsilon}} |u^\varepsilon(x,\omega) - \mathbb{E}u^\varepsilon(x)| > t) \leq C\exp(-cR^{\hat{\beta}}t^2 + C\log\tfrac{1}{\varepsilon}\}), \tag{5.4.1}$$

and the expected value converges to $\overline{u}$ *with the estimate*

$$\sup_{x \in U_{R\varepsilon}} |\mathbb{E}u^\varepsilon(x) - \overline{u}(x)| \lesssim \varepsilon^{\frac{1}{3}}R^{\frac{2}{3}} + (\log\tfrac{1}{\varepsilon})^{1/2}R^{-\hat{\beta}/2} + (\log\tfrac{1}{\varepsilon})^{1/2}(\varepsilon R)^{\frac{\hat{\beta}}{2(1+\hat{\beta})}}, \tag{5.4.2}$$

with constants that depend on $\lambda, \Lambda, d, \rho$, *and the upper bound for* $R\varepsilon^{\frac{2}{4+3\hat{\beta}}}$. *In particular, choosing* $R = \varepsilon^{-\frac{2}{4+3\hat{\beta}}}$ *and* $t = (\log\frac{1}{\varepsilon})^{1/2}\varepsilon^{\frac{1}{4+3\hat{\beta}}}$ *yields Theorem 5.2.16.*

If we assigned boundary data by "lifting up from the hyperplane", then the homogenized boundary condition would be

$$\overline{g}(x) := \tilde{\mu}(\psi, F, D\zeta(\zeta^{-1}(x))),$$

where $\tilde{\mu}$ comes from the alternate cell problem (5.2.42). The only difference in the estimates obtained would be in the continuity of the homogenized boundary condition, which corresponds to the third term in (5.4.2).

We give first an outline of the strategy of the proof of Theorem 5.4.1, which follows from a series of Lemmas. We begin by rescaling at a point $x_0 \in \partial U$ to $u^\varepsilon(x_0+\varepsilon y)$ and prove an estimate on the difference, in a large box of size R around the origin, between the rescaled solution in the general domain and the solution of the corresponding cell problem in $P_{\sigma_{x_0}}$. Then we use one of the cell problem homogenization results, that is either Theorem 5.2.13 or Theorem 5.2.14, to prove an estimate of the concentration of $u^\varepsilon(x_0+\varepsilon y)$ about its mean and the convergence of $\mathbb{E}u^\varepsilon(x_0+\varepsilon y)$ to $\overline{g}(x_0)$ on a strip of size $\sim R$, which is $\sim R$ away from $\partial P_{\sigma_{x_0}}$. Rescaling back to the ε scale, we use this estimate on a finite subset of ∂U, which is "$R\varepsilon$ dense" in ∂U. This implies that u^ε concentrates about its mean on the boundary of the slightly smaller domain $U_{R\varepsilon}$. Then the comparison principle gives the concentration in the interior of $U_{R\varepsilon}$. Finally, the proof of the almost sure pointwise convergence involves a careful use of the Borel-Cantelli Lemma.

We turn now to the full details. We fix $x_0 \in \partial U$ and define the local rescaling of u^ε near x_0 by

$$v^\varepsilon_{x_0}(y) := u^\varepsilon(x_0+\varepsilon y),$$

which solves

$$\begin{cases} F(D^2 v^\varepsilon_{x_0}) = 0 & \text{in} \quad \varepsilon^{-1}(U-x_0), \\ v^\varepsilon_{x_0}(y,\omega) = g(x_0+\varepsilon y, y, \tau_{x_0/\varepsilon}\omega) & \text{on} \quad \varepsilon^{-1}\partial(U-x_0). \end{cases} \tag{5.4.3}$$

Let v_σ, with $\sigma = \sigma_{x_0}$, be the solution to (5.1.5) with boundary data

$$\psi(y,\omega) := g(x_0,y,\omega) \text{ on } \partial P_\sigma,$$

and observe that, in view of the assumptions our on g, ψ satisfies the hypotheses of Theorem 5.2.13 or Theorem 5.2.14 and Lemma 5.3.2.

The next lemma provide an estimate for the difference between $v^\varepsilon_{x_0}$ and v_σ; for its statement recall the definition (5.3.17) of the sets $K(R,\sigma)$.

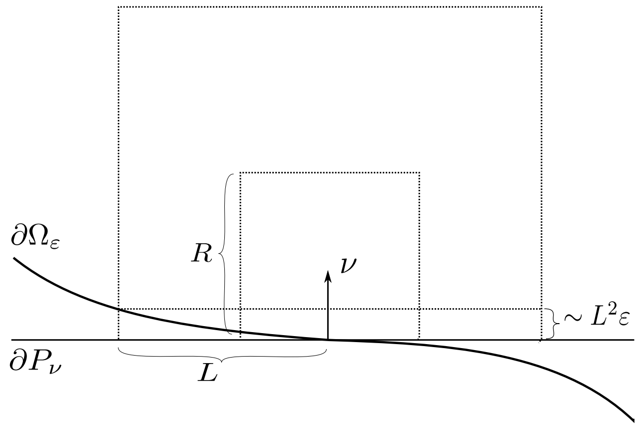

Figure 5.1: Comparing with the solution of the cell problem.

Lemma 5.4.2. *Assume the hypotheses of Theorem 5.2.16. For any* $R > 1$ *and almost surely*

$$|v^\varepsilon_{x_0}(\cdot,\omega) - v_{\sigma_{x_0}}(\cdot,\tau_{x_0/\varepsilon}\omega)| \le C\varepsilon^{1/3}R^{2/3} \ \text{ in } \ K(R,\sigma_{x_0}) \cap \varepsilon^{-1}(U - x_0).$$

We remark that this is the main place where the proof of Theorem 5.2.16 would change, if we assigned boundary data in the general domain by "lifting up from the hyperplane", in which case the estimate, with the notation of the previous section, would instead be

$$|v^\varepsilon_{x_0}(\cdot,\omega) - v_{D\zeta(\zeta^{-1}(x_0))}(\cdot,\tau_{\varepsilon^{-1}\zeta^{-1}(x_0) \bmod \mathbb{Z}^{d-1}}\omega)| \le C\varepsilon^{1/3}R^{2/3} \ \text{ in } \ K(R,\sigma_{x_0}) \cap \varepsilon^{-1}(U - x_0).$$

Proof. We omit reference to x_0, since it is fixed for now.

We estimate the difference between v^ε and v_σ by a localization argument in a region where ∂U is close to its tangent hyperplane at x_0. For a succint presentation we call $U^\varepsilon = \varepsilon^{-1}(U - x_0)$.

It follows from (5.2.7) that there exists some sufficiently large $C_0 = C_0(U)$ depending on the C^2-regularity of ∂U such that, for any $L > R$,

$$\{x \in \mathbb{R}^d : x\cdot\sigma \ge C_0L^2\varepsilon\} \subset U^\varepsilon \cap B_L(0) \subset \{x \in \mathbb{R}^d : x\cdot\sigma \ge -C_0L^2\varepsilon\}.$$

Note that, due to the up to the boundary Lipschitz continuity of v^ε, we have

$$|v^\varepsilon(y,\omega) - g(x_0 + \varepsilon y, y, \tau_{x_0/\varepsilon}\omega)| \le CL^2\varepsilon \ \text{ on } \ \{x \in \mathbb{R}^d : x\cdot\sigma = C_0L^2\varepsilon\}, \qquad (5.4.4)$$

and

$$|v_\sigma(y,\tau_{x_0/\varepsilon}\omega)-\psi(y,\tau_{x_0/\varepsilon}\omega)|\leq CL^2\varepsilon \text{ on } \{x\in\mathbb{R}^d : x\cdot\sigma = C_0L^2\varepsilon\}. \tag{5.4.5}$$

Using the previous two inequalities as well as, the, uniform in in (y,ω), Lipshitz continuity of $g(\cdot,y,\omega)$ and the definition of ψ we obtain

$$|v_\sigma(\cdot,\tau_{x_0/\varepsilon}\omega)-v^\varepsilon(\cdot,\omega)|\leq CL^2\varepsilon + C\varepsilon \lesssim L^2\varepsilon \text{ on } C_0L^2\Pi\cap U_\varepsilon. \tag{5.4.6}$$

Fix $R>0$ with $R\ll L$. Then using (5.4.6) and the localization result (Lemma 5.2.8), we conclude that

$$|v^\varepsilon(\cdot,\omega)-v_\sigma(\cdot,\tau_{x_0/\varepsilon}\omega)|\lesssim L^2\varepsilon+\tfrac{R}{L} \text{ in } K(R,\sigma)\cap U^\varepsilon,$$

and choosing $L=R^{\frac{1}{3}}\varepsilon^{-\frac{1}{3}}$, so that both terms in the above estimate are the same size, we find

$$|v^\varepsilon(\cdot,\omega)-v_\sigma(\cdot,\tau_{x_0/\varepsilon}\omega)|\lesssim \varepsilon^{\frac{1}{3}}R^{\frac{2}{3}} \text{ in } K(R,\sigma)\cap U^\varepsilon. \tag{5.4.7}$$

□

Next we combine (5.4.7) with the estimate for the cell problem given in either Theorem 5.2.13 or Theorem 5.2.14 depending on our assumptions.

From the previous estimates we have, for $y\in K(R,\sigma_{x_0})\cap\varepsilon^{-1}(U-x_0)$,

$$|v^\varepsilon(y,\omega)-\mathbb{E}v^\varepsilon(y)|\leq C_1\varepsilon^{\frac{1}{3}}R^{\frac{2}{3}}+|v_\sigma(y,\tau_{x_0/\varepsilon}\omega)-\mathbb{E}v_\sigma(y)| \tag{5.4.8}$$

and

$$|\mathbb{E}v^\varepsilon(y)-\overline{g}(x_0)|\leq|\mathbb{E}v^\varepsilon(y)-\mathbb{E}v_\sigma(y)|+|\mathbb{E}v_\sigma(y)-\overline{g}(x_0)|\lesssim\varepsilon^{\frac{1}{3}}R^{\frac{2}{3}}+(\log R)^{1/2}R^{-\hat{\beta}/2}. \tag{5.4.9}$$

Using (5.4.8) we derive, for $t>2C_1\varepsilon^{1/3}R^{2/3}$, the following uniform in $K(R,\sigma_{x_0})$ concentration estimate of v^ε about its mean:

$$\begin{aligned}\mathbb{P}(\{\omega:\sup_{y\in K(R,\sigma_{x_0})}|v^\varepsilon(y,\omega)-\mathbb{E}v^\varepsilon(y)|>t\})&\leq\mathbb{P}(\{\omega:\sup_{y\in K(R,\sigma_{x_0})}|v_\sigma(y,\tau_{x_0/\varepsilon}\omega)-\mathbb{E}v_\sigma(y)|>t/2\})\\&\leq CR^{d\hat{\beta}/2}\exp(-cR^{\hat{\beta}}t^2).\end{aligned}$$

On the other hand, if $t \leq 2C_1\varepsilon^{1/3}R^{2/3}$, then, for $R \leq M\varepsilon^{-\frac{2}{4+3\hat{\beta}}}$, we have

$$C\exp(-cR^{\hat{\beta}}t^2) \geq C\exp(-4cC_1^2M^{\hat{\beta}+4/3}) \geq C\exp(-4cC_1^2M^{\hat{\beta}+4/3})\mathbb{P}(\sup_{y\in K(R,\sigma_{x_0})}|v^\varepsilon(y)-\mathbb{E}v^\varepsilon(y)|>t).$$

Combining the two last inequalities gives the following Lemma.

Lemma 5.4.3. *Assume the hypotheses of Theorem 5.2.16. For every $x_0 \in \partial U$ and $1 < R \leq M\varepsilon^{-\frac{2}{4+3\hat{\beta}}}$, v^ε concentrates about its mean uniformly in $K(R,\sigma_{x_0})$ with rate*

$$\mathbb{P}(\{\omega : \sup_{y\in K(R,\sigma_{x_0})}|v^\varepsilon(y,\omega)-\mathbb{E}v^\varepsilon(y)|>t\}) \leq C(M,d,\lambda,\Lambda)R^{d\hat{\beta}/2}\exp(-cR^{\hat{\beta}}t^2),$$

and its expectation $\mathbb{E}v^\varepsilon$ converges, again uniformly in $K(R,\sigma_{x_0})$, to $\overline{g}(x_0)$ with rate

$$\sup_{y\in K(R,\sigma_{x_0})}|\mathbb{E}v^\varepsilon(y)-\overline{g}(x_0)| \lesssim \varepsilon^{\frac{1}{3}}R^{\frac{2}{3}}+(\log R)^{1/2}R^{-\hat{\beta}/2}.$$

The next step is to put a finite net of points on ∂U and use the concentration estimate of the cell problem along with (5.4.8) and a union bound to get a concentration estimate on the entire boundary of a subdomain of U.

For each $\delta > 0$, choose a finite subset Γ_δ of ∂U such that

$$\partial U \subset \cup_{x\in\Gamma_\varepsilon}B(x,\delta) \text{ and } |\Gamma_\delta| \lesssim \delta^{-(d-1)}. \tag{5.4.10}$$

The following lemma provides a cover of ∂U_δ consisting of sets centered at points on Γ_δ.

Lemma 5.4.4. *Assume* (5.2.7)*. For sufficiently small $\delta > 0$, depending on the C^2-regularity of U, and Γ_δ as in* (5.4.10)*,*

$$\partial U_\delta \subset \cup_{x_0\in\Gamma_\delta}[x_0+\delta K(1,\sigma_{x_0})].$$

Proof. For each $x \in \partial U_\delta$, there exists $\bar{x} \in \partial U$ such that $x-\bar{x} = \delta\sigma_{x'}$.

Let $\pi_{\Gamma_\delta}x := \operatorname{argmin}\{y \in \Gamma_\delta : |y-\bar{x}|\}$, and observe, that in view of (5.4.10), $|\pi_{\Gamma_\delta}x-\bar{x}| \leq 2\delta$.

Then

$$x-\pi_{\Gamma_\delta}x = (x-\bar{x})+(\bar{x}-\pi_{\Gamma_\delta}x) = \delta\sigma_{\pi_{\Gamma_\delta}x}+\delta(\sigma_{\bar{x}}-\sigma_{\pi_{\Gamma_\delta}x})+(\bar{x}-\pi_{\Gamma_\delta}x).$$

Since, in view of (5.2.7), $\sigma \in C^1(\partial U; S^{d-1})$, for δ sufficiently small depending on the C^2- regularity of U,

$$[\delta(\sigma_{\bar{x}} - \sigma_{\pi_{\Gamma_\delta} x}) + (\bar{x} - \pi_{\Gamma_\delta} x)] \cdot \sigma_{\pi_{\Gamma_\delta} x} \leq C\delta^2 \text{ and } |\delta(\sigma_{\bar{x}} - \sigma_{\pi_{\Gamma_\delta} x}) + (\bar{x} - \pi_{\Gamma_\delta} x)| \leq 3\delta,$$

and, hence, always for sufficiently small δ,

$$x \in \pi_{\Gamma_\delta} x + \delta K(1, \sigma_{\pi_{\Gamma_\delta} x}).$$

$\square$

Letting $\delta = \varepsilon R$ in the previous lemma and using the concentration estimate at each point of Γ_δ, we get

$$\begin{aligned}\mathbb{P}(\{\omega : \sup_{x \in \partial U_\delta} |u^\varepsilon(x,\omega) - \mathbb{E}u^\varepsilon(x)| > t\}) &\leq \mathbb{P}(\{\omega : \sup_{x_0 \in \Gamma_\delta} \sup_{y \in K(R,\sigma_{x_0})} |u^\varepsilon(x_0 + \varepsilon y, \omega) - \mathbb{E}u^\varepsilon(x_0 + \varepsilon y)| > t\}) \\ &\lesssim (R\varepsilon)^{-(d-1)} \sup_{x_0 \in \Gamma_\varepsilon} \mathbb{P}(\{\omega : \sup_{y \in K(R,\sigma_{x_0})} |u^\varepsilon(x_0 + \varepsilon y, \omega) - \mathbb{E}u^\varepsilon(x_0 + \varepsilon y)| > t\}) \\ &\lesssim (R\varepsilon)^{-(d-1)} R^{d\hat{\beta}/2} \exp(-cR^{\hat{\beta}} t^2).\end{aligned}$$

Then using Lemma 5.4.3 as well as the up to the boundary modulus of continuity of $\overline{u}$, which is controlled by the modulus of continuity of $\overline{g}$ from Lemma 5.2.15, we find

$$\begin{aligned}\sup_{x \in \partial U_\delta} |\mathbb{E}u^\varepsilon(x) - \overline{u}(x)| &\leq \sup_{x \in \partial U_\delta} \big(|\mathbb{E}u^\varepsilon(x) - \overline{g}(\pi_{\Gamma_\delta} x)| + |\overline{g}(\pi_{\Gamma_\delta} x) - \overline{u}(x)|\big) \\ &\leq \sup_{x_0 \in \Gamma_\delta} \sup_{y \in K(R,\sigma_{x_0})} |\mathbb{E}u^\varepsilon(x_0 + \varepsilon y) - \overline{g}(x_0)| + C(\log \tfrac{1}{\delta})^{1/2} \delta^{\alpha'} \\ &\lesssim \varepsilon^{\frac{1}{3}} R^{\frac{2}{3}} + (\log R)^{1/2} R^{-\hat{\beta}/2} + (\log \tfrac{1}{\varepsilon})^{1/2} (\varepsilon R)^{\alpha'};\end{aligned}$$

recall that $\pi_{\Gamma_\delta} x$ is any point $x_0 \in \Gamma_\delta$ such that $x \in x_0 + \delta K(1, \sigma_{x_0})$, which, in view of Lemma 5.4.4, is well defined on ∂U_δ.

We consider next the boundary value problem

$$\begin{cases} F(D^2 \widetilde{u}^\varepsilon) = 0 & \text{in} \quad U_\delta, \\ \widetilde{u}^\varepsilon(x) = \mathbb{E}u^\varepsilon(x,\omega) & \text{on} \quad \partial U_\delta. \end{cases} \tag{5.4.11}$$

Its solution $\widetilde{u}^\varepsilon$ can be thought as the "extension by F" of $\mathbb{E}u^\varepsilon\chi_{\partial U_\delta}$ to the interior of U_δ,

Employing the comparison principle in U_δ yields

$$\mathbb{P}(\{\omega : \sup_{x\in U_\delta} |u^\varepsilon(x,\omega) - \widetilde{u}^\varepsilon(x)| > t\}) \le \mathbb{P}(\{\omega : \sup_{x\in\partial U_\delta} |u^\varepsilon(x,\omega) - \mathbb{E}u^\varepsilon(x,\omega)| > t\}) \lesssim (R\varepsilon)^{-(d-1)} R^{d\hat{\beta}/2}\exp(-cR^{\hat{\beta}}t^2),$$

and

$$\sup_{x\in U_\delta} |\widetilde{u}^\varepsilon(x) - \overline{u}(x)| \lesssim \varepsilon^{\frac{1}{3}} R^{\frac{2}{3}} + (\log R)^{1/2} R^{-\hat{\beta}/2} + (\log \tfrac{1}{\varepsilon})^{1/2} (\varepsilon R)^{\alpha'}.$$

To complete the proof, it is only necessary to replace in the above two estimates $\widetilde{u}^\varepsilon(x)$ by $\mathbb{E}u^\varepsilon(x)$.

For this we argue as follows: For each $x \in U_\delta$ and $A > 1$ to be chosen, we have

$$\begin{aligned}
|\mathbb{E}u^\varepsilon(x) - \widetilde{u}^\varepsilon(x)| \le \mathbb{E}|u^\varepsilon(x) - \widetilde{u}^\varepsilon(x)| &= \int_0^\infty \mathbb{P}(\{\omega : |u^\varepsilon(x,\omega) - \widetilde{u}^\varepsilon(x)| > t\})dt \\
&\lesssim \int_0^\infty [(R\varepsilon)^{-(d-1)} R^{d\hat{\beta}/2} \exp(-cR^{\hat{\beta}}t^2)] \wedge 1 dt \\
&\lesssim AR^{-\hat{\beta}/2}(\log \tfrac{1}{\varepsilon})^{1/2} + \int_{AR^{-\hat{\beta}/2}(\log\frac{1}{\varepsilon})^{1/2}}^\infty (R\varepsilon)^{-(d-1)} R^{d\hat{\beta}/2} \exp(-cR^{\hat{\beta}}t^2)dt \\
&\lesssim AR^{-\hat{\beta}/2}(\log \tfrac{1}{\varepsilon})^{1/2} + \int_{A(\log\frac{1}{\varepsilon})^{1/2}}^\infty (R\varepsilon)^{-(d-1)} R^{d\hat{\beta}/2} R^{-\hat{\beta}/2} \exp(-ct^2)dt \\
&\lesssim AR^{-\hat{\beta}/2}(\log \tfrac{1}{\varepsilon})^{1/2} + (R\varepsilon)^{-(d-1)} R^{d\hat{\beta}/2} R^{-\hat{\beta}/2} \varepsilon^{cA} \\
&\lesssim R^{-\hat{\beta}/2}(\log \tfrac{1}{\varepsilon})^{1/2},
\end{aligned}$$

where for the last inequality to hold we chose A large depending only on universal constants.

Finally we discuss the almost sure convergence.

Lemma 5.4.5. *Assume the hypotheses of Theorem 5.2.16. There exists a measurable $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that, as $\varepsilon \to 0$, $u^\varepsilon(x,\omega) \to \overline{u}(x)$ for all $x \in U$ and all $\omega \in \Omega_0$.*

Proof. In view of the fact that we already know that $\mathbb{E}u^\varepsilon \to \overline{u}$, it suffices to show that $|u^\varepsilon(x,\omega) - \mathbb{E}u^\varepsilon| \to 0$ almost surely, a fact that typically follows from a Borel-Cantelli-type argument.

Since, however, we can apply the latter only along sequences $\varepsilon_n \to 0$, we first need to measure the dependence of u^ε on ε.

The assumptions on g and U yield a universal constant C such that, for any $\varepsilon, \varepsilon' > 0$ and for all $x \in \partial U$,

$$|g(x, \frac{x}{\varepsilon}, \omega) - g(x, \frac{x}{\varepsilon'}, \omega)| \leq C|x||\tfrac{1}{\varepsilon} - \tfrac{1}{\varepsilon'}| \leq C|\tfrac{1}{\varepsilon} - \tfrac{1}{\varepsilon'}|,$$

and, hence, using the comparison principle, we find that, for all $x \in U$,

$$|u^{\varepsilon}(x) - u^{\varepsilon'}(x)| \leq C|\tfrac{1}{\varepsilon} - \tfrac{1}{\varepsilon'}|.$$

Let $\delta(\varepsilon) = \varepsilon^{1-2\alpha_0/\hat{\beta}}$. Then, as long as $|\frac{1}{\varepsilon} - \frac{1}{\varepsilon'}| \leq c\varepsilon^{\alpha_0}$ and $\frac{\varepsilon'}{\varepsilon} \geq (2/3)^{1/\alpha_0}$ with c universal and independent of p, and, without loss of generality, $M_p > 1$, the estimate proved above yields

$$\begin{aligned}\mathbb{P}(\{\omega : \sup_{U_\delta} |u^{\varepsilon'}(x,\omega) - \mathbb{E}u^{\varepsilon'}(x)| > 3M_p\varepsilon'^{\alpha_0}\}) &\leq \mathbb{P}(\{\omega : \sup_{U_\delta} |u^{\varepsilon}(x,\omega) - \mathbb{E}u^{\varepsilon}(x)| > 2M_p\varepsilon^{\alpha_0} - C|\tfrac{1}{\varepsilon} - \tfrac{1}{\varepsilon'}|\}) \\ &\leq \mathbb{P}(\{\omega : \sup_{U_\delta} |u^{\varepsilon}(x,\omega) - \mathbb{E}u^{\varepsilon}(x)| > M_p\varepsilon^{\alpha_0}\}) \lesssim \varepsilon^p.\end{aligned}$$

Now we just choose a sequence $\varepsilon_k \to 0$ such that $\varepsilon_{k+1}^{-1} - \varepsilon_k^{-1} \leq c\varepsilon_k^{\alpha_0}$. For example, we take $\varepsilon_k = k^{-\gamma}$, since, as long as $\gamma \leq \min\{c, \frac{1}{1+\alpha_0}\}$,

$$\varepsilon_{k+1}^{-1} = (k+1)^\gamma \leq k^\gamma + \gamma k^{\gamma-1} = \varepsilon_k^{-1} + \gamma\varepsilon_k^{\frac{1}{\gamma}-1} \leq \varepsilon_k^{-1} + c\varepsilon_k^{\alpha_0} \text{ and } k^\gamma/(k+1)^\gamma \to 1 \text{ as } k \to \infty,$$

and thus $\varepsilon_{k+1}/\varepsilon_k > (2/3)^{1/\alpha_0}$ for k large enough universal.

Then, for $p > \frac{1}{\gamma}$, we find

$$\sum_k \mathbb{P}(\{\omega : \sup_{U_\delta} |u^{\varepsilon_k}(x,\omega) - \mathbb{E}u^{\varepsilon_k}(x)| > M_p\varepsilon_k^{\alpha_0}\}) \leq \sum_k k^{-\gamma p} < +\infty,$$

which, by the Borel-Cantelli lemma, yields

$$\mathbb{P}(\{\omega : \sup_{U_\delta} |u^{\varepsilon_k}(x,\omega) - \mathbb{E}u^{\varepsilon_k}(x)| > M_p\varepsilon_k^{\alpha_0} \text{ for infinitely many } k\}) = 0,$$

and, by the previous estimates,

$$\begin{aligned}&\mathbb{P}(\{\omega : \sup_{U_\delta} |u^{\varepsilon}(x,\omega) - \mathbb{E}u^{\varepsilon}(x)| > 3M_p\varepsilon^{\alpha_0} \text{ for arbitrarily small } \varepsilon\}) \leq \\ &\qquad \mathbb{P}(\{\omega : \sup_{U_\delta} |u^{\varepsilon_k}(x,\omega) - \mathbb{E}u^{\varepsilon_k}(x)| > M_p\varepsilon_k^{\alpha_0} \text{ for infinitely many } k\}) = 0.\end{aligned}$$

Note that, in fact, we have now proven the following stronger result: There exists an almost surely positive random variable $\varepsilon_0 : \Omega \to \mathbb{R}_+$ such that

$$\sup_{U_{\delta(\varepsilon)}} |u^{\varepsilon}(x,\omega) - \mathbb{E}u^{\varepsilon}(x)| \leq 3M_p\varepsilon^{\alpha_0} \text{ for } \varepsilon \leq \varepsilon_0(\omega).$$

□

5.5 Neumann problem

Here we prove the results about the Neumann problem with random oscillatory data, that is the existence of the ergodic constant (Theorem 5.2.18 and Theorem 5.2.19), its continuity (Theorem 5.2.20) and the homogenization in general domains (Theorem 5.2.21). The methods are very similar to those used for the Dirichlet problem, but there are a few differences. Where the proofs parallel or the same as the Dirichlet case, we simply outline the arguments or refer to the previous sections.

We note that throughout the section we take $\beta(\lambda,\Lambda) = \frac{\lambda}{\Lambda}(d-1)-1$ to be the homogeneity exponent of the fundamental solution of the Neumann problem ((5.2.31)).

5.5.1 The Discretized Cell Problem.

Similarly to the Dirichlet cell problem, we consider the solution of the Neumann cell problem (5.1.8) as a function of the boundary data and prove Lipschitz estimates in the appropriate norms and for discretized data.

For simplicity we take again $\sigma = e_d$, and, for $X \in \Xi = C(\bar{Q})^{\mathbb{Z}^{d-1}}$, let $u_R(y) = u_R(y;X)$ be the solution of

$$\begin{cases} F(D_y^2 u_R) = 0 & \text{in} \quad \Pi_{e_d}^{2R}, \\ \partial_{e_d} u_R = \Sigma_{i\in\mathbb{Z}^{d-1}} X_i(\cdot - i)\chi_{i+Q} & \text{on} \quad \partial P_{e_d}, \\ u_R = 0 & \text{on} \quad \partial P_{e_d} + 2Re_d; \end{cases} \tag{5.5.1}$$

notice that since, in general, (5.5.1) does not have a unique solution due to the discontinuity of the boundary data, here we choose a unique u_R, using an approximation procedure similar to the one in Section 2, so that it satisfies a comparison principle.

With the u_R as above, let $f_R : \Xi \to \mathbb{R}$ be given by

$$f_R(X) := \frac{1}{R} u_R(Re_d; X). \tag{5.5.2}$$

In the next lemma, which similar to Lemma 5.3.1, we discuss the continuity properties of f_R.

Lemma 5.5.1. *Assume* (5.2.10) *and* (5.2.11) *and let* β *as in* (5.2.31)*. Then* f_R *has the following continuity properties for* $X, Y \in C(\overline{Q})^{\mathbb{Z}^{d-1}}$ *and some universal constant* $C =: C(d, \lambda, \Lambda) > 0$*:*

(i) $$|f_R(X) - f_R(Y)| \leq |X - Y|_{\ell^\infty}. \tag{5.5.3}$$

(ii) $$|f_R(X) - f_R(Y)| \leq CR^{-(1+\beta)}|X - Y|_{\ell^1}. \tag{5.5.4}$$

(iii) $$|f_R(X) - f_R(Y)| \leq CR^{-(1+\beta)/2}|X - Y|_{\ell^2}. \tag{5.5.5}$$

Proof. The first estimate follows from the maximum principle, since the difference $u_R(y;X) - u_R(y,Y)$ can be bounded by the linear profiles with slopes $\pm|X-Y|_{\ell^\infty}$. The third claim follows from interpolating of (i) and (ii) as in the proof of Lemma 5.3.1, and thus to conclude we need to prove (ii).

Let

$$m = \min[-\partial_d \Phi(x + e_d) : x \in Q] > 0$$

where Φ is given by (5.2.31), and define the barrier

$$\phi_0(x) = 2m^{-1}\Phi(x + e_d),$$

so that $-\partial_{e_d}\phi_0 \geq 1$ on Q.

Then for $X, Y \in \ell^1(\mathbb{Z}^{d-1}; C(\bar{Q}))$ we sum over the sites where $X_j \neq Y_j$,

$$\phi(x) = \sum_{j \in \mathbb{Z}^{d-1}} (\sup_Q |X_j - Y_j|)\phi_0(x - j).$$

Then, as in the proof of Lemma 5.3.1,

$$F(D^2 u_R(\cdot;X) + D^2\phi) \geq F(D^2 u_R(\cdot;Y) - \mathscr{P}^+_{\lambda,\Lambda}(D^2\phi) = 0 \text{ in } \Pi^{2R}_{e_d},$$

$$-\partial_{e_d}(u_R(\cdot;X) + \phi) \geq -\partial_{e_d} u_R(\cdot;Y) \text{ on } \partial P_{e_d},$$

and, since $\phi > 0$,

$$u_R(\cdot;X) + \phi \geq u_R(\cdot;Y) \text{ on } \partial P_{e_d} + 2Re_d.$$

The comparison principle then yields $u_R(\cdot;Y) \leq u_R(\cdot;X) + \phi$ in $\Pi^{2R}_{e_d}$, and, in particular, for some universal $C > 0$,

$$u_R(Re_d;Y) \leq u_R(Re_d;X) + \phi(Re_d) \leq u_R(Re_d;X) + m^{-1}C \sum_{j \in \mathbb{Z}^{d-1}} (|j|^2 + R^2)^{-\beta/2} \sup_Q |X_j - Y_j|,$$

and the desired estimate for $f_R = R^{-1}u_R$ follows. $\square$

5.5.2 Solving the cell problem.

We use the above estimates to solve the Neumann cell problem.

As before we take $\sigma = e_d$, assume $|\psi| \leq 1$ almost surely and consider the solution $v_{e_d,R}$ to the cell problem,

$$\begin{cases} F(D^2 v_{e_d,R}(\cdot,\omega)) = 0 & \text{in} \quad \Pi^{2R}_{e_d}, \\ v_{e_d,R}(\cdot,\omega) = 0 & \text{on} \quad \partial P_{e_d} + 2R\sigma, \\ \partial_d v_{e_d,R}(\cdot,\omega) = \psi(\cdot,\omega) & \text{on} \quad \partial P_{e_d}. \end{cases} \tag{5.5.6}$$

We transform to the setting of the previous subsection by considering $\Psi : \Omega \to \Xi$ defined, for each $j \in \mathbb{Z}^{d-1}$ as an element of $C(\overline{Q})$,

$$\Psi_j(\omega)(\cdot) := \psi(x + \cdot, \omega),$$

and using the uniqueness of solutions we can identify,

$$v_{e_d,R}(\cdot,\omega) = u(\cdot;\Psi).$$

Next we sketch the argument leading to the concentration inequalities as it was described in more detail before. Since the $u(Re_d;\cdot)$ is Lipschitz on Ξ and, in view of (5.2.18), Ψ is a ϕ-mixing random field on $\mathbb{Z}^{d-1}$ with

$$\sum_{j \in \mathbb{Z}^{d-1}} \phi_\Psi(|j|)^{1/2} \lesssim \rho,$$

we apply Theorem 5.2.1 or Theorem 5.2.2 to get concentration of $R^{-1}u(Re_d;\cdot)$ and, hence, of $R^{-1}v(Re_d,\cdot)$ as well about their means.

In the convex case, we have

$$\mathbb{P}(\{\omega : R^{-1}|v_{e_d}(Re_d,\omega) - \mathbb{E}v_{e_d}(Re_d)| \geq t\}) \leq C\exp\left(-cR^{\beta}t^2\right) \text{ for all } t > 0,$$

while in the non-convex case,

$$\mathbb{P}(\{\omega : R^{-1}|v_{e_d}(Re_d,\omega) - \mathbb{E}v_{e_d}(Re_d)| \geq t\}) \leq C\exp\left(-cR^{2\beta-(d-1)}t^2\right) \text{ for all } t > 0.$$

Define

$$\hat{\beta} := \frac{\lambda}{\Lambda}(d-1) \text{ if } F \text{ is convex or concave and, otherwise, } \hat{\beta} := 2(\frac{\lambda}{\Lambda} - \frac{1}{2})(d-1), \tag{5.5.7}$$

and note that, if we assume $\frac{\lambda}{\Lambda} > 1/2$, then $\hat{\beta} > 0$ in the non-convex case; this corresponds to (5.3.11) for the Dirichlet problem.

Following the arguments in the Dirichlet case, we prove Theorem 5.2.18 and Theorem 5.2.19:

Proof of Theorem 5.2.18 and Theorem 5.2.19. For $N \in 2^{\mathbb{N}}$, a large universal constant $A = A(d,\lambda,\Lambda) >> 1$ and $k \in \mathbb{Z}^{d-1}$ we consider the events

$$E_k^N = \{\omega \in \Omega : |N^{-1}v_N(N\sigma + N^{1-\hat{\beta}/2}k,\omega) - \mu_N| \geq A^{1/2}(\log N)^{1/2}N^{-\hat{\beta}/2}\}. \tag{5.5.8}$$

It follows from the stationarity and either Theorem 5.2.1 or 5.2.2 that

$$\mathbb{P}(E_k^N) = \mathbb{P}(E_0^N) = \mathbb{P}(\{\omega : |N^{-1}v_N(N\sigma,\omega) - \mu_N| \geq A^{1/2}(\log N)^{1/2}N^{-\hat{\beta}/2}\}) \leq C\exp(-cA\log N).$$

If $E^{\geq M} = \bigcup_{N \geq M} E^N$, a simple union bound yields

$$\mathbb{P}(E^N) \leq CN^{3\hat{\beta}(d-1)/4}\exp(-cA\log N).$$

Let $E^{\geq M} = \bigcup_{N \geq M} E^N$. It follows that, for some large $M \in 2^{\mathbb{N}}$ and as long as $A > 3\hat{\beta}(d-1)/(4c)$,

$$\mathbb{P}(E^{\geq M}) \leq C\sum_{N \geq M} N^{3\hat{\beta}(d-1)/4-cA} < +\infty.$$

Then, for M sufficiently large in a universal way, $P(\Omega \setminus E^{\geq M}) > 0$, and, on $\Omega \setminus E^{\geq M}$,

$$|N^{-1}v_N(N\sigma + N^{1-\hat{\beta}/2}k,\omega) - \mu_N| \lesssim (\log N)^{1/2}N^{-\hat{\beta}/2} \text{ for all } N \geq M \text{ and } |k| \leq N^{3\hat{\beta}/4}.$$

We work with $\omega \in \Omega \setminus E^{\geq M}$. It follows from the interior oscillation decay estimates (Lemma 5.2.12) that, for every $N \geq M$ and $y' \in \partial P_\sigma \cap B_{N^{1+\hat{\beta}/4}}$,

$$|N^{-1}v_N(N\sigma + y', \omega) - \mu_N| \lesssim N^{1-\hat{\beta}/2}N^{-1} + (\log N)^{1/2}N^{-\hat{\beta}/2} \lesssim (\log N)^{1/2}N^{-\hat{\beta}/2},$$

while the localization estimate in Lemma 5.2.10 gives, for $0 \leq t \leq N$,

$$|v_N(t\sigma, \omega) - \tfrac{1}{2}\mu_N(2N - t)| \lesssim (\log N)^{1/2}N^{1-\hat{\beta}/2} + NN^{2(1-(1+\hat{\beta}/4))} \lesssim (\log N)^{1/2}N^{1-\hat{\beta}/2}. \quad (5.5.9)$$

Next we use again a localization estimate to compare v_N with v_{2N} in their common domain. To this end, notice that $2N\mu_{2N} + v_N$ and v_{2N} have the same Neumann boundary data on ∂P_σ, while, since $v_N(y' + 2N\sigma, \omega) = 0$ for $y' \in \partial P_\sigma$,

$$|2N\mu_{2N} + v_N(y' + 2N\sigma, \omega) - v_{2N}(y' + 2N\sigma, \omega)| \lesssim (\log N)^{1/2}N^{1-\hat{\beta}/2} \text{ on } (P_\sigma + 2N\sigma) \cap \overline{B}_{N^{1+\hat{\beta}/4}}.$$

It then follows from Lemma 5.2.10 that

$$\begin{aligned} |2N\mu_{2N} + v_N(N\sigma, \omega) - v_{2N}(N\sigma, \omega)| & \\ &\lesssim \sup_{|y'| \leq N^{1+\hat{\beta}/2}} |2N\mu_{2N} + v_N(y' + 2N\sigma, \omega) - v_{2N}(y' + 2N\sigma, \omega)| + N^{-\hat{\beta}/2} \\ &\lesssim (\log N)^{1/2}N^{1-\hat{\beta}/2}. \end{aligned} \quad (5.5.10)$$

Combining the previous estimates, using (5.5.10) and (5.5.9) for v_{2N} to estimate the three terms below and with the choice of ω above, for every $N, L \geq M$ in $2^{\mathbb{N}}$, we get

$$\begin{aligned} |\mu_N - \mu_{2N}| \leq |N^{-1}v_N(N\sigma, \omega) - \mu_N| + |2\mu_{2N} + N^{-1}v_N(N\sigma, \omega) - N^{-1}v_{2N}(N\sigma, \omega)| \\ + 2|(2N)^{-1}v_{2N}(N\sigma, \omega) - \tfrac{3}{2}\mu_{2N}| \lesssim (\log N)^{1/2}N^{-\hat{\beta}/2}. \end{aligned}$$

Therefore, for every $N, L \geq M$ in $2^{\mathbb{N}}$,

$$|\mu_N - \mu_L| \lesssim \sum_{K \geq M} (\log K)^{1/2}K^{-\hat{\beta}/2} \lesssim (\log M)^{1/2}M^{-\hat{\beta}/2}.$$

It follows that $(\mu_N)_{2^{\mathbb{N}}}$ is a Cauchy sequence and, therefore, has a limit μ.

The extension to an estimate of $R^{-1}\mathbb{E}v_R(R\sigma)-\mu$, for all $R>1$, is omitted as it is just a combination of the above arguments with the ideas from the Dirichlet case. $\square$

For the proof for general domains, we need the following spatially uniform concentration estimate. Since the notation and proof parallel that of Lemma 5.3.2 we omit the details.

Lemma 5.5.2. *Let v_R be as given above. Then, for any $t>0$,*

$$\mathbb{P}(\{\omega : \sup_{\Pi_\sigma^{2R}\cap\{|y'|\le 3R\}} R^{-1}|v_R(y,\omega)-\mathbb{E}v_R(y)|>t\}) \le CR^{d\hat{\beta}/2}\exp(-CR^{\hat{\beta}}t^2),$$

5.5.3 The continuity of the homogenized boundary condition.

We sketch here the proof of Theorem 5.2.20.

Proof. The first assertion is a direct consequence of the comparison principle. To prove the second, we first assume, without any loss of generality, that $|\psi|\le 1$.

Fix $\sigma_1,\sigma_2\in S^{d-1}$ and let v_1,v_2 and μ_1,μ_2 be respectively the solutions to the corresponding Neumann cell problems and the associated ergodic constants.

Similarly to the proof of Lemma 5.2.15, we define

$$E=\{y\in\mathbb{R}^d : L|\sigma_1-\sigma_2|<y\cdot\sigma_1<2R-L|\sigma_1-\sigma_2|, \quad |y-(y\cdot\sigma_1)\sigma_1|\le L\}\subset P_{\sigma_1}\cap P_{\sigma_2},$$

and using the up to the boundary $C^{1,\alpha}-$ regularity of the v_1 and v_2 (Lemma 5.2.11), we find that, for every ω,

$$\sup_{y\in\partial E\cap\{y\cdot\sigma=L|\sigma_1-\sigma_2|\}}|\partial_{\sigma_i}v_i(y,\omega)-\psi(y,\omega)|\lesssim L^\alpha|\sigma_1-\sigma_2|^\alpha$$

and

$$|\partial_{\sigma_1}v_1-\partial_{\sigma_2}v_2|\le\sup|Dv_i||\sigma_1-\sigma_2|\le C\|\psi\|_{C^\alpha}|\sigma_1-\sigma_2|.$$

Therefore, for $i=1,2$, we have

$$\sup_{y\in\partial E\cap\{y\cdot\sigma_1=L|\sigma_1-\sigma_2|\}}|\partial_{\sigma_1}v_i(y,\omega)-\psi(y,\omega)|\le C(\|\psi\|_{C^\alpha}|\sigma_1-\sigma_2|+L^\alpha|\sigma_1-\sigma_2|^\alpha),$$

and, moreover, since $|\psi| \leq 1$, $|v_i| \leq C|\sigma_1 - \sigma_2|L$ on $\{y \in \mathbb{R}^d : y \cdot \sigma_1 = 2R - L|\sigma_1 - \sigma_2|\}$ and $|v_i| \leq 2R$ in E.

Finally the localization Lemma 5.2.10 that, for $|y'| \leq R$,

$$\frac{1}{R}|v_1(y,\omega) - v_2(y,\omega)| \leq C(|\psi|_{C^\alpha}|\sigma_1 - \sigma_2| + L^\alpha|\sigma_1 - \sigma_2|^\alpha + CR^{-1}L|\sigma_1 - \sigma_2| + R^2L^{-2}).$$

From here the proof follows that of Lemma 5.2.15. Without loss we can assume that $L|\sigma_1 - \sigma_2| \leq 1$, since otherwise $\frac{1}{R}|v_1(y,\omega) - v_2(y,\omega)| \leq 2$ is a better bound.

Using this observation to consolidate terms and the cell problem homogenization result after taking expectations on both sides, we obtain

$$|\mu_1 - \mu_2| \lesssim (1 + |\psi|_{C^\alpha})L^\alpha|\sigma_1 - \sigma_2|^\alpha + R^2L^{-2} + (\log R)^{1/2}R^{-\hat{\beta}/2}.$$

Choosing R, L in terms of $|\sigma_1 - \sigma_2|$ to optimize the bound above gives the desired result. $\square$

5.5.4 The homogenization in general domains.

The proof and statement of Theorem 5.2.21 are a bit easier than that of Theorem 5.2.16. In contrast to the Dirichlet case the convergence rate of u^ε to $\overline{u}$ is uniform in U. In particular the boundary layer, represented by the parameter R in Theorem 5.2.16, does not appear in the Neumann setting.

In spite of this difference, the proof of Theorem 5.2.21 parallels the one of Theorem 5.2.16 and consists of two main steps, namely approximating u^ε in the general domain with the solution in half-space and in a local neighborhood of the "base points", and then using the results on the half-space solutions to get the homogenization in each neighborhood.

We do not calculate the optimal convergence rate allowed by the method in this case. It will be evident from what follows that a more careful analysis, as in the case of the Dirichlet problem, will give the full statement of Theorem 5.2.21 with explicit exponents.

The goal is to show that, if u^ε and $\overline{u}$ are respectively the solution of the general domain Neumann problem 5.1.2 and the homogenized equation (5.1.4) with boundary data as in the statement of

Theorem 5.2.21, then, for every $p > 0$ and any $k' < k := \min(\frac{2}{3}\frac{\alpha}{3+\alpha}, \frac{\alpha^2}{3+\alpha}, \frac{\hat{\beta}}{6})$, with $\alpha < \alpha'(\hat{\beta}, \lambda, \Lambda)$ from Lemma 5.2.20, there exists C, which depends on universal constants p and k', such that,

$$\mathbb{P}(\sup_{x \in U \setminus K} |u^\varepsilon(x,\omega) - \overline{u}(x)| > \varepsilon^{k'}) \leq C\varepsilon^p. \tag{5.5.11}$$

Proof of Theorem 5.2.21. Fix $\varepsilon > 0$ and $t > 0$, select a set $\Gamma_{\varepsilon^{2/3}} \subset \partial U$ of at most $C\varepsilon^{-2/3(d-1)}$ boundary points such that every $\varepsilon^{2/3}$-neighborhood of a point on ∂U contains at least one point in $\Gamma_{\varepsilon^{2/3}}$, let

$$\Omega_\varepsilon^t := \cup_{x \in \Gamma_{\varepsilon^{2/3}}} E_\varepsilon^t(\sigma_x), \tag{5.5.12}$$

where

$$E_\varepsilon^t(\sigma) := \{\omega : R^{-1} \sup_{y \in \Pi_\sigma^{2R} \cap \{|y'| \leq 3R\}} |v_{R,\sigma}(y) - \mathbb{E}v_{R,\sigma}(y)| > t\} \text{ with } R = R_\varepsilon = \varepsilon^{-1/3}.$$

and note that, in view of Lemma 5.5.2,

$$P(\Omega_\varepsilon^t) \leq C\varepsilon^{-(2/3(d-1)+d\hat{\beta}/6)} \exp(-c\varepsilon^{-\hat{\beta}/3} t^2). \tag{5.5.13}$$

Choose $t_\varepsilon = c_0 \varepsilon^{k'}$, with $k' < k$ and a universal c_0 to be chosen small. In particular $k' < k \leq \hat{\beta}/6$ and, thus,

$$\mathbb{P}(\Omega_\varepsilon) \lesssim \varepsilon^p \text{ for every } p < \infty \text{ as } \varepsilon \to 0. \tag{5.5.14}$$

We prove (5.5.11) by showing that, for $\omega \in \Omega \setminus \Omega_\varepsilon$,

$$\phi^- \leq u^\varepsilon(\cdot, \omega) \leq \phi^+,$$

where $\phi^\pm$ solve (5.1.2) with the modified Neumann boundary data $\mu(g(x,\cdot), F, \sigma_x) \pm c_1 \varepsilon^{k'}$ for $k' < k$ given in the statement of the theorem. Here c_1 can be chosen universally small so that $\phi^+ - \phi^- \leq \varepsilon^{k'}$, since $\phi^+ - \phi^- \leq c\varepsilon^{k'} h$, where h is the solution of $\mathscr{P}^+_{\lambda,\Lambda}(D^2 h) = 0$ in $U \setminus K$, $h = 0$ on ∂K and Neumann data identically 1 on ∂U, and, therefore, has a universal upper bound.

The concentration estimate (5.5.11) then follows since, for $\omega \in \Omega \setminus \Omega_\varepsilon$,

$$\sup_{x \in U \setminus K} |u^\varepsilon(x,\omega) - \overline{u}(x)| \leq \sup_{x \in U \setminus K} |\phi^+ - \phi^-| \leq \varepsilon^{k'},$$

or, in other words,

$$\mathbb{P}(\{\omega : \sup_{x\in U\setminus K} |u^\varepsilon(x,\omega) - \overline{u}(x)| > \varepsilon^{k'}\}) \le \mathbb{P}(\Omega_\varepsilon) \lesssim \varepsilon^p.$$

Below we only prove that $u^\varepsilon \le \phi^+$, since the proof of $u^\varepsilon \ge \phi^-$ is similar. We argue by contradiction observing that, if not, then $m := \max_U(u^\varepsilon - \phi^+) > 0$. By the maximum principle the maximum must be attained at a boundary point $x_\varepsilon \in \partial U$, which must belong to the $\varepsilon^{2/3}$- neighborhood of one of "grid points" $x_0 \in \Gamma_{\varepsilon^{2/3}}$ on ∂U.

Let $\sigma = \sigma_0$ and, for $z_0 = x_0 + \varepsilon^{2/3}\sigma$ and $z' := z - (z\cdot\sigma)\sigma$,

$$T(x) := \phi(z_0) + D\phi(z_0)\cdot(x-z_0)'.$$

Note that, in view of Lemma 5.2.20 and the fact that $g(\cdot,y,\omega) \in C^{0,1}(\mathbb{R}^d)$, for any $\alpha < \alpha'(\hat{\beta})$ from Lemma 5.2.20 we have

$$\overline{g}(x) = \mu(g(x,\cdot),F,\sigma_x) \in C^{0,\alpha}(\partial U).$$

Then Lemma 5.2.11 yields that $\phi^+ \in C^{1,\alpha}(\bar{U}\setminus K)$, and, in particular,

$$|\phi^+(x) - T(x)| \le C|(x-z_0)\cdot\sigma| + C|(x-z_0)'|^{1+\alpha} \quad \text{in } U\setminus K. \tag{5.5.15}$$

Let $U^\varepsilon := \{|(x-x_0)\cdot\sigma| \le \varepsilon^{2/3}\}\cap U$ and fix $L>1$ to be chosen later in terms of ε and consider the solution w_ε to

$$\begin{cases} F(D^2 w_\varepsilon) = 0 & \text{in} \quad U^\varepsilon \cap B_{L\varepsilon^{2/3}}, \\ \partial_\sigma w_\varepsilon(\cdot,\omega) = g(x_0,\cdot/\varepsilon,\omega) & \text{on} \quad \partial U \cap B_{L\varepsilon^{2/3}}(x_0), \\ w_\varepsilon(\cdot,\omega) = 0 & \text{on} \quad (\partial P_\sigma + z_0)\cap B_{L\varepsilon^{2/3}}(x_0), \\ w_\varepsilon(\cdot,\omega) = 0 & \text{on} \quad U^\varepsilon \cap \partial B_{L\varepsilon^{2/3}}(x_0). \end{cases}$$

It then follows from (5.5.15) and the zero Dirichlet condition for w_ε that

$$u^\varepsilon \le m + \phi^+ \le m + T + w_\varepsilon + CL^{1+\alpha}\varepsilon^{2/3(1+\alpha)} \quad \text{on} \quad (\partial P_\sigma + z_0)\cap U,$$

and, similarly,

$$u^\varepsilon \le m + T + w_\varepsilon + C\varepsilon^{2/3} \quad \text{on} \quad U^\varepsilon \cap \partial B_{L\varepsilon^{2/3}}(x_0).$$

Choose L so that $L^{1+\alpha}\varepsilon^{2/3(1+\alpha)} \leq \varepsilon^{2/3}$ holds and observe that, in view of the continuity of g,

$$\sup_{x\in B_{L\varepsilon^{2/3}}} |g(x,x/\varepsilon,\omega) - g(x_0,x/\varepsilon,\omega)| \leq CL\varepsilon^{2/3}.$$

Arguing as in Lemma 5.2.10 we estimate the difference of u^ε and $m+T(x)+w_\varepsilon(x)$ using a rotated version of the barrier

$$\varphi(x) = CL^{1+\alpha}\varepsilon^{2/3(1+\alpha)} + CL\varepsilon^{2/3}(\varepsilon^{2/3} - x_d) + CL^{-2}\varepsilon^{-4/3}\varepsilon^{2/3}(|x'|^2 + (d-1)\frac{\Lambda}{\lambda}(1-x_d^2)),$$

and we get

$$u^\varepsilon \leq m + T + w_\varepsilon + C(L^{1+\alpha}\varepsilon^{2/3(1+\alpha)} + L^{-2}\varepsilon^{2/3} + L\varepsilon^{4/3}) \text{ on } U\cap B_{\varepsilon^{2/3}}(x_0).$$

Choosing $L = \varepsilon^{-\frac{2\alpha}{9+3\alpha}}$ and, hence, $L\varepsilon = \varepsilon^{\frac{2}{3+\alpha}}$ gives

$$u^\varepsilon \leq m + T + w_\varepsilon + C\varepsilon^{\frac{2}{3}+k} \text{ on } U\cap B_{\varepsilon^{2/3}}(x_0), \tag{5.5.16}$$

since $k \leq \frac{2\alpha}{3+\alpha}$.

Next we compare a rescaled version of w_ε and the solution $v_R = v_{R,\sigma}$ of the cell problem (5.1.8) with boundary data

$$\psi(y,\omega) := g(x_0, y, \tau_{x_0/\varepsilon}\omega).$$

Since, in view of (5.2.7), for c sufficiently small depending on the C^2-regularity of ∂U,

$$d(\partial U - x_0, \partial P_\sigma \cap B_{c\varepsilon^{\frac{2}{3+\alpha}}}(x_0)) \leq \varepsilon^{\frac{4}{3+\alpha}},$$

it follows from the up to the boundary Hölder regularity of v_R and w_ε, Lemma 5.2.11, that

$$|\partial_\sigma w_\varepsilon(x_0+\cdot,\omega) - \partial_\sigma v_R(\cdot,\omega)| \leq C\varepsilon^{\frac{\alpha^2}{3+\alpha}} \text{ on } (\partial P_\sigma + \varepsilon^{\frac{2}{3+\alpha}}\sigma)\cap B_{c\varepsilon^{3/5}}. \tag{5.5.17}$$

Now we rescale to

$$\tilde{w}_\varepsilon(y,\omega) := \varepsilon^{-1} w_\varepsilon(x_1 + \varepsilon y, \omega),$$

and observe that, due to the facts that $|g| \le 1$, $R = \varepsilon^{-1/3}$ and (5.5.17), $h := \tilde{w}_\varepsilon - v_R$ solves

$$\begin{cases} -\mathscr{P}^+_{\lambda,\Lambda}(D^2 h) \le 0, & \text{in} \quad \Pi^R_\sigma \cap B_{R^{\frac{3(1+\alpha)}{3+\alpha}}}, \\ |\partial_\sigma h| \le C\varepsilon^{\frac{3+3\alpha}{1+\alpha}} & \text{on} \quad \partial P_\sigma \cap B_{R^{\frac{3(1+\alpha)}{3+\alpha}}}, \\ h = 0 & \text{on} \quad (\partial P_\sigma + R\sigma) \cap B_{R^{\frac{3(1+\alpha)}{3+\alpha}}}, \\ h(x) \le 2R & \text{on} \quad \partial B_{R^{\frac{3(1+\alpha)}{3+\alpha}}} \cap \Pi^R_\sigma. \end{cases}$$

Using a rotated version of the barrier

$$\varphi(x) = C\varepsilon^{\frac{\alpha^2}{3+\alpha}}(R - x_d) + 2R^{1-\frac{3(1+\alpha)}{3+\alpha}}\left(|x'|^2 - (d-1)\tfrac{\Lambda}{\lambda}((x_d)^2 - R^2)\right),$$

we conclude that, for some $C > 0$ which is independent of ε and x_0,

$$|\tilde{w}_\varepsilon - v_R| \le CR(\varepsilon^{\frac{\alpha^2}{3+\alpha}} + R^{-\frac{2\alpha}{3+\alpha}}) = CR(\varepsilon^{\frac{\alpha^2}{3+\alpha}} + \varepsilon^{\frac{2}{3}\frac{\alpha}{3+\alpha}}) \le CR\varepsilon^k \quad \text{in} \quad U \cap P_\sigma \cap B_R. \tag{5.5.18}$$

Note that, since $\omega \notin \Omega_\varepsilon$, the definition of Ω_ε yields that, for $\mu = \mu(g(x_0,\cdot), F, \sigma)$,

$$|R^{-1} v_R(\cdot,\omega) + \mu(\frac{y}{R}\cdot\sigma - 1)| \le c_0\varepsilon^{k'} \quad \text{in} \quad \Pi^R_\sigma \cap \{|y'| \le 3R\}. \tag{5.5.19}$$

Rewriting (5.5.18) and (5.5.19) in terms of the original variable yields that, for ε sufficiently small depending on $k - k'$ and c_0,

$$\varepsilon^{-2/3}|w_\varepsilon + \mu((\cdot - x_0)\cdot\sigma - \varepsilon^{2/3})| \le 3c_0\varepsilon^{k'} \quad \text{in} \quad U \cap B_{\varepsilon^{2/3}}(x_0). \tag{5.5.20}$$

Finally, combining (5.5.16) and (5.5.20), we obtain, again for ε sufficiently small, the estimate

$$u^\varepsilon \le m + T - \mu((\cdot - x_1)\cdot\sigma - \varepsilon^{2/3}) + 4c_0\varepsilon^{2/3+k'} \quad \text{in} \quad U \cap B_{\varepsilon^{2/3}}.$$

Recall that $-\partial_\sigma\phi^+ \ge \mu + c_1\varepsilon^{k'}$. Then, for sufficiently small $\varepsilon > 0$ and $c_0 < c_1/8$, for $x \in \partial U \cap B_{\varepsilon^{2/3}}(x_0)$ we have

$$\begin{aligned} m + \phi^+(x) + &\ge m + T(x) - \mu(x\cdot\sigma - \varepsilon^{2/3}) + c_1\varepsilon^{2/3+k'} - C\varepsilon^{2/3(1+\alpha)} \\ &\ge m + T(x) - \mu(x\cdot\sigma - \varepsilon^{2/3}) + \tfrac{1}{2}c_1\varepsilon^{2/3+k'} \\ &> u^\varepsilon(x), \end{aligned}$$

which is a contradiction.

$\square$

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