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ABSTRACT

The CPT theorem and the normal connection between spin and statistics are shown to be consequences of postulates of the S-matrix approach to elementary particles physics. The postulates are much weaker than those of field theory. Neither local fields nor any reference to space-time points are used. Quantum commutation relations and properties of the vacuum play no role. Completeness of the asymptotic states and positive definiteness of the metric are not required, though certain weaker asymptotic conditions prevail. The proofs depend on unitarity, macroscopic relativistic invariance, and a very weak analyticity requirement on the mass-shell scattering functions. The proofs are in the framework of the new S-matrix approach to elementary particle physics, which is herein established on a formal basis.

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DERIVATION OF THE CPT THEOREM AND THE CONNECTION BETWEEN SPIN AND STATISTICS FROM POSTULATES OF THE S-MATRIX THEORY*

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I. INTRODUCTION

The two most important general physical consequences of relativistic field theory are the CPT theorem[1-4] and the connection between spin and statistics.[5-10] The CPT theorem states that for every process occurring in nature there is an allowed dual process in which the particles of the first are replaced by their respective antiparticles, all spins are reversed, and paths are changed to their images under inversion through the origin in space-time. Relationships between probabilities are stated to be the same for a process and its dual. The proved connection between spin and statistics is that wave functions are symmetric under the interchange of variables referring to two identical integral-spin particles and antisymmetric for the half-integral spin case.

These important results are derived from the postulates of local field theory, which, however, are subject to considerable doubt. In the first place it is not known whether the postulates are sufficiently realistic to include any theories except trivial ones in which the scattering matrix is unity.

Secondly, the postulates are very specialized and restrictive, in that they assign a fundamental role to hypothetical local field operators defined over the field of space-time points. Experience does not entail

the existence of such points, and the restriction to theories in which they play a fundamental role may immediately exclude all theories connected to physical reality. Because space-time points are experimentally inaccessible, both in practice and in principle, their introduction runs counter to the philosophy of quantum mechanics. This philosophic inconsistency appears to have its analogue in the mathematical structure in which related inconsistencies seem to arise.[11,12,13]

Even within the general framework of local field theories, some of the postulates are so restrictive that many reasonable theories are excluded. In particular, the requirements of the completeness of the asymptotic states and the positive definiteness of the metric are assumed to hold, not only asymptotically, but also throughout the course of the interaction. But added states of negative metric not among those observed asymptotically seem to be exactly what is needed to remove the apparent inconsistencies from field theory. A theory based on this possibility is among those being most vigorously pursued today.[14,15] The postulate requiring the existence of a nondegenerate vacuum also excludes certain important theories,[14-16] and the fact that one must be concerned with the properties of nothing lends an unwholesome air to the whole business, particularly in the light of previous similar experience.

These abstract objections to the field theoretic approach are reinforced by an examination of the course of events in the practical sphere. The trend today in the practical study of elementary particle interactions is away from the solving of equations conjectured to be satisfied by hypothetical field operators. Rather, the focus is shifting directly to the S matrix[17] and to the consequences of the constraints

imposed upon it by unitarity, macroscopic relativistic invariance, and assumed analyticity properties.[18] This S-matrix approach to elementary particle physics is, in practical work, approaching the status of an independent theory, its connections to field theory gradually being dissolved. It becomes appropriate, therefore, to formalize this trend and to explore the consequences of the altered framework.[19] History encourages the casting away of formal substructures whose ingredients have no counterparts in experience and which are not relevant to practical calculations, and the focusing directly on relationships between experimental quantities. The new approach, since it involves only observable quantities and their analytic continuations, has a claim to probable physical relevance much greater than that of field theory, with its sundry hypothetical ingredients of dubious status.

The calculations needed to confront the new approach with physical fact are, as for the field theoretic case, far from complete. But the question arises whether the general results derived from field theory, and in particular the CPT theorem and the connection between spin and statistics, can be derived also on the basis of the new approach. An affirmative answer would be interesting for several reasons. First, it would show that the restrictive assumptions of local field theory are necessary only to guarantee much less stringent asymptotic properties, which will probably remain valid also in possible future forms of field theory. Second, the proof would be likely to apply to theories such as those of Heisenberg and Nambu, which are not in the class covered by the usual postulates. Third, and most important, if the new S-matrix approach is to constitute an independent approach to elementary particle physics, replacing unworkable field theory, then proofs of these important theorems in the framework of the new theory are required.

It is the purpose of this paper to provide these proofs. Because the aim is also at the widest range of generality the postulates have been taken in a form that avoids all mention of space and time. The CPT theorem is consequently proved in its momentum-space form. The way in which the concept of macroscopic location would be introduced is briefly discussed.

In the next section, postulates for the S-matrix approach are stated in words. Their mathematical forms will be introduced as they are needed in the proofs.

II. THE POSTULATES

A. Quantum Theory

If the normalized relative frequencies (probabilities) of the various possible outcomes of two complete experiments are denoted by P_i and P_j' respectively, and if these necessarily positive numbers are written in the forms $P_i = |a_i|^2$ and $P_j' = |a_j'|^2$, then a_i and a_j' are linearly related; for all a and a',

$$a_j' = \Sigma S_{ji} a_i ,$$

where S is independent of the a_i.

B. Macroscopic Relativistic Invariance (Weak Form)

The relationships between the probabilities of the possible outcomes of two experiments are the same as the relationships between the corresponding outcomes of two experiments related to the first two by a (real) orthochronous proper Lorentz transformation (i.e., the real Lorentz transformations connected to unity).

B'. Macroscopic Relativistic Invariance (Strong Form)

The relationship stated in postulate B for probabilities is also valid for the corresponding amplitudes, provided the freedom in the choice of phases is properly exploited.

C. Particles

The measurement of the momentum, spin, and particle type of all particles present constitutes a complete experiment, in the sense used in postulate A. Such a measurement may be considered possible only in a limiting sense, not necessarily, for instance, during a reaction or at finite times. Projections on linear combinations of spin states are permissible observables, as are projections on self-conjugate combinations of particle-antiparticle amplitudes.

D. Conservation Laws

The physical interpretation of the mathematical quantities shall be such that translation and rotation invariance imply conservation of momentum-energy and angular momentum respectively.

From postulates A through D, including B', a set of spinor functions, called M functions, satisfying unitarity relations can be constructed. A consequence of the unitarity relations is the existence of certain singularities whose positions are determined by the masses of the (stable) particles. Cuts defined by relativistic scalar equations, and terminating at these singularities, can be constructed by using a scale transformation on the masses, and a distinguished sheet, the physical sheet, specified.

E. Maximal Analyticity

Except at singularities required by unitarity the M functions are regular analytic functions in the interior and on the boundary of their physical sheets.

Postulate E, though presumably needed for the derivation of dynamical relations, is much stronger than what is needed for the proofs. The following much weaker postulate is sufficient.

E'. Minimal Analyticity

For each M function, a physical sheet bounded by cuts defined by relativistic scalar equations can be defined. This sheet contains a domain of regularity that includes among its boundary points all physical-type points.

A physical-type point is a point corresponding to real momentum-energy vectors and it is to be approached with positive imaginary physical energies and momentum magnitudes. It is this type of limit that is expected to give

physically interpretable M functions. The physical sheet of a given M function is that sheet for which the physical value of the M function is assumed at the physical-type limit points along the cut that runs over the points corresponding to the process described by the particular M function. These matters are discussed in the section on analytic structure.

F. Physical Connection

Physically interpretable functions obtained by analytic continuation from functions describing physical phenomena also describe physical phenomena; they are not mere mathematical chimeras. Specifically, the M functions at all physical-type points of a physical sheet correspond to processes actually occurring in nature. Regarding interpretation, if a simple connection can be set up permitting a consistent interpretation of the quantities appearing in the theory, and also those that could be obtained by analytic continuation, then this interpretation accords uniformly to reality if it accords at all.

III. REMARKS CONCERNING THE POSTULATES

Postulate A is the basic premise of quantum theory, the aspect to be used herein being the superposition principle.[20] Quantum commutation relations and the quantization of action (Planck's constant), or their equivalents, are not implied by this postulate.

The relativistic postulate is stated as a relationship between experimental quantities. From this postulate, and others, objects satisfying spinor transformation laws will be constructed. No hypothetical objects with spinor transformation properties are arbitrarily introduced. This procedure is the reverse of that in field theory in which objects satisfying spinor transformation laws are hypothesized and the relativistic invariance of experimental results deduced. In this latter approach the relativistic invariance of the experimental results might be said to be derived from more "fundamental" requirements, but is probably a delusion to think that the objects of a mathematical model are more "fundamental" than the experimental results the model is designed to describe. In any case, by simply requiring the invariance of the experimental relationships the relativistic requirement is placed in its weakest form; all possible ways of achieving this end are included.

The particle postulate also constitutes a certain completeness requirement. This requirement has force only in the asymptotic limit, and is therefore much weaker and more satisfactory than the completeness postulate in field theory.

The term "particle" appearing in the particle postulate means that with every momentum vector, $\underset{\sim}{k}_i$, is associated an energy component $k_i^{\,o}$, fixed by the mass condition $(k_i^{\,o})^2 - (\underset{\sim}{k}_i)^2 = (m_{t_i})^2$, where m_{t_i} is a

constant called the mass of the particle of type t_i. The spin states referred to in the postulate shall, to eliminate possible ambiguity, refer, for the case of nonmassless particles, to rest frames of the particles. In such a frame the spin states are to be basis vectors of a finite dimensional representation of the rotation group. The theory of this group is classical.[21] Only nonmassless particles are treated in the body of the text. Massless particles are easily included, as shown in Appendix B.

That projections on linear combinations of spin states are permissible observables is a basic feature of quantum mechanics. It is in this way that spin states referring to different directions are obtained. That the projection on the self-conjugate combinations of particle-antiparticle amplitudes be a permissible observable of the theory is a requirement that is needed in the present proof of the connection between spin and statistics, but it is believed unnecessary and should eventually be eliminated.

In the conservation postulate the notion of translational invariance appears for the first and only time. One may completely avoid the introduction of space-time coordinates by simply taking energy-momentum conservation to be exactly the requirement of translational invariance. However, it is apparent that one could perform a formal Fourier transformation on the momentum-energy variables, introducing thereby formal space-time coordinates. Formal translation invariance is then equivalent to energy-momentum conservation.

It is by this Fourier transformation, using wave packets, that the notion of macroscopic location will be introduced into the theory. The restriction of momentum-energy variables to the mass shell implies that the wave packets will move as free-particle wave packets. Wave packets approximating space-time points cannot be constructed because of the

mass-shell constraint. The unit of action, Planck's constant, enters the theory for the first time as the scale constant relating physical distances to the formal coordinates introduced by this Fourier transformation.

The connection between momentum-energy and translation operators introduced above leads to a connection between angular momentum conservation and rotational invariance. However, the connection between invariance properties and conservation laws is much more general, following also, for instance, in classical theory. The postulate only asserts that this general connection is maintained and, avoiding specific reference to space-time coordinates, requires that which is important to the proofs, the conservation laws.

Postulate E (Maximal Analyticity) asserts that the only singularities in the physical sheets are those required by unitarity. (These include the usual anomalous and complex singularities, as will be shown.) This analyticity requirement evidently contains some extraction of the usual locality requirement. But since it applies only to mass-shell functions, it is much weaker than its field theoretic counterpart. It may in another way be stronger: the postulated domain of analyticity may be larger than the one that can be deduced using field theory. In this case the two theories would be different, perhaps mutually incompatible, systems. Indeed, this is the expectation. The apparent inconsistencies of field theory are expected to be removed as a result of the weakening of the locality requirement. What relations between masses and coupling constants are imposed by the postulated analyticity is the exciting but still open question.

The postulate of maximal analyticity, though not used directly in the proofs, is important to the general philosophy. The object is to start

with a set of postulates that have significance in their own right as a basis for a proposed theory for elementary particle reactions, not to prove the theorems starting from postulates chosen specifically to provide a sufficient basis for the proofs. However, this latter procedure has a certain merit if the postulates are such that an extensive class of possible theories are included. For this reason, the weaker postulate E' (Minimal Analyticity) is used in the proofs rather than postulate E. The proofs thereby become applicable not only to the S-matrix theory, but also to field theory, and to varieties of field theory to which the standard postulates do not apply.

The postulate of physical connection states that interpretable functions obtained by analytic continuation have physical significance. This principle is the basis of the present work. As there is no field theoretic substructure, related physical processes enter only via analytic continuation. The principle has its analogue in field theory, where the formalism set up to describe particle processes is found to have a natural place for antiparticle processes and the interpretation of the theory is correspondingly extended. In order to state the CPT theorem as a statement having physical relevance, the connection to physical reality of the extended interpretation must be accepted. It is the purpose of this postulate to explicitly state that certain naturally occurring mathematical quantities do have physical significance, and to specify the conditions under which an interpretation qualifies as "natural".

The qualification "simple" in simple connection specifies that interpretation of a variable referring to one particle is not to depend on the numerical value of variables referring to the other particles. A connection of this kind is implicit in field theory where each particle

has its own field operator, and analytic continuation in one momentum variable does not alter the interpretation of variables associated with the other particles.

For the postulate of physical connection to be operative, the connection referred to is required to give a consistent interpretation of quantities that could be obtained by analytic continuation. Specifically, the manner of interpretation should be consistent with regions of analyticity as large as those given by postulate E'; postulate F should not conflict with postulate E'.

Postulates A through D assert principles similar to those used in field theory. Postulates E and F enunciate two general principles characteristic of the new S-matrix formalism: the physical sheets of the scattering functions contain only singularities required by unitarity, and the analytic continuation of a scattering function to various physical-type boundaries of the physical sheet gives functions having physical significance.

We proceed to the application of these postulates.

IV. THE M FUNCTIONS AND THEIR PROPERTIES

Using postulates A through D, including B', one can construct a set of functions $M(k_i, \mu_i, t_i)$ with properties similar to the Fourier transforms of the vacuum expectation values of time-ordered products of field operators, such as occur in field theory. These M functions are defined for certain real values of the momentum-energy variables k_i that correspond to physical processes, and only for real values satisfying the mass conditions $k_i^2 = m_{t_i}^2$, and the conservation law $\Sigma k_i = 0$. For each momentum-energy variable k_i, there is an associated index t_i that specifies the type of particle, and a $(2 S_{t_i} + 1)$ valued spin index μ_i built up out of products of $2 S_{t_i}$ spinor indices. Here S_t is the spin of the particle of type t. The spinor indices can be taken to be lower dotted and undotted spinor indices, corresponding to antiparticles and particles respectively. The three quantities k, μ and t, taken as a unit, will be called the variable corresponding to a given particle.

These M functions have the following properties:

A. Spinor Character

For real values of the k_i corresponding to physical processes the $M(k_i, \mu_i, t_i)$ satisfy the usual transformation law for spinor functions. Specifically, for these k_i,

$$M(k_i, \mu_i, t_i) = \prod_i \sum_{\mu_i'} \Lambda_s^{-1}(\mu_i, \mu_i') M(\Lambda k_i, \mu_i', t_i) ,$$

where Λ_s is the spinor transformation matrix corresponding to any element of the real orthochronous proper homogeneous Lorentz group and Λ is the related Lorentz transformation operator.

B. Unitarity

For real values of k_i corresponding to real processes the M functions satisfy the unitarity relations

$$M(K', -\tilde{K}'') + M^*(K'', -\tilde{K}') = -\sum_K \int M(K', -\tilde{K}) K\cdot\tilde{\sigma} M^*(K'', -\tilde{K}) = -\sum_K \int M(K, -\tilde{K}'') K\cdot\tilde{\sigma} M^*(K, -\tilde{K}') \quad .$$

Here * designates complex conjugate, K, K', and K'' represent normal-ordered sets of variables and $-\tilde{K}$ is the normal-ordered set obtained from the set K by reversing the signs on all the momentum-energy vectors, reversing the order of the variables, dotting all spinor indices and changing the type designations to those of the respective antiparticles (see below). The normal-ordering convention for variables with real momentum-energy vectors k_i requires $k_i^0/|k_i^0| \geqslant k_{i+1}^0/|k_{i+1}^0|$, $k_i^1 \geqslant k_{i+1}^1$, $k_i^2 \geqslant k_{i+1}^2$, and $k_i^3 \geqslant k_{i+1}^3$, the various conditions in the set being operative if and only if the equality parts of the preceding conditions are realized. The summation is over all normal-ordered sets K, and the integrations are over the invariant momentum space elements $d^4k_i\, 2m_i\, \delta(k_i^2 - m_i^2)/(2\pi)^3$. The symbol $K\cdot\tilde{\sigma}$ represents a product of the (Pauli) spin matrices $k_i^\mu\, \sigma_\mu^{\dot{A}B}/m_i$, one for each spinor index of the set K. The contraction rule for the spinor indices is such as to ensure that the unitarity relation is a proper spinor equation.

C. Antiparticle Processes

For certain real values of the k_i (always consistent with mass constraints and the conservation laws) the $M(k_i, \mu_i, t_i)$ relate the

amplitudes corresponding to possible outcomes of two (complete) experiments, termed the initial and final experiments. If, for fixed μ_i and t_i, the $M(k_i, \mu_i, t_i)$ is an analytic function regular in a domain that includes two values of some k_i that are negatives of each other, then the only relationship between the interpretations of the $M(k_i, \mu_i, t_i)$ at these two points, consistent with the postulates, is this: If the variable in one case specifies a particle occurring in the final (initial) configuration, then in the other case it specifies the corresponding antiparticle occurring in the initial (final) configuration with the same physical momentum-energy but with opposite spin. It is essential that this connection is not dependent on the conventions adopted, but follows directly from the postulates themselves, chiefly the conservation postulate.

D. Superselection Rule

The number of spinor indices is even on all nonzero M functions.

E. Order of Variables

The interchange of two variables changes an M function by at most a phase.

With the inclusion of postulate E' one obtains:

F. Symmetry

The interchange of two variables of the same type changes an M function by at most a sign. For a given type of variable this sign is always the same, irrespective of the numerical values of the variables, the relative positions of the variables, or of the particular M function in which the variables occur.

The construction of the M functions and the derivations of these properties are given after the main body of the proofs, to which we now proceed.

V. EXTENSION TO COMPLEX LORENTZ TRANSFORMATIONS

According to property (A) the functions $M(K', -\tilde{K}'')$ satisfy the spinor transformation law

$$M(K', -\tilde{K}'') = \Lambda_s^{-1} M(\Lambda K', -\Lambda\tilde{K}'') , \qquad (5.1)$$

provided the K' and K'' specify real momentum-energy vectors corresponding to physical processes, and provided the Λ_s and Λ are taken to be the transformations associated with the real orthochronous proper homogeneous Lorentz transformations.

By virtue of the constraints imposed by the conservation laws and mass conditions, the $M(K', -\tilde{K}'')$ can be considered functions over a reduced set of variables, the constraints being then identically satisfied. And the reduced variables can be selected so as to be real over the (original) domain of definition where the momentum-energy vectors in K' and K'' are real.

Real Lorentz transformations are generated by unimodular spinor transformation matrices subject to the constraint that dotted and undotted indices are transformed by matrices that are complex conjugates of each other. If this constraint is relaxed the corresponding Lorentz transformation matrices Λ are no longer constrained to be real. However, the invariance of scalar products of four-vectors continues to be maintained under this enlarged class of (complex) Lorentz transformations.

By the method of Hall and Wightman[22] one can show that the validity of Eq. (5.1) for real k_i and real orthochronous proper homogeneous Lorentz transformations implies its validity also for the complex Lorentz transformations continuously connected to unity, the k_i ranging then over the domain generated from the original one by the complex Lorentz

transformations. Specifically, since Λ_s and Λ are given analytic functions of parameters that specify the Lorentz transformation, Eq. (5.1) can be used to extend the definition of $M(K', -\tilde{K}'')$ over the range of variables generated from the original region by the complex Lorentz transformations connected to unity. By the method of Hall and Wightman this extended definition may be shown to coincide with the analytic continuation of $M(K', -\tilde{K}'')$ into this region, provided that $M(K', -\tilde{K}'')$ was regular in a neighborhood of the original region.

VI. THE CPT THEOREM

The special class of complex Lorentz transformations of interest to us are those in which the undotted indices are transformed by unity and the dotted indices by

$$\Lambda_s = \exp [\, i \phi \sigma_z \,] . \tag{6.1}$$

For real ϕ, these Λ_s form a set of unimodular transformations continuously connected to unity. For $\phi = \pi$, Eq. (6.1) gives $\Lambda_s = -1$. The corresponding Λ is also minus unity and all four-vectors are carried to their negatives. The application of (6.1), with $\phi = \pi$, to (5.1) gives

$$M(K', -\tilde{K}'') = (-1)^N M(-K', \tilde{K}'') , \tag{6.2}$$

where N is the number of dotted indices of $M(K', -\tilde{K}'')$, and $-K'$ represents the set K' with the signs of all momentum-energy vectors reversed. Because the phase drops out in the calculation of probabilities Eq. (6.2) is, in the light of properties (C) and (D), just the statement of the CPT theorem: If a scattering process is invariant under the proper orthochronous Lorentz transformations and if analytic continuations of scattering functions to all other physically interpretable boundary points of the physical sheet give functions corresponding to physical reality, then for every process occurring in nature there is a CPT inverse process, and relationships between corresponding probabilities are identical.

The requirement, stated in postulate E', that the boundaries of the physical sheet are defined by relativistic scalar equations, and hence depend on scalar invariants, guarantees that the CPT transformation takes a boundary point of the physical sheet to another boundary point of this sheet.

The development above is similar to Jost's[4] in its essential use of the complex Lorentz transformation. Here, however, the transformation is applied directly to the physically interpretable mass-shell scattering functions, and the question of whether certain vacuum expectation values of local field operators are identically equal is not raised.

If a field theoretic substructure were assumed, then the original connection between the various related processes would be conventionally expressed in terms of transformations on field operators, rather than via the analytic continuation of the scattering functions. Consequently, the present remarks do not constitute a proof when considered in a field theoretic context. They show that the CPT theorem is valid in the class of field theories in which analytic continuations of mass shell scattering functions lead to physically interpretable quantities, as specified in postulate F. Relativistic invariance and the conservation laws guarantee the existence of the required continuation, the necessary interpretations, and the needed numerical relationships.

VII. THE CONNECTION BETWEEN SPIN AND STATISTICS

The unitarity condition reads

$$\begin{aligned} & M(K', -\tilde{K}'') + M^*(K'', -\tilde{K}') \\ & \quad = -\sum_K \int M(K', -\tilde{K})\, K\cdot\tilde{\sigma}\, M^*(K'', -\tilde{K}) \\ & \quad = -\sum_K \int M(K, -\tilde{K}'')\, K\cdot\tilde{\sigma}\, M^*(K, -\tilde{K}') \,. \end{aligned} \tag{7.1}$$

For the case that $K' \equiv K''$ designates self-conjugate combinations of particle-antiparticle amplitudes, application of the CPT transformation, followed by an inversion of the order of variables gives

$$\begin{aligned} & M(K', -\tilde{K}') + M^*(K', -\tilde{K}') \\ & \quad = -(-1)^N \sigma_{K'} \sum_K \int M(K, -\tilde{K}')\, K\cdot\tilde{\sigma}\, M^*(K, -\tilde{K}') \\ & \quad = -(-1)^N \sigma_{K'} \sum_K \int M(K', -\tilde{K})\, K\cdot\tilde{\sigma}\, M^*(K', -\tilde{K}) \,. \end{aligned} \tag{7.2}$$

Here N is the number of spinor indices on the variables of K which, by virtue of the superselection rule, can be replaced by the number of spinor indices on the variables of the set K'. The factor $\sigma_{K'}$ is the sign coming from the complete reversal of order of the variables of $M(K', -\tilde{K}')$. The fact is used that for K' corresponding to measurements of self-conjugate combinations of particle-antiparticle amplitudes the sets K' and $\tilde{K}'$ are identical except for a reversal of order (see Sec. X).

Since $K\cdot\tilde{\sigma}$ is a positive definite Hermitian-form, comparison of Eqs. (7.2) and (7.1), for the case $K' = K''$, implies either $\sigma_{K'} = (-1)^N$

or $M(K', -\tilde{K}) = M(K, -\tilde{K}') = 0$ for all K. But if K' contains an odd number of variables with abnormal symmetries then $\sigma_{K'} = -(-1)^N$ and hence $M(K', -\tilde{K})$ and $M(K, -\tilde{K}')$ must vanish. By postulate E' the variables can be switched from one side to the other and consequently all M functions containing variables having abnormal symmetries must vanish.

The variables of the set K' are, in the above, required to designate self-conjugate particles. However, the set K is arbitrary. If there are conservation laws that distinguish the particle and antiparticle parts of the self-conjugate combination of amplitudes, these separate contributions can be distinguished by appropriate choice of the variables of K. It follows that the symmetry of the self-conjugate combination is the same as the symmetry of the individual particle and antiparticle parts, and the normal connection between spin and statistics also obtains for these latter.

The above proof is similar to the recent proofs of the connection between spin and statistics by Lüders and Zumino,[8] and Burgoyne,[9] in that it rests on a conflict in the abnormal case between the CPT transformation, which follows from relativistic invariance, and certain positive definiteness requirements. However, the arguments here involve only mass shell quantities and the positive definiteness requirements arise directly from unitarity, which is much more secure than the general requirement of positive definite metric.

The essential result embodied in the above parts of the proofs is that the CPT theorem and the connection between spin and statistics are contained already in the asymptotic properties of the S matrix. We proceed now to the derivation of these properties from our postulates, without

reference to field theory. It is necessary to show that the abstract postulates themselves ensure the existence of functions with the stated unitarity and spinor transformation properties, that the interpretations of the functions arising from analytic continuations of these originally defined functions are, by virtue of the conservation laws, uniquely determined, and that the analyticity requirement implies the symmetry property under interchange of like variables stated above.

VIII. UNITARITY

Let K' and K'' be normal ordered sets of variables labelling two possible outcomes of an experiment called the final experiment. Let $-\tilde{K}$ be a normal ordered set labelling a possible outcome of an experiment called the initial experiment. Then postulate A implies the unitarity conditions (see Appendix A):

$$\Sigma_{(-\tilde{K})} \; S(K', -\tilde{K})\, S^{*}(K'', -\tilde{K}) = \delta_{K'K''} \; . \tag{8.1}$$

The summation is over normal ordered sets $-\tilde{K}$, the $*$ denotes complex conjugation, and $\delta_{K'K''}$ is unity if $K' \equiv K''$ and otherwise zero. (Discrete variables are assumed temporarily.) Let the possible outcomes of the two experiments be placed in a one to one correspondence so that for every final outcome labelled by K there is a "corresponding" initial outcome labelled by $-\tilde{K}$. The correspondence will be the physical correspondence of "no scattering" which will be discussed below.

A convention will be adopted whereby if a particle of the final configuration is labelled by a momentum-energy vector k_i, the corresponding particle in the "corresponding" initial configuration will be labelled by $-k_i$. Then the conservation law for momentum energy takes the form $\Sigma\, k_i = 0$. This negative sign for the initial k_i is represented by the minus sign in front of $-\tilde{K}$. In order not to prejudice the arguments, the particle-type indices will be taken to have opposite signs for the corresponding variables of K and $\tilde{K}$.

In terms of $R(K', -\tilde{K}'') \equiv S(K', -\tilde{K}'') - \delta_{K'K''}$ the unitarity condition reads

$$R(K', -\tilde{K}'') + R^*(K'', -\tilde{K}') = -\Sigma_K R(K', -\tilde{K}) R^*(K'', -\tilde{K}) . \qquad (8.2)$$

If $R'(K', -\tilde{K}'') \equiv R(K', -\tilde{K}'')(\rho(K') \rho(K''))^{1/2}$ is introduced, where $\rho(K)$ represents the number of values of K per unit element of the product of the invariant momentum elements $d^4k(2m) \delta(k^2 - m^2)/(2\pi)^3$, then the summation may be replaced in part by integrations over these elements, with R' replacing R. The prime on R' will generally be suppressed.

The subtraction of the $\delta_{K'K''}$ has a physical basis. If we were to consider processes in a large finite volume V, then over a time T one expects an initial momentum eigenstate to gradually decay, and other momentum eigenstates of the same energy to gradually grow at a rate that tends to zero as V increases. This characteristic difference in time dependences allows a particular final momentum state to be correlated with each initial momentum state. This physical correlation is the basis of the correspondence between initial and final configurations labelled by $-\tilde{K}$ and K respectively. The states are correlated so that the subtraction of $\delta_{K'K''}$ just removes from $S(K', -\tilde{K}'')$ the part that remains finite as V becomes infinite. It is the remainder, $R(K', -\tilde{K}'')$, which when multiplied by $(\rho(K') \rho(K''))^{1/2}$, is expected to pass over a smooth well-defined continuum limit as V goes to infinity. This expectation is embodied in postulate E' which requires the M functions, which are spinor forms of the R functions, to be analytic functions. The S functions are neither expected nor required to be analytic.

It is the procedure at this point that characterizes the present development as strictly an S-matrix approach, and which inserts an essential physical ingredient into the present proof of the connection between spin and statistics.

IX. RELATIVISTIC INVARIANCE

Postulate B requires, for the case of spinless particles, that

$$| R(K', -\tilde{K}") |^2 = | R(\Lambda K', -\Lambda\tilde{K}") |^2 , \tag{9.1}$$

where ΛK is the set of variables obtained from the set K by replacing each momentum-energy vector, k_i, of the set by Λk_i, the vector obtained from k_i by the real orthochronous proper homogeneous Lorentz transformation Λ. For the case with spins let $P(S', -S")$ be the projection operator for the spin state specified by the set of axial four-vectors $(s_i', -s_j")$. Then postulate B requires

$$|P(S', -S") R(K', -\tilde{K}")|^2 = |P(\Lambda S', -\Lambda S") R(\Lambda K', -\Lambda\tilde{K}")|^2 , \tag{9.2}$$

where $P(\Lambda S', -\Lambda S")$ represents the projection operator corresponding the set of spin vectors $(\Lambda s_i', -\Lambda s_j")$. Using the hermiticity and idempotent ($P^2 = P$) character of projection operators, one may write Eq. (9.2) in the form

$$P(S', -S") R(K', -\tilde{K}") R^*(K', -\tilde{K}") = P(\Lambda S', -\Lambda S")R(\Lambda K', -\Lambda\tilde{K}")R^*(\Lambda K', -\Lambda\tilde{K}") , \tag{9.3}$$

where $P(S', -S")$ now acts between corresponding indices of R and R^*. The order in which the indices are contracted is irrelevant.

The projection operator $P(S', -S")$ is a product of the elementary projection operators for the individual particles. The actual form of these operators depends upon the physical significance of the various spin states. Or conversely, the physical significance of the various spin states is determined by the form of the projection operators. One is free, consequently, in the case of nonmassless particles, to take the elementary

projection operators to be

$$P(s) = \frac{1}{2}(\pm \underline{s}' \cdot \underline{\sigma} + 1) , \tag{9.4}$$

where $\underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli matrix vector and $\underline{s}'$ is the value of s in some rest frame of the particle. To eliminate possible arbitrary rotations, this rest frame will be taken to be the one obtained from the general coordinate system by the Hermitian spinor Lorentz transformation.

If Eq. (9.4) is substituted into (9.3) and the known behavior of $\underline{s}'$ under rotations is used, one obtains

$$P(S', -S'')\, R(K', -\tilde{K}'')\, R^*(K', -\tilde{K}'') = P(S', -S'')\Lambda_s^{-1}\, R(\Lambda K', -\Lambda\tilde{K}'') \times \left[\Lambda_s^{-1}\, R(\Lambda K', -\Lambda\tilde{K}'')\right]^* , \tag{9.5}$$

where Λ_s^{-1} are certain of the usual spinor transformation matrices corresponding to the rotation Λ. Since Eq. (9.5) is valid for all S' and S'', the projection operator $P(S', -S'')$ may be removed and the resulting equation states that RR^* is a spinor function relative to the rotation subgroup. The indices of RR^* that are contracted in Eq. (9.3) with the left-hand index of a factor (9.4) have the transformation character appropriate to an upper undotted or lower dotted spinor index, these being the same for rotations. The right-hand index of Eq. (9.4) is contracted with the corresponding complex-conjugated factor of RR^* and these indices accordingly have the transformation characters of upper dotted or lower undotted indices under rotations.

In order to construct a true spinor function we introduce the operator $\mathcal{L}(K', -\tilde{K}'')$, defined to be the product of the Lorentz transformations that would take spinors associated with various momentum-energy

vectors k_i from their values in the general coordinate system to their values in the respective rest frames, in which the momentum-energy vectors become pure time-like. To eliminate possible arbitrary rotations we again take those Lorentz transformations represented by the Hermitian spinor transformations. Then with the definitions

$$M(K', -\tilde{K}'') \equiv \mathcal{L}^{-1}(K', -\tilde{K}'')\, R(K', -\tilde{K}'') , \tag{9.6}$$

and

$$P(S', -S'', K', -\tilde{K}'') \equiv P(S', -S'')\, \mathcal{L}(K', -\tilde{K}'')\, \mathcal{L}^{*}(K', -\tilde{K}'') , \tag{9.7}$$

condition (9.3) becomes

$$P(S', -S'', K', -\tilde{K}'')\, M(K', -\tilde{K}'')\, M^{*}(K', -\tilde{K}'') = P(\Lambda S', -\Lambda S'', \Lambda K', -\Lambda\tilde{K}'')\, M(\Lambda K', -\Lambda\tilde{K}'')\, M(\Lambda K', -\Lambda\tilde{K}'') . \tag{9.8}$$

If the Lorentz transformations are taken to be the ones appropriate to the transformation characters of the indices of R obtained above, then in $P(S', -S'', K', -\tilde{K}'')$ the elementary projection operators are, according to Eqs. (9.4) and (9.7), by direct calculation,

$$P(s, k) = \frac{1}{2}[(k\cdot\sigma/m) \pm s\cdot\sigma] = \frac{1}{2}[\, k^{\mu}/m \pm s^{\mu}]\, \sigma_{\mu A\dot{B}} , \tag{9.9}$$

or

$$\tilde{P}(s, k) = \frac{1}{2}[(k\cdot\tilde{\sigma}/m) \mp s\cdot\tilde{\sigma}] = \frac{1}{2}[\, k^{\mu}/m \mp s^{\mu}]\, \sigma_{\mu}^{\dot{A}B} , \tag{9.10}$$

where

$$v\cdot\sigma \equiv v_0 + \underset{\sim}{v}\cdot\underset{\sim}{\sigma} \equiv v^{0} + v^{i}\sigma_{i} ; \tag{9.11}$$

$$v\cdot\tilde{\sigma} \equiv v_0 - \underset{\sim}{v}\cdot\underset{\sim}{\sigma} \equiv v^{0} + v^{i}\tilde{\sigma}_{i} . \tag{9.12}$$

The two cases (9.9) and (9.10) correspond to the alternate choices of upper undotted or lower dotted for the transformation character of the indices of RR^* contracted with the left-hand index of Eq. (9.4). The first parts of the two equations (9.9) and (9.10) follow by calculation and the derived expressions have the transformation characters indicated on the far right. Making use of these transformation properties one obtains

$$P(S', -S'', +K', -\tilde{K}'')\, M(K', -\tilde{K}'')\, M^*(K', -\tilde{K}'')$$

$$= P(S', -S'', +K', -\tilde{K}'')\Lambda_s^{-1}\, M(\Lambda K', -\Lambda\tilde{K}'')[\Lambda_s^{-1}\, M(\Lambda K', -\Lambda\tilde{K}'')]^* , \tag{9.13}$$

which, being true for all $(S', -S'')$, implies

$$M(K', -\tilde{K}'') = \Lambda_s^{-1}\, M(\Lambda K', -\Lambda\tilde{K}'') \times \exp i\alpha . \tag{9.14}$$

Apart from possible phase factors, the $M(K', -\tilde{K}'')$ constructed in this way are spinor functions. Postulate B' is invoked to permit the choice $\exp i\alpha = 1$, and the $M(K', -\tilde{K}'')$ are then true spinor functions:

$$M(K', -\tilde{K}'') = \Lambda_s^{-1}\, M(\Lambda K', -\Lambda\tilde{K}'') . \tag{9.15}$$

The superselection rule follows immediately from Eq. (9.15), applied to rotations of 2π.

Substitution of Eq. (9.6) into Eq. (8.2) gives

$$\mathcal{L}(K', -\tilde{K}'')\, M(K', -\tilde{K}'') + [\mathcal{L}(K'', -\tilde{K}')\, M(K'', -\tilde{K}')]^*$$

$$= -\oint_K \mathcal{L}(K', -\tilde{K})\, M(K', -\tilde{K})[\mathcal{L}(K'', -\tilde{K})\, M(K'', -\tilde{K})]^* . \tag{9.16}$$

There is complete freedom in the choice between the two alternative possibilities for the Lorentz transformations, the construction being

equally valid for either case. We will choose to use lower spinor indices, for R and R^*, and, consequently, for M and M^*. For a lower undotted spinor index the Lorentz transformation in $\mathcal{L}(K', -\tilde{K}'')$ is

$$\mathcal{L}(k) = \exp\left[-\frac{\theta}{2}\, \underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{k}} \right] = \cosh\frac{\theta}{2} - \underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{k}} \sinh\frac{\theta}{2} \equiv \sqrt{k\cdot\tilde{\sigma}/m} \quad . \tag{9.17}$$

This matrix is Hermitian, as required, so that the complex conjugate transformation, which operates on the lower dotted indices, is the transpose matrix.

From the form (8.2) the requirement that the indices associated with $-\tilde{K}''$ transform as the complex conjugates of the corresponding ones of K'' follows. Equation (9.15) is then conveniently written in the form

$$\sqrt{K'\cdot\tilde{\sigma}}\ M(K', -\tilde{K}'')\sqrt{K''\cdot\tilde{\sigma}} + \sqrt{K'\cdot\tilde{\sigma}}\ M^*(K'', -\tilde{K}')\sqrt{K''\cdot\tilde{\sigma}}$$

$$= -\sum_K \sqrt{K'\cdot\tilde{\sigma}}\ M(K', -\tilde{K})\sqrt{K\cdot\tilde{\sigma}}\ \sqrt{K\cdot\tilde{\sigma}}\ M^*(K'', -\tilde{K})\sqrt{K''\cdot\tilde{\sigma}} \quad , \tag{9.18}$$

where

$$K\cdot\tilde{\sigma} \equiv \prod_{i=1}^{N} \left(\frac{k_i}{m_i}\right)\cdot\tilde{\sigma}^{(i)}$$

$$\equiv \prod_{i=1}^{N} \left(\frac{k_i{}^{\mu}}{m_i}\right)\tilde{\sigma}_{\mu}^{(i)}$$

$$\equiv \prod_{i=1}^{N} \left(\frac{1}{m_i}\right)(k_i{}^{o} - \underset{\sim}{k}_i\cdot\underset{\sim}{\sigma}^{(i)}) \quad . \tag{9.19}$$

The superscript (i) on $\sigma^{(i)}$ indicates that $\sigma^{(i)}$ is to operate on the ith spinor index of K, and k_i is the momentum-energy vector associated

with this index. The order in which the factors are written down in Eq. (9.18) is such that if K' contains only undotted indices and $\tilde{K}''$ contains only dotted ones, and similarly for K and $-\tilde{K}$, then the contraction is always with adjacent indices. For, variables that do not satisfy these conditions, the contraction is with the nonadjacent index of the Lorentz-transformation matrix.

The reciprocal of $\mathcal{L}(k)$ is

$$\mathcal{L}^{-1}(k) = \exp\left[\frac{\theta}{2}\,\underline{\sigma}\cdot\underline{\hat{k}}\right] = \cosh\frac{\theta}{2} + \underline{\sigma}\cdot\underline{\hat{k}}\sinh\frac{\theta}{2} = \sqrt{(k\cdot\sigma)/m}\ . \tag{9.20}$$

Multiplication by the appropriate inverses brings Eq. (9.18) to the desired form:

$$M(K', -\tilde{K}'') + M^{*}(K'', -\tilde{K}') = -\sum_{K} M(K', -\tilde{K})K\cdot\tilde{\sigma}\ M^{*}(K'', -\tilde{K})\ . \tag{9.21}$$

The contraction rule is as stated above, and Eq. (9.21) has the form of a spinor equation, as inspection of Eq. (9.10) shows.

The above development is independent of the choice of sign of $\pm\,\underline{s}'$ in Eq. (9.4); the transformation properties are independent of this sign. If one wishes to identify $\underline{s}'$ with the physical spin, then the requirement that conservation of angular momentum be a consequence of rotational invariance (postulate D) demands that in the operators $P(s)$ acting on the same type of indices, the opposite signs of $\pm\,\underline{s}'$ must be used for initial and final configurations, since it is the <u>difference</u> of the initial and final physical spins that must enter into the conservation law. It is for this reason that the minus sign was placed before the second argument of $P(S', -S'')$; then the same mathematical function of the arguments may be used for both initial and final spin operators.

The application of the formalism for the case of particles of spin greater than 1/2 is discussed in more detail in an appendix.

X. ANTIPARTICLES

The functions $M(K', \tilde{K}'')$ have been defined at real values of the momentum-energy variables corresponding to real processes. Because of the mass constraints and the energy-momentum conservation laws, these functions are defined only over a restricted subspace of the space of energy-momentum variables. Variables appropriate to this subspace may be introduced.

Consider a possible analytic continuation in these variables from a region corresponding to a physical process to a new region where the energy-momentum vectors are again real, but with perhaps some different signs. For definiteness suppose only one of the energy-momentum vectors has a changed sign. Is it possible to give the function in the new region a physical interpretation, and if so, is this interpretation unique?

Because the continuation is in the subspace corresponding to the mass constraints and the energy-momentum conservation law, these conditions will be formally satisfied also in the new region. But since the sign of k is reversed, the contribution of this term in the formal conservation law is reversed. If the interpretation of the other momentum-energy vectors (with unchanged signs) is to remain unaltered (see postulate F), then the only way to reconcile the formal conservation law with the physical law of conservation of energy-momentum (postulate D) is to associate the new value of the momentum-energy variable with an initial particle if it was formerly associated with a final particle, and vice versa, and to reverse the sign of the connection between the mathematical energy-momentum vector and the physical quantity. This is, of course, the same connection that one obtains in field theory. Here it is the only interpretation consistent with the postulates.

If a final particle carries a non-zero unit of any additive constant of the motion, then the initial particle associated with it by the analytic continuation described above must carry the negative unit; otherwise, the conservation laws would require one of the two processes to vanish, and by analytic continuation both would vanish. The two particles related by this continuation are, consequently, not the same particle, in general, although their masses are the same. They will be called antiparticles, in accordance with the usual terminology.

The type designations of the corresponding variables of K and $\tilde{K}$ have been taken as negatives of each other. If a variable is called a particle or antiparticle variable, according to which of these two it specifies when its energy-sign is positive (i.e. when it refers to the final experiment), then variables of oppositely signed type designation are particle and antiparticle variables respectively.

For a fixed spin index the corresponding physical spin is opposite for the cases in which the variable refers to a particle of the initial or final configuration respectively, as mentioned before.

The formalism arrived at is of the Pauli rather than the Dirac type: spin $\frac{1}{2}$ particles are represented by two component variables. Particles and antiparticles are represented by undotted and dotted indices, respectively. Because particle and antiparticle variables have different spinor characters and, correspondingly, are contracted differently with the spin operators, it is not the simple sum of the amplitudes that is measured when the self-conjugate combination of particle-antiparticle amplitudes combination is measured. A certain (charge conjugation) operation is required to bring the amplitudes to a form suitable for direct addition. In the proof of the connection between spin and statistics one can first apply the CPT

transformation to the various parts that will form the self-conjugate combinations, then apply the transformations needed to bring the various terms to additive form, next use the symmetry rules to invert the order of the combined variables and finally transform the parts back to their original forms. Thus the fact that the amplitudes corresponding to dotted and undotted indices do not directly combine does not have an important effect on the proof. If both upper and lower indices had been used the unitarity relation would have taken a more complicated form, and the fact that the symmetry rules apply only to proper combinations would then play a role.

An alternative formalism in which the no-scattering part of the S-matrix is taken to be the charge-conjugation matrix

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in spinor space is convenient in this respect. With this choice, particles and antiparticles can be taken to have the same transformation character. The same physical spins are then represented by the same spin operators for particles and antiparticles, and the amplitudes are directly additive. The unitarity relation in this formalism contains explicit factors of C, and it is the relation $C^T = -C$ that leads to the factor $(-1)^N$. The derivation of the connection between spin and statistics using this formalism is given in Appendix C.

XI. ORDER OF VARIABLES

A set of variables with real momentum-energy vectors k_i is normal ordered only if the following conditions are satisfied: $(k_i)^2 \geqslant (k_{i+1})^2$, $k_i^0/|k_i^0| \geqslant k_{i+1}^0/|k_{i+1}^0|$, $k_i^1 \geqslant k_{i+1}^1$, $k_i^2 \geqslant k_{i+1}^2$, and $k_i^3 \geqslant k_{i+1}^3$, each of the conditions of the set being operative if and only if the equality parts of the preceding conditions are realized. Except for cases of relative zero measure, for which $k_i = k_{i+1}$ for some i, which can be treated as limiting cases, the normal ordering condition gives a well-defined order for the variables corresponding to any physical process. For definiteness we specify that when the variables are in this order the M function describes the physical process. Stated differently, the value of some (say original) M function that describes the physical processes corresponding to some set of k_i will be taken to be the definition of the value of the (standard) M function when these variables are placed in normal order. If postulates F and E' were true for the original M function they would also be true for the standard one.

Postulate E' requires all the real points to be (physical type) boundary points of a single analytic function. This allows the M function to be extended by analytic continuation and defined for all orders of the variables. Observables calculated by using the values of the M functions obtained by analytic continuation must give the physical answer, according to postulate F. Hence the various M functions obtained by reorderings of the variables can differ only by phases as stated in property (E). If postulate E' were not valid and certain ranges of variables could not be reached from others by analytic continuation, then one is free to establish property (E) by decree in cases where it does not follow from analytic continuation and postulate F.

For the case of variables with the same spin and type indices and the same energy-sign, one can obtain a more stringent condition. This is because the interchange of such variables, which will be called like variables, carries an M function back to itself, though at a new value of the arguments.

Let us denote by $M(x)$ and $M(-x)$ two M functions related by the interchange of two like variables, and by $|M(x)|^2$ and $|M(-x)|^2$ some experimental relationship calculated using these M functions. That is, we suppress the remaining variables. According to postulate F,

$$|M(x)|^2 = |M(-x)|^2 ,$$

because $M(-x)$ is the analytic continuation of $M(x)$ at a new physical-type boundary point of the physical sheet, and the two points have identical physical interpretations.

Separating $M(x)$ into even and odd parts with respect to x one obtains

$$|M(x)|^2 - |M(-x)|^2 = 4 \operatorname{Re} M_e(x) M_0^*(x) = 0 .$$

This implies that $M(x)$ is either even or odd or that the even and odd parts are relatively imaginary.

The condition that $M_e(x)$ and $M_0(x)$ be relatively imaginary, which we take to include the case in which either one vanishes, must obtain for all choices of the remaining (suppressed) observables. However, by appropriate choice of the observables (Hermitian operators) corresponding to the other (suppressed) variables one can adjust arbitrarily the phases of each orbital angular momentum state of the two-particle subsystem. This is easy to see for the simplest case of the scattering of two spinless

particles, since by choosing the operator for (say) the initial state to project onto some chosen combination of the initial orbital states the phases of the final states can be fixed arbitrarily. It is not hard to show that this result can be generalized to arbitrary reactions, and that the phases of orbital states corresponding to the two-particle subsystem can be fixed arbitrarily by appropriate choice of the Hermitian operators corresponding to the observables for the remaining variables. Thus the general vanishing of $4 \text{ Re } M_e(x) M_0^*(x)$ implies the vanishing of either $M_e(x)$ or $M_0(x)$, and $M(x)$ must be either symmetric or antisymmetric under the interchange of like variables. Essential to the proof is the assumption, stated in postulate C, that only the variables specifying momentum, spin and particle type are needed to label the complete set of amplitudes. No other "hidden" variable specifying, for instance, "which one of the various identical particles is measured" is permitted. This assumption is quite natural when viewed from the S-matrix standpoint.

By the application of the fact that the M functions must be either symmetric or antisymmetric under the interchange of any two like variables to both sides of an equation representing this same fact, one immediately finds that a single rule, either symmetry or antisymmetry, holds for the interchange of any two like variables of a fixed type in a given M function with a fixed order of the spin and type indices. Using postulates E' and F, as above, one can extend the rule also to the case in which the ordering of the spin and type indices differ and show that the characteristic sign for the interchange of like variables depends, for a given M function, only on the spin and type designations and possibly on the energy signs.

The above derivation of the fact that the M functions are either symmetric or antisymmetric under the interchange of like variables is based

upon analyticity in the physical region and on the principle, contained in postulate F, that different expressions corresponding to the same physical quantity must give the same answer. This is an argument involving indistinguishability. In ordinary quantum mechanics, the analogous symmetry property of the wave fcuntion is postulated as a boundary condition. In field theory, it is the immediate consequence of the postulate that field operators must either commute or anticommute for space-like separations. This locality postulate of field theory draws its support from the principle of microcausality; the postulated commutation relations ensure that signals never propagate faster than the speed of light, even over very small distances. This support of the postulate is rather unsatisfactory both because of the questionability of the principle of microcausality and because of the particularity of the way in which it is achieved. For instance, the possibility that the commutation relations depend on the states between which the fields act is not considered. The present derivation seems more satisfactory because these various possibilities are included, and because it is based directly on indistinguishability and does not intertangle the logically separate questions of symmetry and relativistic invariance.

The symmetry rule can immediately be extended from the case of equal energy-signs to the case of unequal energy-signs using postulate E', because both sides of the equation representing the symmetry rule can be analytically continued from the region where the energy-signs are equal to the region where they are unequal. This consequence of analyticity is the root of the connection between spin and statistics.

The methods used above do not lead to symmetry rules for the case in which the spin indices differ. Here an interchange of variables carries

an M function over to another M function in which the order of the indices is different. This is a different function not connected to the first by analytic continuation. From postulate F one obtains a relation like

$$|M_{12}(x)|^2 = |M_{21}(-x)|^2 ,$$

instead of

$$|M_{11}(x)|^2 = |M_{11}(-x)|^2 ,$$

the spin indices now being displayed, and the arguments used above lead to no symmetry rules.

If it is assumed that measurements of linear combinations of spin state amplitudes are possible, one may obtain relations of the second type with the spin state "1" now representing a linear combination of the original states. One obtains, for example, relations which, when expressed in terms of the original spin states, read

$$(\cos^2 \delta)\, M_{11}(x) + \cos \delta \sin \delta (M_{12}(x) + M_{21}(x)) + \sin^2 \delta\, M_{22}(x)$$

$$= \pm [\cos^2 \delta\, M_{11}(-x) + \cos \delta \sin \delta (M_{12}(-x) + M_{21}(-x))$$

$$+ \sin^2 \delta\, M_{22}(-x)] ,$$

where the linearity asserted in postulate A is invoked. Since the relation is true for several values of δ one obtains

$$M_{11}(x) = \pm M_{11}(-x) ,$$

$$M_{22}(x) = \pm M_{22}(-x) ,$$

and

$$M_{12}(x) + M_{21}(+ x) = \pm [M_{12}(-x) + M_{21}(-x)] ,$$

where the same sign holds for the three equations. The first two equations show that the same symmetry rule holds for both cases of like spin indices, and by the third line this symmetry also extends to the symmetric combination of the spin states. The third line is generally inconsistent with the assumption that the interchange of variables with different spin indices gives a sign change opposite to the one for the case of equal spin indices. Indeed, using the fact that $M_{12}(x)$ and $M_{21}(-x)$ can differ by only a phase the relationship $M_{12}(x) = \pm M_{21}(-x)$ is the only one generally consistent with the requirement that the M functions be analytic functions of their variables, as one sees by examination of the other solution:

$$M_{21}(-x) = \pm \exp 2 i\alpha M^{*}_{12}(x) ,$$

$$M_{12}(-x) = \pm \exp 2 i\alpha M^{*}_{21}(x) ,$$

and

$$\alpha = \arg (M_{12}(x) + M_{21}(x)) .$$

As in the case of different spin indices, the analyticity requirements alone lead to no symmetry rules for the interchange of variables of different types. The symmetries that prevail are direct consequences of phase conventions regarding the orders of the variables of the M functions.

In the development above there is the apparently arbitrary phase convention that the singular part of the S-matrix (i.e., $\rho(K') \delta_{K'K''}$) be taken as real and positive when the variables are normal-ordered. It is because of this convention that the normal-ordering requirement appears in the unitarity relations we use. Making use of this particular form of the unitarity requirement, one can carry through an argument quite analogous to the one given in the body of the proof of the connection between spin

and statistics, but now without using self-conjugate particles. If one assumes that the symmetry under interchange of antiparticle-type variables is the same as for the corresponding particle-type variables, then one can deduce that the symmetry rule for the interchange of a particle variable with a corresponding antiparticle variable is in accordance with the normal connection between spin and statistics.

This relationship is of no physical significance. All that is established is that the symmetry under interchange of particle and antiparticle variables is in accordance with the normal rule. But nothing is said regarding the rules for the interchange of two-particle variables or two-antiparticle variables. It is these rules that are relevant to the connection between spin and statistics. Moreover, the rule that is established is a direct consequence of the original choice of phase conventions and can change if other conventions are adopted.

In field theory there is an analogous situation. It is the commutation relation between two like fields that determines the connection between spin and statistics. The commutation relation between a field and its adjoint is not relevant unless it is shown that this commutation relation must be the same as for a field and itself. In recent studies of the connection between spin and statistics[8,9] it is only the commutation relation of a field and its adjoint that is shown to be normal, and the arguments consequently do not prove the connection between spin and statistics. Earlier proofs have been objected to because they either apply only to the nonphysical free field case or assume restrictive symmetry requirements.

In the extension of the symmetry rule for the interchange of like variables to the case of differing spin indices the physical requirement,

stipulated in postulate C, that projections on linear combinations of spin-state amplitudes be, in principle, observable quantities played an essential role. In the analogous extension to particle and antiparticle variables the corresponding situation prevails; the requirement that projections on combinations of the particle-antiparticle amplitudes be, in principle, observables is again essential. This requirement underlies our proof, which depends critically on the fact that one can choose variables referring to the self-conjugate combinations of particle-antiparticle amplitudes. When these variables are used, the CPT transformation followed by an inversion of order takes the arguments of $M(K', -\tilde{K}')$ into identically themselves and there is no phase ambiguity. Also, the symmetry is derived directly for the self-conjugate combinations, and from this the symmetries of the particle and antiparticle parts are obtained.

To complete the discussion of order of variables we must show that the symmetry rules for the interchange of like variables are the same for all M functions, as stated in property (F). This follows from the unitarity relation. For real k_i, the unitarity relation reads

$$M(K', -\tilde{K}'') + M^*(K'', -\tilde{K}') = -\int_K \Sigma\, M(K', -\tilde{K})\, K\cdot\tilde{\sigma}\, M^*(K'', -\tilde{K}) \quad ,$$

which, with the introduction of

$$\tilde{M}(K', -\tilde{K}'') \equiv M^*(K''^*, -\tilde{K}'^*) \quad ,$$

becomes

$$M(K', -\tilde{K}'') + \tilde{M}(K', -\tilde{K}'') = -\int_K \Sigma\, M(K', -\tilde{K})\, K\cdot\tilde{\sigma}\, \tilde{M}(K, -\tilde{K}'') \quad .$$

This form permits analytic continuation in the variables k_i.

Suppose K' contains two like variables with real k_i. According to postulate E', these may be interchanged by an analytic continuation. An application of the symmetry rule for like variables then gives

$$\sigma(K'')\, M(K', -\tilde{K}'') + \sigma'(K'')\, \tilde{M}(K', -K'') = -\int_K \sigma(K)\, M(K', -\tilde{K})\, K\cdot\tilde{\sigma}\, \tilde{M}(K, -\tilde{K}'') ,$$

where $\sigma(K)$ is the sign change induced in $M(K', -\tilde{K})$ by the interchange of the two like variables, and $\sigma'(K'')$ is the sign change for the two corresponding like variables of $M^*(K'', -\tilde{K}')$. For the case $K' = K''$ we have $\sigma(K') = \sigma'(K')$, since an application of both transformations to $M(K', -\tilde{K}')$ must leave the sign unchanged in order not to conflict with the sign on the right of the unitarity relation. This gives

$$\sigma(K')[\, M(K', -\tilde{K}') + M^*(K'', -\tilde{K}')\,] = -\int_K \sigma(K)\, M(K', -\tilde{K})\, K\cdot\tilde{\sigma}\, M^*(K, -\tilde{K}') .$$

The integrand factor $M(K', -\tilde{K})\, K\cdot\tilde{\sigma}\, M^*(K', -\tilde{K})$ is a positive definite Hermitian form, as mentioned earlier, so that comparison of the above equation to the original form of the unitarity relation implies

$$\sigma(K) = \sigma(K') ,$$

and the symmetry rule under interchange of like variables is the same for all M functions that can be brought to the form $M(K', -\tilde{K})$. One can take K' to contain just the two variables in question, and then all M functions containing these variables can be brought to this form using postulate E'.

XII. ANALYTIC STRUCTURE

Postulate E requires the M functions to be regular in the interior of a physical sheet whose boundaries are fixed by unitarity. This needs explaining.

The essential idea is that there are some singularities of the M functions which are direct consequences of the unitarity relations; if the M functions occurring on the right of the unitarity relations were assumed free of singularities, those on the left would nonetheless have singularities arising from limitations in the range of integrations occurring on the right.

Specifically, the unitarity relation contains on the right terms each containing a factor of the form

$$\int \prod_i \left[\frac{d^4 k_i}{(2\pi)^4} \; (2m_i)\; (2\pi)\; \delta(k_i^2 - m_i^2)\; \theta(k_i^0) \right] (2\pi)^4 \delta(\Sigma\; k_i - T)\;, \tag{12.1}$$

which is multiplied by other functions that may or may not have singularities. Here T is the sum of the initial (or final) momentum-energy vectors of the problem. As is well-known, the delta functions place limits on the (real) range of integration and the above factor introduces a singularity at $T^2 = (\Sigma\, m_i)^2$ that corresponds to the threshold for the reaction involving the process described by the k_i.

Equation (12.1) gives singularities that would occur in one or both of the M functions occurring on the left of the unitarity relation even if the M functions occurring on the right were free of singularities. Possible singularities of the M functions on the right can lead to additional singularities on the left. To get some of these, one may substitute the

expression for the singular part of the M function obtained from the first application of unitarity back into the right hand side of the unitarity relation. Proceeding by iteration, one obtains structures of the form

$$\int \prod_i \left[\frac{d^4k}{(2\pi)^4} \, (2m_i)(2\pi)\, \delta(k_i^2 - m_i^2)\, \theta(k_i^0) \right] \prod_j (2\pi)^4 \, \delta(\Sigma_j - T_j) \quad , \tag{12.2}$$

where Σ_j is a sum over some subset of the k_i, and T_j is a sum over some subset of the (external) momentum-energy vectors of the M function. Various structures may be correlated with various Feynman-like diagrams, with the lines associated with the mass conservation delta-functions and the vertices now involving the arbitrary numbers of particles entering into a reaction. The conservation laws are maintained at vertices.

The variables k_i and T_i are originally constrained to be real, but one can eliminate the δ-functions in favor of contour integrations and thereby obtain expressions that can be analytically continued in the variables T_i. The set of singularities obtained in this way will be called the singularities required by unitarity. It is shown in an appendix that the set of singularities required by unitarity coincides with the set of singularities obtained in the terms of the perturbation solution of field theory. This allows the extensive body of results concerning singularities of the terms of perturbation theory[23-32] to become immediately available. A possible alternative approach of simply asserting the singularities to be the same as in the terms of the perturbation solution is less satisfactory, since field theory is at once rejected and placed in a fundamental role.

The equations defining the locations of the singularities required by unitarity are relativistically invariant and define manifolds in the space of the scalar invariants of the M function of dimension generally one (complex variable) less than the full dimensionality of the space. The locations of the singularities depend only on the masses m_i corresponding to the various internal lines of the diagram. Cuts terminating at these singularities can be defined by the locus of singularities obtained with the m_i replaced by αm_i, with α a real parameter, independent of i, varying between one and infinity. These cuts separate various sheets of the M function. Postulate E asserts that one particular sheet of each M function, the physical sheet, is free of singularities.

The physical sheet is defined in the following way: Among the cuts there will be one preferred cut that runs over values corresponding to the real momentum-energy vectors for the process described by the M function. The physical values of the M function for the process are required to coincide with the boundary values of the M function on the physical sheet for the "physical-type" limit along this cut. The physical-type limit is the limit approached using points for which the imaginary parts of the physical energy and the physical momentum magnitudes $k = \sqrt{\underline{k} \cdot \underline{k}}$ are positive imaginary. The physical arguments are, of course, the negatives of the mathematical arguments for variables referring to the initial momentum-energy vectors.

The definition of the physical sheet is not arbitrary but is closely tied to unitarity, analyticity and the connection to space-time variables. In order to acquire an orientation, consider the simple case of the scattering of a nonrelativistic particle from an everywhere-finite potential that vanishes for $r > R$. For a particular channel (partial wave),

the radial wave function $f(-k, r)$ that goes asymptotically like $\exp[ikr]$ can be shown to exist for all k, real or complex.[33] Generally, these functions do not vanish at the origin $r = 0$ as is required for a solution, but the combination

$$f(k, r) - (f(k, 0)/f(-k, 0))\ f(-k, r)$$

has this property, and the S-matrix is

$$S(k) = (-1)^{\ell}\ f(k, 0)/f(-k, 0)\ .$$

This function is regular for all values of k for which $f(-k, 0)$ is nonzero. At points of the upper-half k plane, $(\mathrm{Im}\ k > 0)$ for which $f(-k, 0)$ vanishes, the $f(-k, r)$ are (normalizable) eigenfunctions, and the energy eigenvalues are required, by unitarity, to be real. The singularities of the S-matrix in the upper-half k plane are therefore confined to the imaginary axis.[34]

If one transforms to the E-plane and defines there the physical sheet to be the one containing the positive imaginary k axis (on the real E axis), then the physical sheet of the E-plane is free of singularities except for those on the real axis. These singularities have a direct relationship to bound states and physical thresholds, and their positions are fixed by the masses of the stable particles. The corresponding statement for the other sheets is not expected to be valid; resonances and unstable particles lead to singularities in the unphysical sheets.[35-38]

If the condition that the potential vanishes for $r > R$ is replaced by the more realistic requirement that it be representable by a sum of Yukawa potentials, then the numerator function $f(k, 0)$ can have singularities, but these are just those coming from unitarity in the crossed channels[39,40] and are therefore among those required by unitarity.

The definition of the physical sheet for the relativistic many-particle problem is taken to be the generalization of the one occurring for the simple potential scattering problem, and in particular, the momentum vectors are required to move to the region corresponding to exponentials that decrease at large radius as one goes from the physical point into the interior of the physical sheet. This ensures that bound states give singularities below the physical threshold and lying on the physical sheet. The unitarity relation likewise leads to singularities at locations corresponding to these bound states, since they are stable particles. The unitarity relation is therefore expected to hold below the physical thresholds on the physical sheet.

The fact that, for the potential scattering problem, the physical sheet is free of singularities depends jointly on unitarity and the fact that the scattering was from a local potential. Postulate E can therefore be considered some extraction from a locality requirement. The question of exactly what coordinate space conditions are necessary or sufficient for the postulated momentum-space analyticity property lies outside the scope of the present paper.

XIII. CONCLUDING REMARKS

The present study provides two useful results apart from the proofs that were the primary object. First, it has been shown how the appeal to field theoretic concepts can be completely avoided and the new S-matrix formalism built up from simple principles that are relatively secure. This development, which was necessary to the proofs, leads also to a considerable simplification in practical problems, since the shuttling between field theoretic expressions and the scattering functions needed in field theory is eliminated; one works only with the directly interpretable scattering functions. Second, the treatment of spin is in terms of the simple Pauli matrices, and the redundant variables associated with the use of Dirac matrices are not introduced. Thus, for instance, the scattering of two spin $\frac{1}{2}$ particles is described by a spinor function with sixteen matrix elements rather than 256. The troublesome projection operators needed to eliminate the redundant variables of the Dirac theory are not required. Conditions imposed on the M functions by the separate requirements of invariance under spatial reflection, antiparticle conjugation and time reversal are easily handled, as shown in some appendices.

APPENDIX A

Derivation of Unitarity from Postulate A

From postulate A ,

$$(a', a') = (Sa, Sa) = (a, a)$$

for all a . But the quadratic form determines the bilinear form,[41] and so

$$(Sa, Sb) \equiv (a, S^\dagger S\, b) = (a, b)$$

for all a and b . Hence $S^\dagger S = 1$. Therefore $S^\dagger a' = a$ and $SS^\dagger = 1$ is derived from

$$(a, a) = (S^\dagger a', S^\dagger a') = (a', a') .$$

In effect, the completeness requirement may be taken to mean that the metric preserving transformation S is unitary.[42]

APPENDIX B

Massless Particles

For massless particles, which have no rest frame, the procedure used in the text breaks down. In this case the assignment of spin quantum numbers can be related to the direction of motion of the particle. For the spin $\frac{1}{2}$ case, the projection operators analogous to Eqs. (9.9) and (9.10) can be taken as

$$P(k) = \frac{1}{2k_0} (k\cdot\sigma) \quad , \tag{9.9'}$$

and

$$\tilde{P}(k) = \frac{1}{2k_0} (k\cdot\tilde{\sigma}) \quad . \tag{9.10'}$$

These have the same transformation properties as the operators to which they correspond. Unlike the $P(s, k)$ and $\tilde{P}(s, k)$, these operators are idempots and they can therefore be directly interpreted as the projection operators for the two states. This allows the transformation to rest frames to be eliminated. Unlike the nonmassless case, where both spin states could be described using either type of spinor, upper dotted or lower undot, here we have one spin state corresponding to one assignment, and the other spin state corresponding to the other assignment. If we continue to use only the lower indices, then the neutrino will have only one spin state, as is experimentally observed. The V-A interaction is represented by $(\sigma_\mu)\,(\sigma_\mu)$ in the present notation.

The projection operators (9.9') or (9.10') should be inserted for the intermediate states of the unitarity relation, and the symbol $K\cdot\tilde{\sigma}$ appearing in the unitarity relation will include this factor for the case of massless particles.

APPENDIX C

Related Formalisms

In order that the term $\delta_{K'K''}$ be invariant under rotations, the corresponding indices of K and $-\tilde{K}$ must be transformed by inverse matrices; if particle variables transform as lower undotted or upper dotted indices (which transform the same for rotations) then the corresponding antiparticle variables must transform as lower dotted or upper undotted indices. We have arbitrarily chosen to use always lower indices. For a lower undotted index the Lorentz transformation applied to R in constructing M is $(k\cdot\sigma/m)^{1/2}$. Had we chosen to treat the index as an upper dotted index, then the Lorentz transformation would have been rather $(k\cdot\tilde{\sigma}/m)^{1/2}$, which is just the inverse of $(k\cdot\sigma/m)^{1/2}$. Thus one can transform an M function with an upper dotted index to the value it would have taken if the index had been treated as a lower undotted index by multiplying it by $(k\cdot\sigma/m)$. It is purely a matter of choice which one is used. Indeed, in the Dirac formalism both choices are carried along in parallel.

There is no compulsion to take the no-scattering part of S to be $\delta_{K'K''}$. Another rather useful choice is $C_{K'K''}$, the product of the charge conjugation matrices $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. With this choice, particle and antiparticle variables have the same transformation character under rotations and one can, for instance, take all the indices of M to be lower undotted indices. In this case the unitarity relations take the forms

$$\mathcal{C}K''\cdot\tilde{\sigma}\, M(K', -\tilde{K}'') \;+\; [\, \mathcal{C}K'\cdot\tilde{\sigma}\; M(K'', -\tilde{K}') \,]^{*}$$

$$= \; - \; \int\!\!\!\!\!\!\sum M(K', -\tilde{K})\; K\cdot\tilde{\sigma}\; M^{*}(K'', -\tilde{K}) \quad ,$$

and

$$\mathcal{C}^{T}\,(K'\cdot\tilde{\sigma})\; M(K', -\tilde{K}'') \;+\; [\, \mathcal{C}^{T}\,K''\cdot\tilde{\sigma}\; M(K'', -\tilde{K}') \,]^{*}$$

$$= \; - \; \int\!\!\!\!\!\!\sum M(K, -\tilde{K}'')\; K\cdot\tilde{\sigma}\; M^{*}(K, -\tilde{K}') \quad .$$

For $K' \equiv K''$ representing self-conjugate combinations of amplitudes, the CPT transformation followed by an inversion of order takes the right-hand sides of these equations into each other. The left-hand sides are also transformed into each other, except for a factor $(-1)^{N}$ needed to change $\mathcal{C}$ to $\mathcal{C}^{T}$ and a factor $\sigma_{K'}$ needed to reverse the ordering. The connection between spin and statistics then follows as before.

APPENDIX D

Higher Spins

The formulas given in the text apply directly to spin zero and spin $\frac{1}{2}$ particles. For particles of higher spin, one constructs in the rest frame an appropriate combination of the elementary spin $\frac{1}{2}$ spinors using Clebsch-Gordan coefficients or spin state projection operators. The Lorentz transformations that take the R matrix, which refers to the rest frame spin states, to the M matrix, commute with $\underset{\sim}{S} \cdot \underset{\sim}{S}$ and hence with the projection operators. The projection operators can be considered either as acting directly on the M functions, or as contained in the M functions as factors. In the latter case the equations of the text apply unaltered, but with the index μ simply a set of spinor indices. In the former case the $K \cdot \tilde{\sigma}$ appearing in the unitarity relation is replaced by $K \cdot \tilde{\sigma}\ P(S)$ where $P(S)$ is the projection operator. For instance, $P(S) = \frac{1}{4}\ (3 + \underset{\sim}{\sigma}^{(1)} \cdot \underset{\sim}{\sigma}^{(2)})$ for the case of a spin-1 particle. The projection operators commute with the $K \cdot \tilde{\sigma}$ and can be placed on either side, or on both sides of it.

APPENDIX E

Invariance Conditions

The transformations

$$(k_1^{\,o}, \underset{\sim}{k}_i, \mu_i, t_i) \rightarrow (k_i^{\,o}, \underset{\sim}{k}_i, \mu_i, t_i)_p \equiv (k_i^{\,o}, -\underset{\sim}{k}_i, \mu_i, t_i) \quad ,$$

$$(k_i^{\,o}, \underset{\sim}{k}_i, \mu_i, t_i) \rightarrow (k_i^{\,o}, \underset{\sim}{k}_i, \mu_i, t_i)_c \equiv (k_i^{\,o}, \underset{\sim}{k}_i, \mu_i, -t_i) \quad ,$$

and

$$(k_i^{\,o}, \underset{\sim}{k}, \mu_i, t_i) \rightarrow (k_i^{\,o}, \underset{\sim}{k}_i, \mu_i, t_i)_T \equiv (-k_i^{\,o}, \underset{\sim}{k}, \mu_i, -t_i)$$

take the variables associated with a given experimental measurement to those associated with certain transformed experimental measurements connected to the original ones by spatial reflection, antiparticle conjugation, and time reversal, respectively. Correspondingly, we can define the transformations

$$R(K', -\tilde{K}'') \rightarrow R_p(K', -\tilde{K}'') \equiv R(K'_p, -\tilde{K}''_p) \quad ,$$

$$R(K', -\tilde{K}'') \rightarrow R_c(K', -\tilde{K}'') \equiv R(K'_c, -\tilde{K}''_c) \quad ,$$

and

$$R(K', -\tilde{K}'') \rightarrow R_T(K', -\tilde{K}'') \equiv R(K'_T, -\tilde{K}''_T) \quad .$$

If the equations

$$R(K', -\tilde{K}'') = \sigma_p R_p(K', -\tilde{K}'') \quad ,$$

$$R(K', -\tilde{K}'') = \sigma_c R_c(K', -\tilde{K}'') \quad ,$$

and

$$R(K', -\tilde{K}'') = \sigma_T R_T(K', -\tilde{K}'')$$

are satisfied, where σ_p, σ_c and σ_T are phase factors, then the relationships between probabilities for the transformed measurements are

the same as for the original experiments.

In the time-reversal operation the change of sign of the $k_i^{\,o}$ effects the required interchange between initial and final states and the reversal of physical momentum and spin. The transformation to antiparticles that it also induces is compensated by the change $t \to -t$.

In the operations of charge conjugation and time reversal the change $t \to -t$ can be generated by an (up to a phase) equivalent change $k^a \leftrightarrow k^b$, $\mu^a \leftrightarrow \dot{\mu}^b$ for the class of R functions in which the particle and antiparticle variables can be grouped in corresponding pairs. The k^a and μ^a are the particle momenta and spin indices, and k^b and $\dot{\mu}^b$ are the antiparticle quantities. If the R functions are considered matrices with the undotted indices on the left and the dotted indices on the right, then the spin transformation is generated by

$$R \to \mathcal{C}^{-1}\, R^{tr}\, \mathcal{C} ,$$

where R^{tr} is the transpose of R and $\mathcal{C}$ is the product of the elementary charge conjugation operators C defined by

$$C\, \underset{\sim}{\sigma}\, C^{-1} = -\underset{\sim}{\sigma}^{tr} .$$

The operator C effects the transformation between dotted and undotted indices. When this form of the antiparticle conjugation transformation is used the three operations are, to within phases, given by[43]

P: $k_i^{\,o} \leftrightarrow k_i^{\,o}$, $\underset{\sim}{k}_i \to -\underset{\sim}{k}_i$, $\underset{\sim}{\sigma} \to \underset{\sim}{\sigma}$;

C: $k_i^{\,a} \leftrightarrow k_i^{\,b}$, $\underset{\sim}{\sigma} \to -\underset{\sim}{\sigma}$, T. of O. ;

T: $(k_i^{\,o})^a \leftrightarrow -(k_i^{\,o})^b$, $\underset{\sim}{k}_i^{\,a} \leftrightarrow \underset{\sim}{k}_i^{\,b}$, $\underset{\sim}{\sigma} \to -\underset{\sim}{\sigma}$, T. of O.

Here "T. of O." represents an inversion of the order of occurrence of all Pauli spin matrices σ, and k_i^a and k_i^b are corresponding particle and antiparticle momentum-energy vectors.

The requirement of invariance under antiparticle conjugation imposes constraints on the M functions describing processes in which both members of each particle-antiparticle pair occur in the same configuration, initial or final. Time-reversal invariance imposes constraints on the M functions describing processes in which each particle occurs in both the initial and final configurations. The analyticity properties allow these constraints to be carried over to the M functions for processes in which these constraints are removed.

It is sometimes convenient to express the variables $(K', -\tilde{K}")$ in the form $(K^a; K^b)$, where the physical particle variables are in the first group and the antiparticle variables are in the second. One may then write

$$M(K^a; K^b) = \sqrt{K^a \cdot \sigma}\, R(K^a; K^b) \sqrt{K^b \cdot \sigma} \quad ,$$

where the contractions are now always on adjacent indices. The variables represented by the set $(K^a; K^b)$ will be taken to be normal-ordered.

The symmetry operations and conditions have so far been expressed only in terms of the R functions. Let the applications of these same operations to the M functions be represented by $M_p(K^a; K^b)$, $M_c(K^a; K^b)$ and $M_T(K^a; K^b)$, where the new way of writing the variables is introduced for clarity in what follows. We define now the more complicated quantities

$$M_{\overline{p}}(K^a; K^b) \equiv K_p^{a}\cdot\tilde{\sigma}\ M_p(K^a; K^b) K_p^{b}\cdot\tilde{\sigma}$$

$$\equiv K_p^{a}\cdot\tilde{\sigma}\ \sqrt{K_p^{a}\cdot\sigma}\ R_p(K^a; K^b)\sqrt{K_p^{b}\cdot\sigma}\ K_p^{b}\cdot\tilde{\sigma}$$

$$= \sqrt{K_p^{a}\cdot\tilde{\sigma}}\ R_p(K^a; K^b)\ \sqrt{K_p^{b}\cdot\tilde{\sigma}}$$

$$= \sqrt{K^a\cdot\sigma}\ R_p(K^a; K^b)\ \sqrt{K^b\cdot\sigma}\quad ,$$

$$M_{\overline{c}}(K^a; K^b) \equiv K^a\cdot\sigma\ M_c(K^a; K^b)\ K^b\cdot\sigma$$

$$= K^a\cdot\sigma\ (\sqrt{K^a\cdot\sigma}\ R(K^a; K^b)\sqrt{K^b\cdot\sigma}\ )_c\ K^b\cdot\sigma$$

$$= K^a\cdot\sigma\ (\sqrt{K^a\cdot\tilde{\sigma}}\ R_c(K^a; K^b)\sqrt{K^b\cdot\tilde{\sigma}}\ )\ K^b\cdot\sigma$$

$$= \sqrt{K^a\cdot\sigma}\ R_c(K^a; K^b)\ \sqrt{K^b\cdot\sigma}\quad ,$$

and

$$M_{\overline{T}}(K^a; K^b) \equiv M_T(K^a; K^b)$$

$$= (\sqrt{K^a\cdot\sigma}\ R(K^a; K^b)\sqrt{K^b\cdot\sigma}\ )_T$$

$$= \sqrt{K^a\cdot\sigma}\ R_T(K^a; K^b)\ \sqrt{K^b\cdot\sigma}\quad .$$

Inspection of the last lines in each of these equations shows that the conditions of invariance $R = \sigma_p R_p$, $R = \sigma_c R_c$ and $R = \sigma_T R_T$ imply $M = \sigma_p M_{\overline{p}}$, $M = \sigma_c M_{\overline{c}}$ and $M = \sigma_T M_{\overline{T}}$, respectively. These invariance conditions, which might at first appear cumbersome for the cases of parity and antiparticle conjugation, are in fact quite easy to apply and lead immediately to simple forms for the M functions satisfying the various

conditions, as examples given in subsequent appendices show.

A double application of any of the symmetry operations shows that the phase factors σ can only be ± 1. The phase factors need not be strictly independent of the variables $(K', -\tilde{K}'')$; parts corresponding to amplitudes that cannot interfere may have different factors. Both signs may occur therefore, and the set of M functions separate into four parts corresponding to the two signs of σ_p and σ_T . The sign of σ_c is fixed by the CPT theorem and the connection between spin and statistics.

APPENDIX F

Form of the M Function for a Scattering of Spin $\frac{1}{2}$ Particle by Spin Zero Particle

Relativistic invariance requires the M functions to be a sum of terms each of which is the product of an invariant scalar times a product of spin operators $k_i \cdot \sigma$ and $k_j \cdot \tilde{\sigma}$. These two types of operators must appear alternately and the former type must appear on each end. Letting k^a and k^b be the physical momentum-energy vectors associated with the particle and antiparticle variables of the spin $\frac{1}{2}$ particle, and letting the corresponding mass be unity, one obtains as the condition for spatial reflection invariance:

$$\begin{aligned} M_{\overline{p}}(K^a; K^b) &\equiv k_p{}^a \cdot \tilde{\sigma}\ M_p(K^a; K^b)\ k_p{}^b \cdot \tilde{\sigma} \\ &= k^a \cdot \sigma\ M(K_p{}^a; K_p{}^b)\ k^b \cdot \sigma \\ &= \pm M(K^a; K^b) \quad . \end{aligned}$$

Using the relations

$$k_p \cdot \sigma = k \cdot \tilde{\sigma} \qquad \text{and} \qquad k \cdot \sigma\ k \cdot \tilde{\sigma} = k \cdot \tilde{\sigma}\ k \cdot \sigma = 1 ,$$

one immediately obtains as the general solution:

$$M(K^a, K^b) = v \cdot \sigma \pm (k^a \cdot \sigma)(v \cdot \tilde{\sigma})(k^b \cdot \sigma) ,$$

where v is a combination of the momentum-energy vectors of the problem. The relations

$$a \cdot \sigma\ b \cdot \tilde{\sigma} + b \cdot \sigma\ a \cdot \tilde{\sigma} = a \cdot \tilde{\sigma}\ b \cdot \sigma + b \cdot \tilde{\sigma}\ a \cdot \sigma = 2a \cdot b$$

and

$$a \cdot \sigma\ b \cdot \tilde{\sigma}\ c \cdot \sigma = i[abc] \cdot \sigma + a \cdot \sigma(b \cdot c) - b \cdot \sigma(a \cdot c) + c \cdot \sigma(a \cdot b) ,$$

where $[abc]_\lambda = a^\mu b^\nu c^\rho \epsilon_{\mu\nu\rho\lambda}$ are useful ($\epsilon_{0123} = +1$). The elementary combinations with well-defined σ_p and σ_T are

$$s_1 = k^a \cdot \sigma + k^b \cdot \sigma \ , \qquad \sigma_T = +1 \ , \qquad \sigma_p = +1 \ ;$$

$$s_2 = k^a \cdot \sigma - k^b \cdot \sigma \ , \qquad \sigma_T = -1 \ , \qquad \sigma_p = -1 \ ;$$

$$s_3 = n \cdot \sigma + k^a \cdot \sigma \, n \cdot \tilde{\sigma} \, k^b \cdot \sigma \ , \qquad \sigma_T = +1 \ , \qquad \sigma_p = +1 \ ;$$

$$s_4 = n \cdot \sigma - k^a \cdot \sigma \, n \cdot \tilde{\sigma} \, k^b \cdot \sigma \ , \qquad \sigma_T = +1 \ , \qquad \sigma_p = -1 \ ;$$

where $n = p^a + p^b$ is the sum of the physical energy-momentum of the other particle. The contribution with n replaced by $(p^b - p^a) = (k^a - k^b)$ gives zero for s_3, and s_4 becomes twice s_2 so that nothing new is added. One can also use

$$n' = n - \frac{(k^a + k^b)[(k^a + k^b) \cdot n]}{(k^a + k^b) \cdot (k^a + k^b)}$$

if the denominator is nonvanishing. The vector n' is normal to $k^a + k^b$ and $k^a - k^b$ and hence to k^a and k^b. All other combinations can be expressed in terms of these four forms and we may write

$$M = \Sigma \ a_i \ s_i = \Sigma \ a'_i \ s'_i \ ,$$

where s'_i represents the s_i with n' is used in place of n. Then, with $\tilde{s}_i$ the operator obtained from s_i by the transformation $\underline{\sigma} \to -\underline{\sigma}$ and T. of O. one obtains

$$\frac{1}{2} \ \mathrm{Tr} \ \tilde{s}'_i \ M = a'_i \ N'_i \ ,$$

where

$$\frac{}{N_i} = \begin{cases} (k^a + k^b)(k^a + k^b) , \\ (k^a - k^b)(k^a - k^b) , \\ (n' \cdot n')(k^a - k^b)(k^a - k^b) , \\ (n' \cdot n')(k^a + k^b)(k^a + k^b) . \end{cases}$$

The expressions have been written down in terms of the physical momentum energy vectors in the case where the particles occur in both the initial and final configuration. The vectors k^b and p^b occurring here should generally be replaced by the negatives of the corresponding mathematical momentum-energy vectors occurring in the arguments of M. The expressions are then valid for all the processes described by the M function.

APPENDIX G

The System of Two Identical Spin $\frac{1}{2}$ Particles

There are six combinations of the elementary spin operators s_i that are invariant under spatial reflection and time reversal:

$$T_1 = s_1 s_1 ,$$

$$T_2 = s_2 s_2 ,$$

$$T_3 = s_3 s_3 ,$$

$$T_4 = s_4 s_4 ,$$

$$T_5 = s_1 s_3 + s_3 s_1 ,$$

and

$$T_6 = s_1 s_3 - s_3 s_1 .$$

The remaining ten combinations change sign under at least one of the two operations. The final combination T_6 changes sign under a simultaneous interchange of <u>both</u> the initial and final particles and is ruled out for identical particles.

There is also the possibility of using β-decay forms, in which there is a contraction on the vector index of operators σ_μ or $\tilde{\sigma}_\nu$ appearing in the spin spaces of the different particles. In terms of the combinations

$$t = k^a \cdot \sigma + p^a \cdot \sigma = k^b \cdot \sigma + p^b \cdot \sigma ,$$

$$d^a = k^a \cdot \sigma - p^a \cdot \sigma ,$$

and

$$d^b = k^b \cdot \sigma - p^b \cdot \sigma ,$$

convenient forms are

$$(\sigma_\mu)(\sigma_\mu) \ , \qquad \sigma_T = +1 , \quad \epsilon = -1 ;$$

$$(t\ \tilde{\sigma}_\mu\ t)(t\ \tilde{\sigma}_\mu\ t) \ , \qquad \sigma_T = +1 , \quad \epsilon = -1 ;$$

$$(d^a\ \tilde{\sigma}_\mu\ d^b)(d^a\ \tilde{\sigma}_\mu\ d^b) \ , \qquad \sigma_T = +1 , \quad \epsilon = +1 ;$$

$$(t\ \tilde{\sigma}_\mu\ d^b)(t\ \tilde{\sigma}_\mu\ d^b) + (d^a\ \tilde{\sigma}_\mu\ t)(d^a\ \tilde{\sigma}_\mu\ t) \ , \qquad \sigma_T = +1 , \quad \epsilon = -1 ;$$

and

$$(d^a\ \tilde{\sigma}_\mu\ d^b)(t\ \tilde{\sigma}_\mu\ t) + (d^a\ \tilde{\sigma}_\mu\ t)(t\ \tilde{\sigma}_\mu\ d^b)$$

$$+ \ (t\ \tilde{\sigma}_\mu\ d^b)(d^a\ \tilde{\sigma}_\mu\ t) + (t\ \tilde{\sigma}_\mu\ t)(d^a\ \tilde{\sigma}_\mu\ d^b)$$

$$\sigma_T = +1 , \quad \epsilon = +1 ;$$

where the metric $(+,-,-,-)$ is assumed and where ϵ is the sign change under interchange of two initial (or two final) particles. This interchange induces the transformation $k^b \leftrightarrow p^b$ (or $k^a \leftrightarrow p^a$) and an interchange of the initial (final) spin state. The effect of the latter can be obtained using the rearrangement theorem

$$(\langle a\ |\sigma_\mu|\ b\rangle)(\langle c\ |\sigma_\mu|\ d\rangle) \ = \ -(\langle a|\sigma_\mu|\ d\rangle)(\langle c\ |\sigma_\mu|\ b\rangle) \ .$$

The sum and difference of each of these terms with the one obtained from it by application of the transformation $\bar{p}$ are elementary spin functions that have well-defined values for σ_p , σ_T and ϵ .

APPENDIX H

The Location of the Singularities Required by Unitarity

With the substitution of the definition

$$\tilde{M}(K', -\tilde{K}'') \equiv M^*(K''^*, -\tilde{K}'^*)$$

the unitarity relation becomes

$$M(K', -\tilde{K}'') + \tilde{M}(K', -\tilde{K}'') = -\sum \int M(K', -\tilde{K}) \, K\cdot\tilde{\sigma} \, \tilde{M}(K, -\tilde{K}'') \, ,$$

a form that is suitable for analytic continuation from the originally real values of the momentum-energy vectors. The right-hand side contains a sum of terms each of which contains a factor of the form

$$\int \prod_i \left[\frac{d^4k_i}{(2\pi)^4} \, 2m_i \, 2\pi \, \theta(k_i^{\,o}) \, \delta(k_i^{\,2} - m_i^{\,2}) \right] (2\pi)^4 \, \delta^4(\Sigma k_i - T) \, .$$

The delta functions represent the mass constraints and the energy-momentum conservation law, and T is the sum of the (external) momentum-energy vectors in K' or K''.

Because the internal momentum vectors k_i are constrained to be real, the factor above vanishes for $T^2 < (\Sigma m_i)^2$ but not for $T^2 > (\Sigma m_i)^2$ and introduces, generally, a singularity at $T^2 = (\Sigma m_i)^2$ into one or both of the terms on the left of the unitarity relation. It may be assumed for the purpose of the construction that the singularity occurs in both terms on the left. The other alternative would lead only to a smaller set of singularities and we wish to obtain the largest possible set.

The singularities occurring in the various expressions of the above form arise purely from the limits of integration in the unitarity relation--those that would occur even if the M functions on the right of

the unitarity relation were free of singularities. Singularities in these latter would lead generally to additional singularities in the M functions on the left. If one substitutes the singular parts of the M functions obtained by the above first application of unitarity back into the right-hand side of the unitarity relation and proceeds by iteration, expressions of the form

$$\int \prod_i \left[\frac{d^4 k_i}{(2\pi)^4} \, 2m_i \, \theta(k_i^{\,o}) \, \delta(k_i^{\,2} - m_i^{\,2}) \right] \prod_j (2\pi)^4 \, \delta(\Sigma_j - T_j)$$

are obtained. Here Σ_j and T_j are subsums over the internal and external momentum-energy vector respectively, and the $\delta(\Sigma_j - T_j)$ give momentum-energy conservation in the various intermediate states.

The iterative processes may be represented diagrammatically. For instance, Fig. H-1 gives as one possible sequence. The lines represent the particles (on their mass shells) and the steps represent the substitution of particular terms from the unitarity relation. In the second step, contributions to the new M functions in which certain of the particles do not scatter have been displayed. The singularities corresponding to all possible finite sequences will be called the singularities required by unitarity.

The momentum-energy variables are originally constrained to be real. As we are interested in singularities also for complex T_i, the expressions above must be put in a form permitting analytic continuation. In particular, the delta functions must be removed.

Near the point $T_i = 0$ the form is, for nonmassless particles ($m_i > 0$), clearly regular. The analytic continuation will be started from this region. Because $N \geqslant M$, the M momentum-space delta functions

can be used to eliminate the last M momentum integrations (by an appropriate reordering of variables, if needed). The energy-conservation delta-functions may be replaced using the relation $2\pi\ \delta(x) = \lim\ (\epsilon \to 0^+)\ 2\epsilon/(x^2 + \epsilon^2)$, and the energy integrations over $2\pi\ \theta(k_i^{\,o})\ \delta(k_i^{\,2} - m_i^{\,2})$ replaced by contour integrations clockwise about the poles at $k_i^{\,o} = \sqrt{m_i^{\,2} + \underset{\sim}{k}_i^{\,2}} \equiv \omega_i$ of $i/(k_i^{\,2} - m_i^{\,2})$.

The functions $2\epsilon/(x^2 + \epsilon^2)$ give dipoles in the planes of the various energy variables $k_i^{\,o}$, the locations of which depend on the positions of the contours in the remaining energy variables, and on the energy parts of the external variables T_i . For $T_i = 0$ these conservation dipoles lie in the left-half energy planes, as long as the contours are all confined to the right-half planes. The contours may therefore be deformed to run up just right of the imaginary axis, and be completed with large semicircles to the right. The dipoles then lie just to the left of the imaginary axis, with extensions to the left corresponding to the semicircles to the right. If the energies of the T_i are now increased, the dipoles move right and the contours must be shifted right to avoid these advancing singularities. Singularities in the <u>form</u> cannot occur until the T_i becomes such that it is no longer possible to distort the contours to avoid both the advancing dipole singularities and the fixed mass singularities, for arbitrarily small ϵ .[25-27]

For a visual understanding of the boundary of the singularity free region it is helpful to bear in mind a plot of the mass singularities. These lie at the simultaneous solutions of

$$(\mathrm{Re}\ k^o)^2 + (\mathrm{Im}\ \underset{\sim}{k})^2 = m^2 + (\mathrm{Re}\ \underset{\sim}{k})^2 + (\mathrm{Im}\ k^o)^2$$

$$\mathrm{Re}\ k^o\ \mathrm{Im}\ k^o = \mathrm{Re}\ \underset{\sim}{k} \cdot \mathrm{Im}\ \underset{\sim}{k} \quad .$$

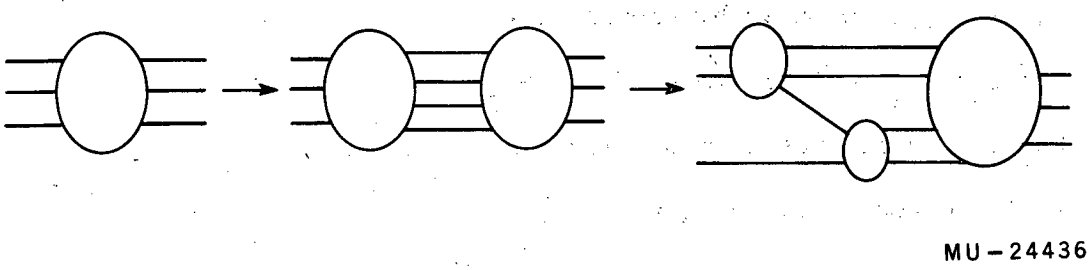

Fig. H-1. Typical development of a diagram by iterative substitution of singular parts.

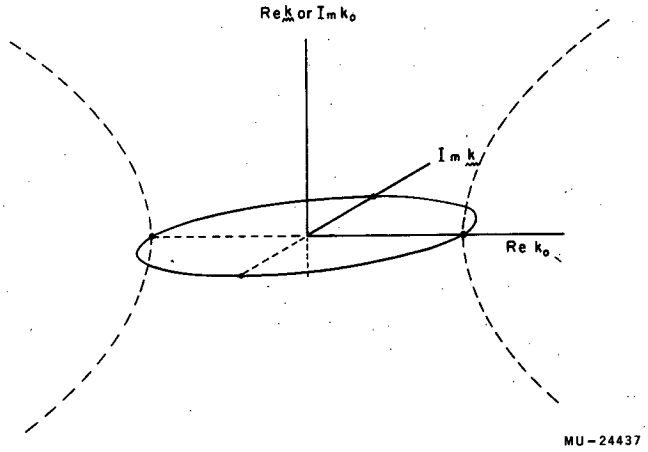

Fig. H-2. Boundary of regular region.

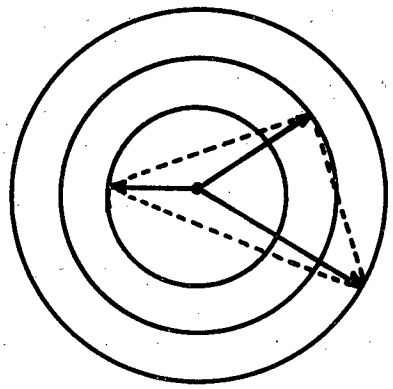

Fig. H-3. Intersection of contours with $\mathrm{Im}\ k_0 = \mathrm{Re}\ \underset{\sim}{k} = 0$ manifold at a vertex singularity.

Figure H-2 represents an hyperbola of revolution about the vertical axis, which represents either $\text{Im } k_0$ or $\text{Re } \underset{\sim}{k}$. The interior, or funnel, region of the diagram is free of mass singularities. For $T_i = 0$ the important parts of the contours run up the vertical axis. If, with increasing T_i, the contours can be kept inside their respective funnels, for arbitrarily small ϵ, then the forms remain singularity free.

For the special case in which the vector parts of the T_i can be taken to vanish, it is possible to simultaneously keep all contours in the vertical planes $\text{Im } \underset{\sim}{k}_i = 0$, for which the mass singularities extend least far to the left. The point in T_i space at which the contours can no longer be confined to the funnels is at the threshold $T_j = \Sigma_j m_i$. If, on the other hand, the imaginary vector parts of the T_i are nonvanishing, the various contours cannot all be confined to the planes $\text{Im } \underset{\sim}{k}_i = 0$, and mass singularities extending below the threshold energy $k_i^{\,o} = m_i$ can become important. For instance, for the simplest vertex part, the limiting point is represented in Fig. H-3. The solid vectors represent, to within a sign, the points where the contours pass through the region $\text{Im } k^o = \text{Re } \underset{\sim}{k} = 0$, and they are required to lie inside their respective circles. The dotted lines represent the external momenta. If the dotted figure can, by an appropriate translation, be made to lie with each of its vertices inside its associated circle, then the form remains nonsingular. This provides a direct geometric derivation of the triangle condition of Karplus, Sommerfield and Wichmann.[23] The same arguments carry over to more complicated diagrams.

In the limit $\epsilon \to 0$ only points simultaneously in the neighborhood of all the conservation dipoles can contribute. One can deform the contours so that they always stay close to the points at which all the conservation dipoles overlap and consider, therefore, the important parts of the contours

to be constrained by the conservation laws. It is when the contours so constrained cannot be kept away from the mass singularities that singularities in the forms can occur.

The conditions on the k_i that must be satisfied at a singularity are the conservation laws, the mass constraints and the condition that it is not possible to distort the contours so as to maintain the conservation laws but avoid the mass singularities. Variations of the k_i consistent with the conservation laws are conveniently expressed using Feynman loops (closed loops in the diagram). It is possible to choose the signs of momentum-energy vectors so that the vectors point in the direction of the closed loops, if these are also chosen appropriately. If the variation of the momentum of the jth loop is δq_j, then the variation of the vector k_i is

$$\delta k_i = \Sigma \epsilon_{ij} \delta q_j ,$$

where the ϵ_{ij} are coefficients equal to one or zero, depending on whether or not the jth loop passes along the ith line. The variation of quantity k_i^2 is then

$$\delta k_i^2 = 2\Sigma k_i \epsilon_{ij} \delta q_j ,$$

where the δq_j can be chosen arbitrarily.

The δk_i^2 can be fixed arbitrarily by an appropriate choice of the δq_j, unless it is possible to find some α_i such that

$$\sum_i \alpha_i k_i \epsilon_{ij} = 0 \qquad \text{(all j)} ,$$

$$\alpha_i \neq 0 \qquad \text{(some i)} .$$

(condition A)

This is the condition that the variations δk_i^2 be linearly dependent. If condition A cannot be satisfied, then it is possible to fix the δk_i^2 arbitrarily and thereby to distort the contours away from the mass singularities without conflicting with the conservation requirements. The necessary condition for a singularity is, therefore, the simultaneous validity of the conservation laws, the mass constraints, and condition A. This is just the result obtained by Landau[24] and others[27] as the necessary condition for singularities in the terms of the perturbation expansion of field theory.

If condition A is satisfied, the δk_i^2 cannot be arbitrarily fixed. However, it may still be possible to distort the contours away from the singularities. For instance, if the contours are pinched by conservation singularities coming from the side $k_i^2 < m_i^2$, then the singularity of the form can occur only if it is not possible to make all the δk_i^2 simultaneously negative. This will be the case if it is possible to satisfy condition A with $\alpha_i \geqslant 0$.

The dimensionality of the manifolds of singularities are determined by counting unknowns and equations. There are N unknown vectors k_i, and M vector equations from the conservation laws. There are N-M vector equations in condition A, so the number of vector equations equals the number of vector unknowns. There are N scalar equations, the mass conditions, and N variables α_i, of which one is an arbitrary scale factor. Thus there is one more equation than determinable unknowns, and one constraint is placed on the external variables. Since the equations are relativistically invariant the locations of the singularities depend only on scalar invariants and the manifold of singularities is a manifold whose position is defined in terms of scalar invariants of dimension one complex variable less than that of full space.[44]

APPENDIX I

The Physical Sheet

The equations giving the locations of the singularities required by unitarity are defined over the entire space of the T_i. However, the solutions of physical interest are those in the manifold defined by the mass constraints. Thus instead of starting the continuation of the function giving the singularities from $T_i = 0$, which may not be in the physical manifold, it is more appropriate to consider only those T_i in the physical manifold. In this manifold the cuts specifying the physical sheet are defined as the locus of singular points under the scale transformation $m_i \to \alpha m_i$, where m_i are the internal masses and α is a scale parameter ranging from infinity to unity. As α decreases the mass singularities converge, and the points $T_i(\alpha)$ are defined as the values of T_i for which the pinching of contours cannot be avoided as the value α is assumed. The cuts are the locus of points $T_i(\alpha)$ for α real and greater than unity.

In defining the M functions, the no-scattering parts of the S functions were identified by their dependences on the volume of space-time, and were separated out. More generally, the S function can be separated into many parts on the basis of volume dependence. The contribution to the scattering function corresponding to processes in which various subgroups of particles interact only among themselves will be expected to have one added factor of the volume of space-time for each division into subgroups. This comes from the integration of the interaction center of each group over all of space-time. Correspondingly, in momentum-energy space there will be a separate conservation law constraint corresponding to each subgroup. The M function is then separable, generally, into parts constrained

by different added conservation laws. These parts are defined over different manifolds and the physical sheet is a collection of physical sheets, one for each manifold.

The analyticity requirement allows the unitarity equations for the various parts of the M functions to be isolated and treated separately. Thus the M function appearing on the left of the unitarity relation can be considered the nonseparable part of the indicated M function, and the contributions on the right correspondingly limited. The remaining contributions to the unitarity relation will be identically satisfied if the unitarity relations for all nonseparable parts are satisfied, provided the separable parts of an M function are the products of the nonseparable parts of the M functions corresponding to the appropriate subgroups. This decomposition law can be considered either an anzatz or an added postulate. But it probably follows from the postulates already given.

In the construction of the singularities required by unitarity the δ-function expressions for the density of states factors have been expressed as a limit $\epsilon \to 0$ of functions defined for all T_i. This alternate expression gives the basis for practical calculations based on unitarity.

The unitarity relation given in the text was derived using the condition that $S_0(K', -\tilde{K}')$, the no-scattering part of the normal ordered S function, was unity. More generally, the unitarity relations would read

$$M(K', -\tilde{K}'')\, S_0^*(K'', -\tilde{K}'') + S_0(K', -\tilde{K}')\, M^*(K'', -\tilde{K}')$$

$$= -\int M(K', -\tilde{K})\, K\cdot\tilde{\sigma}\, M^*(K'', -\tilde{K}) ,$$

and

$$M(K', -\tilde{K}'')\, S_0^*(K', -\tilde{K}') + S_0(K'', -\tilde{K}'')\, M^*(K'', -\tilde{K}')$$

$$= - \int\!\!\!\!\!\!\sum M(K, -\tilde{K}'')\, K\cdot\tilde{\sigma}\, M^*(K, -\tilde{K}') \ ,$$

where $S_0(K', -\tilde{K}')$ is now permitted to be an arbitrary phase factor. For $K' = K''$, application of the CPT transformation and inversion of order to each M function takes the first form into

$$(-1)^N \sigma_{K'} \Big[M(\tilde{K}'_T, -K'_T)\, S_0^*(K', -\tilde{K}') + S_0(K', -\tilde{K}')\, M^*(\tilde{K}'_T, -K'_T)$$

$$= - \int\!\!\!\!\!\!\sum M(\tilde{K}_T, -K'_T)\, K\cdot\tilde{\sigma}\, M^*(\tilde{K}_T, -K'_T) \Big] \ ,$$

where K_T represents the set K with transposed order. This is almost the same as a special case of the second form of the unitarity relation and one can deduce that

$$S_0(\tilde{K}'_T, -K'_T) = S_0(K', -\tilde{K}')\, (-1)^N \sigma(K') \ .$$

If one takes the $S_0(K', -\tilde{K}')$ to be unity, when K' contains only particle variables--no antiparticle variables--then the only solution consistent with the decomposition law is

$$S_0(K', -\tilde{K}') = (-1)^{A(K')} \ ,$$

where $A(K')$ is the number of abnormal antiparticles in the set K'. Here it has been assumed that either the symmetric or antisymmetric case obtains and the abnormal particles are those having the abnormal symmetry relative to the interchange of the particle and antiparticle variables. This form of the unitarity relation, with the $(-1)^{A(K')}$ on the left, leads to a completely different type of analytic structure. Consider an analytic continuation from the case where $K' = K''$ contains only particle

variables, to the case where one abnormal final particle has been carried over to an initial antiparticle. In this second case $A(K')$ and $A(K'')$ differ by unity and it is the difference rather than the sum of the two M functions that appears on the left. If there is a region below both thresholds where the right-hand sides of both equations vanish, as for instance occurs in the scattering of the smallest mass particles, then both the sum and difference of the two M functions vanish and hence so must the M functions themselves. Also, on the right of the unitarity relation the analytic continuation takes various contributions into contributions to the new unitarity relation, but sometimes with the wrong sign. These remarks suggest that the abnormal statistics are inconsistent with the postulates, even without the requirement that self-conjugate combinations of amplitudes can be considered observable.

If one uses an indefinite metric of the form $(-1)^{A(K')}$, then the above-mentioned factors $(-1)^{A(K')}$ occurring in the (pseudo) unitarity relation are moved to the right-hand side where they produce no conflict with analyticity.

APPENDIX J

1. Notes on Spinor Analysis

The following notes on spinor analysis were valuable to readers of a preliminary draft.

Four-vectors and 2-by-2 matrices can be placed in a one-to-one correspondence by the relation

$$V = v^{\mu} \sigma_{\mu} \equiv v^{o} \sigma_{o} + \underset{\sim}{v} \cdot \underset{\sim}{\sigma} ,$$

where $\underset{\sim}{\sigma}$ is the Pauli spin matrix vector and σ_o is the unit matrix. The determinant of V is

$$\det V = (v^o)^2 - \underset{\sim}{v} \cdot \underset{\sim}{v} \equiv v^{\mu} v_{\mu} ,$$

where $v_0 = v^o$ and $v_i = -v^i$. Consider a transformation $V \rightarrow V' = A V B$. If $\det A = \det B = 1$ (A and B unimodular), then $\det V = \det V'$, and $v^{\mu} v_{\mu} = v'^{\mu} v'_{\mu}$. The transformation leaves squares of all four vectors invariant. It consequently leaves inner products $v^{\mu} w_{\mu}$ invariant, as one sees by considering the squares of $v + w$ and $v - w$.

If the matrix V is Hermitian, then the vector v is real. The requirement that Hermitian matrices stay Hermitian under $V \rightarrow V'$ demands $B = \overline{A}$, where bar denotes Hermitian conjugate. The transformations generated by $V \rightarrow V'$ with $B = \overline{A}$ are the <u>real</u> Lorentz transformations. If the constraint $B = \overline{A}$ is relaxed, one obtains the complex Lorentz transformations.

The Lorentz transformation matrix $L^{\mu}{}_{\nu}(A, B)$ is defined by

$$v'^{\mu} = L^{\mu}{}_{\nu}(A, B)\, v^{\nu} ,$$

and it is clearly a linear transformation.

Unimodular 2-by-2 matrices have six degrees of freedom and can be written in the form

$$A = \exp\left(\frac{1}{4}\,\theta^{\mu\nu}\,\sigma_\mu\,\sigma_\nu\right) = \exp\left(\frac{i}{2}\,\underset{\sim}{\Omega}\cdot\underset{\sim}{\sigma} + \frac{1}{2}\,\underset{\sim}{\lambda}\cdot\underset{\sim}{\sigma}\right),$$

where

$$\theta^{\mu\nu} = -\theta_{\nu\mu}, \qquad \lambda^i = 2\theta^{i0}, \quad \text{and} \quad \Omega^i = \epsilon^{ijk}\,\theta_{jk}.$$

The matrix B will be defined as

$$B = \exp\left(-\frac{i}{2}\,\underset{\sim}{\Omega}\cdot\underset{\sim}{\sigma} + \frac{1}{2}\,\underset{\sim}{\lambda}\cdot\underset{\sim}{\sigma}\right).$$

Then for real $\theta^{\mu\nu}$ one has $B = \overline{A}$, and real Lorentz transformations are generated by real $\theta^{\mu\nu}$. With complex $\theta^{\mu\nu}$ the A and B can be arbitrary independent unimodular matrices.

The real Lorentz transformation generated by the unitary $A_r = \exp\left[\frac{i}{2}\,\underset{\sim}{\Omega}\cdot\underset{\sim}{\sigma}\right]$, with real $\underset{\sim}{\Omega}$, generate pure rotations. For example,

$$A_r = \exp\left[\frac{i}{2}\,\theta\,\sigma_3\right] = \cos\frac{\theta}{2} + i\,\sigma_3\sin\frac{\theta}{2}$$

gives

$$v'_1 = v_1\cos\theta + v_2\sin\theta,$$

and

$$v'_2 = v_2\cos\theta - v_1\sin\theta.$$

The pure time-like real Lorentz transformations are generated by the Hermitian (not unitary) matrices

$$A_t = \exp\left[\frac{\theta}{2}\,\underset{\sim}{\sigma}\cdot\underset{\sim}{n}\right] = \cosh\frac{\theta}{2} + (\underset{\sim}{\sigma}\cdot\underset{\sim}{n})\sinh\frac{\theta}{2}.$$

This gives

$$v^{o'} = v^{o} \cosh\theta + \underset{\sim}{v} \cdot \underset{\sim}{n} \sinh\theta \ ,$$

and

$$\underset{\sim}{v'} \cdot \underset{\sim}{n} = \underset{\sim}{v'} \cdot \underset{\sim}{n} \cosh\theta + v^{o} \sinh\theta \ ,$$

the other components being unchanged. The square of A_t is

$$A_t^2 = \exp[\,\theta\, \underset{\sim}{\sigma} \cdot \underset{\sim}{n}\,] = \cosh\theta + \underset{\sim}{\sigma} \cdot \underset{\sim}{n} \sinh\theta = \tau^{\mu}\, \sigma_{\mu} \ ,$$

where $\tau = (\cosh\theta, \underset{\sim}{n} \sinh\theta)$ is the vector into which a unit time-like vector transforms under the action of the transformation. The notation $A_t = \sqrt{\tau \cdot \sigma}$ is often convenient.

Introducing the matrix

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \qquad C^{-1} = -C \ ,$$

one verifies (Pauli Lectures, University of California, 1958) for any 2-by-2 matrix,

$$M = (\,\xi\ \eta\,) = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix} ,$$

the identity

$$M^{tr}\, C\, M = C \det M = C(\,\eta\, C\, \xi\,) \ ,$$

where $(\eta\ C\ \xi)$ is the real inner product. It follows that under $\xi \to \xi' = A\xi$, $\eta \to \eta' = A\eta$ the form $(\eta\ C\ \xi)$ is invariant.

It is conventional to write C^{-1} with upper indices and to define

$$\xi^a = (C^{-1})^{ab} \xi_b \equiv \epsilon^{ab} \xi_b \ .$$

Then the above invariant takes the form $-\eta_a \xi^a$. This provides the basis for a spinor analysis formally similar to tensor analysis. The two-valued indices are called spinor indices, and those associated with the transformation matrix B according to $\hat{\xi} \to \hat{\xi}' = \hat{\xi} B$ are dotted:

$$\hat{\xi}_{\dot{a}} \to \hat{\xi}_{\dot{a}}' = \hat{\xi}_{\dot{a}} B^{\dot{b}}_{\dot{a}} \ .$$

Thus the original matrix V will have matrix elements labelled $V_{a\dot{b}}$, and the Pauli spin matrices σ_μ out of which it was constructed will have matrix elements $\sigma_{\mu a\dot{b}}$.

Carrying along the indices is rather tedious. It is convenient to construct an equivalent matrix algebra. For spinors transforming as $\hat{\xi} \to \hat{\xi}' = \hat{\xi} B$ we shall define

$$\tilde{\xi} = C^{-1} \hat{\xi} \ .$$

Then $\tilde{\xi}' = C^{-1} B^{tr} \hat{\xi} = C^{-1} B^{tr} C \tilde{\xi} \equiv \tilde{B} \tilde{\xi}$, where for any matrix M,

$$\tilde{M} \equiv C^{-1} M^{tr} C = (M)^{-1} \det M \ ,$$

the last part coming from the Pauli identity. For the matrices B , with $\det B = 1$, $\tilde{B} = B^{-1}$, and for real rotations, which are given by unitary $B = \bar{A}$, $\tilde{B}_r = A_r$. Therefore, ξ and $\tilde{\xi}$ transform the same way under real rotations, although differently in general. In spinor notation, the components of the spinors ξ and $\tilde{\xi}$ would have lower undotted and upper dotted spinor indices respectively. The definitions above also imply that

$$\tilde{\sigma}_\mu \equiv C^{-1} \sigma_\mu^{tr} C = (\sigma_0, -\vec{\sigma}) = \sigma_\mu^{\dot{a}b} \ .$$

2. Spinor Functions

Consider $M(v) \equiv v \cdot \sigma \equiv \sigma_{\mu a\dot{b}} v^{\mu} = V$. Then by definition:

$$\Lambda_s M(v) \equiv A\, M(v)\, B \ .$$

Generally the operator Λ_s applied to a matrix M instructs one to multiply each spinor index of M by the transformation matrix associated with that index. From the relationship

$$A\, M(v)\, B = A\, V\, B = V' = M(v')$$

$$= M(Lv) = \sigma_{\mu a\dot{b}}\, L^{\mu}{}_{\nu}(A, B)\, v^{\nu}$$

it follows that $A\, \sigma\, B = \sigma\, L(A, B)$, which can be considered the definition of $L(A, B)$. Also $\Lambda_s M(v) = M(\Lambda v)$, where now $\Lambda v \equiv Lv$.

Notice that equations of the form

$$\tilde{V}\, \xi = \tilde{\eta} \qquad \text{and} \qquad V\, \tilde{\eta} = \xi$$

are invariant under Lorentz transformations, since

$$\xi \to \xi' = A\, \xi\ , \qquad \tilde{\eta} \to \tilde{\eta}' = \tilde{B}\, \tilde{\eta}\ ,$$

$$V \to V' = A\, V\, B\ , \qquad \tilde{V} \to \tilde{V}' = \tilde{B}\, \tilde{V}\, \tilde{A}\ ,$$

and

$$\tilde{B} = B^{-1}\ , \qquad \tilde{A} = A^{-1}\ .$$

3. Parity

The parity operation is generally represented by going to a 4-by-4 representation. Let

$$\mathcal{V} = \begin{pmatrix} \tilde{V} & 0 \\ 0 & V \end{pmatrix} = v^{\mu} \begin{pmatrix} \tilde{\sigma}_{\mu} & \\ & \sigma_{\mu} \end{pmatrix} \equiv v^{\mu}\ \alpha_{\mu}$$

define v^{μ} . Then space inversion is represented by

$$P = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \beta \ ,$$

as one sees by inspection. That there is no matrix A corresponding to parity follows from the fact that any matrix A commuting with the rotation operators, A_r , must commute all three Pauli σ_i and hence with the time transforms A_t .

4. Dirac Equation

The free-field Dirac equation is

$$\pm k^{\mu} \alpha_{\mu} U^{\pm}(k) = \beta\, m\, U^{\pm}(k) \ ,$$

which in terms of

$$U^{\pm}(k) = \begin{pmatrix} \xi(k) \\ \pm\tilde{\eta}(k) \end{pmatrix}$$

becomes the covariant equations

$$(k \cdot \tilde{\sigma})\ \xi(k) = m\, \tilde{\eta}(k) \quad \text{and} \quad (k \cdot \sigma)\ \tilde{\eta}(k) = m\, \xi(k) \ .$$

The introduction of ϕ defined by

$$\xi(k) \equiv A_t(k)\, \phi \equiv \sqrt{(k \cdot \sigma) m^{-1}}\ \phi \ ,$$

gives for the solution

$$\tilde{n}(k) = \tilde{A}_t(k)\,\phi = \sqrt{(k \cdot \tilde{\sigma})m^{-1}}\quad \phi$$

in virtue of the identity

$$(k \cdot \tilde{\sigma})(k \cdot \sigma) = (k \cdot \sigma)(k \cdot \tilde{\sigma}) = m^2 \ .$$

The free field solutions of the Dirac equation are therefore

$$U^{\pm}(k) = \begin{pmatrix} A_t(k)\phi \\ \\ \pm \tilde{A}_t(k)\phi \end{pmatrix} \equiv \begin{pmatrix} \sqrt{(k \cdot \sigma)m^{-1}} & \phi \\ \\ \pm\sqrt{(k \cdot \tilde{\sigma})m^{-1}} & \phi \end{pmatrix} = \mathcal{L}^{-1}(k)\, U^{\pm}(0)$$

where $\mathcal{L}^{-1}(k)$ is the Lorentz transformation that takes spinors from the frame in which the particle is at rest to the frame in which it has momentum-energy k. The charge conjugate solution is

$$V(k) \equiv \begin{pmatrix} 0 & C \\ \\ C^{-1} & 0 \end{pmatrix} \hat{U}(k) \equiv E\,\hat{U}(k) = \begin{pmatrix} \eta(k) \\ \\ \tilde{\xi}(k) \end{pmatrix} = \mathcal{L}^{-1}(k) \begin{pmatrix} C & \phi \\ \\ -C & \phi \end{pmatrix},$$

which has the same transformation properties as $U(k)$. If only real Lorentz transformations are allowed, $\hat{U}(k)$ can be taken to be $U^*(k)$. The spin vectors ϕ are not spinors. They are the spin vectors in that rest frame of the particle obtained from the general coordinate frame by the Lorentz transformation $\mathcal{L}(k)$.

5. Connection between Field Theory and the M-Function Formalism

In field theory one writes

$$\psi_\alpha(k) = \int \frac{d^4k}{(2\pi)^4}\, 2\pi\, \delta(k^2 - m^2)\, \theta(k_0)\, 2m \times \sum_{\sigma=1,2} \left[U_\alpha(k,\sigma) e^{-ikx}\, a(k,\sigma) + V_\alpha(k,\sigma) e^{+ikx}\, \bar{b}(k,\sigma) \right],$$

where

$$\langle\, a(k,\sigma)\ \bar{a}(k',\sigma')\,\rangle_0 = \frac{2\omega}{2m}\ (2\pi)^3\ \delta(\underline{k} - \underline{k}')\delta_{\sigma\sigma'}$$

$$= \langle\, b(k,\sigma)\ \bar{b}(k',\sigma')\,\rangle_0 \ ,$$

and

$$\langle\, a(k,\sigma)\ \bar{\psi}(x)\,\rangle_0 = U^*(k,\sigma)\ e^{+ikx}$$

$$\langle\, \bar{\psi}(x)\ \ \bar{b}(x,\sigma)\,\rangle_0 = V^*(k,\sigma)\ e^{-ikx} \ .$$

The dependence of the covariant Feynman scattering function on $\bar{\psi}(x)$ is

$$\int \bar{\psi}(x)\ \beta\ N(x)\ d^4x \ ,$$

and the corresponding contribution to the scattering matrix for a final particle (k,σ) is

$$\langle a\ (k,\sigma) \int\ \bar{\psi}(x)\,\rangle_0\ \beta\ N(x)\ d^4x$$

$$\equiv U^*(k,\sigma)\ \beta\ N(k) = \begin{pmatrix} \xi^*(k,\sigma) \\ \tilde{\eta}^*(k,\sigma) \end{pmatrix}\ \beta \begin{pmatrix} \xi'(k) \\ \tilde{\eta}'(k) \end{pmatrix}$$

$$= \xi^*(k,\sigma)\ \tilde{\eta}'(k) + \tilde{\eta}^*(k,\sigma)\ \xi'(k)$$

$$= \xi^*_{\dot{a}}(k,\sigma)\ \tilde{\eta}^{\dot{a}'}(k) + \tilde{\eta}^a(k,\sigma)\ \xi_a(k) =$$

$$\phi^*(\sigma)\ \left[\ \sqrt{k\cdot\sigma}\ \ \tilde{\eta}'(k) + \sqrt{k\cdot\tilde{\sigma}}\ \ \xi'(k)\right]$$

where we have taken $m = 1$. Thus

$$R(k) = \sqrt{k \cdot \sigma}\ \tilde{\eta}'(k) + \sqrt{k \cdot \tilde{\sigma}}\ \xi'(k) ,$$

and

$$M(k) = \sqrt{k \cdot \sigma}\ R(k) = \xi'(k) + k \cdot \sigma\ \eta'(k) .$$

The $\xi'(k)$ and $\eta'(k)$ are analytic functions of k , and continuation gives

$$M(-k) = \xi'(-k) - k \cdot \sigma\ \eta'(-k) .$$

The R function for the antiparticle is therefore

$$\sqrt{k \cdot \tilde{\sigma}}\ M(-k) = \sqrt{k \cdot \tilde{\sigma}}\ \xi'(-k) - \sqrt{k \cdot \sigma}\ \eta'(-k) .$$

This is to be compared to

$$\int \langle\, \bar{\Psi}(x)\, \beta\, N(x)\, \bar{b}(k, \sigma) \,\rangle_0 d^4x = - V^*(k,\sigma)\, \beta\, N(-k)$$

$$= - \begin{pmatrix} \sqrt{k \cdot \sigma} & C\, \phi^*(\sigma) \\ - \sqrt{k \cdot \tilde{\sigma}} & C\, \phi^*(\sigma) \end{pmatrix} \beta\, N(-k)$$

$$= \phi^*(\sigma)\, C^{tr} \left[\sqrt{k \cdot \tilde{\sigma}}\ \xi'(-k) - \sqrt{k \cdot \sigma}\ \eta'(-k) \right] .$$

The function occurring in field theory is interpreted by dotting from the left on $\underset{\sim}{s} \cdot \underset{\sim}{\sigma}$, whereas ours is interpreted by dotting from the right on $-\underset{\sim}{s} \cdot \underset{\sim}{\sigma}$. The extra factor C^{tr} compensates for this difference.

FOOTNOTES AND REFERENCES

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