Title

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SYNTHESIS OF A STEPHENSON II FUNCTION GENERATOR FOR EIGHT PRECISION POSITIONS

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ABSTRACT

This paper presents a synthesis methodology for a Stephenson II six-bar function generator that provides eight accuracy points, that is it ensures the input and output angles of the linkage match at eight pairs of specified values. A complex number formulation of the two loop equations of the linkage and a normalization condition are used to form the synthesis equations, resulting in 22 equations in 22 unknowns. These equations are solved using the numerical homotopy continuation solver, Bertini. The approach is illustrated with an example.

INTRODUCTION

This paper presents the synthesis equations for the design of a Stephenson II function generator, which are solved using the numerical homotopy continuation solver, Bertini. The task of the function generator is defined by a list of eight input and output angle values, $(\phi_j, \psi_j),\ j = 1, \ldots, 8$. Three constraint equations are obtained for each input-output pair, or accuracy point, yielding 24 equations. Two of these equations are used to eliminate two unknowns resulting in 22 equations in 22 unknowns. Bertini computed the degree for this polynomial system to be 4,194,304, and 88 hours of calculation running in parallel on a Mac Pro with two 2.93GHz 6-Core processors lead to 38 useful designs.

The loop equations of the Stephenson II are formulated using complex numbers to identify the coordinates of pivots and the dimensions of links following Myszka et al. [1]. This formulation is described by Wampler [2,3] as "isotropic coordinates" because the equations are formulated using complex numbers and their conjugates rather than separating the complex numbers into real and imaginary parts. The result is a convenient set of bilinear synthesis equations. Our initial results indicate that this procedure may be used to design six-bar function generators of other topologies, as well.

LITERATURE REVIEW

Mechanical computers are linkage systems used to calculate an output for a given input, also known as function generators. Svoboda [4] designed function generators by fitting the input-output functions of a given set of linkages to the desired function. In 1954, Freudenstein [5] [6] took a different approach using the loop equations of the four-bar linkage, which he solved directly for a given set of accuracy points to obtain the linkage.

Recently, Hwang and Chen [7] formulated the design of six-bar function generators following Freudenstein and solved the equations using optimization to obtain useful linkages. Sancibrian [8] takes a similar approach using the Generalized Reduced Gradient to find the linkage parameters that minimize the difference between the input-output function of the linkage and the desired function.

Our approach to the design of the Stephenson II six-bar function generator follows Freudenstein, in that we seek a di-

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rect solution for the linkage that achieves a specific set of accuracy points–see Watanabe and Katoh [9] for a description of the properties of the Stephenson II linkage. This synthesis procedure requires the solution of a large polynomial system using numerical homotopy continuation methods described by Sommese and Wampler [10].

The design of function generators can be achieved by inverting the linkage and using the input-output accuracy points to define task positions. This allows the techniques of Burmester theory [11] to be used to design the linkage. Plecnik and McCarthy [12] used this approach to design crank-slider function generators. Kinzel et al. [13] show how this can be done using constraint programming. While Soh and McCarthy [14] have applied Burmester theory to the synthesis of six-bar linkages, it is not clear how their techniques can be reformulated design of six-bar function generators.

COMPLEX VECTORS AND ISOTROPIC COORDINATES

Erdman et al. [15] formulate planar kinematics using complex numbers to represent the coordinates of points in the plane. In this formulation, the coordinates $P = (P_x, P_y)$ of a point are formulated as the complex number,

$$P = P_x + iP_y. \tag{1}$$

The component-wise sum of a complex number is the same as for coordinate vectors, and the product of complex numbers performs rotation and scaling operations. In particular, the exponential $\exp(i\theta)$ is a rotation operator on complex vectors that yields the same result as a 2x2 rotation matrix operating on vectors, that is,

$$\begin{aligned} P =& \exp(i\theta)p, \\ P_x + iP_y =& (\cos\theta + i\sin\theta)(p_x + ip_y), \\ P_x + iP_y =& (\cos\theta p_x - \sin\theta p_y) + i(\sin\theta p_x + i\cos\theta p_y). \end{aligned} \tag{2}$$

Wampler [2] shows that it is convenient to consider the complex number P and its conjugate $\bar{P}$ as components of "isotropic coordinates", $(P, \bar{P})$, because the original components are obtained from the linear transformations,

$$P_x = \frac{1}{2}(P + \bar{P}), \quad P_y = \frac{1}{2i}(P - \bar{P}). \tag{3}$$

For our purposes, the reference to isotropic coordinates means that both the complex and complex conjugate loop equations are used in the formulation of the design equations for the Stephenson II function generator.

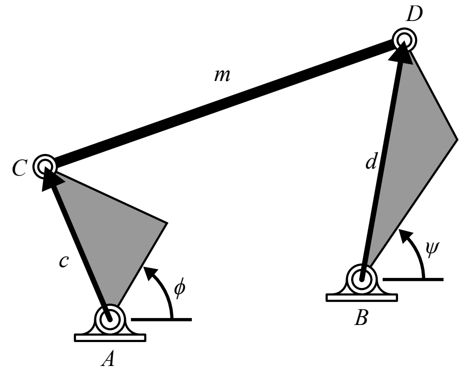


FIGURE 1. A FOUR-BAR LINKAGE FUNCTION GENERATOR.

TABLE 1. ACCURACY POINTS FOUR-BAR FUNCTION GENERATOR.

j	ϕ_j	ψ_j
1	2.763367°	4.339005°
2	21.988925°	33.698463°
3	48.226892°	67.120988°
4	71.414168°	85.306253°
5	87.549520°	89.917699°

FOUR-BAR FUNCTION GENERATOR

In order to illustrate the isotropic coordinate formulation of the synthesis equations, we reproduce the synthesis of function generators presented by Hartenberg and Denavit [6]. The four-bar shown in Fig. 1 has fixed pivots located at A and B and moving pivots located at C and D.

Let the complex numbers that define the coordinates of each of the pivots in the ground frame be $A = A_x + iA_y$, $B = B_x + iB_y$, $C = C_x + iC_y$, and $D = D_x + iD_y$.

Notice that the input link frame is positioned so its x-axis defines the input function value relative to the ground frame, while the segment AC that forms the input link is defined by the complex vector c. Similarly, notice that the output link frame is positioned so that its x-axis provides the output function value relative to the ground frame, while the segment BD is defined by the complex vector d. Hartenberg and Denavit show that the introduction of these crank and follower angles allow the synthesis of a four-bar function generator for five accuracy points.

The task of the four-bar function generator is defined by a list of five pairs of angles $(\phi_j, \psi_j), j = 1, \ldots, 5$. These angles define the sets of complex numbers that represent rotations about the pivots A and B,

$$(Q_j, \bar{Q}_j) = (\exp(i\phi_j), \exp(-i\phi_j)),$$
$$(S_j, \bar{S}_j) = (\exp(i\psi_j), \exp(-i\psi_j)), \quad j = 1, \ldots, 5. \tag{4}$$

The segment CD that defines the coupler of the four-bar linkage can be written in terms of the complex vectors around the loop to yield,

$$C - D = A + cQ_j - B - dS_j \quad j = 1, \ldots, 5. \tag{5}$$

These are the loop equations of the four-bar in each of the accuracy points of the function generator.

Let the magnitude of CD be m, then multiply the closure equation by its complex conjugate to obtain,

$$(A + cQ_j - B - dS_j)(\bar{A} + \bar{c}\bar{Q}_j - \bar{B} - \bar{d}\bar{S}_j) = m^2. \quad j = 1, \ldots, 5 \tag{6}$$

This equation defines the constraint that the linkage must move so the distance between the pivots C and D remains constant. We use this constraint in each of the accuracy points as the synthesis equations for the four-bar linkage function generator.

The five synthesis equations are used to compute the five dimensional parameters, $(c, \bar{c}, d, \bar{d}, m)$. Notice that computing $(c, \bar{c})$ is equivalent to computing the components $c = c_x + ic_y$. In this formulation, the position of the ground pivots A and B are free to be specified by the designer.

The synthesis equations can be simplified by eliminating m by subtracting the first equation from the remaining equations to obtain,

$$K_{1j}c\bar{d} + \bar{K}_{1j}\bar{c}d + K_{2j}c + \bar{K}_{2j}\bar{c} + K_{3j}d + \bar{K}_{3j}\bar{d} = 0, \quad j = 2, \ldots 5, \tag{7}$$

where

$$K_{1j} = -Q_j\bar{S}_j + Q_1\bar{S}_1,$$
$$K_{2j} = (\bar{A} - \bar{B})(Q_j - Q_1),$$
$$K_{3j} = -(\bar{A} - \bar{B})(S_j - S_1). \tag{8}$$

The bilinear structure of the synthesis equations (7) becomes apparent when written in the form.

$$\begin{pmatrix} c \\ d \\ 1 \end{pmatrix}^T \begin{bmatrix} 0 & K_{1j} & K_{2j} \\ \bar{K}_{1j} & 0 & K_{3j} \\ \bar{K}_{2j} & \bar{K}_{3j} & 0 \end{bmatrix} \begin{pmatrix} \bar{c} \\ \bar{d} \\ 1 \end{pmatrix} = 0, \quad j = 2, \ldots, 5. \tag{9}$$

TABLE 2. SOLUTIONS TO THE FOUR-BAR SYNTHESIS EQUATIONS.

Solution 1	$c = 0$	$d = 0$
	$\bar{c} = 0$	$\bar{d} = 0$
Solution 2	$c = 0.7745 - 1.6628i$	$d = -0.2228 - 0.6569$
	$\bar{c} = 0.7745 + 1.6628i$	$\bar{d} = -0.2228 + 0.6569$
Solution 3	$c = -1.5672 + 0.4924i$	$d = -1.0873 - 0.8579$
	$\bar{c} = -3.3387 - 0.2869i$	$\bar{d} = -1.9719 + 0.1892$
Solution 4	$c = -3.3387 + 0.2869i$	$d = -1.9719 - 0.1892$
	$\bar{c} = -1.5672 - 0.4924i$	$\bar{d} = -1.0873 + 0.8579$

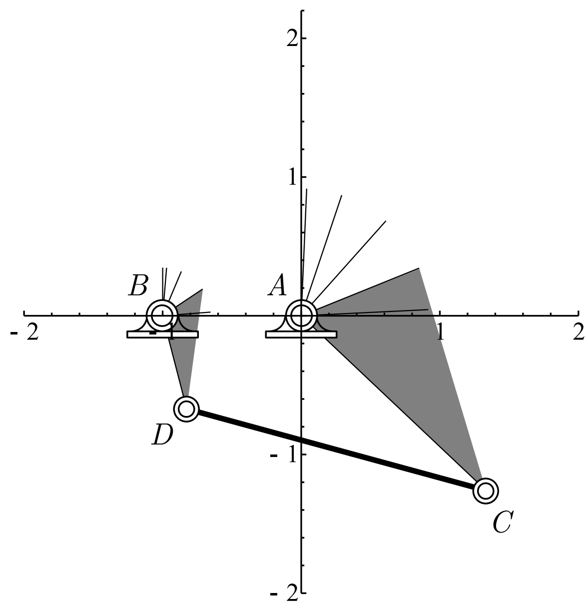


FIGURE 2. FOUR-BAR FUNCTION GENERATOR, THE INPUT ANGLE IS ϕ AND OUTPUT ANGLE IS ψ.

In order to verify our synthesis equations, we compare our results to Hartenberg and Denavit [6]. Hartenberg and Denavit design a four-bar linkage to generate the function

$$\psi(\phi) = 90 \sin\phi, \tag{10}$$

using the accuracy points shown in Tab. 1. These accuracy points define the four self-adjoint matrices of Eqn. 7 that form the synthesis equations.

We solve the synthesis equations using the numerical solver in Mathematica. The results are shown in Tab. 2. It is useful to

TABLE 3. COMPARISON TO HARTENBERG AND DENAVIT

Link	Isotropic Formulation	Hartenberg and Denavit
AC	1.8343529	1.8343688
CD	2.2385372	2.2385439
BD	0.6936395	−0.6936456
AB	1.0	1.0

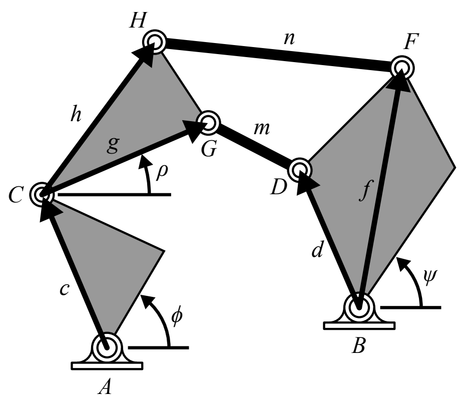


FIGURE 3. THE STEPHENSON II FUNCTION GENERATOR, THE INPUT ANGLE IS ϕ AND OUTPUT ANGLE IS ψ.

note that the trivial solution of a linkage with zero link lengths always exist for these synthesis equations. Solutions 3 and 4 do not relate to physically realizable linkage dimensions because $(c,\bar{c})$ and $(d,\bar{d})$ are not complex conjugate pairs as required by our isotropic coordinate formulation.

The link lengths computed using our isotropic formulation can be compared to the link lengths reported by Hartenberg and Denavit in [6] (Table 10-3, pg. 314). A comparison of the results in Tab. 3 show that our isotropic formulation matches the results of Freudenstein's equations. The resulting linkage is shown in Fig. 2.

A STEPHENSON II FUNCTION GENERATOR

A Stephenson II six-bar linkage, Fig. 3, can be viewed as a four-bar loop stacked on top of a five-bar loop. Our formulation of the loop equations follows Myszka et al. [1].

Let the task specification for the function generator be eight pairs of angles (ϕ_j, ψ_j), $j = 1,\ldots,8$ that are to be the accuracy points, and introduce three sets of complex numbers to represent rotations about the fixed pivots A and B, and the orientation of the segment CG measured relative to the ground frame,

$$\begin{aligned}(Q_j,\bar{Q}_j) &= (\exp(i\phi_j), \exp(-i\phi_j)),\\ (R_j,\bar{R}_j) &= (\exp(i\rho_j), \exp(-i\rho_j)),\\ (S_j,\bar{S}_j) &= (\exp(i\psi_j), \exp(-i\psi_j)), \quad j = 1,\ldots,8. \end{aligned} \tag{11}$$

The complex vectors $(Q_j,\bar{Q}_j)$ and $(S_j,\bar{S}_j)$ are known from the task specification, while $(R_j,\bar{R}_j)$ are to be determined in the synthesis process.

The loop equations of the Stephenson II can be formulated to define the two floating links GD and HF, as

$$\begin{aligned}\mathcal{L}_1 :\quad & G - D = A + cQ_j + gR_j - B - dS_j,\\ \mathcal{L}_2 :\quad & H - F = A + cQ_j + hR_j - B - fS_j, \quad j = 1,\ldots,8. \end{aligned} \tag{12}$$

The constraints that the the lengths $m = |GD|$ and $n = |HF|$ be constant for the movement of the six-bar linkage yield the equations,

$$\begin{aligned}&\mathcal{C}_1 : (A + cQ_j + gR_j - B - dS_j)(\bar{A} + \bar{c}\bar{Q}_j + g\bar{R}_j - \bar{B} - \bar{d}\bar{S}_j) = m^2,\\ &\mathcal{C}_2 : (A + cQ_j + hR_j - B - fS_j)(\bar{A} + \bar{c}\bar{Q}_j + \bar{h}\bar{R}_j - \bar{B} - \bar{f}\bar{S}_j) = n^2,\\ &\quad j = 1,\ldots,8. \end{aligned} \tag{13}$$

The 16 constraint equations (13) together with the eight equations defining $(R_j,\bar{R}_j)$ as a unit vector,

$$R_j\bar{R}_j = 1, \quad j = 1,\ldots,8, \tag{14}$$

yield 24 synthesis equations. The unknowns that we determine from these equations are the 16 values of $(R_j,\bar{R}_j)$ and the eight unknowns, $(c,\bar{c},d,\bar{d},f,\bar{f},m,n)$. The designer is free to specify the fixed pivots A and B and the dimensions g and $(h,\bar{h})$ of the link CGH.

ELIMINATION OF m AND n

The unknowns m and n in the synthesis equations can be eliminated by subtracting the first equations of $\mathcal{C}_1$ and $\mathcal{C}_2$ from the rest of the equations in their respective sets. Once this is done, we can define the seven sets of vectors of unknowns,

$$\begin{aligned}\mathbf{v}_j &= (c,d,R_j,R_1,1), \quad j = 2,\ldots,8,\\ \mathbf{w}_j &= (c,f,R_j,R_1,1), \quad j = 2,\ldots,8. \end{aligned} \tag{15}$$

Then, the synthesis equations can be written in the form,

$$\mathbf{v}_j^T[M_j]\bar{\mathbf{v}}_j = \begin{pmatrix} c \\ d \\ R_j \\ R_1 \\ 1 \end{pmatrix}^T \begin{bmatrix} 0 & M_{1j} & M_{2j} & M_3 & M_{6j} \\ \bar{M}_{1j} & 0 & M_{4j} & M_5 & M_{7j} \\ \bar{M}_{2j} & \bar{M}_{4j} & 0 & 0 & M_8 \\ \bar{M}_3 & \bar{M}_5 & 0 & 0 & -M_8 \\ \bar{M}_{6j} & \bar{M}_{7j} & \bar{M}_8 & -\bar{M}_8 & 0 \end{bmatrix} \begin{pmatrix} \bar{c} \\ \bar{d} \\ \bar{R}_j \\ \bar{R}_1 \\ 1 \end{pmatrix} = 0,\ j = 2,\ldots,8, \tag{16}$$

where

$$\begin{aligned} &M_{1j} = -Q_j\bar{S}_j + Q_1\bar{S}_1, \quad M_{2j} = gQ_j, \quad M_3 = -gQ_1, \\ &M_{4j} = -gS_j, \quad M_5 = gS_1, \quad M_{6j} = (\bar{A}-\bar{B})(Q_j - Q_1), \\ &M_{7j} = -(\bar{A}-\bar{B})(S_j - S_1), \quad M_8 = g(\bar{A}-\bar{B}), \end{aligned} \tag{17}$$

and

$$\mathbf{w}_j^T[N_j]\bar{\mathbf{w}}_j = \begin{pmatrix} c \\ f \\ R_j \\ R_1 \\ 1 \end{pmatrix}^T \begin{bmatrix} 0 & N_{1j} & N_{2j} & N_3 & N_{6j} \\ \bar{N}_{1j} & 0 & N_{4j} & N_5 & N_{7j} \\ \bar{N}_{2j} & \bar{N}_{4j} & 0 & 0 & N_8 \\ \bar{N}_3 & \bar{N}_5 & 0 & 0 & -N_8 \\ \bar{N}_{6j} & \bar{N}_{7j} & \bar{N}_8 & -\bar{N}_8 & 0 \end{bmatrix} \begin{pmatrix} c \\ f \\ R_j \\ R_1 \\ 1 \end{pmatrix} = 0,\ j = 2,\ldots,8. \tag{18}$$

where

$$\begin{aligned} &N_{1j} = -Q_j\bar{S}_j + Q_1\bar{S}_1, \quad N_{2j} = \bar{h}Q_j, \quad N_3 = -\bar{h}Q_1, \\ &N_{4j} = -\bar{h}S_j, \quad N_5 = \bar{h}S_1, \quad N_{6j} = (\bar{A}-\bar{B})(Q_j - Q_1), \\ &N_{7j} = -(\bar{A}-\bar{B})(S_j - S_1), \quad N_8 = h(\bar{A}-\bar{B}). \end{aligned} \tag{19}$$

The result is 22 synthesis equations for the Stephenson II function generator,

$$\begin{aligned} \mathbf{v}_j^T[M_j]\bar{\mathbf{v}}_j = 0, &\quad j = 2,\ldots,8, \\ \mathbf{w}_j^T[N_j]\bar{\mathbf{w}}_j = 0, &\quad j = 2,\ldots,8, \\ R_j\bar{R}_j - 1 = 0, &\quad j = 1,\ldots,8. \end{aligned} \tag{20}$$

where the first two sets of seven equations are obtained from the constraint equations for Stephenson II, and the last set of eight require that the complex vectors R_j have unit magnitude. These equations have the bilinear monomial structure

$$\begin{aligned} \langle c,d,R_j,R_1,1\rangle\langle\bar{c},\bar{d},\bar{R}_j,\bar{R}_1,1\rangle, &\quad j = 2,\ldots,8, \\ \langle c,f,R_j,R_1,1\rangle\langle\bar{c},\bar{f},\bar{R}_j,\bar{R}_1,1\rangle, &\quad j = 2,\ldots,8, \\ \langle R_j,1\rangle\langle\bar{R}_j,1\rangle, &\quad j = 1,\ldots,8. \end{aligned} \tag{21}$$

The Bezout degree of this system is $2^{22} = 4,194,304$, and we solve these equations using the numerical homotopy solver Bertini (`http://www3.nd.edu/~sommese/bertini/`).

TABLE 4. ACCURACY POINTS STEPHENSON II SIX-BAR

j	ϕ_j	ψ_j
1	145°	0°
2	150°	−23.44°
3	155°	−43.75°
4	160°	−60.94°
5	165°	−75.00°
6	170°	−85.94°
7	175°	−93.75°
8	180°	−98.44°

EXAMPLE SYNTHESIS OF A STEPHENSON II SIX-BAR FUNCTION GENERATOR

In order to demonstrate the synthesis process, we use the eight accuracy points shown in Tab. 4, which are selected to approximate the function,

$$\psi(\phi) = \frac{(\phi - 185)^2}{16} - 100. \tag{22}$$

The coordinates of the fixed pivots are specified to be $A = (8,0)$ and $B = (0,6.5)$, and the dimensions of the link CGH are selected as $g = 2$ and $h = (1,\sqrt{3})$ in the link frame. Thus, the specified linkage parameters are

$$\begin{aligned} &A = 8, \quad \bar{A} = 8, \quad B = i6.5, \quad \bar{B} = -i6.5, \\ &g = 2, \quad h = 1 + i\sqrt{3}, \qquad \bar{h} = 1 - i\sqrt{3}. \end{aligned} \tag{23}$$

The synthesis equations were solved using Bertini on a Mac Pro running two 2.93GHz 6-Core Intel Xeon processors with 24

TABLE 5. STEPHENSON II SIX-BAR BERTINI RESULTS.

Dimension	Real	Imaginary
c	8.51864488652	$-0.75552523181i$
d	-0.11852946778	$+2.63898905823i$
f	2.55351461627	$-1.17851717131i$
R_1	0.99699789277	$+0.07742868856i$
R_2	0.96656438960	$-0.25642402530i$
R_3	0.93153985897	$-0.36363923215i$
R_4	-0.87383680374	$+0.48621933367i$
R_5	-0.72187867603	$+0.69201963635i$
R_6	0.98726796553	$-0.15906591160i$
R_7	0.99915998624	$+0.04097953015i$
R_8	0.94537737951	$+0.32597792916i$

GB of RAM. Bertini generated $2^{22} = 4,194,304$ paths and ran for 88 hours to determine 64,858 solutions to the synthesis equations of which 63,755 were nonsingular. Of these solutions there were 3,195 which corresponded to physically realizable dimensions. Physically realizable dimensions are those that satisfy the relations shown in Eqn. 3.

Each of the 3,195 physically realizable solutions were analyzed to determine if they correspond to a Stephenson II function generator that moves smoothly through the specified accuracy points. Mechanisms that accomplish this are called useful. The analysis procedure is outlined in the next section. This procedure identified 38 useful linkages. A useful solution appears in Tab. 5. The corresponding Stephenson II function generator appears in Fig. 4.

ANALYSIS OF A STEPHENSON II SIX-BAR LINKAGE

The objective of this analysis is to determine whether a Stephenson II linkage is capable of producing a smooth trajectory that moves the input and output links through all required accuracy points. Our notion of a smooth trajectory is defined as a continuous set of configurations that does not pass through a singular point, what others have referred to as a mechanism branch [16] [17]. In this section, we first determine all the smooth trajectories a linkage is capable of producing, we then examine each trajectory to determine whether or not it produces the desired accuracy points. Smooth trajectories are pieced together by solving for all assembly configurations for a series of input angles that represents a full revolution of the input link AC.

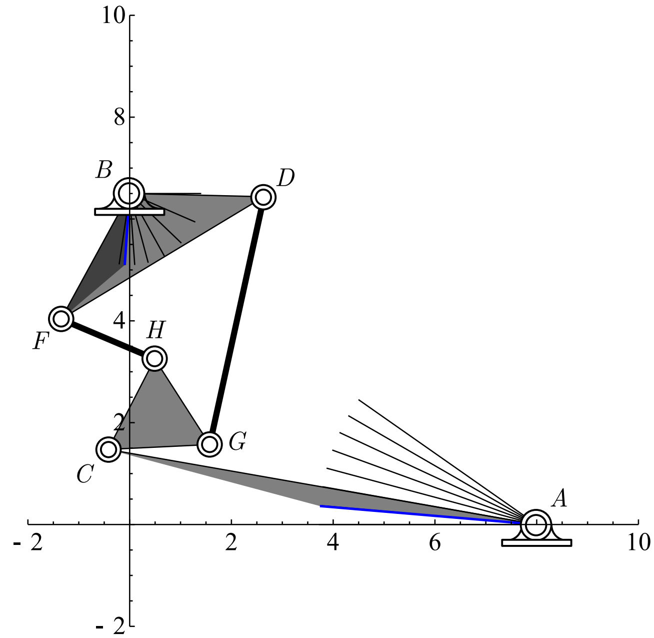


FIGURE 4. ONE EXAMPLE OF THE 38 USEFUL SOLUTIONS TO THE STEPHENSON II SYNTHESIS EQUATIONS.

Each configuration is sorted into a trajectory according to a sorting algorithm.

Forward Kinematics

In order to find all assembly configurations for a given input value ϕ, we formulate and solve the forward kinematics equations. The constraint equations (13) that were used for the synthesis of the Stephenson II also form the base of its forward kinematics equations,

$$\begin{aligned} C_1(Q,\bar{Q},R,\bar{R},S,\bar{S}) &= \\ &(A+cQ+gR-B-dS)(\bar{A}+\bar{c}\bar{Q}+g\bar{R}-\bar{B}-\bar{d}\bar{S}_j)-m^2=0, \\ C_2(Q,\bar{Q},R,\bar{R},S,\bar{S}) &= \\ &(A+cQ+hR-B-fS)(\bar{A}+\bar{c}\bar{Q}+\bar{h}\bar{R}-\bar{B}-\bar{f}\bar{S})-n^2=0, \end{aligned} \tag{24}$$

together with the normalizing conditions

$$N_1(R,\bar{R}) = R\bar{R}-1=0, \quad N_2(S,\bar{S}) = S\bar{S}-1=0,. \tag{25}$$

Note that the input angle ϕ is represented by $(Q,\bar{Q})$ and the configuration angles ρ and ψ are represented by $(R,\bar{R})$ and $(S,\bar{S})$, respectively. All other parameters are known linkage dimensions.

Eqns. 24 and 25 form four bilinear equations in the unknowns $\{R, \bar{R}, S, \bar{S}\}$. McCarthy [11] uses an algebraic elimination procedure to solve equations of this form for the synthesis of a spherical RR chain. His procedure results in a degree six polynomial resultant, the roots of which result in the dimensions of six RR chains. For the case of Eqns. 24 and 25, these roots represent six assembly configurations of a Stephenson II six-bar actuated from link AC.

Sorting Assembly Configurations

In order to determine the movement of a Stephenson II six-bar, we solve the forward kinematics equations for an array of input angles ϕ_k to obtain $\rho_{k,l}$ and $\psi_{k,l}$, where $l = 1, \ldots, 6$ identifies the configurations for that input angle. As ϕ_k is incremented, the solutions to these equations do not appear in any order. It is the goal of this section to sort each configuration $(\phi_k, \rho_{k,l}, \psi_{k,l})$ into a smooth trajectory curve.

Let us define the input of our mechanism at position k as a vector $\mathbf{x}_k$ and the output as a vector $\mathbf{y}_{k,l}$, such that

$$\mathbf{x}_k = \begin{pmatrix} Q_k \\ \bar{Q}_k \end{pmatrix} \quad \text{and} \quad \mathbf{y}_{k,l} = \begin{pmatrix} R_{k,l} \\ S_{k,l} \\ \bar{R}_{k,l} \\ \bar{S}_{k,l} \end{pmatrix}, \quad l = 1, \ldots, 6. \tag{26}$$

As well the kinematics equations (24) and (25) form a vector $\mathbf{F}$ such that

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \begin{Bmatrix} C_1(\mathbf{x}, \mathbf{y}) \\ C_2(\mathbf{x}, \mathbf{y}) \\ N_1(\mathbf{y}) \\ N_2(\mathbf{y}) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{27}$$

The Jacobian of this function is

$$[J_\mathbf{F}] = \left[\frac{\partial \mathbf{F}}{\partial R}, \frac{\partial \mathbf{F}}{\partial S}, \frac{\partial \mathbf{F}}{\partial \bar{R}}, \frac{\partial \mathbf{F}}{\partial \bar{S}} \right] \tag{28}$$

The objective of this algorithm at an input step k is to connect each configuration from set $\mathcal{A}$ to a configuration in set $\mathcal{B}$ where

$$\mathcal{A} = \{(\mathbf{x}_k, \mathbf{y}_{k,l}) \mid l = 1, \ldots, 6\} \tag{29}$$

and

$$\mathcal{B} = \{(\mathbf{x}_{k+1}, \mathbf{y}_{k+1,p}) \mid p = 1, \ldots, 6\}. \tag{30}$$

Note that in general a connecting pair will have $l \neq p$. In order to connect each pair, we use a Taylor series expansion of the kinematics equations at position $k+1$ given by

$$\mathbf{F}(\mathbf{x}_{k+1}, \mathbf{y}_{k+1,l}) \approx \mathbf{F}(\mathbf{x}_{k+1}, \mathbf{y}_{k,l}) + [J_\mathbf{F}]_k (\mathbf{y}_{k+1,l} - \mathbf{y}_{k,l}). \tag{31}$$

We want to estimate the value of $\mathbf{y}_{k+1,l}$ that yields $\mathbf{F} = 0$, therefore we compute the approximate values,

$$\tilde{\mathbf{y}}_{k+1,l} = \mathbf{y}_{k,l} - [J_\mathbf{F}]_k^{-1} \mathbf{F}(\mathbf{x}_{k+1}, \mathbf{y}_{k,l}) \tag{32}$$
$$l = 1, \ldots, 6$$

These values form the set $\tilde{\mathcal{B}}$ of approximations where

$$\tilde{\mathcal{B}} = \{(\mathbf{x}_{k+1}, \tilde{\mathbf{y}}_{k+1,l}) \mid l = 1, \ldots, 6\}. \tag{33}$$

If the l^{th} element of $\tilde{\mathcal{B}}$ is sufficiently close to the p^{th} element of $\mathcal{B}$, then the p^{th} element of $\mathcal{B}$ is taken as the neighbor of the l^{th} element of $\mathcal{A}$ on a smooth trajectory. Once each element of $\mathcal{A}$ is connected to a element of $\mathcal{B}$, the algorithm increments to the next step. Cases in which one to one correspondence do not occur are described in the following section.

Singularities

The technique described above for sorting the roots of the kinematics equations among assembly configurations can fail at singular and near-singular configurations. That is where

$$\det[J_\mathbf{F}(\mathbf{x}_k, \mathbf{y}_{k,l})] \approx 0. \tag{34}$$

Near singular configurations can be present even if there is no singularity. The near singular cases are troublesome because the tracking algorithm can jump from one smooth trajectory to another.

Unlike other methods that explicitly solve for all singular points beforehand [1], our algorithm attempts to sort through the singular points with no prior knowledge to their location. Because singularities mark the input limits of a mechanism, our algorithm sorts configurations whether or not they are entirely physically realizable.

In particular, the tracked curves consist of the elements $(R, S, \bar{R}, \bar{S})$ parameterized by ϕ. The behavior of curves in this space is related to the behavior of curves in a similar space shown in Fig. 5. This figure plots the real and imaginary components of the output S against the independent input parameter ϕ. Configurations are physically realizable at locations where $|S| = 1$, that is where the curves lie on the cylinder. Computed singular locations are marked with purple dots. Note that the curves in Fig. 5 can cross at nonsingular locations because this graph does not include information about R.

It is near singularities that our algorithm can fail to find a one to one correspondence between sets $\mathcal{A}$ and $\mathcal{B}$ described in Eqns. 29 and 30. However, this is addressed by the following logic.

The set $\mathcal{A}$ contains the last elements of the trajectories arriving at position k, and $\mathcal{B}$ contains the first elements of the trajectories departing from position $k+1$. At each position we apply the following logic:

1. If each element of $\mathcal{A}$ connects to each element of $\mathcal{B}$, the algorithm moves on to the next position;
2. If an element of $\mathcal{B}$ is not connected to an element of $\mathcal{A}$, that element of $\mathcal{B}$ begins a new trajectory;
3. If an element of $\mathcal{A}$ connects to multiple elements of $\mathcal{B}$, the trajectory associated with that element of $\mathcal{A}$ is duplicated and each duplicate connects to a matching element of $\mathcal{B}$;
4. If an element of $\mathcal{A}$ does not connect to an element of $\mathcal{B}$, the trajectory associated with that element of $\mathcal{A}$ is concluded.

Physically realizable sections are separated out upon conclusion of the path tracking algorithm. These sections are then split at points where the sign of the Jacobian determinant changes in order to find cases where the algorithm may jump between trajectories. It is possible for such a jump between trajectories to occur without change in sign of the Jacobian [16]. Our algorithm cannot detect these jumps.

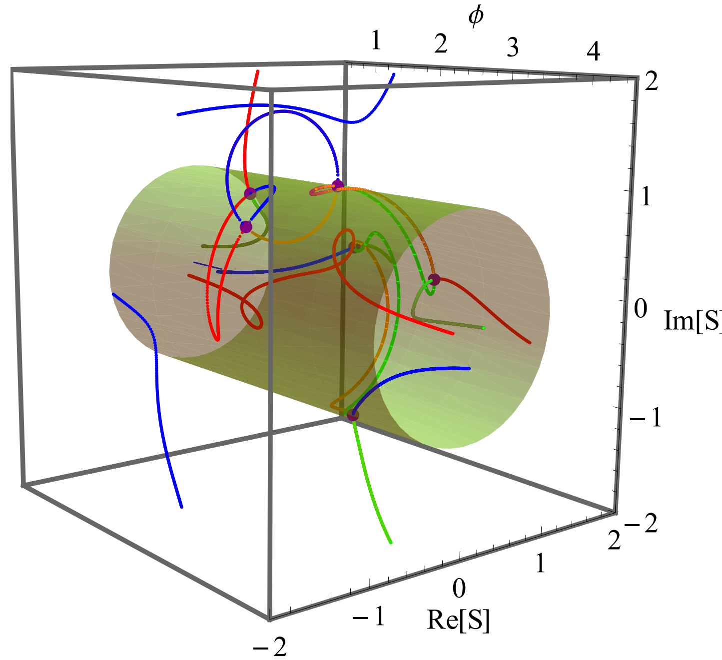


FIGURE 5. EXAMPLE OF TRAJECTORY SORTING AND THE PRESENCE OF SINGULARITIES.

Determining Useful Designs

Once a Stephenson II six-bar function generator has been designed and the trajectories have been generated, we then determine if the eight accuracy points lie on a single trajectory. This ensures that the function generator can move through all eight accuracy points, and that the linkage is useful.

In order to determine that an accuracy point is on a trajectory, we must decide if the accuracy point is within a specified distance of the list of points that define a trajectory. To do this we determine whether an accuracy point is contained in a box defined by two neighboring trajectory points in the ϕ-ψ plane as shown in Fig. 6. Six-bar linkages with non-zero error at the accuracy points can satisfy this criterion.

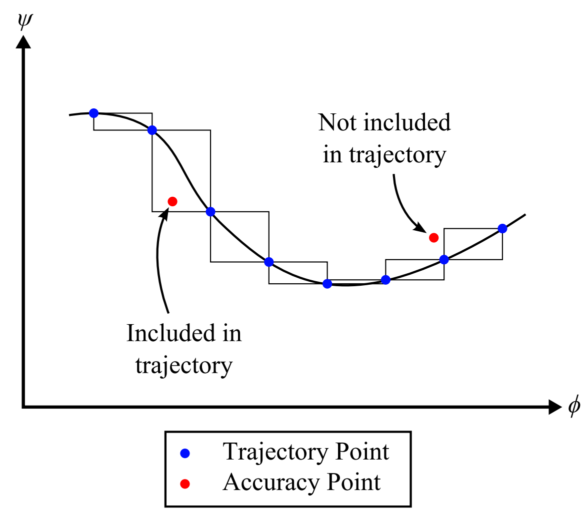
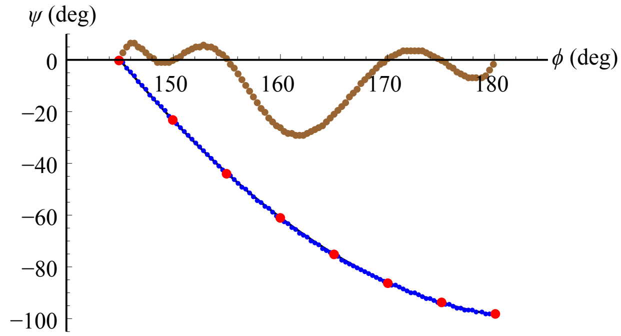


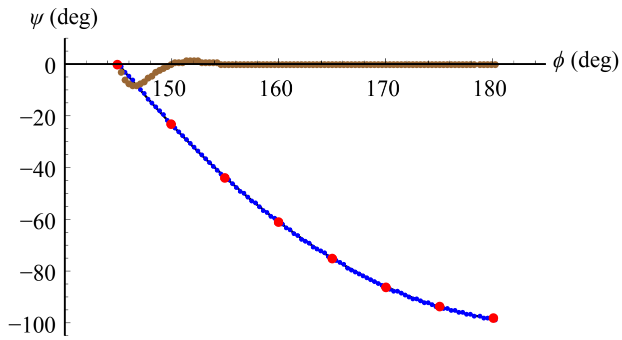
FIGURE 6. CRITERION USED FOR DETERMINING WHETHER A TRAJECTORY CONTAINS AN ACCURACY POINT.

Fig. 7 plots the trajectories of Six-bar 1 and Six-bar 18 of the 38 useful designs with their error curves sampled over 98 points. Each error point is computed as the difference between the output link angle ψ as depicted in Fig. 3 and the desired value of ψ as computed from the from Eqn. 22. For visual clarity, the error curves of Fig. 7 are multiplied by 100. Note that a stricter criterion would be to require that an error curve cross the ϕ-axis at every accuracy point.

Figure 8 shows the input-output functions of eight of the 38 useful designs, including Six-bars 1 and 18. An animation of the useful Stephenson II six-bar function generator shown in Fig. 4 can be viewed at the web page: `http://mechanicaldesign101.com`.



(a) Six-bar 1 of the 38 useful function generators that has a non-zero error at two accuracy points. The error is scaled by 100.



(b) Six-bar 18 of the 38 useful function generators that has zero error at each accuracy point. The error is scaled by 100.

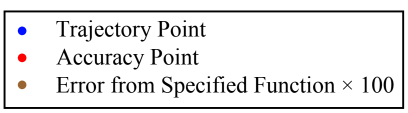


FIGURE 7. TWO EXAMPLES OF SIX-BAR FUNCTION GENERATORS TOGETHER WITH PLOTS OF THE STRUCTURAL ERROR SCALED BY 100.

CONCLUSIONS

This paper presents a design methodology for the synthesis of six-bar function generators based on the Stephenson II topology. A complex vector formulation is used to develop the constraint equations of the linkage. These equations are then formulated in terms of complex numbers and their conjugates, known as isotropic coordinates, rather than in terms of real and imaginary components. This approach provides a convenient formulation for both the synthesis equations and the analysis of the resulting designs to verify their usefulness.

An example of a six-bar function generator for eight accuracy points that approximates a parabolic function yielded 64,858 roots. Among those roots were the solutions for 38 useful linkages. The computation required 88 hours on a Mac Pro and traced over 4 million paths using Bertini. This approach can be applied to the synthesis of Watt II and Stephenson III six-bar function generators as well. However, there is room to increase the efficiency of this approach.

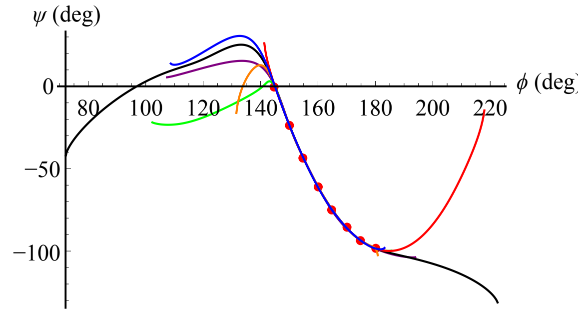


FIGURE 8. THE INPUT-OUTPUT FUNCTIONS OF EIGHT OF THE USEFUL LINKAGES. THE BLUE LINE IS FOR SIX-BAR 1 AND THE RED LINE IS FOR SIX-BAR 18.

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