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Modularity of nearly ordinary 2-adic residually dihedral Galois representations

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Patrick Brodie Allen

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Abstract of the Dissertation

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Professor Chandrashekhar Khare, Chair

We prove modularity of some two dimensional 2-adic Galois representations over a totally real field that are nearly ordinary at all places above 2 and that are residually dihedral. We do this by employing the strategy of Skinner and Wiles using Hida families together with the 2-adic patching method of Khare and Wintenberger. As an application we deduce modularity of some elliptic curves over totally real fields that have good ordinary or multiplicative reduction at places above 2.

The dissertation of Patrick Brodie Allen is approved.

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To my father and mother, David and Michele, and my brothers, Kyle, Aaron, and Sean

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CHAPTER 1

Introduction

This thesis concerns the connection between Galois representations and automorphic forms. Galois representations arise quite naturally in algebraic geometry, and often properties of the Galois representations allow one to deduce properties of the geometric object. Automorphic forms are certain complex analytic functions on homogeneous spaces for Lie groups, the most classical example of which being elliptic modular forms. A priori, automorphic forms and Galois representations don't appear to have anything to do with one another. But in fact, there is a deep connection that has many consequences.

In the 1970s, Shimura canonically associated elliptic curves to certain modular forms. Shimura and Taniyama then conjectured that all elliptic curves over $\mathbb{Q}$ arise in this manner. This was proven for a large class of elliptic curves by Wiles and Taylor in the 1990s. They proved this by showing that the Galois representations arising from a large class of elliptic curves are in fact canonically associated to modular forms. Fermat's Last Theorem, a problem in elementary number theory that had remained unsolved for 350 years, is a corollary of that work. It is remarkable that the proof of Fermat's Last Theorem rests on showing that the Galois representations coming from an object in algebraic geometry are associated to objects in complex analysis.

The idea that arithmetic objects should have automorphic forms associated to them was expounded by Langlands in the 1970s, and it has since become a central theme in number theory. The main tools for proving that Galois representations are automorphic, i.e. arise from automorphic forms, are what is known as modularity lifting theorems. Given a Galois representation on a finite dimensional vector space over a p-adic field, one has its residual representation obtained by reducing a Galois stable lattice modulo p. A modularity lifting theorem is a theorem of the form: "If the residual representation is automorphic, then the original representation is automorphic." The Taylor-Wiles method is the most useful method for proving modularity lifting theorems.

Fontaine and Mazur have made a very precise conjecture predicting when certain odd, "geometric", absolutely irreducible p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ are isomorphic to ones coming from modular forms, and this conjecture has a natural generalization to p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$, for any totally real field F. There has been great progress on this conjecture, and for p odd and $F = \mathbb{Q}$, it has been resolved in almost all cases. When $p = 2$, less is known because the Taylor-Wiles method encounters technical difficulties. Khare and Wintenberger, [KW], and Kisin, [K3], developed an extension of the Taylor-Wiles method to prove modularity of a wide class of 2-adic representations and this was essential in their proof of Serre's conjecture. It's interesting to note that, since proofs of the known cases of the Fontaine-Mazur conjecture use Serre's conjecture as an important ingredient, 2-adic modularity lifting theorems have had applications in proving modularity of p-adic representations even when p is odd.

Due to their technical nature, the current 2-adic modularity lifting theorems require stronger assumptions than their $p > 2$ counterparts. One such assumption is that the residual representation has non-solvable image. It is desirable to have a 2-adic modularity lifting theorem in the residually solvable case for a number of reasons. One of the most important of which is that the 2-adic representations arising from elliptic curves, a very natural source of Galois representations, are always residually solvable as they have image in $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$, a solvable group. Hence, for F a totally real field, modularity lifting theorems for 2-adic, residually solvable representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ prove modularity of elliptic curves over F, a generalization of the work of Wiles, Taylor and Wiles, and Breuil, Conrad, Diamond and Taylor. The main result of my thesis is such a theorem.

This is done by employing a strategy of Skinner and Wiles. They showed, in the $p > 2$ case, that certain representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ for (most) totally real fields, obeying a hypothesis called ordinarily, were modular assuming only that the residual representation is absolutely irreducible, as opposed to a the more restrictive condition of assuming the residual representation is absolutely irreducible when restricted to the finite index subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/F(\zeta_p))$, which is the condition usually imposed. Their strategy is to use families of modular forms known as Hida families in order to move to a new "residual" representation where one can assume stronger conditions. In this thesis we carry out the Skinner and Wiles method in the 2-adic case to prove modularity of representations that are ordinary at places above 2 and whose reductions are absolutely irreducible but have solvable image.

1.1 The Main Theorem

The main result of this thesis work is the following theorem. Before stating it we remark that in assumption (2), the isomorphism of local class field theory is normalized so that uniformizers correspond to arithmetic Frobenii.

Theorem. *Let F be a totally real subfield of $\overline{\mathbb{Q}}$. Let I_F denote set set of embeddings $F \hookrightarrow \overline{\mathbb{Q}}$. Fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_2$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Via these embeddings we view I_F as the set of embeddings $\{F \hookrightarrow \mathbb{R}\}$ as well as the set of embeddings $\{F \hookrightarrow \overline{\mathbb{Q}}_2\}$. For any $v|p$ in F, let $I_{F,v} \subseteq I_F$ denote the subset of τ that give rise to v. We identify $I_{F,v}$ with the set of embedding $F_v \hookrightarrow \overline{\mathbb{Q}}_2$.*

Let

$$\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$$

be a continuous representation unramified outside finitely many primes. Assume there is some $(\mathbf{k}, \mathbf{w}) \in I_F^2$, such that $k_\tau \geq 2$ for each $\tau \in I_F$ and $w = k_\tau + 2w_\tau$ is independent of τ, and such that

1. $\det \rho = \phi \epsilon_2^{w-1}$*, with ϕ a finite order character and ϵ_2 the 2-adic cyclotomic character;*
2. *for each $v|2$, $\rho|_{G_v} \cong \begin{pmatrix} * & * \\ & \chi_v \end{pmatrix}$ such that viewing χ_v as a character of $F_v^\times$ via class*

field theory, $\chi_v(y) = \prod_{\tau \in I_{F,v}} y^{-w_\tau}$ *on some open subgroup of* $\mathcal{O}_{F_v}^\times$*;*

3. *for each choice of complex conjugation* c*,* $\det \rho(c) = -1$.

Let $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}})$ *denote the residual representation associated to* ρ*. We also assume*

(4) $\overline{\rho}$ *is absolutely irreducible dihedral;*

(5) If L/F *is a CM extension such that* $\overline{\rho}|_{G_L}$ *is abelian, then there is some* $v|2$ *in* F *that does not split in* L *and such that if* w *denotes the place in* L *above* v*,* $L_w \neq F_v(\zeta_4)$*, with* ζ_4 *a primitive* 4*-th root of unity;*

Under these assumptions ρ *is modular, i.e. there is a* 2*-nearly ordinary regular algebraic cuspidal automorphic representation* π *of* $\mathrm{GL}_2(\mathbb{A}_F)$ *such that* $\rho \cong \rho_\pi$*.*

The main theorem will follow from 5.2.1 which proves modularity for such ρ assuming that the residual representation is absolutely irreducible but admits a 2-nearly ordinary modular lift, together with an application of a theorem of Wiles that allows us to produce ordinary lifts in the residually dihedral case, cf. 5.1.2.

Let me comment on the assumptions in the theorem. Condition (2) is the *near-ordinarity* condition, and is essential to the method as we use Hida families. The condition (4), together with allowing more general totally real fields F in the case $p = 2$, is the main improvement in this thesis. As mentioned above, both the results of [KW] and [K3] exclude the residually dihedral case, as well as assuming F is unramified above 2 in the potentially ordinary case. Condition (5) is technical and is related to the fact that when the extension L/F is CM and every $v|2$ in F splits in L, Hida's universal nearly ordinary Hecke algebra has CM components. We will elaborate on this in the next section.

The two main difficulties in combining the methods of Skinner and Wiles with those of Khare and Wintenberger are in proving the existence of Taylor-Wiles primes and in performing the patching. In proving the existence of Taylor-Wiles primes, the calculations of [SW1] bounding the size of a certain cohomology group do not carry over to the $p = 2$

case. To overcome this we use a variant of the so called "nice" prime ideals in [SW1]. These prime ideals, $\mathfrak{p}$, are such that they have dimension 1, are of characteristic 2, and when the universal deformation is specialized at $\mathfrak{p}$, its image is non-dihedral and contains unipotent elements. This allows the use of a result of Pink, [P], to show that the image of the Galois representation attached to one of these prime ideals is open in $\mathrm{SL}_2(K_0)$, where K_0 is a field that is of finite index in the residue field of $\mathfrak{p}$. This fact allows one to explicitly compute the cohomology groups in question, and to show the existence of the Taylor-Wiles primes. In performing the patching argument, care has to be taken to keep track of irreducible components. In the Skinner and Wiles argument they carry out the Taylor-Wiles method, in a certain sense, with respect to local deformation ring that has been localized and completed at a dimension one prime ideal. In general the process of completion causes an increase in irreducible components. In order to avoid this, we ensure that our "nice" primes satisfy properties that ensure the resulting localization of the local deformation ring is normal.

Our main theorem has the following immediate corollary.

Corollary. *Let F be a totally real field and E an elliptic curve over F with discriminant Δ. Assume*

1. *for every $v|2$ in F, there is a finite extension F'_v/F_v such that E attains either good ordinary reduction or multiplicative reduction over F'_v;*
2. *E had no 2-torsion defined over F and Δ is not a square in F;*
3. *if Δ is totally negative then there is some $v|2$ in F such that $F_v[\sqrt{\Delta}] \not\subseteq F_v(\zeta_4)$.*

Then E is modular.

1.2 Outline

We elaborate on the strategy of the proof and then discuss the organization of the thesis.

1.2.1

We consider a certain large deformation ring where at places above 2 we enforce no p-adic Hodge theory conditions but demand that the representations be reducible. We identify this large deformation ring with a 2-nearly ordinary Hecke algebra. Hida's classicality theorem then implies that the representation we wish to show is modular is in fact a classical point of this large Hecke algebra.

We show that there is a surjective morphism from our large deformation ring R (tensored with a certain Iwasawa algebra) to the large Hecke algebra $\mathbf{T}$, and we call a prime ideal of R *pro-modular* if it is the pullback of one from $\mathbf{T}$. In order to show that all prime ideals of R are pro-modular, following Skinner and Wiles, we prove two things.

1. There exist certain dimension 1 characteristic 2 pro-modular prime ideals $\mathfrak{p}$ of the Hecke algebra such that we can adapt the 2-adic Taylor-Wiles method to prove the localizations and completions of R and $\mathbf{T}$ are isomorphic. We call these "nice" primes.
2. That every minimal prime of R contains one of the prime ideals in (1).

This then implies $R^{\text{red}} \cong \mathbf{T}$, which is enough to prove the modularity of ρ.

We expand on (1). The reason why the 2-adic Taylor-Wiles method fails in the residually dihedral case is due to the fact that the auxiliary primes don't exist in general because the image is too small. We thus need to ensure that the prime ideals in part (1) do not generate dihedral deformations, i.e. we need to avoid the dihedral locus. Given such a prime ideal, under some additional technical hypothesis, we show in §3.6 using a result of Pink, that the image of these representation have (essentially) open image. Using this, we compute the necessary cohomology groups and show the existence of the auxiliary primes necessary. As in the work of Skinner and Wiles, there is a technicality that must be dealt with all the while. Namely, we cannot adapt the Taylor-Wiles patching argument directly to the localized deformation rings and Hecke algebras. This is because at the heart of the patching is a simple diagonalization argument in the spirit of Cantor, based on the fact that we have infinitely many objects, each of which are finite. The finiteness is lost if one tries to perform patching after localizing. Hence we must perform the patching integrally, and then localize after taking limiting objects. Because things are done integrally, we must take care to control torsion submodules of the cohomology groups we compute. In particular, we will have to ensure that the torsion subgroups of certain cohomology groups and Hecke modules do not depend on the choice of our auxiliary data.

After carrying out the patching there is another complication that arises. In the Taylor-Wiles argument as improved by Kisin, the point of the patching is to create a limiting object that for dimension reasons is isomorphic to a certain power series ring over a local deformation ring. It is important that we can describe the minimal primes of this local deformation ring in order to show that the Hecke module in question is supported on the component of the universal deformation ring containing the point corresponding to the representation we wish to prove is modular. In the Skinner-Wiles method, the dimension argument can only be applied after localizing and completing the appropriate rings at our fixed "nice" prime ideal. The completion may increase the number of minimal primes of our local deformation ring, and if we cannot describe them anymore, then we cannot hope to say our Hecke module is supported on the ones we want. In order to avoid this, we show that if the "nice" primes are chosen appropriately, in particular that at places above 2 they lie over the distinguished locus of the local deformation ring, then the localized deformation ring is normal. This implies its completion is again normal and we can carry out the argument.

In order to show that the local deformation ring is normal at nice primes, it is not enough to analyze its characteristic zero points. We must know something about its mod 2 points as well. In the case of the local deformation ring at places above 2, we use the Deumuškin relation to explicitly compute the deformation ring on the locus of distinguished deformations, i.e. the locus where the action on the Galois stable line is not equal to one on the quotient. For the local deformation rings at primes not above 2, we use recent ideas of Snowden that allows us to show the deformation ring mod 2 may be thought of as representing a certain moduli space of unipotent matrices, whose structure he describes in detail.

The step (2) is carried out exactly as in Skinner and Wiles. We first show that a "nice" prime exists. Using step (1) we know that any minimal prime prime containing it is pro-modular. Then we assume that the set of irreducible components of the deformation ring has been partitioned into two subsets, $\mathcal{C}_1 \sqcup \mathcal{C}_2$, such that all elements in $\mathcal{C}_1$ are known to be pro-modular. We use a connectivity result of Raynaud that tells us there are irreducible components $C_i \in \mathcal{C}_i$, such that there is a prime ideal of large dimension in $C_1 \cap C_2$. If we can show that there is a "nice" prime in $C_1 \cap C_2$, then we prove that C_2 is pro-modular, and continue.

In order to show that $C_1 \cap C_2$ containing a prime of large dimension implies that it contains a "nice" prime, we need to show that the dihedral locus has small dimension. This is why we need the extra condition in the CM case. If every place above 2 splits in the quadratic extension L/F, the dihedral locus will form an irreducible component of the Hecke algebra, hence conjecturally also of the deformation ring, and so Raynaud's connectivity theorem does not guarantee the existence of non-dihedral primes in $C_1 \cap C_2$. If our extra assumption is satisfied, i.e. that there is some $v|2$ that does not split in L, then having large dimension implies that our representation is distinguished at this v, which then implies its not dihedral. We add the extra assumption that the corresponding local extension L_w/F_v is not the extension $F_v(\zeta_4)$ because we use base change to calculate the local deformation rings, and our calculation is done under the assumption that the local field contains a 4-th root of unity. In the case that L/F is not CM, we use known cases of Leopoldt's conjecture to base change to a situation where the dihedral locus has dimension small enough to guarantee that our large dimension prime ideal in $C_1 \cap C_2$ is non-dihedral.

1.2.2

The plan of the thesis is as follows. In section Chapter 2 we start by recalling some commutative algebra facts about complete Noetherian local rings. In particular we recall facts about their completed tensor product. We also record Raynaud's result and use it to prove a variant of the connectivity theorem in the appendix of [SW1]. This variant will be necessary for us as we do not show that our local deformation rings are Cohen-Macaulay. We then recall the notion of group actions and group chunk actions from [KW] that will be necessary for the patching argument. After this we turn to deformation theory, first recalling some general facts. We then study the nearly-ordinary deformation rings at places above p, the main points being to show that it is a domain of the correct dimension, that over a certain locus its reduction mod p is still a domain, and that a certain locus of its characteristic zero points are smooth. We then recall some facts about the local deformation rings away from p and show how one of Snowden's results implies that they are domains mod p. After that we turn to global deformations. The main points of this section are to recall that our global deformation ring has an appropriate presentation over the local one, in order to later apply the connectivity result of Raynaud, and to show that a certain group action on the universal deformation ring is free. Lastly we prove some small lemmas regarding deformations of dihedral representations, which will be useful in proving certain deformations are non-dihedral as well as to determine some properties of non-dihedral deformations.

In Chapter 3 we recall Hida's theory of nearly ordinary automorphic forms in the quaternionic case. In the first section we state definitions, analyse certain neatness properties of the open subgroups that will comprise our level, and state the relation to cuspidal automorphic representations of GL_2. In the next section we define the nearly ordinary Hecke algebra, and in the following section construct the universal nearly ordinary Hecke algebra. If the following section we recall the Galois representations associated to eigenforms, and show how they give a Galois representation into the universal Hecke algebra such that the induced map from the universal deformation ring is surjective and factors through the quotient defined in 2.6.5. In the final section we augment the level with auxiliary primes. The auxiliary primes we use to augment the level may have the property that the corresponding Frobenii do not have distinct eigenvalues under the residual representation. Because of this we cannot prove the standard control theorem for Taylor-Wiles primes, as there may be lifts of $\overline{\rho}$ that are Steinberg at these primes. We show however, that the obstruction to the usual control theorem is annihilated by an element that becomes invertible after localizing at one of our "nice" prime ideals.

In Chapter 3.6 we show the existence of the set of auxiliary primes associated to a representation into $\mathrm{GL}_2(A)$, with A the ring of integers in a characteristic 2 local field K, satisfying some technical hypotheses. The main input is Pink's result, [P], that allows us to conclude that the intersection of the image with $\mathrm{SL}_2(A)$ is conjugate to an open subgroup in $\mathrm{SL}_2(A_0)$, where A_0 is the ring of integers of a characteristic 2 local subfield $K_0 \subset K$. This allows us to compute explicitly with cocycles. A subtle point that must be taken into consideration is that the cohomology groups in question must be computed integrally, and their torsion subgroups must be bounded independently of the choice of auxiliary primes.

In Chapter 4, we prove the $R^{\mathrm{red}} = \mathbf{T}$ theorem. This chapter together with Chapter 3.6 comprise the technical heart of the work. The patching argument is a synthesis of the proof of Proposition 9.3 of [KW] and §5 of [SW1]. The idea is to mimic the proof of Proposition 9.3 in [KW], but to define the maps from a power series over the local deformation ring $R_{\mathrm{loc}}[[x_1, \ldots, x_k]]$ to our augmented level global deformation rings R_n, in such a way that the $x_1, \ldots, x_k$ are mapped to the pullback to R_n of our fixed "nice" prime ideal, instead of the maximal ideal. In this way, we get a surjection after localizing and completing at the "nice" prime. When defining these maps, we need to ensure that certain cokernels are finite of bounded size, so that when we take a projective limit the ranks of the resulting limiting modules do not grow. It is due to this reason that we needed to ensure that the torsion subgroups of the cohomology groups computed in Chapter 3.6 did not depend on the auxiliary primes. After proving the localized $R^{\mathrm{red}} = \mathbf{T}$ theorem, we apply the connectivity argument to conclude that $R^{\mathrm{red}} = \mathbf{T}$.

The last chapter, Chapter 5, proves the main theorem. We first recall some congruences proved in [K2] and [KW] necessary to show the existence of appropriate automorphic lifts after base change. We also prove a small lemma that shows the existence of ordinary lifts in the residually dihedral case, using a result of Wiles, [W], that allows one to insert an ordinary Hilbert modular form of parallel weight 1 into a p-adic family. We then prove the main theorem, by applying base change putting together the aforementioned congruences together with known cases of Leopoldt's conjecture so that we satisfy the assumptions of the $R^{\mathrm{red}} = \mathbf{T}$ theorem of Chapter 4.

1.3 Notation and conventions

We state some notation and conventions used in this thesis. We will denote by p a rational prime throughout. In the later sections we will take $p = 2$. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For any subfield $L \subseteq \overline{\mathbb{Q}}$ we set $G_L = \mathrm{Gal}(\overline{\mathbb{Q}}/L)$. Throughout F will denote a totally real field inside $\overline{\mathbb{Q}}$. Given a finite set S of places of F we denote by $G_{F,S} = \mathrm{Gal}(F_S/F)$, where F_S is the maximal Galois extension of F contained in $\overline{\mathbb{Q}}$ that is unramified outside S. We let I_F denote the set of embeddings $F \hookrightarrow \overline{\mathbb{Q}}$.

For each rational prime ℓ, fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ and embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. We can then view I_F as the set of embedding of F into any of the $\overline{\mathbb{Q}}_\ell$ or $\mathbb{C}$. For any place of F we denote by F_v the completion of F at v inside $\overline{\mathbb{Q}}_\ell$, or $\mathbb{C}$. In the case that v is non-archimedean we write $G_v = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/F_v)$ and let I_v denote the inertia subgroup. For v archimedean we let $G_v = \mathrm{Gal}(\mathbb{C}/F_v)$. In either case we identify G_v with a decomposition group of G_F at v via the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, or $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

As usual $\mathbb{A}_F$ will denote the ring of adeles of F and $\mathbb{A}_F^\infty$ will denote the subring of finite adeles. We normalize the isomorphism of local class field theory so that uniformizers correspond to arithmetic Frobenii, and normalize global class field theory compatibly.

We will denote by ϵ_p the p-adic cyclotomic character and $\overline{\epsilon_p}$ is reduction mod p. We use homological conventions for our Galois representations. For example, the Galois representation attached to an elliptic curve is the one coming from its Tate module, not cohomology. With this convention, a representation is ordinary at v if the local representation is reducible with an unramified quotient.

We will let E denote a finite extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$. We will let $\mathcal{O}$ denote its ring of integers and $\mathbb{F}$ its residue field. We will occasionally enlarge E if necessary. We let $\mathrm{CNL}_{\mathcal{O}}$ denote the category of complete Noetherian local $\mathcal{O}$-algebra A such that the structure morphism $\mathcal{O} \to A$ induces an isomorphism of residue fields. The morphisms in $\mathrm{CNL}_{\mathcal{O}}$ are local $\mathcal{O}$-algebra morphisms. Given an object B in $\mathrm{CNL}_{\mathcal{O}}$ we let CNL_B denote the subcategory whose objects are B-algebra and whose morphisms are B-algebra morphisms. Given any local ring A we denote by $\mathfrak{m}_A$ its maximal ideal.

Given a separable field extension L/K, we let $\mathrm{Nm}_{L/K}$ denote the norm from L/K. We refer the reader to the beginning of each section for more notation and conventions that will be employed in that particular section.

CHAPTER 2

Deformation Theory

In this chapter we develop the deformation theory necessary. In §2.1 we recall some commutative algebra facts for complete Noetherian local rings with finite residue field. Of particular importance to our applications later will be recalling some facts regarding the completed tensor product of such rings, cf. 2.1.4, and a connectivity result, cf. 2.1.8, which is an application of a theorem of Raynaud and is a variant of Corollary A.2 of [SW1].

The following section recalls some facts from [KW] regarding group actions in the category $\mathrm{CNL}_{\mathcal{O}}$. In particular the notion of group chunks and building free actions using them is essential to the 2-adic Taylor-Wiles method developed in [KW] and we will use these results in §4. The following section recalls some general definitions and results about deformation theory.

The subsequent section deals with determining the local deformation rings at the prime p. We follow the construction of Geraghty [G1]. However as we deal only with dimension 2, we can be more explicit and use the Demuškin relation to determine properties of the local deformation ring. In particular, we use the Demuškin relation to prove that a certain open subset of the special fibre is reduced. This will be important for showing that a certain localization of a completed tensor product of local deformation rings is normal §4. Also important to establish this normality is to prove a certain open subset of the generic fibre is smooth, and we prove this following [G1].

The next section deals with determining the local deformation rings at the places not equal to p. Most of this is simply recalling results proved in [K2, K3, KW]. However we will also need information about the special fibre of such rings. This is easy in the archimedean case, and to do this in the case $l \neq p$ we use a recent result of Snowden [S]. This knowledge of the special fibre will be important for showing that a certain localization of a completed tensor product of local deformation rings is normal §4.

In the next section we deal with deformations of global Galois groups. We recall some definitions and properties and show that a certain quotient of the universal deformation ring (tensored with an Iwasawa algebra) can be presented as a complete Noetherian domain modulo "few" relations, which will be necessary to apply the connectivity result 2.1.8 later. We also recall some properties of twisting the universal deformation by characters as in [KW] which is essential to the 2-adic method.

In the last section we prove some easy facts regarding deformations of dihedral representations, that will be useful for determining the images of non-dihedral deformations to characteristic p local fields as well as to establish criteria for a deformation to be non-dihedral.

We now set up some notation. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. Fix a choice of uniformizer ϖ_E for E. We denote by $\mathrm{CNL}_{\mathcal{O}}$ the category of complete local Noetherian $\mathcal{O}$-algebras A such that the structure map $\mathcal{O} \to A$ is local and induces an isomorphism on residue fields. The morphisms in $\mathrm{CNL}_{\mathcal{O}}$ are local $\mathcal{O}$-algebra morphisms. We will refer to objects in $\mathrm{CNL}_{\mathcal{O}}$ as $\mathrm{CNL}_{\mathcal{O}}$-*algebras*. Given a $\mathrm{CNL}_{\mathcal{O}}$-algebra B, we denote by CNL_B the subcategory of $\mathrm{CNL}_{\mathcal{O}}$ whose objects are B-algebras and whose morphisms are B-algebra morphims. We let Ar_B denote the full subcategory of CNL_B consisting of Artinian objects. Given a finite extension E'/E, we also let $\mathrm{Ar}_{E'}$ denote the category of Artinian local rings with residue field E' with topology given by its structure as a finite dimensional E'-vector space. Note that such rings are canonically E'-algebras, and the morphisms in $\mathrm{Ar}_{E'}$ are local E'-algebra morphisms. Given an object A in any of the above categories we let $\mathfrak{m}_A$ denote its maximal ideal.

2.1 Some commutative algebra

We recall some facts about objects in $\mathrm{CNL}_{\mathcal{O}}$ and their completed tensor products that will be useful to us later.

2.1.1

Recall a scheme X is called *Jacobson* if for any closed subset $Z \subseteq X$, the set of closed points in Z is dense. We say a ring R is Jacobson if $\operatorname{Spec} R$ is Jacobson. If R is a $\mathrm{CNL}_{\mathcal{O}}$-algebra, Corollary 10.5.8 of [G4] shows that $\operatorname{Spec} R \smallsetminus \{\mathfrak{m}_R\}$ is Jacobson, and if p is not nilpotent in R then $R[1/p]$ is Jacobson. The following Proposition is taken from §2.3 of [KW].

Proposition 2.1.2. *Let R be an $\mathcal{O}$-flat* $\mathrm{CNL}_{\mathcal{O}}$*-algebra.*

1. *There is a finite extension E'/E with ring of integers $\mathcal{O}$' and a local $\mathcal{O}$-algebra morphism $R \to \mathcal{O}$.*
2. *Ever maximal ideal of $R[1/p]$ is the image of the generic point of* $\operatorname{Spec} \mathcal{O}' \to \operatorname{Spec} R$, *with $R \to \mathcal{O}'$ as in (1).*
3. *Let I is an ideal of R and let $X(I)$ denote the set of all morphisms $R \to \mathcal{O}'$ as in (1) whose kernel contains I. If R/I is $\mathcal{O}$-flat and reduced, then $I = \cap_{x \in X(I)} \ker(x)$*

Proof. Take $x_1, \ldots, x_d \in R$ such that their image in $R/\varpi_E R$ is a system of parameters for $R/\varpi_E R$. Then $\varpi_E, x_1, \ldots, x_d$ is a system of parameters for R, and so $R/(x_1, \ldots, x_d)$ is of dimension 1. Let $R' = R/(x_1, \ldots, x_d)$ and let $\mathfrak{q}$ be a minimal prime of R' such that $R'/\mathfrak{q}$ has dimension 1. Since $R'/(\mathfrak{q} + \varpi_E R')$ has finite length, $R'/\mathfrak{q}$ is a finite $\mathcal{O}$-module. Since $R'/\mathfrak{q}$ is a dimension 1 finite $\mathcal{O}$-algebra domain, its normalization is the ring of integers $\mathcal{O}'$ in a finite extension E'/E. This proves (1).

Let $\mathfrak{m}$ be a maximal ideal of $R[1/p]$ and let $\mathfrak{p} = \mathfrak{m} \cap R$. Since $R/\mathfrak{p}$ is $\mathcal{O}$-flat, part (1) of the proposition implies that we have a local $\mathcal{O}$-algebra morphism $x : R/\mathfrak{p} \to \mathcal{O}'$, for $\mathcal{O}'$

the ring of integers in some finite extension E'/E. The image of the generic point of x in $\operatorname{Spec} R[1/p]$ is $\mathfrak{m}$. This proves (2).

It remains to prove (3). Let $J = \cap_{x \in X(I)} \ker(x)$. Since R/I is $\mathcal{O}$-flat, it suffices to prove that $(J/I)[1/p] = 0$. This follows from the fact that $(R/I)[1/p]$ is Jacobson and reduced. $\square$

2.1.3

Let R_1 and R_2 be $\mathrm{CNL}_{\mathcal{O}}$-algebras. Then their completed tensor product over $\mathcal{O}$ is again a $\mathrm{CNL}_{\mathcal{O}}$-algebra. To see this, let $\mathfrak{m}$ denote the kernel of the natural map $R_1 \hat{\otimes}_{\mathcal{O}} R_2 \to \mathbb{F} \otimes_{\mathcal{O}} \mathbb{F} = \mathbb{F}$. Since $R_1 \hat{\otimes}_{\mathcal{O}} R_2$ is complete for the $\mathfrak{m}$-adic topology and we have an inection $\mathbb{F}^\times \hookrightarrow (R_1 \hat{\otimes}_{\mathcal{O}} R_2)^\times$, we see that any element which does not belong to $\mathfrak{m}$ is invertible, and so $R_1 \hat{\otimes}_{\mathcal{O}} R_2$ is local. We can write R_i as a quotient of a power series ring $\mathcal{O}[[x_1, \ldots, x_{d_i}]]$, and so $R_1 \hat{\otimes}_{\mathcal{O}} R_2$ can be written as a quotient of the power series ring $\mathcal{O}[[x_1, \ldots, x_{d_1+d_2}]]$, hence is Noetherian.

If R_1 and R_2 are flat $\mathrm{CNL}_{\mathcal{O}}$-algebras then $R_i \to R_1 \hat{\otimes}_{\mathcal{O}} R_2$ is flat, cf. Lemma 19.7.1.2 of Chapter 0 of [G2]. In particular $R_1 \hat{\otimes}_{\mathcal{O}} R_2$ is $\mathcal{O}$-flat. The proofs of parts (1) and (2) of the proposition below are taken from Proposition 2.3 of [KW] and Lemma 3.4.12 of [K2].

Proposition 2.1.4. *Let R_1 and R_2 be $\mathrm{CNL}_{\mathcal{O}}$-algebras and let $R = R_1 \hat{\otimes}_{\mathcal{O}} R_2$.*

1. *Let E'/E be a finite extension and for each $i = 1, 2$ let $x_i : R_i[1/p] \to E'$ be an E'-point that is formally smooth over E. The E'-valued point $(x_1, x_2) : R[1/p] \to E'$ is formally smooth over E.*

2. *If for each $i = 1, 2$, R_i is $\mathcal{O}$-flat and $R_i[1/p]$ is geometrically integral, then so is $R[1/p]$. In particular, R is a domain.*

3. *Assume that each R_i is $\mathcal{O}$-flat and that $R_i/\mathfrak{q}_i[1/p]$ is geometrically integral. Then any minimal prime of R is of the form $\mathfrak{q}_1 \hat{\otimes} R_1 + R_1 \hat{\otimes} \mathfrak{q}_2$.*

4. *Assume each R_i is an $\mathbb{F}$-algebra and let $\mathrm{Nil}(R_i)$ denote the nilradical of R_i. The nil-radical of R is $\mathrm{Nil}(R_1) \hat{\otimes}_{\mathbb{F}} R_2 + R_1 \hat{\otimes}_{\mathbb{F}} \mathrm{Nil}(R_2)$.*

Proof. Let $\mathcal{O}'$ be the ring of integers of E', for E' as in (1). Since $R[1/p] \otimes_E E' \cong (R \otimes_{\mathcal{O}} \mathcal{O}')[1/p]$ and $R \otimes_{\mathcal{O}} \mathcal{O}' \cong (R_1 \otimes_{\mathcal{O}} \mathcal{O}')\hat{\otimes}_{\mathcal{O}'}(R_2 \otimes_{\mathcal{O}} \mathcal{O}')$, it suffices to prove (1) after replacing $\mathcal{O}$ with $\mathcal{O}'$, R_i with $R_i \otimes_{\mathcal{O}} \mathcal{O}'$, and R with $R \otimes_{\mathcal{O}} \mathcal{O}'$, and so we may assume x_1 and x_2 are E-valued points. For $d_i = \dim(R_i)_{x_i}$, we have isomorphisms $(R_i)^{\wedge}_{x_i} \cong E[[x_1, \ldots, x_{d_i}]]$, and we then have isomorphisms

$$R^{\wedge}_{(x_1,x_2)} \cong (R_1)^{\wedge}_{x_1} \hat{\otimes}_E (R_2)^{\wedge}_{x_2} \cong E[[x_1, \ldots, x_{d_1+d_2}]],$$

which proves (1).

We now show (2). Take an arbitrary finite extension E'/E. We wish to show that $R[1/p] \otimes_E E'$ is a domain. By what was noted at the beginning of the proof of part (1), it suffices to replace $\mathcal{O}$ with $\mathcal{O}'$, the ring of integers in E', and each R_i by $R_i \otimes_{\mathcal{O}} \mathcal{O}'$, and then show that $R[1/p]$ is a domain. Note that by assumption each of the $R_i[1/p]$ remain geometrically integral after this base extension. By enlarging E if necessary, we may assume that each $R_i[1/p]$ has an E-point x_i that is formally smooth over E. By part (1), we know that the completion of $R[1/p]$ at (x_1, x_2) is formally smooth over E, and so to prove (2) it suffices to show there is an injection $R \hookrightarrow R^{\wedge}_{(x_1,x_2)}$. By part (1) of 2.1.2, there are local $\mathcal{O}$-algebra morphisms $y_i : R_i \to \mathcal{O}$ such that $x_i = y_i[1/p]$. Let $\mathfrak{p}_i = \ker y_i$, and for every $n \geq 1$ let $\mathfrak{a}_n$ be the kernel of the surjection $R \to R_1/\mathfrak{p}_1^n \otimes_{\mathcal{O}} R_2/\mathfrak{p}_2^n$. The ring R is complete for the topology defined by $\{\mathfrak{a}_n\}$. For each $i = 1, 2$, let $\mathfrak{q}_{i,n} = \mathfrak{p}_i^n[1/p] \cap R_i$. A theorem of Chevalley, cf.Theorem 13 of Chapter 8.5 of [ZS], R_i is complete for the topology defined by $\{\mathfrak{q}_{i,n}\}$. If we then let $\mathfrak{b}_n$ denote the kernel of the surjection $R \to R_1/\mathfrak{q}_{1,n} \otimes_{\mathcal{O}} R_2/\mathfrak{q}_{2,n}$, we see that R is complete for the topology defined by $\{\mathfrak{b}_n\}$. Finally, since each R_i is $\mathcal{O}$-flat, we have injections $R_1/\mathfrak{q}_{1,n} \otimes_{\mathcal{O}} R_2/\mathfrak{q}_{2,n} \hookrightarrow (R/\mathfrak{p}^n)[1/p] \otimes_E (R/\mathfrak{p}_2^n)[1/p]$ for all n. But then we have

$$R \cong \varprojlim_n R_1/\mathfrak{q}_{1,n} \otimes_{\mathcal{O}} R_2/\mathfrak{q}_{2,n} \hookrightarrow \varprojlim_n (R/\mathfrak{p}_1^n)[1/p] \otimes_E (R/\mathfrak{p}_2^n)[1/p] \cong R^{\wedge}_{(x_1,x_2)},$$

which is what we wanted to prove.

Let $\mathfrak{q}$ be a minimal prime of R and let $\mathfrak{q}_i$ be its pullback to R_i. We have $\mathfrak{q}_1\hat{\otimes}_{\mathcal{O}}R_2 + R_1\hat{\otimes}_{\mathcal{O}}\mathfrak{q}_2 \subset \mathfrak{q}$. Note that $(R_1/\mathfrak{q}_1)\hat{\otimes}_{\mathcal{O}}(R_2/\mathfrak{q}_2)$ is a domain by (2), and has the same dimension as R. The natural surjection $R \to (R_1/\mathfrak{q}_1)\hat{\otimes}(R_2/\mathfrak{q}_2)$ then must have kernel $\mathfrak{q}$.

We now prove (4). Let $\mathrm{Nil}(R)$ denote the nilradical of R. Note that $\mathrm{Nil}(R_1)\hat{\otimes}_{\mathbb{F}}R_2 = \mathrm{Nil}(R_1)\otimes_{\mathbb{F}} R_2$ and $R_1\hat{\otimes}_{\mathbb{F}}\mathrm{Nil}(R_2) = R_1\otimes_{\mathbb{F}}\mathrm{Nil}(R_1)$ since each $\mathrm{Nil}(R_i)$ is finite length. By considering simple tensors we see that $\mathrm{Nil}(R_1)\otimes_{\mathbb{F}} R_2 + R_1\otimes_{\mathbb{F}}\mathrm{Nil}(R_2)\subseteq \mathrm{Nil}(R)$. Consider the surjection

$$R \longrightarrow R/(R_1\otimes_{\mathbb{F}}\mathrm{Nil}(R_2) + R_1\otimes_{\mathbb{F}}\mathrm{Nil}(R_1)) \cong (R_1/\mathrm{Nil}(R_1))\hat{\otimes}_{\mathbb{F}}(R_2/\mathrm{Nil}(R_2)).$$

A theorem of Chevalley, cf. Corollary 7.5.7 of [G3], shows that $(R_1/\mathrm{Nil}(R_1))\hat{\otimes}_{\mathbb{F}}(R_2/\mathrm{Nil}(R_2))$ is reduced, and so $\mathrm{Nil}(R)\subseteq \mathrm{Nil}(R_1)\otimes_{\mathbb{F}} R_2 + R_1\otimes_{\mathbb{F}}\mathrm{Nil}(R_2)$. □

2.1.5

We prove a connectivity result similar to Proposition A.2 of [SW1]. As in [SW1], our proposition relies crucially on Corollary 4.2 in [R1]. The difference between Proposition A.2 of [SW1] and our proposition is that the ring R in the statement of our proposition is assumed to be a complete Noetherian local domain, whereas in [SW1] the ring R is assumed to be a local Cohen-Macaulay ring. This is necessary for us as we will present our global deformation ring as a quotient of the completed tensor product of the local deformation rings, and we are not able to show that the local deformations rings are Cohen-Macaulay. We first quote an application of Raynaud's theorem in the guise of the following proposition.

Proposition 2.1.6. *Let R be an excellent local ring of dimension d satisfying Serre's property (S_2). Let I be a proper ideal of R generated by r elements. If $d-r\geq 2$, then* $\mathrm{Spec}\,(R/I)\smallsetminus\{\mathfrak{m}_R\}$ *is connected.*

Proof. Let $R^\wedge$ be the completion of R, and let $I^\wedge = IR^\wedge$. Since $\dim R^\wedge = \dim R$ and $R\to R^\wedge$ is faithfully flat, it suffices to show that $\mathrm{Spec}\,(R^\wedge/I^\wedge)\smallsetminus\{\mathfrak{m}_{R^\wedge}\}$ is connected. Since R is excellent, $R^\wedge$ also satisfies (S_2), cf. part (v) of Scholie 7.8.3 of [G3]. Raynaud's result, Corollary 4.2 of [R1], shows that the condition $\mathrm{Lef}(X,Y)$ of Expose X of [G5] is satisfied for $X = \mathrm{Spec}\,R^\wedge\smallsetminus\{\mathfrak{m}_{R^\wedge}\}$ and $Y = \mathrm{Spec}\,(R^\wedge/I^\wedge)\smallsetminus\{\mathfrak{m}_{R^\wedge}\}$. Then Corollary 2.4 of Expose X of [G5] implies that $\mathrm{Spec}\,(R^\wedge/I^\wedge)\smallsetminus\{\mathfrak{m}_{R^\wedge}\}$ is connected if and only if $\mathrm{Spec}\,R^\wedge\smallsetminus\{\mathfrak{m}_{R^\wedge}\}$ is connected. The fact that the latter is connected is a special case of a theorem of Hartshorne, cf. Theorem 3.6 of Expose III of [G5]. □

Corollary 2.1.7. *Let R be an excellent local ring of dimension d satisfying Serre's property (S_2). Let I be a proper ideal of R generated by r elements. Let $\mathcal{C}$ denote the set of irreducible components of* $\operatorname{Spec} R/I$*, and let $\mathcal{C} = \mathcal{C}_1 \sqcup \mathcal{C}_2$ be a partition of $\mathcal{C}$ with each $\mathcal{C}_i$ is non-empty. If $d-r \geq 2$, then there is $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$ such that $C_1 \cap C_2$ contains a prime of dimension $d-r-1$.*

Proof. We prove this by induction on $d-r$. If $d-r = 2$, then this is 2.1.6. Now suppose $d-r > 2$. Applying 2.1.6 again, we know that there is $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$ such that $C_1 \cap C_2$ contains a prime $\mathfrak{p}$ of dimension 1. Let $\mathcal{C}_{\mathfrak{p}}$ denote the set of irreducible components of $\operatorname{Spec}(R/I)_{\mathfrak{p}}$. By our choice of $\mathfrak{p}$, the nontrivial partition $\mathcal{C} = \mathcal{C}_1 \sqcup \mathcal{C}_2$ induces a nontrivial partition $\mathcal{C}_{\mathfrak{p}} = \mathcal{C}_{\mathfrak{p},1} \sqcup \mathcal{C}_{\mathfrak{p},2}$. The induction hypothesis implies there is $C_1 \in \mathcal{C}_{\mathfrak{p},1}$ and $C_2 \in \mathcal{C}_{\mathfrak{p},2}$ such that $C_1 \cap C_2$ contains a prime $\mathfrak{q}$ of dimension $d-r-2$. As a prime of R/I it has dimension $d-r-1$, which establishes the inductive step, and hence the proposition. □

Corollary 2.1.8. *Let R be a complete Noetherian local domain of dimension d and let I be a proper ideal of R generated by r elements. Let $\mathcal{C}$ denote the set of irreducible components of* $\operatorname{Spec} R/I$*, and let $\mathcal{C} = \mathcal{C}_1 \sqcup \mathcal{C}_2$ be a partition of $\mathcal{C}$ with each $\mathcal{C}_i$ is non-empty. If $d-r \geq 2$, then there is $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$ such that $C_1 \cap C_2$ contains a prime of dimension $d-r-1$.*

Proof. Let R' denote the normalization of R, and let $I' = IR'$. Note that $\dim R' = d$ and that I' can be generated by r elements. Since R is a complete Noetherian local domain, R' is finite over R and is also local, cf. part (vii) of Scholie 7.8.3 of [G3]. The finite surjection $\operatorname{Spec} R'/I' \to \operatorname{Spec} R/I$ induces a non-trivial partition $\mathcal{C}' = \mathcal{C}'_1 \sqcup \mathcal{C}'_2$ of the set of irreducible components of $\operatorname{Spec} R'/I'$. By 2.1.7, there is $C'_1 \in \mathcal{C}'_1$ and $C'_2 \in \mathcal{C}'_2$ such that $C'_1 \cap C'_2$ contains a prime $\mathfrak{p}'$ of dimension $d-r-1$. Its image in $\operatorname{Spec} R/I$ satisfies the conclusion of the proposition. □

We conclude this subsection with a lemma that we will have use for later, cf. Exercise 16.10 of [M1].

Lemma 2.1.9. *Let R be a local Noetherian ring and let $r_1, \ldots, r_n \in R$, $n \geq 1$. If $r_1, \ldots, r_n$ generate a prime ideal of height n then R is a domain and for any $1 \leq i \leq r$, $r_1, \ldots, r_i$ generates a prime ideal of height i.*

Proof. By induction on n it suffices to show that if $r \in R$ generates a prime ideal of height 1, then R is a domain. Take a minimal prime $\mathfrak{q} \subset rR$. For any $x \in \mathfrak{q}$ we have $x = ry$ for some y. Then $r \notin \mathfrak{q}$ implies that $y \in \mathfrak{q}$, and so $r\mathfrak{q} = \mathfrak{q}$. Nakayama's lemma implies $\mathfrak{q} = 0$. □

2.2 Group actions on $\mathrm{CNL}_{\mathcal{O}}$

We quote some results and definitions regarding group action on the category $\mathrm{CNL}_{\mathcal{O}}$ from [KW], which will be necessary for the patching argument in §4. This material is taken directly from §2.4-2.6 of [KW], and we refer the reader there for proofs.

2.2.1

Let G be a formal $\mathcal{O}$-group scheme represented by an object in $\mathrm{CNL}_{\mathcal{O}}$. We will call G a $\mathrm{CNL}_{\mathcal{O}}$*-group*. Leting $A(G)$ denote the $\mathrm{CNL}_{\mathcal{O}}$-algebra representing G, we note that the group structure on G is defined in the same way as the Hopf algebra structure on an affine algebraic group except that our comultiplication takes values in the completed tensor product $A(G) \to A(G)\hat{\otimes}_{\mathcal{O}}A(G)$.

Let $X = \mathrm{Spf}A(X)$ for $A(X)$ a $\mathrm{CNL}_{\mathcal{O}}$-algebra. We similarly define an action of G on X, i.e. a $\mathrm{CNL}_{\mathcal{O}}$-morphism $\gamma : A(X) \to A(G)\hat{\otimes}_{\mathcal{O}}A(X)$ making $G(A) \times X(A) \to X(A)$ a group action functorial in A. An action is *free* if the map $G \times X \to X \times X$ given by $(g, x) = (gx, x)$ on points is a closed immersion. This is equivalent to requiring that for any $\mathrm{CNL}_{\mathcal{O}}$-algebra A, $G(A)$ acts freely on $X(A)$.

Let Orb denote the functor that sends a $\mathrm{CNL}_{\mathcal{O}}$-algebra A to the set of orbits of $X(A)$

under G. Let $A(X)_0$ be the subalgebra of $A(X)$ consisting of elements a such that $\gamma(a) = 1 \otimes a$, where $\gamma : A(X) \to A(G)\hat{\otimes}_{\mathcal{O}}A(X)$ is the $\mathrm{CNL}_{\mathcal{O}}$-morphism defining the group action. The elements $a \in A(X)_0$ are the functions on X that are constant on orbits, i.e. $a(gx) = a(x)$ for any $\mathrm{CNL}_{\mathcal{O}}$-algebra A, $x \in X(A)$, $g \in G(A)$. The following proposition is (part of) Proposition 2.5 of [KW].

Proposition 2.2.2. *Let G be a $\mathrm{CNL}_{\mathcal{O}}$-group acting freely on X as above. Assume that G is smooth and the action of G on X is free. Then $A(X)_0$ is a $\mathrm{CNL}_{\mathcal{O}}$-algebra and represents* Orb*, and the map $X \to \mathrm{Orb}$ is smooth. Letting $G_{\mathrm{Orb}} = \mathrm{Orb} \times G$, the map $G_{\mathrm{Orb}} \times_{\mathrm{Orb}} X \to X$ makes X a torsor over* Orb *for G_{Orb}. This torsor is trivial and there is an isomorphism of X with $\mathrm{Orb} \times G$.*

2.2.3

Let H be a finitely generated abelian group whose torsion is a power of p. We call the *diagonalizable* $\mathrm{CNL}_{\mathcal{O}}$*-group* associated to H, denoted H^*, the $\mathrm{CNL}_{\mathcal{O}}$-group defined by completing the diagonalizable group associated to H as in Exposé VIII of [G5] at the identity element of the special fibre. Concretely, we have $\mathbb{Z}^* = \mathbb{G}_m^{\wedge}$, the formal torus on $\mathrm{CNL}_{\mathcal{O}}$, and $(\mathbb{Z}/p^r\mathbb{Z})^* = \mu_{p^r}$. All other H^* are products of these two examples. Note that a surjection of abelian groups $H \to H'$ induces a closed immersion $(H')^* \to H^*$. The following is proposition 2.6 of [KW].

Proposition 2.2.4. *Let X be a representable functor on $\mathrm{CNL}_{\mathcal{O}}$ and let H be a finitely generated group such that H^* acts freely on X.*

1. *A quotient $H^*\backslash X$ exists in $\mathrm{CNL}_{\mathcal{O}}$. The morphism $X \to H^*\backslash X$ makes X a torsor over $H^*\backslash X$ for $(H^*\backslash X) \times H^*$.*

2. *If $H \to H'$ is a surjective morphism of abelian groups, then $(H')^*\backslash X$ has a natural free action of $H^*/(H')^*$ and $H^*\backslash X$ is naturally isomorphic to the quotient of $(H')^*\backslash X$ by the action of $H^*/(H')^*$*

2.2.5

Let $m \geq 1$ be an integer. Denote by $\mathrm{CNL}_{\mathcal{O}}^{[m]}$ the full subcategory of $\mathrm{CNL}_{\mathcal{O}}$ consisting of objects A such that $\mathfrak{m}_A^m = 0$. For a $\mathrm{CNL}_{\mathcal{O}}$-algebra A, we denote by $A^{[m]}$ the $\mathrm{CNL}_{\mathcal{O}}^{[m]}$-algebra $A/\mathfrak{m}_A^m$. Note that $A \mapsto A^{[m]}$ defines a functor from $\mathrm{CNL}_{\mathcal{O}}$ to $\mathrm{CNL}_{\mathcal{O}}^{[m]}$, and we call $A^{[m]}$ the *truncation to level* m. For A a $\mathrm{CNL}_{\mathcal{O}}$-algebra, the restriction of $\mathrm{Spf}A$ to $\mathrm{CNL}_{\mathcal{O}}^{[m]}$ is isomorphic to $\mathrm{Spf}A^{[m]}$. If X is a representable functor on $\mathrm{CNL}_{\mathcal{O}}$, $X = \mathrm{Spf}A(X)$, we let $X^{[m]} = \mathrm{Spf}A(X)^{[m]}$ on $\mathrm{CNL}_{\mathcal{O}}^{[m]}$, and call $X^{[m]}$ the *truncation to level* m of X. If $X = X^{[m]}$, then any $\mathrm{CNL}_{\mathcal{O}}$-map $X \to Y$ factors through $Y^{[m]}$. A map $X \to Y$ is a closed immersion if and only if $X^{[m]} \to Y^{[m]}$ is a closed immersion for each m. Note that if A_1 and A_2 are $\mathrm{CNL}_{\mathcal{O}}^{[m]}$ algebras, then $A_1 \hat{\otimes}_{\mathcal{O}} A_2$ may not be, and the restriction of $\mathrm{Spf}A_1 \times \mathrm{Spf}A_2$ to $\mathrm{CNL}_{\mathcal{O}}^{[m]}$ is represented by $(A_1 \hat{\otimes}_{\mathcal{O}} A_2)^{[m]}$.

We define a *group chunk of level* m to be $G = \mathrm{Spf}A(G)$, where $A(G)$ is a $\mathrm{CNL}_{\mathcal{O}}^{[m]}$-algebra equiped with $\mathrm{CNL}_{\mathcal{O}}^{[m]}$-morphisms $A(G) \to (A(G)\hat{\otimes}_{\mathcal{O}} A(G))^{[m]}$, $A(G) \to A(G)$, and $A(G) \to \mathcal{O}^{[m]}$ satisfying the usual diagrams defining a Hopf algebra. A group chunk defines group functor on $\mathrm{CNL}_{\mathcal{O}}^{[m]}$. Note that if we are given a $\mathrm{CNL}_{\mathcal{O}}$-group G, $G^{[m]}$ is a group chunk of level m for all $m \geq 1$.

Let $X = \mathrm{Spf}A(X)$ with $A(X)$ a $\mathrm{CNL}_{\mathcal{O}}^{[m]}$-algebra, and let G be a group chunk of level m. We define a *group action chunk of level* m, to be a morphism $(G \times X)^{[m]} \to X$ defining a functorial group action on $\mathrm{CNL}_{\mathcal{O}}^{[m]}$. Note that the map $(G \times X)^{[m]} \to X \times X$ given on points by $(g, x) \mapsto (gx, x)$ factors through $(X \times X)^{[m]}$. We call the group chunk action *free* if $(G \times X)^{[m]} \to (X \times X)^{[m]}$ is a closed immersion. Given a $\mathrm{CNL}_{\mathcal{O}}$-action $G \times X \to X$, $(G^{[m]} \times X^{[m]})^{[m]} \to X^{[m]}$ is a group action chunk of level m.

We record Proposition 2.7 of [KW].

Proposition 2.2.6. *Let G be a $\mathrm{CNL}_{\mathcal{O}}$ group. Suppose for each $m \geq 1$, we have $\mathrm{CNL}_{\mathcal{O}}^{[m]}$-objects A_m and $\mathrm{CNL}_{\mathcal{O}}$-morphisms $A_{m+1} \to A_m$, such that $A_\infty = \varprojlim A_m$ is in $\mathrm{CNL}_{\mathcal{O}}$ (i.e. is Noetherian). Assume that for each m, we have a group action chunk of $G^{[m]}$ on $\mathrm{Spf}A_m$.*

1. *Then there is a unique $\mathrm{CNL}_{\mathcal{O}}$-group action $G \times \mathrm{Spf}A_\infty \to \mathrm{Spf}A_\infty$ such that for each m,*

the group action chunk of $G^{[m]}$ *on* $\mathrm{Spf} A_\infty^{[m]}$ *is compatible with the group action chunk of* $G^{[m]}$ *on* A_m *via the closed immersion* $\mathrm{Spf} A_m \to \mathrm{Spf} A_\infty^{[m]}$.

2. *If the group action chunks of* $G^{[m]}$ *on* $\mathrm{Spf} A_m$ *are free, then so is the action of* G *on* $\mathrm{Spf} A_\infty$.

2.3 Some general deformation theory

We first introduce some notation and state some useful facts. Our references for this section are [M2] and section 2 of [KW].

2.3.1

Let G be a profinite group and let

$$\overline{\rho} : G \longrightarrow \mathrm{GL}_n(\mathbb{F})$$

be a continuous homomorphism. Denote by $V_{\mathbb{F}}$ the representation space of $\overline{\rho}$. Given $A \in \mathrm{Ob}(\mathrm{CNL}_{\mathcal{O}})$, a *lift* of $\overline{\rho}$ to A is a continuous homomorphism

$$\rho : G \longrightarrow \mathrm{GL}_n(A)$$

such that

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{GL}_n(A) \\ & \overset{\overline{\rho}}{\searrow} & \downarrow \text{mod } \mathfrak{m}_A \\ & & \mathrm{GL}_n(\mathbb{F}) \end{array}$$

commutes. A *deformation* of $V_{\mathbb{F}}$ to A is a pair (V_A, ϕ_A), where V_A is a free rank two A-module with continuous G-action, and ϕ is an isomorphism $V_A \otimes_A \mathbb{F} \xrightarrow{\sim} V_{\mathbb{F}}$. We will usually drop ϕ_A from the notation. Note that a deformation is an equivalence class of lifts, two lifts begin equivalent if they are conjugate by an element of $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F}))$. Note also that a deformation V_A of $V_{\mathbb{F}}$ to A together with a choice of basis for V_A lifting our fixed basis of $V_{\mathbb{F}}$ determines a lift of $\overline{\rho}$. For this reason we will also call lifts *framed deformations*.

We define set valued functors $\mathcal{D}$ and $\mathcal{D}^{\square}$ on $\mathrm{CNL}_{\mathcal{O}}$ by letting $\mathcal{D}(A)$, respectively $\mathcal{D}^{\square}(A)$, denote the set of deformations of $V_{\mathbb{F}}$ to A, respectively the set of lifts of $\overline{\rho}$ to A. Note that sending a lift to its equivalence class of deformations gives a natural morphism $\mathcal{D}^{\square} \to \mathcal{D}$. We say that G satisfies the *p-finiteness condition* if $\mathrm{Hom}(G', \mathbb{Z}/p\mathbb{Z})$ is finite for all finite index subgroups G' of G. When this holds $\mathcal{D}^{\square}$ is representable. We will give a proof of this fact bellow in 2.3.3. If further $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, then $\mathcal{D}$ is also representable. This can be checked using Schlessinger's critia, cf. [M2]. If R and $R^{\square}$ denote the objects representing $\mathcal{D}$ and $\mathcal{D}^{\square}$, respectively, then the natural morphism $R \to R^{\square}$ is formally smooth of relative dimension $n^2 - 1$, cf. §2.2 of [KW].

Let ϕ denote an $\mathcal{O}$-valued character of G whose reduction is equal to $\det \overline{\rho}$. We define a subfunctor $\mathcal{D}^{\phi}$ of $\mathcal{D}$ by letting $\mathcal{D}^{\phi}(A)$ be the subset of $\mathcal{D}(A)$ consisting of deformations V_A such that $\det V_A = \phi$. We define $\mathcal{D}^{\square,\phi}$ similarly. Let $R^{\square}$ and ρ^{univ} denote the object representing $\mathcal{D}^{\square}$ and its universal lift. If I is the ideal generated by the elements $\det(\rho^{\mathrm{univ}}(\sigma)) - \phi(\sigma)$, it is easy to see that $R^{\square}/I$ represents $\mathcal{D}^{\square,\phi}$. Similarly if $\mathcal{D}$ is representable, then so is $\mathcal{D}^{\phi}$. We caution the reader that later when we specialize to G a galois group of a number field or one of its completions, the notation $\mathcal{D}^{\psi}$ and $\mathcal{D}^{\square,\psi}$ will denote the subfunctors corresponding to deformations and lifts, respectively, with determinant $\psi\epsilon_p$, with ϵ_p the p-adic cyclotomic character, not those with determinant ψ.

2.3.2

We extend the functor $\mathcal{D}^{\square}$ to a larger category. Let $\mathrm{Top}_{\mathcal{O}}$ be the category whose objects are pairs (A, I), where A is a topological $\mathcal{O}$-algebra and I is an ideal of A defining the topology of A such that I contains the image of ϖ_E under the structure map $\mathcal{O} \to A$ and such that A is I-adically complete. The morphisms $(A, I) \to (A', I')$ in $\mathrm{Top}_{\mathcal{O}}$ are $\mathcal{O}$-algebra morphisms $\varphi : A \to A'$ such that $\varphi(I) \subseteq I'$. Note that the map $A \mapsto (A, \mathfrak{m}_A)$ embeds $\mathrm{CNL}_{\mathcal{O}}$ as a full subcategory of $\mathrm{Top}_{\mathcal{O}}$, and that for any (A, I) in $\mathrm{Top}_{\mathcal{O}}$ the map $\mathcal{O} \to A$ induces an injection $\mathbb{F} \to A/I$.

Let G and $\overline{\rho}$ be as in 2.3.1, and assume that G satisfies the p-finiteness condition. We define a functor $\mathcal{D}^{\square}$ on $\mathrm{Top}_{\mathcal{O}}$ by letting $\mathcal{D}^{\square}(A, I)$ denote the set of continuous homomorphisms

$$\rho : G \longrightarrow \mathrm{GL}_n(A)$$

such that

$$\begin{array}{ccc} G & \xrightarrow{\ \rho\ } & \mathrm{GL}_n(A) \\ {\scriptstyle \overline{\rho}}\downarrow & & \downarrow{\scriptstyle \mathrm{mod} I} \\ \mathrm{GL}_n(\mathbb{F}) & \longrightarrow & \mathrm{GL}_n(A/I) \end{array}$$

commutes. Note that under the embedding $A \mapsto (A, \mathfrak{m}_A)$ of $\mathrm{CNL}_{\mathcal{O}}$ into $\mathrm{Top}_{\mathcal{O}}$ the restriction of $\mathcal{D}^{\square}$ on $\mathrm{Top}_{\mathcal{O}}$ to $\mathrm{CNL}_{\mathcal{O}}$ coincides with $\mathcal{D}^{\square}$ as defined in 2.3.1.

Proposition 2.3.3. *There is a* $\mathrm{CNL}_{\mathcal{O}}$*-algebra* $R^{\square}$ *such that* $(R^{\square}, \mathfrak{m}_{R^{\square}})$ *represents* $\mathcal{D}^{\square}$ *on* $\mathrm{Top}_{\mathcal{O}}$*. In particular* $\mathcal{D}^{\square}$ *is representable on* $\mathrm{CNL}_{\mathcal{O}}$*.*

Proof. Let $G' = \ker \overline{\rho}$, $G'(p)$ it maximal pro-p quotient and H the kernel of the natural surjection $G' \to G'(p)$. Note that H is normal in G as G' is normal in G and H is a characteristic subgroup of G'. Since G satisfies the p-finiteness condition, G'/H is topologically finitely generated, and hence so is G/H. Fix a set of topological generators $\gamma_1, \ldots, \gamma_m$ of G/H. Let F denote the free group on the set $\{\gamma_1, \ldots, \gamma_m\}$ and let $F^{\wedge}$ denote its profinite completion. We have a natural surjection $F^{\wedge} \to G/H$ and we denote by K its kernel.

For each γ_k, let $[\overline{\rho}(\gamma_k)] \in \mathrm{GL}_n(\mathcal{O})$ be the matrix whose entries are the Teichmüller lifts of the entries of $\overline{\rho}(\gamma_k)$. Consider the power series ring $\mathcal{O}[[a_{ij}^k]]$ where $1 \leq i, j \leq n$ and $1 \leq k \leq m$. We define a continuous homomorphism

$$\varrho : F^{\wedge} \longrightarrow \mathrm{GL}_n(\mathcal{O}[[a_{ij}^k]])$$

by $\varrho(\gamma_k) = [\overline{\rho}(\gamma_k)] + (a_{ij}^k)$. Let J denote the ideal of $\mathcal{O}[[a_{ij}^k]]$ generated by the elements of the matrices $\varrho(r) - 1$ for all $r \in K$, and set $R^{\square} = \mathcal{O}[[a_{ij}^k]]/J$. Then the pushforward of ϱ along $\mathcal{O}[[a_{ij}^k]] \to R^{\square}$ defines a continuous homomorphism $G/H \to \mathrm{GL}_n(R^{\square})$, and we let

$$\rho^{\mathrm{univ}} : G \to \mathrm{GL}_n(R^{\square})$$

be the continuous homomorphism given by precomposing with the surjection $G \to G/H$. Note that $R^\square$ is an object in $\mathrm{CNL}_\mathcal{O}$. We will now show that $(R^\square, \mathfrak{m}_{R^\square})$ represents $\mathcal{D}^\square$ on $\mathrm{Top}_\mathcal{O}$ and that ρ^{univ} is the universal lift.

Let (A, I) be an object in $\mathrm{Top}_\mathcal{O}$ and let

$$\rho : G \to \mathrm{GL}_n(A)$$

be an element of $\mathcal{D}^\square(A, I)$. Since $G \to \mathrm{GL}_n(A) \to \mathrm{GL}_n(A/I)$ has kernel G', and I^n/I^{n+1} is p-torsion for all n, the morphism ρ factors through G/H. The morphism ρ is equivalent to giving matrices $X_k \in \mathrm{GL}_n(A)$ for each $1 \leq k \leq m$, such that their reduction modulo I is equal to $\overline{\rho}(\gamma_k)$, and such that the induced homomorphism $F^\wedge \to \mathrm{GL}_n(A)$ is trivial on the subgroup K. By viewing ρ as a specialization of ϱ, this is then equivalent to giving an $\mathcal{O}$-algebra morphism $\mathcal{O}[[a_{ij}^k]] \to A$ whose kernel contains J and such that the maximal ideal of $\mathcal{O}[[a_{ij}^k]]$ is mapped to I, i.e to give an $\mathcal{O}$-algebra morphism $\varphi : R^\square \to A$ such that $\varphi(\mathfrak{m}_{R^\square}) \subseteq I$ and we see that ρ is the pushforward of ρ^{univ} under this map. $\square$

This proposition has the following immediate consequence. If E'/E is a finite extension with ring of integers $\mathcal{O}'$ and residue field $\mathbb{F}'$, then $R^\square \otimes_\mathcal{O} \mathcal{O}'$ represents the lifting functor $\mathcal{D}^\square$ for $\overline{\rho} \otimes_\mathbb{F} \mathbb{F}'$ on $\mathrm{CNL}_{\mathcal{O}'}$.

2.3.4

Let E'/E be finite with ring of intergers $\mathcal{O}'$ and residue field $\mathbb{F}'$. Assume we are given a continuous homomorphism $\rho_{\mathcal{O}'} : G \to \mathrm{GL}_n(\mathcal{O}')$ such that

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{GL}_n(\mathcal{O}') \\ {\scriptstyle \overline{\rho}}\downarrow & & \downarrow \\ \mathrm{GL}_n(\mathbb{F}) & \longrightarrow & \mathrm{GL}_n(\mathbb{F}') \end{array}$$

commutes. Let $\rho_{E'}$ denote the induced morphism $\rho_{E'} : G \to \mathrm{GL}_n(E')$. Let $\mathcal{D}^\square_{\rho_{E'}}$ denote the functor on $\mathrm{Ar}_{E'}$ that sends an object B of $\mathrm{Ar}_{E'}$ to the set of continuous homomorphisms

$\rho_B : G \to \mathrm{GL}_n(B)$ such that

$$\begin{array}{ccc} G & \xrightarrow{\rho_B} & \mathrm{GL}_n(B) \\ & {\scriptstyle \rho_{E'}} \searrow & \downarrow {\scriptstyle \bmod \mathfrak{m}_B} \\ & & \mathrm{GL}_n(E') \end{array}$$

commutes. The following Proposition will be useful in determining the generic fibre of the local deformation ring in §**??** and the argument is due to Kisin, cf. Proposition 9.5 of [K1].

Proposition 2.3.5. *For any B in Ar_E and $\rho_B \in \mathcal{D}^{\square}_{\rho_{E'}}$, there is a unique $\mathcal{O}$-algebra morphism $R^{\square} \to B$ such that ρ_B is the pushforward of ρ^{univ} via this morphism.*

Proof. Choose a surjection $E'[[x_1, \ldots, x_n]] \to B$ and let A_0 denote the image of $\mathcal{O}'[[x_1, \ldots, x_n]]$ under this surjection. Note that since the representation $\rho_{E'}$ takes values in $\mathrm{GL}_n(\mathcal{O}')$, the representation ρ_B takes values in $\mathrm{GL}_n(A_0 + \mathfrak{m}_B)$. Let $\mathfrak{n} = A_0 \cap \mathfrak{m}_B$. Since B is Artinian we can define for any $m \geq 1$, $A_m = A_0 + \sum_{j=1}^{\infty} p^{-mj}\mathfrak{n}^j$. For each $m \geq 1$, A_m is a $\mathrm{CNL}_{\mathcal{O}'}$-algebra subring of B. Note that $A_0 + \mathfrak{m}_B = \cup_{m\geq 0}^{\infty} A_m$. Since G is compact, a standard Baire Category argument implies that ρ_B takes values in A_m for some m. By 2.3.3, there is a unique $\mathrm{CNL}_{\mathcal{O}}$-morphism $R^{\square} \to A_m$ such that $G \to \mathrm{GL}_n(A_m)$ is the pushforward of ρ^{univ} via $R^{\square} \to A_m$.

It remains to show uniqueness. Let $\phi, \phi' : R^{\square} \to B$ be two such $\mathcal{O}$-algebra morphisms. Since the image of $R^{\square}$ under either of these maps lies in $A_0 + \mathfrak{m}_B = \cup_{m\geq 0} A_m$ and $R^{\square}$ is compact, again by Baire Category its image via either ϕ or ϕ' must lie in some A_m for m sufficiently large. By the universal property of $R^{\square}$ on $\mathrm{Top}_{\mathcal{O}}$, we must have $\phi = \phi'$. $\square$

2.4 Local deformation rings at p

Throughout this section, F_v will denote a finite extension of $\mathbb{Q}_p$. We fix an algebraic closure $\overline{F_v}$ of F_v and let $G_v = \mathrm{Gal}(\overline{F_v}/F_v)$. We assume that E contains all embedding of F_v into an algebraic closure of E.

For a given a ring A, we call an A-submodule $L \subset A^2$ a *line* if both L and A^2/L are projective of rank one.

2.4.1

We fix a continuous homomorphism

$$\overline{\rho} : G_v \longrightarrow \mathrm{GL}_2(\mathbb{F})$$

and a continuous character $\psi : G_v \to \mathcal{O}^\times$ such that $\det \overline{\rho} = \overline{\psi\epsilon_p}$. Let $V_{\mathbb{F}}$ denote the representation space of $\overline{\rho}$. We will assume throughout this section that there is some line $L_{\mathbb{F}}$ in $V_{\mathbb{F}}$ that is stable by the action of G_v. Let $L_{\mathbb{F}}$ be one such line and let $\overline{\chi}$ denote the character of G_v giving the action on $V_{\mathbb{F}}/L_{\mathbb{F}}$. Note that the choice of $\overline{\chi}$ is unique unless $V_{\mathbb{F}}$ is the direct sum of two distinct characters. In this case we simply make a choice of one of these characters.

We let $\mathcal{D}_v^{\square}$ and $\mathcal{D}_v^{\square,\psi}$ denote the functor of lifts of $\overline{\rho}$ and the subfunctor consisting of lifts with determinant $\psi\epsilon_p$, repsectively. We denote the corresponding representing objects by $R_v^{\square}$ and $R_v^{\square,\psi}$, respectively.

Let G_v^{ab} denote the abelianization of G_v and $G_v^{\mathrm{ab}}(p)$ the maximal pro-p quotient of the abelianization. Set $\Lambda(G_v) = \mathcal{O}[[G_v^{\mathrm{ab}}(p)]]$. Note that $G_v^{\mathrm{ab}}(p)$ is isomorphic to $\mu_{p^s} \times \mathbb{Z}_p^{d+1}$, where $d = [F_v : \mathbb{Q}_p]$ and μ_{p^s} is the group of p-power roots of unity in F_v. Hence $\Lambda(G_v)$ has p^s minimal primes, corresponding to the p^s distinct $\mathcal{O}$-valued characters of μ_{p^s} (recall we have assumed E contains all embeddings of F_v into $\overline{E}$), and its quotient by any of these minimal primes is isomorphic to a power series over $\mathcal{O}$ in $d+1$ variables.

Let $\tilde{\chi}$ denote the Teichmüller lift of $\overline{\chi}$. If A is a $\mathrm{CNL}_{\mathcal{O}}$-algebra and $\chi : G_v \to A^\times$ is a continuous character, then writing $\chi = \tilde{\chi}\chi'$ with χ' factoring through the natural projection $G_v \to G_v^{\mathrm{ab}}(p)$, we see that $\Lambda(G_v)$ represents the set valued functor that assigns to each $\mathrm{CNL}_{\mathcal{O}}$-algebra A the set of continuous characters $\{\chi_A : G_v \to A^\times\}$ that lift $\overline{\chi}$. We let $\chi^{\mathrm{univ}} : G_v \to \Lambda(G_v)$ denote the universal $\Lambda(G_v)$-valued character.

We will need to consider quotients of $\Lambda(G_v)$ by its minimal primes in order to ensure that our local lifting ring is a domain. Recall μ_{p^s} is identified with the the p-power torsion subgroup of G_v^{ab}. Fix a character $\eta : \mu_{p^s} \to \mathcal{O}^\times$, and let $\mathfrak{q}_\eta$ denote the corresponding minimal prime of $\Lambda(G_v)$, and set $\Lambda(G_v, \eta) = \Lambda(G_v)/\mathfrak{q}_\eta$. We let $\chi_\eta^{\mathrm{univ}}$ denote the character obtained by composing χ^{univ} with the natural surjection $\Lambda(G_v) \to \Lambda(G_v, \eta)$. Then $\Lambda(G_v, \eta)$ represents the functor that assigns to each $\mathrm{CNL}_{\mathcal{O}}$-algebra A the set of characters $\{\chi_A : G_v \to A^\times\}$ that lift $\overline{\chi}$ and whose restriction to the p-power torsion subgroup of G_v^{ab} is equal to η.

Set $R^{\square,\psi}_{\Lambda(G_v,\eta)} = R_v^{\square,\psi}\hat{\otimes}_{\mathcal{O}}\Lambda(G_v,\eta)$, and consider $\mathbb{P}^1_{\mathcal{O}} \otimes_{\mathcal{O}} R^{\square,\psi}_{\Lambda(G_v,\eta)}$. If A is an $\mathcal{O}$-algebra, then an A-point of this scheme is a triple (α, β, L) where $\alpha : R_v^{\square,\psi} \to A$ and $\beta : \Lambda(G_v,\eta) \to A$ are $\mathcal{O}$-algebra morphisms, and L is a line in A^2. By pushing forward the universal $R_v^{\square,\psi}$ valued lift via α we get a homomorphism $\rho_A : G_v \to \mathrm{GL}_2(A)$, and by pushing forward $\chi_\eta^{\mathrm{univ}}$ we get a character $\chi_A : G_v \to A^\times$. We define a functor $\mathcal{DL}^{\square,\psi,\eta}$ on the category of $\mathcal{O}$-algebras by letting $\mathcal{DL}^{\square,\psi}(A)$ be the subset of such triples (α,β,L) such that ρ_A leaves L invariant and G_v acts on A^2/L via χ_A.

Lemma 2.4.2. *$\mathcal{DL}^{\square,\psi}$ is represented by a closed subscheme $\mathscr{L}$ of $\mathbb{P}^1_{\mathcal{O}} \otimes_{\mathcal{O}} R^{\square,\psi}_{\Lambda(G_v,\eta)}$.*

Proof. For ease of notation, set $R = R^{\square,\psi}_{\Lambda(G_v,\eta)}$. Let ϕ denote the tautological morphism

$$\phi : \mathcal{O}^2_{\mathbb{P}^1_R} \longrightarrow \mathcal{O}_{\mathbb{P}^1_R}(1),$$

and let $\mathcal{L}$ denote its kernel. Let $U = \operatorname{Spec} A$ be an open affine subset of $\mathbb{P}^1_R$ such that $\mathcal{O}_{\mathbb{P}^1_R}(1)(U)$ is free of rank one over A. Fix a generator e of $\mathcal{O}_{\mathbb{P}^1_R}(1)(U)$. Let

$$\mathcal{I}(U) = \{r \in A : \text{there is some } \sigma \in G_v \text{ and } x \in A^2 \text{ with } \phi(\rho_A(\sigma)x - \chi_A(\sigma)x) = re\}.$$

Then $\mathcal{I}(U)$ is an ideal in A and does not depend on the choice of e. It is easy to see that $\mathcal{I}$ defines a sheaf of ideals of $\mathbb{P}^1_R$, and that $\mathcal{DL}^{\square,\psi}$ coincides with the functor of points of the closed subscheme of $\mathbb{P}^1_R$ defined by $\mathcal{I}$. □

2.4.3

We now let $R^{\triangle,\psi}_{\Lambda(G_v,\eta)}$ denote the image of the homomorphism

$$R^{\square,\psi}_{\Lambda(G_v,\eta)} \longrightarrow \mathcal{O}_{\mathscr{L}}(\mathscr{L}),$$

i.e. $R^{\triangle,\psi}_{\Lambda(G_v,\eta)}$ is the affine algebra of the scheme theoretic image of $\mathscr{L}$ in $\operatorname{Spec} R^{\square,\psi}_{\Lambda(G_v,\eta)}$.

Proposition 2.4.4. *Let E'/E be a finite extension with ring of integers $\mathcal{O}'$. Let $\rho : G_v \to \mathrm{GL}_2(\mathcal{O}')$ be a lift of $\overline{\rho}$ with determinant $\psi\epsilon_p$, and let $\chi : G_v \to (\mathcal{O}')^\times$ be a character whose restriction to the p-power torsion subgroup of G_v^{ab} is equal to η. The point of $R^{\square,\psi}_{\Lambda(G_v,\eta)}$ determined by the pair (ρ,χ) factors through $R^{\triangle,\psi}_{\Lambda(G_v,\eta)}$ if and only if there is a G_v-stable line in $(\mathcal{O}')^2$ such that G_v acts on the quotient via χ.*

Proof. Let f denote the morphism $\mathscr{L} \to \operatorname{Spec} R^{\square,\psi}_{\Lambda(G_v,\eta)}$. The point determined by (ρ,χ) satisfies the conclusion if and only if it is in the image of $\mathscr{L}[1/p]$. Since f is proper, so is $f[1/p]$, and so the topological image of $f[1/p]$ is equal to scheme theoretic image of $f[1/p]$, which is $\operatorname{Spec} R^{\triangle,\psi}_{\Lambda(G_v,\eta)}[1/p]$. □

In order to determine the structure of $R^{\triangle,\psi}_{\Lambda(G_v,\eta)}$ more precisely, we will relate it to $\mathscr{L}$ via the following lemma.

Lemma 2.4.5. *Let Z denote the closed subscheme of $\Lambda(G_v,\eta)$ defined by $(\chi_\eta^{\mathrm{univ}})^2 = \psi\epsilon_p$, and let V denote its complement. The map*

$$\mathscr{L} \times_{\operatorname{Spec}\Lambda(G_v,\eta)} V \longrightarrow \operatorname{Spec} R^{\triangle,\psi}_{\Lambda(G_v,\eta)} \times_{\operatorname{Spec}\Lambda(G_v,\eta)} V$$

is an isomorphism.

Proof. Since scheme theoretic image commutes with flat base change,

$$\mathscr{L} \times_{\operatorname{Spec}\Lambda(G_v,\eta)} V \longrightarrow \operatorname{Spec} R^{\triangle,\psi}_{\Lambda(G_v,\eta)} \times_{\operatorname{Spec}\Lambda(G_v,\eta)} V$$

has injective structural morphism, and so to prove it is an isomorphism it suffices to show it is a closed immersion. To show this it suffices to show that if A is a local ring and $(\rho_A,\chi_A,L_A) \in (\mathscr{L} \times_{\operatorname{Spec}\Lambda(G_v,\eta)} V)(A)$ is an A-point, the line L_A is unique and that it is defined over B, where B is the image of $R^{\square,\psi}_{\Lambda(G_v,\eta)}$ in A under the morphism determined by (ρ_A,χ_A). Indeed, this implies that the fibres are all singletons and that the corresponding maps on local rings are surjective.

Let $(\rho_A,\chi_A,L_A) \in (\mathscr{L} \times_{\operatorname{Spec}\Lambda(G_v,\eta)} V)(A)$, with A a local ring. Take $\sigma \in G_v$ such that $\chi_A^2(\sigma) \neq \psi\epsilon_p(\sigma) \mod \mathfrak{m}_A$. Consider the matrix $M = \rho_A(\sigma) - \psi\epsilon_p\chi_A^{-1}(\sigma)$. Since G_v acts on

L_A via $\psi\epsilon_p\chi_A^{-1}$ we see that $\det M = 0$. But, by our assumption on σ, its reduction modulo the maximal ideal of A has rank 1, hence one of the entries of the matrix M is a unit. This implies that the line L_A is unique and its projective coordinates can be defined using the elements of M. This proves the claim. □

2.4.6

Let B_2 denote the Borel subgroup of upper triangular matrices in GL_2. Fix a continuous homomorphism $\overline{\varrho} : G_v \to \mathrm{B}_2(\mathbb{F})$ with $\det \varrho = \overline{\psi\epsilon_p}$ and such that

$$\varrho = \begin{pmatrix} * & * \\ & \overline{\chi} \end{pmatrix}$$

with $\overline{\chi}$ our fixed character. Define a functor $\mathcal{D}_{\overline{\varrho}}^{\mathrm{Bor},\psi}$ on $\mathrm{CNL}_{\mathcal{O}}$ by letting letting $\mathcal{D}_{\varrho}^{\mathrm{Bor},\psi}(A)$ be the set of continuous morphisms $\varrho_A : G_v \to \mathrm{B}_2(A)$ that reduce to $\overline{\varrho}$ modulo $\mathfrak{m}_A$, have determinant $\psi\epsilon_p$ and such that, writing

$$\varrho_A = \begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix},$$

χ_2 coincides with η on the p-power torsion subgroup of G_v^{ab}.

Lemma 2.4.7. *$\mathcal{D}_{\overline{\varrho}}^{\mathrm{Bor},\psi}$ is representable by a $\mathrm{CNL}_{\mathcal{O}}$-algebra $R_{\overline{\varrho}}^{\mathrm{Bor},\psi}$.*

Moreover if we assume that F_v contains a 4-th root of unity if $p = 2$, and that the image of ϱ is either trivial or has order p, then $R_{\overline{\varrho}}^{\mathrm{Bor},\psi}$ has dimension $3 + 2[F_v : \mathbb{Q}_p]$ and both $R_{\overline{\varrho}}^{\mathrm{Bor},\psi}$ and $R_{\overline{\varrho}}^{\mathrm{Bor},\psi} \otimes_{\mathcal{O}} \mathbb{F}$ are domains.

Proof. The proof of representability in the general case is proved exactly as with $R_v^{\square}$ in 2.3.3. We leave the details to the reader and henceforth assume that if $p = 2$ then F_v contains a 4-th root of unity and that the image of ϱ is either trivial or has order p.

For any $\sigma \in G_v$, we let $[\overline{\varrho}(\sigma)] \in \mathrm{GL}_2(\mathcal{O})$ denote the matrix whose entries are the Teichmüller lifts of the entries of $\overline{\varrho}(\sigma)$. Let $G_v(p)$ be the maximal pro-p quotient of G_v. Our assumption on the image of $\overline{\varrho}$ implies that for any $\varrho_A \in \mathcal{D}_{\overline{\varrho}}^{\mathrm{Bor},\psi}(A)$, ϱ_A factors through $G_v(p)$.

If F_v does not contain a p-th root of unity, then $G_v(p)$ is a free pro-p group of rank $m = 1 + [F_v : \mathbb{Q}_p]$, cf. Theorem 7.5.8 of [NSW], and the p-part of the torsion subgroup of G_v^{ab} (and hence η) is trivial. Fix a set of generators $\gamma_1, \ldots, \gamma_m$ on which $G_v(p)$ is free, and define a lift

$$\varrho^{\mathrm{univ}} : G_v \to \mathrm{B}_2(\mathcal{O}[[a_1, \ldots, a_m, b_1, \ldots, b_m]])$$

by

$$\varrho^{\mathrm{univ}}(\gamma_i) = [\overline{\varrho}(\gamma_i)] \begin{pmatrix} \psi\epsilon_p(\gamma_i)(1 + a_i) & b_i \\ & (1 + a_i)^{-1} \end{pmatrix}.$$

Any lift $\varrho_A \in \mathcal{D}_{\overline{\varrho}}^{\mathrm{Bor},\psi}(A)$ is a specialization of ϱ^{univ} via a unique $\mathrm{CNL}_{\mathcal{O}}$-morphism $\mathcal{O}[[a_1, \ldots, a_m, b_1, \ldots, b_m]] \to A$, and we have $R_{\overline{\varrho}}^{\mathrm{Bor},\psi} = \mathcal{O}[[a_1, \ldots, a_m, b_1, \ldots, b_m]]$. It has dimension $3 + 2[F_v : \mathbb{Q}_p]$ and both it and its reduction modulo the maximal ideal of $\mathcal{O}$ are domains.

If F_v contains a p-th root of unity then, as we are assuming it contains a 4-th root in the case $p = 2$, a theorem of Demuškin, cf. Theorem 7.5.9 of [NSW], shows that $G_v(p)$ can be presented as the free pro-p group on generators $\gamma_1, \ldots, \gamma_m$, with $m = 2 + [F_v : \mathbb{Q}_p]$, modulo the single relation

$$k = \gamma_1^{p^s}(\gamma_1, \gamma_2)(\gamma_3, \gamma_4) \cdots (\gamma_{m-1}, \gamma_m),$$

where s is the largest integer such that $\mu_{p^s} \subset F_v$ and $(\gamma_i, \gamma_{i+1}) = \gamma_i^{-1}\gamma_{i+1}^{-1}\gamma_i\gamma_{i+1}$. We note that the image of $\gamma_1, \ldots, \gamma_m$ in $G_v^{\mathrm{ab}}(p)$ are generators and are subject to the single relation $\gamma^{p^s} = 1$. This shows that γ_1 is a generator of the torsion subgroup of $G_v^{\mathrm{ab}}(p)$. Let $F^\wedge$ be the free group pro-p group on the set $\{\gamma_1, \ldots, \gamma_m\}$. Set $B = \mathcal{O}[[a_2, \ldots, a_m, b_1, \ldots, b_m]]$ and define

$$\varrho_{F^\wedge} : F^\wedge \longrightarrow \mathrm{B}_2(B)$$

by

$$\varrho_{F^\wedge}(\gamma_1) = [\overline{\varrho}(\gamma_1)] \begin{pmatrix} \psi\epsilon_p\eta^{-1}(\gamma_1) & b_1 \\ & \eta(\gamma_1) \end{pmatrix}.$$

and

$$\varrho_{F^\wedge}(\gamma_i) = [\overline{\varrho}(\gamma_i)] \begin{pmatrix} \psi\epsilon_p(\gamma_i)(1 + a_i) & b_i \\ & (1 + a_i)^{-1} \end{pmatrix},$$

for $2 \leq i \leq m$. Note that

$$\varrho_{F^\wedge}(k) - 1 = \begin{pmatrix} 1 & r \\ & 1 \end{pmatrix}.$$

with r in the maximal ideal of B. Then for any $\varrho_A \in \mathcal{D}_{\overline{\varrho}}^{\mathrm{Bor},\psi}(A)$, ϱ_A is the pushforward of a unique $\mathrm{CNL}_{\mathcal{O}}$-morphism $B \to A$ that contains r in its kernel. From this it follows that $R_{\varrho}^{\mathrm{Bor},\psi} \cong B/rB$ and the universal lift ϱ^{univ} is the pushforward of $\varrho_{F^\wedge}$ via the natural surjection.

Using 2.1.9, to show that $R_{\varrho}^{\mathrm{Bor},\psi} \cong B/(r)$ is a domain of dimension $3 + 2[F_v : \mathbb{Q}_p]$ and that $R_{\varrho}^{\mathrm{Bor},\psi} \otimes_{\mathcal{O}} \mathbb{F}$ is a domain, it suffice to show that $B/(r, \varpi_E, a_2, \ldots, a_{m-2}, b_1, \ldots, b_{m-2})$ is a domain of dimension 3, where ϖ_E is a uniformizer for $\mathcal{O}$. Let r_0 denote the image of r in $B/(\varpi_E, a_2, \ldots, a_{m-2}, b_1, \ldots, b_{m-2}) \cong \mathbb{F}[[a_{m-1}, a_m, b_{m-1}, b_m]]$. We are thus reduced to showing that r_0 is irreducible in $\mathbb{F}[[a_{m-1}, a_m, b_{m-1}, b_m]]$.

Let ϱ_0 denote the pushforward of $\varrho_{F^\wedge}$ to $\mathbb{F}[[a_{m-1}, a_m, b_{m-1}, b_m]]$. Then r_0 is given by

$$\varrho_0(\gamma_{m-1})^{-1}\varrho_0(\gamma_m)^{-1}\varrho_0(\gamma_{m-1}\varrho_0(\gamma_m) = \begin{pmatrix} 1 & r_0 \\ & 1 \end{pmatrix}.$$

Our assumption on the image of $\overline{\varrho}$ implies that at most one of $\overline{\varrho}(\gamma_{m-1})$ and $\overline{\varrho}(\gamma_m)$ is non-trivial and that if it is non-trivial, then it is unipotent. Note also that our assumptions imply $\psi\epsilon_p$ is trivial mod ϖ_E. Define β_i, for $i = m-1, m$, by

$$\varrho_0(\gamma_i) = \begin{pmatrix} 1 + a_i & \beta_i \\ & (1+a_i)^{-1} \end{pmatrix} = \overline{\varrho}(\gamma_i) \begin{pmatrix} 1 + a_i & b_i \\ & (1+a_i)^{-1} \end{pmatrix}.$$

A straightforward computation shows that

$$r_0 = (1 + a_{m-1})^{-1}(1 + a_m)^{-1}\big((\beta_m(1 + a_{m-1} - (1 + a_{m-1})^{-1}) - \beta_{m-1}(1 + a_m - (1 + a_m)^{-1})\big),$$

and it suffices to show that

$$r_1 = \beta_m(1 + a_{m-1} - (1 + a_{m-1})^{-1}) - \beta_{m-1}(1 + a_m - (1 + a_m)^{-1} \tag{2.1}$$

is irreducible. To do this we use the following easy but useful fact we leaned from [K], pg. 164: if K is a field and $f \in K[[x_1, \ldots, x_n]]$ is reducible, then for any grading $\det(x_i) = n_i > 0$, the lowest degree term of f is reducible in $K[x_1, \ldots, x_n]$.

First consider the case that $\overline{\varrho}(\gamma_{m-1})$ is nontrivial, and write

$$\overline{\varrho}(\gamma_{m-1}) = \begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix},$$

with $\alpha \in \mathbb{F}$ nonzero, so that $\beta_{m-1} = \alpha(1 + a_{m-1})^{-1} + b_{m-1}$. Note then $\beta_m = b_m$ since the image of $\overline{\varrho}$ has order p. If $p \neq 2$, we use the grading $\det(a_i) = \det(b_i) = 1$, and (2.1) becomes

$$2\alpha a_m + \text{ higher order terms}$$

and $2\alpha a_{m-1}$ is irreducible in $\mathbb{F}[a_{m-1}, a_m, b_{m-1}, b_m]$. If $p = 2$, we use the grading $\deg(a_{m-1}) = \deg(b_{m-1}) = 1$ and $\deg(a_m) = \deg(b_m) = 2$. Then (2.1) becomes

$$-\alpha a_m^2 + b_m a_{m-1}^2 + \text{ higher order terms}$$

and $-\alpha a_{m-1}^2 + b_{m-1} a_m^2$ is irreducible in $\mathbb{F}[a_{m-1}, a_m, b_{m-1}, b_m]$. The case when $\overline{\varrho}(\gamma_m)$ is nontrivial is symmetric.

Now assume $\overline{\varrho}(\gamma_{m-1}) = \overline{\varrho}(\gamma_m) = 1$. We use the standard grading $\deg(a_i) = \deg(b_i) = 1$, and (2.1) becomes

$$2a_{m-1}b_m - 2a_m b_{m-1} + \text{ higher order terms}$$

if $p \neq 2$, and

$$-a_{m-1}^2 b_m + a_m^2 b_{m-1} + \text{ higher order terms}$$

if $p = 2$. In either case the leading term is irreducible in $\mathbb{F}[a_{m-1}, a_m, b_{m-1}, b_m]$. $\square$

Writing the universal lift $\varrho^{\text{univ}} : G_v \to \mathrm{B}_2(R_{\overline{\varrho}}^{\text{Bor},\psi})$ as

$$\varrho^{\text{univ}} = \begin{pmatrix} \chi_1^{\text{Bor}} & * \\ & \chi_2^{\text{Bor}} \end{pmatrix}.$$

The character χ_2^{Bor} gives a $\text{CNL}_{\mathcal{O}}$-morphism $\Lambda(G_v, \eta) \to R_{\overline{\varrho}}^{\text{Bor},\psi}$.

2.4.8

Let x be a closed point of $\mathscr{L}$. Then x is simply a choice of $\overline{\rho}$-stable line L_x in the representation space of $\overline{\rho}$ such that G_v acts via $\overline{\chi}$ on $V_{\mathbb{F}}/L_x$.

Consider the set valued functor $\mathcal{DL}_x^{\square,\psi}$ on $\mathrm{CNL}_{\mathcal{O}}$ that sends a $\mathrm{CNL}_{\mathcal{O}}$-algebra A the set of pairs (ρ_A, L_A), where ρ_A is a lift of $\overline{\rho}$ to $\mathrm{GL}_2(A)$ with determinant $\psi\epsilon_p$, and L_A is a G_v-stable line in A^2 lifting L_x such that the action of the p-part of the torsion subgroup of G_v^{ab} on A^2/L_x is given by η. Let $\mathcal{O}^{\wedge}_{\mathscr{L},x}$ denote the completion of the local ring of $\mathscr{L}$ at x. Note that the natural map

$$\operatorname{Spec} \mathcal{O}^{\wedge}_{\mathscr{L},x} \longrightarrow \mathscr{L}$$

yields a lift : $\rho_x^{\wedge} : G_v \to \mathrm{GL}_2(\mathcal{O}^{\wedge}_{\mathscr{L},x})$ of $\overline{\rho}$ and a G_v-stable line $L_x^{\wedge}$ lifting L_x. The following lemma is immediate.

Lemma 2.4.9. *$\mathcal{O}^{\wedge}_{\mathscr{L},x}$ represents $\mathcal{DL}_x^{\square,\psi}$ with universal object $(\rho_x^{\wedge}, L_x^{\wedge})$.*

For any ring A, we will denote by L_A^{std} the A-line in A^2 fixed by the upper-triangular matrices. Let $x = (\overline{\rho}, \overline{\chi}, L_x)$ be a closed point of $\mathscr{L}$. Take $g \in \mathrm{GL}_2(\mathcal{O})$ such that $\overline{g}L_x = L_{\mathbb{F}}^{\mathrm{std}}$. Note that $\overline{g\rho g}^{-1}$ is upper triangular, and so we have the functor $\mathcal{D}^{\mathrm{Bor},\psi}_{\overline{g\rho g}^{-1}}$ on $\mathrm{CNL}_{\mathcal{O}}$ as in 2.4.6, which is represented by $R^{\mathrm{Bor},\psi}_{g\rho_x g^{-1}}$ as in 2.4.7.

Lemma 2.4.10. *There is an isomorphism $\mathcal{O}^{\wedge}_{\mathscr{L},x} \cong R^{\mathrm{Bor},\psi}_{\overline{g\rho g}^{-1}}[[z]]$ of $\Lambda(G_v,\eta)$-algebras.*

Proof. For ease of notation set $R = R^{\mathrm{Bor},\psi}_{\overline{g\rho g}^{-1}}$ and let ϱ denote its universal lift. Define a lift $\rho_{R[[z]]} : G_v \to R[[z]]$ by

$$\rho_{R[[z]]}(\sigma) = g^{-1} \begin{pmatrix} 1 & \\ z & 1 \end{pmatrix} \varrho(\sigma) \begin{pmatrix} 1 & \\ -z & 1 \end{pmatrix} g$$

and we let $L_{R[[z]]}$ denote the $\rho_{R[[z]]}$-stable line $g^{-1} \begin{pmatrix} 1 & \\ z & 1 \end{pmatrix} L_R^{\mathrm{std}}$. Note that $(\rho_{R[[z]]}, L_{R[[z]]})$ determines a local $\Lambda(G_v,\eta)$-algebra morphism $\mathcal{O}^{\wedge}_{\mathscr{L},x} \to R[[z]]$. Given any $\mathrm{CNL}_{\mathcal{O}}$-algebra A

and $(\rho_A, L_A) \in \mathcal{DL}_x^{\square,\psi}(A)$, there is a unique $c \in \mathfrak{m}_A$ such that

$$L_A = g^{-1} \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} L_A^{\mathrm{std}}.$$

We then have a unique local $\mathcal{O}$-algebra morphism $R[[z]] \to A$, which is also a morphism of $\Lambda(G_v, \eta)$-algebras, such that $(\rho_{R[[z]]}, L_{R[[z]]})$ specializes to (ρ_A, L_A). This shows that $\mathcal{O}^{\wedge}_{\mathscr{L},x} \to R[[z]]$ is an isomorphism of $\Lambda(G_v, \eta)$-algebras. □

Proposition 2.4.11. *Assume that if $p = 2$, then F_v contains a 4th root of unity. Assume also the image of ρ is either trivial or has order p.*

1. $R^{\triangle,\psi}_{\Lambda(G_v,\eta)}$ *is an $\mathcal{O}$-flat domain of dimension $4 + 2[F_v : \mathbb{Q}_p]$.*
2. *Let Z denote the closed subscheme of* $\operatorname{Spec} \Lambda(G_v, \eta)$ *defined by $(\chi_\eta^{\mathrm{univ}})^2 = \psi\epsilon_p$, and let V denote its complement. The scheme*

$$(\operatorname{Spec} R^{\triangle,\psi}_{\Lambda(G_v,\eta)} \times_{\operatorname{Spec} \Lambda(G_v,\eta)} V) \otimes_{\mathcal{O}} \mathbb{F}$$

is integral.

Proof. Note that the special fibre of $\mathscr{L}$ over $R^{\triangle,\psi}_{\Lambda(G_v,\eta)}$ is isomorphic to $\mathbb{P}^1_{\mathbb{F}}$ if $\overline{\rho}$ is triival and is a point otherwise, and so $\mathscr{L}$ is connected. By 2.4.10 and 2.4.7, the completed local ring of $\mathscr{L}$ at any closed point is a domain of dimension $4 + 2[F_v : \mathbb{Q}_p]$ and its reduction modulo the maximal ideal of $\mathcal{O}$ is also a domain. We conclude that both $\mathscr{L}$ and $\mathscr{L} \otimes_{\mathcal{O}} \mathbb{F}$ are integral. This implies that $R^{\triangle,\psi}_{\Lambda(G_v,\eta)}$ is a domain as well as part (2) of the proposition by 2.4.5. It is not hard to see that $\operatorname{Spec} R^{\triangle,\psi}_{\Lambda(G_v,\eta)} \times_{\operatorname{Spec} \Lambda(G_v,\eta)} V$ and $\mathscr{L} \times_{\operatorname{Spec} \Lambda(G_v,\eta)} V$ are each of codimension 1, and so $\dim R^{\triangle,\psi}_{\Lambda(G_v,\eta)} = \dim \mathscr{L} = 4 + 2[F_v : \mathbb{Q}_p]$.

□

2.4.12

Fix a finite extensin E'/E with ring of integers $\mathcal{O}'$ and a continuous homomorphism $\varrho : G_v \to \mathrm{B}_2(E')$ with $\det \varrho = \psi\epsilon_p$. We do not assume that ϱ is a lift of $\overline{\rho}$. Define a functor $\mathcal{D}_\varrho^{\mathrm{Bor},\psi}$ on

$\mathrm{Ar}_{E'}$ by letting letting $\mathcal{D}_{\varrho}^{\mathrm{Bor},\psi}(A)$ be the set of continuous morphisms $\varrho_A : G_v \to \mathrm{B}_2(A)$ that reduce to ϱ modulo $\mathfrak{m}_A$ and have determinant $\psi\epsilon_p$.

Lemma 2.4.13. *The functor $\mathcal{D}_{\varrho}^{\mathrm{Bor},\psi}$ is pro-represented by a complete Noetherian E'-algebra $R_{\varrho}^{\mathrm{Bor},\psi}$. If $\varrho \neq \begin{pmatrix} \chi\epsilon_p & \\ & \chi \end{pmatrix}$ for some character χ with $\chi^2 = \psi$, then $R_{\varrho}^{\mathrm{Bor},\psi}$ is formally smooth over E' of dimension $2 + 2[F_v : \mathbb{Q}_p]$.*

Proof. We prove representability in the same way as 2.3.3 and 2.4.7. There is a finite index subgroup G' of G_v such that $\varrho(G')$ is pro-p. Let $G'(p)$ be the maximal pro-p quotient of G' and let H be the kernel of the natural surjection $G' \to G'(p)$. We have that H is normal in G_v, that G_v/H is topologically finitely generated, and that for any $\varrho_A \in \mathcal{D}_{\varrho}^{\mathrm{Bor},\psi}$, A in $\mathrm{Ar}_{E'}$, ϱ_A factors through G_v/H. Fix a set of topological generators $\gamma_1, \ldots, \gamma_m$ for G_v/H. Let F be the free group on $\{\gamma_1, \ldots, \gamma_m\}$ and let $F^\wedge$ denote its profinite completion. Let K denote the kernel of the natural surjection $F^\wedge \to G_v/H$. Define a continuous homomorphism

$$\varrho_{F^\wedge} : F^\wedge \longrightarrow \mathrm{B}_2(E'[[a_1, \ldots, a_m, b_1, \ldots, b_m]])$$

by

$$\varrho_{F^\wedge}(\gamma_i) = \varrho(\gamma_i) \begin{pmatrix} 1 + a_i & b_i \\ & (1 + a_i)^{-1} \end{pmatrix}$$

Now let J denote the ideal of $E'[[a_1, \ldots, a_m, b_1, \ldots, b_m]]$ generated by the entries of the of the matrices $\varrho_{F^\wedge}(k) - 1$ for $k \in K$. Set $R_{\varrho}^{\mathrm{Bor},\psi} = E'[[a_1, \ldots, a_m, b_1, \ldots, b_m]]/J$. The pushforward of $\varrho_{R^\wedge}$ along the surjection $E'[[a_1, \ldots, a_m, b_1, \ldots, b_m]] \to R_{\varrho}^{\mathrm{Bor},\psi}$ gives a continuous morphism $G_v/H \to \mathrm{B}_2(R_{\varrho}^{\mathrm{Bor},\psi})$, and we let

$$\varrho^{\mathrm{univ}} : G_v \longrightarrow \mathrm{B}_2(R_{\varrho}^{\mathrm{Bor},\psi})$$

denote the composite of this morphism with the natural surjection $G_v \to G_v/H$. It is easy to see that $R_{\varrho}^{\mathrm{Bor},\psi}$ pro-represents $\mathcal{D}_{\varrho}^{\mathrm{Bor},\psi}$ with universal object ϱ^{univ}.

Let $E'[\varepsilon] = E'[[x]]/(x^2)$. Let $\mathfrak{b}$ denote the subspace of upper triangular matrices in $\mathrm{M}_{2\times 2}(E')$ with G_v-action given by $\sigma X = \varrho(\sigma) X \varrho(\sigma)^{-1}$, and let $\mathfrak{b}^0$ denote its trace zero subspace. The tangent space of $R_\varrho^{\mathrm{Bor},\psi}$ is given by $\mathcal{D}_\varrho^{\mathrm{Bor},\psi}(E'[\varepsilon])$. Given $\varrho_{E'[\varepsilon]} \in \mathcal{D}_\varrho^{\mathrm{Bor},\psi}(E'[\varepsilon])$, write $\varrho_{E'[\varepsilon]}(\sigma)$ as

$$\varrho_{E'[\varepsilon]}(\sigma) = (1 + \varepsilon c(\sigma))\varrho(\sigma),$$

for each $\sigma \in G_v$, with $c(\sigma) \in \mathfrak{b}$. It is easily checked that $\varrho_{E'[\varepsilon]}$ is a homomorphism with $\det \varrho_{E'[\varepsilon]} = \psi\epsilon_p = \det \varrho$ if and only if $c \in Z^1(G_v, \mathfrak{b}^0)$, the space of 1-cocycles of G_v with coefficients in $\mathfrak{b}^0$. This determines an isomorphism of E'-vector spaces $\mathcal{D}_\varrho^{\mathrm{Bor},\psi}(E'[\varepsilon]) \cong Z^1(G_v, \mathfrak{b}^0)$. Now

$$\begin{aligned} \dim Z^1(G_v, \mathfrak{b}^0) &= \dim H^1(G_v, \mathfrak{b}^0) - \dim H^0(G_v, \mathfrak{b}^0) + 2 \\ &= 2[F_v : \mathbb{Q}_p] + 2 + H^2(G_v, \mathfrak{b}^0), \end{aligned}$$

by Euler-Poincaré characteristic.

Now fix a minimal presentation

$$0 \longrightarrow J \longrightarrow A \longrightarrow R_\varrho^{\mathrm{Bor},\psi} \longrightarrow 0$$

with A a powerseries over E' in $\dim Z^1(G_v, \mathfrak{b}^0)$ variables. As in §1.6 of [M2], we will show that

$$\dim J/\mathfrak{m}_A J \le \dim H^2(G_v, \mathfrak{b}^0).$$

Let $A_n = A/\mathfrak{m}_A^n$, $R_n = R_\varrho^{\mathrm{Bor},\psi}/\mathfrak{m}^n_{R_\varrho^{\mathrm{Bor},\psi}}$, and $J_n = \ker(A_n \to R_n)$. Then for n sufficiently large the natural map $J/\mathfrak{m}_A J \to J_n/\mathfrak{m}_{A_n} J_n$ is an isomorphism of E' vector spaces, and it suffices to show

$$\dim J_n/\mathfrak{m}_{A_n} J_n \le \dim H^2(G_v, \mathfrak{b}^0).$$

Choose a continuous set theoretic lifting

$$\tilde{\varrho} : G_v \longrightarrow \mathrm{B}_2(A_n)$$

such that $\det \tilde{\varrho}(\sigma) = \psi\epsilon_p(\sigma)$ for all $\sigma \in G_v$ and such that its pushforward to R_n is ϱ^{univ} mod $\mathfrak{m}^n_{R_\varrho^{\mathrm{Bor},\psi}}$. To see that this is possible, note that we can take a continuous E'-vector space section $s : R_n \to A_n$ of the quotient $A_n \to R_n$. This determines a continuous set-theoretic section $\mathrm{B}_2(R_n) \to \mathrm{B}_2(A_n)$, and we let $\tilde{\varrho}'$ be the composite of the map $G_v \to \mathrm{B}_2(R_n)$ with this section. Since the map $\sigma \mapsto \psi\epsilon_p(\sigma) \det \tilde{\varrho}'(\sigma)^{-1}$ is continuous on G_v, so is the function $\tilde{\varrho} : G_v \to \mathrm{B}_2(A_n)$ defined by

$$\tilde{\varrho}(\sigma) = \tilde{\varrho}'(\sigma) \begin{pmatrix} \psi\epsilon_p(\sigma) \det \tilde{\varrho}'(\sigma)^{-1} & \\ & 1 \end{pmatrix}$$

Define the continuous function

$$c : G_v \times G_v \longrightarrow \mathfrak{b}^0 \otimes_{E'} J_n/\mathfrak{m}_{A_n} J_n$$

by $c(\sigma_1, \sigma_2) = \tilde{\varrho}(\sigma_1\sigma_2)^{-1}\tilde{\varrho}(\sigma_2)^{-1}\tilde{\varrho}(\sigma_1)^{-1}$. It can be checked that c is a 2-cocyle of G_v with values in $\mathfrak{b}^0 \otimes_{E'} J_n/\mathfrak{m}_{A_n} J_n$ and that is image $[c]$ in $H^2(G_v, \mathfrak{b}^0 \otimes_{E'} J_n/\mathfrak{m}_{A_n} J_n) \cong H^2(G_v, \mathfrak{b}^0) \otimes_{E'} J_n/\mathfrak{m}_{A_n} J_n$ does not depend on the choice of $\tilde{\varrho}$. We get a natural map

$$(J_n/\mathfrak{m}_{A_n} J_n)^* \longrightarrow H^2(G_v, \mathfrak{b}^0)$$

by $\lambda \mapsto (1 \otimes \lambda)([c])$. We will show that this map is injective. Take non-zero $\lambda \in (J_n/\mathfrak{m}_{A_n} J_n)^*$ such that $(1 \otimes \lambda)([c]) = 0$. Let A'_n denote the quotient of A_n by the kernel of λ. Then we have an exact sequence

$$0 \longrightarrow E' \longrightarrow A'_n \longrightarrow R_n \longrightarrow 0 \tag{2.2}$$

and the obstruction class $[c]$ vanishes for this extension. This implies that there is a continuous homomorphism

$$\varrho_{A'_n} : G_v \longrightarrow \mathrm{B}_2(A'_n)$$

with determinant $\psi\epsilon_p$ and whose pushforward to R_n is ϱ^{univ} modulo $\mathfrak{m}^n_{R_\varrho^{\mathrm{Bor},\psi}}$. Then there is a local E'-algebra morphism $R_\varrho^{\mathrm{Bor},\psi} \to A'_n$ inducing $\varrho_{A'_n}$. Since $\mathfrak{m}^n_{A'_n} = 0$, this map factors through R_n and induces a section of (2.2). But this contradicts the fact that $A'_n \to R_n$ induces an isomorphism on tangent spaces.

We deduce that there is a presentation

$$R_\varrho^{\mathrm{Bor},\psi} \cong E'[[x_1, \ldots, x_g]]/(f_1, \ldots, f_r)$$

where $g = 2 + 2[F_v : \mathbb{Q}_p] + \dim H^2(G_v, \mathfrak{b}^0)$ and $r \leq H^2(G_v, \mathfrak{b}^0)$. In particular, we conclude that $\dim R_{\varrho}^{\mathrm{Bor},\psi} \geq 2 + 2[F_v : \mathbb{Q}_p]$. Let $\mathfrak{u}$ be subspace of Ad^0 consisting of upper triangular unipotent matrices. Note that $\mathfrak{u}$ is G_v stable and the trace pairing on Ad^0 induces a G_v-equivariant isomorphism $(\mathfrak{b}^0)^* \cong \mathrm{Ad}^0/\mathfrak{u}$. Local Tate duality gives an isomorphism $H^2(G_v, \mathfrak{b}^0) \cong H^0(G_v, (\mathrm{Ad}^0/\mathfrak{u})(1))$. It is easily checked that this latter space is non-zero if and only if

$$\varrho = \begin{pmatrix} \epsilon_p \chi & \\ & \chi \end{pmatrix}$$

for some character χ, in which case it has dimension 1. □

2.4.14

Let $\mathscr{L}$ be as in 2.4.2. Take a closed point $x \in \mathscr{L}[1/p]$ with residue field E'. Note that since $\mathscr{L}$ is finite type over $R^{\square,\psi}_{\Lambda(G_v,\eta)}$, part (2) of 2.1.2 implies that E'/E is finite. Let ρ_x and χ_x denote the pushforward of ρ^{univ} and $\chi_\eta^{\mathrm{univ}}$, repsectively, to E' and let L_x denote the fixed G_v-stable line in $(E')^2$ such that G_v acts on $(E')^2/L_x$ via χ_x.

Consider the set valued functor $\mathcal{DL}_x^{\square,\psi}$ on $\mathrm{Ar}_{E'}$ that sends an $\mathrm{Ar}_{E'}$-algebra A the set of pairs (ρ_A, L_A), where ρ_A is a lift of ρ_x to $\mathrm{GL}_2(A)$ with determinant $\psi\epsilon_p$, and L_A is a G_v-stable line in A^2 lifting L_x. Let $\mathcal{O}^\wedge_{\mathscr{L},x}$ denote the completion of the local ring of $\mathscr{L}$ at x. Note that the natural map

$$\mathrm{Spec}\, \mathcal{O}^\wedge_{\mathscr{L},x} \longrightarrow \mathscr{L}$$

yields a lift : $\rho_x^\wedge : G_v \to \mathrm{GL}_2(\mathcal{O}^\wedge_{\mathscr{L},x})$ of ρ_x and a G_v-stable line $L_x^\wedge$ lifting L_x.

Lemma 2.4.15. *$\mathcal{O}^\wedge_{\mathscr{L},x}$ pro-represents $\mathcal{DL}_x^{\square,\psi}$ with universal object $(\rho_x^\wedge, L_x^\wedge)$.*

Proof. Let B be an $\mathrm{Ar}_{E'}$-algebra and let $(\rho_B, L_B) \in \mathcal{DL}_x^{\square,\psi}(B)$. Denote by χ_B the character giving the G_v-action on B^2/L_B. By 2.3.5 there are unique continuous $\mathcal{O}$-algebra morphisms $R_v^\square \to B$ and $\Lambda(G_v) \to B$ such that ρ_B is the pushforward of ρ^{univ} and χ_B is the pushforward of χ^{univ}. Since the ρ_B has determinant $\psi\epsilon_p$ and the restriction of χ_B to the torsion subgroup of G_v^{ab} is equal to η, we get a unique continuous $\mathcal{O}$-algebra morphism $R^{\square,\psi}_{\Lambda(G_v,\eta)} \to B$ which gives rise to ρ_B and χ_B. Since the line L_B is stable under ρ_B, we obtain a unique morphism $y : \operatorname{Spec} B \to \mathscr{L}$ such that the image of the closed point of $\operatorname{Spec} B$ is x. As B is Artinian, y factors through $\operatorname{Spec} \mathcal{O}^{\wedge}_{\mathscr{L},x} \to \mathscr{L}$. $\square$

Let $x = (\rho_x, \chi_x, L_x)$ be a closed point of $\mathscr{L}[1/p]$ and denote by E' its residue field. Take $g \in \mathrm{GL}_2(E')$ such that $gL_x = L^{\mathrm{std}}_{E'}$. Note that $g\rho_x g^{-1}$ is upper triangular, and so we have the functor $\mathcal{D}^{\mathrm{Bor},\psi}_{g\rho_x g^{-1}}$ on $\mathrm{Ar}_{E'}$ as in 2.4.12, which is represented by $R^{\mathrm{Bor},\psi}_{g\rho_x g^{-1}}$ as in 2.4.13.

Lemma 2.4.16. *There is an isomorphism* $\mathcal{O}^{\wedge}_{\mathscr{L},x} \cong R^{\mathrm{Bor},\psi}_{g\rho_x g^{-1}}[[z]]$.

Proof. For ease of notation set $R = R^{\mathrm{Bor},\psi}_{g\rho_x g^{-1}}$ and let ϱ denote its universal lift. Define a lift $\rho_{R[[z]]} : G_v \to R[[z]]$ by

$$\rho_{R[[z]]}(\sigma) = g^{-1} \begin{pmatrix} 1 & \\ z & 1 \end{pmatrix} \varrho(\sigma) \begin{pmatrix} 1 & \\ -z & 1 \end{pmatrix} g$$

and we let $L_{R[[z]]}$ denote the $\rho_{R[[z]]}$-stable line $g^{-1} \begin{pmatrix} 1 & \\ z & 1 \end{pmatrix} L^{\mathrm{std}}_R$. Note that $(\rho_{R[[z]]}, L_{R[[z]]})$ determines a local E'-algebra morphism $\mathcal{O}^{\wedge}_{\mathscr{L},x} \to R[[z]]$. Now given any $\mathrm{Ar}_{E'}$-algebra A and $(\rho_A, L_A) \in \mathcal{DL}^{\square,\psi}_x(A)$, there is a unique $c \in \mathfrak{m}_A$ such that

$$L_A = g^{-1} \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} L^{\mathrm{std}}_A.$$

We then have a unique local E'-algebra morphism $R[[z]] \to A$, such that the specialization of $(\rho_{R[[z]]}, L_{R[[z]]})$ is (ρ_A, L_A). This shows that $\mathcal{O}^{\wedge}_{\mathscr{L},x} \to R[[z]]$ is an isomorphism. $\square$

Proposition 2.4.17. *Let $x = (\rho_x, \chi_x)$ be a closed point of* $\operatorname{Spec} R^{\triangle,\psi}_{\Lambda(G_v,\eta)}[1/p]$*. If $\chi_x^2 \neq \psi, \psi\epsilon_p$ then the local ring of $R^{\triangle,\psi}_{\Lambda(G_v,\eta)}$ at x is formally smooth over E.*

Proof. Let E' denote the residue field of x. Since $\chi_x^2 \neq \psi\epsilon_p$, 2.4.5 implies there is a unique E' point of $\mathscr{L}$, which we denote again by x, and an isomorphism $(R^{\triangle,\psi}_{\Lambda(G_v,\eta)})^{\wedge}_x \to \mathcal{O}^{\wedge}_{\mathscr{L},x}$. The result then follows from 2.4.13 and 2.4.16. $\square$

2.5 Local deformation rings away from p

Let F_v be a characteristic zero local field. Fix an algebraic closure $\overline{F_v}$ of F_v and let $G_v = \mathrm{Gal}(\overline{F_v}/F_v)$. We fix a continuous homomorphism

$$\overline{\rho} : G_v \longrightarrow \mathrm{GL}_2(\mathbb{F})$$

and a continuous character $\psi : G_v \to \mathcal{O}^\times$ such that $\det \overline{\rho} = \overline{\psi\epsilon_p}$. We let $V_{\mathbb{F}}$ denote the representation space of $\overline{\rho}$. We let $\mathcal{D}_v^\square$ and $\mathcal{D}_v^{\square,\psi}$ denote the functor of lifts of $\overline{\rho}$ and the subfunctor consisting of lifts with determinant $\psi\epsilon_p$, repsectively. We denote the corresponding representing objects by $R_v^\square$ and $R_v^{\square,\psi}$, respectively.

2.5.1

Assume that F_v is a non-archimedian field of residual characteristic $l \neq p$. Let q denote the cardinality of the residue field of L. Let ρ^{univ} denote the universal lift to $R_v^\square$.

Proposition 2.5.2. *Let $\overline{R}_v^{\square,\psi}$ denote the quotient of $R_v^{\square,\psi}$ by its p-torsion.*

1. *$\overline{R}_v^{\square,\psi}$ is $\mathcal{O}$-flat and equidimensional of relative dimension* 3.
2. *The set of unramified lifts of $\overline{\rho}_v$ form an irreducible component of* $\mathrm{Spec}\,\overline{R}_v^{\square,\psi}$, *which is formally smooth over $\mathcal{O}$. We denote the quotient by the corresponding minimal prime by $R_v^{\square,\psi,\mathrm{ur}}$.*
3. *If E'/E is finite and $x : \overline{R}_v^{\square,\psi} \to E$ is a continuous $\mathcal{O}$-algebra moprhism such that* $\ker(x)$ *is contained in more than one irreducible component, then letting ρ_x denote the corresponding lift, $\rho_x \cong \gamma_v\epsilon_p \oplus \gamma_v$ for some character γ_v of G_v.*

Proof. By definition $\overline{R}_v^{\square,\psi}$ is p-torsion free, hence is $\mathcal{O}$-flat. The dimension of $\overline{R}_v^{\square,\psi}[1/p]$ is shown to be 3 in Proposition 2.5.4 of [K3]. Proposition 2.5.3 of [K3] shows that there is a quotient $R_v^{\square,\psi} \to R_v^{\square,\psi,\mathrm{ur}}$, such that a lift factors through $R_v^{\square,\psi,\mathrm{ur}}$ if and only if it is unramified, and that $R_v^{\square,\psi,\mathrm{ur}}$ is formally smooth over $\mathcal{O}$ of relative dimension 3. The quotient $R_v^{\square,\psi} \to$

$R_v^{\square,\psi,\mathrm{ur}}$ necessarily factors through $\overline{R}_v^{\square,\psi}$. Part (3) follows from the proof of Proposition 2.5.4 of [K3]. More specifically, in the proof of Proposition 2.5.4 of [K3] it is shown that the completed local ring at x is smooth unless $H^2(G_v, \mathrm{Ad}^0(\rho_x)) \neq 0$, where $\mathrm{Ad}^0(\rho_x)$ denotes the trace zero subspace of the adjoint representation of ρ_x, and it is easily checked that if $H^2(G_v, \mathrm{Ad}^0(\rho_x)) \neq 0$, then $\rho_x \cong \gamma_v \epsilon_p \oplus \gamma_v$ for some character γ_v of G_v. □

We now turn our attention to semistable lifts. We will need the following lemma, which is a variant of a result of Snowden [S].

Lemma 2.5.3. *Assume $\overline{\rho}$ is the trivial representation and that $q \equiv 1 \pmod{p}$. Let J be the ideal in $R_v^{\square}$ generated by the equations*

$$\mathrm{tr}\, \rho^{\mathrm{univ}}(\sigma) = \epsilon_p(\sigma) + 1, \quad \det \rho^{\mathrm{univ}}(\sigma) = \epsilon_p(\sigma),$$

for all $\sigma \in G_v$, as well as the entries of the matrix equations

$$(\rho^{\mathrm{univ}}(\sigma) - 1)(\rho^{\mathrm{univ}}(\tau) - 1) = (\epsilon_p(\sigma) - 1)(\rho_v^{\mathrm{univ}}(\tau) - 1)$$

for all $\sigma, \tau \in G_v$. Then $(R_v^{\square}/J_\gamma) \otimes_{\mathcal{O}} \mathbb{F}$ is a normal Cohen-Macaualy domain of dimension 3.

Proof. Set $R = (R_v^{\square}/J) \otimes_{\mathcal{O}} \mathbb{F}$, and let ρ_R be the pushforward of ρ^{univ} to R. Since we have assumed $q \equiv 1 \pmod{p}$, modulo the maximal ideal of $\mathcal{O}$ the equations defining J become

$$\mathrm{tr}\, \rho_R(\sigma) = 2 \;\text{ and }\; \det \rho_R(\sigma) = 1$$

for all $\sigma \in G_v$, and

$$(\rho_R(\sigma) - 1)(\rho_R(\tau) - 1) = 0$$

for all $\sigma, \tau \in G_v$. From this it follows that R represents the functor on $\mathrm{CNL}_{\mathbb{F}}$ that sends a $\mathrm{CNL}_{\mathbb{F}}$-algebra A to the set of families of matrices $\{1 + X_\sigma \in 1 + \mathfrak{m}_A \mathrm{M}_{2\times 2}(A) : \sigma \in G_v\}$ such that

$$\mathrm{tr}\,(1 + X_\sigma) = 2, \quad \det(1 + X_\sigma) = 1, \quad X_\sigma X_\tau = 0$$

for all $\sigma, \tau \in G_v$. For any such family, the elements $1 + X_\sigma$ commute and have order p (or 0), hence the map $G_v \mapsto 1 + X_\sigma$ factors through the maximal abelian quotient of exponent

p. This group is isomorphic to $\mathbb{F}_p^2$. Hence R represents the functor on $\mathrm{CNL}_{\mathbb{F}}$ that assigns to each $\mathrm{CNL}_{\mathbb{F}}$-algebra A, matrices $X, Y \in \mathfrak{m}_A \mathrm{M}_{2\times 2}(A)$ such that

$$\mathrm{tr}\,(1+X) = \mathrm{tr}\,(1+Y) = 2, \quad \det(1+X) = \det(1+Y) = 1, \quad X^2 = Y^2 = XY = YX = 0.$$

Define a functor F on the category of $\mathbb{F}$-algebas that assigns, to an $\mathbb{F}$-algebra A, the set of pairs $(X, Y) \in \mathrm{M}_{2\times 2}(A)$ satisfying

$$\mathrm{tr}\, X = \mathrm{tr}\, Y = 0, \quad \det X = \det Y = 0, \quad X^2 = Y^2 = XY = YX = 0$$

Theorem 3.3.1 of [S] shows that F is represented by a finite type $\mathbb{F}$-algebra domain R_F of dimension 3, which is normal and Cohen-Macaulay (there is a running assumption that $p > 2$ in [S] but the proof of Theorem 3.3.1 still holds when $p = 2$). Let x denote the closed point $(0, 0) \in F(\mathbb{F})$, and let $(R_F)_x^\wedge$ denote the localization and completion of R_F at x. Part (vii) of Scholie 7.8.3 of [G3] implies that $(R_F)_x^\wedge$ is a normal domain and Theorem 17.5 of [M1] implies that it is Cohen-Macaulay. The lemma now follows from the isomorphism of functors on $\mathrm{CNL}_{\mathbb{F}}$

$$\mathrm{Spf} R \xrightarrow{\sim} \mathrm{Spf}(R_F)_x^\wedge$$

given by $(1 + X, 1 + Y) \mapsto (X, Y)$. □

Proposition 2.5.4. *Let $\gamma : G_v \to \mathcal{O}^\times$ be a finitely ramified continuous character such that $\gamma^2 = \psi$ (enlarging $\mathcal{O}$ if necessary), and such that $\gamma(I_v)$ is prime to p. Assume $\overline{\rho}_v$ is an extension of $\overline{\gamma}$ by $\overline{\gamma\epsilon_p}$.*

There is an $\mathcal{O}$-flat reduced quotient $R_v^{\square,\gamma\text{-}st}$ of $R_v^{\square,\psi}$ such that if E'/E is a finite extenstion, a continuous $\mathcal{O}$-algebra morphism $x : R_v^{\square,\psi} \to E'$, factors through $R_v^{\square,\gamma\text{-}st}$ if and only if the corresponding lift ρ_x is an extension of γ by $\gamma\epsilon_p$. Morever $R_v^{\square,\gamma\text{-}st}$ is a Cohen-Macaulay domain of dimension 4, $R_v^{\square,\gamma\text{-}st}[1/p]$ is formally smooth over E, and $R_v^{\square,\gamma\text{-}st} \otimes_{\mathcal{O}} \mathbb{F}$ is a domain.

Proof. Except for the claim about Cohen-Macaulayness and the reduction mod p, this is proved in Corollary 2.6.7 of [K2] for $p > 2$ and γ unramified, and in Theorem 3.1 of [KW] in the remaining cases.

If there is a unique line in $V_{\mathbb{F}}$ on which G_v acs via $\overline{\gamma\epsilon_p}$, then the proof of Theorem 3.1 in [KW] shows that $R_v^{\square,\gamma\text{-st}}$ is formally smooth over $\mathcal{O}$, and so we we may assume that $\overline{\epsilon_p}$ is trivial, i.e. $q \equiv 1 \pmod p$, and that G_v acts on $V_{\mathbb{F}}$ via the character $\overline{\gamma}$. Twisting lifts by γ^{-1} yields an isomorphism of lifting functors $\mathcal{D}_{\overline{\rho}}^{\square} \cong \mathcal{D}_{\overline{\rho}\otimes\overline{\gamma}^{-1}}^{\square}$, and hence an isomorphism of their universal lifting rings. Part (3) of 2.1.2 then shows that this isomorphism of universal lifting rings yields an isomorphism $R_v^{\square,\gamma\text{-st}} \cong R_v^{\square,1\text{-st}}$, and so we may assume $\gamma = 1$ and $\overline{\rho}$ is the trivial homomorphism.

Let J be as in 2.5.3. Let X denote the set of points $x : R_v^{\square} \to \mathcal{O}'$, where $\mathcal{O}'$ is the ring of integers in some finite extension E'/E, such that the induced lift ρ_x is conjugate to an extension of ϵ_p by 1, and set $J_X = \cap_{x\in X}\ker(x)$. By part (3) of 2.1.2, then the surjection $R_v^{\square} \to R_v^{\square,\gamma\text{-st}}$ has kernel J_X. Since the equations defining J hold for any $x \in X$, this surjection factors through $R_v^{\square}/J$. Let $\mathfrak{q}$ denote the kernel of $R_v^{\square}/J \to R_v^{\square,1\text{-st}}$. Note that $\mathfrak{q}$ is prime and $\varpi_E \notin \mathfrak{q}$. Since $R_v^{\square,1\text{-st}}$ is $\mathcal{O}$-flat of relative dimension 3 and $(R_v^{\square}/J)\otimes_{\mathcal{O}}\mathbb{F}$ is a domain of dimension 3 by 2.5.3, the surjection

$$(R_v^{\square}/J)\otimes_{\mathcal{O}}\mathbb{F} \longrightarrow R_v^{\square,1\text{-st}}\otimes_{\mathcal{O}}\mathbb{F}$$

is an isomorphism. Then 2.5.3 implies $R_v^{\square,1\text{-st}}\otimes_{\mathcal{O}}\mathbb{F}$ is Cohen-Macaualy and a domain. The Cohen-Macaulayness of $R_v^{\square,1\text{-st}}$ now follows from corollary to Theorem 23.3 of [M1]. □

2.5.5

Now assume that $F_v \cong \mathbb{R}$ and assume $\overline{\rho}$ is odd, i.e. $\det\overline{\rho}(c) = -1$ for c complex conjugation.

Proposition 2.5.6. *There is an $\mathcal{O}$-flat reduced quotient $R_v^{\square,-1}$ of $R_v^{\square}$ such that if E'/E is finite, a continuous $\mathcal{O}$-morphism $x : R_v^{\square} \to E'$ factors through $R_v^{\square,-1}$ if and only if ρ_x is odd. $R_v^{\square,-1}$ is a complete intersection domain of dimension 3, $R_v^{\square,-1}[1/p]$ is formally smooth over E, and $R_v^{\square,-1}\otimes_{\mathcal{O}}\mathbb{F}$ is a domain.*

Proof. Proposition 3.3 of [KW] or Proposition 2.5.6 of [K3] shows that $R_v^{\square,-1}$ is smooth over

$\mathcal{O}$ if $\overline{\rho}$ is non-trivial and that if $\overline{\rho}$ is trivial then

$$R_v^{\square,-1} \cong \mathcal{O}[[x,y,z]]/(x^2+2y+yz)$$

which is a complete intersection domain of dimension 3. Lastly x^2+yz is irreducible in $\mathbb{F}[[x,y,z]]$, so $R_v^{\square,-1} \otimes_{\mathcal{O}} \mathbb{F}$ is a domain. $\square$

2.6 Global deformations

Let F be a totally real field and let S denote a finite set of places of F containing all the infinite places as well as all the places above p. Denote by F_S the maximal Galois extension of F unramified outside of S and let $G_{F,S} = \mathrm{Gal}(F_S/F)$. Fix an absolutely irreducible continuous representation

$$\overline{\rho} : G_{F,S} \longrightarrow \mathrm{GL}_2(\mathbb{F}).$$

We fix a continuous character $\psi : G_{F,S} \to \mathcal{O}^\times$ such that $\overline{\psi\epsilon_p} = \det\overline{\rho}$. For each $v \in S$, let F_v denote the completion of F at v, and let $G_v = \mathrm{Gal}(\overline{F_v}/F_v)$ for some fixed algebraic closure $\overline{F_v}$ of F_v. Let Q be some (possibly empty) set of places of F disjoint from S.

Throughout this section, all completed tensor products are taken over $\mathcal{O}$ unless noted otherwise.

Let $R_{F,S\cup Q}$, resp. $R^\psi_{F,S\cup Q}$, denote the universal deformation ring, resp. universal deformation ring with determinant $\psi\epsilon_p$, of the $G_{F,S\cup Q}$-representation $\overline{\rho}$, and for each $v \in S$, let $R_v^{\square,\psi}$ denote the universal framed deformation ring for lifts of $\overline{\rho}|_{G_v}$ with determinant $\psi\epsilon_p|_{G_v}$. Consider the set valued functor on $\mathrm{CNL}_{\mathcal{O}}$ that sends an object A to $(V_A, \{\beta_v\}_{v\in S})$, where V_A is a $G_{F,S\cup Q}$-deformation of $\overline{\rho}$ to A such that the determinant of $V_A|_{G_v}$ is equal to $\psi\epsilon_p|_{G_v}$ for all $v \in S$, and β_v is a lift of $\overline{\beta}$ for each $v \in S$. This functor is representable and we denote the representing object by $R^\square_{F,S\cup Q}$. The subfunctor consisting of the tuples $(V_A, \{\beta_v\}_{v\in S})$ such that $\det V_A = \psi\epsilon_p$ is also representable and we denote the representing object by $R^{\square,\psi}_{F,S\cup Q}$. The forgetful functor $(V_A, \{\beta_v\}_{v\in S}) \mapsto V_A$ gives a canonical maps $R_{F,S\cup Q} \to R^\square_{F,S\cup Q}$ and $R^\psi_{F,S\cup Q} \to R^{\square,\psi}_{F,S\cup Q}$, and in the latter case it is formally smooth of relative dimension $4|S|-1$, cf. Proposition 4.1 of [KW]. Note that $R^{\psi}_{F,S\cup Q}$ is a quotient of $R_{F,S}$ and there is a canonical isomorphism $R^{\square,\psi}_{F,S\cup Q} \cong R^{\psi}_{F,S\cup Q} \otimes_{R_{F,S\cup Q}} R^{\square}_{F,S\cup Q}$.

The identity map $R^{\square}_{F,S\cup Q} \to R^{\square}_{F,S\cup Q}$ gives a universal object $(V^{\text{univ}}, \{\beta^{\text{univ}}_v\}_{v\in S})$. For each $v \in S$, $(V^{\text{univ}}|_{G_v}, \beta^{\text{univ}}_v)$ determines a lift of $\overline{\rho}|_{G_v}$ with determinant $\psi\epsilon_p|_{G_v}$, so we have a canonical morphism $R^{\square,\psi}_v \to R^{\square}_{F,S\cup Q}$. Letting $R^{\square,\psi}_S$ denote the completed tensor product $\hat{\otimes}_{v\in S} R^{\square,\psi}_v$, we see that $R^{\square}_{F,S\cup Q}$ is canonically a $R^{\square,\psi}_S$ algebra. This also give an $R^{\square,\psi}_S$-algebra structure to $R^{\square,\psi}_{F,S\cup Q}$.

2.6.1

Let Ad denote the space $\mathrm{M}_{2\times 2}(\mathbb{F})$ with the adjoint $G_{F,S}$-action, and let Ad^0 denote its trace zero subspace. In what follows we will use the following notation. Given a topological group G such that $\mathrm{Hom}_{cts}(G, \mathbb{F})$ is finite, and a a finite $\mathbb{F}[G]$-module M, we denote by M^* the $\mathbb{F}$-linear dual of M with the induced G-action. For any $i \geq 0$, we denote by $h^i(G_{F,S}, M)$ the $\mathbb{F}$-dimension of the cohomology group $H^i(G, M)$. If $G = G_{F,S}$, and W is a finite set of places of F, we let $H^i_W(G_{F,S}, M)$ denote the kernel of the restriction map

$$H^i(G_{F,S}, M) \longrightarrow \prod_{v\in W} H^i(G_v, M)$$

and let $h^i_W(G_{F,S}, M)$ denote its $\mathbb{F}$-dimension.

Lemma 2.6.2. *There is a presentation*

$$R^{\square,\psi}_{F,S} \cong R^{\square,\psi}_S[[x_1, \ldots, x_g]]/(f_1, \ldots, f_r)$$

with

$$g - r \geq |S| - 1 - h^0(G_{F,S}, (\mathrm{Ad}^0)^*(1)).$$

Proof. Let

$$\phi : \mathfrak{m}_{R^{\square,\psi}_S}/((\mathfrak{m}_{R^{\square,\psi}_S})^2, \varpi_E) \longrightarrow \mathfrak{m}_{R^{\square,\psi}_{F,S}}/((\mathfrak{m}_{R^{\square,\psi}_{F,S}})^2, \varpi_E)$$

denote the map on reduced cotangent spaces induced from $R_S^{\square,\psi} \to R_{F,S}^{\square,\psi}$. By Proposition 4.1.4 of [K4], we have a presentation

$$R_{F,S}^{\square,\psi} \cong R_S^{\square,\psi}[[x_1, \ldots, x_g]]/(f_1, \ldots, f_r)$$

where $g - r \geq \dim_{\mathbb{F}} \operatorname{coker}(\phi) - \dim_{\mathbb{F}} \ker(\phi) - h_S^2(G_{F,S}, \mathrm{Ad}^0)$. One can show, cf. proof of Lemma 4.1.5 of [K4],

$$\dim_{\mathbb{F}} \mathfrak{m}_{R_{F,S}^{\square,\psi}}/((\mathfrak{m}_{R_{F,S}^{\square,\psi}})^2, \varpi_E) = 4|S| + h^1(G_{F,S}, \mathrm{Ad}^0) - h^0(G_{F,S}, \mathrm{Ad}^0) - 1$$

and

$$\dim_{\mathbb{F}} \mathfrak{m}_{R_S^{\square,\psi}}/((\mathfrak{m}_{R_S^{\square,\psi}})^2, \varpi_E) = 4|S| + \sum_{v \in S}(h^1(G_v, \mathrm{Ad}^0) - h^0(G_v, \mathrm{Ad}^0) - 1).$$

And so,

$$\begin{aligned}\dim_{\mathbb{F}} \operatorname{coker}\phi - \dim_{\mathbb{F}} \ker\phi &= \dim_{\mathbb{F}} \mathfrak{m}_{R_{F,S}^{\square,\psi}}/((\mathfrak{m}_{R_{F,S}^{\square,\psi}})^2, \varpi_E) - \dim_{\mathbb{F}} \mathfrak{m}_{R_S^{\square,\psi}}/((\mathfrak{m}_{R_S^{\square,\psi}})^2, \varpi_E) \\ &= |S| - 1 + h^1(G_{F,S}, \mathrm{Ad}^0) - h^0(G_{F,S}, \mathrm{Ad}^0) \\ &\quad - \sum_{v \in S}(h^1(G_v, \mathrm{Ad}^0) - h^0(G_v, \mathrm{Ad}^0)). \qquad (2.3)\end{aligned}$$

The Poitou-Tate sequence implies

$$h_S^2(G_{F,S}, \mathrm{Ad}^0) = h^2(G_{F,S}, \mathrm{Ad}^0) - \sum_{v \in S} h^2(G_v, \mathrm{Ad}^0) + h^0(G_{F,S}, (\mathrm{Ad}^0)^*(1)).$$

Combining this with (2.3), we have

$$\begin{aligned}g - r &\geq \dim_{\mathbb{F}} \operatorname{coker}\phi - \dim_{\mathbb{F}} \ker\phi - h_S^2(G_{F,S}, \mathrm{Ad}^0) \\ &\geq |S| - 1 - h^0(G_{F,S}, (\mathrm{Ad}^0)^*(1)) - \chi(G_{F,S}, \mathrm{Ad}^0) + \sum_{v \in S} \chi(G_v, \mathrm{Ad}^0), \qquad (2.4)\end{aligned}$$

where $\chi(G_{F,S}, \mathrm{Ad}^0)$ and $\chi(G_v, \mathrm{Ad}^0)$ denote the global and local Euler characteristics, respectively, as $\mathbb{F}$-vector spaces.

For $v|\infty$ we have $h^1(G_v, \mathrm{Ad}^0) = h^2(G_v, \mathrm{Ad}^0)$ since G_v is cyclic, and so $\chi(G_v, \mathrm{Ad}^0) = h^0(G_v, \mathrm{Ad}^0)$. For v finite the local Euler characteristic formula gives $\chi(G_v, \mathrm{Ad}^0) = 1$, when

$v \nmid p$, and $\chi(G_v, \mathrm{Ad}^0) = -3[F_v : \mathbb{Q}]$, when $v|p$. The global Euler characteristic formula gives $\chi(G_{F,S}, \mathrm{Ad}^0) = -3[F : \mathbb{Q}] + \sum_{v|\infty} h^0(G_v, \mathrm{Ad}^0)$. Equation (2.4) then becomes

$$g - r \geq |S| - 1 - h^0(G_{F,S}, (\mathrm{Ad}^0)^*(1)).$$

□

We will use the above lemma to show that a certain quotient of $R^{\psi}_{F,S}$ (actually a quotient of $R_{F,S}$ tensored with an certain Iwasawa algebra) has an appropriate presentation in order to apply the connectivity result 2.1.8.

2.6.3

Let Σ be a fixed set of places not containing any places above p or ∞. For each $v \in \Sigma$ fix a continuous character $\gamma_v : G_v \to \mathcal{O}^{\times}$ such that $\gamma_v(I_v)$ is finite and prime to p, and such that $\gamma_v^2 = \psi|_{G_v}$ (enlarging $\mathcal{O}$ if necessary). We further assume the following.

- For $v|p$, $\overline{\rho}|_{G_v} \cong \begin{pmatrix} * & * \\ & \overline{\chi}_v \end{pmatrix}$, where we fix the choice $\overline{\chi}_v$ in the case that $\overline{\rho}|_{G_v}$ is the direct sum of two disctinct characters.
- For $v \in \Sigma$, $\overline{\rho}|_{G_v} \cong \begin{pmatrix} \overline{\gamma}_v\overline{\epsilon_p} & * \\ & \overline{\gamma}_v \end{pmatrix}$.
- For any archimedean v, $\psi|_{G_v} = 1$.
- $\overline{\rho}$ is unramified outside $\Sigma \cup \{v|p\} \cup \{v|\infty\}$.

Fix a finite set of places S_{ur} disjoint from $\Sigma \cup \{v|p\} \cup \{v|\infty\}$ and set $S = S_{\mathrm{ur}} \cup \Sigma \cup \{v|p\} \cup \{v|\infty\}$. For each $v|p$, let η_v be a character of the torsion subgroup of $G_v^{\mathrm{ab}}(p)$. Then η_v correpsonds to a unique minimal prime of $\Lambda(G_v) = \mathcal{O}[[G_v^{\mathrm{ab}}(p)]]$, which we denote by $\mathfrak{q}_{\eta_v}$. We denote the quotient $\Lambda(G_v)/\mathfrak{q}_{\eta_v}$ by $\Lambda(G_v, \eta_v)$. Note that giving the tuple $(\eta_v)_{v|p}$ is equivalent to giving a character η on the torsion subgroup of $\prod_{v|p} G_v^{\mathrm{ab}}(p)$. For each $v \in S$, let $\overline{R}_v^{\square,\psi}$ denote the $\mathrm{CNL}_{\mathcal{O}}$-algebra given by

- $\overline{R}_v^{\square,\psi} = R_{\Lambda(G_v,\eta_v)}^{\triangle,\psi}$ as in 2.4.3 for $v|p$;

- $\overline{R}_v^{\square,\psi} = R_v^{\square,\gamma\text{-st}}$ as in 2.5.4 for $v \in \Sigma$,

- $\overline{R}_v^{\square,\psi} = R_v^{\square,-1}$ as in 2.5.6 for $v|\infty$,

- $\overline{R}_v^{\square,\psi} = R_v^{\square,\psi,\mathrm{ur}}$ as in part (2) of 2.5.2 if $v \in S_{\mathrm{ur}}$.

Let $\overline{R}_S^{\square,\psi} = \hat{\otimes}_{v\in S}\overline{R}_v^{\square,\psi}$ (taken over $\mathcal{O}$), and let $\Lambda(G_p) = \hat{\otimes}_{v|p}\Lambda(G_v)$ and $\Lambda(G_p,\eta) = \hat{\otimes}_{v|p}\Lambda(G_v,\eta_v)$. Note that $\overline{R}_S^{\square,\psi}$ is a quotient of

$$(\hat{\otimes}_{v|p}(R_v^{\square}\hat{\otimes}\Lambda(G_v,\eta_v)))\hat{\otimes}(\hat{\otimes}_{v\in S, v\nmid p}R_v^{\square,\psi}) \cong (\hat{\otimes}_{v\in S}R_v^{\square})\hat{\otimes}\Lambda(G_p,\eta),$$

and that $\Lambda(G_p,\eta)$ represents the functor on $\mathrm{CNL}_{\mathcal{O}}$ that sends a $\mathrm{CNL}_{\mathcal{O}}$-algebra A to the set of tuples $(\chi_v)_{v|p}$, where each χ_v is an A-valued character of G_v that reduces to $\overline{\chi}_v$ modulo the maximal ideal of A and whose restriction to the p-power part of the torsion subgroup of G_v^{ab} is eqaul to η_v.

Lemma 2.6.4. *If $p = 2$, assume that F_v contains a 4-th root of unity for each $v|p$. Assume also that $\overline{\rho}|_{G_v}$ is either the trivial representation or that its image has order p. Then $\overline{R}_S^{\square,\psi}$ is an $\mathcal{O}$-flat domain of dimension $1 + 3|S| + [F : \mathbb{Q}]$.*

Proof. By 2.4.11, 2.5.4, 2.5.6, and part (2) of 2.5.2, each of the $\overline{R}_v^{\square,\psi}$ an $\mathcal{O}$-flat domain of relative dimension

- $3 + 2[F_v : \mathbb{Q}_p]$ if $v|p$,

- 3 if $v \in \Sigma$,

- 2 if $v|\infty$,

- 3 if $v \in S_{\mathrm{ur}}$,

and so $\overline{R}_S^{\square,\psi}$ is $\mathcal{O}$-flat of relative dimension

$$\sum_{v|p} 3 + 2[F_v : \mathbb{Q}_p] + \sum_{v\in\Sigma\cup S_{\mathrm{ur}}} 3 + \sum_{v|\infty} 2 = 3|S| + [F : \mathbb{Q}_p].$$

To see that it is a domain, consider a finite extension E'/E with ring of integers $\mathcal{O}'$ and residue field $\mathbb{F}'$. It follows from 2.3.3 that $R_v^\square \otimes_{\mathcal{O}} \mathcal{O}'$ is the universal lifting ring on $\mathrm{CNL}_{\mathcal{O}'}$ for the representation $\overline{\rho}|_{G_v} \otimes_{\mathbb{F}} \mathbb{F}'$, and it follows easily that $\overline{R}_v^{\square,\psi} \otimes_{\mathcal{O}} \mathcal{O}'$ are the corresponding quotients on the category $\mathrm{CNL}_{\mathcal{O}'}$. Applying 2.4.11, 2.5.4 and 2.5.6 to $\overline{R}_v^{\square,\psi} \otimes_{\mathcal{O}} \mathcal{O}'$, we conclude that $\overline{R}_v^{\square,\psi}[1/p]$ is geometrically integral. Then $\overline{R}_S^{\square,\psi}$ is a domain by 2.1.4. $\square$

2.6.5

Now fix a finite set of primes Q disjoint from S. Letting $R_S^\square = \hat{\otimes}_{v \in S} R_v^\square$, define

$$\overline{R}_{F,S\cup Q}^{\square} = R_{F,S\cup Q}^{\square} \hat{\otimes}_{R_S^\square} \overline{R}_S^{\square,\psi}$$

and

$$\overline{R}_{F,S\cup Q}^{\square,\psi} = R_{F,S\cup Q}^{\square,\psi} \hat{\otimes}_{R_S^\square} \overline{R}_S^{\square,\psi}.$$

Note that $\overline{R}_{F,S\cup Q}^{\square}$ and $\overline{R}_{F,S\cup Q}^{\square,\psi}$ are quotients of $R_{F,S\cup Q}^{\square} \hat{\otimes} \Lambda(G_p, \eta)$. We then define $\overline{R}_{F,S\cup Q}^{\psi}$ to be the image of $R_{F,S\cup Q}^{\psi} \hat{\otimes} \Lambda(G_p, \eta)$ under

$$R_{F,S\cup Q}^{\psi} \hat{\otimes} \Lambda(G_p, \eta) \longrightarrow R_{F,S\cup Q}^{\square,\psi} \hat{\otimes} \Lambda(G_p, \eta) \longrightarrow \overline{R}_{F,S\cup Q}^{\square,\psi}.$$

If E'/E is finite with ring of integers $\mathcal{O}'$, a local $\mathcal{O}$-algebra morphism $R_{F,S\cup Q} \hat{\otimes} \Lambda(G_p) \to \mathcal{O}'$ factors through $\overline{R}_{F,S\cup Q}^{\psi}$ if and only if the corresponding deformation $V_{\mathcal{O}'}$ and tuple of characters $(\chi_v)_{v|p}$ satisfies

- $\det V_{\mathcal{O}'} = \psi \epsilon_p$;
- for each $v|p$, there is a G_v-stable line L in $V_{\mathcal{O}'}$ and the action of G_v on $V_{\mathcal{O}'}/L$ is given by χ_v;
- for each $v|p$, the restriction of χ_v to the p-power torsion subgroup of G_v^{ab} is equal to η_v;
- for each $v \in \Sigma$, $V_{\mathcal{O}'}|_{G_v}$ is an extension of γ_v by $\gamma_v \epsilon_p$;
- for each archimedean v, $V_{\mathcal{O}'}|_{G_v}$ is not the trivial representation,

- for each $v \in S_{\mathrm{ur}}$, $V_{\mathcal{O}'}|_{G_v}$ is unramified.

Note the second last condition is redundant, as it is implied by the first. However, we will later have to consider the $\overline{R}_S^{\square,\psi}$-map $\overline{R}_{F,S\cup Q}^{\square} \to \overline{R}_{F,S\cup Q}^{\square,\psi}$, and the oddness condition is not forced on $\overline{R}_{F,S\cup Q}^{\square}$. Also note that if we had omitted the primes S_{ur} from S entirely, the resulting ring $\overline{R}_{F,S\cup Q}^{\psi}$ would have been the same. We include them because it will be useful later in §4 to ensure that the local framed deformation ring $\overline{R}_S^{\square,\psi}$ surjects onto a certain Hecke algebra as well as to ensure that a certain group action is free.

If A is a CNL$_{\mathcal{O}}$-algebra and $x, x' \in \mathrm{Spf}(R_{F,S\cup Q}^{\square,\psi}\hat{\otimes}\Lambda(G_p,\eta))(A)$ are two A-points with the same image in $\mathrm{Spf}(R_{F,S\cup Q}^{\psi}\hat{\otimes}\Lambda(G_p,\eta))(A)$, i.e. give rise to the same deformation, then $x \in \mathrm{Spf}\overline{R}_{F,S\cup Q}^{\square,\psi}(A)$ if and only if $x' \in \mathrm{Spf}\overline{R}_{F,S\cup Q}^{\square,\psi}(A)$. It follows that $\overline{R}_{F,S\cup Q}^{\psi} \to \overline{R}_{F,S\cup Q}^{\square,\psi}$ is formally smooth of relative dimension $4|S|-1$.

The following proposition will allow us to invoke 2.1.8, which is crucial to the Sinner-Wiles strategy.

Proposition 2.6.6. *If $p = 2$ assume that F_v contains a 4th root of unity for each $v|2$. Assume also that $\overline{\rho}|_{G_v}$ is either the trivial representation or that its image has order p for each $v|p$. There is a presentation*

$$\overline{R}_{F,S}^{\psi} \cong A/(f_1, \ldots, f_m)$$

with A a domain, and $\dim A - m \geq 1 + [F : \mathbb{Q}] - h^0(G_{F,S}, (\mathrm{Ad}^0)^*(1))$.

Proof. Since $\overline{R}_{F,S}^{\psi} \to \overline{R}_{F,S}^{\square,\psi}$ is formally smooth of relative dimension $4|S|-1$, $\overline{R}_{F,S}^{\square,\psi}$ is isomorphic to a power series over $\overline{R}_{F,S}^{\psi}$ in $4|S|-1$ variables, and it suffices to show that there is a presentation

$$\overline{R}_{F,S}^{\square,\psi} \cong A/(f_1, \ldots, f_k)$$

with A a domain and $\dim A - k \geq 4|S| + [F : \mathbb{Q}] - h^0(G_{F,S}, (\mathrm{Ad}^0)^*(1))$. By 2.6.4, $\overline{R}_S^{\square,\psi}$ is a domain of dimension $1 + 3|S| + [F : \mathbb{Q}]$. The presentation in 2.6.2 yields a presentation

$$\overline{R}_{F,S}^{\square,\psi} \cong \overline{R}_S^{\square,\psi}[[x_1, \ldots, x_g]]/(f_1, \ldots, f_r)$$

with $g - r \geq |S| - 1 - h^0(G_{F,S}, (\mathrm{Ad}^0)^*(1))$. Taking $A = \overline{R}_S^{\square,\psi}[[x_1, \ldots, x_g]]$ gives the result. □

2.6.7

Let Q be a set of places of F disjoint from S. Let F_Q^S denote the maximal abelian pro-p extension of F unramified outside of Q and split at primes in S. Note that F_Q^S/F is finite since S contains all the primes above p. Let $G_Q = \mathrm{Gal}(F_Q^S/F)$ and let G_Q^* denote the diagonalizable $\mathrm{CNL}_{\mathcal{O}}$-group as in 2.2.3. Note that for any $A \in \mathrm{Ob}(\mathrm{CNL}_{\mathcal{O}})$, $G_Q^*(A)$ is the subgroup $\mathrm{Hom}(G_Q, A^\times)$ that reduce to the trivial morphism modulo $\mathfrak{m}_A$.

If $S \supset Q$, there is an action of G_Q^* on $\mathrm{Spf}\, R_{F,S\cup Q}^{\square}$ given as follows. Let $A \in \mathrm{Ob}(\mathrm{CNL}_{\mathcal{O}})$, and let V_A be a deformation of $V_{\mathbb{F}}$ to A. If $\chi \in G_Q^*(A)$, viewing χ as a character of $G_{F,S\cup Q}$, we have a deformation $V_A \otimes \chi$. In this way we get an action of G_Q^* on $\mathrm{Spf}\, R_{F,S\cup Q}$. This action extends to an action on $\mathrm{Spf}(R_{F,S\cup Q}^{\square} \hat{\otimes} \Lambda(G_p, \eta))$ by $(V_A, \{\beta_v\}_{v\in S}, \{\chi_v\}_{v|p}) \mapsto (V_A \otimes \chi, \{\beta_v\}_{v\in S}, \{\chi_v\}_{v|p})$. Note that since F_Q^S is split at all places in S, the lift of $\overline{\rho}|_{G_v}$ given by $V_A|_{G_v}$ and β_v is equal to the lift given by $(V_A \otimes \chi)|_{G_v}$ and β_v, for every $v \in S$. It follows that the action of G_Q^* commutes with the morphism $\mathrm{Spf}(R_{F,S\cup Q}^{\square} \hat{\otimes} \Lambda(G_p, \eta)) \to \mathrm{Spf}(R_S^{\square,\psi} \hat{\otimes} \Lambda(G_p, \eta))$. Letting $\overline{R}_S^{\square,\psi}$ and $\overline{R}_{F,S\cup Q}^{\square}$ be as in 2.6.3, we get an action of G_Q^* on $\mathrm{Spf}\overline{R}_{F,S\cup Q}^{\square}$ that commutes with $\mathrm{Spf}\overline{R}_{F,S\cup Q}^{\square} \to \mathrm{Spf}\overline{R}_S^{\square,\psi}$. Note that the map $(V_A, \{\beta_v\}_{v\in S}, \{\chi_v\}_{v|p}) \mapsto \det V_A(\psi\epsilon_p)^{-1}$ determines a morphism $\delta_Q : \mathrm{Spf}\overline{R}_{F,S\cup Q}^{\square} \to G_Q^*$ such that for any $\mathrm{CNL}_{\mathcal{O}}$-algebra A, $g \in G_Q^*(A)$ and $x \in \mathrm{Spf}\overline{R}_{F,S\cup Q}^{\square}(A)$, we have $\delta_Q(gx) = g^2 \delta_Q(x)$. The natural surjection $\overline{R}_{F,S\cup Q}^{\square} \to \overline{R}_{F,S\cup Q}^{\square,\psi}$ identifies $\mathrm{Spf}\overline{R}_{F,S\cup Q}^{\square,\psi}$ with the closed sub-formal-scheme of $\mathrm{Spf}\overline{R}_{F,S\cup Q}^{\square}$ given by $\delta_Q = 1$.

If $p = 2$, we denote by $G_{Q,2}^*$ the 2-torsion subgroup of G_Q^*, i.e. the diagonalizable $\mathrm{CNL}_{\mathcal{O}}$-group $(G_Q/2G_Q)^*$. Note that for $\chi \in G_{Q,2}^*(A)$, we have $\chi^2 = 1$, hence the action of G_Q^* on $\mathrm{Spf}\, R_{F,S\cup Q}^{\square}$, resp. $\mathrm{Spf}\overline{R}_{F,S\cup Q}^{\square}$, induces an action of $G_{Q,2}^*$ on $\mathrm{Spf}\, R_{F,S\cup Q}^{\square,\psi}$, resp. on $\mathrm{Spf}\overline{R}_{F,S\cup Q}^{\square,\psi}$.

Note that if $p = 2$ and $\overline{\rho}$ has solvable image, then there is a unique quadratic extension L/F such that $\overline{\rho}|_{G_L}$ is abelian, since in this case the image of $\overline{\rho}$ has order twice an odd number.

Lemma 2.6.8. *Let $p = 2$. Assume that if $\overline{\rho}$ has solvable image, then some $v \in S$ does not split in L/F, where L is the unique quadratic extension for which $\overline{\rho}|_{G_L}$ is abelian. Then if Q is a set of places disjoint from S, the action of G_Q^* on $\mathrm{Spf}\overline{R}_{F,S\cup Q}^{\square}$ is free.*

Proof. This is essentially Lemma 5.1 of [KW]. It suffices to show that the action of G_Q^* on $\mathrm{Spf}\overline{R}_{F,S\cup Q}$ is free. If $\overline{\rho}$ is non-solvable, then its projective image is isomorphic to $\mathrm{SL}_2(\mathbb{F}_{2^r})$, for some $r \geq 1$. If $\overline{\rho}$ is solvable, then by our assumption on S and the fact that $\overline{\rho}(G_L)$ has odd order, we see that the fixed field of the kernel of $\overline{\rho}$ and F_Q^S are disjoint. In either case, if χ is a non-trivial element of $G_Q^*(A)$, we can find $g \in G_{F,S\cup Q}$ such that $\chi(g) \neq 1$ and $\mathrm{tr}\,\overline{\rho}(g) \neq 0$. Then if V_A is a deformation of $\overline{\rho}$ to A, we have $\chi(g)\mathrm{tr}\,\rho_A(g) \neq \mathrm{tr}\,\rho_A(g)$, for any ρ_A in the deformation class of V_A, since $\mathrm{tr}\,\rho_A(g)$ is a unit. $\square$

2.7 Deformations of dihedral representations

Fix continuous and absolutely irreducible

$$\overline{\rho} : G_{F,S} \to \mathrm{GL}_2(\mathbb{F})$$

and let $V_{\mathbb{F}}$ denote its representation space. Let A be a $\mathrm{CNL}_{\mathcal{O}}$-algebra and V_A a deformation of $V_{\mathbb{F}}$. We say V_A is L-dihedral, for a quadratic extension L/F, if G_L acts on V_A through an abelian quotient, but G_F does not. We say V_A is *dihedral* if it is L-dihedral for some quadratic L/F. Note that V_A is L-dihedral only if $V_{\mathbb{F}}$ is L-dihedral. If A is a domain with field of fractions K, then if V_A is L-dihedral and $V_A \otimes_A \overline{K}$ is irreducible, for $\overline{K}$ an algebraic closure of K, then one can show there is a character $\chi : G_L \to A^\times$ such that $V_A \cong \mathrm{Ind}_{G_L}^{G_F}\chi$. A subgroup H of $\mathrm{GL}_2(\mathbb{F})$ is called *dihedral* if it's image in $\mathrm{PGL}_2(\mathbb{F})$ is isomorphic to a dihedral group. It is easy to check that $V_{\mathbb{F}}$ is dihedral if and only if $\overline{\rho}(G_F)$ is dihedral. From this we see that if $p = 2$ there is a unique quadratic L/F such that $V_{\mathbb{F}}$ is L-dihedral, as the order of $\overline{\rho}(G_F)$ is twice an odd number.

2.7.1

We first establish some criteria for determining when deformations of $V_{\mathbb{F}}$ have image that is Zariski dense in SL_2.

Lemma 2.7.2. *Let A be a $\mathrm{CNL}_{\mathcal{O}}$-algebra domain of characteristic p with fraction field K.*

Let V_A be a deformation of $V_{\mathbb{F}}$ to A such that the map $\mathrm{im}\rho_A \to \mathrm{im}\overline{\rho}$ has non-trivial kernel. The image of ρ_A contains an element with an infinite order eigenvalue in $\overline{K}^{\times}$, for some algebraic closure $\overline{K}$ of K.

Proof. Let $\overline{F}$ be the algebraic closure of $\mathbb{F}$ in $\overline{K}$. If an element in the image of ρ_A has finite order eigenvalues in $\overline{K}^{\times}$ then it's charracteristic polynomial has coefficients in $\overline{F} \cap A = \mathbb{F}$, so it suffices to show there is some $g \in \mathrm{im}\rho_A$ with $\mathrm{tr}\, g \notin \mathbb{F}$. Take nontrivial $g_1 \in \ker(\mathrm{im}\rho_A \to \mathrm{im}\overline{\rho})$. If g_1 has finite order eigenvalues then it's eigenvalues bust be 1 and g_1 is unipotent. There is a basis of V_A such that

$$g_1 = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$$

with $0 \neq x \in \mathfrak{m}_A$. Since $V_{\mathbb{F}}$ is irreducible there is $g_2 \in \mathrm{im}\rho$ such that with respect to our fixed basis,

$$g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $c \in A^{\times}$. If $\mathrm{tr}\, g_2 \notin \mathbb{F}$ we are done, otherwise $\mathrm{tr}\, g_1 g_2 = cx + \mathrm{tr}\, g_2 \notin \mathbb{F}$. □

Lemma 2.7.3. *Let A be a $\mathrm{CNL}_{\mathcal{O}}$-algebra domain and V_A be a non-dihedral deformation of $V_{\mathbb{F}}$ such that for some $\sigma \in G_F$, $\rho_A(\sigma)$ has eigenvalues whose ratio is not a root of unity. Then for any finite index subgroup H of G_F, $\rho_A|_H$ is absolutely irreducible.*

Proof. It suffices to consider the case when H is an open normal subgroup of G_F. Let K denote the fraction field of A and let $\overline{K}$ be an algebraic closure. Assume there is an H-stable subspace W of $V_A \otimes_A \overline{K}$. Since H is normal in G_F, gW is also H invariant for any $g \in G_F$. Take $n \geq 1$ such that $\sigma^n \in H$. Then, for each $g \in G_F$, gW is an eigenspace for $\rho_A(\sigma^n)$. Since the ratio of the eigenvalues of $\rho_A(\sigma)$ is not a root of unity, the eigenvalues of $\rho_A(\sigma^n)$ are distinct. Hence any $g \in G_F$ must permute the two one dimensional eigenspaces for $\rho_A(\sigma^n)$, so either W is G_F stable or there is an index two subgroup N of G_F such that W is N stable and G_F/N interchanges the two eigenspaces for $\rho_A(\sigma^n)$. In the first case ρ_A is reducible over $\overline{K}$, contradicting irreducibility of $V_{\mathbb{F}} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$, and in the second it is dihedral, contradicting our assumptions on V_A. □

Proposition 2.7.4. *Let A be a $\mathrm{CNL}_{\mathcal{O}}$-algebra domain with fraction field K of characteristic p and let V_A be a deformation of $V_{\mathbb{F}}$ with finite order determinant. Fix a basis of V_A and let Γ denote the image of G_F in $\mathrm{GL}_2(A)$ with respect to this basis. If $\Gamma \to \mathrm{GL}_2(\mathbb{F})$ has nontrivial kernel then $\Gamma \cap \mathrm{SL}_2(K)$ is Zariski dense in $\mathrm{SL}_{2/K}$.*

Proof. By 2.7.2 there is some $g \in \mathrm{im}\rho_A$ with infinite order eigenvalues. Then since $\det \rho_A$ is finite order, the ratio of the eigenvalues of g is not a root of unity. We see that V_A satisfies the assumptions of 2.7.3, so for any finite index subgroup H of G_F, $\rho_A|_H$ is absolutely irreducible.

Let $\Gamma^1 = \Gamma \cap \mathrm{SL}_2(K)$ and assume Γ^1 is not Zariski dense in $\mathrm{SL}_{2/K}$. Let $\overline{\Gamma^1}$ denote the Zariski closure of Γ^1 in $\mathrm{SL}_{2/K}$ and $(\overline{\Gamma^1})^0$ its connected component at the identity. Our assumption implies $\dim(\overline{\Gamma^1})^0 \leq 2$, and so $(\overline{\Gamma^1})^0$ is solvable. Then $(\overline{\Gamma^1})^0$ acts reducibly on $V_A \otimes_A \overline{K}$, where $\overline{K}$ is an algebraic closure of K. Since the determinant of V_A is finite, $\Gamma^1 \cap (\overline{\Gamma^1})^0$ is finite index in Γ^1, so there is a finite index subgroup H of G_F, such that $\rho_A|_H$ acts reducibly on $V_A \otimes_A \overline{K}$, a contradiction. □

We note that the assumption that $\Gamma \to \mathrm{GL}_2(\mathbb{F})$ has nontrivial kernel is satisfied in particular if $V_{\mathbb{F}}$ is dihedral and V_A is non-dihedral.

Corollary 2.7.5. *Assume $V_{\mathbb{F}}$ is L-dihedral. Let A be $\mathrm{CNL}_{\mathcal{O}}$-algebra domain of characteristic p and let V_A be a non-dihedral deformation of $V_{\mathbb{F}}$ with finite order determinant. Then there is $\sigma \in G_F \smallsetminus G_L$ such that $\rho_A(\sigma)$ has infinite order.*

Proof. Let K denote the fraction field of A and fix a basis of $V_A \otimes_A K$. Let Γ be the image of ρ_A in $\mathrm{GL}_2(K)$ with respect to this basis. If $g \in \Gamma$ is of finite order then its eigenvalues are roots of unity that lie in a degree two extension of K. There is an integer $N \geq 1$ such that any root of unity that lies in a quadratic extension of K has order $\leq N$. Then if $g \in \Gamma$ is finite order, g^N is unipotent. Hence the set of finite order elements of Γ lie in the Zariski closed subset of $GL_{2/K}$ defined by $g^{Np} = I$, which we denote by X. It is not hard to see that $\dim X \leq 2$.

Assume that $\rho_A(\sigma)$ is finite order whenever $\sigma \in G_F \smallsetminus G_L$. Let $H = \rho_A(G_L) \subseteq \Gamma$ and let $g = \rho(\sigma)$ for some $\sigma \in G_F \smallsetminus G_L$. By assumption every element of gH is finite order, so $gH \subseteq X$. Then $\Gamma = H \cup gH \subseteq gX \cup X$, so the Zariski closure of Γ has dimension ≤ 2. By 2.7.4, V_A must be dihedral. □

2.7.6

Let L/F be a quadratic extension such that $\overline{\rho}$ is L-dihedral. Write $\overline{\rho} = \mathrm{Ind}_{G_L}^{G_F}\overline{\chi}$, for $\overline{\chi} : G_L \to \mathbb{F}^\times$, and let $\tilde{\chi} : G_L \to \mathcal{O}^\times$ denote the Teichmuller lift of $\overline{\chi}$. Let L_S^{ab}/L denote the maximal abelian pro-p extension of L unramified outside S. Set $R_{L\text{-di}} = \mathcal{O}[[\mathrm{Gal}(L_S^{\mathrm{ab}}/L)]]$ and let $\Psi : G_L \to R_{L\text{-di}}^\times$ be the canonical character. We have an L-dihedral deformation of $V_{\mathbb{F}}$ to $R_{L\text{-di}}$ given by $\mathrm{Ind}_{G_L}^{G_F}\tilde{\chi}\Psi$. It is easy to see this deformation is universal for L-dihedral deformations and that there is a surjection $R_{F,S} \to R_{L\text{-di}}$, hence the locus of all L-dihedral points in $\mathrm{Spec}\, R_{F,S}$ is closed. As there are only finitely many quadratic extensions L/F such that $\overline{\rho}$ is L-dihedral, the locus of all dihedral points in $\mathrm{Spec}\, R_{F,S}$ is closed. The same is then true for any quotient of $R_{F,S}$.

The following two lemmas record some properties of dihedral deformations and will be used later to ensure certain deformations are non-dihedral.

Lemma 2.7.7. *Let L/F be a quadratic such that $\overline{\rho}$ is L-dihedral. Denote by L_S^- the maximal Galois subextension of L_S^{ab}/L such that the nontrivial element of* $\mathrm{Gal}(L/F)$ *acts on* $\mathrm{Gal}(L_S^-/L)$ *as -1. Let $R_{F,S} \to A$ be a surjection with kernel containing ϖ_E, and let V_A denote the corresponding deformation. If* $\det V_A$ *is the Teichmuller lift of* $\det V_{\mathbb{F}}$ *to $A^\times$ and V_A is L-dihedral, then*

$$\dim A \leq \mathrm{rk}_{\mathbb{Z}_p}\mathrm{Gal}(L_S^-/L).$$

Proof. Write $\overline{\rho} = \mathrm{Ind}_{G_L}^{G_F}\overline{\chi}$, for $\overline{\chi} : G_L \to \mathbb{F}^\times$. Let $\tilde{\chi} : G_L \to \mathcal{O}^\times$ denote the Teichmuller lift of $\overline{\chi}$. Set $R_{L\text{-di}}^- = \mathcal{O}[[\mathrm{Gal}(L_S^-/L)]]$ and let $\Psi^- : G_L \to (R_{L\text{-di}}^-)^\times$ be the canonical character. We have an L-dihedral deformation of $V_{\mathbb{F}}$ to $R_{L\text{-di}}^-$ given by $\mathrm{Ind}_{G_L}^{G_F}\tilde{\chi}\Psi$. This deformation is universal for L-dihedral deformations whose determinant is the Teichmuller lift of $\det V_{\mathbb{F}}$. We have a surjection $R_{F,S} \to R_{L\text{-di}}$. Our assumptions then apply that the surjection $R_{F,S} \to A$ factors through

$$R_{F,S} \to R^-_{L\text{-di}} \to \mathbb{F}[[\mathrm{Gal}(L^-_S/L)]]$$

from which the result follows. ☐

Lemma 2.7.8. *Let L/F be quadratic such that $\overline{\rho}$ is L-dihedral. Assume there is some $v|p$ in F that does not split in L. Let A be a $\mathrm{CNL}_{\mathcal{O}}$-algebra domain and V_A be a deformation of $V_{\mathbb{F}}$ to A. If V_A is dihedral and that there are characters $\chi_1, \chi_2 : G_v \to A^\times$ such that*

$$\mathrm{tr}\, \rho_A(\sigma) = \chi_1(\sigma) + \chi_2(\sigma)$$

for all $\sigma \in G_v$ (and any ρ_A in the deformation class of V_A), then χ_1/χ_2 had order at most two.

This is taken directly from Lemma 2.2.1 of [S3]. We include it here for ease of reference later.

Proof. Let w denote the unique place in L above v, and let $G_w = \mathrm{Gal}(\overline{F_v}/L_w)$. Note that G_w is index two in G_v, and we can find $\sigma \in G_v$ such that σ generates $\mathrm{Gal}(L/F)$. By the theory of pseudo-representations, we see that $(V_A|_{G_v})^{\mathrm{ss}} = \chi_1 \oplus \chi_2$.

By assumption, there is a character $\chi : G_L \to A^\times$ such that $V_A \cong \mathrm{Ind}_{G_L}^{G_F} \chi$. Note that $V_A|_{G_L} \cong \chi \oplus \chi'$, where χ' denotes the conjugate of χ by σ. Replacing χ by χ', if necessary, we can assume $\chi|_{G_w} = \chi_1|_{G_w}$ and $\chi'|_{G_w} = \chi_2|_{G_w}$. But since χ_i are characters of G_v, we have

$$\chi_1|_{G_w} = \chi_1^\sigma|_{G_w} = \chi'|_{G_w} = \chi_2|_{G_w},$$

and so χ_1/χ_2 factors through $\mathrm{Gal}(L_w/F_v)$. ☐

CHAPTER 3

Modular forms

In this chapter we recall Hida's theory of p-adic Hecke algebras in the case of a totally definite quaternion algebra over a totally real field. We recall some facts about the associated Galois representations and define the particular Hecke algebras and Hecke modules that will be used in the patching argument in §4.

In the first section we recall the definition of modular forms on a totally definite quaternion algebra over a totally real field and their connection with cuspidal automorphic representations of GL_2. This is all standard except that we allow the possibility of non-compact subgroups as in §7 of [KW]. We do this for the following reason. If we have a fixed central character for our space of modular forms and f is a eigenform with Iwahori level at v, then then there are two possibilities for the eigenvalue. When p is odd, the choice for the eigenvalue is uniquely determined modulo p, and so the local representation of G_v associated to f is uniquely determined by the reduction of f mod p. When $p = 2$ this is not the case and in order to ensure that our local deformation ring at v is a domain we only want to allow lifts of a fixed mod p eigenform that have a specified Iwahori level eigenvalue. This is done by allowing certain non-compact subgroups.

In the next section we recall the definition of the (finite level) nearly-ordinary Hecke algebras and the corresponding nearly-ordinary subspace of modular forms.

In the following section we construct the universal nearly-ordinary Hecke algebra and a certain avatar of p-adic Hida families. This is done following [H1] except that we deal only with the totally definite case, and so things become much simpler. We also allow certain non-compact subgroups, but this does not pose an issue, provided one assumes that the level is small enough at some fixed place to guarantee a standard neatness condition.

In the next section we recall properties of Galois representations associated to our quaternionic modular forms and the construction of large Galois representations with values in our universal Hecke algebra using pseudo-representations and a theorem of Nyssen and Rouquier.

The final section will deal with augmenting the level of our Hecke modules at auxiliary primes, necessary for the patching argument in §4. Normally, one augments the level at primes v that are congruent to 1 modulo p and such that the fixed residual representation is unramified at v with the Frobenius having distinct eigenvalues. This is to ensure that there are no lifts of the fixed residual representation that are Steinberg at v. As in [SW1] and [SW2], it will be necessary for us augment the level at places v for which the residual representation does not have distinct eigenvalues. Due to this we cannot ensure that there are no Steinberg-at-v lifts. However we do show that any such are annihilated by a particular element, cf. 3.5.3, which allows us to prove a control theorem for these auxiliary primes which is necessary for the patching argument in §4.

We now introduce some notation and assumptions that will be used throughout this chapter. We denote by $F \subset \overline{\mathbb{Q}}$ a totally real field, $\mathcal{O}_F$ its ring of integers, and $\mathbb{A}_F$ its ring of adeles. If S is a finite set of places of F, we let $\mathbb{A}_{F,S}$ denote $\prod_{v \in S} F_v$ and $\mathbb{A}_F^S$ denote $\prod'_{v \notin S} F_v$. If w is a rational place, we will write $\mathbb{A}_{F,w}$ and $\mathbb{A}_F^w$ instead of $\mathbb{A}_{F,\{v|w\}}$ and $\mathbb{A}_F^{\{v|w\}}$, in particular $\mathbb{A}_F^\infty$ denotes the ring of finite adeles.

Recall we have fixed algebraic closures $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, as well as embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let I denote the set of embeddings $F \hookrightarrow \overline{\mathbb{Q}}$. Via our fixed embeddings of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$, we view I as the set of embeddings of F into $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$, respectively. Let $E/\mathbb{Q}_p$ be a finite extension containing all embeddings of $F \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\mathcal{O}$ denote the ring of integers of E. Given an element $z \in F \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{v|p} F_v$, and $\mathbf{k} \in \mathbb{Z}^I$, we let $z^{\mathbf{k}}$ denote $\prod_{\tau \in I} \tau(z)^{k_\tau} \in E$. For each finite place v of F we let $\mathfrak{m}_v$ denote the maximal ideal of $\mathcal{O}_{F_v}$ and k_v the residue field.

We will let $\mathcal{U}_p = (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \cong \prod_{v|p} \mathcal{O}_{F_v}^\times$. For any $a \geq 1$ we let

$$\mathcal{U}_p^a = \{(x_v)_{v|p} \in \mathcal{U}_p : x_v \equiv 1 \bmod \mathfrak{m}_v^a \text{ for each } v|p\}.$$

For $a \geq 0$ we set $\Lambda(\mathcal{U}_p^a) = \mathcal{O}[[\mathcal{U}_p^a]]$. Note that $\Lambda(\mathcal{U}_p^a)$ has dimension $1 + [F : \mathbb{Q}]$, is local if $a \geq 1$, and is isomorphic to a power series over $\mathcal{O}$ if a is sufficiently large.

For v a finite place of F, and $a \geq 0$, we let $\mathrm{Iw}(v^a)$, respectively $\mathrm{Iw}_1(v^a)$, denote the subgroup of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ that are upper triangular, respectively upper triangular unipotent, modulo $\mathfrak{m}_v^a$. Given integers $b \geq a \geq 0$, wet let $\mathrm{Iw}(v^{a,b}) = \mathrm{Iw}_1(v^a) \cap \mathrm{Iw}(v^b)$.

For a topological $\mathcal{O}$-module M we denote by $M^\vee$ the Pontryagin dual of M, i.e. $M^\vee = \mathrm{Hom}_{\mathcal{O}}(M, E/\mathcal{O})$. This assignment is functorial and $f \mapsto f^\vee$ gives an isomorphism $\mathrm{End}_{\mathcal{O}}(M) \cong \mathrm{End}_{\mathcal{O}}(M^\vee)$. In particular if a commutative ring R acts on M, then it acts naturally on $M^\vee$ by $(r\phi)(m) = \phi(rm)$. Pontryagin duality is not an exact functor, but if we restrict to the subcategory of locally compact Hausdorff $\mathcal{O}$-modules and strict morphisms, i.e. i.e morphisms $f : M \to M'$ such that the induced map $M/\ker(f) \to \mathrm{im}(M)$ is an isomorphism of topological $\mathcal{O}$-modules, then if

$$M_1 \xrightarrow{f} M_1 \xrightarrow{g} M_3$$

is exact and $g(M_2)$ is closed in M_3,

$$M_3^\vee \xrightarrow{g^\vee} M_2^\vee \xrightarrow{f^\vee} M_3^\vee$$

is also exact, cf. Proposition 0.20 of [M4]. We will always be in such a situation in what follows so we will occasionally refer to the "exactness of Pontryagin duality" without further comment.

3.1 Quaternionic cusp forms

Let D denote a quaternion algebra with centre F ramified at all infinite places and split at all places above p. Denote by Σ the set of finite places at which D ramifies. Let ν_D denote the reduced norm of D. Fix a maximal order $\mathcal{O}_D$ of D and, for each finite place

v at which D is split, an isomorphism $\mathcal{O}_D \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v} \cong M_2(\mathcal{O}_{F_v})$ of $\mathcal{O}_{F_v}$-algebras. This then determines an isomorphism $D_v \cong \mathrm{GL}_2(F_v)$ sending $(\mathcal{O}_D)_v$ to $\mathrm{GL}_2(\mathcal{O}_{F_v})$. Using this we can identify $(D \otimes_F \mathbb{A}_F^{\infty,\Sigma})^\times$ with $\mathrm{GL}_2(\mathbb{A}_F^{\infty,\Sigma})$. We also fix a locally algebraic character $\psi : F^\times \backslash (\mathbb{A}_F^\infty)^\times \to \mathcal{O}^\times$, and for each $v \in \Sigma$, unramified characters $\gamma_v : F_v \to \mathcal{O}^\times$ such that $\gamma_v^2 = \psi|_{G_{F_v}}$.

3.1.1

Let A be a topological $\mathcal{O}$-module. For each $\tau \in I$ we have an isomorphism $\mathcal{O}_D \otimes_{\mathcal{O}_F,\tau} \mathcal{O} \cong M_2(\mathcal{O})$. Via this isomorphism, given $\tau \in I$, $k_\tau \geq 2$ and $w_\tau \in \mathbb{Z}$, we can view

$$\mathrm{Sym}^{k_\tau-2} A^2 \otimes_A \det{}^{w_\tau} A^2$$

as an $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathrm{M}_{2\times 2}(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$-module which we denote by $W_{k_\tau,w_\tau}(A)$. Given a pair $\kappa = (\mathbf{k}, \mathbf{w}) \in \mathbb{Z}^I \times \mathbb{Z}^I$ we set $W_\kappa(A) = \otimes_{\tau\in I} W_{k_\tau,w_\tau}(A)$. Note $W_\kappa(A) \cong W_\kappa(\mathcal{O}) \otimes_{\mathcal{O}} A$. Concretely we can view $W_\kappa(\mathcal{O})$ as the space of $\mathcal{O}$-linear combinations on the monomials

$$\prod_{\tau\in I} X_\tau^{k_\tau-2-j_\tau} Y_\tau^{j_\tau}$$

for $0 \leq j_\tau \leq k_\tau - 2$, where the action of $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathrm{M}_{2\times 2}(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ is given by

$$\begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix} \prod_{\tau\in I} X_\tau^{k_\tau-2-j_\tau} Y_\tau^{j_\tau} = (a_p d_p - b_p c_p)^{\mathbf{w}} \prod_{\tau\in I} (\tau(a_p)X_\tau + \tau(c_p)Y_\tau)^{k_\tau-2-j_\tau} (\tau(b_p)X_\tau + \tau(d_p)Y_\tau)^{j_\tau}.$$

We call a pair $\kappa = (\mathbf{k}, \mathbf{w}) \in \mathbb{Z}^I \times \mathbb{Z}^I$ an *algebraic weight* if $k_\tau \geq 2$ for all τ and $k_\tau + 2w_\tau$ is independent of $\tau \in I$.

Following [KW], we will consider some non-compact open subgroups $U = \prod_v U_v$ of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ in order to ensure that certain local deformation rings are domains. This assumption is only necessary when $p = 2$, which is our principal focus, and we make the assumption for all p simply to avoid cases and to treat all primes at once. To facilitate this, we extend the action of $(\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ on $W_\kappa(A)$ to an action of

$$(D \otimes_F \mathbb{A}_F^{\infty,p})^\times \times (\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \cong \prod_{v\nmid p} D_v \times \prod_{v|p} (\mathcal{O}_D)_v^\times$$

by letting D_v act via $\gamma_v^{-1} \circ \nu_D$ if $v \in \Sigma$ and trivially if $v \notin \Sigma \cup \{w|p\}$.

Let $\Sigma' \subseteq \Sigma$. We will call an open subgroup $U = \prod_v U_v$ of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ a $(\Sigma' \subseteq \Sigma)$-*open subgroup* if

- $U_v \subseteq \mathrm{GL}_2(\mathcal{O}_{F_v})$ for $v \notin \Sigma$,
- $U_v = D_v$ for $v \in \Sigma'$, and
- $U_v = (\mathcal{O}_D)_v^\times$ for $v \in \Sigma \setminus \Sigma'$.

Let κ be an algebraic weight and let U be a $(\Sigma' \subset \Sigma)$-open subgroup. We let $S_{\kappa,\psi}(U, A)$ denote the space of functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times \longrightarrow W_\kappa(A)$$

such that $f(xu) = u^{-1} f(g)$ for all $u \in U$ and $f(zx) = \psi(z) f(x)$ for all $z \in (\mathbb{A}_F^\infty)^\times$. For this space to be nonzero there must be a submodule of A on which $\psi(z) = z_p^{2-\mathbf{k}-\mathbf{w}}$ for all $z \in U \cap (\mathbb{A}_F^\infty)^\times$. In the case that $\kappa = ((k, \dots, k), (0, \dots, 0))$ for some $k \geq 2$, we will also write $S_{k,\psi}(U, A)$ for $S_{\kappa,\psi}(U, A)$. Note that for $v \in \Sigma'$ we have $f(xg_v) = \gamma_v \circ \nu_D(g_v) f(x)$ for all $g_v \in D_v$.

Choose $t_1, \dots, t_n \in (D \otimes_F \mathbb{A}_F^\infty)^\times$ such that $(D \otimes_F \mathbb{A}_F^\infty)^\times = \sqcup_{i=1}^n D^\times t_i U (\mathbb{A}_F^\infty)^\times$. Then the map $f \mapsto (f(t_1), \dots, f(t_n))$ defines an isomorphism of $\mathcal{O}$-modules

$$S_{\kappa,\psi}(U, A) \xrightarrow{\sim} \bigoplus_{i=1}^n W_\kappa(A)^{(U(\mathbb{A}_F^\infty)^\times \cap t_i^{-1} D^\times t_i)/F^\times}$$

which is also an isomorphism of A-modules if A is an $\mathcal{O}$-algebra. If if for any $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$ we have

$$(U(\mathbb{A}_F^\infty)^\times \cap t^{-1} D^\times t)/F^\times = 1,$$

then $A \mapsto S_{\kappa,\psi}(U, A)$ is an exact functor and $S_{\kappa,\psi}(U, A) \cong S_{\kappa,\psi}(U, \mathcal{O}) \otimes_{\mathcal{O}} A$. Note, however, that if A is $\mathcal{O}$-flat, in particular if $A = E$, we still have $S_{\kappa,\psi}(U, A) \cong S_{\kappa,\psi}(U, \mathcal{O}) \otimes_{\mathcal{O}} A$ without any assumption on U.

3.1.2

We record a few lemmas regarding the structure of the isotropy groups $(U(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times$, for $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$. Let D^1 denote the subgroup of $D^\times$ of elements of reduced norm 1, and let $U' = U \cap \mathcal{O}_D^\times$. Note that U' is open compact. Set $V = \prod_{w<\infty} \mathcal{O}_{F_w}^\times$. Then the reduced norm gives an exact sequence

$$0 \longrightarrow (UV \cap t^{-1}D^1 t)/\{\pm 1\} \longrightarrow (U(\mathbb{A}_F^\infty)^\times \cap t^{-1}Dt)/F^\times \longrightarrow (((A_F^\infty)^\times)^2 V \cap F^\times)/(F^\times)^2, \quad (3.1)$$

and there is an exact sequence

$$0 \longrightarrow \mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2 \longrightarrow (((A_F^\infty)^\times)^2 V \cap F^\times)/(F^\times)^2 \longrightarrow \mathrm{Cl}[2] \longrightarrow 0 \quad (3.2)$$

with Cl the class group of $\mathcal{O}_F$. The first two of the following three lemmas are taken directly from §7 of [KW].

Lemma 3.1.3. *Let U be a $(\Sigma' \subseteq \Sigma)$-open subgroup of $(D \otimes_F \mathbb{A}_F^\infty)^\times$. Let w be a finite place not above p at which D is split. Let N_w be the cardinality of $\mathrm{GL}_2(k_w)$.*

The exponent of a Sylow p-subgroup of $(U(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times$ divides $4N_w$.

Proof. First assume that U is compact. Then $UV \cap t^{-1}D^1 t$ is discrete and compact, hence finite, and injects into $U_w V_w$. Note that any finite subgroup of $U_w V_w$ of exponent a power of p injects into $\mathrm{GL}_2(k_w)$ under reduction modulo $\mathfrak{m}_w$. By (3.2), $(((A_F^\infty)^\times)^2 V \cap F^\times)/(F^\times)^2$ has exponent 2, and so any p-Sylow subgroup of $(U(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times$ has exponent dividing $2N_w$.

Now assume that U is not compact. Let U' denote the maximal compact open subgroup of U. Since $U(\mathbb{A}_F^\infty)^\times/U'(\mathbb{A}_F^\infty)^\times$ is a finite group of exponent 2, the same holds for the quotient of $(U(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times$ by $(U'(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times$. From what we know in the compact case, we get that the exponent of the any p-Sylow subgroup of $(U(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times$ divides $4N_w$. □

Let v be a finite place of F at which D is split. If $U_v \supseteq \mathrm{Iw}(v)$, then any character $\chi_v : k_v^\times \to \mathcal{O}^\times$ can be viewed as a character of U by

$$\chi_v \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi_v(a_v d_v^{-1} \bmod \mathfrak{m}_v).$$

Since χ_v is trivial on $U \cap (\mathbb{A}_F^\infty)^\times$, we can extend χ_v to a character on $U(\mathbb{A}_F^\infty)^\times$ by letting it be trivial on $(\mathbb{A}_F^\infty)^\times$.

Lemma 3.1.4. *Let Q be some fixed set of finite places of F at which D is split. Fix another finite place $w \notin Q$ at which D is split. Let U be a $(\Sigma' \subseteq \Sigma)$-open subgroup of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ such that $U_v \supseteq \mathrm{Iw}(v)$ for every $v \in Q$. Assume that for every $v \in Q$ the order of the p-subgroup of $k_v^\times$ is divisible by the p-part of $2p(4N_w)$, with N_w the cardinality of $\mathrm{GL}_2(k_w)$.*

There is a character $\chi = \prod_{v \in Q} \chi_v : \prod_{v \in Q} k_v^\times \to \mathcal{O}^\times$ such that,

1. *each χ_v is non-trivial of order a power of p and ≥ 4 if $p = 2$;*

2. *viewing χ as a character of $U(\mathbb{A}_F^\infty)^\times$ as above, χ annihilates $(U(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times$ for any $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$.*

Proof. Our hypothesis implies that there is a character $\chi' = \prod_{v \in Q} \chi'_v : \prod_{v \in Q} k_v^\times \to \mathcal{O}^\times$ of order a power of p, with each χ'_v of order divisible by the p-part of $2p(4N_w)$. We then set $\chi_v = (\chi'_v)^{4N_w}$, and $\chi = \prod_{v \in Q} \chi_v$. Note that the order of χ is divisible by the p-part of $2p$. Since the exponent of a p-Sylow subgroup of $(U(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times$ divides the p-part of $4N_w$ by 3.1.3, we see that χ is trivial on $(U(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times$ for any $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$. □

Lemma 3.1.5. *Let v be a finite place of F at which D is split and let U be a $(\Sigma' \subseteq \Sigma)$-open subgroup of $(D \otimes_F \mathbb{A}_F^\infty)^\times$. There is some $n \geq 1$ such that if $U_v \subseteq \mathrm{Iw}_1(v^n)$ then $(U(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times = 1$ for any $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$.*

Proof. Since $D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times / U(\mathbb{A}_F^\infty)^\times$ is finite, it suffices show the existence of such an $n \geq 1$ for fixed $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$. Let μ denote the set of all roots of unity $\zeta \in \overline{\mathbb{Q}}$ such that

$[F(\zeta):F]\leq 2$. Take $k\geq 1$ sufficiently large such that for any $\zeta\in\mu$ with $\zeta\neq\pm 1$, we have $\zeta+\zeta^{-1}\not\equiv\pm 2 \mod \mathfrak{m}_v^k$. We take n such that $n\geq 2k$ and such that $2\notin\mathfrak{m}_v^n$.

Let U' denote the maximal compact subgroup of U, D^1 denote the subgroup of $D^\times$ of elements of reduced norm 1, and set $V=\prod_{w<\infty}\mathcal{O}_{F_w}^\times$. First consider the subgroup $(U'V\cap t^{-1}D^1t)/\{\pm 1\}$ of $(U'(\mathbb{A}_F^\infty)^\times\cap t^{-1}D^\times t)/F^\times$. Note $U'V\cap t^{-1}D^1t$ is finite since it is compact and discrete. Take $uz\in U'V\cap t^{-1}D^1t$ with $u\in U'$ and $z\in V$. Since $uz\in t^{-1}D^1t$ has finite order, $\mathrm{tr}\,(uz)=\zeta+\zeta^{-1}$ for some $\zeta\in\mu$. Now our assumption on U_v implies $\nu_D(u_v)\in 1+\mathfrak{m}_v^n$, and so $1=\nu_D(u)z^2$ implies that $z_v\in\pm 1+\mathfrak{m}_v^k$ by choice of n. Then

$$\zeta+\zeta^{-1}=\mathrm{tr}\,(uz)=\mathrm{tr}\,(u_vz_v)=z_v\mathrm{tr}\,(u_v)\in\pm 2+\mathfrak{m}_v^k,$$

which implies $\zeta=\pm 1$ by choice of k, and thus that $(U'V\cap t^{-1}D^1t)/\{\pm 1\}$ is trivial.

Since $(U'V\cap t^{-1}D^1t)/\{\pm 1\}$ is trivial, the reduced norm induces and injection

$$(U'(\mathbb{A}_F^\infty)^\times\cap t^{-1}D^\times t)/F^\times\to (V((\mathbb{A}_F^\infty)^\times)^2\cap F^\times)/(F^\times)^2,$$

and by (3.2), $(U'(\mathbb{A}_F^\infty)^\times\cap t^{-1}D^\times t)/F^\times$ is a finite 2-group. Since $U(\mathbb{A}_F^\infty)^\times/U'(\mathbb{A}_F^\infty)^\times$ is a finite group of exponent 2, $(U(\mathbb{A}_F^\infty)^\times\cap t^{-1}D^\times t)/F^\times$ is also a finite 2-group. It thus suffices to show that if $g\in U(\mathbb{A}_F^\infty)^\times\cap t^{-1}D^\times t$ is such that $g^2\in F^\times$, then $g\in F^\times$. If $g^2\in F^\times$ and $g\notin F^\times$, then $\mathrm{tr}\,g=0$. But $U(\mathbb{A}_F^\infty)^\times\cap t^{-1}D^\times t$ injects into $U_vF_v^\times$, and since $2\notin\mathfrak{m}_v^n$, elements of $U_vF_v^\times$ have nonzero trace. □

3.1.6

Set $U'=U\cap\mathcal{O}_D^\times$. For $g\in(D\otimes_F\mathbb{A}_F^{\infty,\Sigma})$, if $U'gU'=\sqcup_i g_iU'$ we have $UgU=\sqcup_i g_iU$. If $g\in(D\otimes_F\mathbb{A}_F^{\infty,\Sigma,p})^\times\times(\mathcal{O}_D\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times$ there is a double coset operator

$$[UgU]:S_{\kappa,\psi}(U,A)\longrightarrow S_{\kappa,\psi}(V,A)$$

given by $([UgU]f)(x)=\sum_i g_if(xg_i)$. If A is an E vector space, then this double coset operator is defined for any $g\in(D\otimes_F\mathbb{A}_F^{\infty,\Sigma})^\times$.

If $V \subseteq U$ is another $(\Sigma' \subset \Sigma)$-open subgroup with V normal in U then the group $\Delta = U(\mathbb{A}_F^\infty)^\times / V(\mathbb{A}_F^\infty)^\times$ acts on $S_{\kappa,\psi}(V, A)$ by $(\langle\delta\rangle f)(x) = u_\delta f(x u_\delta)$, where u_δ is any lift to U of $\delta \in U(\mathbb{A}_F^\infty)^\times / V(\mathbb{A}_F^\infty)^\times$. We will have several occasions to use the following lemmas.

Lemma 3.1.7. *Let U and V be $(\Sigma' \subseteq \Sigma)$-open subgroups of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ with V normal in U. If for all $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$ we have $(U(\mathbb{A}_F^\infty)^\times \cap t^{-1} D^\times t)/F^\times = 1$, then $S_{\kappa,\psi}(V, \mathcal{O})$ is a free $\mathcal{O}[\Delta]$-module.*

Proof. Choose $\{t_1, \ldots, t_n\}_I \subset (D \otimes_F \mathbb{A}_F^\infty)^\times$ such that $(D \otimes_F \mathbb{A}_F^\infty)^\times = \sqcup_{i=1}^n D^\times t_i U(\mathbb{A}_F^\infty)^\times$. Our assumption on U implies we have an $\mathcal{O}$-algebra isomorphism

$$\begin{aligned} S_{\kappa,\psi}(U, \mathcal{O}) &\xrightarrow{\sim} \bigoplus_{i \in I} W_\kappa \\ f &\longmapsto (f(t_1), \ldots, f(t_n)). \end{aligned}$$

Choosing a representative $u_\delta \in U$ for each $\delta \in \Delta$, we have $(D \otimes_F \mathbb{A}_F^\infty)^\times = \sqcup_{i=1}^n \sqcup_{\delta \in \Delta} D^\times t_i u_\delta V(\mathbb{A}_F^\infty)^\times$ and an isomorphism of $\mathcal{O}[\Delta]$-modules

$$\begin{aligned} S_{\kappa,\psi}(V, \mathcal{O}) &\xrightarrow{\sim} \bigoplus_{i=1}^n W_\kappa \otimes_{\mathcal{O}} \mathcal{O}[\Delta] \\ f &\longmapsto \big(\sum_{\delta \in \Delta} u_\delta f(t_i) \otimes \delta^{-1}\big)_{1 \leq i \leq n}. \end{aligned}$$

From which it follows that $S_{\kappa,\psi}(V, \mathcal{O})$ is a free $\mathcal{O}[\Delta]$-module. □

Lemma 3.1.8. *Let U and V be $(\Sigma' \subseteq \Sigma)$-open subgroups of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ with V normal in U and $\Delta = U(\mathbb{A}_F^\infty)^\times / V(\mathbb{A}_F^\infty)^\times$ abelian. Assume that for all $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$ we have $(U(\mathbb{A}_F^\infty)^\times \cap t^{-1} D^\times t)/F^\times = 1$.*

$S_{\kappa,\psi}(V, E/\mathcal{O})^\vee$ is a free $\mathcal{O}[\Delta]$-module and for $\iota : S_{\kappa,\psi}(U, E/\mathcal{O}) \to S_{\kappa,\psi}(V, E/\mathcal{O})$ the natural inclusion, $\iota^\vee$ defines an isomorphism from the Δ-coinvariants of $S_{\kappa,\psi}(V, E/\mathcal{O})^\vee$ to $S_{\kappa,\psi}(U, E/\mathcal{O})^\vee$.

Proof. Let $\mathfrak{a}_\Delta$ denote the augmentation ideal of $\mathcal{O}[\Delta]$. We have

$$\begin{aligned} S_{\kappa,\psi}(V, E/\mathcal{O})^\vee/\mathfrak{a}_\Delta &\cong \mathrm{Hom}_{\mathcal{O}}(S_{\kappa,\psi}(V, E/\mathcal{O})[\mathfrak{a}_\Delta], E/\mathcal{O}) \\ &\cong \mathrm{Hom}_{\mathcal{O}}(S_{\kappa,\psi}(U, E/\mathcal{O}), E/\mathcal{O}) \\ &= S_{\kappa,\psi}(U, E/\mathcal{O})^\vee, \end{aligned}$$

which is the second part of the lemma.

Give $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}[\Delta], \mathcal{O})$ the $\mathcal{O}[\Delta]$-module structure $(\delta\lambda)(r) = \lambda(\delta r)$. There is an isomorphism of $\mathcal{O}[\Delta]$-modules $\mathcal{O}[\Delta] \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}[\Delta], \mathcal{O})$ defined by sending $\delta \in \Delta$ to $(\delta^{-1})^*$, where $(\delta^{-1})^*$ takes value 1 on δ^{-1} and 0 on every other $\gamma \in \Delta$. Hence 3.1.7 implies that $\mathrm{Hom}_{\mathcal{O}}(S_{\kappa,\psi}(V, \mathcal{O}), \mathcal{O})$ is a free $\mathcal{O}[\Delta]$-module. By our assumption on U, $S_{\kappa,\psi}(V, E/\mathcal{O}) \cong S_{\kappa,\psi}(V, \mathcal{O}) \otimes_{\mathcal{O}} E/\mathcal{O}$. Moreover, since $S_{\kappa,\psi}(V, \mathcal{O})$ is free over $\mathcal{O}$, we have an $\mathcal{O}$-module isomorphism

$$\mathrm{Hom}_{\mathcal{O}}(S_{\kappa,\psi}(V, \mathcal{O}) \otimes_{\mathcal{O}} E/\mathcal{O}, E/\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(S_{\kappa,\psi}(V, \mathcal{O}), \mathcal{O})$$

given by $\phi \mapsto \phi \otimes 1$. This map is $\mathcal{O}[\Delta]$ equivariant and so the freeness of $\mathrm{Hom}_{\mathcal{O}}(S_{\kappa,\psi}(V, \mathcal{O}), \mathcal{O})$ over $\mathcal{O}[\Delta]$ implies that of $S_{\kappa,\psi}(V, E/\mathcal{O})^\vee$. □

3.1.9

We finish this subsection by recalling the connection between the $\mathcal{O}$-modules $S_{\kappa,\psi}(U, \mathcal{O})$ and cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$. We say an irreducible cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ is *regular algebraic* if there is an algebraic weight $\kappa = (\mathbf{k}, \mathbf{w})$ such that for each $\tau \in I$, letting v denote the corresponding infinite place coming from our fixed embedding $\overline{\mathbb{Q}} \to \mathbb{C}$, π_v is the discrete series representation with lowest weight $k_\tau - 1$ and central character $z \mapsto \mathrm{sgn}(z)^{k_\tau}|z|^{2-k_\tau-2w_\tau}$. Recall that if V is an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$ and $g \in \mathrm{GL}_2(\mathbb{A}_F^\infty)$, there is a double coset operator $[VgV]$ on π^V given by

$$[VgV]w = \sum_i g_i w$$

if $VgV = \sqcup_i g_i V$.

Set $U' = U \cap \mathcal{O}_D^\times$. Assume that $\psi(z) = z_p^{2-\mathbf{k}-2\mathbf{w}}$ for all $z \in U' \cap (\mathbb{A}_F^\infty)^\times$. Fix an isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ that extends our fixed embeddings $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}} \to \mathbb{C}$. Define a character $\psi_\mathbb{C} : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ by $\psi_\mathbb{C}(z) = \psi(z) z_p^{\mathbf{k}+2\mathbf{w}-2} z_\infty^{2-\mathbf{k}-2\mathbf{w}}$. We also define characters $\gamma_{v,\mathbb{C}} : F_v^\times \to \mathbb{C}^\times$ for each $v \in \Sigma$ simply via the isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$. Note that we can view

$$W_\kappa(\mathbb{C}) = \otimes_{\tau \in I} \mathrm{Sym}^{k_\tau - 2} \mathbb{C}^2 \otimes \det{}^{w_\tau} \mathbb{C}^2$$

as a representation of $D_\infty^\times = (D \otimes_F \mathbb{R})^\times$.

Let $\mathfrak{Aut}^D_{\kappa,\psi_\mathbb{C},\Sigma'}$ denote the set of all irreducible automorphic representations π of D with central character $\psi_\mathbb{C}$ such that $\pi_\infty \cong W_\kappa(\mathbb{C})^*$ and $\pi_v \cong \gamma_v \circ \nu_D$ for each $v \in \Sigma'$. For $\pi \in \mathfrak{Aut}^D_{\kappa,\psi_\mathbb{C},\Sigma'}$ and $g \in (D \otimes_\mathbb{F} \mathbb{A}_F^{\infty,\Sigma})^\times$, there is a double coset operator $[U'gU']$ on $\pi^{U'}$ defined in the same way as the GL_2-case. Let $C^\infty_{\psi_\mathbb{C},\Sigma'}(D^\times \backslash (D \otimes_F \mathbb{A}_F)^\times / U', \mathbb{C})$ denote the space of smooth functions

$$\phi : D^\times \backslash (D \otimes_F \mathbb{A}_F)^\times / U' \longrightarrow \mathbb{C}$$

such that $\phi(zx) = \psi_\mathbb{C}(z)\phi(x)$ for all $z \in \mathbb{A}_F^\infty$ and $\phi(xg_v) = \gamma_{v,\mathbb{C}} \circ \nu_D(g_v)\phi(x)$ for all $g_v \in D_v^\times$ with $v \in \Sigma'$. Then

$$\bigoplus_{\mathfrak{Aut}^D_{\kappa,\psi_\infty}} \pi^{U'} = \mathrm{Hom}_{D_\infty^\times}(W_\kappa(\mathbb{C})^*, C^\infty_{\psi_\mathbb{C},\Sigma'}(D^\times \backslash (D \otimes_F \mathbb{A}_F)^\times / U', \mathbb{C}))$$

and the map $f \mapsto (\lambda \mapsto (x \mapsto \lambda(x_\infty^{-1} x_p f(x))))$ defines an isomorphism

$$\mathfrak{A} : S_{\kappa,\psi}(U, E) \otimes_E \mathbb{C} \xrightarrow{\sim} \mathrm{Hom}_{D_\infty^\times}(W_\kappa(\mathbb{C})^*, C^\infty_{\psi_\mathbb{C}}(D^\times \backslash (D \otimes_F \mathbb{A}_F)^\times / U'))$$

such that $\mathfrak{A} \circ [UgU] = [U'gU'] \circ \mathfrak{A}$ for all $g \in (D \otimes_F \mathbb{A}_F^{\infty,\Sigma})^\times$. Applying the theorem of Jacquet-Langlands and Shimizu to $\mathfrak{Aut}^D_{\kappa,\psi_\mathbb{C},\Sigma'}$ we obtain the following.

Proposition 3.1.10. *Define an open compact subgroup V of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$ by $V_v = U_v$ for $v \notin \Sigma$ and $V_v = \mathrm{Iw}(v)$ for $v \in \Sigma$. Let $\Pi^\Sigma_{\kappa,\psi_\mathbb{C},\Sigma'}$ denote the set of all irreducible cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_F)$ such that*

- *π has central character $\psi_\mathbb{C}$,*

- *π is regular algebraic of weight κ*

- π_v *is square integrable for each* $v \in \Sigma$,

- $\pi_v \cong (\gamma_{v,\mathbb{C}} \circ \det) \otimes \mathrm{St}$ *for each* $v \in \Sigma'$, *where* St *denotes the Steinberg representation.*

There is a surjection of $\mathbb{C}$*-vector spaces*

$$\mathrm{JL} : S_{\kappa,\psi}(U, E) \otimes_E \mathbb{C} \longrightarrow \bigoplus_{\Pi^{\Sigma}_{\kappa,\psi_{\mathbb{C}},\Sigma'}} (\pi^\infty)^V$$

such that

1. *for* $g \in (D \otimes_F \mathbb{A}_F^{\infty,\Sigma})^\times \cong \mathrm{GL}_2(\mathbb{A}_F^{\infty,\Sigma})$, *we have*

$$\mathrm{JL} \circ [UgU] = [VgV] \circ \mathrm{JL};$$

2. JL *is an isomorphism unless* $\kappa = ((2, \ldots, 2), \mathbf{w})$ *in which case the kernel of* JL *consists of the functions that factor through the reduced norm.*

3.2 Nearly ordinary Hecke algebras

Keep the assumptions and notation of the previous subsection. We further assume that for each $v|p$, there is $n \geq 0$ such that $U_v \supseteq \mathrm{Iw}_1(v^n)$

3.2.1

Recall that Σ is the set of finite places at which D ramifies. Let $S = \Sigma \cup \{v|p\} \cup \{v : U_v \neq (\mathcal{O}_D)_v^\times\}$. Note that for any $v \notin S$ and uniformizer ϖ_v at v, the double cosets

$$U \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} U \text{ and } U \begin{pmatrix} \varpi_v & \\ & \varpi_v \end{pmatrix} U$$

do not depend on the choice of ϖ_v. We define operators T_v and S_v on $S_{\kappa,\psi}(U, A)$ by setting

$$T_v f = \left[U \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} U \right] f \text{ and } S_v f = \left[U \begin{pmatrix} \varpi_v & \\ & \varpi_v \end{pmatrix} U \right] f,$$

Note that S_v is simply multiplication by $\psi(\varpi_v)$. If $V \subseteq U$ is another $(\Sigma' \subset \Sigma)$-open subgroup, the natural inclusion $S_{\kappa,\psi}(U,A) \to S_{\kappa,\psi}(V,A)$ is equivariant for the T_v and S_v such that $V_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$.

For each $v|p$ we fix, once and for all, an element $\varpi_v \in F$ such that ϖ_v is a uniformizer for F_v and lies in $\mathcal{O}_{F_w}^\times$ for all $w|p$ with $w \neq v$. We choose our uniformizers for F_v in this way because, following Hida, we modify the usual Hecke operators at places above p in order to define the nearly ordinary subspace of $S_{\kappa,\psi}(U,\mathcal{O})$. This modification will involve a multiplication by a power of ϖ_v, and having ϖ_v belong to F allows us to compare the nearly ordinary subspace of $S_{\kappa,\psi}(U,\mathcal{O})$ with nearly ordinary cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$.

Note first that for any $g \in \mathrm{M}_{2\times 2}(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ with $\det(g) \neq 0$ the endomorphism $\tau(\det(g))^{-w_\tau} g$ of $W_{k_\tau,w_\tau}(E)$ stabilizes the $\mathcal{O}$-lattice $W_{k_\tau,w_\tau}(\mathcal{O})$, hence defines an endomorphism of $W_{k_\tau,w_\tau}(A)$ for any $\mathcal{O}$-module A. For each $v|p$ we then define the operator T_{ϖ_v} on $S_{\kappa,\psi}(U,A)$ by

$$T_{\varpi_v} f = \varpi_v^{-\mathbf{w}} \left[U \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} U \right] f,$$

Note that T_{ϖ_v} depends on the choice of ϖ_v as we are not assuming $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$. If $V \subset U$ is another $(\Sigma' \subseteq \Sigma)$-open subgroup and $\mathrm{Iw}_1(v^n) \subseteq V_v \subseteq U_v \subseteq \mathrm{Iw}(v)$, for some $n \geq 1$, the natural inclusion $S_{\kappa,\psi}(U,A) \to S_{\kappa,\psi}(V,A)$ is a equivariant for T_{ϖ_v}.

For $y \in \mathcal{U}_p = (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$, we define two related operators. First we define $\langle y \rangle$ by

$$\langle y \rangle f = \left[U \begin{pmatrix} y & \\ & 1 \end{pmatrix} U \right] f.$$

We then define a normalized version $\langle y \rangle^{\mathrm{no}}$ by

$$\langle y \rangle^{\mathrm{no}} = y^{-\mathbf{w}} \langle y \rangle.$$

The point of introducing the normalized version is that later we will define an isomorphism between certain spaces of modular forms of different weights, and this isomorphism will not be $\langle y \rangle$-equivariant, but will be $\langle y \rangle^{\mathrm{no}}$-equivariant. Note that an $\mathcal{O}$-subalgebra of

$\mathrm{End}_{\mathcal{O}}(S_{\kappa,\psi}(U,\mathcal{O}))$ containing $\langle y \rangle$ also contains $\langle y \rangle^{\mathrm{no}}$ and vice-versa. If $V \subseteq U$ is another $(\Sigma' \subseteq \Sigma)$-open subgroup then the natural inclusion $S_{\kappa,\psi}(U,A) \to S_{\kappa,\psi}(V,A)$ respects the $\langle y \rangle$ and $\langle y \rangle^{\mathrm{no}}$-actions on each space.

We let $\mathbf{T}_{\kappa,\psi}(U,\mathcal{O})$ denote the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(S_{\kappa,\psi}(U,A))$ generated by T_v for each $v \notin S$, T_{ϖ_v} for each $v|p$, and $\langle y \rangle$ (equivalently $\langle y \rangle^{\mathrm{no}}$) for each $y \in \mathcal{U}_p$. If A is a commutative $\mathcal{O}$-algebra we denote by $\mathbf{T}_{\kappa,\psi}(U,A)$ the A-subalgebra of $\mathrm{End}_A(S_{\kappa,\psi}(U,A))$ generated by the aforementioned Hecke operators. If A is a finite $\mathcal{O}$-module, or if $A = E/\mathcal{O}$, then $\mathbf{T}_{\kappa,\psi}(U,\mathcal{O})$ is a finite commutative $\mathcal{O}$-algebra.

3.2.2

Let A be one of the following: $\mathcal{O}$, a finite quotient of $\mathcal{O}$, $E/\mathcal{O}$, or a finite submodule of $E/\mathcal{O}$. We call a maximal ideal $\mathfrak{m}$ of $\mathbf{T}_{\kappa,\psi}(U,A)$ *nearly ordinary* if the image of T_{ϖ_v} in $\mathbf{T}_{\kappa,\psi}(U,A)/\mathfrak{m}$ is non-zero for each $v|p$. Note that since $\mathbf{T}_{\kappa,\psi}(U,A)$ is finite over $\mathcal{O}$, we have a decomposition

$$\mathbf{T}_{\kappa,\psi}(U,A) = \prod_{\mathfrak{m}} \mathbf{T}_{\kappa,\psi}(U,A)_{\mathfrak{m}}$$

where the product runs over the set of maximal ideals of $\mathbf{T}_{\kappa,\psi}(U,A)$. We define the *nearly ordinary Hecke algebra* (of weight $\mathbf{k}$, character ψ and level U with coefficients in A) by

$$\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U,A) = \prod_{\mathfrak{m}\ \mathrm{no}} \mathbf{T}_{\kappa,\psi}(U,A)_{\mathfrak{m}}$$

where the product runs over all nearly ordinary maximal ideals of $\mathbf{T}_{\kappa,\psi}(U,A)$. Note that the projection

$$\mathbf{T}_{\kappa,\psi}(U,A) \longrightarrow \mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U,A)$$

corresponds to an idempotent e_{H} in $\mathbf{T}_{\kappa,\psi}(U,A)$, and it is known as *Hida's idempotent.* Letting

$$T_p = \prod_{v|p} T_{\varpi_v},$$

it can be checked that

$$e_{\mathrm{H}} = \lim_{n\to\infty} T_p^{n!}.$$

We also define the nearly ordinary subspace of $S_{\kappa,\psi}(U,A)$ by

$$S^{\mathrm{no}}_{\kappa,\psi}(U,A) = e_{\mathrm{H}} S_{\kappa,\psi}(U,A).$$

3.2.3

Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight κ. For each $v|p$, since ϖ_v is an element of F, we can make sense of the operator

$$T_{\varpi_v} = \varpi_v^{-\mathbf{w}} \left[V \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} V \right].$$

on π^V for any compact open subgroup V, viewing I as the set of embedding of $F \to \mathbb{C}$. We say π is *p-nearly ordinary* (or just *nearly ordinary* if the p is clear from the context) if there is a open compact subgroup V of $\mathrm{GL}_2(\mathbb{A}_F)$ and a non-zero vector $x \in \pi^V$ such that for each $v|p$, T_{ϖ_v} acts on x by an element of $\overline{\mathbb{Q}}$ that is a unit in the ring of integers of $\overline{\mathbb{Q}}_p$. Note this does in general depend on our choice of embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. If π is nearly ordinary, then for each $v|p$, π_v is either a principal series representation $\pi(\eta_v|\ |_v^{1/2}, \mu_v|\ |_v^{1/2})$ or a special representation $\sigma(\eta_v|\ |_v^{1/2}, \eta_v|\ |_v^{-1/2})$ with $\varpi_v^{-\mathbf{w}}\eta_v(\varpi_v)$ an element of $\overline{\mathbb{Q}}$ that is a unit in $\overline{\mathbb{Q}}_p$, cf. Corollary 2.2. of [H2]. If π is nearly ordinary and $\pi^V \neq 0$, then there is in fact a unique line in $\pi_v^{V_v}$ on which T_{ϖ_v} acts via a unit in $\overline{\mathbb{Q}}_p$.

Using the notation of 3.1.10, if we let $\Pi^{\Sigma,\mathrm{no}}_{\kappa,\psi_{\mathbb{C}},\Sigma'}$ denote the subset of $\Pi^{\Sigma}_{\kappa,\psi_{\mathbb{C}},\Sigma'}$ consisting of nearly ordinary representations, then the map JL of 3.1.10 restricts to

$$\mathrm{JL} : S^{\mathrm{no}}_{\kappa,\psi}(U,E) \otimes_E \mathbb{C} \longrightarrow \bigoplus_{\pi \in \Pi^{\Sigma,\mathrm{no}}_{\kappa,\psi_{\mathbb{C}},\Sigma'}} \pi^V.$$

Note that if $U_v \subseteq \mathrm{Iw}(v)$ for each $v|p$, then any function factoring through the reduced norm of D is not nearly-ordinary, and so in this case JL is an isomorphism on nearly ordinary subspaces. Using this we can identify $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U,E)$ with a subspace of endomorphism of $\oplus\, \pi^V$.

Take $\pi \in \Pi^{\Sigma,\mathrm{no}}_{\kappa,\psi_{\mathbb{C}},\Sigma'}$ with $\pi^V \neq 0$, and fix a nonzero vector $x = \otimes_v x_v \in \pi^V$, on which T_{ϖ_v} acts via a unit in $\overline{\mathbb{Q}}_p$ for each $v|p$. The line generated by x is $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U,E)$-stable, and if

$x' = \otimes_v x_v$ is another such vector, the $\mathbf{T}^{\text{no}}_{\kappa,\psi}(U, E)$-eigensystem given by x is equal to that of x'. It follows that we have an E-algebra injection

$$\mathbf{T}^{\text{no}}_{\kappa,\psi}(U, E) \longrightarrow \prod_{\pi} \overline{\mathbb{Q}}_p$$

where the product is taken over all $\pi \in \Pi^{\Sigma,\text{no}}_{\kappa,\psi_{\mathbb{C}},\Sigma'}$ with $\pi^V \neq 0$. In particular $\mathbf{T}^{\text{no}}_{\kappa,\psi}(U, \mathcal{O})$ is reduced.

3.3 Universal nearly ordinary Hecke algebras

Keep all assumptions and notation of the previous subsection. We will further assume henceforth that A is one of the following: $\mathcal{O}$, some finite quotient of $\mathcal{O}$, $E/\mathcal{O}$, or some finite submodule of $E/\mathcal{O}$. We will also assume henceforth that $U_v = \text{Iw}(v)$ for all $v|p$.

For integers $b \geq a \geq 0$ we let $U(p^{a,b})$ denote the open subgroup of U given by $U(p^{a,b})_v = U_v$ if $v \nmid p$, and $U(p^{a,b})_v = U_v \cap \text{Iw}(v^{a,b})$ for $v|p$. Note that $U(p^{a,a})$ is not the subgroup of U consisting of elements that are upper triangular unipotent modulo p^a, but rater modulo ϖ_p^a, where ϖ_p is the finite idèle with $(\varpi_p)_v = \varpi_v$ for $v|p$ and equal to 1 elsewhere.

3.3.1

For $y \in \mathcal{U}_p$, $\begin{pmatrix} y & \\ & 1 \end{pmatrix}$ normalizes $U(p^{a,b})$, so we see that

$$\langle y \rangle f = \begin{pmatrix} y & \\ & 1 \end{pmatrix} f.$$

Since we have fixed a central character ψ, the natural inclusion $S_{\kappa,\psi}(U(p^{0,b}), A) \to S_{\kappa,\psi}(U(p^{a,b}), A)$ identifies $S_{\kappa,\psi}(U(p^{0,b}), A)$ with the $\mathcal{U}_p$-invariants of $S_{\kappa,\psi}(U(p^{a,b}), A)$. Similarly for $S^{\text{no}}_{\kappa,\psi}(U(p^{0,b}), A) \to S^{\text{no}}_{\kappa,\psi}(U(p^{a,b}), A)$.

Lemma 3.3.2. *For any $a \geq 0$ and $b \geq \max\{a, 1\}$, the inclusion $S^{\text{no}}_{\kappa,\psi}(U(p^{a,b}), A) \to S^{\text{no}}_{\kappa,\psi}(U(p^{a,b+1}), A)$ is an isomorphism.*

Proof. Recall that $T_p = \prod_{v|p} T_{\varpi_v}$. Then for $b \geq 1$, the action of T_p on $S_{\kappa,\psi}(U(p^{a,b}), A)$ is given by the double coset operator

$$\left[U(p^{a,b}) \begin{pmatrix} \varpi_p & \\ & 1 \end{pmatrix} U(p^{a,b}) \right]$$

where $\varpi_p \in (\mathbb{A}_F^\infty)^\times$ is the idèle equal to ϖ_v at $v|p$ and is 1 elsewhere. It then easy to check that, for $b \geq 1$,

$$U(p^{a,b+1}) \begin{pmatrix} \varpi_p & \\ & 1 \end{pmatrix} U(p^{a,b+1}) = U(p^{a,b}) \begin{pmatrix} \varpi_p & \\ & 1 \end{pmatrix} U(p^{a,b+1}).$$

Hence $T_p S_{\kappa,\psi}(U(p^{a,b+1}), A) \subseteq S_{\kappa,\psi}(U(p^{a,b}), A)$, where we have identified $S_{\kappa,\psi}(U(p^{a,b}), A)$ with its image in $S_{\kappa,\psi}(U(p^{a,b+1}), A)$. It follows that $e_{\mathrm{H}} S_{\kappa,\psi}(U(p^{a,b+1}), A) = e_{\mathrm{H}} S_{\kappa,\psi}(U(p^{a,b}), A)$ for any $a \geq 0$ and $b \geq \max\{a, 1\}$. □

The Hecke-equivariant injections $S_{\kappa,\psi}(U(p^{a,a}), A) \to S_{\kappa,\psi}(U(p^{b,b}), A)$ and $S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}), A) \to S^{\mathrm{no}}_{\kappa,\psi}(U(p^{b,b}), A)$, for $b \geq a$, induce surjections $\mathbf{T}_{\kappa,\psi}(U(p^{b,b}), A) \to \mathbf{T}_{\kappa,\psi}(U(p^{a,a}), A)$ and $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^{b,b}), A) \to \mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}), A)$. We define

$$\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^\infty), A) = \varprojlim_a \mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}), A),$$

and

$$S^{\mathrm{no}}_{\kappa,\psi}(U(p^\infty), E/\mathcal{O}) = \varinjlim_a S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}), E/\mathcal{O}).$$

We have a faithful action of $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^\infty), E/\mathcal{O})$ on $S^{\mathrm{no}}_{\kappa,\psi}(U(p^\infty), E/\mathcal{O})$.

If M is a topological $\mathcal{O}$ module, then Pontryagin duality $M \mapsto M^\vee = \mathrm{Hom}_{\mathcal{O}}(M, E/\mathcal{O})$ induces an isomorphism $\mathrm{End}_{\mathcal{O}}(M) \cong \mathrm{End}_{\mathcal{O}}(M^\vee)$. Hence we have a faithful action of $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}), E/\mathcal{O})$ on $S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}), E/\mathcal{O})^\vee$. Now take $a_0 \geq 1$ large enough so that $(U(p^{a_0,a_0})(\mathbb{A}_F^\infty)^\times \cap t^{-1} D^\times t)/F^\times = 1$ for all $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$, which we can do by 3.1.5. Then for any $a \geq a_0$, $S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}), \mathcal{O})$ is a finite free $\mathcal{O}$-module and

$$S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}), E/\mathcal{O}) \cong S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}), \mathcal{O}) \otimes_{\mathcal{O}} E/\mathcal{O},$$

hence there is an isomorphism

$$\mathrm{Hom}_{\mathcal{O}}(S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),\mathcal{O},\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),E/\mathcal{O}),E/\mathcal{O}).$$

This isomorphism is equivariant for the action of the the operators T_v, with $v \notin S$, T_{ϖ_v}, with $v|p$, and $\langle y\rangle^{\mathrm{no}}$, with $y \in \mathcal{U}_p$, on each side. We then have an isomorphism, for any $a \geq a_0$,

$$\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),\mathcal{O}) \cong \mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),E/\mathcal{O})$$

which sends T_v to T_v, etc. And so there is a faithful action of $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),\mathcal{O})$ on $S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),E/\mathcal{O})^{\vee}$. We then have a faithful action of $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^{\infty}),\mathcal{O})$ on

$$\varprojlim_{a\geq a_0} S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),E/\mathcal{O})^{\vee} \cong \big(\varinjlim_{a\geq a_0} S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),E/\mathcal{O})\big)^{\vee} = S^{\mathrm{no}}_{\kappa,\psi}(U(p^{\infty}),E/\mathcal{O})^{\vee}.$$

We view $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^{\infty}),A)$ as a $\Lambda(\mathcal{U}_p) = \mathcal{O}[[\mathcal{U}_p]]$-algebra by letting $y \in \mathcal{U}_p$ acting via $\langle y\rangle^{\mathrm{no}}$. Denote by $\mathfrak{p}_\kappa$ the prime ideal in $\Lambda(\mathcal{U}_p)$ corresponding to the kernel of the $\mathcal{O}$-algebra homomorphism that sends $y \in \mathcal{U}_p$ to $y^{-\mathbf{w}} \in \mathcal{O}$. With some mild abuse of notation, we also denote by $\mathfrak{p}_\kappa$ it's pullback of to $\Lambda(\mathcal{U}_p^a)$ for any $a \geq 0$. We then have a version of Hida's control theorem.

Proposition 3.3.3. *Let $a \geq 1$ be such that $(U(p^{a,a})(\mathbb{A}_F^{\infty})^{\times} \cap t^{-1}D^{\times}t)/F^{\times} = 1$ for all $t \in (D \otimes_F \mathbb{A}_F^{\infty})^{\times}$.*

$S^{\mathrm{no}}_{\kappa,\psi}(U(p^{\infty}),E/\mathcal{O})^{\vee}$ is finite free over $\Lambda(\mathcal{U}_p^a)$ of rank equal to the $\mathcal{O}$-rank of $S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),\mathcal{O})$. Moreover the natural surjection $S^{\mathrm{no}}_{\kappa,\psi}(U(p^{\infty}),E/\mathcal{O})^{\vee} \to S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),E/\mathcal{O})^{\vee}$ has kernel $\mathfrak{p}_\kappa S^{\mathrm{no}}_{\kappa,\psi}(U(p^{\infty}),E/\mathcal{O})$

Proof. We define a different $\Lambda(\mathcal{U}_p^a)$-module structure on $S^{\mathrm{no}}_{\kappa,\psi}(U(p^{\infty}),E/\mathcal{O})^{\vee}$, by letting $y \in \mathcal{U}_p$ act via $\langle y\rangle$. Note that the two $\Lambda(\mathcal{U}_p^a)$-module structures differ by an automorphism of $\Lambda(\mathcal{U}_p^a)$ that sends $\mathfrak{p}_\kappa$ to the augmentation ideal $\mathfrak{a}$ of $\Lambda(\mathcal{U}_p^a)$. It thus suffices to prove the lemma with this new $\Lambda(\mathcal{U}_p^a)$-module structure and $\mathfrak{p}_\kappa$ replaced by $\mathfrak{a}$.

The second part follows easily from 3.3.2,

$$\begin{aligned} S^{\mathrm{no}}_{\kappa,\psi}(U(p^\infty),E/\mathcal{O})^\vee/\mathfrak{a}S^{\mathrm{no}}_{\kappa,\psi}(U(p^\infty),E/\mathcal{O})^\vee &\cong \mathrm{Hom}_{\mathcal{O}}(S^{\mathrm{no}}_{\kappa,\psi}(U(p^\infty),E/\mathcal{O})^{\mathcal{U}_p^a},E/\mathcal{O}) \\ &\cong \big(\varinjlim_{b\geq a} S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,b}),E/\mathcal{O})\big)^\vee \\ &= S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),E/\mathcal{O})^\vee. \end{aligned}$$

For the first part, it suffices to show, by the second part, that for every $b \geq a$, $S^{\mathrm{no}}_{\kappa,\psi}(U(p^{b,b}),E/\mathcal{O})^\vee$ is a finite free $\mathcal{O}[\mathcal{U}_p^a/\mathcal{U}_p^b]$-module of rank equal to the $\mathcal{O}$-rank of $S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),\mathcal{O})$. Since the projections

$$S^{\mathrm{no}}_{\kappa,\psi}(U(p^{b,b}),E/\mathcal{O})^\vee \longrightarrow S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),\mathcal{O})^\vee$$

are Hecke equivariant and the direct summand of a free module over a local ring is free, it suffices to prove this without having applied Hida's idempotent. This then follows from 3.1.8. □

Corollary 3.3.4. *For any $a \geq 0$ both $S^{\mathrm{no}}_{\kappa,\psi}(U(p^\infty),E/\mathcal{O})^\vee$ and $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^\infty),\mathcal{O})$ are finite over $\Lambda(\mathcal{U}_p^a)$.*

Proof. It suffices to show this for some $a \geq 0$. That it holds for $S^{\mathrm{no}}_{\kappa,\psi}(U(p^\infty),E/\mathcal{O})^\vee$ is part of 3.3.3. Since there is an injection

$$\mathbf{T}_{\kappa,\psi}(U(p^\infty),\mathcal{O}) \to \mathrm{End}_{\Lambda(\mathcal{U}_p^a)}(S^{\mathrm{no}}_{\kappa,\psi}(U(p^\infty),E/\mathcal{O})^\vee)$$

the same is true for $\mathbf{T}_{\kappa,\psi}(U(p^\infty),\mathcal{O})$. □

3.3.5

There is a unique free rank one $\mathcal{O}$-submodule of $W_\kappa(\mathcal{O})$ on which the diagonal subgroup of $GL_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ acts via the character

$$\begin{pmatrix} a_p & \\ & d_p \end{pmatrix} \mapsto a_p^{\mathbf{w}} d_p^{\mathbf{k}+\mathbf{w}-2},$$

which we will denote by χ_κ. Fix a generator v_l of this submodule. By abuse of notation we will also denote by v_l the vector $v_l \otimes 1$ of $W_\kappa(\mathcal{O}/\mathfrak{m}_\mathcal{O}^r)$ for any $r \geq 1$. Then for $A = \mathcal{O}$ or $\mathcal{O}/\mathfrak{m}_\mathcal{O}^r$, the choice of v_l gives a Borel equivariant map

$$W_\kappa(A) \longrightarrow A(\chi_\kappa).$$

If $A = \mathcal{O}/\mathfrak{m}_\mathcal{O}^r$ and $a \geq r$, this yields a $U(p^{a,a})$-equivariant map

$$\mathrm{pr}_\kappa : W_\kappa \longrightarrow A.$$

Recall that, for $v \in \Sigma$, U_v may act nontrivially on both sides of above. However it does so via the same $\mathcal{O}^\times$-valued character $\gamma^{-1} \circ \nu_D$.

Lemma 3.3.6. *Let $a \geq r \geq 1$ and recall that ϖ_p is the finite idèle defined by $(\varpi_p)_v = \varpi_v$ for $v|p$ and $(\varpi_p)_v = 1$ for $v \nmid p$.*

1. *For any $v|p$ and $g \in U(p^{a,a}) \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} U(p^{a,a})$, we have $\varpi_v^{-\mathbf{w}} g_p v_l = v_l + w$ with $w \in \ker(\mathrm{pr}_\kappa)$.*
2. *For any $w \in \ker(\mathrm{pr}_\kappa)$ and $g \in U(p^{a,a}) \begin{pmatrix} \varpi_p^r & \\ & 1 \end{pmatrix} U(p^{a,a})$, we have $\varpi_p^{-r\mathbf{w}} g_p w = 0$.*

Proof. Let α be either ϖ_v or ϖ_p^r. Note that for $u \in U(p^{a,a})$, u_p acts on $W_\kappa(\mathcal{O}/\mathfrak{m}_\mathcal{O}^r)$ through its image modulo $\mathfrak{m}_\mathcal{O}^r$, which is upper triangular unipotent. Hence, for any

$$g \in U(p^{a,a}) \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} U(p^{a.a})$$

we can assume

$$g = \begin{pmatrix} \alpha & x \\ & 1 \end{pmatrix}$$

for some $x \in \mathbb{A}_F^\infty$ with $x_p \in \mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_p$.

Recall we have an isomorphism of $W_\kappa(\mathcal{O}/\mathfrak{m}_\mathcal{O}^r)$ with the space of $\mathcal{O}/\mathfrak{m}_\mathcal{O}^r$-linear combinations on the monomials

$$\prod_{\tau \in I} X_\tau^{k_\tau - 2 - j_\tau} Y_\tau^{j_\tau}, \tag{3.3}$$

with each $0 \leq j_\tau \leq k_\tau - 2$, on which $\mathrm{GL}_2(\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$-action by

$$\begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix} \prod_{\tau \in I} X_\tau^{k_\tau - 2 - j_\tau} Y_\tau^{j_\tau} = (a_p d_p - c_p d_p)^{\mathbf{w}} \prod_{\tau \in I} (\tau(a_p) X_\tau + \tau(c_p) Y_\tau)^{k_\tau - 2 - j_\tau} (\tau(b_p) X_\tau + \tau(d_p) Y_\tau)^{j_\tau}.$$

We see that $\ker(\mathrm{pr}_\kappa)$ is the span of all monomials (3.3) with some $j_\tau < k_\tau - 2$ and we may assume $v_l = \prod_I Y_\tau^{k_\tau - 2}$

For $\alpha = \varpi_v$ we have

$$\varpi_v^{-\mathbf{w}} \begin{pmatrix} \varpi_v & x \\ & 1 \end{pmatrix} \prod_{\tau \in I} Y_\tau^{k_\tau - 2} = \prod_{\tau \in I} (\tau(x_p) X_\tau + Y_\tau)^{k_\tau - 2} = v_l + w$$

with $w \in \ker(\mathrm{pr}_\kappa)$.

For $\alpha = \varpi_p^r$, since $\tau(\varpi_p^r) \in \mathfrak{m}_{\mathcal{O}}^r$ for any $\tau \in I$,

$$\varpi_p^{-r\mathbf{w}} \begin{pmatrix} \varpi_p^r & x \\ & 1 \end{pmatrix} \prod_{\tau \in I} X_\tau^{k_\tau - 2 - j_\tau} Y_\tau^{j_\tau} = \prod_{\tau \in I} (\tau(\varpi_p^r) X_\tau)^{k_\tau - 2 - j_\tau} (\tau(x_p) X_\tau + Y_\tau)^{j_\tau} = 0,$$

if some $j_\tau < k_\tau - 2$. □

If $a \geq r$, the map $\mathrm{pr}_\kappa : W_\kappa(\mathcal{O}/\mathfrak{m}_{\mathcal{O}}^r) \to \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^r$ induces a morphism of $\mathcal{O}$-modules

$$\mathrm{pr}_\kappa^* : S_{\kappa,\psi}(U(p^{a,a}), \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^r) \longrightarrow S_{2,\psi}(U(p^{a,a}), \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^r).$$

Proposition 3.3.7. *For any algebraic weight κ and $a \geq r \geq 1$, pr_κ^* is equivariant for all T_v with $v \notin S$, T_{ϖ_v} with $v|p$, and $\langle y \rangle^{\mathrm{no}}$ with $y \in \mathcal{U}_p$. Moreover pr_κ^* induces an isomorphism*

$$S_{\kappa,\psi}^{\mathrm{no}}(U(p^{a,a}), \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^r) \xrightarrow{\sim} S_{2,\psi}^{\mathrm{no}}(U(p^{a,a}), \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^r).$$

Proof. It is easy to see that pr_κ is equivariant for T_v with $v \notin S$ and $\langle y \rangle^{\mathrm{no}}$ for $y \in \mathcal{U}_p$. For $v|p$, writing

$$U(p^{a,a}) \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} U(p^{a,a}) = \bigsqcup_i g_i U(p^{a,a})$$

we have, by part (1) of 3.3.6

$$\mathrm{pr}_\kappa(\varpi_v^{-\mathbf{w}} \sum_i g_i f(x g_i)) = \sum_i \Big(\prod_{v \in \Sigma'} \gamma_v^{-1} \circ \nu_D(g_{i,v}) \Big) \mathrm{pr}_\kappa(f(x g_i)) = \sum_i g_i \mathrm{pr}_\kappa(f(x g_i)).$$

from which it follows that pr_κ^* is equivariant for T_{ϖ_v}.

The equivariance of pr_κ^* implies that it induces a morphism

$$S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),\mathcal{O}/\mathfrak{m}^r_{\mathcal{O}}) \longrightarrow S^{\mathrm{no}}_{2,\psi}(U(p^{a,a}),\mathcal{O}/\mathfrak{m}^r_{\mathcal{O}}).$$

Take $f \in S_{2,\psi}(U(p^{a,a}),\mathcal{O}/\mathfrak{m}^r_{\mathcal{O}})$. Write

$$U(p^{a,a})\begin{pmatrix} \varpi_p^r & \\ & 1 \end{pmatrix}U(p^{a,a}) = \bigsqcup_i g_i U(p^{a,a})$$

and define a function $s_\kappa(f) : D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times \to W_\kappa(\mathcal{O}/\mathfrak{m}^r_{\mathcal{O}})$ by

$$s_\kappa(f)(x) = \sum_i \varpi_p^{-r\mathbf{w}} g_i f(xg_i)v_l.$$

We first show that $s(f)$ is independent of the choice of $\{g_i\}$. Indeed, choosing $u_i \in U(p^{a,a})$, we have

$$\sum_i \varpi_p^{-r\mathbf{w}} g_i u_i f(xg_iu_i)v_l = \sum_i \varpi_p^{-r\mathbf{w}} g_i u_{i,p} f(xg_i)v_l \tag{3.4}$$

since $f \in S_{2,\psi}(U(p^{a,a}),\mathcal{O}/\mathfrak{m}^r_{\mathcal{O}})$ implies $f(xg_iu_i) = (\prod_{v\in\Sigma'} \gamma_v \circ \nu_D(u_{i,v}))f(xg_i)$. For each i, since $u_{i,p}$ is upper-triangular unipotent mod ϖ_p^a, we have $u_{i,p}v_l = v_l + w_i$, with $w_i \in \ker(\mathrm{pr}_\kappa)$. Then (3.4) becomes

$$\sum_i \varpi_p^{-r\mathbf{w}} g_i f(xg_i)v_l + \sum_i \varpi_p^{-r\mathbf{w}} g_i f(xg_i)w_i = \sum_i \varpi_p^{-r\mathbf{w}} g_i f(xg_i)v_l,$$

by part (2) of 3.3.6. So $s(f)$ is independent of the choice of $\{g_i\}$.

Now we check $s_\kappa(f) \in S_{\kappa,\psi}(U(p^{a,a}),\mathcal{O}/\mathfrak{m}^r_{\mathcal{O}})$. Set $F = s_\kappa(f)$. The fact that $F(xz) = \psi(z)F(x)$ for $z \in (\mathbb{A}_F^\infty)^\times$ is immediate. Take $u \in U(p^{a,a})$. Now

$$(uF)(x) = u\big(\sum_i \varpi_p^{-r\mathbf{w}} g_i f(xug_i)v_l = \sum_i \varpi_p^{-r\mathbf{w}} ug_i f(xug_i)v_l. \tag{3.5}$$

For each i, we can write $ug_i = g_ju_j$ and (3.5) becomes

$$(uF)(x) = \sum_j \varpi_p^{-r\mathbf{w}} g_ju_j f(xg_ju_j)v_l = F(x).$$

Note that for $f \in S_{2,\psi}(U(p^{a,a}), \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^r)$, since $T_p^r = \prod_{v|p} T_{\varpi_v}^r$, part (1) of 3.3.6 implies that

$$(\mathrm{pr}_\kappa \circ s_\kappa)(f) = T_p^r f.$$

Conversely, given $F \in S_{\kappa,\psi}(U(p^{a,a}), \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^r)$, and writing $F(x) = f(x)v_l + F'(x)$ with F' a function taking values in $\ker(\mathrm{pr}_\kappa)$, part (2) of 3.3.6 implies that

$$((s_\kappa \circ \mathrm{pr}_\kappa)(F))(x) = \sum_i \varpi_p^{-r\mathbf{w}} g_i f(xg_i) v_l = \sum_i \varpi_p^{-r\mathbf{w}} g_i (f(xg_i)v_l + F'(xg_i)) = (T_p^r F)(x).$$

Now if $f \in S_{2,\psi}^{\mathrm{no}}(U(p^{a,a}), \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^r)$, then $s_\kappa(f) \in S_{\kappa,\psi}^{\mathrm{no}}(U(p^{a,a}), \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^r)$. Indeed, since each space is finite, there is some $n \geq r$ such that on each space $e_H = T_p^n$. Then

$$T_p^n s_\kappa(f) = (s_\kappa \circ \mathrm{pr}_\kappa)(T_p^{n-r} s_\kappa(f)) = s_\kappa(T_p^{n-r}(\mathrm{pr}_\kappa \circ s_\kappa)(f)) = s_\kappa(T_p^n f) = s_\kappa(f).$$

We then see that pr_κ and s_κ restrict to morphisms between the nearly ordinary subspaces whose composites are automorphisms. It follows that pr_κ and s_κ are isomorphisms on the nearly ordinary subspaces. □

Since $\mathfrak{m}_{\mathcal{O}}^{-r}/\mathcal{O} \cong \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^r$, the following corollary follows from 3.3.7 upon taking direct limits and Pontryagin duals.

Corollary 3.3.8. *For any algebraic weight κ, there is an $\mathcal{O}$-module isomorphism, equivariant for all T_v with $v \notin S$, T_{ϖ_v} with $v|p$, and $\langle y\rangle^{\mathrm{no}}$ with $y \in \mathcal{U}_p$,*

$$S_{\kappa,\psi}^{\mathrm{no}}(U(p^\infty), E/\mathcal{O})^\vee \cong S_{2,\psi}^{\mathrm{no}}(U(p^\infty), E/\mathcal{O})^\vee.$$

Henceforth we denote $S_{2,\psi}(U(p^\infty), E/\mathcal{O})^\vee$ by $S_\psi(U)$. By 3.3.4, $S_\psi(U)$ is a finite $\Lambda(\mathcal{U}_p)$-module that is free over $\Lambda(\mathcal{U}_p^a)$ for sufficiently large $a \geq 1$. We let $\mathbf{T}_\psi(U)$ denote the $\Lambda(\mathcal{U}_p)$-subalgebra of $\mathrm{End}_{\Lambda(\mathcal{U}_p)}(S_\psi(U))$ generated by T_v for all $v \notin S$ and T_{ϖ_v} for $v|p$. Note that $T_\psi(U)$ is finite a finite $\Lambda(\mathcal{U}_p)$-algebra, and is reduced. The following corollary is immediate from 3.3.8.

Corollary 3.3.9. *Let $\kappa = (\mathbf{k}, \mathbf{w})$ be an algebraic weight such that $U \cap (\mathbb{A}_F^\infty)^\times$ acts on $W_\kappa(\mathcal{O})$ via ψ^{-1}. We have an $\Lambda(\mathcal{U}_p)$-algebra isomorphism*

$$\mathbf{T}_{\kappa,\psi}^{\mathrm{no}}(U(p^\infty), \mathcal{O}) \cong \mathbf{T}_\psi(U)$$

identifying the T_v for $v \notin S$, and the T_{ϖ_v} for $v|p$

Recall that if κ is an algebraic weight, then for any $a \geq 1$ we denote by $\mathfrak{p}_\kappa$ the kernel of the $\mathcal{O}$-algebra morphism $\Lambda(\mathcal{U}_p^a) \to \mathcal{O}$ corresponding to the character $y \mapsto y^{-\mathbf{w}}$ of $\mathcal{U}_p^a$. Combining 3.3.9 with 3.3.3 yields the following.

Corollary 3.3.10. *Let $\kappa = (\mathbf{k}, \mathbf{w})$ be an algebraic weight such that $U \cap (\mathbb{A}_F^\infty)^\times$ acts on $W_\kappa(\mathcal{O})$ via ψ^{-1}. Let $a \geq 1$ be such that $(U(p^{a,a})(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times = 1$ for all $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$.*

The isomorphism $\mathbf{T}_\psi(U) \cong \mathbf{T}_{\kappa,\psi}(U(p^\infty), \mathcal{O})$ of 3.3.9 combined with the natural projection $\mathbf{T}_{\kappa,\psi}(U(p^\infty), \mathcal{O}) \to \mathbf{T}_{\kappa,\psi}(U(p^{a,a}), \mathcal{O})$ has kernel $\mathfrak{p}_\kappa \mathbf{T}_\psi(U)$.

We say that a prime $\mathfrak{p}$ of $\mathbf{T}_\psi(U)$ is an *arithmetic prime* if there is some $a \geq 1$, and some algebraic weight κ such that $\mathfrak{p} \cap \Lambda(\mathcal{U}_p^a) = \mathfrak{p}_\kappa$.

Corollary 3.3.11. *For any minimal prime $\mathfrak{q}$ of $\mathbf{T}_\psi(U)$, the set of arithmetic primes in $\mathfrak{q}$ is Zariski dense.*

Proof. It is easy to see that the set of primes $\mathfrak{p} \in \operatorname{Spec} \Lambda(\mathcal{U}_p)$ such that $\mathfrak{p} \cap \Lambda(\mathcal{U}_p^a) = \mathfrak{p}_\kappa$ for some a and some κ, is Zariski dense in $\Lambda(\mathcal{U}_p)$. The result now follows from the fact that $\mathbf{T}_\psi(U)/\mathfrak{q}$ is finite over $\Lambda(\mathcal{U}_p)$ by 3.3.4. $\square$

3.4 Modular Galois representations

We keep all assumptions and notations of the previous subsection. In particular κ is an algebraic weight and U is a $(\Sigma' \subseteq \Sigma)$-open subgroup of $(D \otimes_F \mathbb{A}_F)^\times$, and S denotes the finite set of places containing Σ, all primes at which U_v is not maximal compact, as well as all places above p and ∞.

3.4.1

Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. There is an absolutely irreducible representation

$$\rho_\pi : G_F \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

such that $\rho_\pi|_{G_v}$ and π_v satisfy the (suitably normalized) local Langlands correspondence for every place v of F. The existence of such a ρ_π was shown in [T1] building on [W] and [C1], and an alternate construction is given in [BR]. The compatibility with the local Langlands correspondence was shown for places away from p in [C1] and [T1], and for places above p in [BR], [S1] and [S2].

In the case that π is p-nearly ordinary, we can say more. Let $(\mathbf{k}, \mathbf{w})$ denote the weight of π. Take an open compact subgroup $V \subset \mathrm{GL}_2(\mathbb{A}_F^\infty)$, with $V_v \subseteq \mathrm{Iw}(v)$ for each $v|p$, such that there is $0 \neq x \in \pi^V$ with $T_{\varpi_v} x = \alpha_v x$, where $\alpha_v \in \overline{\mathbb{Q}}$ is a p-adic unit under out fixed embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Then the line generated by x is also stable under $\langle y \rangle$ for every $y \in \mathcal{U}_p$, and the eigenvalues are algebraic. Letting χ'_v denote the resulting $\overline{\mathbb{Q}}$-valued character, we define a character $\chi_v : F_v^\times \to \overline{\mathbb{Q}}_p^\times$ by $\chi_v(y) = \chi'_v(y) y^{-\mathbf{w}}$ and $\chi_v(\varpi_v) = \alpha_v$. It is shown in [W] and [H2] that

$$\rho_\pi|_{G_v} \cong \begin{pmatrix} * & * \\ & \chi_v \end{pmatrix}.$$

3.4.2

Let $f \in S^{\mathrm{no}}_{\kappa,\psi}(U, \mathcal{O})$, be an eigenfunction for $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U, \mathcal{O})$, and let

$$\lambda_f : \mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U, \mathcal{O}) \longrightarrow \overline{\mathbb{Q}}_p$$

denote the resulting homomorphism. The image of f under the Jacquet-Langlands-Shimizu correspondence, c.f. 3.1.10, generates an irreducible π_f. Letting $\rho_f = \rho_{\pi_f}$ with ρ_{π_f} as in 3.4.1, the discussion in 3.4.1 applied to

$$\rho_f : G_F \longrightarrow \mathrm{GL}_2(\mathbb{A}_F)$$

implies

1. ρ_f is unramified outside S and $\operatorname{tr}\rho_f(\mathrm{Frob}_v) = \lambda_f(T_v)$ for each $v \notin S$;

2. $\det\rho_f = \psi\epsilon_p$, in particular $\det\rho_f(c) = -1$ for any choice c of complex conjugation;

3. for every $v \in \Sigma$, $\rho_f|_{G_v} \cong \begin{pmatrix} \theta_v\epsilon_p & * \\ & \theta_v \end{pmatrix}$ with θ_v an unramified character, and if $v \in \Sigma'$ then $\theta_v = \gamma_v$;

4. for every $v|p$, $\rho_f|_{G_v} \cong \begin{pmatrix} * & * \\ & \chi_v \end{pmatrix}$, where, via the isomorphism of local class field theory, $\chi_v(\varpi_v) = \lambda_f(T_{\varpi_v})$, and $\chi_v(y) = \lambda_f(\langle y\rangle^{\mathrm{no}})$ for all $y \in \mathcal{O}_{F_v}^\times$.

3.4.3

We call an ideal $\mathfrak{a}$ of $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U,\mathcal{O})$ *Eisenstein* if there is an abelian extension L/F such that for all but finitely many finite places v of F that split completely in L, we have $T_v - 2 \in \mathfrak{a}$. We call an ideal $\mathfrak{a}$ of $\mathbf{T}_\psi(U)$ *Eisenstein* if it is the pullback of an Eisenstein ideal under a surjection $\mathbf{T}_\psi(U) \to \mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),\mathcal{O})$ as in 3.3.10.

Let f be an eigenform for $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U,\mathcal{O})$, and let

$$\lambda_f : \mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U,\mathcal{O}) \longrightarrow \overline{\mathbb{Q}}_p,$$

denote the corresponding $\mathcal{O}$-algebra morphism. The kernel of λ_f is contained in a unique maximal ideal $\mathfrak{m}$. Choosing a $\overline{\mathbb{Z}}_p$-lattice for ρ_f and reducing modulo the maximal ideal of $\overline{\mathbb{Z}}_p$, we obtain a representation

$$\overline{\rho}_{\mathfrak{m}} : G_{F,S} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}).$$

If $\mathfrak{m}$ is non-Eisenstein, $\overline{\rho}_{\mathfrak{m}}$ is irreducible and (up to isomorphism) $\overline{\rho}_{\mathfrak{m}}$ does not depend on the choice of $\overline{\mathbb{Z}}_p$-lattice in the representation space of ρ_f, nor on the choice of f.

Given the non-Eisenstein maximal ideal $\mathfrak{m}$ of $\mathbf{T}_\psi(U)$ and $v|p$, we define a character

$$\chi^{\mathrm{univ}}_{v,\mathfrak{m}} : G_v \longrightarrow \mathbf{T}_\psi(U)^\times_{\mathfrak{m}}$$

by composing the isomorphism of class field theory with the character of $F_v^\times$ that sends ϖ_v to T_{ϖ_v}, and on $\mathcal{O}_{F_v}^\times$ is equal to the canonical character

$$\mathcal{O}_{F_v}^\times \longrightarrow \Lambda(\mathcal{U}_p)^\times \longrightarrow \mathbf{T}_\psi(U)_\mathfrak{m}^\times.$$

Proposition 3.4.4. *Let* $\mathfrak{m}$ *be a non-Eisenstein maximal ideal of* $\mathbf{T}_\psi(U)$. *There exists a continuous representation*

$$\rho_{U,\mathfrak{m}} : G_{F,S} \longrightarrow \mathrm{GL}_2(\mathbf{T}_\psi(U)_\mathfrak{m})$$

such that

1. *for any finite place* $v \notin S$, $\operatorname{tr} \rho_{U,\mathfrak{m}}(\mathrm{Frob}_v) = T_v$.

Moreover, this representation satisfies

(2) $\det \rho_{U,\mathfrak{m}} = \psi\epsilon_p$,

(3) for $v|p$ *and* $\sigma \in G_v$, $\operatorname{tr} \rho(\sigma) = \psi\epsilon_p(\chi_{v,\mathfrak{m}}^{\mathrm{univ}})^{-1}(\sigma) + \chi_{v,\mathfrak{m}}^{\mathrm{univ}}(\sigma)$.

Proof. Take an algebraic weight κ and $a \geq 1$ be such that $U \cap (\mathbb{A}_F^\infty)^\times$ acts on $W_\kappa(\mathcal{O})$ by ψ^{-1} and such that for any $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$, $(U(p^{a,a})(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times = 1$. Then $\mathfrak{m}$ is the pullback of a non-Eisenstein maximal ideal of $\mathbf{T}_{\kappa,\psi}^{\mathrm{no}}(U(p^{a,a}), \mathcal{O})$. Let

$$\lambda : \mathbf{T}_{\kappa,\psi}^{\mathrm{no}}(U(p^{a,a}), \mathcal{O})_\mathfrak{m} \longrightarrow \overline{\mathbb{Q}}_p$$

by an $\mathcal{O}$-algebra morphism corresponding to an eigenform f, and let ρ_f denote the corresponding representation as in 3.4.2. Since $\operatorname{tr} \rho_\lambda(\mathrm{Frob}_v) = \lambda(T_v)$, for every $v \notin S$, the injection

$$\mathbf{T}_{\kappa,\psi}^{\mathrm{no}}(U(p^{a,a}), \mathcal{O})_\mathfrak{m} \longrightarrow \prod_f \overline{\mathbb{Q}}_p,$$

as in 3.2.3 implies there is a is a pseudo-representation

$$r_a : G_{F,S} \longrightarrow \mathbf{T}_{\kappa,\psi}(U(p^{a,a}), \mathcal{O})_\mathfrak{m}$$

with $r_a(\mathrm{Frob}_v) = T_v$ for every $v \notin S$. We also get a pseudo-representation

$$r_b : G_{F,S} \longrightarrow \mathbf{T}_{\kappa,\psi}(U(p^{b,b}), \mathcal{O})_\mathfrak{m}$$

for every $b \geq a$, and

$$\begin{array}{ccc} G_{F,S} & \xrightarrow{r_b} & \mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^{b,b}), \mathcal{O})_{\mathfrak{m}} \\ & \searrow^{r_a} & \downarrow \\ & & \mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}), \mathcal{O})_{\mathfrak{m}} \end{array}$$

commutes. We then get a pseudo representation

$$r = \varprojlim_{b \geq a} r_b : G_{F,S} \longrightarrow \mathbf{T}_{\psi}(U)_{\mathfrak{m}},$$

such that $r(\mathrm{Frob}_v) = T_v$ for any $v \notin S$. Since r modulo $\mathfrak{m}$ is the trace of an absolutely irreducible representation, namely $\overline{\rho}_{\mathfrak{m}}$, a theorem of Nyssen and Rouqier, [N], [R2], implies that r is the trace of a representation

$$\rho_{U,\mathfrak{m}} : G_{F,S} \longrightarrow \mathrm{GL}_2(\mathbf{T}_{\psi}(U)_{\mathfrak{m}}),$$

and a theorem of Carayol, [C2] implies this representation is unique. To see (2) and (3), note that the specialization of $\rho_{U,\mathfrak{m}}$ at any arithmetic prime satisfies the corresponding properties, hence so does $\rho_{U,\mathfrak{m}}$ by Zariski density of arithmetic primes, cf. 3.3.11, and reducedness of $\mathbf{T}_{\psi}(U)_{\mathfrak{m}}$. □

Corollary 3.4.5. *Let S' be any finite set of places of F containing S. Then there exist finite places $v_1, \ldots, v_k \notin S'$ such that $\mathbf{T}_{\psi}(U)_{\mathfrak{m}} = \Lambda(\mathcal{U}_p^1)[T_{v_1}, \ldots, T_{v_k}][T_{\varpi_v}]_{v|p}$.*

Proof. Let μ denote the prime to p-torsion subgroup of $\mathcal{U}_p$. By definition, $\mathbf{T}_{\psi}(U)$ is generated over $\Lambda(\mathcal{U}_p^1)$ by the the operators T_v for $v \notin S$, T_{ϖ_v} for $v|p$, as well as $\langle y \rangle$ for $y \in \mu$. As we have assumed E contains all embeddings $F_v \to \overline{\mathbb{Q}}_p$, the projection $\mathbf{T}_{\psi}(U) \to \mathbf{T}_{\psi}(U)_{\mathfrak{m}}$ sends each $\langle y \rangle$ with $y \in \mu$, to elements of $\mathcal{O}$. By 3.4.4 and Chebotarev density, $\mathbf{T}_{\psi}(U)_{\mathfrak{m}}$ is generated over $\Lambda(\mathcal{U}_p^1)[T_{\varpi_v}]_{v|p}$ by the operators T_v for $v \notin S'$. By 3.3.4, we can chose a finite subset $\{v_1, \ldots, v_k\}$ that generate $\mathbf{T}_{\psi}(U)_{\mathfrak{m}}$ over $\Lambda(\mathcal{U}_p^1)[T_{\varpi_v}]_{v|p}$. □

3.4.6

Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbf{T}_{\psi}(U)$, and denote by

$$\overline{\rho}_{\mathfrak{m}} : G_{F,S} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}})$$

the corresponding residual representation. For $v|p$, let $\Lambda(I_v) = \mathcal{O}[[I_v^{\mathrm{ab}}(p)]]$ and $\Lambda(I_p) = \hat{\otimes}_{v|p}\Lambda(I_v)$ Also let, for $v|p$, $\Lambda(G_v) = \mathcal{O}[[G_v^{\mathrm{ab}}(p)]]$ and let $\Lambda(G_p) = \hat{\otimes}_{v|p}\Lambda(G_v)$. Local class field theory then gives $\Lambda(I_p) \cong \Lambda(\mathcal{U}_p^1)$. Recall that for each $v|p$, we have a $\mathbf{T}_\psi(U)_\mathfrak{m}$-valued character $\chi_{v,\mathfrak{m}}^{\mathrm{univ}}$ of G_v obtained by sending ϖ_v to T_{ϖ_v}. Hence the $\Lambda(I_p)$-algebra structure on $\mathbf{T}_\psi(U)_\mathfrak{m}$ extends to a $\Lambda(G_p)$-algebra structure. Fix an $\mathcal{O}$-valued character $\eta = (\eta_v)_{v|p}$ of the torsion subgroup of $\mathcal{U}_p^1$ (equivalently a character of the torsion subgroup of $\prod_{v|p} G_v^{\mathrm{ab}}(p)$). The character η determines a minimal primes of $\Lambda(\mathcal{U}_p^1)$ and $\Lambda(G_p)$, each of which denote by $\mathfrak{q}_\eta$. We set $\Lambda(\mathcal{U}_p^1, \eta) = \Lambda(\mathcal{U}_p^1)/\mathfrak{q}_\eta$, $\Lambda(G_p, \eta) = \Lambda(G_p)/\mathfrak{q}_\eta$ and $\mathbf{T}_\psi(U,\eta)_\mathfrak{m} = \mathbf{T}_\psi(U)_\mathfrak{m}/\mathfrak{q}_\eta$.

After enlarging $\mathcal{O}$, if necessary, we may assume that all eigenvalues of $\overline{\rho}_\mathfrak{m}$ are defined over $\mathbb{F}$. Let $R_{F,S}$ denote the universal deformation ring for $G_{F,S}$-deformations of $\overline{\rho}_\mathfrak{m}$, as in 2.6. Proposition 3.4.4 gives a local $\mathcal{O}$-algebra morphism $R_{F,S} \to \mathbf{T}_\psi(U)_\mathfrak{m}$. We then get a local $\Lambda(G_p)$-algebra morphism $R_{F,S}\hat{\otimes}\Lambda(G_p) \to \mathbf{T}_\psi(U)_\mathfrak{m}$, which is surjective by 3.4.5. We let $\overline{R}_{F,S}^\psi$ denote the quotient of $R_{F,S}\hat{\otimes}\Lambda(G_p)$ defined in 2.6.5 (with $S_{\mathrm{ur}} = \emptyset$). Recall that for a finite extension E'/E with ring of integers $\mathcal{O}'$, a local $\mathcal{O}$-algebra morphism $R_{F,S}\hat{\otimes}\Lambda(G_p) \to \mathcal{O}'$ factors through $\overline{R}_{F,S}^\psi$ if and only if the corresponding deformation $V_{\mathcal{O}'}$ and characters $(\chi_v)_{v|p}$ satisfy the following

- $\det V_{\mathcal{O}'} = \psi\epsilon_p$;
- for each $v|p$, there is a G_v-stable line L in $V_{\mathcal{O}'}$ such that G_v acts on $V_{\mathcal{O}'}/L$ via χ_v;
- for each $v|p$, the restriction of χ_v to the torsion subgroup of $G_v^{\mathrm{ab}}(p)$ is equal to η_v;
- for each $v \in \Sigma$, $V_{\mathcal{O}'}|_{G_v}$ is an extension of γ_v by $\gamma_v\epsilon_p$.

Proposition 3.4.7. *Let $\mathfrak{m}$ and $\overline{R}_{F,S}^\psi$ be as above and assume $\Sigma' = \Sigma$. The $\Lambda(G_p)$-algebra morphism $R_{F,S}\hat{\otimes}\Lambda(G_p) \to \mathbf{T}_\psi(U,\eta)_\mathfrak{m}$ factors through $\overline{R}_{F,S}^\psi$.*

Proof. Let $\mathfrak{p}$ be an arithmetic prime of $\mathbf{T}_\psi(U,\eta)_\mathfrak{m}$. The pushforward of the representation in 3.4.4 along $\mathbf{T}_\psi(U,\eta)_\mathfrak{m}/\mathfrak{p}$ is an integral model for some ρ_f as in 3.4.2. By 3.4.2, the map

$$R_{F,S}\hat{\otimes}\Lambda(G_p) \to \mathbf{T}_\psi(U,\eta)_\mathfrak{m} \to \mathbf{T}_\psi(U,\eta)_\mathfrak{m}/\mathfrak{p}$$

factors through $\overline{R}^{\psi}_{F,S}$. The result now follows from the Zariski density of arithmetic primes, c.f. 3.3.11, and the fact that $\mathbf{T}_{\psi}(U,\eta)_{\mathfrak{m}}$ is reduced. $\square$

3.5 Auxiliary primes and freeness

We keep the notations and assumptions of the previous sections. In particular D is a totally definite quaternion algebra with centre F, U is a $(\Sigma' \subseteq \Sigma)$-open subgroup of $(D \otimes_F \mathbb{A}_F^{\infty})^{\times}$, $\psi : F^{\times}\backslash(\mathbb{A}_F^{\infty})^{\times} \to \mathcal{O}^{\times}$ is a continuous character such that $\psi(z) = z_p^{2-\mathbf{k}-2\mathbf{w}}$ on $U \cap (\mathbb{A}_F^{\infty})^{\times}$ for some algebraic weight $(\mathbf{k},\mathbf{w})$, and S denote the finite set of places at which either D is ramified, $U_v \neq \mathrm{GL}_2(\mathcal{O}_{F_v})$, $v|p$, or $v|\infty$.

3.5.1

Fix a finite place $w \notin S$, and let U' be the open subgroup of U such that $U'_v = U_v$ if $v \neq w$ and $U_w = \mathrm{Iw}(w)$. Given $a \geq 0$, we define a map

$$\xi_w^a : S_{2,\psi}(U(p^{a,a}), E/\mathcal{O})^2 \longrightarrow S_{2,\psi}(U'(p^{a,a}), E/\mathcal{O})$$
$$(f,g) \longmapsto f + \begin{pmatrix} 1 & \\ & \varpi_w \end{pmatrix} g$$

This map is equivariant for all Hecke operators outside w, hence induces a map on the nearly ordinary subspaces, which we also denote by ξ_w^a. We then set

$$\xi_w = \varinjlim_{a\geq 1} \xi_w^a : S^{\mathrm{no}}_{2,\psi}(U(p^{\infty}), E/\mathcal{O})^2 \longrightarrow S^{\mathrm{no}}_{2,\psi}(U'(p^{\infty}), E/\mathcal{O})$$

and $\xi_w^{\vee} : S_{\psi}(U') \to S_{\psi}(U)^2$ it Pontryagin dual. These are both maps of $\Lambda(\mathcal{U}_p)$-modules and respect the action of T_v, for $v \notin S \cup \{w\}$, and T_{ϖ_v} for $v|p$.

Lemma 3.5.2. *Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbf{T}_{\psi}(U)$ and denote its pullback to $\mathbf{T}_{\psi}(U')$ also by $\mathfrak{m}$. The localization of $\xi_w^{\vee}$ at $\mathfrak{m}$, $S_{\psi}(U')_{\mathfrak{m}} \to S_{\psi}(U)^2_{\mathfrak{m}}$, is surjective.*

Proof. It suffices to show that

$$\xi_w : S^{\mathrm{no}}_{2,\psi}(U(p^{\infty}), E/\mathcal{O})^2_{\mathfrak{m}} \longrightarrow S^{\mathrm{no}}_{2,\psi}(U'(p^{\infty}), E/\mathcal{O})_{\mathfrak{m}}$$

is injective. For this it suffices to show that for any $a \geq r \geq 1$, that

$$\xi_w^a : S_{2,\psi}^{\mathrm{no}}(U(p^{a,a}), \mathfrak{m}_{\mathcal{O}}^{-r}/\mathcal{O})^2_{\mathfrak{m}} \longrightarrow S_{2,\psi}^{\mathrm{no}}(U'(p^{a,a}), \mathfrak{m}_{\mathcal{O}}^{-r}/\mathcal{O})_{\mathfrak{m}}$$

is injective. If (f, g) belongs to the kernel of

$$\xi_w^a : S_{2,\psi}(U(p^{a,a}), \mathfrak{m}_{\mathcal{O}}^{-r}/\mathcal{O})^2 \longrightarrow S_{2,\psi}(U'(p^{a,a}), \mathfrak{m}_{\mathcal{O}}^{-r}/\mathcal{O}),$$

then f is invariant under $U(p^{a,a})\mathrm{SL}_2(F_w)$. Letting $(D \otimes_F \mathbb{A}_F^\infty)^1$ denote the subgroup of elements of reduced norm 1, strong approximation implies that f is invariant under $(D \otimes_F \mathbb{A}_F^\infty)^1$, hence f factors through the reduced norm. Then f is not in the support of $\mathfrak{m}$, since $\mathfrak{m}$ is non-Eisenstein. □

For the remainder of this section fix $\mathfrak{p} \in \operatorname{Spec} \mathbf{T}_\psi(U)$ contained in a non-Eisenstein maximal ideal $\mathfrak{m}$, and denote by $\rho_{\mathfrak{p}}$ the $G_{F,S}$-representation into $\mathrm{GL}_2(\mathbf{T}_\psi(U)_{\mathfrak{m}}/\mathfrak{p})$ induced from 3.4.4. Note that this implies $\rho_{\mathfrak{p}}$ is unramified at w. Denote again by $\mathfrak{p}$ and $\mathfrak{m}$ the pullbacks of $\mathfrak{p}$ and $\mathfrak{m}$ to $\mathbf{T}_\psi(U')$.

Lemma 3.5.3. *Let $\sigma_w \in G_w$ be some lift of* Frob_w*. Let*

$$\rho_{U',\mathfrak{m}} : G_F \longrightarrow \mathrm{GL}_2(\mathbf{T}_\psi(U')_{\mathfrak{m}})$$

be as in Proposition 3.4.4, and let $y = (\operatorname{tr} \rho_{U',\mathfrak{m}}(\sigma_w))^2 - \psi(\varpi_w)(1 + \mathrm{Nm}(w))^2$.

We have $y(\ker \xi_w^\vee) = 0$. Moreover, if $p \in \mathfrak{p}$, $\mathrm{Nm}(v) \equiv 1 \pmod p$ and $\rho_{\mathfrak{p}}(\mathrm{Frob}_w)$ has distinct eigenvalues, then $y \notin \mathfrak{p}$.

Proof. Take $a \geq 1$ and an algebraic weight κ such that

- $(U(p^{a,a})(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times = 1$ for all $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$;
- the action of $U(p^{a,a}) \cap (\mathbb{A}_F^\infty)^\times$ on $W_\kappa(\mathcal{O})$ is given by ψ^{-1};
- $\mathfrak{m}$ is the pullback of a maximal ideal of $\mathbf{T}_{\kappa,\psi}(U(p^{a,a}), \mathcal{O})$ under the projection of 3.3.10.

For A an $\mathcal{O}$-module, let

$$\begin{aligned}\xi^a_{w,\kappa,A} : S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),A)^2_{\mathfrak{m}} &\longrightarrow S^{\mathrm{no}}_{\kappa,\psi}(U'(p^{a,a}),A)_{\mathfrak{m}} \\ (f,g) &\longmapsto f + \begin{pmatrix} 1 & \\ & \varpi_w \end{pmatrix} g.\end{aligned}$$

We know that $\mathrm{coker}(\xi_{w,\kappa,\overline{\mathbb{Q}}_p})$ has a basis $\{f_i\}$ consisting of eigenforms which are new at v. Letting ρ_{f_i} denote the Galois representation associated to f_i, local-global compatibility, cf [C1], implies that

$$\rho_{f_i}|_{G_w} \cong \begin{pmatrix} \epsilon_p \chi_i & * \\ & \chi_i \end{pmatrix}$$

where χ_i is an unramified character of G_w such that $\chi_i^2 = \psi|_{G_w}$. The eigenform f_i defines an $\mathcal{O}$-algebra morphism $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U'(p^{a.a}),\mathcal{O})_{\mathfrak{m}} \to \overline{\mathbb{Q}}_p$ such that, precomposing with the projection $\mathbf{T}_\psi(U') \to \mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U'(p^{a,a}),\mathcal{O})_{\mathfrak{m}}$,

$$\begin{array}{ccc} G_F & \xrightarrow{\rho_{U',\mathfrak{m}}} & \mathrm{GL}_2(\mathbf{T}_\psi(U')_{\mathfrak{m}}) \\ & \searrow^{\rho_{f_i}} & \downarrow \\ & & \mathrm{GL}_2(\overline{\mathbb{Q}}_p) \end{array}$$

commutes. In particular, the image of $(\mathrm{tr}\,\rho_{U',\mathfrak{m}}(\sigma_w))^2$ under this map is $\psi(\varpi_w)(1+\mathrm{Nm}(w))^2$. We see that $y(\mathrm{coker}(\xi^a_{w,\kappa,\overline{\mathbb{Q}}_p})) = 0$, hence $y(\mathrm{coker}(\xi^a_{w,\kappa,E})) = 0$.

If V is any $(\Sigma' \subseteq \Sigma)$-open subgroup with $(V(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times = 1$ for all $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$, we have $S_{\kappa,\psi}(V,E) \cong S_{\kappa,\psi}(V,\mathcal{O}) \otimes_{\mathcal{O}} E$ and $S_{\kappa,\psi}(V,E/\mathcal{O}) \cong S_{\kappa,\psi}(V,\mathcal{O}) \otimes_{\mathcal{O}} E/\mathcal{O}$, and so there is a natrual surjection $S_{\kappa,\psi}(V,E) \to S_{\kappa,\psi}(V,E/\mathcal{O})$. This yields a Hecke-equivariant commutative diagram

$$\begin{array}{ccccccc} S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),E)^2_{\mathfrak{m}} & \xrightarrow{\xi^a_{w,\kappa,E}} & S^{\mathrm{no}}_{\kappa,\psi}(U'(p^{a,a}),E)_{\mathfrak{m}} & \longrightarrow & \mathrm{coker}(\xi^a_{w,\kappa,E}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}),E/\mathcal{O})^2_{\mathfrak{m}} & \xrightarrow{\xi^a_{w,\kappa,E/\mathcal{O}}} & S^{\mathrm{no}}_{\kappa,\psi}(U'(p^{a,a}),E/\mathcal{O})_{\mathfrak{m}} & \longrightarrow & \mathrm{coker}(\xi^a_{w,\kappa,E/\mathcal{O}}) & \longrightarrow & 0 \end{array}$$

with exact rows. Since the first two vertical maps are surjections, so is the third and we deduce that $y(\mathrm{coker}(\xi^a_{w,\kappa,E/\mathcal{O}})) = 0$.

We then have $y(\operatorname{coker}(\varinjlim_a \xi^a_{w,\kappa,E/\mathcal{O}})) = 0$. Noting that $\xi^a_{w,\kappa,E/\mathcal{O}} = \varinjlim_{r\geq 1} \xi^a_{w,\kappa,\mathfrak{m}_{\mathcal{O}}^{-r}/\mathcal{O}}$ and using the ismorphism of 3.3.7, we see that $\xi_w = \varinjlim_a \xi^a_{w,\kappa,E/\mathcal{O}}$, in particular $y(\operatorname{coker}(\xi_w)) = 0$. By exactness of Pontryagin duality we get $y(\ker(\xi_w^\vee)) = 0$.

Assume $p \in \mathfrak{p}$, $\mathrm{Nm}(w) \equiv 1 \pmod{p}$, and $y \in \mathfrak{p}$. Then since $\rho_{U,\mathfrak{m}}$ is unramified at w and $\operatorname{tr} \rho_{U,\mathfrak{m}}(\sigma) = T_w$, our assumptions imply $T_w^2 - 4\psi(\varpi_w) \in \mathfrak{p}$. The characteristic polynomial of $\rho_{\mathfrak{p}}(\mathrm{Frob}_w)$ is $X^2 - T_w X + \psi(\mathrm{Frob}_w)\mathrm{Nm}(v) = X^2 - T_w X + \psi(\varpi_w)$ does not have distinct roots if $T_w^2 = 4\psi(\varpi_w)$ modulo $\mathfrak{p}$. □

3.5.4

Let Q be a finite set of primes of F such that for each $w \in Q$, $w \nmid p$, D is split at w, and $U_w = \mathrm{GL}_2(\mathcal{O}_{F_w})$. For each $w \in Q$, let k_w denote the residue field of F_w and fix a p-power order quotient Δ_w of $k_w^\times$. Set $\Delta_Q = \prod_{w\in Q} \Delta_w$. Define the open subgroups U' and U_{Δ_Q} of U by $U'_v = U_v$ if $v \notin Q$, and $U_w = \mathrm{Iw}(w)$ for $w \in Q$, and

$$U_{\Delta_Q} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U' : a_w d_w^{-1} \mapsto 1 \text{ in } \Delta_w \text{ for each } w \in Q \right\}$$

Lemma 3.5.5. *Let Q be as above. Let $a \geq 1$ be such that $(U(p^{a,a})(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times = 1$ for all $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$, and such that there is an algebraic weight κ with $U(p^{a,a}) \cap (\mathbb{A}_F^\infty)^\times$ acting on $W_\kappa(\mathcal{O})$ via ψ^{-1}.*

$S_\psi(U_{\Delta_Q})$ is a free $\Lambda(\mathcal{U}_p^a)[\Delta_Q]$-module and the natural surjection $S_\psi(U_{\Delta_Q}) \to S_\psi(U')$ has kernel $\mathfrak{a}_Q S_\psi(U_{\Delta_Q})$, where $\mathfrak{a}_Q$ is the Δ_Q-augmentation ideal of $\Lambda(\mathcal{U}_p^a)[\Delta_Q]$. In particular the $\Lambda(\mathcal{U}_p^a)[\Delta_Q]$-rank of $S_\psi(U_{\Delta_Q})$ is eqaul to the $\Lambda(\mathcal{U}_p^a)$-rank of $S_\psi(U')$.

Proof. Let κ and $a \geq 1$ be as in the statement of the lemma. Take $b \geq a$. Applying 3.1.8 to the groups $U'(p^{a,a}) \subseteq U'(p^{b,b})$, $U'(p^{a,a}) \subset U_{\Delta_Q}(p^{b,b})$ and $U'(p^{b,b}) \subset U_{\Delta_Q}(p^{b,b})$, we deduce that $S_{\kappa,\psi}(U'(p^{b,b}), E/\mathcal{O})^\vee$ and $S_{\kappa,\psi}(U_{\Delta_Q}(p^{b,b}), E/\mathcal{O})^\vee$ are free over $\mathcal{O}[\mathcal{U}_a/\mathcal{U}_b]$ and $\mathcal{O}[\mathcal{U}_a/\mathcal{U}_b][\Delta_Q]$, respectively and that the natural surjection

$$S_{\kappa,\psi}(U_{\Delta_Q}(p^{b,b}), E/\mathcal{O})^\vee \longrightarrow S_{\kappa,\psi}(U'(p^{b,b}), E/\mathcal{O})^\vee$$

induces an isomorphism of $S_{\kappa,\psi}(U'(p^{b,b}), E/\mathcal{O})^\vee$ with the Δ_Q coinvariants of $S_{\kappa,\psi}(U_{\Delta_Q}(p^{b,b}), E/\mathcal{O})^\vee$. Applying Hida's idempotent and passing to the limit over $b \geq a$ gives the result. □

Lemma 3.5.6. *Let $w \in Q$, and let σ_w be a generator of the p-part of the tame inertia subgroup of I_w. Note that under $I_w \to \mathcal{O}_{F_w}^\times \to k_w^\times \to \Delta_w$, given by class field theory, σ_w is mapped to a generator of Δ_w.*

If $\mathfrak{m}$ is a non-Eisenstein maximal ideal of $\mathbf{T}_\psi(U_{\Delta_Q})$, and $\rho_{U_{\Delta_Q},\mathfrak{m}}$ denotes the representation in 3.4.4, then $\operatorname{tr} \rho_{U_{\Delta_Q},\mathfrak{m}}(\sigma_w) = \delta_w + \delta_w^{-1}$.

Proof. Let $\lambda : \mathbf{T}_\psi(U_{\Delta_Q})_\mathfrak{m} \to \overline{\mathbb{Q}}_p$ denote an arithmetic point. By the definition of $(U_{\Delta_Q})_v$, the automorphic representation associated to λ via Jacquet-Langlands is not cuspidal at w. Local global compatibility then shows that $\operatorname{tr} \rho_\lambda(\sigma_w) = \lambda(\delta_w + \delta_w^{-1})$. The result now follows from Zariski density of arithmetic points and the fact that $\mathbf{T}_\psi(U_{\Delta_Q})$ is reduced. □

3.6 Galois Cohomology and Auxiliary Primes

Crucial to the patching method is the existence of so called Taylor-Wiles primes or auxiliary primes. The proof of their existence is the main result of this section.

The first section uses some of the lemmas proved in §2.7 together with a result of Pink to prove that certain non-dihedral deformations to characteristic p local fields have open image (up to finite index subfields).

Using this, in the next section we compute (almost) the cohomology of the image acting on the adjoint representation. We do this by explicit cocycle computation.

In the final section we use the result from the previous one to show the existence of auxiliary primes analogous to those in Lemma 5.10 of [KW]. As in §6 of [SW1] some care has to be taken. In particular it is not sufficient to compute the cohomology with coefficients in our local field, one must do the computations integrally. Moreover one must make sure that the size of the torsion subgroups do not depend on the auxiliary primes chosen. This is due to the fact that when performing the patching in §4, we must consider finite quotients of our universal deformation ring and Hecke modules. In order to ensure that the limits of the resulting projective systems have the correct rank, we must ensure that the alluded to torsion subgroups do not grow. It is because of this that we must be careful to control all error terms in this section.

We recall and introduce some notation and assumptions that will be used throughout this chapter. Recall $F \subset \overline{\mathbb{Q}}$ is a totally real number field and $G_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. For any extension M/F inside $\overline{\mathbb{Q}}$, let $G_M = \mathrm{Gal}(\overline{\mathbb{Q}}/M)$. Let K be a characteristic 2 local field with ring of integers A and residue field $\mathbb{F}$. Let q denote the cardinality of $\mathbb{F}$. Let $\mathfrak{m}$ denote the maximal ideal of A and let ϖ be a fixed choice of uniformizer. Fix an algebraic closure $\overline{K}$ of K.

We fix a continuous $\rho : G_F \to \mathrm{GL}_2(A)$ satisfying:

A1 ρ unramified outside a finite set of places S;

A2 ρ is not dihedral and $\mathrm{im}\rho \to \mathrm{GL}_2(\mathbb{F})$ has nontrivial kernel;

A3 $\det \rho$ is finite order;

A4 the image of ρ contains a non-trivial unipotent element.

A5 if $\overline{\rho}$ is L-dihedral there is some $\tau_0 \in G_F \smallsetminus G_L$ such that $\rho(\tau_0)$ has distinct infinite order A-rational eigenvalues.

Let V denote the free rank two A module on which G acts via ρ. Let Ad denote the space of endomorphisms of V with the adjoint action of G_F, and Z its centre. For any A-algebra R (in particular K, $\overline{K}$, $\mathbb{F}$) we set $V_R = V \otimes_A R$, $\mathrm{Ad}_R = \mathrm{Ad} \otimes_A R$, and $Z_R = Z \otimes_A R$. For $m \geq 1$ we also write, for notational convenience, Ad_m and Z_m for $\mathrm{Ad}_{A/\mathfrak{m}^m} = \mathrm{Ad}/\mathfrak{m}^m\mathrm{Ad}$ and $Z_{A/\mathfrak{m}^m} = Z/\mathfrak{m}^m Z$, respectively.

3.7 The image

The main result of this section is to establish an openness result, 3.7.2, on the image of a representation ρ satisfying our assumptions A1-A4 and then to compute the H^1 of its action on the adjoint representation. Set $\mathcal{G} = \mathrm{im}\rho$ and $\mathcal{G}^1 = \mathcal{G} \cap \mathrm{SL}_2(V)$. Note that by 2.7.4 we know that $\mathcal{G}^1$ is Zariski dense in $\mathrm{SL}_{2/K}$.

Lemma 3.7.1. *Let Γ be a Zariski dense compact subgroup of $\mathrm{SL}_2(K)$. Then there is a subfield K_0 of K with K/K_0 finite and a quaternion algebra D over K_0, split over K, such that if D^1 denotes the algebraic group over K_0 defined by the norm one elements of D, there is an isomorphism $\tilde{\varphi} : D^1 \times_{K_0} K \xrightarrow{\sim} \mathrm{SL}_{2/K}$ with $\Gamma \subseteq \tilde{\varphi}(D^1(K_0))$ and both $\tilde{\varphi}^{-1}(\Gamma)$ and $\tilde{\varphi}^{-1}([\Gamma, \Gamma])$ are open in $D^1(K_0)$.*

Proof. Note that $\tilde{\varphi}^{-1}([\Gamma, \Gamma])$ open in $D^1(K_0)$ implies that $\tilde{\varphi}^{-1}(\Gamma)$ is open in $D^1(K_0)$, so we only have to prove the former.

Applying Theorem 0.2 of [P] to the image of Γ in $\mathrm{PGL}_2(K)$, there is a subfield K_0 of K with K/K_0 finite, an absolutely simple adjoint group H over K_0, and an isogeny $\varphi : H \times_{K_0} K \to \mathrm{PGL}_{2/K}$ with nonvanishing derivative such that $\Gamma \subseteq \varphi(H(K_0))$ and the associated isogeny $\tilde{\varphi} : \tilde{H} \times_{K_0} K \to \mathrm{SL}_{2/K}$ of simply connected covers identifies $[\Gamma, \Gamma]$ with an open subgroup of $\tilde{H}(K_0)$.

Since PGL_2 does not admit nonstandard isogenies and the derivative of φ is nonzero, φ is a central isogeny. Since H is adjoint, φ and $\tilde{\varphi}$ are isomorphisms. All K_0 forms of $\mathrm{SL}_{2/K}$ are inner, so $\tilde{H}$ is the algebraic group defined by the norm one elements of some quaternion algebra D defined over K_0 that splits over K. □

Proposition 3.7.2. *Fix a basis of V and identify $\mathcal{G}^1$ with a subgroup of $\mathrm{SL}_2(A)$ using this basis. There is a subfield $K_0 \subseteq K$ such that K/K_0 is finite and and element $g \in \mathrm{PGL}_2(K)$ such that both $g\mathcal{G}^1 g^{-1}$ and $g[\mathcal{G}^1, \mathcal{G}^1]g^{-1}$ are open in $\mathrm{SL}_2(K_0)$.*

Proof. Since $\mathcal{G}^1$ is Zariski dense in $\mathrm{SL}_{2/K}$ by 2.7.4, we can apply 3.7.1. Let D and $\tilde{\varphi}$ be as in 3.7.1. By assumption A4, $\mathcal{G}^1$ contains a non-trivial unipotent element, hence D must be split over K_0. The lemma now follows upon noting that any automorphism of $\mathrm{SL}_{2/K}$ is inner, hence $\tilde{\varphi}$ is given by conjugation by an element of $\mathrm{PGL}_2(K)$. □

We finish this section with a lemma concerning the elements of Ad fixed by a compact Zariski dense subgroup of $\mathrm{SL}_2(A)$.

Corollary 3.7.3. *Let* Γ *be a Zariski dense compact subgroup of* $\mathrm{SL}_2(A)$. *Then* $(\mathrm{Ad}/Z)^\Gamma = \{0\}$.

Proof. Take $X \in \mathrm{Ad}$ such that the image of X in Ad/Z is Γ-invariant. Let $W = KX + Z_K \subset \mathrm{Ad}_K$ and let N be the kernel of $\Gamma \to \mathrm{Aut}(W)$. Then X is an endomorphism of V_K that commutes with the action of N. Since $0 \subset Z \subseteq W$ is a Γ-stable filtration of W and $\dim_K W \leq 2$, Γ/N is solvable. This together with the Zariski density of Γ implies N is Zariski dense. It follows that $X \in Z$. □

3.7.4

Let K_0 be as in Proposition 3.7.2. Denote by A_0 it's ring of integers, $\mathfrak{m}_0$ its maximal ideal and ϖ_0 a choice of uniformizer. Let e be the ramification index of K/K_0. The main goal of this section is to describe the cohomology group $H^1(\Gamma, \mathrm{Ad})$ for Γ an open subgroup of $\mathcal{G} = \mathrm{im}(\rho)$.

Ideally one would want a proposition similar to Lemma 6.9 of [SW1], which in our context would be to prove that $H^1(\Gamma, \mathrm{Ad})$ is finite (they are actually more precise and consider not only the splitting field of $\mathrm{im}(\rho)$ but also ajoin all p-power roots of unity). This is not true in our case because of the presence of the centre in Ad and the fact that if $\overline{\rho}$ is dihedral the image of ρ is pro-solvable. For example, if $\overline{\rho}$ is dihedral and L denotes the unique field from which $\overline{\rho}$ is induced (it is unique since $p = 2$), then we have an A-module of rank one inside $H^1(\mathcal{G}, Z)$ given by the surjection $\mathcal{G} \to \mathrm{Gal}(L/F)$ composed with the map sending the nontrivial element of $\mathrm{Gal}(L/F)$ to any nonzero element in Z. It seems likely that the natural map $H^1(\mathcal{G}, Z) \to H^1(\mathcal{G}, \mathrm{Ad})$, which is injective by Lemma 3.7.3, is surjective. We fall short of proving this. We can however describe the cokernel (if it exists) in enough detail for our purposes in §3.8.

Lemma 3.7.5. *Let* Γ *be an open subgroup of* $\mathcal{G}$. *The* A *rank of* $\operatorname{coker}(H^1(\Gamma, Z) \to H^1(\Gamma, \mathrm{Ad}))$ *is at most one. Moreover if it is one there is a positive integer* N_0 *depending only on* Γ *such that if* $\gamma \in H^1(\Gamma, \mathrm{Ad})$ *maps to a non-torsion element of this cokernel, there is a cocycle* $\kappa : \Gamma \to \mathrm{Ad}$ *representing* $\varpi^{N_0}\gamma$ *such that for infinitely many* $g \in \Gamma$ *with distinct* A*-rational eigenvalues* $\kappa(g) \in Z \smallsetminus \{0\}$

Proof. Fix an A-basis for V, and identify Ad with $\mathrm{M}_{2\times 2}(A)$ using this basis. Let $g \in \mathrm{PGL}_2(K)$ be as in 3.7.2. Set $\Gamma' = g\Gamma g^{-1}$ and $\mathrm{Ad}' = g\mathrm{Ad} \subset \mathrm{M}_{2\times 2}(K)$. We have a isomorphisms $H^1(\Gamma', Z) \cong H^1(\Gamma, Z)$ and $H^1(\Gamma', \mathrm{Ad}') \cong H^1(\Gamma, \mathrm{Ad})$ compatible with the maps $H^1(\Gamma, Z) \to H^1(\Gamma, \mathrm{Ad})$ and $H^1(\Gamma', Z) \to H^1(\Gamma', \mathrm{Ad}')$. And so it suffices to prove the proposition for $H^1(\Gamma', Z)$ and $H^1(\Gamma', \mathrm{Ad}')$. Since Ad' is open compact in $\mathrm{M}_{2\times 2}(K)$ there is $l \geq 0$ such that $\varpi^l \mathrm{M}_{2\times 2}(A) \subseteq \mathrm{Ad}' \subseteq \varpi^{-l}\mathrm{M}_{2\times 2}(A)$. For $k \geq 1$, set $J_k = (I + \varpi_0^k \mathrm{M}_{2\times 2}(A_0)) \cap \mathrm{SL}_2(K_0)$. By 3.7.2, Γ' contains an open subgroup of the form J_k for some k. Set $N_0 = 3ke + 4l$. We will show that for any $\gamma \in H^1(\Gamma', \mathrm{Ad}')$ there is a cocycle κ representing $\varpi^{N_0}\gamma$ such that, letting $\kappa' : \Gamma \to \mathrm{Ad}'/Z$ denote the cocycle obtained by composing κ with the projection $\mathrm{Ad}' \to \mathrm{Ad}'/Z$, the restriction $\kappa'|_{J_{3k}}$ is uniquely determined by some $b \in A$ and that

- if $b = 0$, then $\kappa|_{J_{3k}} \in Z$;
- if $b \neq 0$, then there are infinitely many diagonal $g \in J_k$ with $\kappa(g) \in Z \smallsetminus \{0\}$.

To see that this implies the lemma, let N be an open normal subgroup of Γ' contained in J_{3k}. Since N is open in Γ' it is Zariski dense in $\mathrm{SL}_{2/K}$, hence so is $g^{-1}Ng$, where $g \in \mathrm{PGL}_2(K)$ is as above. Then 3.7.3 implies $(\mathrm{Ad}'/Z)^N = (\mathrm{Ad}/Z)^{g^{-1}Ng} = \{0\}$. By inflation-restriction this implies $H^1(\Gamma', \mathrm{Ad}'/Z) \to H^1(N, \mathrm{Ad}'/Z)$ is injective. Since the map $H^1(\Gamma', \mathrm{Ad}') \to H^1(N, \mathrm{Ad}'/Z)$ factors as

$$H^1(\Gamma', \mathrm{Ad}') \longrightarrow H^1(\Gamma', \mathrm{Ad}'/Z) \longrightarrow H^1(J_{3k}, \mathrm{Ad}'/Z) \longrightarrow H^1(N, \mathrm{Ad}'/Z),$$

we have

$$\begin{aligned}\ker(H^1(\Gamma', \mathrm{Ad}') \longrightarrow H^1(J_{3k}, \mathrm{Ad}'/Z)) &= \ker(H^1(\Gamma', \mathrm{Ad}') \longrightarrow H^1(\Gamma', \mathrm{Ad}'/Z)) \\ &= H^1(\Gamma', Z).\end{aligned}$$

The remainder of the proof will comprise of some somewhat laborious cocycle computations using relations between elements in $\mathrm{SL}_2(K_0)$. We first introduce some notation. For $\alpha \in 1 + \mathfrak{m}_0^k$ and $x \in \mathfrak{m}_0^k$ set

$$t(\alpha) = \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix},\ u^+(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix},\ u^-(x) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix},$$

and define the subgroups

$$T_k = \{t(\alpha) \in J_k\},\ U_k^+ = \{u^+(x) \in J_k\},\ U_k^- = \{u^-(x) \in J_k\}.$$

Fix $\gamma \in H^1(\Gamma, \mathrm{Ad}')$, with $\kappa_1 : \Gamma \to \mathrm{Ad}'$ a cocycle representing γ. Take $t(\alpha), t(\beta) \in T_k$ and write

$$\kappa_1(t(\alpha)) = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \text{ and } \kappa_1(t(\beta)) = \begin{pmatrix} a_\beta & b_\beta \\ c_\beta & d_\beta \end{pmatrix}.$$

Then using $t(\alpha)t(\beta) = t(\beta)t(\alpha)$ and the cocycle relation, we find that

$$(\alpha^2 - 1)b_\beta = (\beta^2 - 1)b_\alpha \text{ and } (\alpha^{-2} - 1)c_\beta = (\beta^{-2} - 1)c_\alpha. \tag{3.6}$$

Now assume $\beta \in (1 + \mathfrak{m}_0^k) \smallsetminus (1 + \mathfrak{m}_0^{k+1})$. Since $\mathrm{Ad}' \subseteq \varpi^{-l}\mathrm{M}_{2\times 2}(A)$ we have $b_\beta, c_\beta \in \mathfrak{m}^{-l}$. Then, letting

$$X = \begin{pmatrix} & \frac{\varpi^{2ke+2l}}{1-\beta^2} b_\beta \\ \frac{\varpi^{2ke+2l}}{1-\beta^{-2}} c_\beta & \end{pmatrix},$$

we have $X \in \varpi^l \mathrm{M}_{2\times 2}(A) \subseteq \mathrm{Ad}'$ and we can define a cocycle $\kappa_2 : \Gamma \to \mathrm{Ad}'$ by

$$\kappa_2(g) = \varpi^{2ke+2l}\kappa_1(g) - gX + X.$$

This cocycle represents the cohomology class $\varpi^{2ke+2l}\gamma$ and $\kappa_2(t(\beta))$ is diagonal. Using (3.6) we see that

$$\kappa_2(t(\alpha)) \in \left\{ \begin{pmatrix} a & \\ & d \end{pmatrix} \right\} \tag{3.7}$$

for all $t(\alpha) \in T_k$.

Before proceeding we prove a sub-lemma.

Lemma 3.7.6. *Let $\kappa' : J_k \to \mathrm{Ad}'$ be a 1-cocyle and let $g = \begin{pmatrix} w & x \\ y & w \end{pmatrix} \in J_k$ have order two. Then*

$$\kappa'(g) = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$$

with $xc = yb$.

Proof. First note that our assumptions on g and J_k impliy w and at least one of x, y are non-zero. Since g has order two, the cocycle relation implies $g\kappa'(g) = \kappa'(g)$. This yields equations

$$w^2a + wyb + wxc + xyd = a \tag{3.8}$$

$$wxa + w^2b + x^2c + wxd = b \tag{3.9}$$

$$wya + y^2b + w^2c + wyd = c \tag{3.10}$$

$$xya + wyb + wxc + w^2d = d \tag{3.11}$$

Using $w^2 = 1 + xy$, equations (3.8) and (3.11) both become

$$xy(a + d) + w(xc + yb) = 0, \tag{3.12}$$

equation (3.9) becomes

$$wx(a+d) + x(xc+yb) = 0, \tag{3.13}$$

and equation (3.10) becomes

$$wy(a+d) + y(xc+yb) = 0. \tag{3.14}$$

If $x = 0$, then (3.12) implies $b = 0$ and (3.14) implies $a = d$. If $x \neq 0$, then (3.13) implies $xc + by = w(a+d)$. Substituting this into (3.12) we have $(w^2 + xy)(a+d) = 0$, i.e. $a = d$. Equation (3.12) then implies $xc = yb$.

□

By 3.7.6 we can write

$$\kappa_2(u^+(\varpi_0^k)) = \begin{pmatrix} a & b \\ & a \end{pmatrix}.$$

Let $u = \varpi^e \varpi_0^{-1} \in A^\times$. Since $\mathrm{Ad}' \subseteq \varpi^{-l}\mathrm{M}_{2\times 2}(A)$, $b \in \mathfrak{m}^{-l}$, so letting

$$X = \begin{pmatrix} u^k \varpi^{2l} b & \\ & 0 \end{pmatrix}.$$

we have $X \in \varpi^l \mathrm{M}_{2\times 2}(A) \subseteq \mathrm{Ad}'$. We can then define a cocycle $\kappa : \Gamma \to \mathrm{Ad}'$ by

$$\kappa(g) = \varpi^{ke+2l}\kappa_2(g) - gX + X.$$

The cocycle κ represents the cohomology class $\varpi^{3ke+4l}\gamma = \varpi^{N_0}\gamma$ and an easy computation shows $\kappa(u^+(\varpi_0^k)) \in Z$. Note that since every $t(\alpha) \in T_k$ commutes with X, $\kappa(t(\alpha)) = \varpi^{ke+2l}\kappa_2(t(\alpha))$ is diagonal for every $t(\alpha) \in T_k$ by (3.7). Applying Lemma 3.7.6 to κ we have

$$\kappa(u^+(x)) = \begin{pmatrix} a_x & b_x \\ & a_x \end{pmatrix} \quad \text{and} \quad \kappa(u^-(x)) = \begin{pmatrix} d_x & \\ c_x & d_x \end{pmatrix}, \tag{3.15}$$

with $b_{\varpi_0^k} = 0$. We will now show that the restriction of this cocycle to J_{3k} is uniquely determined modulo Z by the value $b_{\varpi_0^{k+1}}$.

First I claim that for any $\alpha \in \mathfrak{m}_0^k$, writing

$$\kappa(t(\alpha)) = \begin{pmatrix} a_\alpha & \\ & d_\alpha \end{pmatrix},$$

we have $a_\alpha = d_\alpha$, and for any $x \in \mathfrak{m}_0^k$, writing $\kappa(u^+(x))$ and $\kappa(u^-(x))$ as in (3.15), we have $b_x = c_x$. This will be shown simultaneously by setting $\alpha = 1+x$ and considering the relation $u^+(x)u^-(x) = t(\alpha)u^-(x)u^+(x)t(\alpha)$.

Let $g = u^+(x)u^-(x) = t(\alpha)u^-(x)u^+(x)t(\alpha)$. Applying the cocycle relation to $u^+(x)u^-(x)$ we have

$$\kappa(g) = \begin{pmatrix} a_x + d_x + xc_x & b_x + x^2 c_x \\ c_x & a_x + d_x + xc_x \end{pmatrix}$$

Applying the cocycle relation to $t(\alpha)u^-(x)u^+(x)t(\alpha)$ we have

$$\kappa(g) = \begin{pmatrix} a_x + d_x + xb_x + x^2(a_\alpha + d_\alpha) & \alpha^2 b_x + \alpha^2 x(a_\alpha + d_\alpha) \\ \alpha^{-2}(c_x + x^2 b_x) + x(a_\alpha + d_\alpha) & a_x + d_x + xb_x + x^2(a_\alpha + d_\alpha) \end{pmatrix}.$$

Comparing top left entries,

$$c_x = b_x + x(a_\alpha + d_\alpha). \tag{3.16}$$

Comparing top right entries and using (3.16)

$$\begin{aligned} b_x + x^2 c_x &= \alpha^2(b_x + x(a_\alpha + d_\alpha)) \\ &= \alpha^2 c_x = (1 + x^2)c_x, \end{aligned}$$

hence $b_x = c_x$. Then (3.16) gives $a_\alpha = d_\alpha$.

We use the following notation in what follows. For $\alpha \in 1 + \mathfrak{m}_0^k$ and $x \in \mathfrak{m}_0^k$ write

$$\kappa(t(\alpha)) = \begin{pmatrix} a_\alpha & \\ & a_\alpha \end{pmatrix}, \quad \kappa(u^+(x)) = \begin{pmatrix} a_x & b_x \\ & a_x \end{pmatrix}, \quad \kappa(u^-(x)) = \begin{pmatrix} d_x & \\ b_x & d_x \end{pmatrix}. \tag{3.17}$$

Since $J_k = U_k^- T_k U_k^+$, κ is uniquely determined modulo Z by the values b_x, for $x \in \mathfrak{m}_0^k$. I claim that the values b_x for $x \in \mathfrak{m}_0^{3k}$ are uniquely determined by $b_{\varpi_0^{k+1}}$.

First we establish some easy relations. For any $x \in \mathfrak{m}_0^k$, U_k^+ act trivially on $\kappa(u^+(x))$ and U_k^- acts trivially on $\kappa(u^-(x))$. Hence, for any $x, y \in \mathfrak{m}_0^k$ we have

$$b_{x+y} = b_x + b_y, \quad a_{x+y} = a_x + a_y, \quad d_{x+y} = d_x + d_y. \tag{3.18}$$

Let $\alpha \in 1 + \mathfrak{m}_0^k$ and $x \in \mathfrak{m}_0^k$. Using $u^+(\alpha^2 x) = t(\alpha)u^+(x)t(\alpha)^{-1}$ and the cocycle relation we find

$$\kappa(u^+(\alpha^2 x)) = \begin{pmatrix} a_x & \alpha^2 b_x \\ & a_x \end{pmatrix},$$

and similarly for $u^-(\alpha^2 x)$. So

$$b_{\alpha^2 x} = \alpha^2 b_x, \quad a_{\alpha^2 x} = a_x, \quad d_{\alpha^2 x} = d_x. \tag{3.19}$$

Now, if $y \in \mathfrak{m}_0^k$, setting $\alpha = 1 + y$ and applyng (3.18) and (3.19) to $u^+(y^2 x) = u^+(x)u^+(\alpha^2 x)$ we have

$$b_{y^2 x} = y^2 b_x, \quad a_{y^2 x} = 0, \quad d_{y^2 x} = 0. \tag{3.20}$$

Let $\mathbb{F}_0$ denote the resdiue field of A_0. Take $z \in \mathbb{F}_0$ and $n \geq 3k$. Let $\sqrt{z}$ denote the unique element of $\mathbb{F}_0$ such that $(\sqrt{z})^2 = z$. If k and n have the same parity then, setting $y = \sqrt{z}\varpi_0^{\frac{n-k}{2}} \in \mathfrak{m}_0^k$ in (3.20) gives

$$b_{z\varpi_0^n} = z\varpi_0^{n-k} b_{\varpi_0^k} = 0.$$

If k and n have different parity then, setting $y = \sqrt{z}\varpi_0^{\frac{n-k-1}{2}} \in \mathfrak{m}_0^k$ in (3.20) gives

$$b_{z\varpi_0^n} = z\varpi_0^{n-k-1} b_{\varpi_0^{k+1}}.$$

Combining these two equations with (3.18) we know the value of b_x for any $x \in \mathfrak{m}_0^{3k}$ of the form $x = z_{3k}\varpi_0^{3k} + \cdots z_n \varpi_0^n$ with $z_{3k}, \ldots, z_n \in \mathbb{F}_0$. For arbitrary $x \in \mathfrak{m}_0^{3k}$ the value b_x is then determined by the continuity of κ.

It remains to show that if $b_{\varpi_0^{k+1}} \neq 0$ then there are infinitely many $g \in T_k$ with $\kappa(g) \in Z \setminus \{0\}$. Take $x, y \in \mathfrak{m}_0^k$ and set $\alpha = 1 + xy$. We have the relation

$$u^+(y)u^-(x) = u^-(\alpha^{-1}x)u^+(\alpha y)t(\alpha).$$

The cocycle relation gives

$$\kappa(u^+(y)u^-(x)) = \begin{pmatrix} a_y + d_x + yb_x & b_y + y^2 b_x \\ b_x & a_y + d_x + yb_x \end{pmatrix},$$

and

$$\kappa(u^-(\alpha^{-1}x)u^+(\alpha y)t(\alpha)) = \begin{pmatrix} a_\alpha + a_{\alpha y} + d_{\alpha^{-1}x} + \alpha^{-1}xb_{\alpha y} & b_{\alpha y} \\ \alpha^{-2}x^2 b_{\alpha y} + b_{\alpha^{-1}x} & a_\alpha + a_{\alpha y} + d_{\alpha^{-1}x} + \alpha^{-1}xb_{\alpha y} \end{pmatrix},$$

which, by comparing the diagonal entries, gives

$$a_y + d_x + yb_x = a_\alpha + a_{\alpha y} + d_{\alpha^{-1}x} + \alpha^{-1}xb_{\alpha y}. \tag{3.21}$$

Using (3.18), (3.19) and (3.20) we see

$$b_{\alpha y} = b_{y+y^2x} = b_y + y^2 b_x, \quad a_{\alpha y} = a_{y+y^2x} = a_y, \quad d_{\alpha^{-1}x} = d_{\alpha x} = d_{x+y^2x} = d_x.$$

And so (3.21) becomes

$$a_\alpha = yb_x - \alpha^{-1}x(b_y + y^2 b_x)$$
$$= \alpha^{-1}(yb_x - xb_y). \tag{3.22}$$

Let n be an odd integer such that $n-k-1 \geq 3k$. Since $n-k-1$ has the same parity as k we know $b_{\varpi_0^{n-k-1}} = 0$ and so, putting $x = \varpi_0^{k+1}$ and $y = \varpi_0^{n-k-1}$ in (3.22), we have

$$a_{1+\varpi_0^n} = (1+\varpi_0^n)^{-1}\varpi_0^{n-k-1} b_{\varpi_0^{k+1}},$$

which is zero if and only if $b_{\varpi_0^{k+1}}$ is. In particular if $b_{\varpi_0^{k+1}} \neq 0$ then there are infinitely many n such that $\kappa(t(1+\varpi_0^n)) \in Z \setminus \{0\}$.

□

3.8 Auxiliary Primes

We now use the results in the previous section to prove the existence of auxiliary primes similar to Lemma 5.10 of [KW].

3.8.1

For each $n \geq 1$, let $F_n = F(\mu_{2^n})$. Let $\mathbb{Q}(\mu_{2^n})^+$ denote the maximal totally real subfield of $\mathbb{Q}(\mu_{2^n})$ and let n_0 be the largest integer $n \geq 2$ such that $\mathbb{Q}(\mu_{2^n})^+ \subseteq F$. For each $n \geq 1$ let ζ_n be a primitive 2^n-th root of unity such that $\zeta_{n+1}^2 = \zeta_n$. Let $x_n = \frac{1}{2}(\zeta_n + \zeta_n^{-1})$ and $y_n = \frac{1}{2}(x^n + 1)$. For $n > n_0$ let $\tilde{F}_n = F_n(y_{n_0}^{1/2^n})$, $\tilde{y}_n \in F^\times/(F^\times)^{2^n}$ denote the image of y_{n_0}, and $\kappa_n \in H^1(G_F, \mu_{2^n})$ be the image of $\tilde{y}_n$ by the Kummer map.

Lemma 3.8.2. *Let $n > n_0$. We have*

1. $F_{n_0}(y_{n_0}^{1/4})$ *is a dihedral extension of F of degree 8;*

2. *the extension $\tilde{F}_n/F_n$ is cyclic of degree 2^{n-1} and its cyclic subextension of degree 2 is $F_n(y_{n_0}^{1/4})$;*

3. $\kappa_n \in H^1(G_{F,S}, \mu_{2^n})$ *and its order is divisible by* 2^{n-1}*;*

4. any quadratic subextension of $\tilde{F}_n/F$ *is contained in* F_n*;*

Proof. Parts (1), (2) and (3) are proved in Lemmas 5.8 and 5.9 of [KW]. Part (4) is a consequence of the first two. Indeed, if M/F is a quadratic subextension of $\tilde{F}_n/F$ we have either $MF_n = F_n$ or $MF_n = F_n(y_{n_0}^{1/4})$, by part 2. Since MF_n/F is abelian, part (1) implies $MF_n = F_n$. □

For any $n \geq 1$ let S_n, resp. $\tilde{S}_n$, denote the places above S in F_n, resp. $\tilde{F}_n$.

Lemma 3.8.3. *For any* $n > n_0$*,*

$$\ker(H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_{\tilde{F}_n,\tilde{S}_n}, \mathrm{Ad})) = \ker(H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_{F_{n_0+1},S_{n_0+1}}, \mathrm{Ad})) \cong A^2.$$

Proof. Let $\gamma \in \ker(H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_{\tilde{F}_n,\tilde{S}_n}, \mathrm{Ad}))$ and let γ_{F_n} denote the image of γ in $H^1(G_{F_n,S_n}, \mathrm{Ad})$. Since $\tilde{F}_n/F_n$ is Galois and $\mathrm{Ad}^{G_{\tilde{F}_n}} = Z$, we see that $\gamma_{F_n} \in H^1(\mathrm{Gal}(\tilde{F}_n/F_n), Z)$ by inflation-restriction. Noting that the map $H^1(\mathrm{Gal}(\tilde{F}_n/F_n), Z) \to H^1(G_{F_n}, \mathrm{Ad})$ factors through $H^1(G_{F_n}, Z)$ and considering the commutative diagram

$$\begin{array}{ccccc}
H^1(G_{F,S}, Z) & \longrightarrow & H^1(G_{F,S}, \mathrm{Ad}) & \longrightarrow & H^1(G_{F,S}, \mathrm{Ad}/Z) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(G_{F_n,S_n}, Z) & \longrightarrow & H^1(G_{F_n,S_n}, \mathrm{Ad}) & \longrightarrow & H^1(G_{F_n,S_n}, \mathrm{Ad}/Z)
\end{array}$$

we see that γ_{F_n} maps to zero in $H^1(G_{F_n,S_n}, \mathrm{Ad}/Z)$. Hence, the image of γ in $H^1(G_{F,S}, \mathrm{Ad}/Z)$ restricts to zero in $H^1(G_{F_n,S_n}, \mathrm{Ad}/Z)$. By 3.7.3, $(\mathrm{Ad}/Z)^{G_{F_n}} = 0$, and so the inflation restriction exact sequence implies $H^1(G_{F,S}, \mathrm{Ad}/Z) \to H^1(G_{F_n,S_n}, \mathrm{Ad}/Z)$ is injective and the image of γ in $H^1(G_{F,S}, \mathrm{Ad}/Z)$ is zero, hence γ is in the image of $H^1(G_{F,S}, Z)$. Then γ is a homomorphism from $G_{F,S}^{\mathrm{ab}}$ to Z that is trivial on $G_{\tilde{F}_n,\tilde{S}_n}$. Part (4) of 3.8.2 implies that γ is trivial on G_{F_n,S_n} and we have

$$\ker(H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_{\tilde{F}_n,\tilde{S}_n}, \mathrm{Ad})) = \ker(H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_{F_n,S_n})).$$

Now note that, for any $n > n_0$, $\mathrm{Gal}(F_n/F)$ is the product of $\mathbb{Z}/2\mathbb{Z}$ and a cyclic 2-group. Then

$$\begin{aligned}\ker(H^1(G_{F,S},\mathrm{Ad}) \to H^1(G_{F_n,S_n})) &= \mathrm{Hom}(\mathrm{Gal}(F_n/F), Z)\\ &= \mathrm{Hom}(\mathrm{Gal}(F_{n_0+1}/F), Z)\\ &= \ker(H^1(G_{F,S},\mathrm{Ad}) \to H^1(G_{F_{n_0+1},S_{n_0+1}})),\end{aligned}$$

and $\mathrm{Hom}(\mathrm{Gal}(F_{n_0+1}/F), Z) \cong A^2$. □

3.8.4

We know, by 3.7.1, that $[\mathcal{G},\mathcal{G}]$ is open in $\mathcal{G}$. By 3.7.1 again, we see that the commutator subgroup of the commutator subgroup of $\mathcal{G}$, i.e $[[\mathcal{G},\mathcal{G}],[\mathcal{G},\mathcal{G}]]$, is open in $[\mathcal{G},\mathcal{G}]$, hence also in $\mathcal{G}$. Since $\tilde{F}_n/F_n$ and F_n/F are both abelian extension, it follows that $\rho|_{G_{\tilde{F}_n}}$ contains $[[\mathcal{G},\mathcal{G}],[\mathcal{G},\mathcal{G}]]$ for all $n \geq 1$. This implies that there is n_1, which we can assume is greater than n_0, and a finite extension M/F such that if F_ρ denotes the subfield of $\overline{\mathbb{Q}}$ fixed by $\ker(\rho)$, we have $F_\rho \cap \tilde{F}_n = M$ and $\rho(G_{\tilde{F}_n}) = \rho(G_M)$ for all $n \geq n_1$. Set $\Gamma = \rho(G_M)$. Since Γ is open in $\mathcal{G}$, there is, by 3.7.2, a subfield K_0 of K with K/K_0 finite and a basis for V_K such that, letting A_0 denote the ring of integers of K_0 and ϖ_0 a choice of uniformizer, Γ contains $\mathrm{SL}_2(A_0) \cap (I + \varpi_0^k M_2(A_0))$ with respect to this basis. Fix such k for the remainder of this section. Let e be the ramification index of K/K_0 and write $\mathfrak{m}_0 = \varpi_0 A_0$.

Lemma 3.8.5. *Let B be a Γ-stable subgroup of* Ad *(not necessarily an A-module). Let $m \geq 1$ be such that $B \not\subseteq \varpi^m\mathrm{Ad}$. There is a non-negative integer N_0, independent of B, such that some $X \in B$ satisfies either*

- $\mathrm{val}(\mathrm{tr}\, X) < m + N_0$ *or*
- $X = zI + Y$ *with* $\mathrm{val}(z) < m + N_0$ *and* $Y \in \varpi^{\mathrm{val}(z)+1}\mathrm{Ad}$.

Proof. Fix a basis of V and identify Ad with $\mathrm{M}_{2\times 2}(A)$ using this basis. There is some $g \in \mathrm{PGL}_2(K)$ such that $\Gamma' = g\Gamma g^{-1}$ contains $SL_2(A_0) \cap (I + \varpi_0^k \mathrm{M}_{2\times 2}(A_0))$, with k as above.

Set $\mathrm{Ad}' = g\mathrm{Ad} \subset \mathrm{M}_{2\times 2}(K)$ and $B' = gB$. Then B' is a Γ' stable subgroup of Ad'. To prove the lemma we use the fact that Γ' contains an open subgroup of $\mathrm{SL}_2(A_0)$ and so we have a very explicit description of Γ'. Although g does not preserve the filtration of $\mathrm{M}_{2\times 2}(K)$, there is some non-negative integer l such that $\varpi^l \mathrm{M}_{2\times 2}(A) \subseteq \mathrm{Ad}' \subseteq \varpi^{-l}\mathrm{M}_{2\times 2}(A)$. We set $N_0 = l + 4ke$.

Since $B \not\subseteq \varpi^m \mathrm{Ad}$, there is $X \in B$ such that, writing

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

one of a, b, c, d is not in $\mathfrak{m}^m$. If $\mathrm{val}(\mathrm{tr}\, X) < m + N_0$ we are done, so we assume otherwise. Write

$$X' = gX = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

We know that $\mathrm{val}(a' + d') = \mathrm{val}(\mathrm{tr}\, X') = \mathrm{val}(\mathrm{tr}\, X) \geq m + N_0 > m + l$. Then if both $b', c' \in \mathfrak{m}^{m+l}$ we can write $X' = a'I + Y'$ with $Y' \in \varpi^{m+l}\mathrm{M}_{2\times 2}(A) \subseteq \varpi^m \mathrm{Ad}'$. Then $X = a'I + g^{-1}Y'$ and $g^{-1}Y' \in \varpi^m \mathrm{Ad}$. Since $X \notin \varpi^m \mathrm{Ad}$ we must have $a' \notin \mathfrak{m}^m$ and we can take $z = a'$ and $Y = g^{-1}Y'$.

We are left with the case that one of b', c' does not belong to $\mathfrak{m}^{m+l}$. Assume $b \notin \mathfrak{m}^{m+l}$. Since B' is Γ' stable, B' contains

$$\begin{pmatrix} 1+\varpi_0^k & \\ & (1+\varpi_0^k)^{-1} \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} (1+\varpi_0^k)^{-1} & \\ & 1+\varpi_0^k \end{pmatrix} - \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$
$$= \begin{pmatrix} & \varpi_0^{2k} b' \\ ((1+\varpi_0^{2k})^{-1} - 1)c' & \end{pmatrix}. \qquad (3.23)$$

And so B' also contains

$$\begin{pmatrix} 1 & \\ \varpi_0^k & 1 \end{pmatrix} \begin{pmatrix} & \varpi_0^{2k} b' \\ ((1+\varpi_0^{2k})^{-1} - 1)c' & \end{pmatrix} \begin{pmatrix} 1 & \\ \varpi_0^k & 1 \end{pmatrix} - \begin{pmatrix} & \varpi_0^{2k} b' \\ ((1+\varpi_0^{2k})^{-1} - 1)c' & \end{pmatrix}$$
$$= \begin{pmatrix} \varpi_0^{3k} b' & \\ \varpi_0^{4k} b' & \varpi_0^{3k} b' \end{pmatrix},$$

as well as

$$\begin{pmatrix} (1+\varpi_0^k)^{-1} & \\ & 1+\varpi_0^k \end{pmatrix} \begin{pmatrix} \varpi_0^{3k}b' & \\ \varpi_0^{4k}b' & \varpi_0^{3k}b' \end{pmatrix} \begin{pmatrix} 1+\varpi_0^k & \\ & (1+\varpi_0^k)^{-1} \end{pmatrix} = \begin{pmatrix} \varpi_0^{3k}b' & \\ (\varpi_0^{4k}+\varpi_0^{6k})b' & \varpi_0^{3k}b' \end{pmatrix}. \qquad (3.24)$$

Then, using (3.23) again, we see B' contains

$$\begin{pmatrix} 1 & \\ \varpi_0^k+\varpi^{2k} & 1 \end{pmatrix} \begin{pmatrix} & \varpi_0^{2k}b' \\ ((1+\varpi_0^{2k})^{-1}-1)c' & \end{pmatrix} \begin{pmatrix} 1 & \\ \varpi_0^k+\varpi_0^{2k} & 1 \end{pmatrix} - \begin{pmatrix} & \varpi_0^{2k}b' \\ ((1+\varpi_0^{2k})^{-1}-1)c' & \end{pmatrix} = \begin{pmatrix} (\varpi_0^{3k}+\varpi_0^{4k})b' & \\ (\varpi_0^{4k}+\varpi_0^{6k})b' & (\varpi_0^{3k}+\varpi_0^{4k})b' \end{pmatrix}. \qquad (3.25)$$

Subtracting (3.24) from (3.25) we have that B' contains

$$\begin{pmatrix} \varpi_0^{4k}b' & \\ & \varpi_0^{4k}b' \end{pmatrix}.$$

Taking $z = \varpi_0^{4k}b'$, $g^{-1}(zI) = zI \in B$ and $\mathrm{val}(z) = \mathrm{val}(b') + 4ke < m + l + 4ke = m + N_0$. The case of $b' \in \mathfrak{m}^{m+l}$ but $c' \notin \mathfrak{m}^{m+l}$ is similar. $\square$

Lemma 3.8.6. *Let $g \in \mathrm{GL}_2(V)$ have distinct A rational eigenvalues α, β and set $w = \mathrm{val}(\alpha - \beta)$. For $z \in A$ non-zero and $Y \in \varpi^{\mathrm{val}(z)+w+1}\mathrm{Ad}$, if*

$$zI + Y \in (g-1)\mathrm{Ad} + \varpi^m \mathrm{Ad}$$

then $m \leq \mathrm{val}(z) + w$.

Proof. Let α and β denote the eigenvalues of g. First let V_α denote the α eigenspace for g, e_α a generator of V_α and e_β be any element of V mapping to a generator of V/V_α. Write $ge_\beta = \beta e_\beta + xe_\alpha$ with $x \in A$. Note a splitting of

$$0 \longrightarrow V_\alpha \to V \longrightarrow V/V_\alpha \longrightarrow 0,$$

exists if and only if $\mathrm{val}(x) \geq \mathrm{val}(\alpha+\beta)$ and so we cannot, in general, assume $x = 0$. Identify Ad with $\mathrm{M}_{2\times 2}(A)$ using the basis $\{e_\alpha, e_\beta\}$ of V.

Assume there is $X \in \mathrm{Ad}$ such that $zI + Y \in (g-1)X + \varpi^m\mathrm{Ad}$. Set $j = \min\{m, \mathrm{val}(z) + w + 1\}$. Then $zI - (g-1)X \in \varpi^j\mathrm{Ad}$. Writing

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$\begin{aligned}(g-1)X &= \begin{pmatrix} \alpha & x \\ & \beta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha^{-1} & \alpha^{-1}\beta^{-1}x \\ & \beta^{-1} \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} \alpha^{-1}xc & * \\ (\alpha^{-1}\beta - 1)c & \alpha^{-1}xc \end{pmatrix}.\end{aligned}$$

Then $zI - (g-1)X \in \varpi^j\mathrm{Ad}$ implies $(\alpha^{-1}\beta - 1)c \in \mathfrak{m}^j$, so $\mathrm{val}(c) \geq j - w$. Then the upper left entry implies $z - \alpha^{-1}xc \in \mathfrak{m}^j$, so $\mathrm{val}(z) \geq j - w$. Then $j \leq \mathrm{val}(z) + w$ implies, by definition of j, that $j = m$ and $m \leq \mathrm{val}(z) + w$. □

3.8.7

We introduce some notation that will be used for the remainder of this chapter. Let M be a finitely generated A module. We denote by M_{tor} the torsion submodule of M and by M_{free} the free A-module M/M_{tor}.

We now fix certain submodules of $H^1(G_{F,S}, \mathrm{Ad})$ that will be used in the proof of auxiliary primes below. Let F_ρ denote subfield of $\overline{\mathbb{Q}}$ fixed by $\ker(\rho)$ and $M = F_\rho \cap \tilde{F}_{n_1}$. Recall that n_1 was chosen so that $F_\rho \cap \tilde{F}_n = M$ for any $n \geq n_1$. Let

$$r = \mathrm{rk}_A H^1(G_{F,S}, \mathrm{Ad}) - \mathrm{rk}_A H^1(\mathrm{Gal}(\tilde{F}_{n_1}/F), Z) = \mathrm{rk}_A H^1(G_{F,S}, \mathrm{Ad}) - 2,$$

and

$$r_0 = \mathrm{rk}_A(\mathrm{im}(H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_M, \mathrm{Ad})) \cap H^1(\mathrm{Gal}(F_\rho/M), \mathrm{Ad})).$$

Set $\Gamma = \rho(G_M)$. We know by 3.7.5, that the A-rank of the cokernel of the map $H^1(\Gamma, Z) \to H^1(\Gamma, \mathrm{Ad})$ is at most one. For the remainder of this section we will assume that it has A-rank one and that there is some $\gamma \in H^1(G_{F,S}, \mathrm{Ad})$ whose image in $H^1(G_M, \mathrm{Ad})$ lands in $H^1(\mathrm{Gal}(F_\rho/M), \mathrm{Ad}) = H^1(\Gamma, \mathrm{Ad})$ and maps to a non-torsion element in $\mathrm{coker}(H^1(\Gamma, Z) \to H^1(\Gamma, \mathrm{Ad}))$. The case when every element of $H^1(G_{F,S}, \mathrm{Ad})$ that lands in $H^1(\Gamma, \mathrm{Ad})$ maps to a torsion element of $\mathrm{coker}(H^1(\Gamma, Z) \to H^1(\Gamma, \mathrm{Ad}))$, which includes the case when the cokernel is torsion, is easier as there is one fewer "type" of cohomology class to consider below (in particular one does not need case (b) 3.8.8) and it will be obvious to the reader how to adjust the arguments.

We fix $W_1 \subset \cdots \subset W_r$ of $H^1(G_{F,S}, \mathrm{Ad})$ such that

1. W_i is free of rank i for each i;

2. W_r intersects the image of $H^1(\mathrm{Gal}(\tilde{F}_{n_1}/F), Z)$ trivially;

3. the image W_{r_0} in $H^1(G_M, \mathrm{Ad})$ is contained in $H^1(\mathrm{Gal}(F_\rho/M), \mathrm{Ad}) = H^1(\Gamma, \mathrm{Ad})$;

4. the image of W_{r_0-1} in $H^1(G_M, \mathrm{Ad})$ is contained in $H^1(\mathrm{Gal}(F_\rho/M), Z) = H^1(\Gamma, Z)$.

We note

- for all $n \geq n_1$ the map $W_r \to \mathrm{im}(H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_{\tilde{F}_n, \tilde{S}_n}, \mathrm{Ad}))_{\mathrm{free}}$ is injective with finite cokernel by (1), (2), and 3.8.3;

- for all $n \geq n_1$ the image of W_{r_0} under $H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_{F_\rho \tilde{F}_n}, \mathrm{Ad})$ zero by (3);

- for all $n \geq n_1$ the map $W_r/W_{r_0} \to \mathrm{im}(H^1(G_{F,S}, \mathrm{Ad}) \to H^1(G_{F_\rho \tilde{F}_n}, \mathrm{Ad}))$ is injective by (2), (3) and the definition of r_0.

There are three different types of cohomology classes we will need to consider based on whether an element $\gamma \in W_r$ does not belongs to W_{r_0}, belongs to W_{r_0} but not to W_{r_0-1}, or belongs to W_{r_0-1}. The main tool for guaranteeing the existence of auxiliary primes is the following lemma.

Lemma 3.8.8. *There are non-negative integers w and N such that if $\gamma \in W_r$ and $s \geq 0$ are such that one of the following hold*

(a) $\gamma \in W_{r_0-1}$ *but* $\gamma \notin \varpi^s W_{r_0-1}$,

(b) $\gamma \in W_{r_0}$ *but* $\gamma \notin \varpi^s W_{r_0} + W_{r_0-1}$,

(c) $\gamma \notin \varpi^s W_r + W_{r_0}$,

then for any $n \geq n_1$ there is a nonempty open set $U \subset G_{\tilde{F}_n}$ such that

1. *for every $\sigma \in U$, $\rho(\sigma)$ has distinct A-rational eigenvalues α, β with* $\mathrm{val}(\alpha - \beta) \leq w$, *and*

2. *for any cocycle κ representing γ, $\sigma \in U$ and $m \geq 1$, if*

$$\varpi^j \kappa(\sigma) \in (\sigma - 1)\mathrm{Ad} + \varpi^m \mathrm{Ad}$$

then $j > m - s - N$.

Proof. We prove the three different cases (a), (b) and (c) separately in the next three sub-lemmas.

Lemma 3.8.9. *There are non-negative integer w_a and N_a such that if $\gamma \in W_r$ and $s \geq 0$ are such that*

(c) $\gamma \in W_{r_0-1}$ *but* $\gamma \notin \varpi^s W_{r_0-1}$

then for any $n \geq n_1$ there is a nonempty open set $U \subset G_{\tilde{F}_n}$ such that

1. *for every $\sigma \in U$, $\rho(\sigma)$ has distinct A-rational eigenvalues α, β with* $\mathrm{val}(\alpha - \beta) \leq w_a$, *and*

2. *for any cocycle κ representing γ, $\sigma \in U$ and $m \geq 1$, if*

$$\varpi^j \kappa(\sigma) \in (\sigma - 1)\mathrm{Ad} + \varpi^m \mathrm{Ad}$$

then $j > m - s - N_a$.

Proof. If $\overline{\rho}$ is L-dihedral, fix τ_0 as in assumption A5 made at the very beginning of the chapter. So $\tau_0 \in G_F \smallsetminus G_L$ and $\rho(\tau_0)$ has distinct infinite order A-rational eigenvalues. Set $w_a = \mathrm{val}(\mathrm{tr}\,\rho(\sigma))$.

The image of W_{r_0-1} in $H^1(G_M, \mathrm{Ad})$ is contained in $H^1(\Gamma, Z)$ and we identify it with it's image. Since $H^1(\Gamma, Z)$ is a separated A-module (in fact it is finitely generated), there is $N_1 \geq 1$ such that $\delta \in W_{r_0-1}$ with $\delta \notin \varpi W_{r_0-1}$ implies $\delta \notin \varpi^{N_1} H^1(\Gamma, Z)$. If $\overline{\rho}$ is dihedral we set $N_a = N_1 + w_a$, otherwise we set $N_a = N_1$.

Since $W_{r_0-1} \subseteq H^1(\Gamma, Z)$ there is a cocyle κ_0 representing γ such that κ_0 is given by a continuous homomorphism $G_M \to \Gamma \to Z$. Let $n \geq n_1$. We will show below that there is some $\sigma_0 \in G_{\tilde{F}_n}$ such that

(i) $\rho(\sigma_0)$ has distinct A-rational eigenvalues and $\mathrm{val}(\mathrm{tr}\,\rho(\sigma)) \leq w_a$;

(ii) $\det \rho(\sigma_0) = 1$;

(iii) $\kappa_0(\sigma_0) = zI \in Z$ with $z \notin \mathfrak{m}^{s+N_1}$.

Granting the existence of such σ_0 we can define our open set U. Let U be the non-empty open subset of $G_{\tilde{F}_n}$ consisting of elements σ such that

- $\mathrm{tr}\,\rho(\sigma) - \mathrm{tr}\,\rho(\sigma_0) \in \mathfrak{m}^{2w_a+1}$;
- $\det \rho(\sigma) = 1$;
- $\kappa_0(\sigma) - \kappa_0(\sigma_0) \in \varpi^{s+N_a}\mathrm{Ad}$.

We first show that any element of the set U satisfies the conclusion of the lemma. To check (1), let $t^2 + xt + 1 = (t - \alpha_0)(t - \beta_0)$ be the characteristic polynomial for $\rho(\sigma_0)$. Take $\sigma \in U$ and write $f(t) = t^2 + yt + 1$ for the characteristic polynomial of $\rho(\sigma)$. Since $x = \mathrm{tr}\,\rho(\sigma_0)$, $y = \mathrm{tr}\,\rho(\sigma)$ we know $y - x \in \mathfrak{m}^{2w_b+1}$. Then $f(\alpha) = (y-x)\alpha \in \mathfrak{m}^{2w_b+1} = x^2\mathfrak{m} = f'(\alpha)^2\mathfrak{m}$ implies the roots of $f(t)$ are A-rational by Hensel's Lemma, cf. [B], Chapter III, §4, n°5, Corollary 1 to Theorem 2. Denoting them by α, β, we have $\mathrm{val}(\alpha - \beta) = \mathrm{val}(y) = \mathrm{val}(x) = w_b$, and so (1) holds.

To see that (2) of the lemma is satisfied, let κ be any cocycle representing γ. Then

$$\varpi^j \kappa(\sigma) \in (\sigma - 1)\mathrm{Ad} + \varpi^m \mathrm{Ad}$$

implies

$$\varpi^j \kappa_0(\sigma) \in (\sigma - 1)\mathrm{Ad} + \varpi^m \mathrm{Ad}$$

as $\kappa(\sigma) - \kappa_0(\sigma) \in (\sigma - 1)\mathrm{Ad}$. Then, since $\kappa_0(\sigma) = zI + Y$ with $\mathrm{val}(z) < s + N_1$ and $Y \in \varpi^{\mathrm{val}(z)+w_a+1}\mathrm{Ad}$, 3.8.6 gives $j > m - s - N_a$.

We now show the existence of $\sigma_0 \in G_{\tilde{F}_n}$ satisfying (i), (ii) and (iii) above. By choice of N_1, $\gamma \notin \varpi^{s+N_1} H^1(\Gamma, Z)$ and so there is some $\tau \in G_{\tilde{F}_n}$ such that $\kappa_0(\tau) \notin \varpi^{s+N_1} Z$.

First assume that $\overline{\rho}$ is non-dihedral. Then by Dickson's classification of subgroups of $\mathrm{PGL}(\mathbb{F})$, we must have $\overline{\rho}(G_F) \cong \mathrm{SL}_2(\mathbb{F}')$ with $|\mathbb{F}'| \geq 4$. Since this group is simple the same is true of $\overline{\rho}(G_{\tilde{F}_n})$. Then since $\mathrm{Hom}(\overline{\rho}(G_{\tilde{F}_n}), Z) = 0$, we can find $\tau \in \ker(\overline{\rho}|_{G_{\tilde{F}_n}})$ such that $\kappa_0(\tau) \notin \varpi^{s+N_1} Z$. Fix some $\sigma \in G_{\tilde{F}_n}$ such that $\overline{\rho}(\sigma)$ has distinct eigenvalues and determinant 1. If $\kappa_0(\sigma) \notin \varpi^{s+N_1} Z$ then we set $\sigma_0 = \sigma$. Otherwise we set $\sigma_0 = \sigma\tau$. Note that $\overline{\rho}(\sigma_0) = \overline{\rho}(\sigma)$ satisfies (i) and (ii) and $\kappa_0(\sigma_0) = \kappa_0(\sigma) + \kappa_0(\tau) \notin \varpi^{s+N_1} Z$, so σ_0 satisfies (iii).

Now we assume $\overline{\rho}$ is dihedral and let L denote the unique quadratic extension of F such that $\overline{\rho}$ is induced from a character on G_L. We first assume that there is some $\tau \in G_{L\tilde{F}_n}$ such that $\kappa_0(\tau) \notin \varpi^{s+N_1} Z$. Then, by replacing τ by τ^j for j odd and sufficiently large, we can assume that $\tau \in \ker(\overline{\rho})$. Let σ be any element of $G_{\tilde{F}_n}$ such that $\overline{\rho}(\sigma)$ has distinct eigenvalues and determinant one. If $\kappa_0(\sigma) \notin \varpi^{s+N_1} Z$ then we set $\sigma_0 = \sigma$. Otherwise we set $\sigma_0 = \sigma\tau$. Note that $\overline{\rho}(\sigma_0) = \overline{\rho}(\sigma)$ satisfies (i) and (ii) and $\kappa_0(\sigma_0) = \kappa_0(\sigma) + \kappa_0(\tau) \notin \varpi^{s+N_1} Z$, so σ_0 satisfies (iii).

Now assume that all $\sigma \in G_{L\tilde{F}_n}$ have $\kappa_0(\sigma) \in \varpi^{s+N_1} Z$. In particular note that this implies $L \not\subseteq \tilde{F}_n$. Let τ be such that $\kappa_0(\tau) \notin \varpi^{s+N_1}$. Then τ maps to the non-trivial element of $\mathrm{Gal}(L\tilde{F}_n/\tilde{F}_n)$. Let $\tau' \in G_{\tilde{F}_n}$ be any other element mapping to the non-trivial element of $\mathrm{Gal}(L\tilde{F}_n/\tilde{F}_n)$. By assumption $\kappa(\tau'\tau^{-1}) \in \varpi^{s+N_1} Z$, so $\kappa(\tau') \notin \varpi^{s+N_a} Z$. Hence we may assume $\tau = \tau_0$, where τ_0 is the element fixed at the beginning of the proof. Then $\sigma_0 = \tau_0$ satisfies (i),(ii) and (iii) above. $\square$

Lemma 3.8.10. *There are non-negative integer w_b and N_b such that if $\gamma \in W_r$ and $s \geq 0$ are such that*

(b) $\gamma \in W_{r_0}$ but $\gamma \notin \varpi^s W_{r_0} + W_{r_0-1}$,

then for any $n \geq n_1$ there is a nonempty open set $U \subset G_{\tilde{F}_n}$ such that

1. *for every $\sigma \in U$, $\rho(\sigma)$ has distinct A-rational eigenvalues α, β with $\mathrm{val}(\alpha - \beta) \leq w_b$, and*

2. *for any cocycle κ representing γ, $\sigma \in U$ and $m \geq 1$, if*

$$\varpi^j \kappa(\sigma) \in (\sigma - 1)\mathrm{Ad} + \varpi^m \mathrm{Ad}$$

then $j > m - s - N_b$.

Proof. Fix $\delta \in H^1(\Gamma, \mathrm{Ad})$ such that δ is a generator for $\mathrm{coker}(H^1(\Gamma, Z) \to H^1(\Gamma, \mathrm{Ad}))_{\mathrm{free}}$. By 3.7.5 there is $N_0 \geq 1$ and a cocycle κ_0 representing $\varpi^{N_0}\delta$, such that there are infinitely many $g \in \Gamma$ that have distinct A-rational eigenvalues with $\kappa_0(g) \in Z$ but $\kappa_0(g) \neq 0$. As $[\Gamma, \Gamma]$ is open in Γ there is some $\sigma_0 \in G_{\tilde{F}_n}$, which we fix for the remainder of the proof, such that

(i) $\rho(\sigma_0) \in [\Gamma, \Gamma]$,

(ii) $\rho(\sigma_0)$ has distinct A-rational eigenvalues,

(iii) $\kappa_0(\sigma_0) \in Z$ but $\kappa(\sigma_0) \neq 0$.

We now set $w_b = \mathrm{val}(\mathrm{tr}\,\rho(\sigma_0))$. Fix an element γ' of W_{r_0} that maps to a generator of W_{r_0}/W_{r_0-1} and let $N_1 \geq 0$ be such that $\gamma' - \varpi^{N_1}\delta$ maps to a torsion element of $\mathrm{coker}(H^1(\Gamma, Z) \to H^1(\Gamma, \mathrm{Ad}))$. Writing $\kappa_0(\sigma_0) = zI$ we set $N_b = N_1 + \mathrm{val}(z) + w_b + 1$.

Take $n \geq n_1$ and let U be the non-empty open subset of $G_{\tilde{F}_n}$ consisting of elements σ such that

- $\mathrm{tr}\,\rho(\sigma) - \mathrm{tr}\,\rho(\sigma_0) \in \mathfrak{m}^{2w_b+1}$;

- $\rho(\sigma) \in [\Gamma, \Gamma]$;

- $\kappa_0(\sigma) - \kappa_0(\sigma_0) \in \varpi^{N_b}\mathrm{Ad}$.

We now check that every $\sigma \in U$ satisfies (1) and (2) of the Lemma.

Part (1) follows from Hensel's Lemma as in 3.8.9 (note $\rho(\sigma) \in [\Gamma, \Gamma]$ implies $\det \rho(\sigma) = 1$). Now let κ be a cocycle representing γ and let $\sigma \in U$. Let $m, j \geq 1$ and be such that

$$\varpi^j \kappa(\sigma) \in (\sigma - 1)\mathrm{Ad} + \varpi^m \mathrm{Ad}. \tag{3.26}$$

We have $\gamma = \varpi^{s'}\gamma' + \gamma''$, where γ' is as above, $\gamma'' \in W_{r_0 - 1}$ and $s' < s$. Let $l \geq 0$ be such that $\mathfrak{m}^l$ is the annihilator of $\mathrm{coker}(H^1(\Gamma, Z) \to H^1(\Gamma, \mathrm{Ad}))_{\mathrm{tor}}$. Then, by choice of l and N_1 above,

$$\varpi^l \gamma = \varpi^{s'+l}\gamma' + \varpi^l \gamma'' = \varpi^{s'+l+N_1}\delta + \delta'$$

with $\delta' \in H^1(\Gamma, Z) = \mathrm{Hom}(\Gamma^{\mathrm{ab}}, Z)$. Equation (3.26) implies

$$\varpi^{j+l+N_0}\kappa(\sigma) \in (\sigma - 1)\mathrm{Ad} + \varpi^{m+l+N_0}\mathrm{Ad}. \tag{3.27}$$

Now $\varpi^{j+l+N_0}\kappa$ represents the cohomology class $\varpi^{j+l+N_0}\gamma = \varpi^{j+s'+l+N_1+N_0}\delta + \varpi^{j+N_0}\delta'$. Then since κ_0 is a cocyle representing δ and $\delta'(\sigma) = 0$, as $\sigma \in [\Gamma, \Gamma]$, there is $X \in \mathrm{Ad}$ such that

$$\varpi^{j+l+N_0}\kappa(\sigma) = \varpi^{j+s'+l+N_1+N_0}\kappa_0(\sigma) + (\sigma - 1)X.$$

This together with (3.27) yields

$$\varpi^{j+s'+l+N_1+N_0}\kappa_0(\sigma) \in (\sigma - 1)\mathrm{Ad} + \varpi^{m+l+N_0}\mathrm{Ad}. \tag{3.28}$$

Writing $\kappa_0(\sigma) = zI + Y$ with $Y \in \varpi^{\mathrm{val}(z)+1}\mathrm{Ad}$, 3.8.6 and (3.28) imply

$$m + l + N_0 \leq j + s' + l + N_1 + N_0 + \mathrm{val}(z) + w_b$$

hence

$$j \geq m - s' - (N_1 + \mathrm{val}(z) + w_b) > m - s - N_b,$$

which is 2 of the lemma. $\square$

Lemma 3.8.11. *There are non-negative integers w_c and N_c such that if $\gamma \in W_r$ and $s \geq 0$ are such that*

(c) $\gamma \notin \varpi^s W_r + W_{r_0}$

then for any $n \geq n_1$ there is a nonempty open set $U \subset G_{\tilde{F}_n}$ such that

1. *for every $\sigma \in U$, $\rho(\sigma)$ has distinct A-rational eigenvalues α, β with $\mathrm{val}(\alpha - \beta) \leq w_c$, and*

2. *for any cocycle κ representing γ, $\sigma \in U$ and $m \geq 1$, if*

$$\varpi^j \kappa(\sigma) \in (\sigma - 1)\mathrm{Ad} + \varpi^m \mathrm{Ad}$$

then $j > m - s - N_c$.

Proof. Since, for all $n \geq n_1$, W_r/W_{r_0} injects into $H^1(G_{F_\rho \tilde{F}_n}, \mathrm{Ad})^{\mathrm{Gal}(F_\rho \tilde{F}_n/\tilde{F}_n)}$ and ρ identifies $\mathrm{Gal}(F_\rho \tilde{F}_n/\tilde{F}_n)$ with Γ we have, setting $\tilde{F}_\infty = \cup_{n \geq 1} \tilde{F}_n$, an injection

$$W/W_{r_0} \longrightarrow H^1(G_{E\tilde{F}_\infty}, \mathrm{Ad})^\Gamma = \mathrm{Hom}_\Gamma(G_{E\tilde{F}_\infty}, \mathrm{Ad}),$$

and we identify W_r/W_{r_0} with it's image under this map. Since $\mathrm{Hom}_\Gamma(G_{E\tilde{F}_\infty}, \mathrm{Ad})$ is separated, there is $N_1 \geq 0$ such that $\delta \notin \varpi W_r/W_{r_0}$ implies $\delta \notin \varpi^{N_1} H^1(G_{E\tilde{F}_\infty}, \mathrm{Ad})$. We set $N_c = N_0 + N_1$ where N_0 is as in 3.8.5. We simply take $w_c = 0$.

Since, for any two cocyles κ and κ' representing γ, $\kappa(\sigma)$ and $\kappa'(\sigma)$ differ by an element of $(\sigma - 1)\mathrm{Ad}$, it suffices to show 2 holds for one particular choice of cocycle representing γ. We will show below that there is some $\sigma_0 \in G_{\tilde{F}_\infty}$ and a cocycle κ representing γ such that

(i) $\overline{\rho}(\sigma_0)$ has distinct eigenvalues and

(ii) either $\mathrm{tr}\,\kappa(\sigma_0) \notin \mathfrak{m}^{s+N_c}$ or $\kappa(\sigma_0) = zI + Y$ with $z \notin \mathfrak{m}^{s+N_c}$ and $Y \in \varpi^{\mathrm{val}(z)+1}\mathrm{Ad}$.

Granting the existence of such a σ_0 and κ we can define our open set U to be the non-empty (assuming σ_0 exists) open subset of $G_{\tilde{F}_n}$ consisting of σ such that

- $\overline{\rho}(\sigma) = \overline{\rho}(\sigma_0)$ and

- $\kappa(\sigma) - \kappa(\sigma_0) \in \varpi^{s+N_c}\mathrm{Ad}$.

Any $\rho(\sigma)$ with $\sigma \in U$ has distinct eigenvalues mod $\mathfrak{m}$, so σ satisfies (1) by Hensel's Lemma. To see that the elements of U satisfy (2) of the lemma first consider the case that $\mathrm{tr}\,\kappa(\sigma_0) \notin \mathfrak{m}^{s+N_c}$. Then if $\sigma \in U$ we have $\mathrm{tr}\,\kappa(\sigma) \notin \mathfrak{m}^{s+N_c}$ and, since $\mathrm{tr}\,X = 0$ for any $X \in (\sigma-1)\mathrm{Ad}$, $\varpi^j\kappa(\sigma) \in (\sigma-1)\mathrm{Ad}+\varpi^m\mathrm{Ad}$ implies $\varpi^j\mathrm{tr}\,\kappa(\sigma) \geq m$ so $j > m-s-N_c$. Now assume $\kappa(\sigma_0) = zI+Y$ with $z \notin \mathfrak{m}^{s+N_c}$ and $Y \in \varpi^{\mathrm{val}(z)+1}\mathrm{Ad}$. Then $\kappa(\sigma) = zI+Y'$ with $Y' \in \varpi^{\mathrm{val}(z)+1}\mathrm{Ad}$. The eigenvalues of $\rho(\sigma)$ are distinct mod $\mathfrak{m}$ so 3.8.6 implies that, if $\varpi^j\kappa(\sigma) \in (\sigma-1)\mathrm{Ad}+\varpi^m\mathrm{Ad}$, then $m \leq j + \mathrm{val}(z)$, so $j > m - s - N_c$.

It remains to show there exists some $\sigma_0 \in G_{\tilde{F}_\infty}$ and some cocycle representing γ satisfying (i) and (ii) above. First let κ be any cocycle representing γ. Since $\tilde{F}_\infty/F$ is a pro-2 extension, there is $\sigma \in G_{\tilde{F}_\infty}$ such that $\overline{\rho}(\sigma)$ has distinct eigenvalues. Then, by Hensel's Lemma, $\rho(\sigma)$ has distinct A-rational eigenvalues and we denote them by α and β. Since α and β are distinct mod $\mathfrak{m}$ we can find an eigenbasis of V for $\rho(\sigma)$. Identify Ad with $\mathrm{M}_{2\times 2}(A)$ using this basis. Write

$$\kappa(\sigma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $\alpha/\beta - 1$ and $\beta/\alpha - 1$ are units, we can adjust κ by the coboundary

$$g \mapsto g \begin{pmatrix} & (\frac{\alpha}{\beta}-1)^{-1}b \\ (\frac{\beta}{\alpha}-1)^{-1}c & \end{pmatrix} g^{-1} - \begin{pmatrix} & (\frac{\alpha}{\beta}-1)^{-1}b \\ (\frac{\beta}{\alpha}-1)^{-1}c & \end{pmatrix},$$

and assume $b = c = 0$. If at least one of a, d is not in $\mathfrak{m}^{s+N_c}$ then we have either $\mathrm{tr}\,\kappa(\sigma) = a + d \notin \mathfrak{m}^{s+N_c}$ or

$$\kappa(\sigma) = \begin{pmatrix} a & \\ & a \end{pmatrix} + \begin{pmatrix} 0 & \\ & a+d \end{pmatrix}$$

with $a \notin \mathfrak{m}^{s+N_c}$ and $\mathrm{val}(a+d) > \mathrm{val}(a)$. In either case can then take $\sigma_0 = \sigma$.

Now assume that both $a, d \in \mathfrak{m}^{s+N_c}$, i.e. $\kappa(\sigma) \in \varpi^{s+N_a}\mathrm{Ad}$. By the choice of N_0 and s we know that the restriction of γ to $H^1(G_{E\tilde{F}_\infty}, \mathrm{Ad})^{\mathrm{Gal}(E\tilde{F}_\infty/F)} = \mathrm{Hom}_\Gamma(G_{E\tilde{F}_\infty}, \mathrm{Ad})$ does not belong to $\varpi^{s+N_1}\mathrm{Hom}_\Gamma(G_{E\tilde{F}_\infty}, \mathrm{Ad})$. So $\kappa(G_{E\tilde{F}_\infty})$ is a Γ stable subgroup of Ad which is not contained in $\varpi^{s+N_1}\mathrm{Ad}$. By 3.8.5, there is some $\tau \in G_{E\tilde{F}_\infty}$ such that either $\mathrm{tr}\,\kappa(\tau) \notin \mathfrak{m}^{s+N_0+N_1} = \mathfrak{m}^{s+N_c}$ or $\kappa(\tau) = zI + Y$ with $z \notin \mathfrak{m}^{s+N_0+N_1} = \mathfrak{m}^{s+N_c}$ and $Y \in \varpi^{\mathrm{val}(z)+1}\mathrm{Ad}$. Set $\sigma_0 = \tau\sigma$. Since $\rho(\tau\sigma) = \rho(\sigma)$, σ_0 satisfies (i). Since $\kappa(\sigma_0) = \kappa(\tau)+\kappa(\sigma)$ and $\kappa(\sigma) \in \varpi^{s+N_c}\mathrm{Ad}$, σ_0 satisfies (ii) by choice of τ. □

Setting $w = \max\{w_a, w_b, w_c\}$ and $N = \max\{N_a, N_b, N_c\}$, 3.8.8 then follows from 3.8.9, 3.8.10 and 3.8.11. □

3.8.12

We now apply 3.8.8 to find our sets of auxiliary primes. Some care has to be taken in choosing these primes. In order to make sure certain dual Selmer groups are small, we need to make sure that the image of $H^1(G_{F,S}, \mathrm{Ad})$ in the the product of the local cohomology groups is as large as possible. It is not enough to only ensure that the rank of this image is as large as possible. This is because we will need to consider Selmer groups with coefficients in Ad_m and we need to ensure that the dual Selmer groups have size q^{2m} asymptotically in m. To this end, given a cohomology class $\gamma \in H^1(G_{F,S}, \mathrm{Ad})$, we not only need to find a prime v of F such that the image of γ is non-torsion in $H^1(G_v, \mathrm{Ad})$ but we need to ensure that it does not lie too "deep" in $H^1(G_v, \mathrm{Ad})$. The way we make sure the cohomology class does not lie too "deep" in $H^1(G_v, \mathrm{Ad})$ is by using property (2) of 3.8.8. There is a complication that arises here. In order to use property (2) effectively we need to make sure the value of s in the assumption of the lemma stays bounded.

Let us elaborate here. We remarked above that we require the dual Selmer groups with coefficients in Ad_m to have size asymptotic to q^{2m}. In other words, we want the size of the these dual Selmer groups to be close to q^{2m}, where "close" is independently of m. But we also need to construct the sets Q_n of auxiliary primes for each $n \geq n_1$, and it will be important that the size of the dual Selmer group with coefficients in Ad_m is close to q^{2m} in a way that also does not depend on n. The way one usually constructs auxiliary primes, and the way we will do it here, is inductively. One first chooses a cohomology class $\gamma \in H^1(G_{F,S}, \mathrm{Ad})$ and then finds a prime v_1 that kills γ. Then we take γ_2 that lives in the dual Selmer group for the Selmer structure given by the single prime $\{v_1\}$ and find a prime $\{v_2\}$ that kills γ_2, etc. The problem is that if the value of s for which γ_2 satisfies (a), (b), or (c) of 3.8.8 depends on v_1 then the "depth" for which γ_2 lies in the local cohomology group will depend on v_1 and so potentially on n. We must be careful to avoid this in the proof of the following lemma.

Lemma 3.8.13. *There are non-negative integers w and N such that for each $n \geq n_1$ there is a set of primes Q_n of F, of cardinality $r = \mathrm{rank}_A H^1(G_{F,S}, \mathrm{Ad}) - 2$, satisfying*

1. *for each $v \in Q_n$, ρ is unramified at v and $\rho(\mathrm{Frob}_v)$ has distinct A-rational eigenvalues α_v, β_v with $\mathrm{val}(\alpha_v - \beta_v) \leq w$;*
2. *each $v \in Q_n$ splits in the extension $\tilde{F}_n/F$;*
3. *if the image of $\gamma \in W_r$ under the map*
$$W_r \longrightarrow H^1(G_{F,S}, \mathrm{Ad}) \longrightarrow \prod_{v \in Q_n} H^1(G_v, \mathrm{Ad}) \longrightarrow \prod_{v \in Q_n} H^1(G_v, \mathrm{Ad})_{\mathrm{free}}$$
lies in $\varpi^{(2^r-1)N} \prod_{v \in Q_n} H^1(G_v, \mathrm{Ad})_{\mathrm{free}}$ then $\gamma \in \varpi W_r$.

Proof. Let w and N be as in 3.8.8. Fix elements $\gamma_1, \ldots, \gamma_r \in W_r$ such that for each $1 \leq i \leq r$, $\{\gamma_1, \ldots, \gamma_i\}$ is a basis for W_i. We will inductively construct a set of primes $\{v_1, \ldots, v_i\}$, for $1 \leq i \leq r$, of F such that each v_j satisfies (1) and (2) above as well as the following:

(IND_i) if the image of $\gamma \in W_i$ under the map
$$W_i \longrightarrow H^1(G_{F,S}, \mathrm{Ad}) \longrightarrow \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad}) \longrightarrow \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}}$$
lies in $\varpi^{(2^i-1)N} \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}}$ then $\gamma \in \varpi W_i$.

Taking $Q_n = \{v_1, \ldots, v_r\}$ then establishes the lemma. In what follows, given primes $v_1, \ldots, v_i$ of F, we will denote the map
$$H^1(G_{F,S}, \mathrm{Ad}) \longrightarrow \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad}) \longrightarrow \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}}$$

by res_i.

First take $i = 1$. Then $W_1 = A\gamma_1$ and γ_1 satisfies either (a) or (b) of 3.8.8 (depending on whether $r_0 = 1$ or $r_0 > 1$) with $s = 0$. Hence there is an open subset U of $G_{\tilde{F}_n,\tilde{S}_n}$ such that

(a) for every $\sigma \in U$, $\rho(\sigma)$ has distinct A-rational eigenvalues α, β with $\mathrm{val}(\alpha - \beta) \leq w$, and

(b) for any cocycle κ representing γ_1 and $\sigma \in U$,

$$\varpi^j \kappa(\sigma) \in (\sigma - 1)\mathrm{Ad} + \varpi^m \mathrm{Ad}.$$

implies $j > m - N$.

Viewing U as a subset of $G_{F,S}$ and applying Chebotarev density we obtain a prime v_1 of F satisfying (1) and (2) of the lemma. To see that (IND_1) holds take $\gamma = a\gamma_1 \in W_1$ such that $\mathrm{res}_1(\gamma) \in \varpi^N H^1(G_{v_1}, \mathrm{Ad})_{\mathrm{free}}$. Take $l \geq 0$ such that ϖ^l annihilates $H^1(G_{v_1}, \mathrm{Ad})_{\mathrm{tor}}$. Then $\varpi^l \gamma \in \varpi^{l+N} H^1(G_{v_1}, \mathrm{Ad})$. So for any choice of cocycle κ representing γ_1, we have

$$a\varpi^l \kappa(\mathrm{Frob}_v) \in (\mathrm{Frob}_v - 1)\mathrm{Ad} + \varpi^{l+N}\mathrm{Ad},$$

and (b) implies $\mathrm{val}(a) + l > l$, i.e. $\gamma \in \varpi W_1$, which is (IND_1).

Now assume we have primes $v_1, \ldots, v_i$, with $1 \leq i < r$, of F satisfying (1) and (2) of the proposition as well as (IND_i). If there is no $\gamma \in W_{i+1} \smallsetminus \varpi W_{i+1}$ such that

$$\mathrm{res}_i(\gamma) \in \varpi^{(2^{i+1}-1)N} \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}},$$

then (IND_{i+1}) is automatically satisfied for any choice of v_{i+1}. In this case we can apply 3.8.8 to γ_{i+1} with $s = 0$, and we obtain a non-empty open set U of $G_{\tilde{F}_n,\tilde{S}_n}$ to which we can apply Chebotarev density as in the $i = 1$ case to we get a prime v_{i+1} of F satisfying (1) and (2) of the lemma.

Now assume there is some $\gamma \in W_{i+1} \smallsetminus \varpi W_{i+1}$ such that

$$\mathrm{res}_i(\gamma) \in \varpi^{(2^{i+1}-1)N} \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}}.$$

The idea is now to replace $\{\gamma_1, \ldots, \gamma_{i+1}\}$ with a basis for W_{i+1} that includes γ and to apply 3.8.8 to γ.

Write $\gamma = a_1\gamma_1 + \cdots + a_{i+1}\gamma_{i+1}$. Since $\gamma \notin \varpi W_{i+1}$ there is at least one $1 \leq j \leq i+1$ such that a_j is a unit. Let j_0 be the largest such index. Then note that $\{\gamma_1, \ldots, \gamma_{j_0-1}, \gamma, \gamma_{j_0+1}, \ldots, \gamma_{i+1}\}$ is a basis for W_{i+1}. We first show that (IND$_i$) also holds with W_i replaced by the A span of $\{\gamma_1, \ldots, \gamma_{j_0-1}, \gamma_{j_0+1}, \ldots, \gamma_{i+1}\}$. Let $\gamma' = b_1\gamma_1 + \cdots + b_{i+1}\gamma_{i+1}$ with $b_{j_0} = 0$. We claim that if

$$\mathrm{res}_i(\gamma') \in \varpi^{(2^i-1)N} \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}}$$

then $b_j \in \mathfrak{m}$ for all $1 \leq j \leq i+1$.

First assume that $\mathrm{val}(b_{i+1}) \leq \mathrm{val}(a_{i+1})$. Then $\gamma - \frac{a_{i+1}}{b_{i+1}}\gamma' \in W_i$ and when written in terms of the basis $\{\gamma_1, \ldots, \gamma_i\}$ we see that the j_0 coefficient is $a_{j_0} - \frac{a_{i+1}}{b_{i+1}}b_{j_0} = a_{j_0}$, which is a unit. Thus $\gamma - \frac{a_{i+1}}{b_{i+1}} \notin \varpi W_i$. But

$$\mathrm{res}_i\left(\gamma - \frac{a_{i+1}}{b_{i+1}}\gamma'\right) = \mathrm{res}_i(\gamma) - \frac{a_{i+1}}{b_{i+1}}\mathrm{res}_i(\gamma') \in \varpi^{(2^i-1)N} \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}}, \qquad (3.29)$$

contradicting (IND$_i$). So we must have $\mathrm{val}(b_{i+1}) > \mathrm{val}(a_{i+1})$. Then $\gamma' - \frac{b_{i+1}}{a_{i+1}}\gamma \in W_i$ and, similar to (3.29), we see that

$$\mathrm{res}_i\left(\gamma' - \frac{b_{i+1}}{a_{i+1}}\gamma\right) \in \varpi^{(2^i-1)N} \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}}.$$

By (IND$_i$) we have $\gamma' - \frac{b_{i+1}}{a_{i+1}}\gamma \in \varpi W_i$. Now $\gamma' - \frac{b_{i+1}}{a_{i+1}}\gamma = (b_1 - \frac{b_{i+1}}{a_{i+1}}a_1)\gamma_1 + \cdots (b_i - \frac{b_{i+1}}{a_{i+1}}a_i)\gamma_i$, so $b_j - \frac{b_{i+1}}{a_{i+1}}a_j \in \mathfrak{m}$ for each $1 \leq j \leq i$. But then $\mathrm{val}(b_{i+1}) > \mathrm{val}(a_{i+1})$ implies $b_j \in \mathfrak{m}$ for all $1 \leq j \leq i+1$, which is what we wanted to show.

We let $\{\delta_1, \ldots, \delta_{i+1}\} = \{\gamma_1, \ldots, \gamma_{j_0-1}, \gamma, \gamma_{j_0+1}, \ldots, \gamma_{i+1}\}$ but ordered such that $\delta_{i+1} = \gamma$. By the above claim, if

$$\mathrm{res}_i(b_1\delta_1 + \cdots + b_i\delta_i) \in \varpi^{(2^i-1)N} \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}} \text{ then } \mathrm{val}(b_j) \geq 1 \text{ for all } 1 \leq j \leq i. \qquad (3.30)$$

We wish to apply 3.8.8 to δ_{i+1} but first we need a little more information. In particular we need to know for what value of s does δ_{i+1} satisfy either (a), (b) or (c) of 3.8.8. Recall we have written $\delta_{i+1} = a_1\gamma_1 + \cdots + a_{i+1}\gamma_{i+1}$. We claim that $\mathrm{val}(a_{i+1}) < (2^i - 1)N$. Indeed if a_{i+1} is not a unit then $j_0 \le i$ and (IND_i) implies

$$\mathrm{res}_i(a_1\gamma_1 + \cdots + a_i\gamma_i) \notin \varpi^{(2^i-1)N} \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}}.$$

Then

$$\mathrm{res}_i(a_1\gamma_1 + \cdots + a_{i+1}\gamma_{i+1}) \in \varpi^{(2^i-1)N} \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}},$$

gives

$$\mathrm{res}_i(a_{i+1}\gamma_{i+1}) \notin \varpi^{(2^i-1)N} \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}},$$

hence $\mathrm{val}(a_{i+1}) < (2^i - 1)N$.

By the way $\gamma_1, \ldots, \gamma_r$ were chosen, the above claim implies δ_{i+1} satisfies one of (a), (b) or (c) of 3.8.8 for some $s = (2^i - 1)N$. Let U be the open subset of $G_{\tilde{F}_n, \tilde{S}_n}$ given by applying 3.8.8 to δ_{i+1}. Applying Chebotarev density to U we get a prime v_{i+1} of F that, as in explained in the case $i = 1$, satisfies (1) and (2) of the lemma and such that for any cocycle κ representing δ_{i+1} if

$$\varpi^j \kappa(\mathrm{Frob}_{i+1}) \in (\mathrm{Frob}_{i+1} - 1)\mathrm{Ad} + \varpi^m \mathrm{Ad}$$

then $j > m - (2^i - 1)N - N = m - 2^i N$. As explained in the $i = 1$ case, this implies that the image of δ_{i+1} in $H^1(G_{v_{i+1}}, \mathrm{Ad})_{\mathrm{free}}$ does not belong to $\varpi^{2^i N} H^1(G_{v_{i+1}}, \mathrm{Ad})_{\mathrm{free}}$.

It remains to show (IND_{i+1}) holds. Take $\delta \in W_{i+1}$ and assume that

$$\mathrm{res}_{i+1}(\delta) \in \varpi^{(2^{i+1}-1)N} \prod_{j=1}^{i+1} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}}.$$

Write $\delta = b_1\delta_1 + \cdots + b_{i+1}\delta_{i+1}$. We will show $\mathrm{val}(b_j) \ge 1$ for all $1 \le j \le i+1$. Since

$$\mathrm{res}_i(\delta_{i+1}) \in \varpi^{(2^{i+1}-1)N} \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}}$$

we have

$$\mathrm{res}_i(b_1\delta_1 + \cdots + b_i\delta_i) \in \varpi^{(2^{i+1}-1)N} \prod_{j=1}^{i} H^1(G_{v_j}, \mathrm{Ad})_{\mathrm{free}}.$$

By (3.30) we know that $\mathrm{val}(b_j) > (2^{i+1}-1)N-(2^i-1)N = 2^iN$ for each $1 \le j \le i$. But then the image of $\varpi^{(2^i-1)N}(b_1\delta_1+\cdots+b_i\delta_i)$ in $H^1(G_{v_{i+1}}, \mathrm{Ad})_{\mathrm{free}}$ lands in $\varpi^{(2^{i+1}-1)N}H^1(G_{v_{i+1}}, \mathrm{Ad})_{\mathrm{free}}$ and so $\varpi^{(2^i-1)N}b_{i+1}\delta_{i+1}$ also maps to $\varpi^{(2^{i+1}-1)N}H^1(G_{v_{i+1}}, \mathrm{Ad})_{\mathrm{free}}$. Since the image of δ_{i+1} in $H^1(G_{v_{i+1}}, \mathrm{Ad})_{\mathrm{free}}$ does not belong to $\varpi^{2^iN}H^1(G_{v_{i+1}}, \mathrm{Ad})_{\mathrm{free}}$ we must have

$$(2^i-1)N + \mathrm{val}(b_{i+1}) > (2^{i+1}-1)N - 2^iN,$$

which implies $\mathrm{val}(b_{i+1}) \ge 1$. This establishes ($\mathrm{IND}_{i+1}$). □

3.8.14

If $V \subseteq W$ are finite sets of primes in F and M is an A-module with a continuous G_F action which is unramified outside Σ, we denote by $H^1_V(G_{F,W}, M)$ the subgroup of $H^1(G_{F,W}, M)$ consisting of elements whose image in $\prod_{v\in V} H^1(G_v, M)$ under the restriction map is trivial.

We introduce some notation as in [SW1]. For each $n, m \ge 1$ let $C_{n,m}$ and $D_{n,m}$ be non-negative integers. We write

$$C_{n,m} \asymp D_{n,m}$$

if there are constants $0 < a < b$ such that

$$a < \frac{C_{n,m}}{D_{n,m}} < b$$

for all $n, m \ge 1$.

Proposition 3.8.15. *For each $n \ge n_1$ there is a set of primes Q_n of F disjoint from S and of cardinality $r = \mathrm{rank}_A H^1(G_{F,S}, \mathrm{Ad}) - 2$, such that*

1. *for $v \in Q_n$, ρ is unramified at v, $\rho(\mathrm{Frob}_v)$ has distinct A-rational eigenvalues, and $\mathrm{val}(\mathrm{tr}\,\rho(\mathrm{Frob}_v)) < w$, with w not depending on n or m;*
2. *v splits in the extension $\tilde{F}_n/F$;*
3. *for each $v \in Q_n$, $|H^0(G_v, \mathrm{Ad}_m)| \asymp q^{2m}$;*
4. *$|H^1_{Q_n}(G_{F,S\cup Q_n}, \mathrm{Ad}_m)| \asymp q^{2m}$;*

5. *letting $F^S_{Q_n}$ denote the maximal abelian extension of F of degree a power of 2 which is unramified outside Q_n and split at primes in S, $G_n = \mathrm{Gal}(F^S_{Q_n}/F)$, and $s = |S|$, we have $G_n/2^{n-2}G_n \cong (\mathbb{Z}/2^{n-2}\mathbb{Z})^t$ with $t = 2 - s + r$.*

Proof. We let Q_n be the set of primes given by 3.8.13. Then (1) and (2) of the proposition are given by (1) and (2) of 3.8.13. In particular the bound on the valuation of the trace follows from (1) of 3.8.13 since our characteristic is 2.

Take $v \in Q_n$ and let α and β denote the eigenvalues of $\rho(\mathrm{Frob}_v)$. As was noted in the proof of 3.8.6, we may not be able to find a basis of V so that $\rho(\mathrm{Frob}_v)$ is diagonal. However if we fix any basis of V, there is some $g \in \mathrm{PGL}_2(K)$ such that

$$g\rho(\mathrm{Frob}_v)g^{-1} = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}.$$

Identify Ad with $\mathrm{M}_{2\times 2}(A)$ using our fixed basis of V and set $\mathrm{Ad}' = g\mathrm{Ad} \subset \mathrm{M}_{2\times 2}(K)$. Letting Frob_v act on Ad' via $g\rho(\mathrm{Frob}_v)g^{-1}$ and setting $\mathrm{Ad}'_m = \mathrm{Ad}'/\varpi^m\mathrm{Ad}'$, we see that $H^0(G_v, \mathrm{Ad}_m) \cong H^0(G_v, \mathrm{Ad}'_m)$. There is some $l \geq 0$ such that

$$\varpi^l\mathrm{Ad} \subseteq \mathrm{Ad}' \subseteq \varpi^{-l}\mathrm{Ad}. \tag{3.31}$$

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_{2\times 2}(K)$,

$$\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha^{-1} & \\ & \beta^{-1} \end{pmatrix} = \begin{pmatrix} a & \alpha\beta^{-1}b \\ \alpha^{-1}\beta c & d \end{pmatrix}. \tag{3.32}$$

Then $\mathrm{Ad}' \supseteq \varpi^l\mathrm{Ad}$ and (3.32) imply

$$\left\{\begin{pmatrix} a & \\ & d \end{pmatrix} + \varpi^m\mathrm{Ad}' : a, d \in \mathfrak{m}^l\right\}/\varpi^m\mathrm{Ad}' \subseteq H^0(G_v, \mathrm{Ad}'_m).$$

Now $\varpi^m \mathrm{Ad}' \subseteq \varpi^{m-l}\mathrm{Ad}$ then implies

$$\left|\left\{\begin{pmatrix} a & \\ & d \end{pmatrix} + \varpi^m \mathrm{Ad}' : a, d \in \mathfrak{m}^l\right\} / \varpi^m \mathrm{Ad}'\right|$$
$$\geq \left|\left\{\begin{pmatrix} a & \\ & d \end{pmatrix} + \varpi^{m-l} \mathrm{Ad} : a, d \in \mathfrak{m}^l\right\} / \varpi^{m-l} \mathrm{Ad}\right| = q^{2m-4l},$$

hence

$$|H^0(G_v, \mathrm{Ad}_m)| = |H^0(G_v, \mathrm{Ad}'_m)| \geq q^{2m-4l}. \tag{3.33}$$

To get a lower bound, recall that by 1 of 3.8.13 there is an integer w that does not depend on v such that $\mathrm{val}(\alpha\beta^{-1} - 1) = \mathrm{val}(\alpha^{-1}\beta - 1) \leq w$. Using this, together with (3.32) and (3.31), we see that

$$H^0(G_v, \mathrm{Ad}'_m) \subseteq \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \varpi^m \mathrm{Ad}' : a, d \in \mathfrak{m}^{-l} \text{ and } b, c \in \mathfrak{m}^{m-l-w}\right\} / \varpi^m \mathrm{Ad}'.$$

Now $\varpi^m \mathrm{Ad}' \supseteq \varpi^{m+l}\mathrm{Ad}$ then implies

$$\left|\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \varpi^m \mathrm{Ad}' : a, d \in \mathfrak{m}^{-l} \text{ and } b, c \in \mathfrak{m}^{m-l-w}\right\} / \varpi^m \mathrm{Ad}'\right|$$
$$\leq \left|\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \varpi^{m+l} \mathrm{Ad} : a, d \in \mathfrak{m}^{-l} \text{ and } b, c \in \mathfrak{m}^{m-l-w}\right\} / \varpi^{m+l} \mathrm{Ad}\right| = q^{2m+8l+2w},$$

hence

$$|H^0(G_v, \mathrm{Ad}_m)| = |H^0(G_v, \mathrm{Ad}'_m)| \leq q^{2m+8l+2w}. \tag{3.34}$$

Since l and w do not depend on n or m, (3.34) and (3.33) imply (3) of the proposition.

We now check (4) of the proposition. Since ρ is unramified at each $v \in Q_n$ the injection

$$H^1_{Q_n}(G_{F,S\cup Q_n}, \mathrm{Ad}) \longrightarrow H^1(G_{F,S\cup Q_n}, \mathrm{Ad})$$

factors through $H^1(G_{F,S}, \mathrm{Ad})$. Similarly with Ad_m in place of Ad. From the exact sequence

$$0 \longrightarrow \mathrm{Ad} \xrightarrow{\varpi^m} \mathrm{Ad} \longrightarrow \mathrm{Ad}_m \longrightarrow 0 \tag{3.35}$$

we have

$$0 \longrightarrow H^1(G_{F,S}, \mathrm{Ad})/\varpi^m \longrightarrow H^1(G_{F,S}, \mathrm{Ad}_m) \longrightarrow H^2(G_{F,S}, \mathrm{Ad}).$$

Since the size of the torsion subgroups of $H^i(G_{F,S}, \mathrm{Ad})$ do not depend on n or on m we have

$$|H^1(G_{F,S}, \mathrm{Ad}_m)| \asymp |H^1(G_{F,S}, \mathrm{Ad})/\varpi^m| \asymp |(A/\mathfrak{m}^m)^{r+2}| = q^{m(r+2)}. \tag{3.36}$$

Consider our fixed submodule W_r of $H^1(G_{F,S}, \mathrm{Ad})$. Say $\gamma \in W_r$ is such that γ maps to $\varpi^m \prod_{v\in Q_n} H^1(G_v, \mathrm{Ad})$. Then γ maps to $\varpi^m \prod_{v\in Q_n} H^1(G_v, \mathrm{Ad})_{\mathrm{free}}$. Writing $\gamma = \varpi^j\gamma'$ with $\gamma' \notin \varpi W_r$, part (3) of 3.8.13 implies $j > m - (2^r - 1)N$. It follows that

$$|\mathrm{im}(W_r \longrightarrow \prod_{v\in Q_n} H^1(G_v, \mathrm{Ad})/\varpi^m)| \geq q^{r(m-(2^r-1)N)}. \tag{3.37}$$

Applying local cohomology to (3.35) we have an injection

$$0 \longrightarrow \prod_{v\in Q_n} H^1(G_v, \mathrm{Ad})/\varpi^m \longrightarrow \prod_{v\in Q_n} H^1(G_v, \mathrm{Ad}_m).$$

Combining this with (3.37) and the commutativity of

$$\begin{array}{ccccc} W_r & \hookrightarrow & H^1(G_{F,S}, \mathrm{Ad}) & \longrightarrow & \prod_{v\in Q_n} H^1(G_v, \mathrm{Ad}) \\ & & \downarrow & & \downarrow \\ & & H^1(G_{F,S}, \mathrm{Ad}_m) & \longrightarrow & \prod_{v\in Q_n} H^1(G_v, \mathrm{Ad}_m) \end{array}$$

we conclude

$$|\mathrm{im}(H^1(G_{F,S}, \mathrm{Ad}_m) \longrightarrow \prod_{v\in Q_n} H^1(G_v, \mathrm{Ad}_m))| \geq q^{r(m-(2^r-1)N)}. \tag{3.38}$$

But, since each $v \in Q_n$ splits in $\tilde{F}_n$, we also have

$$(A/\mathfrak{m}^m)^2 \cong H^1(\mathrm{Gal}(\tilde{F}_n/F), Z_m) \longrightarrow H^1(\mathrm{Gal}(\tilde{F}_n/F), (\mathrm{Ad}_m)^{G_{\tilde{F}_n}}) \longrightarrow H^1_{Q_n}(G_{F,S\cup Q_n}, \mathrm{Ad}_m),$$

so

$$|H^1_{Q_n}(G_{F,S\cup Q_n}, \mathrm{Ad}_m)| \geq q^{2m}.$$

This combined with (3.36) and (3.38) imply

$$|H^1_{Q_n}(G_{F,S\cup Q_n}, \mathrm{Ad}_m)| \asymp q^{2m},$$

which is (4) of the proposition.

It remains to show (5). Using (3) of 3.8.2, it is shown in [KW] that (5) holds with

$$t = \dim_{\mathbb{F}} H^1_{Q_n}(G_{F,S\cup Q_n}, \mathbb{F}) - s + r$$

and so we only have to show $\dim_{\mathbb{F}} H^1_{Q_n}(G_{F,S\cup Q_n}, \mathbb{F}) = 2$. Note if M is any A-module on which G_F acts trivially $H^1_{Q_n}(G_{F,S\cup Q_n}, M)$ is the group of continuous homomorphisms from $\mathrm{Gal}(F_S^{Q_n}/F)$ to M, where $F_S^{Q_n}$ is the maximal abelian Galois extension of F of exponent 2, unramified outside S and split at the primes in Q_n, hence

$$\dim_{\mathbb{F}} H^1_{Q_n}(G_{F,S\cup Q_n}, \mathbb{F}) = \mathrm{rk}_A H^1_{Q_n}(G_{F,S\cup Q_n}, Z).$$

We have a series of injections

$$H^1(\mathrm{Gal}(\tilde{F}_n/F), Z) \longrightarrow H^1_{Q_n}(G_{F,S\cup Q_n}, Z) \longrightarrow H^1_{Q_n}(G_{F,S\cup Q_n}, \mathrm{Ad}),$$

where the last inclusion comes from the fact that $H^1(G_{F,S}, Z) \to H^1(G_{F,S}, \mathrm{Ad})$ since $(\mathrm{Ad}/Z)^{G_F} = \{0\}$. Since $\mathrm{rk}_A H^1(\mathrm{Gal}(F_n/F), Z) = 2$, $\mathrm{rk}_A H^1_{Q_n}(G_{F,S\cup Q_n}, Z) \geq 2$. Part (3) of 3.8.13 implies W_r intersects $H^1_{Q_n}(G_{F,S\cup Q_n}, \mathrm{Ad})$ trivially, so $\mathrm{rk}_A H^1_{Q_n}(G_{F,S\cup Q_n}, \mathrm{Ad}) \leq \mathrm{rk}_A H^1(G_{F,S}, \mathrm{Ad}) - \mathrm{rk}_A W_r = 2$. Part (5) of the proposition now follows. □

CHAPTER 4

Promodularity

The purpose of this section is to prove a certain $R^{\mathrm{red}} = \mathbf{T}$ theorem, where R is a quotient of a universal deformation ring tensored with an Iwasawa algebra as in §2.6, and $\mathbf{T}$ is a quotient of the universal nearly ordinary Hecke algebra as in §3.3 by a minimal prime of the Iwasawa algebra.

In the first section we state assumptions on our field and residual representation, recall notation and properties of the deformation rings, Hecke algebras, and Hecke modules, and state the localized "$R = T$" theorem, cf. 4.1.8. An important technical point is 4.1.6, where we prove normality of certain localizations of our local deformation ring. This will be important in the patching argument because we will have to perform a completion after localizing at a dimension one prime. The normality implies that the completed local deformation ring is still a domain. If we did not have this, one would lose control of the minimal primes of the local deformation ring and in this case it does not seem clear how to show the completed Hecke module is supported on the whole deformation ring.

In the next section we state some reductions, introduce the auxiliary level data, and recall its relevant properties.

In the following section we perform the patching argument to prove the localized "R=T" theorem. The initial patching is carried out in a similar way as Proposition 9.3 of [KW], except that we must control "error terms" generated from the fact that our auxiliary data is associated to a dimension one primes ideal, as opposed to the maximal ideal. After performing the patching we may localize and complete the limiting objects at our fixed dimension one prime ideal. It is worth pointing out that we must perform the patching first and then the localization and completion second. This is due to the fact that the backbone of the patching argument is the pigeonhole principle, i.e. one has infinitely many finite objects and hence can extract a projective system. The remainder of the argument is then still quite similar to Proposition 9.3 of [KW], due to the fact that we can ensure the completed local deformation ring remains a domain.

In the last section we complete the proof of $R^{\mathrm{red}} = \mathbf{T}$ using the localized version together with our connectivity result 2.1.8. The argument is almost exactly the same as in Proposition 4.1 of [SW2].

Throughout this chapter we take $p = 2$.

4.1 Notation and statement of the localized $R^{\mathrm{red}} = \mathbf{T}$ theorem

4.1.1

Let $F \subset \overline{\mathbb{Q}}$ denote a totally real number field and let $G_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. We assume that $[F : \mathbb{Q}]$ is even and that for each $v|2$, F_v contains a 4-th root of unity. For each place v of F, we let $G_v = \mathrm{Gal}(\overline{F_v}/F)$. Let E be a finite extension of $\mathbb{Q}_2$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. We assume that for any $v|2$ the image of any embedding $F_v \to \overline{\mathbb{Q}}_2$ is contained in E.

Fix an absolutely irreducible continuous representation

$$\overline{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F}).$$

We assume that all egienvalues of elements of $\overline{\rho}(G_F)$ are defined over $\mathbb{F}$. We assume that for all $v|2$, $\overline{\rho}|_{G_v}$ is trivial or has order 2.

We fix a continuous character $\psi : F^\times \backslash (\mathbb{A}_F^\infty)^\times \to \mathcal{O}^\times$ such that

- ψ is totally even and unramified outside $\{v|2\}$;
- on some open subgroup of $(\mathbb{A}_F^\infty)^\times$, $\psi(z) = \mathrm{Nm}_{F/\mathbb{Q}}(z_2)^{1-w}$ for some $w \in \mathbb{Z}$;
- $\overline{\psi\epsilon_2} = \det \overline{\rho}$.

Fix a finite set of finite places Σ of F of even cardinality not containing any places above p. For each $v \in \Sigma$ we fix unramified characters $\gamma_v : \mathrm{Gal}(\overline{F_v}/F_v) \to \mathcal{O}^\times$, and we assume

- for each $\in \Sigma$, $\overline{\rho}|_{G_v} \cong \begin{pmatrix} \overline{\gamma_v \epsilon_2} & * \\ & \overline{\gamma_v} \end{pmatrix}$;

- for each $v \in \Sigma$, $\gamma_v^2 = \psi|_{G_v}$;

- $\overline{\rho}$ is unramified outside of $\Sigma \cup \{v|2\} \cup \{v|\infty\}$.

We fix a finite place v_0 of F disjoint from $\Sigma \cup \{v|2\} \cup \{v|\infty\}$. This place will be used to ensure a certain neatness property below.

4.1.2

Let D denote the quaternion algebra with centre F, ramified at all archimedean places as well as all the places in Σ. Fix a maximal order $\mathcal{O}_D$ of D and an algebraic weight $\kappa = (\mathbf{k}, \mathbf{w})$ for F. Let U be the open subgroup of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ given by

- $U_v = \mathrm{Iw}_1(v)$ for $v|2$;

- $U_v = D_v^\times$ for $v \in \Sigma$;

- $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ for v not above 2 and not in Σ.

We choose an open subgroup U_0 of U by letting $(U_0)_v = U_v$ for $v \neq v_0$ and letting $(U_0)_{v_0} = \mathrm{Iw}_1(v_0^n)$ with n sufficiently large so that $(U_0(\mathbb{A}_F^\infty)^\times \cap t^{-1}Dt)/F^\times = 1$ for every $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$, cf. 3.1.5.

We let U act on $W_\kappa(\mathcal{O})$ as in §3.1. In particular for $v \in \Sigma$, $D_v^\times$ acts on $W_\kappa(\mathcal{O})$ as $\gamma_v^{-1} \circ \nu_D$, where ν_D is the reduced norm of D. We assume that $U \cap (\mathbb{A}_F^\infty)^\times$ acts on $W_\kappa(\mathcal{O})$ via ψ^{-1}, and let $S_{\kappa,\psi}^{\mathrm{no}}(U, \mathcal{O})$ denote the corresponding nearly ordinary space of quaternionic modular forms, cf. §3.1 and §3.2. We let $\mathbf{T}_{\kappa,\psi}^{\mathrm{no}}(U, \mathcal{O})$ denote the nearly ordinary Hecke algebra as in §3.2, and $\mathbf{T}_\psi(U)$ the universal nearly ordinary Hecke algebra as in §3.3. We also let $S_\psi(U)$ be the universal family of nearly ordinary modular forms as in §3.3, i.e. $S_\psi(U) = (\varinjlim_a S_{2,\psi}(U(p^{a,a}), E/\mathcal{O}))^\vee$. We have similar algebras and modules for U_0 in place of U.

Say we have a finite set of places Q of F disjoint from S_0. Note that $\mathrm{Nm}(v) \equiv 1 \pmod 2$ for each $v \in Q$. For each $v \in Q$ let Δ_v be the maximal 2-power quotient of $k_v^\times$. We then define U_Q to be the open subgroup of U_0 given by $(U_Q)_v = (U_0)_v$ if $v \notin Q$ and for $v \in Q$

$$(U_Q)_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Iw}(v) : ad^{-1} \mapsto 1 \in \Delta_v \right\}.$$

We then define $S^{\mathrm{no}}_{\kappa,\psi}(U_Q, \mathcal{O})$, $\mathbf{T}_{\kappa,\psi}(U_Q, \mathcal{O})$, $\mathbf{T}_\psi(U_Q)$, and $S_\psi(U_Q)$ as before. Recall that for V any of U, U_0, or U_Q, that $\mathbf{T}_\psi(V)$ is a $\Lambda(\mathcal{U}_2^1)$-algebra and the natural maps between them are $\Lambda(\mathcal{U}_2^1)$-algebra morphisms, where $\Lambda(\mathcal{U}_2^1) = \mathcal{O}[[\mathcal{U}_2^1]]$ with $\mathcal{U}_2^1 = \ker(\prod_{v|2} \mathcal{O}_{F_v}^\times \to \prod_{v|2}(\mathcal{O}_{F_v}/\varpi_v \mathcal{O}_{F_v})^\times)$.

We assume that there is some eigenform $f \in S^{\mathrm{no}}_{\kappa,\psi}(U, \mathcal{O})$ such that $\overline{\rho_f} \cong \overline{\rho}$, with ρ_f the Galois representation as in §3.4. Denote by $\mathfrak{m}$ the corresponding maximal ideal of $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U, \mathcal{O})$ and again denote by $\mathfrak{m}$ its pullback to any of $\mathbf{T}_\psi(U)$, $\mathbf{T}_\psi(U_0)$, $\mathbf{T}_\psi(U_Q)$.

Recall we have $\Lambda(I_v) = \mathcal{O}[[I_v^{\mathrm{ab}}(2)]]$, where $I_v^{\mathrm{ab}}(2)$ is the maximal pro-2 quotient of the abelianization of the inertia group at v. We set $\Lambda(I_2) = \hat{\otimes}_{v|2} \Lambda(I_v)$. Local class field theory gives an isomorphism $\Lambda(I_2) \cong \Lambda(\mathcal{U}_2^1)$. Also recall $\Lambda(G_2) = \hat{\otimes}_{v|2} \Lambda(G_v)$, where $\Lambda(G_v) = \mathcal{O}[[G_v^{\mathrm{ab}}(p)]]$. For V any of U, U_0, or U_Q as above, the $\Lambda(\mathcal{U}_2^1)$-algebra structure on $\mathbf{T}_\psi(U)_\mathfrak{m}$ extends to a $\Lambda(G_2)$-algebra structure, cf. 3.4.6.

Let μ denote the torsion subgroup of $\mathcal{U}_2^1$. Since E contains all embedding $F_v \to \overline{\mathbb{Q}}_2$ for all $v|2$, the minimal primes of $\Lambda(\mathcal{U}_2^1)$ and $\Lambda(G_2)$ are in one to one correspondence with $\mathcal{O}^\times$ valued characters of μ. We let η be the character of μ given by our fixed eigenform f above, and denote by $\mathfrak{q}_\eta$ the corresponding minimal primes of $\Lambda(\mathcal{U}_2^1)$ and $\Lambda(G_2)$. Set $\Lambda(\mathcal{U}_2^1, \eta) = \Lambda(\mathcal{U}_2^1)/\mathfrak{q}_\eta$ and $\Lambda(G_2, \eta) = \Lambda(G_2)/\mathfrak{q}_\eta$. Note that $\Lambda(\mathcal{U}_2^1, \eta)$ and $\Lambda(G_2, \eta)$ are isomorphic to power series rings in $[F : \mathbb{Q}]$ and $|\{v|2\}| + [F : \mathbb{Q}]$ variables, respectively.

4.1.3

We now specify some finite places at F at which our deformation problem will be unramified. This may seem redundant but is important for two reasons. The first is that below we will chose a dimension one characteristic 2 prime of the Hecke algebra and it will be important that (after a choice of framing) the local deformation ring surjects onto the Hecke algebra modulo the prime ideal in order to compare tangent spaces of the localized local and global deformation rings. The second is to guarantee the freeness of a certain group action, cf. 2.6.8, which is necessary for the 2-adic patching method. To these ends first choose a finite set of places $\{v_1, \ldots, v_k\}$ disjoint from $\Sigma \cup \{v|2\} \cup \{v|\infty\} \cup \{v_0\}$ such that $\mathbf{T}_\psi(U,\eta)_\mathfrak{m} \cong \Lambda(\mathcal{U}_2^1)[T_{v_1}, \ldots, T_{v_k}][T_{\varpi_v}]_{v|2}$, cf. 3.4.5. By enlarging $\{v_1, \ldots, v_k\}$ if necessary we can assume that if $\overline{\rho}$ is dihedral, and L denote the unique quadratic field for which $\overline{\rho}|_{G_L}$ is abelian, there is some $v_i \in \{v_1, \ldots, v_k\}$ that is inert in L. We set $S_{\mathrm{ur}} = \{v_1, \ldots, v_k\}$ and

$$S = \Sigma \cup \{v|2\} \cup \{v|\infty\} \cup S_{\mathrm{ur}}$$

We then set $S_0 = S \cup \{v_0\}$.

4.1.4

Let Q be a (possibly empty) set of places of F disjoint from S. We let $R^{\psi}_{F,S\cup Q}$ denote the universal deformation ring for $G_{F,S\cup Q}$-deformations of $\overline{\rho}$ with determinant $\psi\epsilon_2$. We let $R^{\square,\psi}_{F,S\cup Q}$ denote the universal framed deformation ring for framed $G_{F,S\cup Q}$-deformations of $\overline{\rho}$ with determinant $\psi\epsilon_2$ and frames at places in S, cf. §2.6. We let $R^{\square}_{F,S\cup Q}$ denote the universal framed deformation ring for $G_{F,S\cup Q}$-deformations of $\overline{\rho}$ with frames at S, and with determinant $\psi\epsilon_2|_{G_{F_v}}$ for each $v \in S$ but not fixed globally, c.f. §2.6. We similarly define $R^{\psi}_{F,S_0\cup Q}$, $R^{\square,\psi}_{F,S_0\cup Q}$, and $R^{\square}_{F,S_0\cup Q}$.

For each $v \in S_0$, we let $R^{\square,\psi}_v$ denote the universal lifting ring for lifts of $\overline{\rho}|_{G_v}$ with determinant $\psi\epsilon_2$. Set $R^{\square,\psi}_S = \hat{\otimes}_{v\in S} R^{\square,\psi}_v$ and $R^{\square,\psi}_{S_0} = \hat{\otimes}_{v\in S_0} R^{\square,\psi}_v$. For each $v \in S_0$ we define quotients of $\overline{R}^{\square,\psi}_v$ of $R^{\square,\psi}_v$ if $v \nmid p$ and of $R^{\square,\psi}_v \hat{\otimes} \Lambda(G_v, \eta_v)$ for $v|2$, as follows.

- For $v|2$, $\overline{R}_v^{\square,\psi} = R^{\triangle,\psi}_{\Lambda(G_v,\eta_v)}$ with $R^{\triangle,\psi}_{\Lambda(G_v,\eta_v)}$ as in 2.4.3. Since we are assuming $\overline{\rho}|_{G_v}$ is either trivial or has order 2 image, and that F_v contains a 4-th root of unity, we have
  - By 2.4.17, $\overline{R}_v^{\square,\psi}$ is a domain of relative $\mathcal{O}$-dimension $3+2[F_v:\mathbb{Q}_p]$ and if $x = (\rho_x,\chi_x)$ is a closed point of $\Lambda(G_v,\eta_v)$ such that $\chi_x^2 \neq \psi$ or $\psi\epsilon_2$, then $\overline{R}_v^{\square,\psi}$ is formally smooth over E at x.
  - By 2.4.11, if Z_v denotes the closed subscheme of $\operatorname{Spec}\Lambda(G_v,\eta_v)$ defined by $(\chi^{\text{univ}}_{\eta_v})^2 = \psi\epsilon_2$, and V_v denotes its complement,
  $$(\operatorname{Spec}\overline{R}_v^{\square,\psi} \times_{\operatorname{Spec}\Lambda(G_v,\eta_v)} V_v)\otimes_{\mathcal{O}}\mathbb{F}$$
  is integral.
- For $v\in\Sigma$, $\overline{R}_v^{\square,\psi} = R_v^{\square,\gamma\text{-st}}$ as in 2.5.4. It is a domain of relative $\mathcal{O}$-dimension 3, $\overline{R}_v^{\square,\psi}[1/p]$ is formally smooth over E, and $\overline{R}_v^{\square,\psi}\otimes_{\mathcal{O}}\mathbb{F}$ is a domain.
- For $v|\infty$, $\overline{R}_v^{\square,\psi} = R_v^{\square,-1}$ as in 2.5.6. It is a domain of relative $\mathcal{O}$-dimension 2, $\overline{R}_v^{\square,\psi}[1/p]$ is formally smooth over E, and $\overline{R}_v^{\square,\psi}\otimes_{\mathcal{O}}\mathbb{F}$ is a domain.
- For $v\in S_{\text{ur}}$, $\overline{R}_v^{\square,\psi} = R_v^{\square,\psi,\text{ur}}$ as in (2) of 2.5.2. It is formally smooth over $\mathcal{O}$ of relative dimension 3.
- $\overline{R}_{v_0}^{\square,\psi}$ is as in 2.5.2. It is $\mathcal{O}$-flat of relative dimension 3 and there is a minimal prime $\mathfrak{q}_{\text{ur}}$ such that a lift factors through $\overline{R}_{v_0}^{\square,\psi}/\mathfrak{q}_{\text{ur}}$ if and only if it is unramified. The quotient $\overline{R}_{v_0}^{\square,\psi}/\mathfrak{q}_{\text{ur}}$ is formally smooth over $\mathcal{O}$.

We set $\overline{R}_S^{\square,\psi} = \hat{\otimes}_{v\in S}\overline{R}_v^{\square,\psi}$ and $\overline{R}_{S_0}^{\square,\psi} = \hat{\otimes}_{v\in S_0}\overline{R}_v^{\square,\psi}$

Lemma 4.1.5. *$\overline{R}_{S_0}^{\square,\psi}$ is $\mathcal{O}$-flat of relative dimension $3|S_0|+[F:\mathbb{Q}_p]$. Any minimal prime of $\overline{R}_{S_0}^{\square,\psi}$ is of the form $\mathfrak{q}\overline{R}_{S_0}^{\square,\psi}$ with $\mathfrak{q}$ a minimal prime of $\overline{R}_{v_0}^{\square,\psi}$.*

Proof. By 2.4.11, 2.5.2, 2.5.4, and 2.5.6, and (3) of 2.1.4, $\overline{R}_{S_0}^{\square,\psi}$ is $\mathcal{O}$-flat of relative dimension
$$\sum_{v|2} 3+2[F_v:\mathbb{Q}_p] + \sum_{v|\infty} 2 \ + \sum_{v\in\Sigma\cup S_{\text{ur}}\cup\{v_0\}} 3 = 3|S_0|+[F:\mathbb{Q}_p],$$
and the claim regarding minimal primes follows from part (4) of 2.1.4. □

In particular, for $\mathfrak{q}_{\mathrm{ur}}$ the minimal prime of $R_{v_0}^{\square,\psi}$ corresponding to unramified lifts, 4.1.5 implies that $\mathfrak{q}_{\mathrm{ur}}$ generates a minimal prime of $\overline{R}_{S_0}^{\square,\psi}$ which we again denote by $\mathfrak{q}_{\mathrm{ur}}$.

Let $\chi_{\eta_v}^{\mathrm{univ}} : G_v \to \Lambda(G_v, \eta_v)^\times$ denote the universal $\Lambda(G_v, \eta_v)$-valued character.

Lemma 4.1.6. *Let Z_1 denote the closed subscheme of $\operatorname{Spec}\Lambda(G_2, \eta)$ defined by $(\chi_{\eta_v}^{\mathrm{univ}})^2 = \psi$ for some $v|2$, and Z_2 the closed subscheme defined by $(\chi_{\eta_v}^{\mathrm{univ}})^2 = \psi\epsilon_2$ for some $v|2$. Let U denote the complement of $Z_1 \cup Z_2$, and let $f : \operatorname{Spec}\overline{R}_{S_0}^{\square,\psi}/\mathfrak{q}_{\mathrm{ur}} \to \operatorname{Spec}\Lambda(G_2, \eta)$ denote the natural morphism. If $\mathfrak{p} \in f^{-1}(U)$, then $(\overline{R}_{S_0}^{\square,\psi}/\mathfrak{q}_{\mathrm{ur}})_{\mathfrak{p}}$ is normal.*

Proof. For ease of notation we set $R = \overline{R}_{S_0}^{\square,\psi}/\mathfrak{q}_{\mathrm{ur}}$, $R_v = \overline{R}_v^{\square,\psi}$ for $v \in S$ and $R_{v_0} = R_{v_0}^{\square,\psi}/\mathfrak{q}_{\mathrm{ur}}$. Note that $R \cong \hat{\otimes}_{v\in S_0} R_v$. We check Serre's conditions (S_2) and (R_1). Since $f^{-1}(U)$ is open it suffices to check

(a) for any $\mathfrak{p} \in f^{-1}(U)$, we have $\operatorname{depth} R_{\mathfrak{p}} \geq \min\{i, \operatorname{ht}\mathfrak{p}\}$ and

(b) for any $\mathfrak{p} \in f^{-1}(U)$ with $\operatorname{ht}\mathfrak{p} = 1$, $R_{\mathfrak{p}}$ is regular.

Take $\mathfrak{p} \in f^{-1}(U)$. First assume that $2 \notin \mathfrak{p}$. Since $R[1/2]$ is Jacobson, we can find a finite extension E'/E and an E' valued point $x : R[1/2] \to E'$ whose kernel lies in $f^{-1}(U)$ and contains $\mathfrak{p}$. Write $x = (x_v)$ for $x_v : R_v \to E'$, for each $v \in S_0$. Part (2) of 2.5.2, 2.5.4, and 2.5.6, imply that R_v is smooth over E for each $v \in S_0$ with $v \nmid 2$. For $v|2$, since the kernel of x lies in $f^{-1}(U)$, 2.4.17 implies that R_v is smooth over E at x_v. Part (1) of 2.1.4 shows that $R[1/2]$ is smooth over E at x. Since $\mathfrak{p}$ is contained in the kernel of x, $\mathfrak{p}$ is regular. This establishes both (a) and (b) for primes in $f^{-1}(U)[1/2]$.

Now take $\mathfrak{p} \in f^{-1}(U)$ with $2 \in \mathfrak{p}$. Let ϖ_E denote a uniformizer of E. Let N denote the nilradical of $R/\varpi_E R$ and, for each $v \in S_0$, N_v denote the nilradical of R_v. Part (2) of 2.5.2, 2.5.4, and 2.5.6, imply that $N_v = 0$ for each $v \in S_0$ with $v \nmid 2$. For $v|2$, 2.4.11 implies that the support of N_v is contained in the closed subscheme of $\Lambda(G_v, \eta_v)$ defined by $(\chi_{\eta_v}^{\mathrm{univ}}) = \psi\epsilon_2$. Then part (4) of 2.1.4 implies that the support of N is contained in $f^{-1}(Z_2)$, and so $R_{\mathfrak{p}}/\varpi_E R_{\mathfrak{p}} \cong (R/\varpi_E R)_{\mathfrak{p}}$ is reduced. If $\operatorname{ht}\mathfrak{p} = 1$, then $R_{\mathfrak{p}}/\varpi_E R_{\mathfrak{p}}$ is a field, and $\mathfrak{p}$ is regular. If $\mathrm{ht}\mathfrak{p} > 1$, then $R_{\mathfrak{p}}/\varpi_E R_{\mathfrak{p}}$ reduced and of dimension ≥ 1 implies the existence of a non-zero divisor in its maximal ideal, hence $\mathrm{depth} R_{\mathfrak{p}} \geq 2$. $\square$

As in 2.6.3, we define quotients

- $R^{\square,\psi}_{F,S\cup Q}\hat{\otimes}\Lambda(G_2,\eta) \to \overline{R}^{\square,\psi}_{F,S\cup Q}$,
- $R^{\square,\psi}_{F,S_0\cup Q}\hat{\otimes}\Lambda(G_2,\eta) \to \overline{R}^{\square,\psi}_{F,S_0\cup Q}$, and
- $R^{\square}_{F,S_0\cup Q}\hat{\otimes}\Lambda(G_2,\eta) \to \overline{R}^{\square}_{F,S_0\cup Q}$

by letting

- $\overline{R}^{\square,\psi}_{F,S\cup Q} = \overline{R}^{\square,\psi}_{S} \otimes_{R^{\square,\psi}_{S}} R^{\square,\psi}_{F,S\cup Q}$,
- $\overline{R}^{\square,\psi}_{F,S_0\cup Q} = \overline{R}^{\square,\psi}_{S_0} \otimes_{R^{\square,\psi}_{S_0}} R^{\square,\psi}_{F,S_0\cup Q}$, and
- $\overline{R}^{\square}_{F,S_0\cup Q} = \overline{R}^{\square,\psi}_{S_0} \otimes_{R^{\square,\psi}_{S_0}} R^{\square}_{F,S_0\cup Q}$.

We define a quotient $R^{\psi}_{F,S\cup Q}\hat{\otimes}\Lambda(G_2,\eta) \to \overline{R}^{\psi}_{F,S\cup Q}$ by letting $\overline{R}^{\psi}_{F,S\cup Q}$ denote the image of $R^{\psi}_{F,S\cup Q}\hat{\otimes}\Lambda(G_2,\eta)$ under the natural map

$$R^{\psi}_{F,S\cup Q}\hat{\otimes}\Lambda(G_2,\eta) \to R^{\square,\psi}_{F,S\cup Q} \to \overline{R}^{\square,\psi}_{F,S\cup Q},$$

and similarly with S_0 in place of S. Note that if E'/E is finite with ring of integers $\mathcal{O}_{E'}$, a local $\mathcal{O}$-algebra morphism $\overline{R}^{\square,\psi}_{F,S_0} \to \mathcal{O}_{E'}$ has kernel lying over $\mathfrak{q}_{\mathrm{ur}} \in \operatorname{Spec} R^{\square,\psi}_{S_0}$ if and only if the induced morphism $\overline{R}^{\psi}_{F,S_0} \to \mathcal{O}_{E'}$ factors through $\overline{R}^{\psi}_{F,S}$.

By 3.4.6, the existence of f in 4.1.2 yields surjective morphisms $\overline{R}^{\psi}_{F,S} \to \mathbf{T}_{\psi}(U,\eta)_{\mathfrak{m}}$, $\overline{R}^{\psi}_{F,S_0} \to \mathbf{T}_{\psi}(U_0,\eta)_{\mathfrak{m}}$, and $\overline{R}^{\psi}_{F,S_0\cup Q} \to \mathbf{T}_{\psi}(U_Q,\eta)_{\mathfrak{m}}$. These are all morphisms of $\Lambda(G_2,\eta)$-algebras, and the natural diagrams all commute.

If R is either of $\overline{R}^{\psi}_{F,S\cup Q}$ or $\overline{R}^{\psi}_{F,S_0\cup Q}$, A is a $\mathrm{CNL}_{\mathcal{O}}$ object, and $x \in \mathrm{Spf} R(A)$, we let ρ_x denote the pushforward of the universal deformation by x. If $\mathfrak{p} \in \operatorname{Spec} R$, we denote by $\rho_{\mathfrak{p}}$ the pushforward of the universal deformation by $R \to R/\mathfrak{p}$.

4.1.7

As in [SW1] and [SW2], we say a prime $\mathfrak{p} \in \operatorname{Spec} \overline{R}^{\psi}_{F,S}$ is *pro-modular* if it is the inverse image of a prime of $\mathbf{T}_{\psi}(U,\eta)_{\mathfrak{m}}$. We say an irreducible component of $\operatorname{Spec} \overline{R}^{\psi}_{F,S}$ is *pro-modular* if it's corresponding minimal prime is pro-modular. We say a prime $\mathfrak{p} \in \operatorname{Spec} \overline{R}^{\psi}_{F,S}$ is *nice* if

(a) $\mathfrak{p}$ is pro-modular;

(b) $\mathfrak{p}$ is a dimension one prime containing 2;

(c) $\rho_{\mathfrak{p}}$ is absolutely irreducible and non-dihedral;

(d) for each $v|2$, the image of $\mathfrak{p}$ in $\operatorname{Spec} \Lambda(G_v, \eta_v)$ does not lie in the closed subscheme defined by $(\chi^{\mathrm{univ}}_{\eta_v})^2 = \psi\epsilon_2$;

(e) the image of $\rho_{\mathfrak{p}}$ has a non-trivial unipotent element.

We are now in a position to state the the localized "$R = T$" theorem.

Proposition 4.1.8. *With the notation and assumptions as above, if $\mathfrak{p} \in \operatorname{Spec} \overline{R}^{\psi}_{F,S}$ is a nice prime, then every prime of $\overline{R}^{\psi}_{F,S}$ contained in $\mathfrak{p}$ is pro-modular.*

4.2 The Setup

The proof of 4.1.8 will be carried out in a number of steps. Fix a nice prime $\mathfrak{p}$ and let $A = \overline{R}^{\psi}_{F,S}/\mathfrak{p}$.

4.2.1

We have a commutative diagram

$$\begin{array}{ccc} \overline{R}^{\psi}_{F,S_0} & \longrightarrow & \mathbf{T}_{\psi}(U_0,\eta)_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ \overline{R}^{\psi}_{F,S} & \longrightarrow & \mathbf{T}_{\psi}(U,\eta)_{\mathfrak{m}} \end{array}$$

with all arrows surjective. Pull back $\mathfrak{p}$ to a prime $\mathfrak{p}_0$ of $\mathbf{T}_\psi(U_0,\eta)_\mathfrak{m}$ and denote again by $\mathfrak{p}_0$ its pullback to $\overline{R}^{\psi}_{F,S_0}$. Let X denote the Zariski closure in $\mathrm{Spec}\,\overline{R}^{\psi}_{F,S_0}$ of the set of points $x \in \mathrm{Spf}\overline{R}^{\psi}_{F,S_0}(\mathcal{O}_{E'})$ whose corresponding deformations are unramified at v_0, as E' ranges over all finite extensions of E. Note that X is the image of $\mathrm{Spec}\,(\overline{R}^{\square,\psi}_{F,S_0}/\mathfrak{q}_{\mathrm{ur}})$ under $\mathrm{Spec}\,\overline{R}^{\square,\psi}_{F,S_0} \to \mathrm{Spec}\,\overline{R}^{\psi}_{F,S_0}$. From this it follows that X is also equal to the image of $\mathrm{Spec}\,\overline{R}^{\psi}_{F,S} \to \mathrm{Spec}\,\overline{R}^{\psi}_{F,S_0}$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\,\mathbf{T}(U,\eta)_\mathfrak{m} & \longrightarrow & \mathrm{Spec}\,\overline{R}^{\psi}_{F,S} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,\mathbf{T}(U_0,\eta)_\mathfrak{m} & \longrightarrow & \mathrm{Spec}\,\overline{R}^{\psi}_{F,S_0}. \end{array}$$

Let $\mathfrak{q}$ be a minimal prime of $\mathbf{T}_\psi(U_0,\eta)_\mathfrak{m}$ that lies in S. We know that any arithmetic prime contained in $\mathfrak{q}$ is in the image of $\mathrm{Spec}\,\mathbf{T}_\psi(U,\eta)_\mathfrak{m}$. By Zariski density of arithmetic primes, cf. 3.3.11, $\mathfrak{q}$ must be in the image of $\mathrm{Spec}\,\mathbf{T}_\psi(U,\eta)_\mathfrak{m}$. It thus suffices to prove that any element of $X \cap \mathrm{Spec}\,(\overline{R}^{\psi}_{F,S_0})_{\mathfrak{p}_0} \subset \mathrm{Spec}\,\overline{R}^{\psi}_{F,S_0}$ is in the image of $\mathrm{Spec}\,\mathbf{T}(U_0,\eta)_{\mathfrak{p}_0}$.

We have that $\overline{R}^{\square,\psi}_{F,S_0}$ is isomorphic to a power series ring over $\overline{R}^{\psi}_{F,S_0}$ in $4|S_0|-1$ variables, cf. 2.6.5. Set $j = 4|S_0|-1$, and choose a presentation $\overline{R}^{\square,\psi}_{F,S_0} \cong \overline{R}^{\psi}_{F,S_0}[[y_1,\dots,y_j]]$. Let $\overline{R}^{\square,\psi}_{F,S_0} \to \overline{R}^{\psi}_{F,S_0}$ be the map sending each y_i to zero. Set $\mathbf{T}^{\square}_\psi(U_0,\eta)_\mathfrak{m} = \overline{R}^{\square,\psi}_{F,S_0}\hat{\otimes}_{\overline{R}^{\psi}_{F,S_0}}\mathbf{T}_\psi(U_0,\eta)_\mathfrak{m}$. We have a map $\mathbf{T}^{\square}_\psi(U_0,\eta)_\mathfrak{m} \to \mathbf{T}_\psi(U_0,\eta)_\mathfrak{m}$, by sending each y_i to zero. Pull back $\mathfrak{p}_0$ to ideals of $\mathbf{T}^{\square}_\psi(U_0,\eta)_\mathfrak{m}$ and $\overline{R}^{\square,\psi}_{F,S}$ and denote each again by $\mathfrak{p}_0$. In order to show that any element of $X \cap \mathrm{Spec}\,(\overline{R}^{\psi}_{F,S_0})_{\mathfrak{p}_0} \subset \mathrm{Spec}\,\overline{R}^{\psi}_{F,S_0}$ is in the image of $\mathrm{Spec}\,\mathbf{T}(U_0,\eta)_{\mathfrak{p}_0}$ it suffices to show that any element of $\mathrm{Spec}\,(\overline{R}^{\square,\psi}_{F,S_0})_{\mathfrak{p}_0}$ lying over $\mathfrak{q}_{\mathrm{ur}} \in \mathrm{Spec}\,R^{\square,\psi}_{S_0}$ is in the image of $\mathrm{Spec}\,\mathbf{T}_\psi(U_0,\eta)_{\mathfrak{p}_0} \to \mathrm{Spec}\,(\overline{R}^{\square,\psi}_{F,S_0})_{\mathfrak{p}_0}$.

Finally, recall that we have a faithful $\mathbf{T}_\psi(U_0,\eta)_\mathfrak{m}$-module $S_\psi(U_0,\eta)_\mathfrak{m}$. Set $S^{\square}_\psi(U_0,\eta)_\mathfrak{m} = \overline{R}^{\square,\psi}_{F,S_0}\hat{\otimes}_{\overline{R}^{\psi}_{F,S_0}} S_\psi(U_0,\eta)_\mathfrak{m}$, this is a faithful $\mathbf{T}^{\square}_\psi(U_0,\eta)_\mathfrak{m}$-module. Then to prove 4.1.8, it suffices to show that if $\mathfrak{q} \in \mathrm{Spec}\,(\overline{R}^{\square,\psi}_{F,S_0})_{\mathfrak{p}_0}$ lies over $\mathfrak{q}_{\mathrm{ur}} \in \mathrm{Spec}\,\overline{R}^{\square,\psi}_{S_0}$, then $\mathfrak{q}$ is in the support of $S^{\square}_\psi(U_0,\eta)_\mathfrak{m}$. This is what we will prove.

4.2.2

For ease of notation, set $\Lambda = \Lambda(\mathcal{U}_p^1, \eta) = \Lambda(I_p, \eta)$, $R_{\mathrm{loc}} = \overline{R}_{S_0}^{\square,\psi}$, $R_0' = \overline{R}_{F,S_0}^{\square}$, $R_0 = \overline{R}_{F,S_0}^{\square,\psi}$ and $M_0 = S_\psi^\square(U_0, \eta)_{\mathfrak{m}}$. Also set $B = \Lambda[[y_1, \ldots, y_j]]$. Let $\mathfrak{p}_\Lambda$ denote the pullback of $\mathfrak{p}$ to Λ. Note that R_0' and R_0 are B-algebras and the surjection $R_0' \to R_0$ is a B-algebra morphism. We denote by $\mathfrak{p}_0'$ the pullback of $\mathfrak{p}$ to R_0' and $\mathfrak{p}_{\mathrm{loc}}$ its pullback to R_{loc}. Note that by our assumption on $(U_0)_{v_0}$, 3.3.3 and 3.3.8 imply that $S_\psi(U, \eta)_{\mathfrak{m}}$ is finite free over Λ, and thus M_0 is finite free of the same rank over B.

Lemma 4.2.3. *There is a prime $\mathfrak{q} \in \operatorname{Spec} R_0$ contained in $\mathfrak{p}_0$ and in the support of M_0 such that the irreducible component of R_{loc} determined by $\mathfrak{q}_{\mathrm{ur}}$ is the unique irreducible component containing the image of $\mathfrak{q}$.*

Proof. Since $\operatorname{Spec} R_{\mathrm{loc}} \to \operatorname{Spec} \overline{R}_{v_0}^{\square,\psi}$ induces a bijection on irreducible components by 4.1.5, it suffices to show the existence of such a $\mathfrak{q}$ such that the irreducible component of $\operatorname{Spec} \overline{R}_{v_0}^{\square,\psi}$ determined by $\mathfrak{q}_{\mathrm{ur}}$ is the unique irreducible component containing the image of $\mathfrak{q}$ under $\operatorname{Spec} R_0 \to \operatorname{Spec} \overline{R}_{v_0}^{\square,\psi}$.

Let $\mathfrak{q}$ be a minimal prime of $\mathbf{T}_\psi^\square(U)_{\mathfrak{m}}$ contained in $\mathfrak{p}$, and write $\mathfrak{q}$ again for its pullback to R_0. Choose an arithmetic $\mathcal{O}$-morphism $\lambda_f : \mathbf{T}_\psi^\square(U) \to \overline{\mathbb{Q}}_p$, whose kernel contains $\mathfrak{q}$. Since ρ_f is unramified at v_0 we know that the image of $\ker(\lambda_f)$ in $\operatorname{Spec} \overline{R}_{v_0}^{\square,\psi}$ lies in the irreducible component determined by $\mathfrak{q}_{\mathrm{ur}}$. If it was contained in more than one irreducible component, 2.5.2 shows that $\rho_f|_{G_{v_0}} \cong \gamma\epsilon_2 \oplus \gamma$, for some unramified character γ of G_v. But using local-global compatibility, this contradicts the fact that the automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A}_F)$ corresponding to f via the Jacquet-Langlands-Shimizu correspondence is generic at v_0. Since the irreducible component of $\overline{R}_{v_0}^{\square,\psi}$ determined by $\mathfrak{q}_{\mathrm{ur}}$ is the unique one containing $\ker(x)$, it is also the unique one containing $\mathfrak{q}$. □

4.2.4

Note that the map $R_0 \to A$, determines a fixed tuple $(V_A, \{\chi_v\}_{v|2}, \{\beta_{A,v}\}_{v\in S_0})$, with V_A a deformation of $\overline{\rho}$ to A, χ_v an A valued character of G_v for each $v|2$, and $\beta_{A,v}$ a basis for V_A reducing to our fixed basis of $\overline{\rho}$ for each $v \in S_0$.

Fix $n_1 \geq 1$ as in 3.8.4. For each $n \geq n_1$ fix a finite set of finite places Q_n as in 3.8.15. Recall that $|Q_n|$ has cardinality independent of n, and we denote it by h. For each $n \geq 1$, in the notation of 4.1.2 and 4.1.4, we set

$$R'_n = \overline{R}^{\square}_{F,S_0\cup Q_n} \quad \text{and} \quad R_n = \overline{R}^{\square,\psi}_{F,S_0\cup Q_n}.$$

Fix a choice for the framing of $\overline{R}^{\square}_{F,S_0\cup Q_n}$ over $\overline{R}_{F,S_0\cup Q_n}$ compatible with the surjection $\overline{R}^{\square}_{F,S_0\cup Q_n} \to \overline{R}^{\square}_{F,S_0}$. This then gives a B-algebra structure to R'_n and R_n and we have a commutative diagram in $\mathrm{CNL}_{\mathcal{O}}$

$$\begin{array}{ccccc} R_{\mathrm{loc}} & \longrightarrow & R'_n & \longrightarrow & R_n \\ & \searrow & \downarrow & & \downarrow \\ & & R'_0 & \longrightarrow & R_0 \end{array}$$

such that each is a morphism of Λ-algebras, and each of $R'_n \to R_n$, $R'_n \to R'_0$ and $R_n \to R_0$ are surjective morphisms of B-algebras. Let $\mathfrak{p}_n$ and $\mathfrak{p}'_n$ denote the pullback of $\mathfrak{p}$ to R_n and R'_n, respectively.

Lemma 4.2.5. *1. Let $k = 1 + 2h$. For any $n \geq n_1$ there an injection of A-modules*

$$A^k \longrightarrow \mathfrak{p}'_n/((\mathfrak{p}'_n)^2 + \mathfrak{p}_{\mathrm{loc}})$$

whose cokernel is finite of size bounded independent of n.

2. The minimal number of generators of the maximal ideal of R_n is bounded independently of n.

Proof. We first prove (1). Let K denote the fraction field of A. Let $\mathfrak{t}_n = \mathfrak{p}'_n/((\mathfrak{p}'_n)^2 + \mathfrak{p}_{\mathrm{loc}})$. Take $x_1, \ldots, x_{k_n} \in \mathfrak{t}_n$ such that the images of $x_1, \ldots, x_{k_n}$ in $\mathfrak{t}_n/\mathfrak{m}_A$ are linearly independent and such that $x_1, \dots, x_{k_n}$ form a basis for $\mathfrak{t}_n \otimes_A K$. This gives a map $\varphi : A^{k_n} \to \mathfrak{t}_n$. We will show that $k_n = k$ and $\mathrm{coker}(\varphi))$ is finite of size bounded independently of n.

Let A' denote the integral closure of A in K, and set $\mathfrak{t}'_n = \mathfrak{t}_n \otimes_A A'$. I claim that $\mathfrak{t}_n \to \mathfrak{t}'_n$ is an injection. As both are finitely generated over A, it suffices to show injectivity modulo $\mathfrak{m}_A$ by Nakayama's Lemma. This follows from the fact that the composite

$$\mathfrak{t}_n/\mathfrak{m}_A \longrightarrow \mathfrak{t}'_n/\mathfrak{m}_A \longrightarrow \mathfrak{t}'_n/\mathfrak{m}_{A'}$$

is the isomorphism $\mathfrak{t}_n \otimes_A \mathbb{F} \cong (\mathfrak{t}_n \otimes_A A') \otimes_{A'} \mathbb{F}$.

Identifying $x_1, \dots, x_{k_n}$ with their images in $\mathfrak{t}'_n$, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ax_1 + \cdots Ax_{k_n} & \xrightarrow{\varphi} & \mathfrak{t}_n & \longrightarrow & \mathrm{coker}(\varphi) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & A'x_1 + \cdots + A'x_{k_n} & \xrightarrow{\varphi'} & \mathfrak{t}'_n & \longrightarrow & \mathrm{coker}(\varphi') & \longrightarrow & 0 \end{array}$$

and the snake lemma gives an injection $\ker(f) \to (A'/A)^{k_n}$. If $k_n = k$ then $(A'/A)^{k_n}$ is finite of size bounded independently of n. So it suffices to show the A'-rank of $\mathfrak{t}'_n$ is k, and that $\mathrm{coker}(\varphi')$ is finite of size bounded independently of n.

To do this we use the results of §3.6. If we view $\rho_{\mathfrak{p}}$ as taking values in $\mathrm{GL}_2(A')$, then all the assumptions of §3.6 are satisfied (with $\rho = \rho_{\mathfrak{p}}$ and A' in place of A), except possibly for the assumption A5, i.e. that in the case where $\overline{\rho}$ is L-dihedral, there is some $\tau_0 \in G_F \smallsetminus G_L$ such that $\rho_{\mathfrak{p}}(\tau_0)$ has distinct infinite order A'-rational eigenvalues (note that (d) of the definition of nice primes ensures that A2 hold even in the case that $\overline{\rho}$ is not dihedral). Assuming $\overline{\rho}$ is dihedral, 2.7.5 implies there is some $\tau_0 \in G_F$ such that $\overline{\rho}(\tau_0)$ has order 2, but $\rho_{\mathfrak{p}}(\tau_0)$ has infinite order. Note that since $2 \in \mathfrak{p}$, $\det \rho_{\mathfrak{p}}$ is finite and unipotent elements of $\rho_{\mathfrak{p}}(G_F)$ have finite order, hence $\rho_{\mathfrak{p}}(\tau_0)$ has distinct infinite order eigenvalues. Since $\overline{\rho}(\tau_0)$ does not have distinct eigenvalues, the eigenvalues of $\rho_{\mathfrak{p}}$ lie in the ring of integers A'' of a quadratic ramified extension K''/K. Set $\mathfrak{t}''_n = \mathfrak{t}'_n \otimes_{A'} A'' = \mathfrak{t}_n \otimes_A A''$. Since A'' is free of rank 2 over A', it suffices to show that the A''-rank of $\mathfrak{t}''_n$ is k and the cokernel of $\varphi'' : A''x_1 + \cdots A''x_{k_n} \to \mathfrak{t}''_n$ is finite of order bounded independently of n.

Recall some notation from 3.8.14. For each $n, m \geq 1$ given $C_{n,m}$ and $D_{n,m}$ be non-negative integers, we write

$$C_{n,m} \asymp D_{n,m}$$

if there are constants $0 < a < b$ such that

$$a < \frac{C_{n,m}}{D_{n,m}} < b$$

for all $n, m \geq 1$. If M is a finite abelian group with a continuous $G_{F,S_0 \cup Q_n}$-action and $V \subset S_0 \cup Q_n$, we let $H^i_V(G_{F,S_0 \cup Q_n}, M)$ denote the subgroup of $H^i_V(G_{F,S_0 \cup Q_n}, M)$ consisting of elements whose restriction to $H^i(G_v, M)$ is trivial for each $v \in V$. Let Ad denote the set of 2×2 matrices over A'' with the adjoint $G_{F,S_0 \cup Q_n}$-action. Set $\mathrm{Ad}_m = \mathrm{Ad}/\mathfrak{m}^m_{A''}\mathrm{Ad}$. Let $q = |\mathbb{F}|$.

Since $\mathfrak{t}''_n$ is a finitely generated A''-module, we can write $\mathfrak{t}''_n \cong (A'')^{k_n} \oplus T_n$, with T_n finite. By our choice of $x_1, \ldots, x_{k_n}$, $\mathrm{coker}(\varphi'') \cong T_n$. Since T_n is finite, $\mathrm{Hom}_{A''}(T_n, K''/A'') \cong T_n$ and

$$\mathrm{Hom}_{A''}(\mathfrak{t}''_n, K''/A'') \cong (K''/A'')^{k_n} \times T_n.$$

It then suffices to show that $\mathrm{Hom}_{A''}(\mathfrak{t}''_n, A''/\mathfrak{m}^m_{A''}) \asymp (q^m)^k$. Let $\varpi_{K''}$ be a uniformizer for K''. We have

$$\mathrm{Hom}_{A''}(\mathfrak{t}''_n, A''/\mathfrak{m}^m_{A''}) \cong \mathrm{Hom}_A(\mathfrak{t}_n, A''/\mathfrak{m}^m_{A''})$$

and this space is identified with the set of $\phi \in \mathrm{Hom}_{R_{\mathrm{loc}}}(R'_n, A''[\varepsilon]/(\varepsilon^2, \varpi_{K''}\varepsilon))$ such that

$$R'_n \xrightarrow{\phi} A''[\varepsilon]/(\varepsilon^2, \varpi^m_{K''}\varepsilon) \xrightarrow{\varepsilon \mapsto 0} A''$$

is the map $R'_n \to A \to A''$. This hom set is then identified with tuples $(V, \{\beta_v\}_{v \in S_0})$, where

- V is a $G_{F,S_0 \cup Q_n}$-deformation to $A''[\varepsilon]/(\varepsilon^2, \varpi^m_{K''}\varepsilon)$ that lifts the A''-deformation $V_A \otimes_A A''$, where V_A is our fixed deformation corresponding to $\mathfrak{p}$;

- each β_v is a basis of V lifting the basis of $V_A \otimes_A A''$ determined by our fixed basis $\beta_{A,v}$ of V_A, such that for each $v \in S_0$ the lift determined by $V|_{G_v}$ and the basis β_v is equal to the lift given by $V_A|_{G_v} \otimes_A A''[\varepsilon]/(\varepsilon^2, \varpi_{K''}^m \varepsilon)$ and $\beta_{A,v}$. We note that there is no need to specify G_v characters as they are determined by the fact that our morphisms are R_{loc}-morphisms. The set of such tuples surjects onto the set of deformations lifting $V_A \otimes_A A''$ with fixed restriction to G_v for each $v \in S_0$. A standard argument shows that this space of deformations is isomorphic to $H^1_{S_0}(G_{F,S_0 \cup Q_n}, \mathrm{Ad}_m)$. The fibre over any given deformation V is given by sets of basis $\{\beta_v\}_{v \in S_0}$ reducing to our fixed set of bases modulo ε, such that the lift given by $V|_{G_v}$ and β_v is independent of the element in the fibre, up to equivalence by automorphisms of V reducing to the identity modulo ε that commute with the $G_{F,S_0 \cup Q_n}$-action.

Putting this all together, we have

$$|\mathrm{Hom}_A(\mathfrak{t}_n, A''/\mathfrak{m}_{A''}^m)| = \frac{|H^1_{S_0}(G_{F,S_0 \cup Q_n}, \mathrm{Ad}_m)| \prod_{v \in S_0} |H^0(G_v, \mathrm{Ad}_m)|}{|H^0(G_F, \mathrm{Ad}_m)|}. \tag{4.1}$$

Since the trace pairing on Ad_m is perfect and the characteristic of K'' is 2, the Pontryagin dual of Ad_m is $G_{F,S_0 \cup Q_n}$-isomorphic to itself, and the dual Selmer group to $H^1_{S_0}(G_{F,S_0 \cup Q_n}, \mathrm{Ad}_m)$ is $H^1_{Q_n}(G_{F,S_0 \cup Q_n}, \mathrm{Ad}_m)$. Then Wiles's formula, c.f. Theorem 2.19 of [DDT], together with (4.1) yields

$$|\mathrm{Hom}_A(\mathfrak{t}_n, A''/\mathfrak{m}_{A''}^m)| = \frac{|H^1_{Q_n}(G_{F,S_0 \cup Q_n}, \mathrm{Ad}_m)|}{|H^0(G_F, \mathrm{Ad}_m)|} \prod_{v \in Q_n} \frac{|H^1(G_v, \mathrm{Ad}_m)|}{|H^0(G_v, \mathrm{Ad}_m)|}.$$

Local Tate duality and Euler-Poincaré characteristic give

$$|\mathrm{Hom}_A(\mathfrak{t}_n, A''/\mathfrak{m}_{A''}^m)| = \frac{|H^1_{Q_n}(G_{F,S_0 \cup Q_n}, \mathrm{Ad}_m)|}{|H^0(G_F, \mathrm{Ad}_m)|} \prod_{v \in Q_n} |H^0(G_v, \mathrm{Ad}_m)|. \tag{4.2}$$

Since $V_A \otimes_A A''$ is absolutely irreducible, $|H^0(G_F, \mathrm{Ad}_m)| \asymp q^m$. Then parts (3) and (4) of 3.8.15 with (4.2) imply

$$|\mathrm{Hom}_A(\mathfrak{t}_n, A''/\mathfrak{m}_{A''}^m)| \asymp (q^m)^{1+2|Q_n|} = q^{mk}$$

which is what was required to prove.

To see (2), it suffices to show the minimal number of generators of R'_n over R_{loc} is bounded independently of n. Letting $\mathrm{Ad}_{\mathbb{F}}$ denote the space of 2×2 matrices over $\mathbb{F}$ with the adjoint

$G_{F,S_0\cup Q_n}$, a similar argument to above, shows that this minimal number of generators is bounded above by

$$\dim_{\mathbb{F}} H^1_{Q_n}(G_{F,S_0\cup Q_n},\mathrm{Ad}_{\mathbb{F}})-\dim_{\mathbb{F}} H^0(G_F,\mathrm{Ad}_{\mathbb{F}})+\sum_{v\in Q_n}\dim_{\mathbb{F}} H^0(G_v,\mathrm{Ad}_{\mathbb{F}}) \le \dim_{\mathbb{F}} H^1(G_{F,S_0},\mathrm{Ad}_{\mathbb{F}})-1+4h.$$

□

4.2.6

We let $F^{S_0}_{Q_n}$ denote the maximal abelian p-extension unramified outside Q_n and split at all primes in S_0. Set $G_n = \mathrm{Gal}(F^{S_0}_{Q_n}/F)$. Part (5) of 3.8.15 shows that $G_n/2^{n-2}G_n \cong (\mathbb{Z}/2^{n-2}\mathbb{Z})^t$ with $t = 2-|S_0|+|Q_n|$. Let G^*_n denote the diagonalizable $\mathcal{O}$-group as in 2.6.7. Recall that and element of $G^*_n(A)$ is a character $\chi : G_n \to A^\times$ reducing to the trivial character modulo the maximal ideal of A.

By 2.6.8 and our assumptions on S, cf. 4.1.3, there is a free action of G^*_n on $\mathrm{Spf}R'_n$, and the morphism $\mathrm{Spf}R'_n \to \mathrm{Spf}R_{\mathrm{loc}}$ is constant on orbits. We also have an action of $G^*_{n,2}$, the 2-torsion of G^*_n, on $\mathrm{Spf}R_n$ and a the function $\delta_{Q_n} : \mathrm{Spf}R'_n \to G^*_n$ such that that $R'_n \to R_n$ identifies $\mathrm{Spf}R_n$ with the closed sub-formal-scheme of $\mathrm{Spf}R'_n$ defined by $\delta_{Q_n} = 1$, and for any $\mathrm{CNL}_{\mathcal{O}}$ object A, $\alpha \in G^*_n(A)$, and $x \in \mathrm{Spf}R'_n(A)$, we have $\delta_{Q_n}(\alpha x) = \alpha^2\delta_{Q_n}(x)$, cf. 2.6.7. We also note that for any $\mathrm{CNL}_{\mathcal{O}}$ object A, the stucture map $\mathcal{O} \to A$ gives a group homomorphism $G^*_n(\mathcal{O}) \to G^*_n(A)$. In this way the finite group $G^*_n(\mathcal{O})$ acts on $\mathrm{Spf}R'_n$, and hence on R'_n. Similarly, $G^*_{n,2}(\mathcal{O})$ acts on R_n.

4.2.7

For each $n \ge n_1$, let $S_\psi(U_{Q_n},\eta)$ be as in 4.1.2. We set $M_n = \overline{R}^{\square,\psi}_{F,S_0\cup Q_n} \otimes_{\overline{R}^{\psi}_{F,S_0\cup Q_n}} S_\psi(U_{Q_n},\eta)_{\mathfrak{m}}$. Note that M_n is an R_n-module and we have a surjection $M_n \to M_0$ of R_n-modules, where the R_n-module structure on M_0 is via the surjection $R_n \to R_0$.

Write $Q_n = \{v_1,\dots,v_h\}$ and define two power series rings $B[[s_1,\dots,s_h]]$ and $B[[t_1,\dots,t_h]]$. We view $B[[t_1,\dots,t_h]]$ as a subring of $B[[s_1,\dots,s_h]]$ by $t_i \mapsto (1+s_i)+(1+s_i)^{-1}-2$. For each $v_i \in Q_n$, fix a generator σ_i of the p-part of the tame inertia group of F_{v_i}. The map $I_{v_i} \to \mathcal{O}_{F_{v_i}}^\times \to k_{v_i}^\times \to \Delta_{v_i}$ given by class field theory then sends σ_i to a generator δ_i of Δ_{v_i}. Let V be tautological deformation to $\overline{R}_{F,S_0\cup Q_n}$. We define a local B-algebra morphism $B[[t_1,\dots,t_h]] \to R'_n$ by sending $2+t_i$ to the trace of σ_i acting on V. We also define a $B[[s_1,\dots,s_h]]$-module structure on M_n by letting s_i act via the action of δ_i on $S_\psi(U_{Q_n})_\mathfrak{m}$. By 3.5.6 the two $B[[t_1,\dots,t_h]]$-module structures on M_n given by $B[[s_1,\dots,s_h]]$ and R_n coincide. Note that under the surjection $R_n \to R_0$, each t_i is mapped to zero.

We remark that the power series ring $B[[t_1,\dots,t_h]]$ is introduced because we may not be able to define a $B[[s_1,\dots,s_h]]$-algebra structure on R_n. Although $\overline{\rho}$ is unramified at each $v \in Q_n$, its Frobenius eigenvalues may not be distinct, so the local lift to R_n may not be split and in this case we not not have a morphism $\Delta_v \to R_n^\times$. This is why we introduce this subring $B[[t_1,\dots,t_h]]$ of "traces".

Lemma 4.2.8. *1. There is $s \geq 1$, independent of n, such that letting $\mathfrak{b}_n$ denote the annihilator of M_n in $B[[s_1,\dots,s_h]]$,*

$$\mathfrak{b}_n \subseteq ((1+s_1)^{2^n}-1,\dots,(1+s_h)^{2^n}-1)$$

and M_n is free over $B[[s_1,\dots,s_h]]/\mathfrak{b}_n$ of rank s.

2. $(s_1,\dots,s_h)M_n \subseteq \ker(M_n \to M_0)$.

3. There is $\lambda_M \in \Lambda$, $\lambda_M \notin \mathfrak{p}_\Lambda$ and independent of n such that letting $N_n = \ker(M_n/(s_1,\dots,s_h)M_n \to M_0)$, $\lambda_M N_n \subseteq \mathfrak{p}_n N_n$.

4. There is an action of the finite group $G^*_{n,2}(\mathcal{O})$ on M_n, such that for $\alpha \in G^*_{n,2}(\mathcal{O})$, $r \in R_n$, and $m \in M_n$, $r(\alpha m) = \alpha((\alpha r)m)$.

5. Letting $G^*_{n,2}(\mathcal{O})$ act on $B[[s_1,\dots,s_h]]$ by $\chi s_i = \chi(\delta_i)(1+s_i)-1$, we also have $s_i(\alpha m) = \alpha((\alpha s_i)m)$.

Proof. Parts (1) follows immediately from 3.5.5 and our assumption on v_0. Parts (3) and (4) follow by considering twists of elements of $S_{\kappa,\psi}(U_{Q_n}(p^{a,a}), E/\mathcal{O})$ by characters $\chi : G_n/2G_n \to \mathcal{O}^\times$ as in section 7.5 of [KW], and then passing to the limit in a and taking Pontryagin duals.

We now show (2). By (1) of 3.8.15, there is w, independent of n, such that for any $v \in Q_n$, $\mathrm{val}_K(\rho_{\mathfrak{p}}(\mathrm{Frob}_v)) < w$, where val_K denotes the valuation of the fraction field K of A giving a uniformizer valuation 1. Take $\lambda_M \in \Lambda$ be any element whose image in A is non-zero and has valuation at least $2w$. By 3.5.3, there is $y_v \in R_n$ lifting $(\mathrm{tr}\,\rho_{\mathfrak{p}}(\mathrm{Frob}_v))^2$ such that $y_v N_n = 0$. In particular $y_v(N_n/\mathfrak{p}_n N_n) = 0$. Since the valuation of y_v modulo $\mathfrak{p}_n$ is bounded above by $2w$, we have $\lambda_M(N_n/\mathfrak{p}_n N_n) = 0$. $\square$

4.3 Patching and proof of 4.1.8

Denote by $\mathfrak{T}$ the formal $\mathrm{CNL}_{\mathcal{O}}$-torus $(\mathbb{Z}^t)^*$, where t is as in 4.2.6. Recall $t = 2 - |S_0| + |Q_n|$. For any $n \geq 1$, we let $\mathfrak{T}_{2^n}$ denote the 2^n-torsion subgroup of $\mathfrak{T}$.

Lemma 4.3.1. *Let $k = 1 + 4|S_0|$ as in 4.2.5. Let CNL_Λ denote the full subcategory of $\mathrm{CNL}_{\mathcal{O}}$ consisting of Λ-algebras. We have*

- *CNL_Λ objects R'_∞, R_∞, a power series $R_{\mathrm{loc}}[[x_1, \ldots, x_k]]$ and an $R_\infty \times B[[s_1, \ldots, s_h]]$-module M_∞;*
- *CNL_Λ morphisms $R_{\mathrm{loc}}[[x_1, \ldots, x_k]] \to R'_\infty \to R_\infty$, $R'_\infty \to R'_0$ and $R_\infty \to R_0$, and a morphism of $R_\infty \times B[[s_1, \ldots, s_h]]$-modules $M_\infty \to M_0$;*

such that the following hold.

1. *The diagrams*

$$\begin{array}{ccccc} R_{\mathrm{loc}}[[x_1, \ldots, x_k]] & \longrightarrow & R'_\infty & \longrightarrow & R_\infty \\ \uparrow & & \downarrow & & \downarrow \\ R_{\mathrm{loc}} & \longrightarrow & R'_0 & \longrightarrow & R_0 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} B[[t_1, \ldots, t_h]] & \longrightarrow & R_\infty \\ \uparrow & & \downarrow \\ B & \longrightarrow & R_0 \end{array}$$

   both commute and the two $B[[t_1, \ldots, t_h]]$-module structures on M_∞ (coming from R_∞ and $B[[s_1, \ldots, s_h]]$) coincide.

2. *The morphisms $R'_\infty \to R_\infty$, $R'_\infty \to R'_0$, $R_\infty \to R_0$, and $M_\infty \to M_0$ are all surjections.*

3. $(t_1, \dots, t_h)R_\infty \in \ker(R_\infty \to R_0)$ *and* $(s_1, \dots, s_h)M_\infty \in \ker(M_\infty \to M)$.

4. Letting $N_\infty = \ker(M_\infty/(s_1, \dots, s_h) \to M_0)$ *and letting* $\mathfrak{p}_\infty$ *be the pullback of* $\mathfrak{p}_0$ *to* R_∞, $N_\infty/\mathfrak{p}_\infty N_\infty$ *is a torsion* A*-module.*

5. Letting $\mathfrak{p}'_\infty$ *be the pullback of* $\mathfrak{p}'_0$ *to* R'_∞, $(\mathfrak{p}_{\text{loc}}, x_1, \dots, x_k)R'_\infty \subseteq \mathfrak{p}'_\infty$ *and* $\mathfrak{p}'_\infty/((\mathfrak{p}'_\infty)^2 + (\mathfrak{p}_{\text{loc}}, x_1, \dots, x_k)R'_\infty)$ *is a torsion* A*-module.*

6. There is a free action of $\mathfrak{T}$ *on* $\text{Spf}R'_\infty$ *and a map* $\delta_\infty : \text{Spf}R'_\infty \to \mathfrak{T}$ *such that*

- *the morphism* $\text{Spf}R'_\infty \to \text{Spf}R_{\text{loc}}$ *is constant on orbits of* $\mathfrak{T}$*,*
- *the closed immersion* $\text{Spf}R_\infty \to \text{Spf}R'_\infty$ *identifies* $\text{Spf}R_\infty$ *with the closed subfunctor of points* x *with* $\delta_\infty(x) = 1$ *and*
- *for any object* A *in* $\text{CNL}_\mathcal{O}$, $x \in \text{Spf}R'_\infty(A)$ *and* $\alpha \in \mathfrak{T}(A)$*, we have* $\delta_\infty(\alpha x) = \alpha^2\delta_\infty(x)$.

7. The subfunctor $\text{Spf}R_\infty \to \text{Spf}R'_\infty$ *is stable under the action of* $\mathfrak{T}_2$*, and there is an action of* $\mathfrak{T}_2(\mathcal{O})$ *on* M_∞ *satisfying the following compatibility conditions: for any* $m \in M_\infty$, $r \in R_\infty$ *and* $\alpha \in \mathfrak{T}_2(\mathcal{O})$ *we have* $r(\alpha m) = \alpha((\alpha r)m)$.

Proof. The proof is similar to the contruction in Proposition 9.3 of [KW].

For a $\text{CNL}_\mathcal{O}$ algebra A with maximal ideal $\mathfrak{m}_A$ and $r \geq 1$, we let $\mathfrak{m}_A^{(r)}$ denote the ideal of A generated by elements of $\mathfrak{m}_A$ that are r-th powers. Note that if $\mathfrak{m}_A$ can be generated by g elements, then $\mathfrak{m}_A^{gr} \subseteq \mathfrak{m}_A^{(r)}$. Let $s \geq 1$ be as in (1) of 4.2.8. For each $m \geq 1$ set $r_m = sm(h+j)2^m$ and

$$\mathfrak{c}_m = (\mathfrak{m}_\Lambda^m, y_1^{2^m}, \dots, y_j^{2^m}, (1+s_1)^{2^m} - 1, \dots, (1+y_j)^{2^m} - 1) \subset B[[s_1, \dots, s_h]].$$

Let $\mathfrak{d}_m = \mathfrak{c}_m \cap B[[t_1, \dots, t_h]]$. As in the proof of Proposition 3.3.1 of [K2] we can show for any $m \geq 1$ that $M_0/\mathfrak{c}_m M_0$ is an $R_0/(\mathfrak{d}_m R_0 + \mathfrak{m}_{R_0}^{(r_m)})$-module. Similarly, for any $n \geq n_1$ and $m \geq 1$, we can show that $M_n/\mathfrak{c}_m M_n$ is an $R_n/(\mathfrak{d}_m R_n + \mathfrak{m}_{R_n}^{(r_m)})$-module.

Fix $\lambda_R \in \Lambda$ with non-zero image in A, such that λ_R annihilates the cokernel of the map in (1) of 4.2.5 for all $n \geq n_1$. Also let λ_M be as in (3) of 4.2.8.

Let $G_n' = G_n/2^{n-2}G_n$. As in [KW] we fix a surjection $\mathbb{Z}^t \to G_n$ for each $n \geq n_0$. Note that by 4.2.6 this induces an isomorphism $\mathbb{Z}/2^{n-2}\mathbb{Z} \cong G_n'$ and a closed embedding $G_n^* \to \mathfrak{T}$ which identifies $G_n'^*$ with $\mathfrak{T}_{2^{n-2}}$.

Let g by such that the maximal ideal of R_n is genererated by $\leq g$ elements for all $n \geq n_1$, cf. (2) of 4.2.5.

We let $\mathrm{CNL}_{R_{\mathrm{loc}}}$ denote the full subcategory of $\mathrm{CNL}_{\mathcal{O}}$ consisting of R_{loc}-algebras. For $m \geq 1$ a patching datum of level m, denoted (D_m', D_m, L_m), consists of the following.

(a) A surjective $\mathrm{CNL}_{R_{\mathrm{loc}}}$ morphism

$$D_m \to R_0/(\mathfrak{d}_m R_0 + \mathfrak{m}_{R_0}^{(r_m)})$$

which is also a $B[[t_1, \ldots, t_h]]$-algebra morphism, such that $\mathfrak{m}_{D_m}^{(r_m)} = (0)$.

(b) A object D_m' of $\mathrm{CNL}_{R_{\mathrm{loc}}}^{[gr_m]}$ such that $\mathfrak{m}_{D_m'}$ can be generated by g elements, a surjective $\mathrm{CNL}_{R_{\mathrm{loc}}}^{[gr_m]}$ morphism

$$D_m' \to D_m,$$

a free action of $\mathfrak{T}_{2^m}^{[gr_m]}$ on D_m' such that the map $\mathrm{Spf} D_m' \to \mathrm{Spf} R_{\mathrm{loc}}$ is constant on orbits of $\mathfrak{T}_{2^m}^{[gr_m]}$, and a morphism $\delta_m : \mathrm{Spf} D_m' \to \mathfrak{T}$, such that $\delta_m(\alpha x) = \alpha^2 \delta_m(x)$ for any A in $\mathrm{CNL}_O^{[gr_m]}$, $x \in \mathrm{Spf} D_m'(A)$ and $\alpha \in \mathfrak{T}_{p^m}(A)$. Let D_m'' be the object in $\mathrm{CNL}_{\mathcal{O}}^{[gr_m]}$ representing the closed subfunctor of $\mathrm{Spf} D_m'$ given by points x with $\delta_m(x) = 1$. We further demand that the surjection $D_m' \to D_m$ factors through $D_m' \to D_m''$ and that the kernel of $D_m'' \to D_m$ is contained in $\mathfrak{m}_{D_m''}^m$.

(c) A $D_m \times B[[s_1, \ldots, s_h]]$-module L_m, which is finite free of rank s over $B[[s_1, \ldots, s_h]]/\mathfrak{c}_m$ and a surjection of D_m-modules $L_m \to M_0/\mathfrak{c}_m M_0$ (here the D_m-module on $M_0/\mathfrak{c}_m M_0$ is via the surjection $D_m \to R_0/(\mathfrak{d}_m + \mathfrak{m}_{R_0}^{(r_m)})$ in (a)), such that letting $\mathfrak{a}_m$ denote the inverse image in D_m of the image of $\mathfrak{p}_0$ in $R_0/(\mathfrak{d}_m R_0 + \mathfrak{m}_{R_0}^{(r_m)})$,

$$\lambda_M \ker(L_m/(s_1, \ldots, s_h) \to M_0/\mathfrak{c}_m M_0) \subseteq \mathfrak{a}_m \ker(L_m/(s_1, \ldots, s_h) \to M_0/\mathfrak{c}_m M_0).$$

(d) A $\mathrm{CNL}_{R_{\mathrm{loc}}}$ morphism

$$R_{\mathrm{loc}}[[x_1, \ldots, x_k]] \to D'_m$$

such that, letting $\mathfrak{a}'_m$ denote the inverse image in D'_m of the image of $\mathfrak{p}_0$ in $R_0/(\mathfrak{d}_m R_0 + \mathfrak{m}_{R_0}^{(r_m)})$, we have $(\mathfrak{p}_{\mathrm{loc}}, x_1, \ldots, x_k)D'_m \subseteq \mathfrak{a}'_m$ and

$$\lambda_R(\mathfrak{a}'_m/((\mathfrak{a}'_m)^2 + (\mathfrak{p}_{\mathrm{loc}}, x_1, \ldots, x_k)D'_m)) \subseteq \mathfrak{m}_{D'_m}^m.$$

We define a morphism $({D'_m}^1, D_m^1, L_m^1) \to ({D'_m}^2, D_m^2, L_m^2)$ of patching date of level m to be surjective $\mathrm{CNL}_{R_{\mathrm{loc}}}$ morphisms ${D'_m}^1 \to {D'_m}^2$ and $D_m^1 \to D_m^2$ and a surjection of D_m^1-modules $L_m^1 \to L_m^2$ such that

- $D_m^1 \to D_m^2$ is a $B[[t_1, \ldots, t_h]]$-algebra morphism and is compatible with the surjections to $R_0/(\mathfrak{d}_m + \mathfrak{m}_{R_0}^{(r_m)})$ of (a);
- ${D'_m}^1 \to {D'_m}^2$ is compatible with the $\mathfrak{T}_{2^m}^{[gr_m]}$ action and with the morphisms $\delta_m^i : \mathrm{Spf}\, {D'_m}^i \to \mathfrak{T}$ from (b), and

$$\begin{array}{ccc} {D'_m}^1 & \longrightarrow & D_m^1 \\ \downarrow & & \downarrow \\ {D'_m}^2 & \longrightarrow & D_m^2 \end{array}$$

  commutes;
- ${D'_m}^1 \to {D'_m}^2$ is compatible with the morphisms $R_{\mathrm{loc}}[[x_1, \ldots, x_k]] \to {D'_m}^i$ from (d).

Fix $n \geq n_1$ and $n_1 \leq m \leq n$. We now show how to define a patching datum $(D'_{m,n}, D_{m,n}, L_{m,n})$ of level m, from R'_{n+2}, R_{n+2}, and M_{n+2}.

Set $D'_{m,n} = R'_{n+2}/\mathfrak{m}_{R'_{n+2}}^{gr_m}$, $D_{m,n} = R_{n+2}/(\mathfrak{d}_m R_{n+2} + \mathfrak{m}_{R_{n+2}}^{(r_m)})$ and $L_{m,n} = M_{n+2}/\mathfrak{c}_m M_{n+2}$. The surjection $R_n \to R_0$ is an $R_{\mathrm{loc}} \times B[[t_1, \ldots, t_h]]$-algebra morphism, so we get a morphism $D_{m,n} \to R_0/(\mathfrak{d}_m R_0 + \mathfrak{m}_{R_0}^{(r_m)})$ satisfying the properties of (a) in the definition of a patching datum of level m.

As noted above, $L_{m,n}$ is a $D_{m,n}$-module and the surjection $M_n \to M_0$ of R_n-modules induces a surjection $L_{m,n} \to M_0/\mathfrak{c}_m M_0$ of $D_{m,n}$-modules. That $L_{m,n}$ satisfies the required properties follows from 4.2.8.

We now show part (b). By choice of g, $\mathfrak{m}_{D'_{m,n}}$ can be generated by g elements. As noted above, we have $\mathfrak{m}_{R'_n}^{gr_m} \subseteq \mathfrak{m}_{R'_n}^{(r_m)}$, so the surjection $R'_n \to R_n$ induces a surjection $D'_{m,n} \to D_{m,n}$. From

$$\mathfrak{T}_{p^m} \to \mathfrak{T}_{p^n} \cong G'^*_{n+2} \to G^*_{n+2}$$

the free action of G^*_{n+2} on $\mathrm{Spf} R'_{n+2}$ yields a free action of $\mathfrak{T}_{p^m}$ on $\mathrm{Spf} R'_{n+2}$. This gives rise to a free group action chunk, cf. 2.2.5, in $\mathrm{CNL}_{\mathcal{O}}^{[gr_m]}$ of $\mathfrak{T}_{p^m}^{[gr_m]}$ on $\mathrm{Spf} D'_{m,n}$. Now let $\delta : \mathrm{Spf} R'_{n+2} \to G^*_{n+2}$ be as in 4.2.6. With our fixed immersion $G^*_n \to \mathfrak{T}$ we define δ_m to be the composite

$$\mathrm{Spf} D'_{m,n} \to \mathrm{Spf} R'_{n+2} \to G^*_{n+2} \to \mathfrak{T}.$$

The fact that $\delta_m(\alpha x) = \alpha^2 \delta_m(x)$ for any A in $\mathrm{CNL}_{\mathcal{O}}^{[gr_m]}$, $x \in \mathrm{Spf} D'_{m,n}(A)$ and $\alpha \in \mathfrak{T}_{p^m}(A)$ now follows from 4.2.6. Let $D''_{m,n}$ be the object in $\mathrm{CNL}_{\mathcal{O}}^{[gr_m]}$ representing the closed subfunctor of $\mathrm{Spf} D'_{m,n}$ given by points x with $\delta_m(x) = 1$. The surjection $D'_{m,n} \to D_{m,n}$ factors through $D'_{m,n} \to D''_{m,n}$. We also see that, since $\mathfrak{d}_m R'_{n+2} + \mathfrak{m}_{R'_{n+2}}^{(r_m)} \subseteq \mathfrak{m}_{R'_{n+2}}^m$, the kernel of $D''_{m,n} \to D_{m,n}$ is contained in $\mathfrak{m}_{D''_{m,n}}^m$. We have verified all the conditions of (b).

To realize part (d) first note that $\mathfrak{a}_m$ is the image in $D'_{m,n}$ of

$$\mathfrak{p}'_{n+2} + \mathfrak{d}_m R'_{n+2} + \mathfrak{m}_{R'_{n+2}}^{(r_m)} \subseteq \mathfrak{p}'_{n+2} + \mathfrak{m}_{R'_{n+2}}^m.$$

By 4.2.5, we can choose a $\mathrm{CNL}_{R_{\mathrm{loc}}}$ morphism

$$R_{\mathrm{loc}}[[x_1, \ldots, x_k]] \to R'_{n+2}$$

such that $(\mathfrak{p}_{\mathrm{loc}}, x_1, \ldots, x_k) R'_{n+2} \subseteq \mathfrak{p}'_{n+2}$ and such that $\mathfrak{p}'_{n+2}/((\mathfrak{p}'_{n+2})^2 + (\mathfrak{p}_{\mathrm{loc}}, x_1, \ldots, x_k) R'_{n+2})$ is annihilated by λ_R. This then induces the desired morphism of part (d).

Note that for any $n > n_1$ and $n_1 \leq m < n$ we have natural surjections $D'_{m+1,n} \to D'_{m,n}$, $D_{m+1,n} \to D_{m,n}$ and $L_{m+1,n} \to L_{m,n}$ that induce and isormorphism

$$(D'_{m+1,n}/\mathfrak{m}_{D'_{m+1,n}}^{gr_m}, D_{m+1,n}/(\mathfrak{d}_m D_{m+1,n} + \mathfrak{m}_{D_{m+1,n}}^{(r_m)}), L_{m+1,n}/\mathfrak{c}_m L_{m+1,n}) \cong (D'_{m,n}, D_{m,n}, L_{m,n})$$

of patching data of level m. Since, for each $m \geq 1$, there are only finitely many isomorphism classes of patching data of level m, after extracting a subsequence of $(n)_{n \geq n_1}$ we can assume that $(D'_{m,n}, D_{m,n}, L_{m,n}) \cong (D'_{m,m}, D_{m,m}, L_{m,m})$ for all $n \geq m$ and denoting this common isomorphism class by (D'_m, D_m, L_m), we get a projective system in $m \geq n_1$. Set $R'_\infty = \varprojlim D'_m$, $R_\infty = \varprojlim D_m$ and $M_\infty = \varprojlim L_m$. The fact that for each $m \geq a$ the maximal ideal of D'_m can be generated by g objects ensures that both R'_∞ and R_∞ are Noetherian.

Just as in Proposition 9.3 of [KW] we get a commutative diagram in $\mathrm{CNL}_{R_{\mathrm{loc}}}$

$$\begin{array}{ccc} R'_\infty & \longrightarrow & R_\infty \\ \downarrow & & \downarrow \\ R'_0 & \longrightarrow & R_0 \end{array}$$

with $R_\infty \to R_0$ a $B[[t_1, \ldots, t_h]]$-algebra morphism, and an $R_\infty \times B[[s_1, \ldots, s_h]]$-module morphism $M_\infty \to M_0$ satisfying parts (1), (2), (3), and (6) of the lemma. Parts (1), (2), and (3), all follow directly from the analogous statements at finite level. To see (6), first note that the free action of $\mathfrak{T}$ on $\mathrm{Spf} R'_\infty$ follows from 2.2.6 and the fact that the group action chunks of $\mathfrak{T}_{p^m}^{[gr_m]}$ on $\mathrm{Spf} D'_{m,n}$ are free. The morphism δ_∞ is defined by the limit of the δ_m. From the corresponding properties of δ_m, it is immediate that the morphism $\mathrm{Spf} R'_\infty \to \mathrm{Spf} R_{\mathrm{loc}}$ is constant on orbits of $\mathfrak{T}$, and that for any object A in $\mathrm{CNL}_{\mathcal{O}}$, $x \in \mathrm{Spf} R'_\infty(A)$ and $\alpha \in \mathfrak{T}(A)$, we have $\delta_\infty(\alpha x) = \alpha^2 \delta_\infty(x)$. It remains to see that the closed immersion $\mathrm{Spf} R_\infty \to \mathrm{Spf} R'_\infty$ identifies $\mathrm{Spf} R_\infty$ with the closed subfunctor of points x with $\delta_\infty(x) = 1$. Note that at finite level, the subfunctor defined by $\delta_m = 1$, is represented by D''_m and there is a surjection $D''_m \to D_m$ with kernel contained in $\mathfrak{m}^m_{D''_m}$. Then the closed the closed subfunctor defined by $\delta_\infty = 1$ is $\varprojlim D''_m \cong \varprojlim D_m = R_\infty$.

We now show (4) and (5). Let $\mathfrak{a}_m$ denote the ideal of D_m as in (c) of patching datum. Because $\varprojlim$ is exact on systems of finite objects, we have

$$N_\infty / \mathfrak{c}_m N_\infty \cong \ker(L_m/(s_1, \ldots, s_h) \to M_0/\mathfrak{c}_m M_0).$$

Then since $\mathfrak{p}_\infty = \varprojlim \mathfrak{a}_m$, we have $\lambda_M N_\infty \subseteq \mathfrak{p}_\infty N_\infty$, which implies $N_\infty / \mathfrak{p}_\infty N_\infty$ is a torsion A module since λ_M has non-zero image in A.

The morphisms $R_{\mathrm{loc}}[[x_1, \ldots, x_k]] \to D'_m$, for each $m \geq 1$, yield a $\mathrm{CNL}_{R_{\mathrm{loc}}}$ morphism $R_{\mathrm{loc}}[[x_1, \ldots, x_k]] \to R'_\infty$. Let $\mathfrak{a}'_m$ be the ideal of D_m as in part (d) of the definition of a patching datum of level m. Since $\mathfrak{p}'_\infty = \varprojlim \mathfrak{a}'_m$ and

$$\lambda_R(\mathfrak{a}'_m/((\mathfrak{a}'_m)^2 + (\mathfrak{p}_{\mathrm{loc}}, x_1, \ldots, x_k)D'_m)) \subseteq \mathfrak{m}^m_{D'_m},$$

we have

$$\lambda_R(\mathfrak{p}'_\infty/((\mathfrak{p}'_\infty)^2 + (\mathfrak{p}_{\mathrm{loc}}, x_1, \ldots, x_k)R'_\infty)) = (0),$$

which implies part (2) of the lemma since λ_R has non-zero image in A.

It remains to show part (7) of the lemma. The fact that $\mathrm{Spf} R_\infty$ is stable under the action of $\mathfrak{T}_2$ follows immediately from part (6) of the Lemma. Note that for $n > 2$, the isomorphism $G'^*_n \cong \mathfrak{T}_{p^{n-2}}$ induces an isomorphism $G^*_{n,2} \cong \mathfrak{T}_2$. If $m \geq n$ the action of $\mathfrak{T}_2(\mathcal{O})$ on $B[[s_1, \ldots, s_h]]$ as in 4.2.8 stabilizes the ideal $\mathfrak{c}_m$ and the compatible actions of $\mathfrak{T}_2(\mathcal{O})$ on R_{n+2} and M_{n+2} descend to compatible actions on $D_{m,n}$ and $L_{m,n}$. Also note that for $\alpha \in \mathfrak{T}_2(\mathcal{O})$, if we let $\alpha' \in \mathfrak{T}_2^{[gr_m]}(\mathcal{O}/\mathfrak{m}_{\mathcal{O}}^{gr_m})$ denote the corresponding truncated group element, the diagram

$$\begin{array}{ccc} D'_m & \longrightarrow & D_m \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ D'_m & \longrightarrow & D_m \end{array}$$

commutes. It follows that, after taking limits, the action of $\mathfrak{T}_2(\mathcal{O})$ on M_∞ is compatible with the action of $\mathfrak{T}_2(\mathcal{O})$ on R_∞. This establishes (7) and the lemma is proved. □

By Proposition 2.5 of [KW], we have a $\mathrm{CNL}_{\mathcal{O}}$ object R^{inv}_∞ that represents the orbits for the action of $\mathfrak{T}$ on $\mathrm{Spf} R'_\infty$ and the morphism $R^{\mathrm{inv}}_\infty \to R'_\infty$ is formally smooth of relative dimension t. Note that as $\mathrm{Spf} R'_\infty \to \mathrm{Spf} R_{\mathrm{loc}}$ is constant on $\mathfrak{T}$-orbits, we have a $\mathrm{CNL}_{\mathcal{O}}$ morphism $R_{\mathrm{loc}} \to R^{\mathrm{inv}}_\infty$ and $R^{\mathrm{inv}}_\infty \to R'_\infty$ is a morphism of R_{loc}-algebras. Lemma 9.4 of [KW] then shows that the map $R^{\mathrm{inv}}_\infty \to R_\infty$ makes R_∞ a torsor over R^{inv}_∞ with group $\mathfrak{T}_2$. Let $\mathfrak{p}^{\mathrm{inv}}_\infty$ denote pullback to R^{inv}_∞ of $\mathfrak{p}_0$. Note that R_∞ is finite over R^{inv}_∞, so $\dim R^{\mathrm{inv}}_\infty/\mathfrak{p}^{\mathrm{inv}}_\infty = \dim R_\infty/\mathfrak{p}_\infty = \dim R_0/\mathfrak{p}_0$ and $2 \in \mathfrak{p}^{\mathrm{inv}}_\infty$. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two primes of R_∞ that lie over $\mathfrak{p}^{\mathrm{inv}}_\infty \in \mathrm{Spec}\, R^{\mathrm{inv}}_\infty$. Choose a characteristic 2 field L with embeddings $R_\infty/\mathfrak{p}_1 \to L$ and $R_\infty/\mathfrak{p}_2 \to L$. Since R_∞ is a $\mathfrak{T}_2$-torsor over R^{inv}_∞, there is $\alpha \in \mathfrak{T}_2(L)$ such that if x denotes the point $R_\infty \to R_\infty/\mathfrak{p}_1 \to L$ of $\mathrm{Spf} R_\infty(L)$ then αx is the point $R_\infty \to R_\infty/\mathfrak{p}_2 \to L$. But since L has characteristic 2,

$\mathfrak{T}_2(L)$ is trivial, and so $\mathfrak{p}_1 = \mathfrak{p}_2$ and we see that $\mathfrak{p}_\infty$ is the unique prime of R_∞ above $\mathfrak{p}_\infty^{\text{inv}}$. It follows that $(R_\infty)_{\mathfrak{p}_\infty^{\text{inv}}}$ is local and the natural map $(R_\infty)_{\mathfrak{p}_\infty^{\text{inv}}} \to (R_\infty)_{\mathfrak{p}_\infty}$ is an isomorphism. It also follows that the $\mathfrak{p}_\infty^{\text{inv}}$-adic topology on $(R_\infty)_{\mathfrak{p}_\infty}$ is the same as the $\mathfrak{p}_\infty$-adic topology on $(R_\infty)_{\mathfrak{p}_\infty}$, since $(R_\infty)_{\mathfrak{p}_\infty}/\mathfrak{p}_\infty^{\text{inv}}(R_\infty)_{\mathfrak{p}_\infty}$ is a finite local algebra over the field $(R_\infty^{\text{inv}})_{\mathfrak{p}_\infty^{\text{inv}}}/\mathfrak{p}_\infty^{\text{inv}}$, hence is Artinian. Similar statements hold for any R_∞-module, in particular the natural map of $(R_\infty^{\text{inv}})_{\mathfrak{p}_\infty^{\text{inv}}}$-modules $(M_\infty)_{\mathfrak{p}_\infty^{\text{inv}}} \to (M_\infty)_{\mathfrak{p}_\infty}$ is an isomorphism and the topologies defined on $(M_\infty)_{\mathfrak{p}_\infty}$ by $\mathfrak{p}_\infty^{\text{inv}}$ and $\mathfrak{p}_\infty$ are equivalent. Note that this module is non-zero as the surjection $M_\infty \to M$ induces a surjection $(M_\infty)_{\mathfrak{p}_\infty} \to (M_0)_{\mathfrak{p}}$.

Lemma 4.3.2. *If $\mathfrak{q}$ be a prime of R_∞^{inv} contained in $\mathfrak{p}_\infty^{\text{inv}}$ and whose image in* $\operatorname{Spec} R_{\text{loc}}$ *contains $\mathfrak{q}_{\text{ur}}$, then $\mathfrak{q}$ is in the support of M_∞ (as an R_∞^{inv}-module).*

Proof. By 4.1.6 and part (d) of the definition of nice prime, cf. 4.1.7, $(R_{\text{loc}}/\mathfrak{q}_{\text{ur}})_{\mathfrak{p}_{\text{loc}}}$ is normal. It follows that $(R_{\text{loc}}/\mathfrak{q}_{\text{ur}})^\wedge_{\mathfrak{p}_{\text{loc}}}$ is also normal and hence a domain, cf. (vii) of Scholie 7.8.3 of [G3]. This implies that there is a unique irreducible component of $\operatorname{Spec}(R_{\text{loc}})^\wedge_{\mathfrak{p}_{\text{loc}}}$ lying above the irreducible component of $\operatorname{Spec} R_{\text{loc}}$ given by $\mathfrak{q}_{\text{ur}}$. We will denote this component by C_{ur}.

Since $(R_\infty)_{\mathfrak{p}_\infty} \cong (R_\infty)_{\mathfrak{p}_\infty^{\text{inv}}}$ is finite over $(R_\infty^{\text{inv}})_{\mathfrak{p}_\infty^{\text{inv}}}$, $(M_\infty)_{\mathfrak{p}_\infty}$ is finite over $(R_\infty^{\text{inv}})_{\mathfrak{p}_\infty^{\text{inv}}}$ and it suffices to show that for any minimal prime $\mathfrak{Q}$ of $(R_\infty^{\text{inv}})^\wedge_{\mathfrak{p}_\infty^{\text{inv}}}$ over $\mathfrak{q}_{\text{ur}}$ is in the support of $(M_\infty)^\wedge_{\mathfrak{p}_\infty}$

Since M_∞ is finite free over $B[[s_1, \ldots, s_h]]$ and $B[[s_1, \ldots, s_h]]$ is finite free over $B[[t_1, \ldots, t_h]]$, we see that the $(\mathfrak{p}_\Lambda, y_1, \ldots, y_j, t_1, \ldots, t_h) \subset B[[t_1, \ldots, t_h]]$ depth of M_∞, as a $B[[t_1, \ldots, t_h]]$-module, is

$$\operatorname{ht}(\mathfrak{p}_\Lambda, y_1, \ldots, y_j, t_1, \ldots, t_h) = d + j + h.$$

Since the image of $(\mathfrak{p}_\Lambda, y_1, \ldots, y_j, t_1, \ldots, t_h)$ in R_∞ is contained in $\mathfrak{p}_\infty$ we see that

$$\operatorname{depth}_{(R_\infty)^\wedge_{\mathfrak{p}_\infty}}((M_\infty)^\wedge_{\mathfrak{p}_\infty}) \geq \operatorname{depth}_{R_\infty}(\mathfrak{p}_\infty, M_\infty) \geq d + j + h.$$

In particular, letting $\mathfrak{a}$ denote the annihilator of $(M_\infty)^\wedge_{\mathfrak{p}_\infty}$ in $(R_\infty)^\wedge_{\mathfrak{p}_\infty}$, we have

$$\dim(R_\infty)^\wedge_{\mathfrak{p}_\infty}/\mathfrak{a} \geq d + j + h. \tag{4.3}$$

Consider the morphism $R_{\rm loc}[[x_1,\ldots,x_k]] \to R'_\infty$ of 4.3.1. By part (3) of 4.3.1, this map induces a surjection

$$(R_{\rm loc})^\wedge_{\mathfrak{p}_{\rm loc}}[[x_1,\ldots,x_k]] \to (R'_\infty)^\wedge_{\mathfrak{p}'_\infty}.$$

Since R'_∞ is formally smooth over $R^{\rm inv}_\infty$ of relative dimension t, and $R^{\rm inv}_\infty/\mathfrak{p}^{\rm inv}_\infty \cong R'_\infty/\mathfrak{p}'_\infty$ there is an isomorphism $R'_\infty \cong R^{\rm inv}_\infty[[z_1,\ldots,z_t]]$ such that sends $\mathfrak{p}'_\infty$ to $\mathfrak{p}^{\rm inv}_\infty + (z_1,\ldots,z_t)$. It follows that $(R'_\infty)^\wedge_{\mathfrak{p}'_\infty}$ is formally smooth over $(R^{\rm inv}_\infty)^\wedge_{\mathfrak{p}^{\rm inv}_\infty}$ of relative dimension t. Now we have

$$\dim(R_\infty)^\wedge_{\mathfrak{p}_\infty} = \dim(R^{\rm inv}_\infty)^\wedge_{\mathfrak{p}^{\rm inv}_\infty} = \dim(R'_\infty)^\wedge_{\mathfrak{p}'_\infty} - t \le 1 + d + 3|S_0| + k - t = d + j + h.$$

Combining this with (4.3), we see that the support of $(M_\infty)^\wedge_{\mathfrak{p}_\infty}$ in $(R_\infty)^\wedge_{\mathfrak{p}_\infty}$ is a union of irreducible components. The same then follows for the support of $(M_\infty)^\wedge_{\mathfrak{p}_\infty}$ in $(R^{\rm inv}_\infty)^\wedge_{\mathfrak{p}^{\rm inv}_\infty}$. Since $(R'_\infty)^\wedge_{\mathfrak{p}'_\infty}$ is formally smooth over $(R^{\rm inv}_\infty)^\wedge_{\mathfrak{p}^{\rm inv}_\infty}$, their irreducible components are in bijection with each other. The rings $(R_{\rm loc})^\wedge_{\mathfrak{p}_{\rm loc}}[[x_1,\ldots,x_k]]$ and $(R'_\infty)^\wedge_{\mathfrak{p}'_\infty}$ have the same dimension, so the surjection $(R_{\rm loc})^\wedge_{\mathfrak{p}_{\rm loc}}[[x_1,\ldots,x_k]] \to (R'_\infty)^\wedge_{\mathfrak{p}'_\infty}$ shows that each irreducible component of $\operatorname{Spec}(R^{\rm inv}_\infty)^\wedge_{\mathfrak{p}^{\rm inv}_\infty}$ lies over a unique irreducible components of $\operatorname{Spec}(R_{\rm loc})^\wedge_{\mathfrak{p}_{\rm loc}}$. Recall there is a unique irreducible component $C_{\rm ur}$ of $\operatorname{Spec}(R_{\rm loc})^\wedge_{\mathfrak{p}_{\rm loc}}$ lying over $\mathfrak{q}_{\rm ur}$. It then suffices to show that there is some prime $\mathfrak{Q} \in \operatorname{Supp}_{(R^{\rm inv}_\infty)^\wedge_{\mathfrak{p}^{\rm inv}_\infty}}(M_\infty)^\wedge_{\mathfrak{p}_\infty}$ such that $C_{\rm ur}$ is the unique irreducible component containing the image of $\mathfrak{Q}$.

To do this we use 4.2.3. Take $\mathfrak{q}$ as in 4.2.3. Pullback $\mathfrak{q}$ to ideals of R_∞ and R'_∞. Since $\mathfrak{q}$ is contained in $\mathfrak{p}_0$, and is in the support of M_0, it is in the support of $(M_\infty)_{\mathfrak{p}_\infty}$. The irreducible component of $\operatorname{Spec} R_{\rm loc}$ determined by $\mathfrak{q}_{\rm ur}$ is the unique irreducible component containing the image of $\mathfrak{q}$ under $\operatorname{Spec} R_\infty \to \operatorname{Spec} R'_\infty \to R_{\rm loc}[[x_1,\ldots,x_k]] \to \operatorname{Spec} R_{\rm loc}$. Letting $\mathfrak{Q}$ be any prime of $(R^{\rm inv}_\infty)^\wedge_{\mathfrak{p}^{\rm inv}_\infty}$ lying over $\mathfrak{q} \cap R^{\rm inv}_\infty$ then has the required properties. $\square$

4.3.3

We can now complete the proof of Proposition 4.1.8. Let $\mathfrak{Q}$ be a minimal prime of R_∞ contained in $\mathfrak{p}_\infty$ whose image in $\operatorname{Spec} R_{\rm loc}$ is contained in $\mathfrak{q}_{\rm ur}$. Let $\mathfrak{q} = \mathfrak{Q} \cap R^{\rm inv}_\infty$. By 4.3.2, $\mathfrak{q}$ is in the support of M_∞. Then at least one minimal prime $\mathfrak{Q}'$ of R_∞ above $\mathfrak{q}$ is in the support of M_∞. Since R_∞ is a $\mathfrak{T}_2$-torsor on $R^{\rm inv}_\infty$ we see that $\mathfrak{T}_2$ acts transitively on the set of minimal primes above $\mathfrak{q}$. Hence we are reduced to showing that the support of M_∞ is stable under this action. Let $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ be two primes above $\mathfrak{q}$. Note that if A is a $\mathrm{CNL}_{\mathcal{O}}$ domain, then the natural map $\mathfrak{T}_2(\mathcal{O}) \to \mathfrak{T}_2(A)$ is surjective. Using this, one can see that that there is $\alpha \in \mathfrak{T}_2(\mathcal{O})$, such that the automorphism of R_∞ induced by α sends $\mathfrak{Q}_1$ to $\mathfrak{Q}_2$. By part (5) of 4.3.1, there is a compatible action of $\mathfrak{T}_2(\mathcal{O})$ on M_∞. It follows that the annihilator of M_∞ is contained in $\mathfrak{Q}_1$ if and only if it is contained in $\mathfrak{Q}_2$, hence the support of M_∞ is stable under the action of $\mathfrak{T}_2$.

Let $\mathfrak{A}$ denote the annihilator of $(M_\infty)^\wedge_{\mathfrak{p}_\infty}$ in $(R_\infty)^\wedge_{\mathfrak{p}_\infty}$. Since $\mathfrak{A} \cap R_\infty$ is contained in any prime ideal of R_∞ whose image in $\operatorname{Spec} R_{\mathrm{loc}}$ contains $\mathfrak{q}_{\mathrm{ur}}$, we see that $\mathfrak{A}$ is contained in any prime ideal of $(R_\infty)^\wedge_{\mathfrak{p}_\infty}$ whose image in $\operatorname{Spec} R_{\mathrm{loc}}$ contains $\mathfrak{q}_{\mathrm{ur}}$. Let $\mathfrak{q} \in \operatorname{Spec}(R_0)^\wedge_{\mathfrak{p}_0}$ be such that its image in $\operatorname{Spec} R_{\mathrm{loc}}$ contains $\mathfrak{q}_{\mathrm{ur}}$. Pull $\mathfrak{q}$ back to a prime of $(R_\infty)^\wedge_{\mathfrak{p}_\infty}$ and call it $\mathfrak{q}_\infty$. By above, $\mathfrak{q}_\infty$ is in the support of $(M_\infty)^\wedge_{\mathfrak{p}_\infty}$. Since $t_1, \ldots, t_h \in \ker((R_\infty)^\wedge_{\mathfrak{p}_\infty} \to (R_0)^\wedge_{\mathfrak{p}_0})$, we see that $\mathfrak{q}_\infty/(t_1, \ldots, t_h) \in \operatorname{Spec}(R_\infty)^\wedge_{\mathfrak{p}_\infty}/(t_1, \ldots, t_h)$ is in the support of $(M_\infty)^\wedge_{\mathfrak{p}_\infty}/(t_1, \ldots, t_h)$. Since $t_i^2 \in (s_1, \ldots, s_h)(M_\infty)^\wedge_{\mathfrak{p}_\infty}$ for each i, we further deduce that $\mathfrak{q}_\infty/(t_1, \ldots, t_h)$ is in the support of $(M_\infty)^\wedge_{\mathfrak{p}_\infty}/(s_1, \ldots, s_h)$ as an $(R_\infty)^\wedge_{\mathfrak{p}_\infty}/(t_1, \ldots, t_h)$-module. But, by (3) and (4) of 4.3.1, $(M_\infty)^\wedge_{\mathfrak{p}_\infty}/(s_1, \ldots, s_h) \cong (M_0)^\wedge_{\mathfrak{p}_0}$. Then, since the action of $(R_\infty)^\wedge_{\mathfrak{p}_\infty}/(t_1, \ldots, t_h)$ on $(M_0)^\wedge_{\mathfrak{p}_0}$ factors through $(R_0)^\wedge_{\mathfrak{p}_0}$ and this maps sends $\mathfrak{q}_\infty/(t_1, \ldots, t_h)$ to $\mathfrak{q}$, we conclude that $\mathfrak{q}$ is in the support of $(M_0)^\wedge_{\mathfrak{p}_0}$ □

4.4 An $R^{\mathrm{red}} = \mathbf{T}$ theorem

Unless specifically noted otherwise, keep the assumptions and notations of the previous subsections. If $\overline{\rho}$ is dihedral we let L/F denote the unique quadratic extension of F such that $\overline{\rho}_{G_L}$ is Abelian. Let L_S^{ab} denote the maximal pro-2 extension of L unramified outside places above S. Let L_S^- denote the maximal subextension of L_S^{ab}/L such that the nontrivial element of $\mathrm{Gal}(L/F)$ acts on $\mathrm{Gal}(L_S^-/L)$ by -1.

Lemma 4.4.1. *Assume that*

(i) for each $v|2$, $[F_v : \mathbb{Q}_2] \geq 4$;

(i) if L/F is CM, then there is some $v|2$ in F that does not split in L;

(ii) if L/F is not CM, then $\mathrm{rank}_{\mathbb{Z}_p}\mathrm{Gal}(L_S^-/F) < [F:\mathbb{Q}]-3$.

Take $\mathfrak{Q} \in \operatorname{Spec}\mathbf{T}_\psi(U,\eta)_\mathfrak{m}$ with $2 \in \mathfrak{Q}$ such that $\dim \mathbf{T}_\psi(U,\eta)_\mathfrak{m}/\mathfrak{Q} \geq [F:\mathbb{Q}]-3$.

1. *Letting $\chi_{\mathfrak{Q},v}$ denote the resulting $\mathbf{T}_\psi(U,\eta)_\mathfrak{m}/\mathfrak{Q}$ valued character coming from specializing $\chi_{\eta_v}^{\mathrm{univ}}|_{I_v}$, $\chi_{\mathfrak{Q},v}$ is infinite order.*

2. *The the specialization $\rho_\mathfrak{Q}$ of $\rho_{U,\mathfrak{m}}$ at $\mathfrak{Q}$ is non-dihedral.*

Proof. Since $\mathbf{T}_\psi(U,\eta)_\mathfrak{m}$ is finite and torsion-free over $\Lambda(I_p,\eta) = \hat{\otimes}_{v|2}\Lambda(I_v,\eta_v)$, part (1) follows from the fact that $\dim \Lambda(I_v,\eta_v)\otimes_\mathcal{O}\mathbb{F} = [F_v:\mathbb{Q}_2] \geq 4$, and so $\mathfrak{Q}\cap\Lambda(I_v,\eta_v)$ defines a non-maximal prime ideal in $\Lambda(I_v,\eta_v)/(\varpi_E)$.

Part (2) is trivial if $\overline{\rho}$ is not dihedral, so assume $\overline{\rho}$ is L-dihedral. First assume that L/F is CM. Let $v|2$ in F be such that v does not split in L. If $\rho_\mathfrak{Q}$ where dihedral, then part (3) of 3.4.4 together with 2.7.8 contradict the fact that $\chi_{\mathfrak{Q},v}$ is infinite order.

Now assume that L/F is not CM. Let B denote the subring of $\mathbf{T}_\psi(U,\eta)_\mathfrak{m}/\mathfrak{Q}$ generated by traces of $\rho_\mathfrak{Q}$. Note that $\mathbf{T}_\psi(U,\eta)_\mathfrak{m}/\mathfrak{Q}$ is integral over B, and so $\dim B \geq [F:\mathbb{Q}]-3$. Then we know that the image of $R_{F,S}\to\mathbf{T}_\psi(U,\eta)_\mathfrak{m}/\mathfrak{Q}$ contains B, and so has dimension $\geq [F:\mathbb{Q}]-3$. The result now follows from 2.7.7 as we have assumed $\mathrm{rank}_{\mathbb{Z}_p}\mathrm{Gal}(L_S^-/F) < [F:\mathbb{Q}]-3$. □

Proposition 4.4.2. *Assume*

(i) for each $v|2$, $[F_v:\mathbb{Q}_2]\geq 4$;

(i) if L/F is CM, then there is some $v|2$ in F that does not split in L;

(ii) if L/F is not CM, then $\mathrm{rank}_{\mathbb{Z}_p}\mathrm{Gal}(L_S^-/F) < [F:\mathbb{Q}]-3$.

Then every prime of $\overline{R}_{F,S}^\psi$ is pro-modular.

Proof. This proof is essentially the same as Proposition 4.1 of [SW2]. We first show that there exists a nice prime, cf. 4.1.7, in $\operatorname{Spec}\overline{R}_{F,S}^\psi$. For each $v|2$, let Z_v denote the closed subscheme of $\operatorname{Spec}\Lambda(I_v,\eta_v)$ defined by $(\chi^{\mathrm{univ}}_{\eta_v}|_{I_v})^2=\psi\epsilon_2$. Note that for each $v|2$, if $\mathfrak{p}_v\subset Z_v$ and $2\in\mathfrak{p}_v$, then the pushforward of $\chi^{\mathrm{univ}}_{\eta_v}|_{I_v}$ to $\Lambda(I_v,\eta_v)/\mathfrak{p}$ is finite order (in fact trivial by our assumptions on $\overline{\rho}$). For ease of notation, set $R=\overline{R}^{\psi}_{F,S}$, $\mathbf{T}=\mathbf{T}_{\psi}(U,\eta)_{\mathfrak{m}}$, $\Lambda=\Lambda(I_2,\eta)$, and $\Lambda_v=\Lambda(I_v,\eta_v)$.

In order to show R has a nice prime it suffices to show that $\mathbf{T}$ has a prime $\mathfrak{p}$ such that

(a) $\mathfrak{p}$ is a dimension one prime containing 2;

(b) $\rho_{\mathfrak{p}}$ is not dihedral;

(c) for each $v|2$, the image of $\mathfrak{p}$ in $\operatorname{Spec}\Lambda_v$ is not contained in Z_v;

(d) the image of $\rho_{\mathfrak{p}}$ contains a non-trivial unipotent element.

Let $\mathfrak{Q}_0$ be a minimal prime of $\mathbf{T}/(\varpi_E)$. Then $\mathbf{T}/\mathfrak{Q}_0$ if a finite torsion-free $\Lambda/(\varpi_E)$-algebra, hence has dimension d. Fix some $\sigma\in G_F$ such that $\overline{\rho}(\sigma)$ has order two and let $T=\operatorname{tr}\rho_{\mathfrak{Q}_0}(\sigma)$. Note T is contained in the maximal ideal of $\mathbf{T}/\mathfrak{Q}_0$ since $\overline{\rho}(\sigma)$ has order two. Let $\mathfrak{Q}$ be a minimal prime of $\mathbf{T}$ containing $(T,\mathfrak{Q}_0)$ and note that it has dimension $\geq[F:\mathbb{Q}]-1$. By part (1) of 4.4.1, $\mathfrak{Q}\cap\Lambda(I_v,\eta_v)$ is not contained in Z_v for any $v|2$, and by part (2) it is not dihedral. Since the diheral locus is closed, cf. 2.7.6, the locus of primes in $\operatorname{Spec}\mathbf{T}/\mathfrak{Q}$ satisfying (b) and (c) above is open, and we have just seen that it is non-empty. Since $\operatorname{Spec}(\mathbf{T}/\mathfrak{Q})\smallsetminus\{\mathfrak{m}\}$ is Jacobson, this open set contains a dimension 1 prime. Let $\mathfrak{p}$ be such a prime. It remains to show that the image of $\rho_{\mathfrak{p}}$ contains a non-trivial unipotent element. Let $\sigma\in G_F$ be as above. Since $\mathfrak{p}\subseteq\mathfrak{Q}$, and both $\rho_{\mathfrak{Q}}(\sigma)$ and $\overline{\rho}(\sigma)$ have order two, $\rho_{\mathfrak{p}}(\sigma)$ has order two. Since $2\in\mathfrak{p}$, $\rho_{\mathfrak{p}}(\sigma)$ is unipotent.

Let $\mathcal{C}$ denote the set of irreducible components of $\operatorname{Spec}R$. Note that by 4.1.8 any irreducible component containing $\mathfrak{p}$ as above is pro-modular. Hence we can take a partition $\mathcal{C}=\mathcal{C}_1\sqcup\mathcal{C}_2$ of $\mathcal{C}$ into two disjoint subsets such that $\mathcal{C}_1$ is non-empty and every element of $\mathcal{C}_1$ is pro-modular. We will show that if $\mathcal{C}_2$ is non-empty there is $C_1\in\mathcal{C}_1$ and $C_2\in\mathcal{C}_2$ such that $C_1\cap C_2$ contains a nice prime. Applying 4.1.8 again shows C_2 is pro-modular. Continuing in this way we conclude all irreducible components of $\operatorname{Spec}R$ are pro-modular.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be as above. By 2.6.6, there is a presentation $R \cong A/(f_1, \ldots, f_r)$, with A a complete local domain and $\dim A - r \geq 1 + [F : \mathbb{Q}] - \dim_{\mathbb{F}} H^0(G_F, (\mathrm{Ad}^0_{\mathbb{F}})^*(1))$. Since $\overline{\rho}$ is absolutely irreducible $\dim_{\mathbb{F}} H^0(G_F, (\mathrm{Ad}^0_{\mathbb{F}})^*(1)) = \dim_{\mathbb{F}} H^0(G_F, \mathrm{Ad}_{\mathbb{F}}/Z) \leq 1$. Then 2.1.8 implies that there is $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$ such that $C_1 \cap C_2$ contains a prime $\mathfrak{Q}_0$ of dimension $[F : \mathbb{Q}] - 1$. Fix some $\sigma \in G_F$ such that $\overline{\rho}(\sigma)$ has order two and let $T = \mathrm{tr}\,(\rho_{\mathfrak{Q}_0}(\sigma))$. Let $\mathfrak{Q}$ be a minimal prime of $\overline{R}^{\psi}_{F,S}/(\mathfrak{Q}_0, \varpi_E, T)$ and note that the dimension of $\mathfrak{Q}$ is $\geq [F : \mathbb{Q}] - 3$ and that it is pro-modular since C_1 is pro-modular. Then 4.4.1 implies that $\mathfrak{Q} \cap \Lambda_v$ is not contained in Z_v for any $v|2$ and that $\rho_{\mathfrak{Q}}$ is non-dihedral. Since the locus of primes in $\mathrm{Spec}\, R/\mathfrak{Q}$ that are non-dihedral and do not lie over Z_v for any $v|2$ is open and non-empty, it contains a dimension one prime $\mathfrak{p}$, as $\mathrm{Spec}\,(R/\mathfrak{Q}) \smallsetminus \{\mathfrak{m}_R\}$ is Jacobson. As above we also deduce, for our fixed $\sigma \in G_F$, that $\rho_{\mathfrak{p}}(\sigma)$ is non-trivial unipotent. □

CHAPTER 5

The Main Theorem

We are now in a position to prove the main theorem. Before doing so, in the first section we recall some congruences proved in [KW] and [K2] that are necessary to be able to satisfy the assumptions of the $R^{\mathrm{red}} = \mathbf{T}$ theorem. We also prove a small lemma that shows the existence of ordinary lifts in the residually dihedral case. This is an application of a result of Wiles that allows one to insert a p-ordinary Hilbert modular form of parallel weight 1 into a p-adic family.

In the second section we prove the main theorem. This is mostly routine, except that we must use some known cases of Leopoldt's conjecture in our base changes, to ensure the assumptions of 4.4.2 are met.

5.1 Congruences

For simplicity we fix an isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ throughout this section.

5.1.1

We fix a continuous absolutely irreducible

$$\overline{\rho} : G_F \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}})$$

such that for each $v|p$, $\overline{\rho}|_{G_v}$ is reducible. Write

$$\overline{\rho}_{G_v} \cong \begin{pmatrix} \overline{\chi}'_v & * \\ & \overline{\chi}_v \end{pmatrix}.$$

Let $\overline{\chi}$ denote the tuple $\overline{\chi} = (\overline{\chi}_v)_{v|p}$. If we are given any finite field extension F'/F we will still denote by $\overline{\chi}$ the tuple $(\overline{\chi}_v|_{G_w})_{w|v,v|p}$.

Given a cuspical automorphic representaion π of $\mathrm{GL}_2(\mathbb{A}_F)$, we say that π *lifts* $\overline{\rho}$ if, letting ρ_π be the representation as in 3.4.1, there is a G_F-stable lattice in the representation space of ρ_π whose reduction is equal to $\overline{\rho}$ (after extension of scalars, if necessary). We say that π is a $\overline{\chi}$-*good lift* of $\overline{\rho}$ if π is p-nearly ordinary and, for each $v|p$, we have

$$\rho_\pi|_{G_v} \cong \begin{pmatrix} * & * \\ & \chi_v \end{pmatrix}$$

with χ_v a lift of $\overline{\chi}_v$. Similarly if f is an eigenfunction in some $S^{\mathrm{no}}_{\kappa,\psi}(U,\mathcal{O})$ as in 3.2.2, and π_f denotes the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ associated to it by 3.1.10, we say f *lifts* $\overline{\rho}$ if π_f does, and that f is a $\overline{\chi}$-*good lift* of $\overline{\rho}$, if π_f is. We keep track of $\overline{\chi}$-good lifts in what follows. This isn't necessary for our applications to modularity of Galois representations (we will perform a base change to assume that $\overline{\rho}_{G_v}$ is either trivial or unipotent for each $v|p$) however we do so as it entails relatively little extra work.

Lemma 5.1.2. *If $\overline{\rho}$ is dihedral, there is a p-nearly ordinary regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_(\mathbb{A}_F)$ of weight $((k,\dots,k),(0,\dots,0))$, with $k \geq 2$, that is a $\overline{\chi}$-good lift of $\overline{\rho}$*

Proof. We first note that the reducibility of $\overline{\rho}|_{G_v}$ together with the assumption that $\overline{\rho}$ is dihedral implies that $\overline{\rho}|_{G_v}$ implies that either $\overline{\rho}|_{G_v}$ is split or $\overline{\chi}_v = \overline{\chi}'_v$ and $p = 2$. For each $v|p$, let χ_v denote the Teichmüller lift of $\overline{\chi}_v$ and, except in the case that $p = 2$ and $\overline{\rho}|_{G_v}$ is nonsplit, we let χ'_v be the Teichmúller lift of $\overline{\chi}'_v$. In the case that $p = 2$ and $\overline{\rho}|_{G_v}$ is nonsplit, we let w denote the unique prime above v in L, where L is the unique quadratic extension from which $\overline{\rho}$ is induced, and G_w the corresponding index two subgroup G_v. We then let χ'_v be the $\mathcal{O}$-valued character such that $\chi'_v(\sigma) = \chi_v$ for all $\sigma \in G_w$ and $\chi'_v(\sigma) = -\chi_v(\sigma)$ for $\sigma \in G_v \smallsetminus G_w$. Note that this is possible since when $p = 2$, χ_v has odd order and so the field fixed by its kernel is disjoint from L_w/F_v. We view χ_v and χ'_v as taking values in $\overline{\mathbb{Q}}^\times$.

It suffices to prove this in the case that $\overline{\chi}_v$ is unramified for each $v|p$. Otherwise, we take a finite order character $\theta : F^\times\backslash\mathbb{A}_F^\times \to \overline{\mathbb{Q}}^\times$ such that $\theta|_{\mathcal{O}_{F_v}^\times} = \chi_v|_{\mathcal{O}_{F_v}^\times}$ for each $v|p$, cf. Theorem 5, Chapter 10 of [AT]. Then letting π denote the resulting lift of $\overline{\rho}\otimes\overline{\theta}^{-1}$, $\pi\otimes\theta$ will be the desired $\overline{\chi}$-good lift of $\overline{\rho}$.

We first construct a totally odd dihedral representation $\rho_0 : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}})$ lifting $\overline{\rho}$. Write $\overline{\rho} = \mathrm{Ind}_{G_F}^{G_L}\overline{\chi}$ for a quadratic extension L/F and $\overline{\chi} : G_L \to \mathbb{F}^\times$. Let $\chi : G_L \to \overline{\mathbb{Q}}^\times$ denote the Teichmuller lift of χ. If $\overline{\rho}(c) \neq 1$ for any choice c of complex conjugation, we set $\rho_1 = \mathrm{Ind}_{G_L}^{G_F}\chi$. If there is some choice of complex conjugation at which $\overline{\rho}$ is trivial, we use a trick of Serre. Note that in this case $p = 2$. Let $\{\tau_1,\ldots,\tau_k\}$ denote the set of embeddings $F \hookrightarrow \mathbb{R}$ where $\overline{\rho}$ is trivial. Then each τ_i splits in L/F, and we write σ_i, σ_i' for the two embeddings above τ_i. Theorem 5 of Chapter 10 of [AT] implies the existence of a character $\xi : G_L \to \overline{\mathbb{Q}}^\times$ of order either two or four, such that ξ nontrivial at each $\{\sigma_1,\ldots,\sigma_k\}$ and trivial at each $\{\sigma_1',\ldots,\sigma_k'\}$ as well as at every place above 2. Note that if ξ' denotes the conjugate of ξ by $\mathrm{Gal}(L/F)$, we have $\xi(c)\xi'(c) = -1$ if c is the complex conjugation corresponding to any $\tau\in\{\tau_1,\ldots,\tau_k\}$. We then set $\rho_1 = \mathrm{Ind}_{G_L}^{G_F}\chi\xi$. By choice of ξ, ρ_1 is totally odd. Since ξ has order two or four, it is trivial mod 2, and ρ_0 is a lift of $\overline{\rho}$. Also, since ξ is trivial at any place above 2 we have

$$\rho_1|_{G_v} \cong \begin{pmatrix} \chi_v' & \\ & \chi_v \end{pmatrix}$$

for χ_v' and χ_v as above.

A classical construction yields a cuspidal Hilbert modular newform f_1 of weight $((1,\ldots,1),(0,\ldots,0))$, such that $\rho_{f_1}\cong\rho_1$. Let π_{f_1} denote the corresponding automorphic representation. For each $v|p$, $(\pi_{f_1})_v$ is the principal series $\pi(\chi_v',\chi_v)$. If χ_v' is unramified then the double coset operator

$$\left[\mathrm{GL}_2(\mathcal{O}_{F_v}\begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix}\mathrm{GL}_2(\mathcal{O}_{F_v})\right]$$

acts on $\pi(\chi_v',\chi_v)^{\mathrm{GL}_2(\mathcal{O}_{F_v})}$ via $\chi_v'(\varpi_v)+\chi_v(\varpi_v)$. In this case we replace f_1 with the v-stabilized eigenform on which the double coset operator

$$\left[\mathrm{Iw}(v)\begin{pmatrix}\varpi_v & \\ & 1\end{pmatrix}\mathrm{Iw}(v)\right]$$

acts via $\chi_v(\varpi_v)$. We may then assume that $T_{\varpi_v} f_1 = \chi_v(\varpi_v) f_1$ for each $v|p$. Theorem 3 of [W] allows us to insert f_1 in an ordinary p-adic analytic family and letting f_k denote a classical specialization at some parallel weight $k \geq 2$, we have $T_{\varpi_v} f_k = \alpha_v f_k$ for some α_k congruent to $\overline{\chi}_v(\varpi_v)$ for each $v|p$. This automorphic representation generated by this f_k is then a $\overline{\chi}$-good lift of $\overline{\rho}$. □

5.1.3

Let $\kappa = (\mathbf{k}, \mathbf{w})$ be an algebraic weight and let $\psi : F^\times \backslash \mathbb{A}_F^\times \to \mathcal{O}^\times$ be a continuous character such that $\psi(z) = z_p^{2-\mathbf{k}-2\mathbf{w}}$ on some open subgroup of $\mathbb{A}_F^\times$. We denote by $\psi_{\mathbb{C}}$ the character $\psi_{\mathbb{C}} : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ given by $\psi_{\mathbb{C}}(z) = \psi(z) z_p^{\mathbf{k}+2\mathbf{w}-2} z_\infty^{2-\mathbf{k}-2\mathbf{w}}$, using our fixed isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$. Conversely, given a character $\psi_{\mathbb{C}} : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ such that $\psi_{\mathbb{C}}(z) = z_\infty^{2-\mathbf{k}-2\mathbf{w}}$ on some open subgroup of $\mathbb{A}_F^\times$, we let ψ denote the character $\psi : F^\times \backslash \mathbb{A}_F^\times \to \mathcal{O}^\times$ (enlarging $\mathcal{O}$ if necessary) given by $\psi(z) = \psi_{\mathbb{C}}(z) z_\infty^{\mathbf{k}+2\mathbf{w}-2} z_p^{2-\mathbf{k}+2\mathbf{w}}$.

In what follows we can always ensure that the base changes performed are disjoint from any fixed finite extension of F as follows. Let K be a given finite extension of F and let K' denote its normal closure. Let $\{v\}$ be a finite set of places of F, unramified in K' and such that the Frob_v exhaust all conjugacy classes in $\mathrm{Gal}(K'/F)$. Note that this set can always be chosen to be disjoint from any given finite set of places of F. We then demand that our extension F'/F splits at all places in $\{v\}$. We will not repeat this argument in each of the lemmas below.

We will call an extension F'/F an *allowable base change* if F'/F is finite of even degree, solvable, totally real, and disjoint from $\overline{F}^{\ker \overline{\rho}}$.

Lemma 5.1.4. *Let π be a regular algebraic cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_F)$ *of weight κ and central character $\psi_{\mathbb{C}}$. Assume that π is p-nearly ordinary and a $\overline{\chi}$-good lift* *of $\overline{\rho}$. Let $\kappa' = (\mathbf{k}', \mathbf{w}')$ be another algebraic weight and let $\psi'_{\mathbb{C}} : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ be such that $\psi'_{\mathbb{C}}(z) = z_\infty^{2-\mathbf{k}'-2\mathbf{w}'}$ on some open subgroup of $\mathbb{A}_F^\times$.*

If $\psi \cong \psi'$ modulo the maximal ideal of $\mathcal{O}$, then there is an allowable base change F'/F and a p nearly ordinary regular algebraic cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_{F'})$ such that

- *the central character of π' is $\mathrm{Nm}_{F'/F} \circ \psi'_{\mathbb{C}}$;*
- *if w is an archimedean place of F' that extends the embedding $\tau : F \hookrightarrow \mathbb{R}$, then π'_w is discrete series of lowest weight $k'_\tau - 1$ and central character $z_v \mapsto \mathrm{sgn}(z_v)^{k'_\tau} |z_v|^{2-k'_\tau-2w'_\tau}$;*
- *for $v|p$, $(\pi'_v)^{\mathrm{Iw}_1(v)} \neq 0$;*
- *π' is a $\overline{\chi}$-good lift of $\overline{\rho}$.*

Proof. Since π_v is either principal series or Steinberg at each $v|p$, there is a base change F'/F of even degree, disjoint from the splitting field of $\overline{\rho}$, such that letting π_0 denote the base change of π to F', $(\pi_0)_w^{\mathrm{Iw}_1(w)} \neq 0$ for any $w|p$ in F'. We again denote by κ and κ' the weights of F' obtained from κ and κ' in the obvious way. We also again write $\psi_{\mathbb{C}}$ and $\psi'_{\mathbb{C}}$ instead of $\psi_{\mathbb{C}} \circ \mathrm{Nm}_{F'/F}$ and $\psi'_{\mathbb{C}} \circ \mathrm{Nm}_{F'/F}$.

Let D be the quaternion algebra over F' ramified at all infinite places and split at all finite places. Using the Jacquet-Langlands-Shimizu correspondence 3.1.10 and 3.4.1, it suffices to show that we have a Hecke equivariant isomorphism

$$S^{\mathrm{no}}_{\kappa,\psi}(U, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \cong S^{\mathrm{no}}_{\kappa',\psi'}(U, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F}$$

for U an appropriate open subgroup of $\mathrm{GL}_2(\mathcal{O}_{F'} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$. Note in particular that the equivariance of this isomorphism with respect to T_{ϖ_v}, for each $v|p$, and $\langle y \rangle^{\mathrm{no}}$, for each $y \in (\mathcal{O}_{F'} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$, ensures that π' will be a $\overline{\chi}$-good lift of $\overline{\rho}$.

Choose open subgroup $U \subseteq \mathrm{GL}_2(\mathcal{O}_{F'} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$ such that $U_v = \mathrm{Iw}_1(v)$ for all $v|p$, and small enough such that $\pi_0^U \neq 0$ and

$$(U(\mathbb{A}_{F'}^\infty)^\times \cap t^{-1} D^\times t)/(F')^\times = 1 \tag{5.1}$$

for every $t \in \mathrm{GL}_2(\mathbb{A}_{F'}^\infty)$. Since U satisfies (5.1), we have

$$S^{\mathrm{no}}_{\kappa,\psi}(U,\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \cong S^{\mathrm{no}}_{\kappa,\psi}(U,\mathbb{F})$$

and similarly for $S_{\kappa',\psi'}(U,\mathcal{O})$. Since $U = U(p^{1,1})$ and $\psi \cong \psi'$ modulo the maximal ideal of $\mathcal{O}$, 3.3.7 give Hecke equivariant isomorphisms

$$S^{\mathrm{no}}_{\kappa,\psi}(U,\mathbb{F}) \cong S^{\mathrm{no}}_{2,\psi}(U,\mathbb{F}) \cong S^{\mathrm{no}}_{\kappa',\psi'}(U,\mathbb{F}).$$

□

Lemma 5.1.5. *Let K be a fixed finite extension of F. Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with central character $\psi_{\mathbb{C}}$. Assume that π is a p-nearly ordinary $\overline{\chi}$-good lift of $\overline{\rho}$. Let Σ be some (possibly empty) set of finite places of F, disjoint from $\{v|p\}$, such that for each $v \in \Sigma$, $\pi_v \cong (\gamma_v \circ \det) \otimes \mathrm{St}$, for γ_v a character of $F_v^\times$.*

There allowable base change F'/F and a p-nearly ordinary regular algebraic cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_{F'})$, such that

- *the central character of π' is $\psi_{\mathbb{C}} \circ \mathrm{Nm}_{F'/F}$;*
- *for any archimdean place v of F and $w|v$ in F', $\pi_v \cong \pi'_w$ as $(\mathfrak{gl}_2, \mathrm{O}(2))$-modules;*
- *for any finite place w note above p or any of the places of Σ, π'_v is unramified;*
- *for $v \in \Sigma$ and $w|v$, $\pi'_w \cong (\gamma_v \circ \mathrm{Nm}_{F'_w/F_v} \circ \det) \otimes \mathrm{St}$;*
- *$(\pi'_v)^{\mathrm{Iw}_1(v)} \neq 0$ for $v|p$;*
- *π' is a $\overline{\chi}$-good lift of $\overline{\rho}$.*

Proof. For ease of notation, we will throughout denote the base change of a character by the same letter, in particular again write ψ and γ_v for $\psi \circ \mathrm{Nm}_{F'/F}$ and $\gamma_v \circ \mathrm{Nm}_{F'_w/F_v}$, respectively.

Using local-global compatibility and the representation ρ_π, we can find an allowable base change F_1/F, such that, letting π_1 denote the base change of π to F_1, if v is a finite place of F_1 with $v \nmid p$ at which $(\pi_1)_v$ is ramified, then $(\pi_1)_v$ is an unramified twist of the Steinberg representation, and such that $(\pi_1)_v^{\mathrm{Iw}_1(v)} \neq 0$ for any $v|p$. Let Σ_1 denote the set of places of F_1 above Σ. Let Q_1 denote the set of places $\{v\}$ such that $(\pi_1)_v$ is ramified and $v \notin \Sigma_1 \cup \{v|p\}$.

Choose a finite place $w_0 \notin \Sigma_1 \cup Q \cup \{v|p\}$, and let N_{w_0} be the order of $\mathrm{GL}_2(k_{w_0})$. Choose an allowable base change F_2/F_1, split at w_0, such that for any finite place v above a place in Q, the order of the p-subgroup of $k_v^\times$ is divisible by the p-part of $2p(4N_{w_0})$. Let Σ_2 denote the set of places of F_2 above Σ_1 and similarly for Q_2. Let π_2 denote the base change of π_1 to F_2.

Let D denote the quaternion algebra over F_2, ramified at all archimedean places and places in Σ_2, and split elsewhere, and fix a maximal order $\mathcal{O}_D$ of D and an fix isomorphisms $(\mathcal{O}_D)_v \cong \mathrm{M}_{2\times 2}(\mathcal{O}_{F_v})$ for every split v. Let U denote the open subgroup of $\mathcal{O}_D^\times$ such that

- $U_v = \mathrm{GL}_2(\mathcal{O}_{F_{2,v}})$ if $v \notin \Sigma_2^c \cup Q_2\{w|p\}$;
- $U_v = D_v^\times$ if $v \in \Sigma_2$;
- $U_v = \mathrm{Iw}_1(v)$ if $v \in Q$;
- $\mathrm{Iw}(v) \subseteq U_v \subseteq \mathrm{Iw}_1(v^n)$ for some $n \geq 1$ if $v|p$;
- if V denoted the open compact subgroup of $\mathrm{GL}_2(A_{F_2}^\infty)$ given by $V_v = U_v$ for $v \notin \Sigma_2$ and $V_v = \mathrm{Iw}(v)$ for $v \in \Sigma_2$, then $\pi_2^V \neq 0$.

Let κ denote the weight of π_2. Via the Jaquet-Langlands-Shimizu correspondence, π_2 is generated by a Hecke eigenform in $\mathcal{S}_{\kappa,\psi}^{\mathrm{no}}(U, \mathcal{O})$ (for suitable $\mathcal{O}$).

Define the open compact subgroup U' of $\mathrm{GL}_2(\mathbb{A}_{F_2}^\infty)$ be letting $U'_v = U_v$ for $v \notin Q_2$, and for $v \in Q_2$ we set

$$U'_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Iw}(v) : a = d \bmod \mathfrak{m}_{\mathcal{O}_{F_{2,v}}} \right\}.$$

By 3.1.4, we can take a non-trivial character $\chi : \prod_{v \in Q_2} k_v^\times \to \mathcal{O}^\times$ of p-power order, and of order divisible by 4 if $p = 2$, such that when viewed as a character of $U(\mathbb{A}_{F_2}^\infty)^\times$, trivial on U', χ is trivial on $(U(\mathbb{A}_{F_2}^\infty)^\times \cap t^{-1} D^\times t)/F^\times$ for any $t \in (D \otimes_{F_2} \mathbb{A}_{F_2}^\infty)^\times$.

Viewing $W_\kappa(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}(\chi)$ as a representation of $U(\mathbb{A}_F^\infty)^\times$, we let $S_{\kappa\otimes\chi,\psi}(U,\mathcal{O})$ denote the space of functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_{F_2}^\infty)^\times \longrightarrow W_\kappa(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}(\chi)$$

such that $f(xu) = u^{-1} f(x)$ for all $x \in U$ and $f(zx) = \psi(z) f(z)$ for all $z \in (\mathbb{A}_{F_2}^\infty)^\times$. Fixing a set of representatives $\{t\}$ for the double cosets $D^\times \backslash (D \otimes_{F_2} \mathbb{A}_{F_2}^\infty)^\times / U(\mathbb{A}_{F_2}^\infty)^\times$, we have an isomorphism of $\mathcal{O}$-modules

$$S_{\kappa\otimes\chi,\psi}(U,\mathcal{O}) \cong \bigoplus_{\{t\}} W_\kappa(\mathcal{O})(\chi)^{(U(\mathbb{A}_{F_2}^\infty)^\times \cap t^{-1} D^\times t)/F_2^\times}.$$

Since χ is trivial on each of the $(U(\mathbb{A}_{F_2}^\infty)^\times \cap t^{-1} D^\times t)/F_2^\times$, and χ is of p-power order, $S_{\kappa\otimes\chi,\psi}(U,\mathcal{O})$ and $S_{\kappa,\psi}(U,\mathcal{O})$ have the same $\mathcal{O}$-rank and the same image in $S_{\kappa,\psi}(U,\mathbb{F})$. The isomorphism

$$S_{\kappa,\psi}(U,\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \cong S_{\kappa\otimes\chi,\psi}(U,\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F}$$

then implies the existence of a nearly ordinary maximal ideal $\mathfrak{m}$ in the Hecke subalgebra of $\mathrm{End}(S_{\kappa\otimes\chi,\psi}(U,\mathcal{O}))$, with $\overline{\rho}_{\mathfrak{m}} \cong \overline{\rho}$. Note that the nearly ordinary condition is preserved since Q does not containing any places above p (the action of T_p is unchanged).

If π^D is a cupsidal automorphic representation of $(D \otimes_{F_2} \mathbb{A}_{F_2})^\times$, generated by a Hecke eigenform of $S_{\kappa\otimes\chi,\psi}(U,\mathcal{O})_{\mathfrak{m}}$, and π_2' denotes the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{F_2})$ obtained from π^D via the Jacquet-Langlands-Shimizu correspdonance, cf. 3.1.10, then we have

- the central character of π_2' is $\psi_{\mathbb{C}} \circ \mathrm{Nm}_{F_2/F}$;
- for any archimdean place v of F and $w|v$ in F_2, $(\pi_2')_w \cong \pi_v$ as $(\mathfrak{gl}_2, \mathrm{O}(2))$-modules;
- for any finite place $w \notin \Sigma_2 \cup Q_2$ and not above p, $(\pi_2')_w$ is unramified;
- for any $v \in \Sigma_2$, $(\pi_2')_v \cong (\gamma_v \circ \det) \otimes \mathrm{St}$;
- for any $v \in Q_2$, $(\pi_2')_v$ is a ramified principal series;
- π_2' is p-nearly ordinary;

- for each $v|p$, $(\pi'_2)_v^{\mathrm{Iw}_1(v)} \neq 0$;

- π'_2 is a $\overline{\chi}$-good lift of $\overline{\rho}|_{G_{F_2}}$.

We can then find another allowable base change F'/F_2, such that the characters defining the principal series representations $(\pi'_2)_v$, for $v \in Q_2$, become unramified. Letting π' denote the base change of π'_2 to F' gives the result. □

Lemma 5.1.6. *Assume $p = 2$. Let D be a quaternion algebra with centre F, ramified at all archimedean places and at a nonempty set Σ of finite places not containing any places above p. Fix a maximal order $\mathcal{O}_D$ of D. Fix an algebraic weight κ, a continuous character $\psi : F^\times \backslash (\mathbb{A}_F^\infty)^\times \to \mathcal{O}^\times$, and an open subgroup U of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ such that*

- *$U_v \subseteq \mathrm{GL}_2(\mathcal{O}_{F_v})$ for split v;*

- *$U_v = D_v^\times$ for each $v \in \Sigma$,*

- *$U_v \supseteq \mathrm{Iw}(v)$ for each $v|p$,*

- *the action of $U \cap (\mathbb{A}_F^\infty)^\times$ on $W_\kappa(\mathcal{O})$ is given by ψ^{-1},*

- *$(U(\mathbb{A}_F^\infty)^\times \cap t^{-1} D^\times t)/F^\times = 1$ for each $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$.*

Let U' denote the maximal compact subgroup of U. Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U', \mathcal{O})$ and let $f \in S^{\mathrm{no}}_{\kappa,\psi}(U', \mathcal{O})_\mathfrak{m}$ be an eigenform such that for each $v \in \Sigma$, $D_v^\times$ acts on f via $\gamma_v \circ \nu_D$, with $\gamma_v : F_v^\times \to \mathcal{O}^\times$ an unramified character and ν_D the reduced norm of D.

For each $v \in \Sigma$ let $\gamma'_v : F_v^\times \to \mathcal{O}^\times$ be either γ_v or $-\gamma_v$. There is a Hecke eigenform $f' \in S^{\mathrm{no}}_{\kappa,\psi}(U', \mathcal{O})_\mathfrak{m}$ such that $D_v^\times$ acts on f' via γ'_v for each $v \in \Sigma$.

Proof. Let $\Delta = \prod_{v \in \Sigma} F_v^\times/(F_v^\times)^2 \mathcal{O}_{F_v}^\times$. The reduced norm defines an isomorphism

$$U(\mathbb{A}_F^\infty)^\times / U'(\mathbb{A}_F^\infty)^\times \xrightarrow{\sim} \Delta.$$

Fix coset representative $\{t\}$ for $D^\times\backslash(D\otimes_F \mathbb{A}_F^\infty)^\times/U(\mathbb{A}_F^\infty)^\times$. By 3.1.7, our assumption on U implies that we have an isomorphism of $\mathcal{O}[\Delta]$-modules

$$S_{\kappa,\psi}(U'(p^{a,a}),\mathcal{O}) \cong \bigoplus_{i=1}^{n} W_\kappa(\mathcal{O})\otimes_{\mathcal{O}}\mathcal{O}[\Delta],$$

and $S_{\kappa,\psi}(U',\mathcal{O})$ is free over $\mathcal{O}[\Delta]$. Since Δ is a 2-group, $\mathcal{O}[\Delta]$ is a local ring and $S^{\mathrm{no}}_{\kappa,\psi}(U'(p^{a,a}),\mathcal{O})_{\mathfrak{m}}$ is also free over $\mathcal{O}[\Delta]$. Since

$$E[\Delta]\cong \prod_{\chi:\Delta\to\{\pm 1\}} E(\chi)$$

we see that the $\mathcal{O}$-rank of the submodule of $S^{\mathrm{no}}_{\kappa,\psi}(U',\mathcal{O})_{\mathfrak{m}}$ consisting of elements on which $D_v^\times$ acts via γ_v for all $v\in\Sigma$ is equal to the $\mathcal{O}$-rank of the submodule on which $D_v^\times$ acts via γ_v' for all $v\in\Sigma$. □

Lemma 5.1.7. *Let π be a p-nearly ordinary regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ that is a $\overline{\chi}$-good lift of $\overline{\rho}$. Let Σ be a finite set of finite places not containing any above p such that for every $v\in\Sigma$, π_v is unramified and $\overline{\rho}|_{G_v}$ is an extension of $\overline{\gamma}_v$ by $\overline{\gamma}_v\overline{\epsilon_p}$, for some unramified character $\gamma_v: G_v\to\mathcal{O}^\times$ with $\gamma_v^2=\psi_{\mathbb{C}}|_{G_v}$, with $\psi_{\mathbb{C}}$ the central character of π.*

There is an allowable base change F'/F and a p-nearly ordinary regular algebraic cuspidal representation π' of $\mathrm{GL}_2(\mathbb{A}_F')$ such that

- *the central character of π' is $\psi_{\mathbb{C}}\circ \mathrm{Nm}_{F'/F}$;*
- *for any archimdean place v of F and $w|v$ in F', $\pi'_w\cong\pi_v$ as $(\mathfrak{gl}_2,\mathrm{O}(2))$-modules;*
- *π is unramified outside the places above Σ and the places above p;*
- *for any $v\in\Sigma$ and $w|v$ in F', $\pi'_w\cong(\gamma_v\circ\mathrm{Nm}_{F'_w/F_v}\circ\det)\otimes\mathrm{St}$;*
- *π' is a $\overline{\chi}$-good lift of $\overline{\rho}$.*

Proof. We prove this as in Lemma 3.5.3 of [K2]. For ease of notation, we will throughout denote the base change of a character by the same letter, in particular again write ψ and γ_v for $\psi\circ\mathrm{Nm}_{F'/F}$ and γ_v for $\gamma_v\circ\mathrm{Nm}_{F'_w/F_v}$, respectively. Also, given an algebraic weight

$\kappa = (\mathbf{k}, \mathbf{w})$ for F we will again denote by $\kappa = (\mathbf{k}, \mathbf{w})$ the algebraic weight for F' given by letting $(k_{\tau'}, w_{\tau'}) = (k_\tau, w_\tau)$ if $\tau' : F' \hookrightarrow \mathbb{R}$ extends $\tau : F \hookrightarrow \mathbb{R}$.

By first performing a quadratic allowable base change if necessary we may assume that $[F : \mathbb{Q}]$ is even. Let $\kappa = (\mathbf{k}, \mathbf{w})$ denote the weight of π, and ψ denote the $\mathcal{O}^\times$-valued character of $(\mathbb{A}_F)^\times$ corresponding to the central character of π. Let D_0 denote the quaternion algebra ramified at all archimedean places of F and split at all finite places of F. Let U be an open compact subgroup of $\mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$, such that $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ for all finite places v such that π_v is unramified, and such that $\pi^U \neq 0$. Let $a \geq 1$ be sufficiently large such that

$$(U(p^{a,a})(\mathbb{A}_F^\infty)^\times \cap t^{-1}D^\times t)/F^\times = 1$$

for every $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$. By the Jacquet-Langlands-Shimizu correspondence, π is generated by some $\mathbf{T}^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}), \mathcal{O})$-eigenform in $S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}), \mathcal{O})$.

Take $2 \leq k \leq p+1$ and $w \in \mathbb{Z}$ such that $k + 2w = k_\tau + 2w_\tau$, for any $\tau \in I$, and let $\kappa_0 = ((k, \ldots, k), (w, \ldots, w))$. We have a Hecke equivariant isomorphism, cf. 3.3.7

$$S^{\mathrm{no}}_{\kappa,\psi}(U(p^{a,a}), \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \cong S^{\mathrm{no}}_{\kappa_0,\psi}(U(p^{a,a}), \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F},$$

and so there is an eigenfunction $f_0 \in S^{\mathrm{no}}_{\kappa_0,\psi}(U, \mathcal{O})$ that is a $\overline{\chi}$-good lift of $\overline{\rho}$.

Write $\Sigma = \{v_1, \ldots, v_r\}$. Choose a tower of quadratic extensions $F = F_0 \subset F_1 \subset \cdots \subset F_r$, with each F_i/F_{i-1} an allowable base change such that for any $1 \leq j \leq r$, any $w \in F_{i-1}$ above v_j splits in F_i if $i = j$ and is inert in F_i otherwise. Let D_i be the quaternion algebra with centre F_i, ramified at all archimedean places and all places above $v_1, \ldots, v_i$, and split elsewhere. Let $\mathcal{O}_{D_i}$ be a fixed maximal order of D_i. We will show, by induction, that there is an open subgroup U_i of $(D_i \otimes_{F_i} \mathbb{A}_{F_i}^\infty)^\times$, $a_i \geq 1$, and an eigenfunction $f_i \in S^{\mathrm{no}}_{\kappa_0,\psi}(U_i(p^{a_i,a_i}), \mathcal{O})$ such that

(a) for any $1 \leq j \leq i$, if $w | v_j$ then $(U_i)_w = (D_i)_w^\times$ and U_w acts on $W_{\kappa_0}(\mathcal{O})$ by $\gamma_{v_j}^{-1} \circ \nu_{D_i}$, with ν_{D_i} the reduced norm of D_i;

(b) $(U_i)_w = \mathrm{GL}_2(\mathcal{O}_{F_{i,w}})$ for any w above v in F such that $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$,

(c) $(U_i(p^{a_i,a_i})(\mathbb{A}_{F_i}^\infty)^\times \cap t^{-1}D_i^\times t)/F^\times = 1$, for any $t \in (D_i \otimes_{F_i} \mathbb{A}_{F_i}^\infty)^\times$;

(d) f_i is a $\overline{\chi}$-good lift of $\overline{\rho}$.

Having obtained f_r, we can again use 3.3.7, to obtain a $\overline{\chi}$-good lift f_r' of $\overline{\rho}$ of level $U_r(p^{a_r,a_r})$ and weight κ', where $(k_{\tau'}, w_{\tau'}) = (k_\tau, w_\tau)$ for any $\tau' \in I_{F_r}$ extending $\tau \in I$. Then applying Jacquet-Langlands-Shimizu to f_r' gives the required result.

We now prove the induction. Denote by $\mathfrak{m}$ the maximal ideal of $\mathbf{T}^{\mathrm{no}}_{\kappa_0,\psi}(U_i(p^{a_i,a_i}),\mathcal{O})$ corresponding to f_i. Since $2 \le k \le p+1$, there is a perfect pairing $\langle\ ,\ \rangle$ on $W_k(\mathcal{O})$, cf. §1 of [T2], and hence a perfect pairing on $W_{\kappa_0}(\mathcal{O})$, which we also denote by $\langle\ ,\ \rangle$. Fix a set of coset representatives $\{t\}$ for $D_i^\times\backslash(D_i \otimes_{F_i} \mathbb{A}_{F_i}^\infty)^\times/U(\mathbb{A}_{F_i}^\infty)^\times$. Assumption (c) allows us to define a perfect pairing on $S_{\kappa_0,\psi}(U(p^{a_i,a_i}),\mathcal{O})$ by

$$\langle h_1, h_2\rangle_i = \sum_{\{t\}} \langle h_1(t), h_2(t)\rangle \psi(\det t)^{-1}.$$

Define an open subgroup U_i' of $(D_i \otimes_{F_i} \mathbb{A}_{F_i}^\infty)^\times$ by letting $(U_i')_w = (U_i)_w$ if w is not above v_{i+1} and $(U_i')_w = \mathrm{Iw}(w)$ if w is the unique place in F_i above v_{i+1}. By our assumption on $\overline{\rho}|_{G_{v_{i+1}}}$, we see that

$$(T_{v_{i+1}}^2 - (\mathrm{Nm}(v_{i+1})+1)^2\psi(\varpi_{v_{i+1}}))S_{\kappa_0,\psi}(U_i(p^{a_i,a_i}),\mathcal{O}) \subset \mathfrak{m}S_{\kappa_0,\psi}(U_i(p^{a_i,a_i}),\mathcal{O}).$$

Pulling back $\mathfrak{m}$ to a maximal ideal of $\mathbf{T}^{\mathrm{no}}_{\kappa_0,\psi}(U_i'(p^{a_i,a_i}),\mathcal{O})$, Corollary 3.1.11 of [K2] shows there is an eigenfunction $h \in S^{\mathrm{no}}_{\kappa_0,\psi}(U_i'(p^{a_i,a_i}),\mathcal{O})$ that is in the support of $\mathfrak{m}$ and is w-new, for w the unique place of F_i above v_{i+1}. If $(D_i)_{v_{i+1}}^\times$ does not act on h by $\gamma_{v_{i+1}} \circ \nu_{D_i}$, then we must have that $p = 2$ and $(D_i)_{v_{i+1}}^\times$ acts on h via $-\gamma_{v_{i+1}} \circ \nu_{D_i}$. Applying 5.1.6 allows us to assume $(D_i)_{v_{i+1}}^\times$ acts on h by $\gamma_{v_{i+1}} \circ \nu_{D_i}$. Considering the base change of h to F_{i+1} together with the Jacquet-Langlands-Shimizu correspondence then yields the desired f_{i+1}. □

We now group most of the above lemmas into one proposition.

Proposition 5.1.8. *Let π be a p-nearly ordinary regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ that is a $\overline{\chi}$-good lift of $\overline{\rho}$. Let $\kappa = (\mathbf{k}, \mathbf{w})$ be an algebraic weight* *and* $\psi_{\mathbb{C}} : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ *a character such that* $\psi_{\mathbb{C}}(z) = z_\infty^{2-\mathbf{k}-2\mathbf{w}}$ *on some open subgroup of* $\mathbb{A}_F^\times$. *Let* Σ *be a finite set of finite places not containing any places above* p. *For each* $v \in \Sigma$, *fix an unramified character* $\gamma_v : F_v \to \mathcal{O}^\times$ *such that* $\overline{\rho}|_{G_v}$ *is an extension of* $\overline{\gamma}_v$ *by* $\overline{\gamma}_v\overline{\epsilon_p}$ *and such that* $\gamma_v(\varpi_v)^2 = \psi(\varpi_v)$.

There is an allowable base change and a p*-nearly ordinary cuspidal automorphic representation* π' *of* $\mathrm{GL}_2(\mathbb{A}_{F'})$ *such that*

- π *has central character* $\psi_{\mathbb{C}} \circ \mathrm{Nm}_{F'/F}$;
- *if* w *is an archimedean place of* F' *that extends the embedding* $\tau : F \hookrightarrow \mathbb{R}$, *then* π'_w *is discrete series of lowest weight* $k_\tau - 1$ *and central character* $z_w \mapsto \mathrm{sgn}(z_w)^{k_\tau}|z_w|^{2-k_\tau-2w_\tau}$;
- *if* w *is a finite place of* F' *with* $w \nmid p$ *and* w *not above any place in* Σ, *then* π_w *is unramified;*
- *if* w *is a finite place of* F' *lying above some place in* $v \in \Sigma$, *then* $\pi_w \cong (\gamma_{v,\mathbb{C}} \circ \mathrm{Nm}_{F'_w/F_v} \circ \det) \otimes \mathrm{St}$;
- *for any* $w|p$, $(\pi'_w)^{\mathrm{Iw}_1(w)} \neq 0$;
- π' *is a* $\overline{\chi}$*-good lift of* $\overline{\rho}$.

Proof. Note that if p is odd, the unramified characters γ_v are determined uniquely by ψ and $\overline{\rho}$, but if $p = 2$, then they are only determined up to sign. We first apply 5.1.4 to obtain π_1 with the right weight and central character. We then apply 5.1.5 to obtain π_2 that is unramified for every $v \nmid p$, and with Iwahori fixed vectors at each place above p. We then apply 5.1.7 to obtain π_3, such that π_3 is our desired twist of Steinberg at each w above $v \in \Sigma$. However π_3 may no longer have Iwahori fixed vectors at places above p, and so we apply 5.1.5 one more time to get our desired π' Note that when applying 5.1.5 this last time we can ensure that the places above those in Σ remain the desired twist of Steinberg. $\square$

5.2 Proof of the main theorem

We can now prove our main theorem.

Theorem 5.2.1. *Let F be a totally real subfield of $\overline{\mathbb{Q}}$. Let I_F denote set of embeddings $F \hookrightarrow \overline{\mathbb{Q}}$. Fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_2$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Via these embeddings we view I_F as the set of embeddings $\{F \hookrightarrow \mathbb{R}\}$ as well as the set of embeddings $\{F \hookrightarrow \overline{\mathbb{Q}}_2\}$. For any $v|p$ in F, let $I_{F,v} \subseteq I_F$ denote the subset of τ that give rise to v. We identify $I_{F,v}$ with the set of embedding $F_v \hookrightarrow \overline{\mathbb{Q}}_2$.*

Let

$$\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$$

be a continuous representation unramified outside finitely many primes. Assume there is some $(\mathbf{k}, \mathbf{w}) \in I_F^2$, with $k_\tau \geq 2$ for each $\tau \in I_F$ and $w = k_\tau + 2w_\tau$ independent of τ, and such that

1. $\det \rho = \phi \epsilon_p^{w-1}$, *with ϕ a finite order character;*
2. *for each $v|2$, $\rho|_{G_v} \cong \begin{pmatrix} * & * \\ & \chi_v \end{pmatrix}$ such that viewing χ_v as a character of $F_v^\times$ via class field theory, $\chi_v(y) = \prod_{\tau \in I_{F,v}} y^{-w_\tau}$ on some open subgroup of $\mathcal{O}_{F_v}^\times$;*
3. *for each choice of complex conjugation c, $\det \rho(c) = -1$.*

Let $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}})$ denote the residual representation associated to ρ. We assume

(4) $\overline{\rho}$ is absolutely irreducible;

(5) If L/F is a CM extension such that $\overline{\rho}|_{G_L}$ is abelian, then there is some $v|2$ in F that does not split in L and such that if w denotes the place in L above v, $L_w \neq F_v(\zeta_4)$, with ζ_4 a primitive 4-th root of unity;

(6) there is a 2-nearly ordinary regular algebraic cuspidal automorphic representation π_0 of $\mathrm{GL}_2(\mathbb{A}_F)$ with $\overline{\rho_{\pi_0}} \cong \overline{\rho}$.

Under these assumptions ρ is modular, i.e. there is a 2-nearly ordinary regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $\rho \cong \rho_\pi$.

Proof. In the case that $\rho|_{G_v}$ is split for $v|2$, we fix χ_v as in (2). In the case that there is quadratic L/F that is CM, we fix a $v_0|p$ satisfying (5). We can find a totally real solvable extension F'/F of even degree such that

- F'/F is disjoint from the fixed field of $\ker \overline{\rho}$
- for any $v|2$ in F, $[F'_v : \mathbb{Q}_2] \geq 4$;
- for any $v|2$ in F', the image of $\overline{\rho}|_{G_v}$ has order two if v is above our fixed place v_0, and it is trivial otherwise;
- for any $v|2$, F_v contains a 4-th root of unity.

Note that to ensure we can do the last two simultaneously we are using assumption (5) in the case that that L/F is CM.

Now assume $\overline{\rho}$ is dihedral and that L/F is not CM. Let r, resp. s, denote the number of real, resp. complex, embedding of L. Note that $r \geq 1$. Let L_S^{ab} denote the maximal pro-p abelian extension of L unramified outside S. Let L_S^- denote the maximal sub-extension of L_S^{ab}/L such that the nontrivial element of $\mathrm{Gal}(L/F)$ act on $\mathrm{Gal}(L_S^-/L)$ by -1. We distinguish two sub-cases depending on whether or not L is contained in the 2-adic cyclotomic extension of F.

First assume that L/F is not contained in the 2-adic cyclotomic extension. Let F_n denote the totally real subfield of the cyclotomic extension $F(\mu_{2^n})$. Set $L_n = F_n L$ and let r_n, resp s_n, denote the number of real, resp. complex, embeddings of L_n. Note that $r_n \geq [F_n : F]$. The subgroup of $\mathcal{O}_{L_n}^\times$ on which $\mathrm{Gal}(L_n/F_n)$ acts via the nontrivial character has rank

$$r_n + s_n - [F_n : \mathbb{Q}] = \frac{r_n}{2} \geq \frac{[F_n : F]}{2}.$$

The weak Leopoldt conjecture is known to hold for the tower $\{L_n/L\}_{n \geq 1}$, c.f. Theorem 10.3.25 of [NSW]. Hence letting $\overline{\mathcal{O}_{L_n}^\times}$ denote the closure of $\mathcal{O}_{L_n}^\times$ in $(\mathcal{O}_{L_n} \otimes_{\mathbb{Z}} \mathbb{Z}_2)^\times$, there is a constant c such that

$$\mathrm{rank}_{\mathbb{Z}}\mathcal{O}_{L_n}^{\times} - \mathrm{rank}_{\mathbb{Z}_2}\overline{\mathcal{O}_{L_n}^{\times}} < c$$

for all n. Then we have

$$\mathrm{rank}_{\mathbb{Z}_2}\mathrm{Gal}((L_n)_S^-/L_n) \leq [F_n : \mathbb{Q}] - \frac{[F_n : F]}{2} + c$$

for all n. Be replacing F with F_n for n sufficiently large we may assume that $\mathrm{rank}_{\mathbb{Z}_2}\mathrm{Gal}(L_S^-/L) < [F : \mathbb{Q}] - 3$.

In the case that L/F is contained in the 2-adic cyclotomic extension, we use an idea of [S3]. Notice that $L = L_0F$, with $L_0/\mathbb{Q}$ an abelian extension. Set $F_0 \cap L_0$. Choose a prime ℓ such that $\mathbb{Q}(\mu_{\ell^\infty})$ is disjoint from the fixed field of $\ker\overline{\rho}$. Let M_n denote the sub-extension of $\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}$ of degree ℓ^n. Since L_0/F_0 is not CM, there is a subgroup of $\mathcal{O}_{M_nL_0}^{\times}$ of rank at least $\ell^n - 1$ on which $\mathrm{Gal}(M_nL_0/M_nF_0)$ act via the nontrivial character. Since Leopoldt's conjecture is known for abelian extensions of $\mathbb{Q}$, the closure of this subgroup has $\mathbb{Z}_2$-rank at least ℓ^n. Then

$$\mathrm{rank}_{\mathbb{Z}_2}\mathrm{Gal}((M_nL)_S^-/(M_nL)) \leq [M_nF : \mathbb{Q}] - \ell^n.$$

Be replacing F with M_nF for n sufficiently large we may assume that $\mathrm{rank}_{\mathbb{Z}_2}\mathrm{Gal}(L_S^-/L) < [F : \mathbb{Q}] - 3$.

Let Σ denote the set of finite places not above p at which ρ is ramified. Recall that an allowable base change is a totally real finite solvable extension F'/F of even degree that is disjoint from the fixed field of $\ker\overline{\rho}$. By replacing F with an allowable base change we may assume that $[F : \mathbb{Q}]$ and Σ are both even.

By replacing F with an allowable base change we may assume that for any $v \in \Sigma$, ρ is an extension of γ_v by $\gamma_v\epsilon_p$, for some unramified character γ_v. By further replacing F with an allowable base change, 5.1.8 implies we may assume the same properties for ρ_{π_0}, with the same characters γ_v for $v \in \Sigma$, as well as that π_0 has weight $(\mathbf{k}, \mathbf{w})$, and that $\det\rho = \det\rho_{\pi_0}$.

For each $v|2$, let η_v denote the character of the 2-Sylow subgroup of the torsion subgroup of G_v^{ab} given by restriction of χ_v, and let $\eta = (\eta_v)_{v|2}$. For each $v|2$, write

$$\rho_{\pi_0} \cong \begin{pmatrix} * & * \\ & \chi_v^0 \end{pmatrix}$$

such that viewing χ_v^0 as a character of $F_v^\times$ via class field theory, $\chi_v^0(y) = \prod_{\tau \in I_{F,v}} y^{-w_\tau}$ on some open subgroup of $\mathcal{O}_{F_v}^\times$. Let η_v^0 denote the restriction of χ_v^0 to the 2-Sylow subgroup of the torsion subgroup of G_v^{ab} and let $\eta^0 = (\eta_v^0)_{v|2}$. Note that for each $v|2$, the kernels of both η_v and η_v^0 are disjoint from the splitting field of $\overline{\rho}|_{G_v}$. We perform one final base change to assume that both η_v and η_v^0 are trivial for each $v|2$. Since the kernels of these characters were disjoint from the splitting field of $\overline{\rho}|_{G_v}$, we may still assume that if $\overline{\rho}$ is dihedral and L/F is CM, there is some $v_0|2$ in F that does not split in L. Note that the $\overline{\mathbb{Q}}_2$ points of $\Lambda(G_p) = \hat{\otimes}_{v|2}\Lambda(G_v)$ determined by $(\chi_v)_{v|2}$ and $(\chi_v^0)_{v|2}$ lie on the same irreducible component.

Let E be a finite sub-extension of $\overline{\mathbb{Q}}_2/\mathbb{Q}_2$ that contains the image of ρ and ρ_{π_0}, and let $\mathcal{O}$ denote its ring of integers. Let $S = \Sigma \cup \{v|p\} \cup \{v|\infty\}$. Using the Jacquet-Langlands-Shimizu correspondence, the properties of π_0 allow us to transfer π_0 to a nearly ordinary eigenform f as in 4.1.2. The field F and residual representations satisfy the assumptions of 4.1.1 as well as 4.4.2. Then letting $\overline{R}_{F,S}^{\psi}$ be the quotient of $R_{F,S}\hat{\otimes}\Lambda(G_p)$ as in 4.1.4, we see that $(\rho, (\chi_v)_{v|2})$ defines a point of $\mathrm{Spf}\overline{R}_{F,S}^{\psi}$. The theorem now follows from 4.4.2. □

Our theorem from the introduction is a result of 5.2.1 above together with 5.1.2.

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