Nonnegative Smooth Interpolation

By

FUSHUAI JIANG
DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

in the

OFFICE OF GRADUATE STUDIES

of the

UNIVERSITY OF CALIFORNIA

DAVIS

Approved:

Garving Kevin Luli, Chair

John Hunter

Bruno Nachtergaele

Committee in Charge

2022

To Imani and my parents

Contents

Abstract iv

Acknowledgments v

Chapter 1. Introduction 1

Chapter 2. Summary of Main Results 4

Chapter 3. Nonnegative $C^2(\mathbb{R}^2)$ interpolation 11

Chapter 4. On the shape fields Finiteness Principle 118

Bibliography 143

Nonnegative Smooth Interpolation

Abstract

This dissertation presents three peer-reviewed journal articles, published in “Advances in Mathematics” and “International Mathematics Research Notices”, on Whitney-type extension and interpolation problems with nonnegative constraint. The mathematical preliminaries and a detailed summary of results are found in Chapter 2. Information on co-authors and funding acknowledgment for each journal article are found at the beginning of their corresponding chapters.

Acknowledgments

I would like to begin by thanking my wife Imani Morgan, for supporting me almost every step throughout my graduate school, even during some of her own hardest times, and also for kindly tolerating my academic relocation, so that we can build characters together during the East Coast winter.

I would like to thank my advisor Prof. Kevin Luli for introducing me to this wonderful research area, and for his patience and guidance, numerous brilliant ideas, and seemingly unlimited funding.

Then I would like to thank the department staff Vanessa Bravo, Tina Denena, Sarah Driver (former), and Victoria Whistler (former) for making my life significantly easier and ϵ-richer.

I would also like to thank Prof. John Garnett and Prof. Peter Petersen from UCLA for laying down the foundation of my graduate career.

Finally, I would like to thank my friends Jenny Brown, Matt Litman, Alvin Moon, Joseph Pappe, Kyle Phillips, Raag Ramani, Pooja Tripathi, and Arina Ushakova for the times at Sophia's and elsewhere, necessarily and sufficiently.

CHAPTER 1

Introduction

At the basic level, Whitney-type extension (for infinite sets) and interpolation (for finite sets) problems seek efficient ways to estimate global behavior from local data while obeying certain constraints. This dissertation mainly concerns interpolation that preserves differentiability and nonnegativity, with some additional discussion on general shape-preserving interpolation. Other features commonly considered in extension literature include integrability or convexity.

In his seminal works [**32**, **33**, **34**], Whitney posed what we now call the Whitney Extension Problem, which asks whether a continuous function defined on an arbitrary closed set in $\mathbb{R}^n$ can be extended to a globally defined C^m function. He solved the problem for $n = 1$ by taking limits of divided differences, a technique not applicable in higher dimensions due to the lack of natural ordering. The full solution to the Whitney Extension Problem was only obtained less than two decades ago by Fefferman [**6**, **8**, **9**], building on the works of Glaeser [**17**], Shvartsman [**25**, **26**, **28**], Brudnyi-Shvartsman [**4**, **5**], and Bierstone-Milman-Pawłucki [**3**].

The quantitative (i.e. finite-set) version of the Whitney Extension Problem not only serves as a crucial ingredient in the full solution to the original problem, but also shines light on the mathematical aspects of data interpolation. This version of the problem asks for a way to extend a function defined on a finite set in $\mathbb{R}^n$ to a C^m function on $\mathbb{R}^n$ with norm having the smallest possible order of magnitude. This task consists of two components, one is to compute the order of the magnitude of the norm, and the other is to compute the interpolant with norm achieving such order of magnitude.

In [**4**,**5**,**8**,**25**,**26**,**28**], the authors showed that the only obstruction to a global interpolant having small norm is the existence of some local interpolant having large norm over some set of bounded cardinality. This is the essence of the Brudnyi-Shvartsman Finiteness Principle (Finiteness Principle for short), and we will refer to the upper bound on the cardinality as the "finiteness constant". Moreover, in [**7**], the author showed that there exists a bounded linear extension operator, such that the extension at each point (i.e. the coefficients of the Taylor polynomial) is a sparse linear combination of the given function values. We call such linear operator to be of "bounded depth".

Computational advances on the problem were then made by Fefferman-Klartag [**10**, **14**, **15**]. In particular, the authors improved the Finiteness Principle for more efficient computation of the C^m norm, and showed the existence of what we will refer to as the Fefferman-Klartag interpolation algorithms. These algorithms can be run on an idealized computer with von Neumann architecture and likely have the best guaranteed computational complexity possible, using $O(N \log N)$ one-time work and $O(N)$ storage, with N being the size of the given data set and the constants depending only on m and n.

Nonnegativity arises in numerous physical scenarios, such as modeling temperature or chemical concentration, and it is also one of the simplest shape-preserving requirement. Some examples of literature on nonnegative interpolation include [**1**, **22**, **23**, **24**]. It is then natural to pose the nonnegative variant of the quantitative Whitney Extension Problem: Given a nonnegative function on a finite set in $\mathbb{R}^n$, how do we find a globally nonnegative C^m function whose norm has the smallest possible order of magnitude?

The study of the quantitative nonnegative Whitney Extension Problem is pioneered by the works of Fefferman-Israel-Luli [**11**, **12**], in which the authors showed that a similar Finiteness Principle still holds in this context. However, due to the reliance of a sophisticated induction procedure on a fairly abstract object called "shape fields", the construction was not explicit and the finiteness constants were larger than necessary.

This dissertation presents three journal articles, with [**19**, **20**] in Chapter 3 focusing on nonnegative C^2 interpolation and [**21**] in Chapter 4 focusing on general shape-preserving C^m interpolation.

In [**19**], we provided an alternative proof of the Finiteness Principle for nonnegative $C^2(\mathbb{R}^2)$ interpolation that yields a greatly improved finiteness constant. The method employed in the proof also lays the foundation for all subsequent results in [**19**, **20**].

In stark contrast with the non-constrained case [**7**], we showed the nonexistence of a bounded linear (with respect to the positive cone) extension operator [**19**]. On the other hand, we showed the existence of bounded nonlinear extension operator having a similar property as having bounded depth [**20**].

Parallel to the results on extension operators, we also showed the existence of Fefferman-Klartag-type algorithms for nonnegative $C^2(\mathbb{R}^2)$ interpolation [**20**] with comparable complexity.

The techniques in [**19**, **20**] can be modified to yield similar results for nonnegative $C^2(\mathbb{R}^n)$ interpolation.

In [**21**], we proved a reduction-type result for shape fields introduced by Fefferman-Israel-Luli [**11**,**12**], building on a clustering technique in [**2**]. As a consequence, we were able to greatly improve the finiteness constants for various general shape-preserving interpolation, including nonnegative $C^m(\mathbb{R}^n)$ interpolation.

The mathematical preliminaries and a detailed description of the these results can be found in the next chapter.

CHAPTER 2

Summary of Main Results

For $m, n \in \mathbb{N}_0$, we use $C^m(\mathbb{R}^n)$ to denote the vector space of m-times continuously differentiable functions whose derivatives up to order m are bounded and continuous, normed by

$$\|F\|_{C^m(\mathbb{R}^n)} := \max_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha F(x)|,$$

under which it becomes a Banach space. Here and below, we use the Greek letter α to denote multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, and we write ∂^α to denote the differential operator $\partial_{x^1}^{\alpha_1} \cdots \partial_{x^n}^{\alpha_n}$ whose order is $|\alpha| := \sum_{j=1}^n \alpha_j$.

Let $A, B > 0$. We write $A \lesssim B$ if $A \leq CB$ for some constant C depending only on m and n. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$, and in this case, we say A and B have the "same order of magnitude".

For a finite set $X \subset \mathbb{R}^n$, we use $\#X$ to denote the cardinality of X.

We state the main problem of this dissertation.

Problem 2.1. *Let $E \subset \mathbb{R}^n$ be a finite set and let $f : E \to [0, \infty)$.*

(A) Compute the order of magnitude of

$$\|f\|_{C_+^m(E)} := \inf\left\{\|F\|_{C^m(\mathbb{R}^n)} : F = f \text{ on } E \text{ and } F \geq 0 \text{ on } \mathbb{R}^n.\right\}.$$

(B) Compute a function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on E, $F \geq 0$ on $\mathbb{R}^n$, and $\|F\|_{C^m(\mathbb{R}^n)} \lesssim \|f\|_{C_+^m(E)}$.

By computing the order of magnitude of $M > 0$, we mean computing a number $\tilde{M} > 0$ such that $M \approx \tilde{M}$. By "computing a function F" from (E, f), we mean the following: After processing the input (E, f), we are able to accept a query consisting of a point $x \in \mathbb{R}^n$, and produce a list of numbers $(f_\alpha(x) : |\alpha| \leq m)$. The algorithm "computes the function F" if for each $x \in \mathbb{R}^n$, we have $\partial^\alpha F(x) = f_\alpha(x)$ for $|\alpha| \leq m$.

We also content ourselves with an idealized computer with standard von Neumann architecture that is able to process exact real numbers. We refer the readers to [**15**] for discussion on finite-precision computing.

The study of Problem 2.1(A) is pioneered by Fefferman-Israel-Luli [**11**,**12**], in which the authors proved the following Finiteness Principle.

THEOREM 2.1 ([**12**]). *There exists a number* $k^\sharp$ *depending only on* m *and* n *such that for every finite set* $E \subset \mathbb{R}^n$ *and* $f : E \to [0, \infty)$,

$$\|f\|_{C^m_+(E)} \approx \max_{S \subset E, \# S \le k^\sharp} \|f\|_{C^m_+(S)}.$$

The proof of Theorem 2.1 in [**12**] depends on a sophisticated refinement procedure for a collection of abstract object called "shape fields" given in [**11**]. As such, the construction is not very explicit and the number $k^\sharp$ is larger than necessary. For instance, for $m = n = 2$, [**12**] gives $k^\sharp \ge 5^{200}$. We will omit the definition of a shape field in the introduction due to its technicality, and refer the interested readers to [**11**] and Chapter 4 below.

In [**19**], we provide an alternative proof of Theorem 2.1 that greatly improves the number $k^\sharp$.

THEOREM 2.2 ([**19**]). *For* $m = n = 2$, *we may take* $k^\sharp = 64$ *in Theorem 2.1.*

Moreover, the proof of Theorem 2.2 in [**19**] lays the foundation for subsequent results on efficient computation of the norm and explicit construction of the interpolants. Before we state these results, we recall some key definitions in [**20**] which reflect the complexity of a nonlinear map.

DEFINITION 2.1. *Let* $N_0 \ge 1$ *be an integer. Let* $B = \{\xi_1, \cdots, \xi_{N_0}\}$ *be a basis of* $\mathbb{R}^{N_0}$. *Let* $\Omega \subset \mathbb{R}^{N_0}$ *be a subset. Let* X *be a set. Let* $\Xi : \Omega \to X$ *be a map.*

- *We say* Ξ *has depth at most* D *(with respect to the basis* B*) if there exists a* D*-dimensional subspace* $V = \mathrm{span}(\xi_{i_1}, \cdots, \xi_{i_D})$, $\xi_{i_1}, \cdots, \xi_{i_D} \in B$, *such that for all* $z_1, z_2 \in \Omega$ *with* $\pi_V(z_1) = \pi_V(z_2)$, *we have* $\Xi(z_1) = \Xi(z_2)$. *Here,* $\pi_V : \mathbb{R}^{N_0} \to V$ *is the natural projection. We call the set of indices* $\{i_1, \cdots, i_D\}$ *the source of* Ξ.
- *Suppose* Ξ *has depth* D. *Let* $V = \mathrm{span}(\xi_{i_1}, \cdots, \xi_{i_D})$ *and* π_V *be as above. By an efficient representation of* Ξ, *we mean a specification of the index set* $\{i_1, \cdots, i_D\} \subset \{1, \cdots, N_0\}$ *and*

a map $\tilde{\Xi} : \pi_V(\Omega) \to X$ *with* $\Xi = \tilde{\Xi} \circ \pi_V$ *on* Ω, *such that given* $\nu \in \pi_V(\Omega)$, $\tilde{\Xi}(\nu)$ *can be computed using at most* C_D *operations. Here,* C_D *is a constant depending only on* D.

REMARK 2.1. *Suppose* $\Xi : \mathbb{R}^{\bar{N}} \to \mathbb{R}$ *is a linear functional. Recall from [**15**] that a "compact representation" of a linear functional* $\Xi : \mathbb{R}^{\bar{N}} \to \mathbb{R}$ *consists of a list of indices* $\{i_1, \cdots, i_D\} \subset \{1, \cdots, \bar{N}\}$ *and a list of coefficients* $\chi_{i_1}, \cdots, \chi_{i_D}$, *so that the action of* Ξ *is characterized by*

$$\Xi : (\xi_1, \cdots, \xi_{\bar{N}}) \mapsto \sum_{\Delta=1}^{D} \chi_{i_\Delta} \cdot \xi_{i_\Delta}.$$

Therefore, given $\nu \in \mathrm{span}(\xi_{i_1}, \cdots, \xi_{i_D})$, *we can compute* $\Xi(\nu)$ *by the dot product of two vectors of length* D, *which requires* C_D *operations. The present notion of "efficient representation" is a natural generalization of the "compact representation" in [**15**] adapted to the nonlinear nature of constrained interpolation.*

Let $C^2_+(E)$ be the collection of functions $f : E \to [0, \infty)$. We think of $C^2_+(E) \cong [0, \infty)^N$ with $N = \#E$. We use the standard orthonormal frame of $\mathbb{R}^N$ as a basis for the purpose of defining finite depth. We write $\mathcal{P}$ to denote the vector space of polynomials on $\mathbb{R}^2$ with degree no greater than two, and we write $\mathcal{J}_x F$ to denote the second-degree Taylor polynomial of F at x.

We now state the main result in [**20**].

THEOREM 2.3 ([**20**]). *Suppose we are given a finite set* $E \subset \mathbb{R}^2$ *with* $\#(E) = N$. *Then there exists a collection of maps* $\{\Xi_x : x \in \mathbb{R}^2\}$, *where* $\Xi_x : C^2_+(E) \times [0, \infty) \to \mathcal{P}$ *for each* $x \in \mathbb{R}^2$, *such that the following hold.*

(A) There exists a universal constant D *such that for each* $x \in \mathbb{R}^2$, *the map* $\Xi_x(\cdot, \cdot) : C^2_+(E) \times [0, \infty) \to \mathcal{P}$ *is of depth at most* D.

(B) Suppose we are given $(f, M) \in C^2_+(E) \times [0, \infty)$ *with* $\|f\|_{C^2_+(E)} \leq M$. *Then there exists a function* $F \in C^2_+(\mathbb{R}^2)$ *such that*

$$\mathcal{J}_x F = \Xi_x(f, M) \text{ for all } x \in \mathbb{R}^2, \ \|F\|_{C^2(\mathbb{R}^2)} \leq CM, \text{ and } F(x) = f(x) \text{ for } x \in E.$$

(C) There is an algorithm, that takes the given data set E, *performs one-time work, and then responds to queries.*

A query consists of a point $x \in \mathbb{R}^2$, and the response to the query is the map Ξ_x, given in its efficient representation (see Definition 2.1).

The one-time work takes $CN \log N$ operations and CN storage. The work to answer a query is $C \log N$.

If we define a map $\mathcal{E} : C^2_+(E) \times [0, \infty) \to C^2(\mathbb{R}^2)$ by specifying

$$\mathcal{J}_x \mathcal{E}(f, M) := \Xi_x(f, M)$$

with $\left\{\Xi_x : x \in \mathbb{R}^2\right\}$ as in Theorem 2.3, we see that Theorem 2.3 implies the existence of an extension map of bounded depth that preserves nonnegativity.

THEOREM 2.4 ([**18**,**20**]). *Let $E \subset \mathbb{R}^2$ be a finite set. There exist (universal) constants C, D, and a map $\mathcal{E} : C^2_+(E) \times [0, \infty) \to C^2_+(\mathbb{R}^2)$ such that the following hold.*

(A) Let $M \geq 0$. Then for all $f \in C^2_+(E)$ with $\|f\|_{C^2_+(E)} \leq M$, we have $\mathcal{E}(f, M) = f$ on E and $\|\mathcal{E}(f, M)\|_{C^2(\mathbb{R}^2)} \leq CM$.

(B) For each $x \in \mathbb{R}^2$, there exists a set $S(x) \subset E$ with $\#(S(x)) \leq D$ such that for all $M \geq 0$ and $f, g \in C^2_+(E)$ with $\|f\|_{C^2_+(E)}, \|g\|_{C^2_+(E)} \leq M$ and $f|_{S(x)} = g|_{S(x)}$, we have

$$\partial^\alpha \mathcal{E}(f, M)(x) = \partial^\alpha \mathcal{E}(g, M)(x) \text{ for } |\alpha| \leq 2.$$

We note that Theorem 2.4 was independently proven in [**18**] without the use of Theorem 2.3. However, we will not include [**18**] in this dissertation for simplicity of presentation.

To our pleasant surprise, we also proved that the nonlinearity of the operators in Theorems 2.3 and 2.4 is unavoidable in general, which is in sharp contrast with the unconstrained case [**7**].

THEOREM 2.5 ([**19**]). *For each $n \geq 1$, there exists a finite set $E \subset \mathbb{R}^n$ that does not admit a map $\mathcal{E} : \{f : E \to [0, \infty)\} \to C^2(\mathbb{R}^n)$ satisfying both of the following.*

(A) For all $f : E \to [0, \infty)$, $\mathcal{E}(f) = f$ on E, $\mathcal{E}(f) \geq 0$ on $\mathbb{R}^n$, and $\|\mathcal{E}(f)\|_{C^2(\mathbb{R}^n)} \lesssim \|f\|_{C^2_+(E)}$.

(B) For all $f, g : E \to [0, \infty)$, $\mathcal{E}(f + g) = \mathcal{E}(f) + \mathcal{E}(g)$.

Theorems 2.3 and 2.4, together with the Callahan-Kosaraju decomposition (or well separated pairs decomposition in computer science literature), give rise to the following improved Finiteness Principle for efficiently computing the order of magnitude of the norm. Note that a non-computational version was proved independently in [**19**] without the use Theorems 2.3, 2.4, or the Callahan-Kosaraju decomposition.

THEOREM 2.6 ([**20**]). *Let* $E \subset \mathbb{R}^2$ *with* $\#(E) = N < \infty$. *Then there exist universal constants* C_1, C_2, C_3, C_4, C_5 *and a list of subsets* $S_1, S_2, \cdots, S_L \subset E$ *satisfying the following.*

(A) We can compute the list $\{S_\ell : \ell = 1, \cdots, L\}$ *from* E, *using one-time work of at most* $C_1 N \log N$ *operations, and using storage at most* $C_2 N$.

(B) $\#(S_\ell) \leq C_3$ *for each* $\ell = 1, \cdots, L$.

(C) $L \leq C_4 N$.

(D) Given any $f : E \to [0, \infty)$, *we have*

$$\max_{\ell=1,\cdots,L} \|f\|_{C^2_+(S_\ell)} \leq \|f\|_{C^2_+(E)} \leq C_5 \max_{\ell=1,\cdots,L} \|f\|_{C^2_+(S_\ell)}. \tag{2.1}$$

Furthermore, we showed in [**20**] that computing the order of magnitude of each $\|f\|_{C^2_+(S_\ell)}$ in (2.1) amounts to solving a convex quadratic optimization problem with affine constraint. Such minimization problem is readily solvable, for instance, by the method of Lagrange multipliers.

Combining Theorem 2.3 and Theorem 2.6, we can efficiently compute a nonnegative $C^2(\mathbb{R}^2)$ interpolant with norm having the optimal order of magnitude. The one-time work for pre-processing the set E ($\#E = N$) uses at most $CN \log N$ operations and CN storage. After that, computing $\|f\|_{C^2_+(E)}$ uses at most CN operations, and answering a query uses at most $C \log N$ operations. The guaranteed complexity here is also likely the best possible.

The methods employed in [**18, 19, 20**] can be adapted to obtain similar efficient results for nonnegative $C^2(\mathbb{R}^n)$ interpolation. These adaptations were also used in [**13**] to treat C^2 interpolation with both upper and lower range restrictions.

The papers [**19, 20**] can be found in Chapter 3.

There are no known efficient solutions (in the sense of Theorems 2.3 and 2.6) to Problem 2.1 for $m \geq 3$. However, in [**21**], we are able to greatly improve the finiteness constant in Theorem 2.1.

THEOREM 2.7. *We may take $k^\sharp$ to be 2^Δ in Theorem 2.1, where $\Delta = \binom{m+n-1}{m-1}$ is the dimension of vector space of polynomials on $\mathbb{R}^n$ with degree no greater than $m-1$.*

For $m = 2$, Theorem 2.7 yields $k^\sharp = 4 \cdot 2^{n-1}$, which is comparable to the optimal $3 \cdot 2^{n-1}$ shown in [**4**, **5**, **26**, **28**] (without constraint). [1] For general $m \geq 3$ and $n \geq 2$, we do not know what the optimal $k^\sharp$ is, even for interpolation without constraint.

Our paper [**21**] also studies a cousin of Problem 2.1 concerning "smooth selection", which can be viewed either as an interpolation problem with error or as a trajectory problem with obstacles.

PROBLEM 2.2. *Fix $m, n, d \in \mathbb{N}_0$. Let $\mathsf{Conv}(\mathbb{R}^d)$ denote the collection of all convex sets in $\mathbb{R}^d$. Given a finite set $E \subset \mathbb{R}^n$ and a set-valued function $\mathcal{K} : E \to \mathsf{Conv}(\mathbb{R}^d)$, how do we find a function $F \in C^m(\mathbb{R}^n, \mathbb{R}^d)$ such that $F(x) \in \mathcal{K}(x)$ for each $x \in E$ and $\|F\|_{C^m(\mathbb{R}^n,\mathbb{R}^d)}$ has the smallest possible order of magnitude?*

A special variant of Problem 2.2, where C^m-class is replaced by Lipschitz-class and $\mathcal{K}$ is a mapping into hyperplanes, has been extensively studied by Shvartsman [**27**, **29**, **31**], for which the author proved various Helly-type results. In [**7**, **8**, **10**, **14**, **15**], the authors provided efficient algorithmic solutions to another special variant of Problem 2.2, in which $d = 1$ and each $\mathcal{K}(x)$ is a compact interval and is allowed to dilate concentrically and uniformly.

The first progress on the study of Problem 2.2 in the present form was made in [**11**], in which the authors show the validity of a similar Finiteness Principle. Their result again relies on the refinement of shape fields and yields an unnecessarily large finiteness constant. The reduction argument for shape fields in our paper [**21**] then yields the following.

THEOREM 2.8. *Fix $m, n, d \in \mathbb{N}_0$ and let $k^\sharp = d \cdot \binom{m+n-1}{m-1}$. Given a finite set $E \subset \mathbb{R}^n$ and a set-valued function $\mathcal{K} : E \to \mathsf{Conv}(\mathbb{R}^d)$, we have*

$$\inf\left\{\|F\|_{C^m(\mathbb{R}^n,\mathbb{R}^d)} : F(x) \in \mathcal{K}(x)\,\forall x \in E\right\} \approx \max_{\substack{S\subset E\\ \#S\leq k^\sharp}} \inf\left\{\|F\|_{C^m(\mathbb{R}^n,\mathbb{R}^d)} : F(x) \in \mathcal{K}(x)\,\forall x \in S\right\}.$$

[1] That is, for each $\mu \in \mathbb{N}_0$, there exists a finite set $E_\mu \subset \mathbb{R}^n$ and a function $f_\mu : E_\mu \to \mathbb{R}$, such that for every $S \subset E_\mu$ with $\#S \leq 3 \cdot 2^{n-1} - 1$, there exists $F_\mu^S \in C^2(\mathbb{R}^n)$ that interpolates (S, f_μ) with norm 1, but f_μ fails to extend to a $C^2(\mathbb{R}^n)$ function with norm no greater than μ.

Our reduction argument in [**21**] is inspired by a clustering technique developed in [**2**]. See also [**30**] for a different reduction argument based on Lipschitz selection, and see [**16**] for a sharp finiteness constant for Lipschitz selection.

The paper [**21**] can be found in Chapter 4.

CHAPTER 3

Nonnegative $C^2(\mathbb{R}^2)$ interpolation

The first paper "Nonnegative $C^2(\mathbb{R}^2)$ interpolation" was published in Advances in Mathematics, Vol. 375 (2020) [**19**]. The paper is based on joint work with co-author Garving K. Luli at the Department of Mathematics, University of California - Davis. The authors were supported by National Science Foundation Grant DMS-1554733 (F.J. and G.K.L.), UC Davis Summer Graduate Student Researcher Award (F.J.), the Alice Leung Scholarship in Mathematics (F.J.), and the UC Davis Chancellor's Fellowship (G.K.L.).

The second paper "Algorithms for nonnegative $C^2(\mathbb{R}^2)$ interpolation" was published in Advances in Mathematics, Vol. 385 (2021) [**20**]. The paper is based on joint work with co-author Garving K. Luli at the Department of Mathematics, University of California - Davis. The authors were supported by National Science Foundation Grant DMS-1554733 (F.J. and G.K.L.), UC Davis Summer Graduate Student Researcher Award (F.J.), the Alice Leung Scholarship in Mathematics (F.J.), and the UC Davis Chancellor's Fellowship (G.K.L.).

Advances in Mathematics 375 (2020) 107364

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Nonnegative $C^2(\mathbb{R}^2)$ interpolation

Fushuai Jiang, Garving K. Luli *

Department of Mathematics, University of California, Davis, One Shields Ave, Davis, CA 95616, USA

ARTICLE INFO

Article history:
Received 24 June 2020
Received in revised form 20 July 2020
Accepted 4 August 2020
Available online 17 August 2020
Communicated by C. Fefferman

Keywords:
Nonnegative interpolant
Nonnegative interpolation
Constrained interpolation
Whitney extension problems

ABSTRACT

In this paper, we prove two improved versions of the Finiteness Principle for nonnegative $C^2(\mathbb{R}^2)$ interpolation, previously proven by Fefferman, Israel, and Luli. The first version sharpens the finiteness constant to 64, and the second version carries better computational practicality. Along the way, we also provide a detailed construction of nonnegative C^2 interpolants in one-dimension, and prove the nonexistence of a bounded linear C^2-extension operator that preserves nonnegativity.

1. Introduction

For nonnegative integers m, n, we write $C^m(\mathbb{R}^n)$ to denote the Banach space of m-times continuously differentiable real-valued functions such that the following norm is finite

$$\|F\|_{C^m(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n}\left(\sum_{|\alpha|\le m}|\partial^\alpha F(x)|^2\right)^{1/2}.$$

* Corresponding author.
E-mail addresses: fsjiang@math.ucdavis.edu (F. Jiang), kluli@math.ucdavis.edu (G.K. Luli).

https://doi.org/10.1016/j.aim.2020.107364

If S is a finite set, we write $\#(S)$ to denote the number of elements in S. We use C to denote constants that depend only on m and n.

Problem 1. *Let* $E \subset \mathbb{R}^n$ *be a finite set. Let* $f : E \to [0, \infty)$. *Compute the order of magnitude of*

$$\|f\|_{C^m_+(E)} := \inf \left\{ \|F\|_{C^m(\mathbb{R}^n)} : F|_E = f \text{ and } F \geq 0 \right\} . \tag{1.1}$$

By "order of magnitude" we mean the following: Two quantities M and $\tilde{M}$ determined by E, f, m, n are said to have the same order of magnitude provided that $C^{-1}M \leq \tilde{M} \leq CM$, with C depending only on m and n. To compute the order of magnitude of $\tilde{M}$ is to compute a number M such that M and $\tilde{M}$ have the same order of magnitude.

Problem 1 without the nonnegative constraint has been extensively studied, see [3,5,7,9,13,14].

We also consider an open problem posed in [9].

Problem 2. *Let* $E \subset \mathbb{R}^n$ *be a finite set. Let* $f : E \to [0, \infty)$. *Compute a nonnegative function* $F \in C^m(\mathbb{R}^n)$ *such that* $F|_E = f$ *and* $\|F\|_{C^m(\mathbb{R}^n)} \leq C\|f\|_{C^m_+(E)}$.

We will present a brief history of Problem 1 and an overview of our results on Problems 1 and 2.

We start with elementary background. Given a subset $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}$, we define the trace norm of f as

$$\|f\|_{C^m(E)} := \inf \left\{ \|F\|_{C^m(\mathbb{R}^n)} : F|_E = f \right\};$$

we say that $F \in C^m(\mathbb{R}^n)$ is an almost optimal $C^m(\mathbb{R}^n)$ interpolant if $F \in C^m(\mathbb{R}^n)$, $F|_E = f$, and $\|F\|_{C^m(\mathbb{R}^n)} \leq C(m, n)\|f\|_{C^m(E)}$ for some constant $C(m, n)$ depending only on m, n. For nonnegative interpolants, one can define analogously the trace norm by requiring the interpolant to be nonnegative, see (1.1).

We recall the basic finiteness principle of [5].

Theorem 0.A *(Finiteness Principle). For large enough* $k^\sharp$ *and* C, *both depending only on* m *and* n, *the following holds:*

Let $f : E \to \mathbb{R}$ *with* $E \subset \mathbb{R}^n$ *finite. Suppose that for each* $S \subset E$ *with* $\#(S) \leq k^\sharp$ *there exists* $F^S \in C^m(\mathbb{R}^n)$ *with norm* $\left\|F^S\right\|_{C^m(\mathbb{R}^n)} \leq 1$, *such that* $F^S = f$ *on* S. *Then there exists* $F \in C^m(\mathbb{R}^n)$ *with norm* $\|F\|_{C^m(\mathbb{R}^n)} \leq C$, *such that* $F = f$ *on* E.

Theorem 0.A and several related results were first conjectured by Y. Brudnyi and P. Shvartsman [1,2,22]. The first nontrivial case $C^2(\mathbb{R}^n)$ was proven by P. Shvartsman [21,22] with the sharp finiteness constant $k^\sharp = 3 \cdot 2^{n-1}$. Theorem 0.A is further refined to a Sharp Finiteness Principle in [7], which serves as the backbone for efficient algorithms for computing trace norms and almost optimal interpolants.

For nonnegative smooth interpolation, in [12], the authors proved the following theorem.

Theorem 0.B *(Finiteness Principle for Nonnegative Smooth Interpolation). For large enough* $k^\sharp$ *and* C*, both depending only on* m *and* n*, the following holds:*

Let $f : E \to [0, \infty)$ *with* $E \subset \mathbb{R}^n$ *finite. Suppose that for each* $S \subset E$ *with* $\#(S) \leq k^\sharp$*, there exists* $F^S \in C^m(\mathbb{R}^n)$ *with norm* $\|F^S\|_{C^m(\mathbb{R}^n)} \leq 1$*, such that* $F^S = f$ *on* S *and* $F^S \geq 0$ *on* $\mathbb{R}^n$*. Then there exists* $F \in C^m(\mathbb{R}^n)$ *with norm* $\|F\|_{C^m(\mathbb{R}^n)} \leq C$*, such that* $F = f$ *on* E *and* $F \geq 0$ *on* $\mathbb{R}^n$*.*

The proof of Theorem 0.B given in [12] depends on a refinement procedure for shape fields proven in [11]. As such, the construction of the interpolant is not very explicit, and the finiteness constant $k^\sharp$ is larger than it is necessary. For example, for $m = 2, n = 2$, [12] gives $k^\sharp \geq 100 + 5^{l_*+100}$ for some $l_* \geq 100$.

In this paper, we begin by showing that for $m = 2, n = 2$, $k^\sharp = 64$ is sufficient (see Theorem 4). Although not proven sharp here, it is a substantial improvement over the one given by [12].

For a better finiteness constant than [12] and also ours, see [23] (which gives $k^\sharp = 8$); however, the method in [23] assumes the validity of the Finiteness Principle and does not yield a construction for the interpolant.

With a more careful analysis of our proof for the Finiteness Principle, we are able to prove a Sharp Finiteness Principle analogous to the first one proven in [7] without the nonnegative constraint; the Sharp Finiteness Principle reads as follows: Given a finite set $E \subset \mathbb{R}^2$ with $\#(E) = N$, we can produce a list of subsets $S_1, \cdots, S_L$ such that $E = \bigcup_{\ell=1}^{L} S_\ell$, $\#(S_\ell) \leq C$, and $L \leq CN$ such that $\|f\|_{C^2_+(E)}$ and $\max_{\ell=1,\cdots,L} \|f\|_{C^2_+(S_\ell)}$ have the same order of magnitude. Thus, computing the order of magnitude of $\|f\|_{C^2_+(E)}$ amounts to computing each $\|f\|_{C^2_+(S_\ell)}$ for $\ell = 1, \cdots, L$. In the forthcoming papers [19,20], we will use this result to provide efficient algorithms analogous to the Fefferman-Klartag algorithms [13] for solving nonnegative interpolation problems.

Our two-dimensional results in this paper rely on their one-dimensional counterparts. We will provide a detailed analysis of the one-dimensional situation in Section 6. Along the way, we also show the nonexistence of a bounded linear extension operator that preserves nonnegativity. This is the content of Theorem 3. This is in sharp contrast to $C^m(\mathbb{R}^n)$ extensions without the nonnegative constraint, for which there exists a bounded linear extension operator of bounded depth [6].

Our approach is inspired by [6–8,18]. However, we will need new ingredients to apply the machinery adapted from the aforementioned references.

Lastly, we remark that our approach can be adapted to treat nonnegative $C^m(\mathbb{R})$ $(m > 2)$ extensions for finite sets E, and to prove the Finiteness Principle for nonnegative $C^{1,\omega}(\mathbb{R}^2)$ extensions for arbitrary closed sets E.

Next, we sketch the main ideas for our approach, sacrificing accuracy for the ease of understanding.

We begin with interpolation in one-dimension. For nonnegative $C^2(\mathbb{R})$ interpolation, we will show that, if one can interpolate three consecutive points, then one can interpolate any finite set of points by patching consecutive three-point interpolants together.[1] To handle nonnegative $C^2(\mathbb{R}^2)$ interpolation, we will reduce local interpolation problems to the one-dimensional situation.

To illustrate the idea, we assume that $E \subset Q_0 := [0,1) \times [0,1)$. For a square $Q \subset \mathbb{R}^2$, we write $2Q$ to denote the two times concentric dilation of Q, and δ_Q to denote the sidelength of Q. We perform a Calderón-Zygmund decomposition to Q_0, bisecting Q_0 and its children, which we will call Q_ν, until the following conditions are satisfied: Any two nearby squares are comparable in size; $E \cap 2Q_\nu$ lies on a curve with slope $\leq C$ and curvature $\leq C\delta_{Q_\nu}^{-1}$; and any two local solutions near Q_ν are indistinguishable up to a Taylor error on the order of δ_{Q_ν}. We then solve the local interpolation problem by straightening $E \cap 2Q_\nu$ and treating it as a one-dimensional problem. To ensure two nearby local solutions are Whitney compatible when patched together by a partition of unity, we prescribe a collection of Whitney-compatible polynomials, denoted by P_ν, each based at a representative point x_ν near the center of Q_ν, and force the local solution to take P_ν as a jet at x_ν.

The two-dimensional Finiteness Principle is then a consequence of its one-dimensional counterpart and Helly's Theorem from combinatorial geometry.

In order to prove the Sharp Finiteness Principle, we need to localize the dependence of the P_ν's on the given data (E, f). This involves a variant of Helly's Theorem, a careful analysis when f is locally small (on the order of δ_Q^2), and the combinatorial properties of the Calderón-Zygmund squares.

Here we have given an overly simplified account of our approach. In practice, we have to control derivatives on small scales and handle subtraction with great care in order to preserve nonnegativity. The technical matters will be handled in the sections below.

Inspired by [3], we also pose the following question on the best finiteness constant for nonnegative $C^2(\mathbb{R}^2)$ interpolation, and conjecture the answer to be in the positive.

Problem 3. *For nonnegative* $C^2(\mathbb{R}^2)$ *interpolation, can we take* $k^\sharp = 6$*?*

It would be interesting to know more about the connection between the methods employed in this paper and the method of "Lipschitz selection" presented in [3].

We end the introduction by announcing here our solutions to Problems 1 and 2; the detail will be presented in the forthcoming papers [19,20]. For a given $E \subset \mathbb{R}^2$ with $\#(E) = N$, we can process E with at most $CN \log N$ operations and CN storage. After that, we can compute the order of magnitude of $\|f\|_{C^2_+(E)}$ for any $f : E \to [0, \infty)$ using at most CN operations. After preprocessing E using at most $CN \log N$ operations and CN storage, we are able to receive further inputs, consisting of a function $f : E \to [0, \infty)$ and

[1] Here we mention that the finiteness constant $k^\sharp = 3$ is sharp for nonnegative $C^2(\mathbb{R})$ interpolation. See [3].

a number $M \geq 0$. Then, given $x \in \mathbb{R}^2$, we are able to produce a list $(f_\alpha(x) : |\alpha| \leq 2)$ using at most $C \log N$ operations. Suppose an **Oracle** tells us that $\|f\|_{C^2_+(E)} \leq M$. We can then guarantee the existence of a nonnegative function $F \in C^2(\mathbb{R}^2)$ with $\|F\|_{C^2(\mathbb{R}^2)} \leq CM$ and $F|_E = f$, such that $\partial^\alpha F(x) = f_\alpha(x)$ for $|\alpha| \leq 2$.

To the extend of our knowledge, there has been no previously known result on Problem 2.

This paper is part of a literature on extension and interpolation, going back to the seminal works of H. Whitney [15–17]. We refer the interested readers to [1–9,11–14,21, 22,24] and references therein for the history and related problems.

Acknowledgment. We express our gratitude to Charles Fefferman, Kevin O'Neill, and Pavel Shvartsman for their valuable comments. We also thank all the participants in the 11th Whitney workshop for fruitful discussions, and Trinity College Dublin for hosting the workshop.

The first author is supported by the UC Davis Summer Graduate Student Researcher Award and the Alice Leung Scholarship in Mathematics. The second author is supported by NSF Grant DMS-1554733 and the UC Davis Chancellor's Fellowship.

2. Statement of results

First we set up notations. Let $n = 1, 2$. We write $C^2_+(\mathbb{R}^n)$ to denote the collection of all functions $F : \mathbb{R}^n \to [0, \infty)$ whose derivatives up to the second order are continuous and bounded. We write ∂^m to denote the m-th derivative of a single-variable function.

We begin with our results in one-dimension.

Theorem 1.A ***(1-D Finiteness Principle)***. *There exists a constant $C > 0$ such that the following holds.*

Let $E = \{x_1, \dots, x_N\} \subset \mathbb{R}$ be a finite set with $x_1 < \cdots < x_N$ and $N \geq 3$. Let $f : E \to [0, \infty)$. Suppose

- *(i) For every consecutive three points $E_j = \{x_j, x_{j+1}, x_{j+2}\}$ $(j = 1, \dots, N-2)$ there exists a function $F_j \in C^2_+(\mathbb{R})$ such that $F_j\big|_{E_j} = f$; and*
- *(ii) $\|F_j\|_{C^2(\mathbb{R})} \leq M$.*

Then there exists $F \in C^2_+(\mathbb{R})$ with

- *(A) $F|_E = f$, and*
- *(B) $\|F\|_{C^2(\mathbb{R})} \leq CM$.*

Remark 2.1. In the present work, we do not pursue the minimal C. See, e.g. [8] for a discussion on best constants.

We will also need the following variant of Theorem 1.A in the proof of Lemma 5.4.

Theorem 1.B. *There exists a constant* $C > 0$ *such that the following holds.*

Let $E = \{x_1, \ldots, x_N\} \subset \mathbb{R}$ *be a finite set with* $x_1 < \cdots < x_N$ *and* $N \geq 3$. *Let* $f : E \to \mathbb{R}$. *Suppose*

(i) For every consecutive three points $E_j = \{x_j, x_{j+1}, x_{j+2}\}$ $(j = 1, \ldots, N-2)$, *there exists a function* $F_j \in C^2(\mathbb{R})$ *such that* $F_j|_{E_j} = f$;

(ii) $|\partial^m F_j| \leq A_m$ *on* $\mathbb{R}$ *for* $m = 0, 1, 2$.

Then there exists $F \in C^2(\mathbb{R})$ *such that*

(A) $F|_E = f$;

(B) $|\partial^m F| \leq CA_m$ *on* $\mathbb{R}$ *for* $m = 0, 1, 2$.

Remark 2.2. The proofs of Theorems 1.A and 1.B will be given in Section 6.

Let $n = 1, 2$. Given a finite set $E \subset \mathbb{R}^n$, we write $C^2(E)$ to denote all functions $f : E \to \mathbb{R}$, equipped with the trace norm $\|f\|_{C^2(E)} := \inf\left\{\|F\|_{C^2(\mathbb{R}^n)} : F|_E = f\right\}$. We write $C^2_+(E)$ to denote all functions $f : E \to [0, \infty)$, equipped with the "trace norm" $\|f\|_{C^2_+(E)} := \inf\left\{\|F\|_{C^2(\mathbb{R}^n)} : F|_E = f \text{ and } F \geq 0\right\}$.

The proofs of Theorems 1.A and 1.B along with an argument involving quadratic programming immediately give rise to the following results.

Theorem 2.A. *Let* $E \subset \mathbb{R}$ *be a finite set. There exist universal constants* C, D *and an operator* $\mathcal{E} : C^2_+(E) \to C^2_+(\mathbb{R})$ *such that the following hold.*

(A) $\mathcal{E}(f)\big|_E = f$ *for all* $f \in C^2_+(E)$.

(B) $\|\mathcal{E}(f)\|_{C^2(\mathbb{R})} \leq C\|f\|_{C^2_+(E)}$.

(C) Moreover, for each $x \in \mathbb{R}$, *there exists* $S(x) \subset E$ *with* $\#(S(x)) \leq D$, *such that for all* $f, g \in C^2_+(E)$ *with* $f|_{S(x)} = g|_{S(x)}$, *we have*

$$\partial^m(\mathcal{E}(f))(x) = \partial^m(\mathcal{E}(g))(x) \quad \textit{for } m = 0, 1, 2\,.$$

Remark 2.3. In general, $\mathcal{E}$ is not additive. See Theorem 3.

Theorem 2.A holds in the absence of the nonnegative constraint. This is the content of the next theorem.

Theorem 2.B. *Let* $E \subset \mathbb{R}$ *be a finite set. There exist universal constants* C, D *and a linear operator* $\mathcal{E} : C^2(E) \to C^2(\mathbb{R})$ *such that the following hold.*

(A) $\mathcal{E}(f)\big|_E = f$ *for all* $f \in C^2(E)$.

(B) $\|\mathcal{E}(f)\|_{C^2(\mathbb{R})} \leq C\|f\|_{C^2(E)}$.

(C) Moreover, for each $x \in \mathbb{R}$*, there exists* $S(x) \subset E$ *with* $\#(S(x)) \leq D$*, such that for all* $f, g \in C^2(E)$ *with* $f|_{S(x)} = g|_{S(x)}$*, we have*

$$\partial^m(\mathcal{E}(f))(x) = \partial^m(\mathcal{E}(g))(x) \quad \textit{for } m = 0, 1, 2\,.$$

Remark 2.4. The number D in Theorems 2.A and 2.B is called the depth of the operator $\mathcal{E}$. The proofs of Theorems 2.A and 2.B will be given in Section 6. We also remark that the set $S(x)$ takes a particularly simple form.

- Suppose $\#(E) \leq 3$. We take $S(x) = E$.
- Suppose $\#(E) \geq 4$. Enumerate $E = \{x_1, \cdots, x_N\}$ with $x_1 < \cdots < x_N$.
  - If $x < x_1$ or $x > x_N$, we take $S(x)$ to be the three points in E closest to x.
  - If $x \in [x_1, x_2]$, we take $S(x) = \{x_1, x_2, x_3\}$.
  - If $x \in [x_{N-1}, x_N]$, we take $S(x) = \{x_{N-2}, x_{N-1}, x_N\}$.
  - Otherwise, we take $S(x) = \{x'_1, x'_2, x'_3, x'_4\} \subset E$ with $x'_1 < x'_2 < x'_3 < x'_4$ such that $x \in [x'_2, x'_3]$.

It has been shown in [16] the existence of an extension operator satisfying (A,B) of Theorem 2.B. We thank P. Shvartsman for bringing to our attention that an algorithm for constructing $S(x)$ in a more general one-dimensional setting (without nonnegativity) was given in [24], in which the interested readers will also find an informative account of the one-dimensional extension theory (without nonnegativity).

Theorem 3. *There exists a finite set* $E \subset \mathbb{R}$ *that does not admit a map* $\mathcal{E} : C^2_+(E) \to C^2(\mathbb{R})$ *satisfying both of the following.*

(A) For all $f \in C^2_+(E)$*, we have* $\mathcal{E}(f)(x) = f(x)$ *for all* $x \in E$*,* $\mathcal{E}(f) \geq 0$ *on* $\mathbb{R}$*, and* $\|\mathcal{E}(f)\|_{C^2(\mathbb{R})} \leq C\|f\|_{C^2_+(E)}$ *for some universal constant* C*.*

(B) $\mathcal{E}(f + g) = \mathcal{E}(f) + \mathcal{E}(g)$ *for all* $f, g \in C^2_+(E)$*.*

Remark 2.5. By considering finite sets of the form $E \times \underbrace{\{0\} \times \cdots \times \{0\}}_{(n-1) \text{ copies}}$ with E as in Theorem 3, we can further conclude that, for $C^2(\mathbb{R}^n)$ with $n \geq 2$, there does not exist a bounded additive extension operator that preserves nonnegativity. See Section 6 for the proof.

We now turn to our results in two-dimension.

Theorem 4 ***(2-D Finiteness Principle)****. There exists a constant* $C > 0$ *such that the following holds.*

Let $f : E \to [0, +\infty)$ *with* $E \subset \mathbb{R}^2$ *finite. Suppose for each* $S \subset E$ *with* $\#(S) \leq 64$*, there exists* $F^S \in C^2_+(\mathbb{R}^2)$ *such that*

(i) $\|F^S\|_{C^2(\mathbb{R}^2)} \leq M$*, and*
(ii) $F^S\big|_S = f$.

Then there exists $F \in C^2_+(\mathbb{R}^2)$ *such that*

(A) $F|_E = f$*, and*
(B) $\|F\|_{C^2(\mathbb{R}^2)} \leq CM$.

Remark 2.6. The proof of Theorem 4 is given in Section 7.

We also have an improved version of Theorem 4.

Theorem 5 *(**2-D Sharp Finiteness Principle**). Let* $E \subset \mathbb{R}^2$ *with* $\#(E) = N < \infty$*. Then there exist universal constants* C, C', C'' *and a list of subsets* $S_1, S_2, \cdots, S_L \subset E$ *satisfying the following.*

(A) $\#(S_\ell) \leq C$ *for each* $\ell = 1, \cdots, L$.
(B) $L \leq C'N$.
(C) Given any $f : E \to [0, \infty)$*, we have*

$$\max_{\ell=1,\cdots,L} \|f\|_{C^2_+(S_\ell)} \leq \|f\|_{C^2_+(E)} \leq C'' \max_{\ell=1,\cdots,L} \|f\|_{C^2_+(S_\ell)} .$$

Remark 2.7. The proof of Theorem 5 is given in Section 8.

In a forthcoming paper [19], we will prove the following result.

Theorem 6. *Let* $E \subset \mathbb{R}^2$ *be a finite set. There exist (universal) constants* C, D*, and a map* $\mathcal{E} : C^2_+(E) \times [0, \infty) \to C^2_+(\mathbb{R}^2)$ *such that the following hold.*

(A) Let $M \geq 0$*. Then for all* $f \in C^2_+(E)$ *with* $\|f\|_{C^2_+(E)} \leq M$*, we have* $\mathcal{E}(f, M) = f$ *on* E *and* $\|\mathcal{E}(f, M)\|_{C^2(\mathbb{R}^2)} \leq CM$.
(B) For each $x \in \mathbb{R}^2$*, there exists a set* $S(x) \subset E$ *with* $\#(S(x)) \leq D$ *such that for all* $M \geq 0$ *and* $f, g \in C^2_+(E)$ *with* $\|f\|_{C^2_+(E)}, \|g\|_{C^2_+(E)} \leq M$ *and* $f|_{S(x)} = g|_{S(x)}$*, we have*

$$\partial^\alpha \mathcal{E}(f, M)(x) = \partial^\alpha \mathcal{E}(g, M)(x) \text{ for } |\alpha| \leq 2 .$$

We will not use Theorem 6 in this paper.
As a consequence of Theorems 5 and 6, in [20], we will provide the following algorithms.

Algorithm 1. Nonnegative $C^2(\mathbb{R}^2)$ Interpolation Algorithm - Trace Norm.

DATA: $E \subset \mathbb{R}^2$ finite with $\#(E) = N$.
QUERY: $f : E \to [0, \infty)$.
RESULT: The order of magnitude of $\|f\|_{C^2_+(E)}$. More precisely, the algorithm outputs a number $M \geq 0$ such that both of the following hold.
– We guarantee the existence of a function $F \in C^2_+(\mathbb{R}^2)$ such that $F|_E = f$ and $\|F\|_{C^2(\mathbb{R}^2)} \leq CM$.
– We guarantee there exists no $F \in C^2_+(\mathbb{R}^2)$ with norm at most M satisfying $F|_E = f$.
COMPLEXITY:
– Preprocessing E: at most $CN \log N$ operations and CN storage.
– Answer query: at most CN operations.

Algorithm 2. Nonnegative $C^2(\mathbb{R}^2)$ Interpolation Algorithm - Interpolant.

DATA: $E \subset \mathbb{R}^2$ finite with $\#(E) = N$. $f : E \to [0, \infty)$. $M \geq 0$.
ORACLE: $\|f\|_{C^2_+(E)} \leq M$.
RESULT: A query function that accepts $x \in \mathbb{R}^2$ and produces a list of numbers $(f_\alpha(x) : |\alpha| \leq 2)$ that guarantees the following: There exists a function $F \in C^2_+(\mathbb{R}^2)$ with $\|F\|_{C^2(\mathbb{R}^2)} \leq CM$ and $\tilde{F}|_E = f$, such that $\partial^\alpha F(x) = f_\alpha(x)$ for $|\alpha| \leq 2$. The function F does not depend on the query point x.
COMPLEXITY:
– Preprocessing (E, f): at most $CN \log N$ operations and CN storage.
– Answer query: at most $C \log N$ operations.

We will present the proofs for Theorems 1–5 in the sections below. We will start from scratch and introduce the relevant terminologies and notations in the next section.

3. Conventions and preliminaries

Constants

We use $c_*, C_*, C, C' > 0$, etc. to denote “controlled” universal constants. They may be different quantities in different instances. We will label them to avoid confusion when necessary.

Coordinates and norms

We assume that we are given an ordered orthogonal coordinate system $x = (s, t)_{\text{standard}}$ on $\mathbb{R}^2$ a priori. We write $B(x, r)$ to denote the open disc of radius $r > 0$ centered at $x \in \mathbb{R}^2$.

We use $\alpha, \beta \in \mathbb{N}_0^2$ etc. to denote multi-indices. We adopt the partial ordering $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i$ for $i = 1, 2$.

Let $\Omega \subset \mathbb{R}^n$ be a set with nonempty interior Ω^0. For positive integers m, n, we write $C^m(\Omega)$ to denote the vector space of m-times continuously differentiable real-valued functions on Ω^0 such that the following norm is finite:

$$\|F\|_{C^m(\Omega)} := \sup_{x \in \Omega^0} \left(\sum_{|\alpha| \leq m} |\partial^\alpha F(x)|^2 \right)^{1/2}. \tag{3.1}$$

We write $C^m_+(\Omega)$ to denote the collection of functions $F \in C^m(\Omega)$ such that $F \geq 0$ on Ω. This is not a vector space.

Let $E \subset \mathbb{R}^n$ be finite. We define

$$C^m(E) := \{F|_E : F \in C^m(\mathbb{R}^n)\} .$$

$C^m(E)$ is a vector space that can be equipped with a seminorm, which we will called the trace norm of $f \in C^m(E)$:

$$\|f\|_{C^m(E)} := \inf \left\{ \|F\|_{C^m(\mathbb{R}^n)} : F \in C^m(\mathbb{R}^n) \quad \text{and } F|_E = f \right\} .$$

Similarly, we define

$$C^m_+(E) := \left\{ F|_E : F \in C^m_+(\mathbb{R}^n) \right\} .$$

We will abuse terminology and refer to the following as the (nonnegative) trace norm of $f \in C^m_+(E)$:

$$\|f\|_{C^m_+(E)} := \inf \left\{ \|F\|_{C^m(\mathbb{R}^n)} : F \in C^m_+(\mathbb{R}^n) \quad \text{and } F|_E = f \right\} .$$

Jets

We write $\mathcal{P}$ to denote the space of degree one polynomials on $\mathbb{R}^2$. It is a three-dimensional vector space.

For $x_0 = (s_0, t_0) \in \mathbb{R}^2$ and a continuously differentiable function F on $\mathbb{R}^2$, the 1-jet of F at $x_0 \in \mathbb{R}^2$ is given by

$$\mathcal{J}_{x_0} F(x) := F(x_0) + \nabla F(x_0) \cdot (x - x_0) .$$

We use $\mathcal{R}_{x_0}$ to denote the vector space of 1-jets at $x_0 \in \mathbb{R}^2$. $\mathcal{R}_{x_0}$ inherits a norm from $\mathbb{R}^3$ via the identification

$$I_{x_0} \,:\, a(s - s_0) + b(t - t_0) + c \,\mapsto\, (a, b, c). \tag{3.2}$$

Calderón-Zygmund squares

A square $Q \subset \mathbb{R}^2$ is of the form $Q = [s_0, s_0 + \delta) \times [t_0, t_0 + \delta)$, where $\delta > 0$ and $s_0, t_0 \in \mathbb{R}$.

For a square $Q \subset \mathbb{R}^2$, λQ denotes the concentric dilation of Q by a factor of $\lambda > 0$. Let $Q^* = 2Q$. δ_Q denotes the side length of Q.

For a square $Q_0 \in \mathbb{R}^2$, by a dyadic bisection of Q_0, we mean dividing Q_0 into four mutually disjoint congruent squares Q_1, Q_2, Q_3, Q_4 such that $Q_0 = \bigcup_{i=1}^4 Q_i$. Q_0 is called the dyadic parent of $Q_1, \dots, Q_4$. In this case, we write $Q_i^+ = Q_0$ for $i = 1, \dots, 4$. A dyadic parent for a dyadic square is unique if it exists.

Two squares Q and Q' are neighbors if one of the following holds.

- $Q = Q'$; or
- $\mathrm{closure}(Q) \cap \mathrm{closure}(Q') \neq \varnothing$, but $\mathrm{interior}(Q) \cap \mathrm{interior}(Q') = \varnothing$.

If Q and Q' are neighbors, we write $Q \leftrightarrow Q'$.

A collection of mutually disjoint squares $\Lambda = \{Q\}$ is a Calderón-Zygmund (CZ) covering of $\mathbb{R}^2$ if $\mathbb{R}^2 = \bigcup_{Q \in \Lambda} Q$, and

$$\text{if } Q \leftrightarrow Q', \text{ then } \frac{1}{4}\delta_Q \leq \delta_{Q'} \leq 4\delta_Q\,. \tag{3.3}$$

It is easy to see that (3.3) implies that a CZ covering satisfies the bounded intersection property: If $Q \in \Lambda$, then

$$\#\left(\left\{Q' \in \Lambda : \frac{9}{8}Q' \cap \frac{9}{8}Q \neq \varnothing\right\}\right) \leq 21. \tag{3.4}$$

We will only consider *nonnegative* (smooth) cutoff functions and partition of unity. A C^2-partition of unity $\{\theta_Q\}$ subordinate to a CZ covering $\Lambda = \{Q\}$ of $\mathbb{R}^2$ is CZ-compatible with Λ if

$$\theta_Q \geq 0,\ \operatorname{supp}(\theta_Q) \subset \frac{9}{8}Q,\ |\partial^\alpha \theta_Q| \leq C\delta_Q^{-|\alpha|}\ \forall |\alpha| \leq 2, \text{ and } \sum_{Q \in \Lambda} \theta_Q \equiv 1. \tag{3.5}$$

Here C is some universal constant. Such partition of unity exists, see e.g. [15].

4. Basic convex sets and Whitney fields

Definition 4.1. Let $E \subset \mathbb{R}^2$ be a finite set. Let $f : E \to [0, +\infty)$. For a point $x \in \mathbb{R}^2$, a subset $S \subset E$, and a real number $M \geq 0$, we introduce the following objects:

$$\Gamma_+(x, S, M) := \left\{ P \in \mathcal{P} : \begin{array}{l} \text{There exists } F^S \in C^2_+(\mathbb{R}^2) \text{ such that} \\ \|F^S\|_{C^2(\mathbb{R}^2)} \leq M, F^S\big|_S = f, \text{ and } \mathcal{J}_x F^S = P. \end{array} \right\}, \tag{4.1}$$

and

$$\sigma(x, S) := \left\{ P \in \mathcal{P} : \begin{array}{l} \text{There exist } F^S \in C^2(\mathbb{R}^2) \text{ such that} \\ F^S\big|_S = 0, \|F^S\|_{C^2(\mathbb{R}^2)} \leq 1, \text{ and } \mathcal{J}_x F^S = P. \end{array} \right\}. \tag{4.2}$$

Given an integer $k \geq 0$ and a number $M \geq 0$, we define

$$\Gamma_+^\sharp(x,k,M) := \bigcap_{S\subset E,\, \#(S)\le k} \Gamma_+(x,S,M)\,, \tag{4.3}$$

and

$$\sigma^\sharp(x,k) := \bigcap_{S\subset E,\, \#(S)\le k} \sigma(x,S)\,. \tag{4.4}$$

Since $\#(E) < \infty$, for sufficiently large $M \ge 0$ depending on E and f, $\Gamma_+(x,S,M) \neq \varnothing$ for any $S \subset E$. As a consequence, for a specific $k \ge 0$, $\Gamma_+^\sharp(x,k,M) \neq \varnothing$ if M is sufficiently large.

It is easy to see that Γ_+, $\Gamma_+^\sharp$, σ, and $\sigma^\sharp$ are convex and bounded (as subsets of $\mathbb{R}^3$ via the identification (3.2)). We can easily see from (4.2) and (4.4) that σ and $\sigma^\sharp$ are *symmetric about the origin*. Since E is finite, for each fixed $x \in \mathbb{R}^2$ and $M > 0$, there are only finitely many distinct $\sigma(x,S)$ and $\Gamma(x,S,M)$. Therefore, we may apply the finite version of Helly's Theorem (see Section 4.1 for the statement). Both $\sigma^\sharp$ and $\Gamma_+^\sharp$ are monotone decreasing (with respect to set inclusion $\subset$) in k. Furthermore, $\Gamma_+^\sharp$ is monotone increasing in M.

Since σ and $\sigma^\sharp$ contain the zero polynomial, they are never empty.

Understanding the shapes of $\Gamma_+^\sharp$ and $\sigma^\sharp$ is the key to proving Theorems 4, 5, and 6.

We will also be working with the following object.

Definition 4.2. Given $x \in \mathbb{R}^2$ and $\delta > 0$, we introduce the following object

$$\mathcal{B}(x,\delta) := \left\{ P \in \mathcal{P} : |\partial^\alpha P(x)| \le \delta^{2-|\alpha|} \right\}. \tag{4.5}$$

To understand the significance of $\mathcal{B}(x,\delta)$, we point out that Taylor's theorem can be reformulated in the following way: Given $F \in C^2(\mathbb{R}^2)$ with $\|F\|_{C^2(\mathbb{R}^2)} \le M$, then $\mathcal{J}_x F - \mathcal{J}_y F \in CM \cdot \mathcal{B}(x, |x-y|)$ for any $x, y \in \mathbb{R}^2$.

4.1. Lemmas on convex sets

Lemma 4.1. $\Gamma_+^\sharp(x,k,M) - \Gamma_+^\sharp(x,k,M) \subset 2M \cdot \sigma^\sharp(x,k)$. *The minus sign denotes vector subtraction.*

Proof. Let $P_1, P_2 \in \Gamma_+^\sharp(x,k,M)$. For each $S \subset E$ with $\#(S) \le k$, there exist $F_1^S, F_2^S \in C_+^2(\mathbb{R}^2)$ such that for $i = 1,2$, $F_i^S\big|_S = f$, $\|F_i^S\|_{C^2(\mathbb{R}^2)} \le M$, and $\mathcal{J}_x F_i^S = P_i$. Then $(F_1^S - F_2^S)\big|_S = 0$, $\|F_1^S - F_2^S\|_{C^2(\mathbb{R}^2)} \le 2M$, and $\mathcal{J}_x F_1^S - F_2^S = P_1 - P_2$. Since S is arbitrary, $P_1 - P_2 \in \sigma^\sharp(x,k,2M) = 2M \cdot \sigma^\sharp(x,k)$. $\square$

We recall a classical result by Helly, the proof of which can be found in [26].

Helly's Theorem. *Let $\mathcal{F}$ be a finite collection of convex sets in $\mathbb{R}^D$. Suppose every subcollection of $\mathcal{F}$ of cardinality at most $(D+1)$ has nonempty intersection. Then the whole collection has nonempty intersection.*

The following lemma states that we can control polynomials in $\Gamma_+^\sharp$ based at some point by polynomials that are based at a different point but are "less universal" (in the sense that it is the jet for an interpolant for fewer points).

Lemma 4.2. *There exists a universal constant C such that the following holds. Let $x, x' \in \mathbb{R}^2$. Let $k_1 \geq 4k_2$. Let $M \geq 0$. Given $P \in \Gamma_+^\sharp(x, k_1, M)$, there exists $P' \in \Gamma_+^\sharp(x', k_2, M)$ such that*

$$|\partial^\alpha (P - P')(x)| \ , \ |\partial^\alpha (P - P')(x')| \leq CM|x - x'|^{2-|\alpha|} \text{ for } |\alpha| \leq 1.$$

Proof. Fix P and M as in the hypothesis of the lemma. For each $S \subset E$, we define

$$\Gamma_+^{\text{temp}}(S) := \left\{ P' \in \mathcal{P} : \begin{array}{l} \text{There exists } F^S \in C_+^2(\mathbb{R}^2) \text{ such that } \|F^S\|_{C^2(\mathbb{R}^2)} \leq M, \\ F^S\big|_S = f, \mathcal{J}_x F^S = P, \text{ and } \mathcal{J}_{x'} F^S = P'. \end{array} \right\}.$$

Then Γ_+^{temp} is a convex and bounded subset of $\mathcal{P}$. Notice that

$$S \subset \widetilde{S} \text{ implies } \Gamma_+^{\text{temp}}(\widetilde{S}) \subset \Gamma_+^{\text{temp}}(S). \tag{4.6}$$

It also follows from the definition of $\Gamma_+^\sharp(x, k_1, M)$ that

$$\text{if } \#(S) \leq k_1, \text{ then } \Gamma_+^{\text{temp}}(S) \neq \varnothing. \tag{4.7}$$

Let $S_1, ..., S_4 \subset E$ be given with $\#(S_i) \leq k_2$ for each i. Let $S = \bigcup_{i=1}^4 S_i$. Then $\#(S) \leq 4k_2 \leq k_1$. Thanks to (4.7), $\Gamma_+^{\text{temp}}(S) \neq \varnothing$. Since $S_i \subset S$, (4.6) implies that $\Gamma_+^{\text{temp}}(S) \subset \Gamma_+^{\text{temp}}(S_i)$. Therefore,

$$\bigcap_{i=1}^4 \Gamma_+^{\text{temp}}(S_i) \supset \Gamma_+^{\text{temp}}(S) \neq \varnothing.$$

Since $\{S_i\}_{i=1}^4$ are arbitrary, applying Helly's Theorem to the convex sets $\Gamma_+^{\text{temp}}(S_i) \subset \mathcal{P}$ (with $\dim \mathcal{P} = 3$), we have

$$\bigcap_{S \subset E, \#(S) \leq k_2} \Gamma_+^{\text{temp}}(S) \neq \varnothing.$$

Let $P' \in \bigcap_{S \subset E, \#(S) \leq k_2} \Gamma_+^{\text{temp}}(S)$. By definition, $P' \in \Gamma_+^\sharp(x', k_2, M)$. Setting $S = \varnothing$, we see that there exists $F \in C_+^2(\mathbb{R}^2)$ with

- $\|F\|_{C^2(\mathbb{R}^2)} \leq M$; and
- $\mathcal{J}_x F = P$ and $\mathcal{J}_{x'} F = P'$.

By Taylor's theorem, we have

$$|\partial^\alpha (P - P')(x)| = |\partial^\alpha (\mathcal{J}_x F - \mathcal{J}_{x'} F)(x)| \leq |\partial^\alpha (F - \mathcal{J}_{x'} F)(x)| \leq CM|x - x'|^{2-|\alpha|}.$$

The estimate for $|\partial^\alpha (P - P')(x')|$ is similar. □

Lemma 4.3. *Under the hypothesis of Theorem 4, $\Gamma_+^\sharp(x, 16, M) \neq \varnothing$ for all $x \in \mathbb{R}^2$.*

Proof. Recall that $\Gamma_+(\cdot,\cdot,\cdot)$ is a convex set in a three-dimensional vector space $\mathcal{P}$. By Helly's Theorem, it suffices to show that the intersection of any four-element subfamily is nonempty. To this end, fix $x \in \mathbb{R}^2$, let $S_1, \cdots, S_4 \subset E$ with $\#(S_i) \leq 16$, and let $S = \bigcup_{i=1}^4 S_i$. We have

$$\Gamma_+(x, S, M) \subset \bigcap_{i=1}^{4} \Gamma_+(x, S_i, M). \tag{4.8}$$

Since $\#(S) \leq 64$, the hypothesis of Theorem 4 implies that $\Gamma_+(x, S, M) \neq \varnothing$, and hence, the intersection on the right hand side of (4.8) is nonempty. This concludes the proof. □

The following variant of Helly's Theorem can be found in Section 3 of [6].

Lemma 4.4. *Let $\mathcal{F}$ be a finite collection of compact, convex, and symmetric subsets of $\mathbb{R}^D$. Suppose 0 is an interior point for each $K \in \mathcal{F}$. Then there exist $K_1, \cdots, K_{D(D+1)} \in \mathcal{F}$ such that*

$$K_1 \cap \cdots \cap K_{D(D+1)} \subset C_D \cdot \left(\bigcap_{K \in \mathcal{F}} K \right).$$

Here, C_D is a constant that depends only on D.

Lemma 4.5. *There exists a universal constant C such that the following holds. Let $x \in \mathbb{R}^2$. Then given $k \geq 0$, there exist $S_1, \cdots, S_{12} \subset E$, with $\#(S_i) \leq k$ for each i, such that*

$$\bigcap_{i=1}^{12} \sigma(x, S_i) \subset C \cdot \left(\bigcap_{S \subset E, \#(S) \leq k} \sigma(x, S) \right) = C \cdot \sigma^\sharp(x, k).$$

Proof. Let $x \in \mathbb{R}^2$. Note that $\sigma^\sharp(x,k)$ has nonempty interior (in the relative topology of the maximal affine space that it spans). We apply Lemma 4.4 (with $D \leq \dim \mathcal{P} = 3$) to $\mathrm{closure}(\sigma(x, S))$. Thus, there exist $S_1, \cdots, S_{12} \subset E$ with $\#(S_i) \leq k$ for each $i = 1, \cdots, 12$, such that

$$\bigcap_{i=1}^{12} \sigma(x, S_i) \subset C_D \cdot \left(\bigcap_{S\subset E, \#(S)\leq k} \text{closure}\,(\sigma(x,S)) \right).$$

Therefore,

$$\bigcap_{i=1}^{12} \sigma(x, S_i) \subset 2C_D \cdot \left(\bigcap_{S\subset E, \#(S)\leq k} \sigma(x,S) \right) = 2C_D \cdot \sigma^\sharp(x,k). \tag{4.9}$$

This proves the lemma. □

4.2. Whitney fields

In this subsection, we assume $n = 1$ or 2. We use $\mathcal{P}$ to denote the space of polynomials on $\mathbb{R}^n$ with degree no greater than one.

We now recall the notion of a Whitney field.

Let $S \subset \mathbb{R}^n$ be a finite set. We use $W^2(S)$ to denote the (finite dimensional) vector space of sections of $S \times \mathcal{P}$. An element $\vec{P} \in W^2(S)$ is called a Whitney field, and has the form $\vec{P} = (P^x)_{x\in S}$. $W^2(S)$ can be endowed with a norm

$$\|\vec{P}\|_{W^2(S)} := \max_{x\in S} \left(\sum_{|\alpha|\leq 1} |\partial^\alpha P^x(x)|^2 \right)^{1/2} + \max_{\substack{x,y\in S\\ x\neq y}} \left(\sum_{|\alpha|\leq 1} \left(\frac{|\partial^\alpha (P^x - P^y)(x)|}{|x-y|^{2-|\alpha|}} \right)^2 \right)^{1/2}. \tag{4.10}$$

We are interested in jets that can be extended to nonnegative C^2 functions. For $x \in \mathbb{R}^n$ and $M \geq 0$, we define

$$\mathcal{C}_+(x, M) := \left\{ P \in \mathcal{P} : \left(\sum_{|\alpha|\leq 1} |\partial^\alpha P(x)|^2 \right)^{1/2} \leq M,\ P(x) \geq 0,\ \text{and } |\nabla P| \leq \sqrt{4M \cdot P(x)} \right\}. \tag{4.11}$$

The next lemma tells us how to approximate Γ_+.

Lemma 4.6. *There exists a universal constant* C *such that the following holds. Given* $M \geq 0$*, we have*[2]

$$\Gamma_+(x, \varnothing, C^{-1}M) \subset \mathcal{C}_+(x, M) \subset \Gamma_+(x, \varnothing, CM). \tag{4.12}$$

[2] Here, when $n = 1$, $\Gamma_+(x, \varnothing, M)$ is defined to be $\left\{ \mathcal{J}_x F : F \in C^2(\mathbb{R}), F \geq 0, \text{ and } \|F\|_{C^2(\mathbb{R})} \leq M \right\}$, where $\mathcal{J}_x$ is the first degree Taylor expansion about the point x for a single-variable function.

Proof. The statement is clear for $M = 0$.

Suppose $M > 0$.

The first inclusion follows immediately from Taylor's theorem. We prove the second inclusion.

Without loss of generality, we may assume $x = 0$.

Pick $P \in \mathcal{C}_+(0, M)$. We have

$$\sum_{|\alpha|\leq 1} |\partial^\alpha P(0)|^2 \leq M^2$$

and

$$|\nabla P|^2 \leq 4M \cdot P(0). \tag{4.13}$$

Restricting P to each one-dimensional subspace of $\mathbb{R}^n$ and using (4.13), we see that

$$\tilde{P} := M|x|^2 + P(x) = M|x|^2 + \nabla P \cdot x + P(0) \geq 0 \text{ for } x \in \mathbb{R}^n.$$

Let B be the unit disc in $\mathbb{R}^n$. Let $\theta \in C^2_+(\mathbb{R}^n)$ be a cutoff function satisfying

$$\text{supp}\,(\theta) \subset B\,,\ \theta \equiv 1 \text{ near } 0\,,\ |\partial^\alpha \theta| \leq C \text{ for } |\alpha| \leq 2.$$

We define

$$F := \theta \cdot \tilde{P} = M\theta|x|^2 + \theta\nabla P \cdot x + \theta P(0)\,.$$

Immediately, we have $\mathcal{J}_0 F = \mathcal{J}_0 \tilde{P} = \mathcal{J}_0 P = P$ and $F \geq 0$ on $\mathbb{R}^n$. Moreover,

$$|\partial^\alpha F(x)| \leq CM \text{ for } x \in B \text{ and } |\alpha| \leq 2\,. \tag{4.14}$$

Since θ is supported in B, we can conclude that, $\|F\|_{C^2(\mathbb{R}^n)} \leq CM$ and $\mathcal{J}_0 F \in \Gamma_+(0, \varnothing, A)$ for $A = CM$. This concludes the proof. $\square$

Definition 4.3. Recall the definition of $\mathcal{C}_+$ in (4.11). Given a finite set $S \subset \mathbb{R}^n$, we define

$$W^2_+(S) := \left\{ \vec{P} = (P^x)_{x\in S} \in W^2(S) : \begin{array}{l} \text{There exists } M \geq 0 \text{ such that} \\ P^x \in \mathcal{C}_+(x, M) \text{ for each } x \in S. \end{array} \right\}. \tag{4.15}$$

For $\vec{P} \in W^2_+(S)$, we define

$$\|\vec{P}\|_{W^2_+(S)} := \|\vec{P}\|_{W^2(S)} + \mathcal{M}(\vec{P}), \tag{4.16}$$

where $\|\vec{P}\|_{W^2(S)}$ is defined in (4.10) and

$$\mathcal{M}(\vec{P}) := \inf\left\{ M > 0 : |\nabla P^x| \leq \sqrt{4MP^x(x)} \text{ for each } x \in S \right\}. \tag{4.17}$$

Remark 4.1. The definition of $\mathcal{M}$ is motivated by the estimate (4.14).

The following is immediate from Taylor's theorem and Lemma 4.6.

Lemma 4.7. *Let* $F \in C^2_+(\mathbb{R}^n)$. *Let* $S \subset \mathbb{R}^n$ *be a finite set. For each* $x \in S$, *let* $P^x := \mathcal{J}_x F$. *Let* $\vec{P} := (P^x)_{x\in S}$. *Then* $\vec{P} \in W^2_+(S)$ *with* $\|\vec{P}\|_{W^2_+(S)} \le C\|F\|_{C^2(\mathbb{R}^n)}$ *for some constant* C *depending only on* n.

The next lemma follows immediately from Lemma 4.7.

Lemma 4.8. *Let* $S \subset \mathbb{R}^n$ *be a finite set. Given any* $f \in C^2_+(E)$, *there exists* $\vec{P} \in W^2_+(S)$ *such that* $\|\vec{P}\|_{W^2_+(S)} \le C\|f\|_{C^2_+(S)}$ *and* $P^x(x) = f(x)$ *for each* $x \in S$. *The constant* C *depends only on* n.

Lemma 4.9 *(Whitney extension theorem for finite sets). Let* $S \subset \mathbb{R}^n$ *be a finite set. There exist a constant* C *depending only on* n, *and a map* $\mathcal{W}^S : W^2_+(S) \to C^2_+(\mathbb{R}^n)$ *such that the following hold.*

(A) $\|\mathcal{W}^S(\vec{P})\|_{C^2(\mathbb{R}^n)} \le C\|\vec{P}\|_{W^2_+(S)}$.
(B) $\mathcal{J}_x\mathcal{W}^S(\vec{P}) = P^x$ *for each* $x \in S$.

Sketch of proof. We begin by assuming $S = \{y\}$. We write $*$ instead of $\{*\}$ in certain places to avoid cumbersome notation.

Let $\vec{P} = P \in W^2_+(y)$.

Suppose $P(y) = 0$. Since $P \in W^2(y)$, we must have $\nabla P \equiv 0$. Therefore, we simply set

$$\mathcal{W}^y(P) \equiv 0 .$$

Conclusions (A) and (B) are satisfied.

Suppose $P(y) > 0$. By definition,

$$P \in \mathcal{C}_+(y, M), \text{ where } M := \max\left\{ \left(\sum_{|\alpha|\le 1} |\partial^\alpha P(y)|^2 \right)^{1/2}, \frac{|\nabla P|^2}{4P(y)} \right\} .$$

Thus, $\tilde{P}(x) := P(x) + M|x-y|^2 \ge 0$ for all $x \in \mathbb{R}^n$.

Let χ be a cutoff function that satisfies $\chi \equiv 1$ near y, supp $(\chi) \subset B(y,1)$, and $|\partial^\alpha \chi| \le C$ for $|\alpha| \le 2$. Define

$$\mathcal{W}^y(P) := \chi \cdot \tilde{P}. \tag{4.18}$$

It is clear that $\mathcal{W}^y(P) \ge 0$ and $\mathcal{J}_y\mathcal{W}^y(P) = \mathcal{J}_y\tilde{P} = P$. Moreover, for $x \in B(y,1)$ and $|\alpha| \le 2$

$$|\partial^\alpha \mathcal{W}^y(P)| \le CM.$$

Therefore,

$$\|\mathcal{W}^y(P)\|_{C^2(\mathbb{R}^n)} \le CM \le C'\|P\|_{W^2_+(S)}. \tag{4.19}$$

Next, we sketch the proof of the lemma for general S.

Let WC be a Whitney cover of $\mathbb{R}^n$ associated with the set S, and let $\{\theta_Q\}$ be a partition of unity compatible with WC. See [12].

In particular, WC and $\{\theta_Q\}$ satisfy the following properties.

- $\mathbb{R}^n = \bigcup_{Q\in WC} Q$;
- $Q \in WC$ if and only if Q satisfies one of the following:
  - $\delta_Q = 1$ and $S \cap Q^* \le 1$ (recall that $Q^* = 2Q$);
  - $\delta_Q < 1$, $S \cap Q^* \le 1$, and $S \cap (Q^+)^* > 1$ (recall that Q^+ is the dyadic parent of Q).
- If $Q, Q' \in WC$ and $Q \leftrightarrow Q'$ (i.e. the closures of Q and Q' have nonempty intersection), then $C^{-1}\delta_Q \le \delta_{Q'} \le C\delta_Q$.
- $\sum_{Q\in WC} \theta_Q \equiv 1$,
- $\operatorname{supp}(\theta_Q) \in Q^*$ for each $Q \in WC$, and
- $|\partial^\alpha \theta_Q| \le C\delta_Q^{-|\alpha|}$ for $|\alpha| \le 2$ and $Q \in WC$.

For each $Q \in WC$, we consider three different cases.

Case 1 When $S \cap Q^* \ne \varnothing$, we set $\mathcal{W}^Q := \mathcal{W}^y$ where $y \in S \cap Q^*$ and $\mathcal{W}^y$ is defined in (4.18). We set $P^Q := P^y$.

Case 2 When $S \cap Q^* = \varnothing$ and $\delta_Q < 1$, we may pick $y \in S \cap (Q^+)^*$. We set $\mathcal{W}^Q := \mathcal{W}^y$ and set $P^Q := P^y$.

Case 3 When $S \cap Q^* = \varnothing$ and $\delta_Q = 1$, we set $\mathcal{W}^Q \equiv 0$ and $P^Q :\equiv 0$.

Finally, we set

$$\mathcal{W}^S(\vec{P}) := \sum_{Q\in WC} \theta_Q \cdot \mathcal{W}^Q(P^Q).$$

One then verifies that $\mathcal{W}^S(\vec{P}) \ge 0$ and $\|\mathcal{W}^S(\vec{P})\|_{C^2(\mathbb{R}^n)} \le C\|\vec{P}\|_{W^2_+(S)}$ via Lemma 4.6 and a routine argument from the classical Whitney extension theorem. See [25] for details. □

5. Calderón-Zygmund squares

5.1. Calderón-Zygmund decomposition of $\mathbb{R}^2$

Definition 5.1. Let $C_{\text{nice}} > 0$ and $k \ge 1$. Recall the notation $Q^* = 2Q$. We say a dyadic square Q is <u>k-nice</u> if for all $x \in E \cap Q^*$,

$$\operatorname{diam}\left(\sigma^\sharp(x,k)\right) \geq C_{\text{nice}}\delta_Q. \tag{5.1}$$

We now describe our decomposition procedure.

CZ Algorithm. Let Q be a square.

- If Q is k-nice, then return $\Lambda_Q^{(k)} = \{Q\}$;
- otherwise, return

$$\Lambda_Q^{(k)} := \bigcup\left\{\Lambda_{Q'}^{(k)} : Q' \text{ dyadic and } (Q')^+ = Q\right\}.$$

Remark 5.1. The algorithm terminates after finitely many steps for each unit square. To see this, notice that E is finite, and for fixed k and C_{nice}, (5.1) clearly holds for sufficiently small squares containing no more than one point. Moreover, since $\sigma^\sharp$ does not depend on f, the complexity of our algorithm depends solely on the set E.

Definition 5.2. For a particular choice of $C_{\text{nice}} > 0$ and $k \geq 1$, we use $\Lambda_{\text{nice}}^{(k)} = \{Q_i\}$ to denote the collection of k-nice squares obtained from applying the algorithm above to each of the unit squares with their vertices on the integer lattice.

Lemma 5.1. $\Lambda_{\text{nice}}^{(k)}$ *is a CZ covering of* $\mathbb{R}^2$.

Proof. Since we obtain $\Lambda_{\text{nice}}^{(k)}$ by applying the algorithm to each square of the unit grid, $\Lambda_{\text{nice}}^{(k)}$ is indeed a covering of $\mathbb{R}^2$.

Suppose (3.3) fails, i.e., there exist some $Q, Q' \in \Lambda_{\text{nice}}^{(k)}$ with $Q \leftrightarrow Q'$ but

$$\delta_Q \leq \frac{1}{8}\delta_{Q'}.$$

Then $(Q^+)^* \subset (Q')^*$. Since Q^+ is not k-nice, there exists $\widehat{x} \in E \cap (Q^+)^* \setminus Q^*$ such that

$$\operatorname{diam}\left(\sigma^\sharp(\widehat{x},k)\right) < 2C_{\text{nice}}\delta_Q.$$

On the other hand,

$$C_{\text{nice}}\delta_{Q'} \leq \operatorname{diam}\left(\sigma^\sharp(\widehat{x},k)\right).$$

A contradiction is reached once we combine all the inequalities above, because Q' is k-nice. □

Our main goal is to construct a local interpolant for each k-nice square and then to patch these local solutions together. We need several lemmas that guarantee the consistency of our operation.

The following lemma states that polynomials in $\Gamma_+^\sharp$ with the same base point x control each other in the Whitney sense after our decomposition.

Lemma 5.2. *Let* $C_{\text{nice}}, k \geq 1$*,* $Q \in \Lambda^{(k)}_{\text{nice}}$*,* $x \in E \cap Q^*$*, and* $0 \leq |\alpha| \leq 1$*. If* $P, P' \in \Gamma^\sharp_+(x, k, M)$*, then*

$$|\partial^\alpha (P - P')(x)| \leq 14 C_{\text{nice}} M \delta_Q^{2-|\alpha|}. \tag{5.2}$$

Proof. Note that (5.2) is immediate if $\delta_Q = 1$ or $\alpha = (0,0)$. Therefore, we only need to consider the case when $\delta_Q < 1$ and $|\alpha| = 1$. The assumption $\delta_Q < 1$ implies that there exists $y \in E \cap (Q^+)^*$ such that $\operatorname{diam}\left(\sigma^\sharp(y,k)\right) < 2C_{\text{nice}}\delta_Q$. Fix such y.

Suppose toward a contradiction, that we can find a point $x \in E \cap Q^*$ and $P, P' \in \Gamma^\sharp_+(x, k, M)$ such that (5.2) is false for some $|\alpha| = 1$. Fix such α.

By Lemma 4.1, $P - P' \in 2M \cdot \sigma^\sharp(x,k)$. By definition, for any $S \subset E$ with $\#(S) \leq k$, there exists $F^S \in C^2(\mathbb{R}^2)$ such that

- $F^S\big|_S = 0$,
- $\|F^S\|_{C^2(\mathbb{R}^2)} \leq 2M$, and
- $\partial^\alpha(\mathcal{J}_x F^S) = \partial^\alpha(P - P')$.

By assumption, $|\partial^\alpha F^S(x)| > 14 C_{\text{nice}} M \delta_Q$. Since, $x, y \in (Q^+)^*$, we have $|x - y| < 6\delta_Q$. Therefore,

$$|\partial^\alpha \mathcal{J}_y F^S(y)| = |\partial^\alpha F^S(y)| \geq |\partial^\alpha F^S(x)| - \|F^S\|_{C^2(\mathbb{R}^2)}|x-y| \geq 2C_{\text{nice}} M \delta_Q.$$

Since S is arbitrary, we have $\operatorname{diam}\left(\sigma^\sharp(y,k)\right) \geq 2C_{\text{nice}}\delta_Q$. A contradiction. □

Lemma 5.3. *Let* $C_{\text{nice}}, k \geq 1$*. There exists a universal constant* C *such that the following holds. Let* $Q, Q' \in \Lambda^{(k)}_{\text{nice}}$*. Let* $x_Q \in Q$ *and* $x_{Q'} \in Q'$*. Let* $M \geq 0$*. Let* $P_Q \in \Gamma^\sharp_+(x_Q, 4k, M)$ *and* $P_{Q'} \in \Gamma^\sharp_+(x_{Q'}, 4k, M)$*. Then for* $|\alpha| \leq 1$ *and* $x \in 100Q \cup 100Q'$*,*

$$|\partial^\alpha(P_Q - P_{Q'})(x)| \leq CM \cdot \max\{|x_Q - x_{Q'}|, \delta_Q, \delta_{Q'}\}^{2-|\alpha|}. \tag{5.3}$$

Proof. Set

$$\delta_\infty := \max\{|x_Q - x_{Q'}|, \delta_Q, \delta_{Q'}\}.$$

By (3.3), we have

$$|x_Q - x|, |x_{Q'} - x|, |x_Q - x_{Q'}| \leq C\delta_\infty \text{ for } x \in 100Q \cup 100Q'.$$

By Lemma 4.2, there exists a $P_{\text{temp}} \in \Gamma^\sharp_+(x_{Q'}, k, M)$ with

$$|\partial^\alpha(P_Q - P_{\text{temp}})(x_{Q'})| \leq CM|x_Q - x_{Q'}|^{2-|\alpha|} \leq CM\delta_\infty^{2-|\alpha|}. \tag{5.4}$$

Since $P_{Q'} \in \Gamma^\sharp_+(x_{Q'}, 4k, M) \subset \Gamma^\sharp_+(x_{Q'}, k, M)$, Lemma 5.2 applied to $P_{Q'}$ and P_{temp} gives

$$|\partial^\alpha(P_{Q'} - P_{\text{temp}})(x_{Q'})| \le CM\delta_{Q'}^{2-|\alpha|} \le CM\delta_\infty^{2-|\alpha|} \text{ for } |\alpha| \le 1. \tag{5.5}$$

Combining (5.4) and (5.5), we have

$$|\partial^\alpha(P_Q - P_{Q'})(x_{Q'})| \le CM\delta_\infty^{2-|\alpha|} \text{ for } |\alpha| \le 1. \tag{5.6}$$

Since P_Q and $P_{Q'}$ are affine polynomials, (5.3) follows from (5.6) in the case $|\alpha| = 1$. By the fundamental theorem of calculus, we have

$$(P_Q - P_{Q'})(x) = (P_Q - P_{Q'})(x_{Q'}) + \int_{\text{seg}(x_{Q'} \to x)} \nabla(P_Q - P_{Q'}), \tag{5.7}$$

where $\text{seg}(x_{Q'} \to x)$ is the straight line segment from $x_{Q'}$ to x. Note that $\nabla(P_Q - P_{Q'})$ is a constant vector since both P_Q and $P_{Q'}$ are affine. Taking the absolute value of (5.7) and applying (5.6) with $|\alpha| = 1$, we conclude that (5.3) holds for $|\alpha| = 0$. □

5.2. Local geometry

The goal of this section is to show that according to our decomposition, we have partitioned the data points into clusters whose geometry is essentially one-dimensional. To proceed, we introduce some notations.

Note that the C^2 norm we are using in (3.1) is rotationally invariant. Let $\omega \in [-\pi/2, \pi/2]$. We associate with ω a coordinate system obtained by rotating the plane counterclockwise about the origin by an angle of ω. Thus, for $x \in \mathbb{R}^2$,

$$x = (s, t)_{\text{standard}} = (x_\omega^{(1)}, x_\omega^{(2)})_\omega,$$

where $x_\omega^{(1)} = s\cos\omega + t\sin\omega$ and $x_\omega^{(2)} = -s\sin\omega + t\cos\omega$. When the choice of ω is clear, we write ∂_1, ∂_2 to denote the partial derivatives with respect to the first, second variable, respectively. They coincide with the directional derivatives along ω and $\omega^\perp$, if we also treat ω as a unit vector.

If $\phi : I \to \mathbb{R}$ is a function defined on $I \subset \mathbb{R}$, we denote by $\text{Graph}(\phi; I, \omega)$ the graph of ϕ over I (with respect to the standard coordinate system) rotated by the angle ω.

Lemma 5.4. *Let $k \ge 4$ and let C_{nice} be sufficiently large. Suppose $Q \in \Lambda_{\text{nice}}^{(k)}$. Then there exist $\omega \in [-\pi/2, \pi/2]$ and a twice continuously differentiable function $\phi : \mathbb{R} \to \mathbb{R}$ such that*

- $E \cap Q^* \subset \text{Graph}(\phi; \mathbb{R}, \omega)$*;*
- $|\phi'| \le 1$*, and*
- $|\phi''| \le \delta_Q^{-1}$.

The constant C depends only on C_{nice}.

Proof. If $E \cap Q^* = \varnothing$, there is nothing to prove. From now on, we assume $E \cap Q^* \neq \varnothing$.

Fix $x_0 \in E \cap Q^*$. Let $\delta = \delta_Q$. Since $Q \in \Lambda^{(k)}_{\text{nice}}$, we have $\text{diam}\left(\sigma^\sharp(x_0, k)\right) \geq C_{\text{nice}}\delta$. Since $\sigma^\sharp$ is symmetric about the origin, there exist $P^{x_0} \in \sigma^\sharp(x_0, k)$ and $\omega = \frac{I_{x_0}(P^{x_0})}{\|I_{x_0}(P^{x_0})\|}$ (where I_{x_0} is the identification map in (3.2) and $\|\cdot\|$ is the Euclidean norm) such that

$$|\partial_2 P^{x_0}(x_0)| \geq C_{\text{nice}}\delta/2 \tag{5.8}$$

and

$$\partial_1 P^{x_0}(x_0) = 0\,. \tag{5.9}$$

Here, $\partial_i = \partial_{x_\omega^{(i)}}$ for $i = 1, 2$.

Claim 5.1. *Given any $\epsilon_0 > 0$, we may pick $C_{\text{nice}} > 0$ large enough such that the following holds.*

For any $S \subset E \cap Q^*$ containing x_0 with $\#(S) \leq k$, there exists $\phi^S \in C^2(\mathbb{R})$ such that

(i) $S \subset \text{Graph}\left(\phi^S; I^S, \omega\right)$,
(ii) $|\left(\phi^S\right)'| \leq \epsilon_0$ *on* I^S, *and*
(iii) $|\left(\phi^S\right)''| \leq \epsilon_0\delta_Q^{-1}$ *on* I^S.

Proof of Claim 5.1. Let $S \subset E \cap Q^*$ be such that $x_0 \in S$ and $\#(S) \leq k$.

Since $P^{x_0} \in \sigma^\sharp(x_0, k)$, there exists $F^S \in C^2(\mathbb{R}^2)$ such that

(i) $F^S\big|_S = 0$,
(ii) $\|F^S\|_{C^2(\mathbb{R}^2)} \leq 1$, and
(iii) $\mathcal{J}_{x_0} F^S = P^{x_0}$.

By (5.8), we have

$$|\partial_2 F^S(x_0)| = |\partial_2 P^{x_0}| \geq C_{\text{nice}}\delta_Q/2. \tag{5.10}$$

Now, for all $x \in Q^*$, we have $|x_0 - x| \leq 3\delta_Q$. Hence, for all $x \in Q^*$, by (5.9) and property (ii) of F^S, we have

$$|\partial_1 F^S(x)| \leq \|F^S\|_{C^2(\mathbb{R}^2)}|x_0 - x| \leq 3\delta_Q. \tag{5.11}$$

From (5.10), we also have, for all $x \in Q^*$,

$$|\partial_2 F^S(x)| \geq |\partial_2 F^S(x_0)| - \|F^S\|_{C^2(\mathbb{R}^2)}|x_0 - x| \geq (C_{\text{nice}}/2 - 3)\delta_Q.$$

Therefore, if C_{nice} is sufficiently large, the implicit function theorem yields a function $\phi^S \in C^2(I^S)$ for some open interval I^S such that $S \subset \text{Graph}\left(\phi^S; I^S, \omega\right)$.

First we compute the derivatives of ϕ^S:

$$\left(\phi^S\right)'(x^{(1)}_\omega) = -\frac{\partial_1 F^S(x)}{\partial_2 F^S(x)} \tag{5.12}$$

$$\left(\phi^S\right)''(x^{(1)}_\omega) = \frac{-\left(\partial_2 F^S(x)\right)^2 \partial_1^2 F^S(x) + 2\partial_1 F^S(x)\partial_2 F^S(x)\partial_{12}^2 F^S(x) - \left(\partial_1 F^S(x)\right)^2 \partial_2^2 F^S(x)}{\left(\partial_2 F^S(x)\right)^3}. \tag{5.13}$$

From (5.11) - (5.13), we conclude that, for sufficiently large C_{nice},

$$\left|(\phi^S)'\right| \leq \epsilon_0 \text{ on } I^S \quad \text{and} \quad \left|(\phi^S)''\right| \leq \epsilon_0 \delta_Q^{-1} \text{ on } I^S\,.$$

This concludes the proof of the claim. □

Next, we define the projections $\pi_i : \mathbb{R}^2 \to \mathbb{R}$ by $\pi_i((x^{(1)}_\omega, x^{(2)}_\omega)) = x^{(i)}_\omega$, for $i = 1, 2$. By Claim 5.1, we know that $\pi_1|_{E\cap Q^*}$ is a one-to-one map. Therefore, $E \cap Q^*$ lies on a graph with respect to the $x^{(1)}_\omega$-axis.

It remains to see that the graph can be taken to have controlled derivatives.

For simplicity of notation, we suppress ω in the subscript.

Let $x_0 = (x^{(1)}_0, x^{(2)}_0)$. We may assume without loss of generality that $\pi_1(E \cap Q^*) = \left\{x^{(1)}_0, x^{(1)}_1, \ldots, x^{(1)}_{L-1}\right\}$ such that $x^{(1)}_0 < x^{(1)}_1 < \cdots < x^{(1)}_{L-1}$, where $L = \#(E \cap Q^*)$. Let $\pi_2(E \cap Q^*) = \left\{x^{(2)}_0, x^{(2)}_1, \ldots, x^{(2)}_{L-1}\right\}$, where $x^{(2)}_i = \pi_2 \circ \pi_1^{-1}(x^{(1)}_i)$ for $i = 1, \ldots, L-1$.

Let $E_j = \left\{x^{(1)}_j, x^{(1)}_{j+1}, x^{(1)}_{j+2}\right\}$ for $j = 1, \ldots, L-3$. Let $S_j = \pi_1^{-1}(E_j) \cup \{x_0\}$. By Claim 5.1, we know that there exist $\phi^{S_j} \in C^2(I_j)$ and a constant C, depending only on C_{nice}, such that

- $\phi^{S_j}\big|_{E_j} = \pi_2 \circ \pi_1^{-1}$,
- $|(\phi^{S_j})'(x^{(1)})| \leq \epsilon_0$ for all $x^{(1)} \in [x^{(1)}_j, x^{(1)}_{j+2}]$, and
- $|(\phi^{S_j})''(x^{(1)})| \leq \epsilon_0 \delta_Q^{-1}$ for all $x^{(1)} \in [x^{(1)}_j, x^{(1)}_{j+2}]$.

Therefore, by Theorem 1.B and the fact that $\delta_Q \leq 1$, we may choose ϵ_0 sufficiently small such that there exists $\phi \in C^2(\mathbb{R})$ such that

- $\phi|_{E\cap Q^*} = \pi_2 \circ \pi_1^{-1}$,
- $\|\phi'\|_{C^0(\mathbb{R})} \leq 1$, and
- $\|\phi''\|_{C^0(\mathbb{R})} \leq \delta_Q^{-1}$.

This completes the proof of the lemma. □

For future reference, we make the following definition.

Definition 5.3. A pair (k, C_{nice}) **guarantees good geometry** if the following hold:

- $k \geq 4$; and
- C_{nice} is sufficiently large such that Lemma 5.4 holds.

Lemma 5.5. *Let (k, C_{nice}) guarantee good geometry. Let $Q \in \Lambda^{(k)}_{\text{nice}}$. There exist a universal constant C and a diffeomorphism $\Phi = \Phi_Q \in C^2(\mathbb{R}^2, \mathbb{R}^2)$, such that the following hold.*

(A) $\Phi(E \cap Q^*) \subset \mathbb{R} \times \{0\}$;
(B) $\|\nabla\Phi\|, \|\nabla\Phi^{-1}\| \leq 2$; *and*
(C) $\|\nabla^2\Phi\|, \|\nabla^2\Phi^{-1}\| \leq C\delta_Q^{-1}$.

Here, $\|\cdot\|$ denotes the Euclidean norm.

Proof. We may compose on the right by a rotation ω if necessary, and assume $\omega = 0$. Such rotation will not affect the Euclidean norm. Let ϕ be as in Lemma 5.4. Put

$$\Phi(s,t) := (s, t - \phi(s)) \text{ and } \Phi^{-1}(\widehat{s}, \widehat{t}) := (\widehat{s}, \widehat{t} + \phi(\widehat{s})). \tag{5.14}$$

They are clearly inverses of each other and are twice continuously differentiable.

Property (A) follows from how we construct ϕ (see Lemma 5.4).

To see (B), we note that

$$\nabla\Phi(s,t) = \begin{pmatrix} 1 & 0 \\ -\phi'(s) & 1 \end{pmatrix} \text{ and } \nabla\Phi^{-1}(\widehat{s}, \widehat{t}) = \begin{pmatrix} 1 & 0 \\ \phi'(\widehat{s}) & 1 \end{pmatrix}. \tag{5.15}$$

Property (B) then follows from (5.15) and the first derivative estimate of ϕ in Lemma 5.4.

Further differentiating each matrix in (5.15), we see that the only nonzero terms occur when ∂_s is applied to the bottom left entries and yields $\mp\phi''$. Conclusion (C) then follows from the second derivative estimate of ϕ in Lemma 5.4. □

Lemma 5.6. *Let (k, C_{nice}) guarantee good geometry. There exists a universal constant c_{rep} such that the following holds. Let $Q \in \Lambda^{(k)}_{\text{nice}}$. Then there exists $x^\sharp_Q \in Q$ with $\operatorname{dist}\left(x^\sharp_Q, E\right) \geq c_{\text{rep}}\delta_Q$.*

Proof. If $E \cap \frac{1}{2}Q = \varnothing$, we may pick $x^\sharp_Q$ to be the center of Q and let $c_{\text{rep}} = 1/4$.

Suppose $E \cap \frac{1}{2}Q \neq \varnothing$. Fix $\widehat{x} \in E \cap \frac{1}{2}Q$. There exists a universal constant $c_1 > 0$ such that $B(\widehat{x}, c_1\delta_Q) \subset Q$, where $B(\widehat{x}, c_1\delta_Q)$ is the ball of radius $c_1\delta_Q$ centered at $\widehat{x}$. Let Φ be as in Lemma 5.5. (Again, we may assume $\omega = 0$.) By (B) of Lemma 5.5, there exists a constant $c_2 > 0$, depending only on C_{nice}, such that $B(\Phi(\widehat{x}), c_2\delta_Q) \subset \Phi(B(\widehat{x}, c_1\delta_Q))$. Recall that $\Phi(E \cap Q^*) \subset \mathbb{R} \times \{0\}$. Let $\overline{x}^\sharp_Q := \Phi(\widehat{x}) + (0, c_2\delta_Q/2)$. Then $\operatorname{dist}\left(\overline{x}^\sharp_Q, \Phi(E \cap Q^*)\right) \geq$

$c_2\delta_Q/2$. Let $x_Q^\sharp = \Phi^{-1}(\overline{x}_Q^\sharp)$. By (B) of Lemma 5.5 again, $\operatorname{dist}\left(x_Q^\sharp, E \cap Q^*\right) \geq c_3\delta_Q$ for some $c_3 > 0$ depending only on C_{nice}. Finally, since $x_Q^\sharp \in Q$, $\operatorname{dist}\left(x_Q^\sharp, E \setminus Q^*\right) \geq \delta_Q/2$. This concludes the proof of the lemma. □

Recall Definition 4.2.

Lemma 5.7. *Let (k, C_{nice}) guarantee good geometry. Let $Q \in \Lambda_{\text{nice}}^{(k)}$. Let $x_Q^\sharp$ be as in Lemma 5.6. Then*

$$\sigma^\sharp(x_Q^\sharp, 4k) \subset C \cdot \mathcal{B}(x_Q^\sharp, \delta_Q)$$

for some universal constant C.

Proof. If $\delta_Q = 1$, then the lemma follows from the definitions of $\sigma^\sharp$ and $\mathcal{B}$.

Suppose $\delta_Q < 1$. Then Q^+ exists and is not k-nice, meaning that there exists $\widehat{x} \in E \cap (Q^+)^*$ such that

$$\operatorname{diam}\left(\sigma^\sharp(\widehat{x}, k)\right) < 2C_{\text{nice}}\delta_Q \,. \tag{5.16}$$

Fix such $\widehat{x}$. By our choice of $x_Q^\sharp$ (see Lemma 5.6), we have that

$$|\widehat{x} - x_Q^\sharp| \leq C\delta_Q \,.$$

Let $P \in \sigma^\sharp(x_Q^\sharp, 4k)$. The argument for the proof of Lemma 4.2 applied to $\sigma^\sharp$ yields $P' \in \sigma^\sharp(\widehat{x}, k)$ such that

$$P - P' \in C \cdot \left(\mathcal{B}(x_Q^\sharp, \delta_Q) \cap \mathcal{B}(\widehat{x}, \delta_Q)\right) \,. \tag{5.17}$$

Moreover, since $P' \in \sigma^\sharp(\widehat{x}, k)$, by the definition of $\sigma^\sharp$, we have $P'(\widehat{x}) = 0$. Thanks to (5.16), we also have $|\nabla P'| \leq C\delta_Q$. Therefore, we can conclude that

$$P' \in C \cdot \mathcal{B}(\widehat{x}, \delta_Q). \tag{5.18}$$

Taylor's theorem, together with (5.17) and (5.18), implies that $P \in C \cdot \mathcal{B}(x_Q^\sharp, \delta_Q)$. Since P is an arbitrary element in $\sigma^\sharp(x_Q^\sharp, 4k)$, the lemma follows. □

6. 1-D results

In this section, we provide the proofs for our one-dimensional results. First, we will prove Theorem 1.B and indicate how the proof of Theorem 1.A follows. Then, we will sketch a proof for Theorem 2.A. The proof for Theorem 2.B uses the same idea but with easier intermediate steps.

We will use x, y to denote points on $\mathbb{R}$, and ∂^m to denote the m-th derivative of a single-variable function. When $m = 1$, we simply write ∂ instead of ∂^1. We use $\mathcal{P}$ to denote the vector space of one-variable polynomials with degree no greater than one.

6.1. Finiteness principles for $C^2(\mathbb{R})$ *and* $C^2_+(\mathbb{R})$

Proof of Theorem 1.B. For $N \geq 3$, let $I_1 = (-\infty, x_3]$, $I_2 = [x_2, x_4]$, $\ldots$, $I_{N-3} = [x_{N-3}, x_{N-1}]$, and $I_{N-2} = [x_{N-2}, +\infty)$. By assumption, for each j, there exists $F_j \in C^2_+(\mathbb{R})$ with $F_j\big|_{E_j} = f$ and

$$|F_j| \leq A_0\,, |\partial F_j| \leq A_1\,, |\partial^2 F_j| \leq A_2\,. \tag{6.1}$$

We introduce a partition of unity $\{\theta_j\}$ that satisfies

(i) $\sum_{j=1}^{n-2} \theta_j \equiv 1$ on $\mathbb{R}$;
(ii) $\operatorname{supp}(\theta_j) \subset I_j$ for each $j = 1, \ldots, N-2$; and
(iii) [3] for each $1 \leq k \leq 2$ and $1 \leq j \leq N-2$,

$$|\partial^k \theta_j(x)| \leq \begin{cases} C|x_{j+1} - x_j|^{-k} & \text{if } x \in [x_j, x_{j+1}] \\ C|x_{j+2} - x_{j+1}|^{-k} & \text{if } x \in [x_{j+1}, x_{j+2}] \end{cases}. \tag{6.2}$$

Notice that the interior of $I_i \cap I_j$ supports at most two partition functions (θ_i and θ_j).

Define

$$F(x) = \sum_{j=1}^{N-2} \theta_j(x) F_j(x). \tag{6.3}$$

Clearly, $F|_E = f$, F is twice continuously differentiable, and

$$|F| \leq 2A_0. \tag{6.4}$$

Observe that (6.1) and condition (ii) of $\{\theta_j\}$ imply

$$|\partial^m F| \leq A_m \text{ on } (-\infty, x_2] \cup [x_{N-1}, +\infty)\,. \tag{6.5}$$

Suppose $x \in (x_2, x_{N-1})$. Let j be the least integer such that $x \in I_j$. The only partition functions possibly nonzero at x are θ_j and θ_{j+1}. Since $\theta_j(x) + \theta_{j+1}(x) \equiv 1$, we have $\partial^k \theta_j(x) = -\partial^k \theta_{j+1}(x)$ for $k = 1, 2$. Thus,

$$\partial^k F(x) = \partial^k F_j(x)\theta_j(x) + \partial^k F_{j+1}(x)\theta_{j+1}(x) + \sum_{l=0}^{k-1} \binom{k}{l} \partial^l (F_j - F_{j+1})(x) \partial^{k-l} \theta_j(x). \tag{6.6}$$

[3] For the existence of such partition function, see e.g. [15].

Claim 6.1. *Let* $x \in I_j \cap I_{j+1}$. *Then*

$$|(F_j - F_{j+1})(x)| \le 2A_1 |x_{j+1} - x_j| . \tag{6.7}$$

For $l = 0, 1$, *we also have*

$$|\partial^l (F_j - F_{j+1})(x)| \le 2A_2 |x_{j+1} - x_j|^{2-l} . \tag{6.8}$$

Proof of Claim 6.1. Note that by construction, $I_j \cap I_{j+1} = [x_{j+1}, x_{j+2}]$.

Observe that (6.7) is an immediate consequence of the mean value theorem.

It remains to show (6.8).

Observe that $(F_j - F_{j+1})(x_{j+1}) = (F_j - F_{j+1})(x_{j+2}) = 0$. By Rolle's theorem, there exists $\widehat{x}_j \in (x_j, x_{j+1})$ such that $\partial(F_j - F_{j+1})(\widehat{x}_j) = 0$. By the fundamental theorem of calculus and triangle inequality, we have

$$|\partial(F_j - F_{j+1})(x)| \le \int_{\widehat{x}_j}^{x} |\partial^2 (F_j - F_{j+1})(y)| dy \le 2A_2 |x_{j+2} - x_{j+1}| \text{ for all } x \in I_j \cap I_{j+1}.$$

Similar calculations yield the case $l = 0$. (6.8) is proven. □

Now, (6.2), (6.5), (6.6), and (6.7) imply that

$$|\partial F| \le CA_1. \tag{6.9}$$

Likewise, (6.2), (6.5), (6.6), and (6.8) imply that

$$|\partial^2 F| \le CA_2. \tag{6.10}$$

In view of (6.4), (6.9), and (6.10), we conclude the proof of the theorem. □

Proof of Theorem 1.A. We simply take $A_m = 1$ for $m = 0, 1, 2$ in the above proof of Theorem 1.B, and note that $F(x)$ defined by (6.3) is nonnegative if all of the F_j's are nonnegative. □

6.2. $C^2(\mathbb{R})$ *and* $C^2_+(\mathbb{R})$ *extension operators of bounded depth*

Now we explain the proof of Theorem 2.A.

Let $E \subset \mathbb{R}$ be a finite set. We enumerate $E = \{x_1, \cdots, x_N\}$ with $x_1 < \cdots < x_N$. Let $E_i := \{x_i, x_{i+1}, x_{i+2}\}$ for $i = 1, \cdots, N-2$. Suppose for each i, we are given an extension operator $\mathcal{E}_i : C^2_+(E_i) \to C^2_+(\mathbb{R})$ with $\|\mathcal{E}_i(f)\|_{C^2(\mathbb{R})} \le C\|f\|_{C^2_+(E_i)}$ and $(\mathcal{E}_i(f))\big|_{E_i} = f$. Let $\{I_i\}$ and $\{\theta_i\}$ be as in the proof of Theorem 1.A. We define

$$\mathcal{E}(f)(x) := \sum_{i=1}^{N-2} \theta_i(x) \cdot \mathcal{E}_i(f)(x)\,. \tag{6.11}$$

Conclusions (A) and (B) of Theorem 2.A follow from the same argument as in the proof of Theorem 1.A. Moreover, by assumption, $\mathcal{E}_i(f)$ depends only on $\{f(x_i), f(x_{i+1}), f(x_{i+2})\}$ for each i, and the θ_i's have bounded overlap. Therefore, conclusion (C) and Remark 2.4 follow.

Hence, in order to construct a bounded extension operator with bounded depth in dimension one, it suffices to construct a bounded extension operator for every consecutive three points. This is a routine linear algebra problem and is readily solvable via the nonnegative Whitney extension theorem (see Lemma 4.9). We leave the details to the interested readers.

For Theorem 2.B, we simply replace each summand on the right-hand side in (6.11) with $\theta_i \cdot \mathcal{E}_i$, where $\mathcal{E}_i$ is an extension operator associated with E_i without the nonnegative constraints.

6.3. Non-additivity

In this section, we use the following notations

$$\|f\|_{\dot{C}^m(E)} := \inf\left\{\|F\|_{\dot{C}^m(\mathbb{R})} : F \in C^m(\mathbb{R}) \quad \text{and} \quad F|_E = f\right\} \text{ and}$$
$$\|f\|_{\dot{C}^m_+(E)} := \inf\left\{\|F\|_{\dot{C}^m(\mathbb{R})} : F \in C^m_+(\mathbb{R}) \quad \text{and} \quad F|_E = f\right\}.$$

Proof of Theorem 3. Let $\epsilon > 0$ be a sufficiently small number. We use C, C', C_* etc. to denote universal constants.

Consider $E = \{x_1, x_2, x_3\} \subset \mathbb{R}$, where $x_j = (j-1)\epsilon$ for $j = 1, 2, 3$. Suppose toward a contradiction, that $\mathcal{E} : C^2_+(E) \to C^2_+(\mathbb{R})$ is a bounded extension map that is additive. That is, $\mathcal{E}(f+g) = \mathcal{E}(f) + \mathcal{E}(g)$ for all $f, g \in C^2_+(E)$, and

$$C^{-1}\|\mathcal{E}(f)\|_{C^2(\mathbb{R})} \le \|f\|_{C^2_+(E)} \le C\|\mathcal{E}(f)\|_{C^2(\mathbb{R})}\,.$$

For $j = 1, 2, 3$, we define

$$f(x_j) := (j-1)\epsilon \quad \text{and} \quad g(x_j) := 1 - f(x_j)\,.$$

Then $f, g \in C^2_+(E)$, and $f + g \equiv 1$. It is easy to see that

$$\|f+g\|_{C^2_+(E)} = 1\,.$$

In fact,

$$\|f+g\|_{\dot{C}^m(E)} = 0 \text{ for } m = 1, 2\,.$$

Since $\mathcal{E}$ is bounded, we have

$$1 \leq \|\mathcal{E}(f+g)\|_{C^2(\mathbb{R})} \leq C\,. \tag{6.12}$$

We analyze the derivatives of $\mathcal{E}(f)$ and $\mathcal{E}(g)$.

We begin with $\mathcal{E}(f)$. By calculating the divided difference using $(x_1, f(x_1))$ and $(x_2, f(x_2))$, we see that

$$|\partial\mathcal{E}(f)(\tilde{x}_0)| \geq 1 \text{ for some } \tilde{x}_0 \in (0, \epsilon).$$

Since $\mathcal{E}$ preserves nonnegativity and $f(0) = 0$, $\mathcal{E}(f)$ must have a local minimum at 0. Therefore,

$$\partial\mathcal{E}(f)(0) = 0.$$

By calculating the divided difference, using $(0, \partial\mathcal{E}(f)(0))$ and $(\tilde{x}_0, \partial\mathcal{E}(f)(\tilde{x}_0))$, we see that

$$|\partial^2\mathcal{E}f(x_0)| \geq C_0\epsilon^{-1} \tag{6.13}$$

for some $x_0 \in (0, \epsilon)$. Fix such x_0.

Now we turn to $\mathcal{E}(g)$.

Let ψ be a cutoff function such that $\psi \equiv 1$ in a neighborhood of $[0, 2\epsilon]$, $\operatorname{supp}(\psi) \subset [-1, 1]$, and $|\partial^m\chi| \leq C$ for $m = 0, 1, 2$. Consider the function $\tilde{g}$ defined by

$$\tilde{g}(x) := \psi(x) \cdot (1 - x)$$

It is clear that $\tilde{g} \in C^2_+(\mathbb{R})$ with $\tilde{g}|_E = g$. Moreover,

$$\|\tilde{g}\|_{C^2(\mathbb{R})} \leq C\,.$$

Therefore,

$$\|g\|_{C^2_+(E)} \leq C\,.$$

Since $\mathcal{E}$ is bounded, we know that, for x_0 as in (6.13),

$$|\partial^2\mathcal{E}(g)(x_0)| \leq C_1\,. \tag{6.14}$$

Therefore, we have, with C_0 and C_1 as in (6.13) and in (6.14),

$$|\partial^2(\mathcal{E}f + \mathcal{E}g)(x_0)| \geq C_0\epsilon^{-1} - C_1 \geq C\epsilon^{-1}\,.$$

For sufficiently small ϵ, this would contradict (6.12). □

7. 2-D finiteness principle

7.1. Statement of the main local lemma

The goal of this section is to prove a local version of the finiteness principle, which produces a nonnegative local interpolant taking a jet in some prescribed $\Gamma_+^\sharp$ (see (4.3)) at a point sufficiently far away from the data. We will use these jets as transitions in our estimates.

Recall Definition 5.3. Also recall that Lemma 5.6 produces a point $x_Q^\sharp \in Q$ such that

$$\operatorname{dist}\left(x_Q^\sharp, E\right) \geq c_{\text{rep}}\delta_Q \tag{7.1}$$

for each $Q \in \Lambda_{\text{nice}}^{(k)}$ given that (k, C_{nice}) guarantees good geometry. We fix the number c_{rep}.

Lemma 7.1. *Let $E \subset \mathbb{R}^2$ be finite, and let $f : E \to [0, \infty)$. Let (k, C_{nice}) guarantee good geometry and $Q \in \Lambda_{\text{nice}}^{(k)}$. Let $x_Q^\sharp \in Q$ satisfy (7.1). Let $k_{\text{loc}} \geq 3$. Suppose $\Gamma_+^\sharp(x_Q^\sharp, k_{\text{loc}}, M) \neq \varnothing$. Then there exist a universal constant C and a function $F_Q^\sharp \in C_+^2(100Q)$ such that the following hold.*

(A) $F_Q\big|_{E\cap Q^*} = f$,
(B) $\|F_Q\|_{C^2(100Q)} \leq CM$, *and*
(C) $\mathcal{J}_{x_Q^\sharp} F_Q \in \Gamma_+^\sharp(x_Q^\sharp, k_{\text{loc}}, CM)$.

Note that if $\#(E \cap Q^*) \leq k_{\text{loc}}$, the conclusion follows immediately.

Hereafter, we assume $\#(E \cap Q^*) > k_{\text{loc}} \geq 3$.

The main idea of the proof is to treat the local interpolation problem differently depending on whether the local data is big or small. For big local data, we solve the problem as if there were no nonnegative constraints. For small local data, we simply prescribe a zero jet.

Below we give a more detailed overview of our strategy, still without dwelling into the technicalities.

Our approach relies on three crucial lemmas. The first one (Lemma 7.2) describes the relationships among the value, gradient, and zero set of a jet generated by a nonnegative function. The second one (Lemma 7.3) is a perturbation lemma, which specifies the conditions under which we are allowed to modify an element in $\Gamma_+^\sharp(x_Q^\sharp, \cdot, \cdot)$. We emphasize the importance of the choice of base point $x_Q^\sharp$, which is far away enough from all the data points (on the order of δ_Q) so that we have room to modify the interpolants' behavior near $x_Q^\sharp$. The third one (Lemma 7.5) tells us that the local data is either uniformly big or uniformly small (on the order of δ_Q^2).

We begin the proof of Lemma 7.1 by first tackling a one-dimensional interpolation problem. Recall that, thanks to Lemma 5.4, the data points locally lie on a curve. The interpolation problem along this curve is essentially one-dimensional and readily solved, thanks to Theorems 1.A, 1.B, and Lemma 5.5.

We then solve the local problem when the local data is uniformly large, namely, $\min_{x\in E\cap Q^*} f(x) \geq B\delta_Q^2$ for some universal $B > 0$ to be determined. We replace the local data $f|_{E\cap Q^*}$ by $g(x) = f(x) - P^\sharp(x)$ for $x \in E \cap Q^*$, where $P^\sharp$ is a suitable element in $\Gamma_+^\sharp(x_Q^\sharp, k_{loc}, C)$ such that g achieves two zeros and that $P^\sharp \geq B'\delta_Q^2$ on $100Q$ for some $B' > 0$ depending only on B. Thanks to Rolle's theorem, the resulting one-dimensional g-interpolant, although not necessarily nonnegative, will be uniformly small on the order of δ_Q^2, and in particular, bounded from below by $-c\delta_Q^2$. Now, we are in the suitable order of magnitude to force a zero jet at $x_Q^\sharp$. To do this, we simply extend the one-dimensional interpolant in the normal direction by constant, and use a bump function to damp out the function at $x_Q^\sharp$. If we choose B such that B' is bigger than c, we may add $P^\sharp$ back to the zero-jet interpolant while preserving nonnegativity of the sum on $100Q$, and solve the local problem.

Next, we solve the local problem when the data is not uniformly big. Thanks to Lemma 7.5, the local data has to be uniformly small, i.e., $\max_{x\in E\cap Q^*} f(x) \leq B''\delta_Q^2$ for some $B'' > 0$ depending only on B. Therefore, we are in the correct order of magnitude to force a zero jet as in the previous step. Thanks to the perturbation lemma (Lemma 7.3), the zero jet in this case is indeed a k_{loc}-point jet, and the problem is solved.

Sections 7.2 and 7.3 will be devoted to the proof of Lemma 7.1.

7.2. Key lemmas

In this section, we use Cartesian coordinates $x = (s,t)$ on $\mathbb{R}^2$. We also write $x_Q^\sharp = x_Q^\sharp = (s_Q^\sharp, t_Q^\sharp)$.

Lemma 7.2. *There exist universal constants* C, C', C'' *such that the following hold. Suppose* $P \in \Gamma_+(x, \varnothing, M)$*. Then*

$$P(y) + CM|y-x|^2 \geq 0 \text{ for all } y \in \mathbb{R}^2, \tag{7.2}$$

$$|\nabla P| \leq C'\sqrt{MP(x)} \text{ and} \tag{7.3}$$

$$\operatorname{dist}(x, \{P = 0\}) \geq C''M^{-1/2}\sqrt{P(x)}. \tag{7.4}$$

Proof. Inequality (7.2) is a direct consequence of Taylor's Theorem.

To see (7.3), we simply compute the discriminants of the left hand side of (7.2) restricted to the s and t-directions.

Now we prove (7.4). If $P(x) = 0$ or P is a constant polynomial, the inequality is obvious. Assume that $P(x) > 0$ and P is nonconstant.

Since P is an affine function and the gradient points toward the direction of maximal increase, we have

$$|\nabla P| = \frac{P(x)}{\text{dist}\,(x, \{P = 0\})}\,. \tag{7.5}$$

From (7.3) and (7.5), we have the desired estimate (7.4). □

Lemma 7.3. *Let $M > 0$. Let (k, C_{nice}) guarantee good geometry (see Definition 5.3), let $k' \geq 1$, and let $Q \in \Lambda^{(k)}_{\text{nice}}$. Let $x^\sharp_Q$ be as in Lemma 5.6. Suppose $E \cap Q^* \neq \varnothing$. Suppose $\Gamma^\sharp_+(x^\sharp_Q, k', M) \neq \varnothing$.*

(A) There exists a number $B > 0$ exceeding a large universal constant such that the following holds. Suppose $f(x) \geq BM\delta_Q^2$ for each $x \in E \cap Q^*$. Then

$$\Gamma^\sharp_+(x^\sharp_Q, k', M) + M \cdot \mathcal{B}(x^\sharp_Q, \delta_Q) \subset \Gamma^\sharp_+(x^\sharp_Q, k', CM).$$

(B) Let $A > 0$. Suppose $f(x) \leq AM\delta_Q^2$ for some $x \in E \cap Q^*$. Then

$$0 \in \Gamma^\sharp_+(x^\sharp_Q, k', A'M)\,.$$

The number A' depends only on A.

Proof. We prove (A) first.

Let $B > 0$ be sufficiently large.

Claim 7.1. *Under the hypothesis of (A). Given any $P \in \Gamma^\sharp_+(x^\sharp_Q, k', M)$, we have*

$$P(x^\sharp_Q) \geq B_0 M\delta_Q^2\,,$$

where we can take $B_0 = C(\sqrt{B} - 1/2)^2$.

Proof of Claim 7.1. We repeat proof of Claim 7.4 with more control on the parameters.

Let $x \in E \cap Q^*$. Since $P \in \Gamma^\sharp_+(x^\sharp_Q, k', M)$, by definition, there exists $F \in C^2_+(\mathbb{R}^2)$ with $\|F\|_{C^2(\mathbb{R}^2)} \leq M$, $\mathcal{J}_{x^\sharp_Q} F = P$, and

$$F(x) = f(x) \geq BM\delta_Q^2\,. \tag{7.6}$$

Suppose toward a contradiction, that $F(x^\sharp_Q) < B_0 M\delta_Q^2$. We see from (7.3) that $|\nabla F(x^\sharp_Q)| \leq C\sqrt{B_0}M\delta_Q$. By the fundamental theorem of calculus, we have

$$|\nabla F(x)| \leq |\nabla F(x^\sharp_Q)| + C\|F\|_{C^2(\mathbb{R}^2)}\delta_Q \leq C'\left(\sqrt{B_0} + \frac{1}{4}\right) M\delta_Q \text{ on } Q^*\,.$$

By the fundamental theorem of calculus again, we have

$$\begin{aligned} F(x) &\le F(x_Q^\sharp) + C\delta_Q \cdot \sup_{x\in Q^*} |\nabla F(x)| \\ &\le C'\left(B_0 + \sqrt{B_0} + \frac{1}{4}\right) M\delta_Q^2 \\ &= C'\left(\sqrt{B_0} + \frac{1}{2}\right)^2 M\delta_Q^2 \text{ on } Q^*. \end{aligned} \tag{7.7}$$

If we pick B_0 in (7.7) to be so small that $\sqrt{B_0} < \frac{\sqrt{B}-\frac{1}{2}}{C'}$, we will contradict (7.6). This proves the claim. □

Pick $P \in \Gamma_+^\sharp(x_Q^\sharp, k', M)$. By the claim, we know that $P(x_Q^\sharp) \ge B_0 M\delta_Q^2$.

Let $\tilde{P} \in M \cdot \mathcal{B}(x_Q^\sharp, \delta_Q)$. By definition, we have

$$|\partial^\alpha \tilde{P}(x_Q^\sharp)| \le M\delta_Q^{2-|\alpha|} \text{ for } |\alpha| \le 2.$$

Let $S \subset E$ with $\#(S) \le k'$. We want to show that there exists $F \in C_+^2(\mathbb{R}^2)$ with $\|F\|_{C^2(\mathbb{R}^2)} \le CM$, $F(x) = f(x)$ for each $x \in S$, and $\mathcal{J}_{x_Q^\sharp} F = P + \tilde{P}$.

We enumerate $S = \{x_1, \cdots, x_{k'}\}$. We let $\tilde{S} = \{x_0, x_1, \cdots, x_{k'}\}$ with $x_0 := x_Q^\sharp$.

By the definition of $\Gamma_+^\sharp$, there exists

$$F^S \in C_+^2(\mathbb{R}^2) \text{ with } \|F^S\|_{C^2(\mathbb{R}^2)} \le M,\ F^S\big|_S = f, \text{ and } \mathcal{J}_{x_Q^\sharp} F^S = P. \tag{7.8}$$

For $i = 1, \cdots, k'$, we set

$$P^{x_i} := \mathcal{J}_{x_i} F^S.$$

We also set

$$P^{x_0} := P + \tilde{P}. \tag{7.9}$$

We put

$$\vec{P} := (P^{x_i})_{i=0}^{k'} \in W^2(\tilde{S}).$$

Thanks to Lemma 4.7 and Lemma 4.9, it suffices to show that $\vec{P} \in W_+^2(\tilde{S})$ and $\|\vec{P}\|_{W_+^2(S)} \le CM$.

Thanks to (7.8), we have

$$P^{x_i} \in \mathcal{C}_+(x_i, CM) \text{ for all } i = 1, \cdots, k', \tag{7.10}$$

and

$$P^{x_i} - P^{x_j} \in CM \cdot \mathcal{B}(x_i, |x_i - x_j|) \text{ for all } i, j = 1, \cdots, k'. \tag{7.11}$$

On the other hand, thanks to Claim 7.1, we have

$$P^{x_0}(x_0) = P(x_0) + \tilde{P}(x_0) \geq (B_0 - 1)M\delta_Q^2 \geq 0\,,$$

and

$$|\nabla P^{x_0}| \leq |\nabla P| + |\nabla \tilde{P}| \leq C\sqrt{MP(x_Q^\sharp)} + M\delta_Q \leq C'\sqrt{M(P+\tilde{P})(x_Q^\sharp)}\,.$$

This, combined with (7.8), shows that

$$P^{x_0} \in \mathcal{C}_+(x_0, CM)\,. \tag{7.12}$$

It remains to estimate $\|\vec{P}\|_{W^2(\tilde{S})}$.

By Taylor's theorem, we have

$$P^{x_i} - P \in CM \cdot (\mathcal{B}(x_i, |x_i - x_0|) \cap CM \cdot \mathcal{B}(x_0, |x_i - x_0|))\ . \tag{7.13}$$

By Lemma 5.6, we have

$$\text{dist}\left(x_Q^\sharp, E\right) \geq C\delta_Q\,.$$

This, together with Taylor's theorem and the fact that $\tilde{P} \in M\mathcal{B}(x_0, \delta_Q)$, implies

$$\tilde{P} \in CM \cdot \mathcal{B}(x_i, |x_i - x_0|) \text{ for all } i = 1, \cdots, k'\,. \tag{7.14}$$

Therefore,

$$\begin{aligned} P^{x_i} - P^{x_0} &= P^{x_i} - P - \tilde{P} && \text{(by (7.9))} \\ &\in (-\tilde{P}) + CM \cdot (\mathcal{B}(x_i, \delta_Q) \cap \mathcal{B}(x_0, \delta_Q)) && \text{(by (7.13))} \\ &\subset C'M \cdot (\mathcal{B}(x_i, \delta_Q) \cap \mathcal{B}(x_0, \delta_Q)) && \text{(by (7.14))}\,. \end{aligned} \tag{7.15}$$

From (7.10)-(7.15), we can conclude that $\|\vec{P}\|_{W^2_+(S)} \leq CM$. This concludes the proof of (A).

Now we turn to the proof of (B).

Claim 7.2. *Assume the hypothesis of (B). Let* $P \in \Gamma_+^\sharp(x_Q^\sharp, k', M)$. *Then* $P(x_Q^\sharp) \leq C(\sqrt{A}+1)^2 M\delta_Q^2$.

Proof of Claim 7.2. Fix $\widehat{x} \in E \cap Q^*$ such that $f(\widehat{x}) \leq BM\delta_Q^2$. Since $k' \geq 1$, by the definition of $\Gamma_+^\sharp$, there exists a function $F \in C_+^2(\mathbb{R}^2)$ with $F(\widehat{x}) = f(\widehat{x}) \leq AM\delta_Q^2$, $\|F\|_{C^2(\mathbb{R}^2)} \leq M$, and $\mathcal{J}_{x_Q^\sharp} F = P$. By Lemma 7.2, we have

$$|\nabla F(\widehat{x})| = |\nabla \mathcal{J}_{\widehat{x}} F| \leq \sqrt{A} M\delta_Q\,.$$

By Taylor's theorem, we see that

$$|\nabla F(x)| \leq C(\sqrt{A}+1)M\delta_Q \quad \text{for} \quad x \in Q^*.$$

By the fundamental theorem of calculus, we see that

$$P(x_Q^\sharp) = F(x_Q^\sharp) \leq C(A+\sqrt{A}+1)M\delta_Q^2 \leq C'(\sqrt{A}+1)^2 M\delta_Q^2.$$

This finishes the proof of Claim 7.2. □

It remains to show that $0 \in \Gamma_+(x_Q^\sharp, S, A'M)$ for each $S \subset E$ with $\#(S) \leq k'$.

We use A_0, A_1, etc. to denotes quantities that depend only on A.

Fix $P \in \Gamma_+^\sharp(x_Q^\sharp, k', M)$. Let $S \subset E$ satisfy $\#(S) \leq k'$. By definition, there exists $F^S \in C_+^2(\mathbb{R}^2)$, such that $F^S\big|_S = f$, $\|F^S\|_{C^2(\mathbb{R}^2)} \leq 1$, and $\mathcal{J}_{x_Q^\sharp} F^S = P$. By Claim 7.2, we see that $P(x_Q^\sharp) \leq A_0 M\delta_Q^2$ and by (7.3) that $|\nabla P| \leq A_1 M\delta_Q$. In other words,

$$F^S(x_Q^\sharp) \leq A_0 M\delta_Q^2 \text{ and } |\nabla F^S(x_Q^\sharp)| \leq A_1 M\delta_Q.$$

The fundamental theorem of calculus then implies

$$|\nabla F^S(x)| \leq A_2\delta_Q \text{ and } F^S(x) \leq A_2\delta_Q^2 \text{ for all } x \in B(x_Q^\sharp, \frac{c_{\text{rep}}\delta_Q}{100}). \tag{7.16}$$

Let $\psi \in C_+^2(\mathbb{R}^2)$ be a cutoff function such that

$$0 \leq \psi \leq 1\,,\ \psi \equiv 1 \text{ near } x_Q^\sharp\,,\ \text{supp}\,(\psi) \subset B(x_Q^\sharp, \frac{c_{\text{rep}}\delta_Q}{100})\,,\ |\partial^\alpha \psi| \leq C\delta_Q^{-|\alpha|} \text{ for } |\alpha| \leq 2. \tag{7.17}$$

Let

$$\tilde{F}^S := (1-\psi)F^S.$$

We have the following.

- By (7.1) and the fact that $\text{supp}\,(\psi) \subset B(x_Q^\sharp, \frac{c_{\text{rep}}\delta_Q}{100})$, we have $\tilde{F}^S\big|_S = f$.
- By (7.17) and the assumption that $F^S \geq 0$, we have $\tilde{F}^S \geq 0$ on $\mathbb{R}^2$.
- Thanks to (7.16) and (7.17), $\|\tilde{F}^S\|_{C^2(\mathbb{R}^2)} \leq A_2 M$.
- Since $\psi \equiv 1$ near $x_Q^\sharp$, we have $\mathcal{J}_{x_Q^\sharp}\tilde{F}^S \equiv 0$.

Since S is arbitrary, we have $0 \in \Gamma_+^\sharp(x_Q^\sharp, k', A_2 M)$. This completes the proof of (B) and the proof of the lemma. □

Lemma 7.4. *There exists a universal constant* $B > 0$ *such that the following holds. Let* $M > 0$. *Let* (k, C_{nice}) *guarantee good geometry (see Definition 5.3). Let* $Q \in \Lambda_{\text{nice}}^{(k)}$. *Let*

$x^\sharp_Q$ *be as in Lemma 5.6. Suppose* $E \cap Q^* \neq \varnothing$ *and* $f(x) \geq BM\delta_Q^2$ *for all* $x \in E \cap Q^*$. *Let* $k' \geq 0$. *Then*

$$\Gamma_+^\sharp(x_Q^\sharp, k', M) + M \cdot \sigma^\sharp(x_Q^\sharp, 4k) \subset \Gamma_+^\sharp(x_Q^\sharp, k', CM).$$

Proof. This is a direct consequence of Lemma 5.7 and Lemma 7.3. □

Lemma 7.5. *For each* $B_{\min} > 0$*, we can find* $B_{\max}$*, depending only on* $B_{\min}$*, such that the following holds.*

Let $E \subset \mathbb{R}^2$ *be a finite set. Let* $f : E \to [0, \infty)$. *Let* $k' \geq 2$. *Suppose* $\Gamma_+^\sharp(x, k', M) \neq \varnothing$ *for all* $x \in \mathbb{R}^2$. *Let* (k, C_{nice}) *guarantee good geometry. Let* $Q \in \Lambda_{\text{nice}}^{(k)}$. *Then at least one of the following holds.*

(A) $f(x) \leq B_{\max} M\delta_Q^2$ *for all* $x \in E \cap Q^*$.
(B) $f(x) \geq B_{\min} M\delta_Q^2$ *for all* $x \in E \cap Q^*$.

Proof. Fix $B_{\min} > 0$. We use B, B', etc. to denote quantities that depend only on $B_{\min}$.

Without loss of generality, we may assume $M = 1$.

If $\min_{x \in E \cap Q^*} f(x) \geq B_{\min}\delta_Q^2$, there is nothing to prove.

Suppose there exists $x_0 \in E \cap Q^*$ such that $f(x_0) < B_{\min}\delta_Q^2$. Fix such x_0.

Let $S \subset E \cap Q^*$ satisfy $x_0 \in S$ and $\#(S) \leq k'$. Since $\Gamma_+^\sharp(x, k', 1) \neq \varnothing$ for each $x \in \mathbb{R}^2$, there exists $F^S \in C_+^2(\mathbb{R}^2)$ such that $F^S\big|_S = f$ and $\|F^S\|_{C^2(\mathbb{R}^2)} \leq 1$.

Since $F^S(x_0) < B_{\min}\delta_Q^2$, (7.3) implies there exists $B > 0$ such that

$$|\nabla F^S(x_0)| = |\nabla \mathcal{J}_{x_0} F^S| \leq B\delta_Q.$$

Therefore, since $\|F^S\|_{C^2(\mathbb{R}^2)} \leq 1$, we have $|\nabla F^S(x)| \leq B'\delta_Q$ for all $x \in Q^*$. By the fundamental theorem of calculus, since $F^S(x_0) < B_{\min}\delta_Q^2$, we must have $|F^S(x)| \leq B''\delta_Q^2$ for all $x \in Q^*$. In particular,

$$|F^S(x)| \leq B''\delta_Q^2 \text{ for all } x \in S.$$

Let $B_{\max} := B''$. Since S is arbitrary and is allowed to contain more than one point, we may conclude the proof of the lemma once we let S range over all k'-point subsets of $E \cap Q^*$ containing x_0. □

7.3. Solving the local problem

In this subsection, we prove Lemma 7.1. We fix the local data structure for the rest of the section.

<u>Local Data Structure (LDS)</u>

- A lengthscale $\delta \leq 1$.
- A square $Q \subset \mathbb{R}^2$ with $\delta_Q = \delta$.
- A representative point $x^\sharp \in Q$ such that $\operatorname{dist}\left(x^\sharp, E\right) \geq c_{\text{rep}}\delta$.
- A function $\phi \in C^2(\mathbb{R})$ that satisfies $|\phi^{(k)}| \leq \delta^{1-k}$ for $k = 1, 2$.
- A diffeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\Phi(s,t) = (s, t - \phi(s))$.
- $E_{\text{loc}} = E \cap Q^*$ such that $E_{\text{loc}} \subset \{(s, \phi(s)) : s \in \mathbb{R}\}$.

Any $Q \in \Lambda^{(k)}_{\text{nice}}$ with (k, C_{nice}) guaranteeing good geometry admits the local data structure, thanks to Lemmas 5.4, 5.5, and 5.6.

We have shown in Lemma 7.5 that each local interpolation problem belongs to at least one of the two categories: The function's local values are uniformly big ($\min_{x \in E_{\text{loc}}} f(x) \geq B_{\min}\delta^2$), or are uniformly small ($\max_{x \in E_{\text{loc}}} f(x) \leq B_{\max}\delta^2$). The next lemma solves the former case.

Lemma 7.6. *There exists a sufficiently large* $B_{\min} > 0$ *such that the following holds.*

Let LDS be given. Let $k_{\text{loc}} \geq 3$. *Suppose* $\Gamma^\sharp_+(x^\sharp, k_{\text{loc}}, M) \neq \varnothing$, *and* $f \geq B_{\min}\delta^2$ *on* E_{loc}. *Then there exists* $F \in C^2_+(\mathbb{R}^2)$ *with* $F\big|_{E_{\text{loc}}} = f$, $\|F\|_{C^2(100Q)} \leq CM$, *and* $\mathcal{J}_{x^\sharp} F \in \Gamma^\sharp_+(x^\sharp, k_{\text{loc}}, CM)$.

Proof. Without loss of generality, we may assume $M = 1$.

We will use b, B, B', etc. to denote quantities that depend only on $B_{\min}$, and c, C, C', etc. to denote universal constants.

Let $P \in \Gamma^\sharp_+(x^\sharp, k_{\text{loc}}, 1)$. Pick distinct $x_1, x_2 \in E_{\text{loc}}$. Let $P^\sharp$ be the unique affine polynomial that passes through $(x_1, f(x_1)), (x_2, f(x_2))$, and $(x^\sharp, P(x^\sharp))$. We first prove two claims about $P^\sharp$.

Claim 7.3. *We have*

$$|\nabla\left(P - P^\sharp\right)| \leq C \operatorname{diam}\left(\text{Triangle}(x_1, x_2, x^\sharp)\right) \leq C\delta. \tag{7.18}$$

As a consequence, $P^\sharp \in \Gamma^\sharp_+(x^\sharp, k_{\text{loc}}, C)$.

Proof of Claim 7.3. For convenience of notation, we temporarily label $x_0 := x^\sharp$.

Let $S = \{x_1, x_2\}$. Since $P \in \Gamma^\sharp_+(x_0, k_{\text{loc}}, 1)$ with $k_{\text{loc}} \geq 3$, there exists $F^S \in C^2_+(\mathbb{R}^2)$ with $F^S\big|_S = f$, $\|F^S\|_{C^2(\mathbb{R}^2)} \leq 1$, and $\mathcal{J}_{x_0} F^S = P$. In particular, F^S agrees with $P^\sharp$ at x_i for $i = 0, 1, 2$.

Let L_{ij} be the (open) segment connecting x_i and x_j. The L_{ij}'s are the sides of $\text{Triangle}(x_0, x_1, x_2)$. Let $u_{ij} = \frac{x_j - x_i}{|x_j - x_i|}$. Rolle's theorem implies that there exist $\xi_{ij} \in L_{ij}$ such that

$$\nabla(F^S - P^\sharp)(\xi_{ij}) \cdot u_{ij} = 0 \ \text{ for all } \ 0 \leq i, j \leq 2.$$

Since $\|F^S\|_{C^2(\mathbb{R}^2)} \leq 1$ and $P^\sharp$ is an affine polynomial, we have

$$|\nabla(F^S - P^\sharp)(x_0) \cdot u_{ij}| \leq C\mathrm{diam}\,(\mathrm{Triangle}(x_0, x_1, x_2))\,. \tag{7.19}$$

Since $\mathrm{dist}\,(x_0, E) \geq c_{\mathrm{rep}}\delta$ and E_{loc} lies on the graph of ϕ with $|\phi'| \leq 1$, we have

$$\mathrm{Angle}\,(u_{01}, u_{12}) \geq \gamma \tag{7.20}$$

for some $\gamma > 0$ depending only on c_{rep}.

Let ω be any unit vector. (7.20) implies that we can write

$$\omega = R_{\omega,1}u_{01} + R_{\omega,2}u_{12},\ |R_{\omega,i}| \leq C \text{ for } i = 1, 2. \tag{7.21}$$

Here, C is a constant depending only on γ. (7.21) implies that

$$|\nabla(F^S - P^\sharp)(x_0) \cdot \omega| \leq C\mathrm{diam}\,(\mathrm{Triangle}(x_0, x_1, x_2)) \leq C\delta\,.$$

We conclude (7.18) by letting ω range over all unit vectors. Thanks to Lemma 7.3, $P^\sharp \in \Gamma_+^\sharp(x^\sharp, k_{\mathrm{loc}}, C)$. This proves the claim. □

Claim 7.4. *Suppose* $B_{\min}$ *is sufficiently large. Then*

$$P^\sharp(x^\sharp) \geq CB_{\min}\delta^2. \tag{7.22}$$

Proof of Claim 7.4. The proof is identical to the proof of Claim 7.1. □

Recall from LDS that E_{loc} lies on the graph of a C^2 function ϕ. Therefore, we may write $E_{\mathrm{loc}} = \{z_i = (s_i, \phi(s_i)) : 1 \leq i \leq N\}$ with $s_i < s_{i+1}$ for all $i = 1, \cdots, N-1$.

For $i = 1, \cdots, N-2$, let $S_i = \{z_i, z_{i+1}, z_{i+2}\}$. By Claim 7.3, $P^\sharp \in \Gamma_+^\sharp(x^\sharp, k_{\mathrm{loc}}, C)$. By definition, there exists $F^{S_i} \in C_+^2(\mathbb{R}^2)$ such that $F^{S_i}\big|_{S_i} = f$, $\|F^{S_i}\|_{C^2(\mathbb{R}^2)} \leq C$, and $\mathcal{J}_{x^\sharp}F^{S_i} = P^\sharp$.

Define $g : E_{\mathrm{loc}} \to \mathbb{R}$ by

$$g := f - \left(P^\sharp\big|_{E_{\mathrm{loc}}}\right).$$

Note that g is not necessarily nonnegative.

Define $G^{S_i} : \mathbb{R}^2 \to \mathbb{R}$ by

$$G^{S_i} := F^{S_i} - P^\sharp.$$

Then immediately, we have

$$G^{S_i}\big|_{S_i} = g \text{ and} \tag{7.23}$$

$$\mathcal{J}_{x^\sharp}G^{S_i} \equiv 0\,. \tag{7.24}$$

Since $P^\sharp \in \Gamma_+^\sharp(x^\sharp, k_{loc}, C)$, we have $\|P^\sharp\|_{C^2(100Q)} \le C$. From this, together with the condition $\|F^{S_i}\|_{C^2(\mathbb{R}^2)} \le C$, we learn that

$$\|G^{S_i}\|_{C^2(100Q)} \le C\,. \tag{7.25}$$

Thanks to (7.24), (7.25), and the fundamental theorem of calculus, we have

$$|\nabla G^{S_i}(s,t)| \le C\delta \text{ for all } (s,t) \in 100Q. \tag{7.26}$$

Let $I_1 = (-\infty, s_3], I_2 = [s_2, s_4], \cdots, I_{N-3} = [s_{N-3}, s_{N-1}]$, and $I_{N-2} = [s_{N-2}, +\infty)$. Let $\left\{\overline{\theta}_i : \mathbb{R} \to \mathbb{R}\right\}$ be a partition of unity subordinate to $\{I_i\}$ such that

$$\operatorname{supp}\left(\overline{\theta}_i\right) \subset I_i \text{ and } |\partial_s^k \overline{\theta}_i(s)| \le \begin{cases} C|s_{i+1} - s_i|^{-k} & \text{if } s \in [s_i, s_{i+1}] \\ C|s_{i+2} - s_{i+1}|^{-k} & \text{if } s \in [s_{i+1}, s_{i+2}] \end{cases} \text{ for } 1 \le k \le 2\,. \tag{7.27}$$

Note that the interior (in the topology of $\mathbb{R}$) of $I_i \cap I_j$ supports at most two partition functions.

Let

$$\theta_i(s,t) := \overline{\theta}_i(s) \text{ for } i = 1, \cdots, N-2.$$

It follows immediately that the interior of $(I_i \times \mathbb{R}) \cap (I_j \times \mathbb{R})$ supports at most two partition functions. It is also clear that

$$\partial_t \theta_i \equiv 0 \text{ for } i = 1, \cdots, N-2. \tag{7.28}$$

Recall Φ as in LDS. Define

$$G := \left(\sum_{i=1}^{N-2} \left(G^{S_i} \circ \Phi^{-1}\right)\Big|_{t=0} \cdot \theta_i \right) \circ \Phi = \sum_{i=1}^{N-2} \left[G^{S_i} \circ \gamma\right] \cdot [\theta_i \circ \Phi]\,,$$

where $\gamma(s) := (s, \phi(s))$ is a parametrization of the graph of ϕ.

Claim 7.5. *The function* G *satisfies* $G\big|_{E_{loc}} = g$ *and* $\|G\|_{C^2(100Q)} \le C$.

Proof of Claim 7.5. It is clear from (7.23) that $G\big|_{E_{loc}} = g$.

Now we estimate $\|G\|_{C^2(100Q)}$.

Thanks to (7.28), $\operatorname{supp}(\theta_i \circ \Phi) \subset I_i \times \mathbb{R}$. Hence, the support of the $\theta_i \circ \Phi$'s have bounded overlap. Since $0 \le \theta_i \circ \Phi \le 1$, (7.25) implies that

$$|G| \le C \text{ on } 100Q. \tag{7.29}$$

Now we compute the derivatives of G. Thanks to (7.28), we have

$$\partial_t^k G \equiv 0 \text{ for } k = 1, 2; \text{ and } \partial_{st}^2 G \equiv 0\,. \tag{7.30}$$

Therefore, it remains to estimate the pure s-derivatives of G.

First of all, thanks to (7.28), we have

$$\partial_s^k(\theta_i \circ \Phi) = \partial_s^k \theta_i \circ \Phi = \partial_s^k \overline{\theta}_i \text{ for } k = 1, 2\,.$$

It follows from (7.27) that

$$|\partial_s^k(\theta_i \circ \Phi)(s,t)| \le \begin{cases} C|s_{i+1} - s_i|^{-k} & \text{if } (s,t) \in [s_i, s_{i+1}] \times \mathbb{R} \\ C|s_{i+2} - s_{i+1}|^{-k} & \text{if } (s,t) \in [s_{i+1}, s_{i+2}] \times \mathbb{R} \end{cases} \text{ for } 1 \le k \le 2\,. \tag{7.31}$$

Now we compute the s-derivatives of $G^{S_i} \circ \gamma(s)$.

$$\partial_s(G^{S_i} \circ \gamma) = \partial_s G^{S_i} \circ \gamma + \phi' \partial_t G^{S_i} \circ \gamma\,,$$
$$\partial_s^2(G^{S_i} \circ \gamma) = \partial_s^2 G^{S_i} \circ \gamma + (\phi')^2 \partial_t^2 G^{S_i} \circ \gamma + 2\phi' \partial_{st}^2 G^{S_i} \circ \gamma + \phi'' \partial_t G^{S_i} \circ \gamma\,.$$

Recall that $|\phi^{(k)}| \le \delta^{1-k}$ for $k = 1, 2$. Applying (7.26) to the last term of the second identity and (7.25) to the rest of the terms, we conclude that

$$\|G^{S_i} \circ \gamma\|_{C^2[-50,50]} \le C. \tag{7.32}$$

Since $G(s,t) = G^{S_1} \circ \gamma(s)$ or $G(s,t) = G^{S_{N-2}} \circ \gamma(s)$ outside of the strip $[s_2, s_{N-1}] \times \mathbb{R}$, (7.25) implies

$$|\partial_s^k G(s,t)| \le C \text{ for } (s,t) \notin [s_2, s_{N-1}] \times \mathbb{R},\ k = 1, 2\,. \tag{7.33}$$

Suppose $(s,t) \in [s_2, s_{N-1}] \times \mathbb{R}$. Let j be the least integer such that $s \in I_j$. Then

$$\begin{aligned} \partial_s^k G = {} & \left(\partial_s^k(G^{S_j} \circ \gamma)\right) \cdot (\theta_j \circ \Phi) + \left(\partial_s^k(G^{S_{j+1}} \circ \gamma)\right) \cdot (\theta_{j+1} \circ \Phi) \\ & + \sum_{l=0}^{k-l} \binom{k}{l} \left(\partial_s^l(G^{S_j} \circ \gamma - G^{S_{j+1}} \circ \gamma)\right) \cdot \left(\partial_s^{k-l}\theta_j \circ \Phi\right). \end{aligned} \tag{7.34}$$

By an argument similar to the proof of Claim 6.1, combined with estimate (7.32), we have, for $s \in I_j \cap I_{j+1}$,

$$\begin{aligned} &\left|\left(G^{S_j} \circ \gamma - G^{S_{j+1}} \circ \gamma\right)(s)\right| \le C\,|s_{j+1} - s_j| \text{ and} \\ &\left|\partial_s^l \left(G^{S_j} \circ \gamma - G^{S_{j+1}} \circ \gamma\right)(s)\right| \le C\,|s_{j+1} - s_j|^{2-l} \text{ for } l = 0, 1\,. \end{aligned} \tag{7.35}$$

To estimate (7.34), we apply (7.32) to the first two terms (note that $0 \leq \theta_j \circ \Phi \leq 1$), and apply (7.31) and (7.35) to the last term. Hence,

$$\left|\partial_s^k G(s,t)\right| \leq C \text{ for } (s,t) \in [s_2, s_{N-1}] \times \mathbb{R},\ k = 1, 2\,. \tag{7.36}$$

The claim follows from (7.29), (7.30), (7.33), and (7.36). □

Recall that, by construction, $g(x_1) = g(x_2) = 0$. Since G is constant in the t-direction, Rolle's theorem implies that

$$|\partial^\alpha G| \leq C_{\text{lb}}\delta^{2-|\alpha|} \text{ on } 100Q \text{ for } |\alpha| \leq 2\,. \tag{7.37}$$

In particular,

$$G \geq -C_{\text{lb}}\delta^2 \text{ on } 100Q. \tag{7.38}$$

Let $\psi \in C^2_+(\mathbb{R}^2)$ be a cutoff function that satisfies the following.

- $0 \leq \psi \leq 1$ on $\mathbb{R}^2$, $\psi \equiv 1$ near $x^\sharp$, $\operatorname{supp}(\psi) \subset B(x^\sharp, \frac{c_{\text{rep}}\delta}{100})$; and
- $|\partial^\alpha \psi| \leq C\delta^{2-|\alpha|}$ for $|\alpha| \leq 2$.

Define $\tilde{G} := \mathbb{R}^2 \to \mathbb{R}$ by

$$\tilde{G} := (1-\psi)G\,.$$

Then we have the following.

- Thanks to Claim 7.5, (7.37), and the second property of ψ, we have

$$\|\tilde{G}\|_{C^2(100Q)} \leq C\,. \tag{7.39}$$

- $\tilde{G}|_{E_{\text{loc}}} = g$, since $\operatorname{dist}\left(x^\sharp, E\right) \geq c_{\text{rep}}\delta$ and $\operatorname{supp}(\psi) \subset B(x^\sharp, \frac{c_{\text{rep}}\delta}{100})$.
- $\mathcal{J}_{x^\sharp}\tilde{G} \equiv 0$, since $\psi \equiv 1$ near $x^\sharp$.

- Moreover, since $0 \leq \psi \leq 1$, (7.38) implies

$$\tilde{G} \geq -C_{\text{lb}}\delta^2 \text{ on } 100Q. \tag{7.40}$$

Finally, define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F := \tilde{G} + P^\sharp\,.$$

Then the following are immediate.

- $F|_{E_{loc}} = g + \left(P^\sharp|_{E_{loc}}\right) = f$, and
- $\mathcal{J}_{x^\sharp} F = \mathcal{J}_{x^\sharp} \tilde{G} + \mathcal{J}_{x^\sharp} P^\sharp = P^\sharp \in \Gamma_+^\sharp(x^\sharp, k_{loc}, C)$.

It remains to show that $\|F\|_{C^2(100Q)}$ is universally bounded and that F is nonnegative on $100Q$.

Recall that $P^\sharp \in \Gamma_+^\sharp(x^\sharp, k_{loc}, C)$, so we have $\|P^\sharp\|_{C^2(100Q)} \leq C$. It follows from (7.39) that

$$\|F\|_{C^2(100Q)} \leq C.$$

It remains to show that F is nonnegative on $100Q$.

To this end, observe that (7.3) and (7.22) imply

$$\text{dist}\left(x^\sharp, \left\{P^\sharp = 0\right\}\right) \geq C\sqrt{P^\sharp(x^\sharp)} = C\sqrt{B_{min}} \cdot \delta. \tag{7.41}$$

Therefore, for sufficiently large B_{min}, (7.22) and (7.41) yield

$$P^\sharp \geq C_{lb}\delta^2 \text{ on } 100Q. \tag{7.42}$$

Therefore, (7.40) and (7.42) imply that

$$F \geq 0 \text{ on } 100Q.$$

This concludes the proof of Lemma 7.6. □

Fix B_{min} as in Lemma 7.6. The following lemma complements Lemma 7.6.

Lemma 7.7. *Let LDS be given. Let $k_{loc} \geq 3$. Suppose $\Gamma_+^\sharp(x^\sharp, k_{loc}, M) \neq \emptyset$, and that there exists $x \in E_{loc}$ such that $f(x) < B_{min}M\delta^2$. Then there exists $F \in C_+^2(100Q)$ such that $F|_{E_{loc}} = f$, $\|F\|_{C^2(100Q)} \leq BM$, and $\mathcal{J}_{x^\sharp}F \in \Gamma_+^\sharp(x^\sharp, k_{loc}, BM)$. The number B depends only on B_{min}.*

Proof. Without loss of generality, we may assume $M = 1$.

We write B_1, B_2, etc. to denote quantities depending only on B_{min}.

By Lemma 7.5, there exists $B_{max} > 0$, depending only on B_{min} such that

$$f(x) \leq B_{max}\delta^2 \text{ for all } x \in E_{loc}.$$

By Lemma 7.3, we have

$$0 \in \Gamma_+^\sharp(x^\sharp, k_{loc}, B_1). \tag{7.43}$$

Recall that E_{loc} lies on the graph of a C^2 function ϕ. Write $E_{loc} = \{z_i = (s_i, \phi(s_i)) : 1 \leq i \leq N\}$ with $s_i < s_{i+1}$ for all $i = 1, \cdots, N-1$.

For $i = 1, \cdots, N-2$, let $S_i = \{z_i, z_{i+1}, z_{i+2}\}$. By (7.43), there exists $F^{S_i} \in C^2_+(\mathbb{R}^2)$ such that $F^{S_i}\big|_{S_i} = f$, $\|F^{S_i}\|_{C^2(\mathbb{R}^2)} \le B_4$, and $\mathcal{J}_{x^\sharp} F^{S_i} \equiv 0$.

Let $I_1 = (-\infty, s_3], I_2 = [s_2, s_4], \cdots, I_{N-3} = [s_{N-3}, s_{N-1}]$, and $I_{N-2} = [s_{N-2}, +\infty)$. Let $\{\overline{\theta}_i\}$ be a partition of unity subordinate to $\{I_i\}$ such that

$$\operatorname{supp}(\overline{\theta}_i) \subset I_i \text{ and } |\partial_s^k \overline{\theta}_i(s)| \le \begin{cases} C|s_{i+1} - s_i|^{-k} & \text{if } s \in [s_i, s_{i+1}] \\ C|s_{i+2} - s_{i+1}|^{-k} & \text{if } s \in [s_{i+1}, s_{i+2}] \end{cases} \text{ for } 1 \le k \le 2\,.$$

Put

$$\theta_i(s,t) := \overline{\theta}_i(s) \text{ for } i = 1, \cdots, N-2\,.$$

Recall the diffeomorphism Φ in LDS. Define $\overline{F} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\overline{F} := \left(\sum_{i=1}^{N-2} \left(F^{S_i} \circ \Phi^{-1}\right)\Big|_{t=0} \cdot \theta_i \right) \circ \Phi\,.$$

It is clear that $\overline{F} \ge 0$. By the same argument as in the proof of Claim 7.5, we have $\overline{F}|_{E_{loc}} = f$ and $\|\overline{F}\|_{C^2(100Q)} \le B_2$.

Since $f \le B_{max}\delta^2$ on E_{loc} and $\overline{F}$ is constant in the t-direction, we also have

$$|\overline{F}| \le CB_{max}\delta^2 \text{ on } [-50, 50).$$

Let $\psi \in C^2_+(\mathbb{R}^2)$ be a cutoff function such that

- $0 \le \psi \le 1$ on $\mathbb{R}^2$, $\psi \equiv 1$ near $x^\sharp$, $\operatorname{supp}(\psi) \subset B(x^\sharp, \frac{c_{rep}\delta}{100})$; and
- $|\partial^\alpha \psi| \le C\delta^{2-|\alpha|}$ for $|\alpha| \le 2$.

Define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F := (1-\psi)\overline{F}\,.$$

The following hold.

- $F \ge 0$, since $\overline{F} \ge 0$ and $0 \le \psi \le 1$.
- $F\big|_{E_{loc}} = f$, thanks to (7.1) and the fact that $\operatorname{supp}(\psi) \subset B(x^\sharp, \frac{c_{rep}\delta}{100})$.
- $\mathcal{J}_{x^\sharp} F \equiv 0 \in \Gamma^\sharp_+(x^\sharp, k_{loc}, B_4)$, since $\psi \equiv 1$ near $x^\sharp$.
- $\|F\|_{C^2(100Q)} \le B_3$. To see this, we note that since $F \ge 0$ on $\mathbb{R}^2$, $\|F\|_{C^2(100Q)} \le B_2$, and $|F| \le CB_{max}\delta^2$ on $100Q$, (7.3) implies that $|\nabla F| \le B_4\delta$ on $100Q$. Thanks to the second condition on ψ, the conclusion follows.

This proves the lemma. □

Proof of Lemma 7.1. Fix B_{min} as in Lemma 7.6. The lemma follows from Lemma 7.6 and Lemma 7.7. □

7.4. Proof of Theorem 4

Before proceeding to the proof of Theorem 4, we make a brief comment on the finiteness constant 64. Lemma 4.2 and Lemma 5.3 state that jets of 4k-point interpolants based in neighboring squares from $\Lambda_{nice}^{(k)}$ are compatible in the Whitney sense (see (5.3)); Lemma 5.4 states that the geometry of data points in each square of $\Lambda_{nice}^{(k)}$ is sufficiently nice when $k \geq 4$; Lemma 7.1 states that in such case, a local version of the extension problem is readily solved. Hence, if we pick $k = 4$, we may use the jets of $4 \cdot 4 = 16$-point interpolants (if they exist) to guarantee compatibility of nearby local extensions. By Lemma 4.3, such jets exist.

Now, we examine compatibility of the local interpolants constructed in Lemma 7.1.

Proof of Theorem 4. Without loss of generality, we may assume $M = 1$.

Set $k = 4$. Pick C_{nice} so that $(4, C_{nice})$ guarantees good geometry.

By Lemma 5.1 $\Lambda_{nice}^{(4)}$ is a CZ covering of $\mathbb{R}^2$.

By Lemma 4.3, $\Gamma_+^\sharp(x, 16, 1) \neq \varnothing$ for any $x \in \mathbb{R}^2$.

We distinguish three types of squares in $\Lambda_{nice}^{(k)}$.

Type 1. Suppose $E \cap Q^* \neq \varnothing$. Let $F_Q^\sharp := F_Q$, where F_Q is as in Lemma 7.1 with $k_{loc} = 16$. Let $P_Q^\sharp := \mathcal{J}_{x_Q^\sharp} F_Q^\sharp$. We have $P_Q^\sharp \in \Gamma_+^\sharp(x_Q^\sharp, 16, C)$.

Type 2. Suppose $E \cap Q^* = \varnothing$ but $\delta_Q < 1$. Pick $P_Q^\sharp \in \Gamma_+^\sharp(x_Q^\sharp, 16, 1)$, and set $F_Q^\sharp := \mathcal{W}^{\{x_Q^\sharp\}}(P_Q^\sharp)$, where $\mathcal{W}^{\{x_Q^\sharp\}}$ is as in Lemma 4.9 with $S = \{x_Q^\sharp\}$.

Type 3. Suppose $E \cap Q^* = \varnothing$ and $\delta_Q = 1$. Set $F_Q^\sharp \equiv 0$.

By Lemma 7.1 and Lemma 4.9, $F_Q^\sharp \in C_+^2(100Q)$, $F_Q^\sharp\big|_{E\cap Q^*} = f$, and

$$\|F_Q^\sharp\|_{C^2(100Q)} \leq C. \tag{7.44}$$

Claim 7.6. *If* $Q \leftrightarrow Q'$, *then for each* $x \in \frac{9}{8}Q \cup \frac{9}{8}Q'$ *and* $0 \leq |\alpha| \leq 1$,

$$|\partial^\alpha(F_Q^\sharp - F_{Q'}^\sharp)(x)| \leq C\delta_Q^{2-|\alpha|}. \tag{7.45}$$

The constant C *is universal.*

Proof of Claim 7.6. Temporarily fix $x \in \frac{9}{8}Q \cup \frac{9}{8}Q'$ for $Q \leftrightarrow Q'$.

Assume that either Q or Q' is of Type 3, then (7.45) follows from (3.3) and (7.44).

Suppose neither Q nor Q' is of Type 3. Thanks to (3.3) and our choice of $x_Q^\sharp$ and $x_{Q'}^\sharp$ in Lemma 5.6, we have $|x_Q^\sharp - x|, |x_{Q'}^\sharp - x|, |x_Q^\sharp - x_{Q'}^\sharp| \leq C\delta_Q$.

Recall from Lemma 7.1 and Lemma 4.9 that

$$\mathcal{J}_{x_Q^\sharp} F_Q^\sharp = P_Q^\sharp \in \Gamma_+^\sharp(x_Q^\sharp, 16, C) \text{ and } \mathcal{J}_{x_{Q'}^\sharp} F_{Q'}^\sharp = P_{Q'}^\sharp \in \Gamma_+^\sharp(x_{Q'}^\sharp, 16, C).$$

By Taylor's theorem,

$$|\partial^\alpha \left(F_Q^\sharp - P_Q^\sharp\right)(x)| \leq C\delta_Q^{2-|\alpha|} \text{ and } |\partial^\alpha \left(F_{Q'}^\sharp - P_{Q'}^\sharp\right)(x)| \leq C\delta_{Q'}^{2-|\alpha|} \leq C\delta_Q^{2-|\alpha|}. \quad (7.46)$$

By Lemma 5.3,

$$|\partial^\alpha (P_Q^\sharp - P_{Q'}^\sharp)(x)| \leq C\delta_Q^{2-|\alpha|}. \quad (7.47)$$

Now, (7.45) follows from (7.46) and (7.47). □

Let $\{\theta_Q\}$ be a partition of unity that is CZ-compatible with $\Lambda_{\text{nice}}^{(4)}$. Define

$$F(x) := \sum_{Q \in \Lambda_{\text{nice}}^{(4)}} \theta_Q(x) \cdot F_Q^\sharp(x)$$

It is clear that $F \geq 0$, $F|_E = f$, and F is twice continuously differentiable. For $|\alpha| \leq 2$ and $x \in Q$,

$$\partial^\alpha F(x) = \sum_{Q \in \Lambda_{\text{nice}}^{(4)}} \partial^\alpha F_Q^\sharp(x) \cdot \theta_Q(x) + \sum_{Q' \leftrightarrow Q} \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}(F_Q^\sharp - F_{Q'}^\sharp)(x) \cdot \partial^\beta \theta_{Q'}(x). \quad (7.48)$$

Applying (3.4), (3.5), (7.44), and (7.45) to (7.48), we can conclude that

$$\|F\|_{C^2(\mathbb{R}^2)} \leq C. \quad \square$$

8. Sharp finiteness principle

In this section, we give the proof of Theorem 5. Here we remind the readers the statement of the theorem.

Theorem 5 *(2-D Sharp Finiteness Principle). Let $E \subset \mathbb{R}^2$ with $\#(E) = N < \infty$. Then there exist universal constants C, C', C'' and a list of subsets $S_1, S_2, \cdots, S_L \subset E$ satisfying the following.*

(A) $\#(S_\ell) \leq C$ *for each* $\ell = 1, \cdots, L$.
(B) $L \leq C'N$.
(C) Given any $f : E \to [0, \infty)$, *we have*

$$\max_{\ell=1,\cdots,L} \|f\|_{C_+^2(S_\ell)} \leq \|f\|_{C_+^2(E)} \leq C'' \max_{\ell=1,\cdots,L} \|f\|_{C_+^2(S_\ell)}.$$

Before we proceed to the proof, we briefly explain the clusters S_ℓ's in the statement.

For each square $Q \in \Lambda^{(k)}_{\text{nice}}$, we associate to it a basic cluster $S(x^\sharp_Q)$ (see Definition 8.2) that guarantees internal Whitney compatibility.

The clusters in Theorem 5 can be classified into three types.

- The first type is the union of a "consecutive" three-point cluster (since E locally lies on a curve with controlled geometry), nearby basic clusters, and nearby "keystone" clusters (see next bullet point). This is the "largest" type of clusters, since it plays the key role of relaying information about E to various lengthscales.
- The second type is the basic cluster for each "keystone square" (see Definition 8.1). Keystone squares are locally the smallest squares and they play an important role in relaying information to nearby small squares containing no data point.
- The third type is the union of keystone square clusters (see the second bullet point above) that are associated with each "special square" (see Lemma 8.2). This type of clusters is used to eliminate the ambiguity in how these special squares receive information from E.

We now give the full account.

8.1. CZ squares and clusters

Let (k, C_{nice}) guarantee good geometry (Definition 5.3). We fix such (k, C_{nice}) for the rest of the section. We may assume, for instance,

$$k = 4 \text{ and } C_{\text{nice}} = 1000\,.$$

Definition 8.1. We define the following objects.

- We set
$$\Lambda_0 := \Lambda^{(k)}_{\text{nice}} \text{ (see Definition 5.2).} \tag{8.1}$$
- We also set
$$\Lambda^\sharp := \{Q \in \Lambda_0 : E \cap Q^* \neq \varnothing\}\,. \tag{8.2}$$
Note that $\Lambda^\sharp$ coincides with Type 1 squares in the proof of Theorem 4 (Section 7.4).
- We say $Q \in \Lambda_0$ is a keystone square if $\delta_Q < 1$ and for any $Q' \in \Lambda_0$ with $Q' \cap 100Q \neq \varnothing$, we have $\delta_{Q'} \geq \delta_Q$. The collection of keystone squares is denoted by Λ_{KS}.

Keystone squares first appear in the work of Sobolev extension [18]. See also [10] for a more thorough discussion.

Lemma 8.1. *Let Λ_{KS} be as in Definition 8.1. Then*

$$\#(\Lambda_{\mathrm{KS}}) \le C \cdot \#(E).$$

The proof of Lemma 8.1 can be found in Section 4 of [18] and Section 7 of [10].

Next, we define the basic cluster associated with each square in Λ_0.

Definition 8.2. Let $Q \in \Lambda_0$ and let $x^\sharp_Q$ be as in Lemma 5.6. (Note that $x^\sharp_Q$ is a representative point "far" from the data on the lengthscale δ_Q.) Let $S_1, \cdots, S_{12} \subset E$ be as in Lemma 4.5 (with $x = x^\sharp_Q$ and $4k$ in place of k). We define

$$S(x^\sharp_Q) := \bigcup_{i=1}^{12} S_i\,. \tag{8.3}$$

Since $\#(S_i) \le C$ for $i = 1, \cdots, 12$ (see Lemma 4.5), we have

$$\#(S(x^\sharp_Q)) \le C\,. \tag{8.4}$$

By Lemma 4.5, we have

$$\sigma(x^\sharp_Q, S(x^\sharp_Q)) \subset C \cdot \sigma^\sharp(x^\sharp_Q, 4k)\,. \tag{8.5}$$

Next, we state a key lemma that allows us to relay information from keystone squares to small squares in Λ_0 whose neighborhood contains no points from E. The latter requires separate attention for the following reason: Suppose $Q \in \Lambda_0$ with $\delta_Q < 1$ and $E \cap Q^* = \varnothing$. Then $(Q^+)^*$ may intersect an uncontrolled number of squares in $\Lambda^\sharp$. Keystone squares are designed partially to deal with such situations. See [10,18] for further discussion.

Lemma 8.2. *Let $\Lambda_0, \Lambda_{\mathrm{KS}}$ be as in Definition 8.1. We can find a subset $\Lambda_{\mathrm{special}} \subset \Lambda_0$ and a map $\mu : \Lambda_0 \to \Lambda_{\mathrm{KS}}$ such that the following holds for some universal constant C.*

(A) $\#(\Lambda_{\mathrm{special}}) \le C \cdot \#(E)$.
(B) $\mu(Q) \in \Lambda_{\mathrm{KS}}$*, where* Λ_{KS} *is as in Definition 8.1. Moreover,* $\mathrm{dist}\,(Q, \mu(Q)) \le C\delta_Q$.
(C) Suppose $Q, Q' \in \Lambda_0 \setminus \Lambda_{\mathrm{special}}$ *and* $Q \leftrightarrow Q'$*, then* $\mu(Q) = \mu(Q')$.

The proof of Lemma 8.2 can be found in Section 6 of [10].

Definition 8.3. Recall $\Lambda_0, \Lambda^\sharp, \Lambda_{\mathrm{KS}}$ as in Definition 8.1. Recall the representative point $x^\sharp_Q$ as in Lemma 5.6. Let $Q \in \Lambda_{\mathrm{KS}}$. We define

$$S_{\mathrm{KS}}(Q) := S(x^\sharp_Q), \tag{8.6}$$

where $S(x^\sharp_Q)$ is as in (8.3). Recall $\Lambda_{special}, \mu$ as in Lemma 8.2. Let $Q \in \Lambda_{special}$. We define

$$S_{\text{special}}(Q) := \bigcup_{Q' \leftrightarrow Q, Q' \in \Lambda_0} S(x^\sharp_{\mu(Q')}), \tag{8.7}$$

where $x^\sharp_{\mu(Q')}$ is as in Lemma 5.6 and $S(x^\sharp_{\mu(Q')})$ is as in (8.3).

Recall from Lemma 5.1 that Λ_0 is a CZ covering of $\mathbb{R}^2$. In particular, Λ_0 enjoys the bounded intersection property (3.4). Together with (8.4) and the definitions of $S_{\text{KS}}, S_{\text{special}}$ in (8.6), (8.7), we see that

$$\#(S_{\text{KS}}(Q)) \leq C \text{ for each } Q \in \Lambda_{\text{KS}}, \text{ and} \tag{8.8}$$

$$\#(S_{\text{special}}(Q)) \leq C \text{ for each } Q \in \Lambda_{special}. \tag{8.9}$$

Now we turn our attention to clusters associated with $\Lambda^\sharp$.

For convenience, we set, for each $Q \in \Lambda^\sharp$,

$$N(Q) := \#(E \cap Q^*)\,.$$

Thanks to Lemma 5.4, we know that for each $Q \in \Lambda^\sharp$, up to a rotation, $E \cap Q^* \subset \{(s, \phi(s)) : s \in \mathbb{R}\}$, where ϕ is as in Lemma 5.4. We enumerate

$$E \cap Q^* = \{(s_i, \phi(s_i)) : i = 1, \cdots, N(Q)\} \quad \text{such that } s_1 < \cdots < s_{N(Q)}\,. \tag{8.10}$$

Let Φ be as in Lemma 5.5. We also set

$$I_Q := \Phi(100Q)\big|_{\mathbb{R}\times\{t=0\}}\,. \tag{8.11}$$

For the rest of this section, whenever we consider $Q \in \Lambda^\sharp$, we always assume that Q has been rotated so that enumeration of the form (8.10) holds.

The next three definitions describe the objects of interest in this section. Definitions 8.4 and 8.5 concern the clusters, and Definition 8.6 concerns the main polynomial convex sets.

Definition 8.4. Let $Q \in \Lambda^\sharp$. Let $E \cap Q^*$ be enumerated as in (8.10).

- In the case $1 \leq N(Q) \leq 2$, we set

$$\overline{S}(Q, 1) := E \cap Q^* \quad \text{and } \nu(Q) := 1\,. \tag{8.12}$$

- Suppose $N(Q) \geq 3$, we set

$$\overline{S}(Q, \nu) := \{(s_\nu, \phi(s_\nu)), (s_{\nu+1}, \phi(s_{\nu+1})), (s_{\nu+2}, \phi(s_{\nu+2}))\}, \tag{8.13}$$

for $\nu = 1, \cdots, \nu(Q)$, where $\nu(Q) := N(Q) - 2$.

By the bounded intersection property of Λ_0 (see (3.4)), we have

$$\#\left\{\overline{S}(Q,\nu) : Q \in \Lambda^\sharp, \nu \in \{1, \cdots, \nu(Q)\}\right\} \leq C \cdot \#(E)\,. \tag{8.14}$$

Definition 8.5. Let $Q \in \Lambda^\sharp$. Let $\overline{S}(Q,\nu)$ be as in Definition 8.4. Let $x_Q^\sharp$ be as in Lemma 5.6. Let μ be the map in Lemma 8.2. Let $S(\cdot)$ be as in (8.3). For each $\nu = 1, \cdots, \nu(Q)$, we set

$$S(Q,\nu) := \overline{S}(Q,\nu) \cup \bigcup_{\substack{Q' \leftrightarrow Q \\ Q' \in \Lambda_0}} \left(S(x^\sharp_{Q'}) \cup S(x^\sharp_{\mu(Q')})\right) \tag{8.15}$$

Remark 8.1. The cluster $S(Q,\nu)$ associated with each $Q \in \Lambda^\sharp$ is the "largest" among all three types of clusters (the other two being $S_{\mathrm{KS}}(Q)$ in (8.6) and $S_{\mathrm{special}}(Q)$ in (8.7)). This is expected, since each $Q \in \Lambda^\sharp$ satisfies $E \cap Q^* \neq \varnothing$, and must relay information to neighboring squares and their keystone representatives.

Thanks to (8.4), the fact that $\#(\overline{S}(Q,\nu)) \leq 3$, and the bounded intersection property of Λ_0 (see (3.4)), we have

$$\#(S(Q,\nu)) \leq C \text{ for each } \nu = 1, \cdots, \nu(Q)\,. \tag{8.16}$$

Thanks to (8.14), we have

$$\#\left\{S(Q,\nu) : Q \in \Lambda^\sharp, \nu \in \{1, \cdots, \nu(Q)\}\right\} \leq C \cdot \#(E)\,. \tag{8.17}$$

To distinguish the roles of the clusters related to $\Lambda^\sharp$, we make the following definition.

Definition 8.6. Let $Q \in \Lambda^\sharp$ and $M \geq 0$. Let $x_Q^\sharp$ be as in Lemma 5.6. Let $S(Q,\nu)$ be as in Definition 8.5. We define

$$\mathcal{K}(Q,\nu,M) := \Gamma_+(x_Q^\sharp, S(Q,\nu), M). \tag{8.18}$$

8.2. Whitney compatibility

The next lemma is similar to Lemma 5.3.

Lemma 8.3. *There exists a universal constant C such that the following holds. Let $Q, Q' \in \Lambda_0$. Let $x_Q^\sharp, x_{Q'}^\sharp$ be as in Lemma 5.6. Let $S(x_Q^\sharp)$ be as in (8.3). Let $S, S' \subset E$. Suppose*

$$S(x_Q^\sharp) \subset (S \cap S')\,. \tag{8.19}$$

Then given $P \in \Gamma_+(x^\sharp_Q, S, M)$ *and* $P' \in \Gamma_+(x^\sharp_{Q'}, S', M)$, *we have*

$$|\partial^\alpha (P - P')(x^\sharp_Q)| \le CM \left(\delta_Q + |x^\sharp_Q - x^\sharp_{Q'}|\right)^{2-|\alpha|} \text{ for } |\alpha| \le 2. \tag{8.20}$$

Proof. Fix P and P' as in the hypothesis. By definition, there exist $F, F' \in C^2_+(\mathbb{R}^2)$ such that the following hold.

- $F|_S = f$ and $F'|_{S'} = f$.
- $\|F\|_{C^2(\mathbb{R}^2)} \le M$ and $\|F'\|_{C^2(\mathbb{R}^2)} \le M$.
- $\mathcal{J}_{x^\sharp_Q} F = P$ and $\mathcal{J}_{x^\sharp_{Q'}} F' = P'$.

Thanks to (8.19), we see that

$$F - F' = 0 \text{ on } S(x^\sharp_Q).$$

By the definition of σ in Section 4, we see that

$$\mathcal{J}_{x^\sharp_Q}(F - F') = P - \mathcal{J}_{x^\sharp_Q} F' \in 2M \cdot \sigma(x^\sharp_Q, S(x^\sharp_Q)).$$

By Lemma 5.7 and the definition of $S(x^\sharp_Q)$ in (8.3), we see that

$$|\partial^\alpha (P - \mathcal{J}_{x^\sharp_Q} F')(x^\sharp_Q)| \le CM\delta_Q^{2-|\alpha|}. \tag{8.21}$$

By the triangle inequality, we have

$$|\partial^\alpha (P - P')(x^\sharp_Q)| \le |\partial^\alpha (P - \mathcal{J}_{x^\sharp_Q} F')(x^\sharp_Q)| + |\partial^\alpha (\mathcal{J}_{x^\sharp_Q} F' - \mathcal{J}_{x^\sharp_{Q'}} F')(x^\sharp_Q)|.$$

Using (8.21) to estimate the first term, and using Taylor's theorem to estimate the second, we see that (8.20) holds. □

Remark 8.2. We note that Lemma 8.3 is a one-sided estimate, in the sense that the right hand side of (8.20) does not contain the lengthscale $\delta_{Q'}$. However, this is remedied once we know that $Q \leftrightarrow Q'$. This is further examined in the next corollary, which states that suitable choices of clusters give rise to Whitney compatible jets.

Corollary 8.1. *There exists a universal constant* C *such that the following holds. Let* $\Lambda_0, \Lambda^\sharp, \Lambda_{KS}$ *be as in Definition 8.1. Let* $\Lambda_{\mathtt{special}}$ *and* μ *be as in Lemma 8.2. For* $Q \in \Lambda_0$, *let* $x^\sharp_Q$ *be as in Lemma 5.6, and let* $S(x^\sharp_Q)$ *be as in (8.3). Suppose* $Q, Q' \in \Lambda_0$ *with* $Q \leftrightarrow Q'$ *and* $P, P' \in \mathcal{P}$ *satisfy one of the following conditions.*

(A) Suppose $Q, Q' \in \Lambda^\sharp$. *Let* $\nu = 1, \cdots, \nu(Q)$ *and* $\nu' = 1, \cdots, \nu(Q')$ *(Definition 8.5). Let* $\mathcal{K}(\cdot,\cdot,\cdot)$ *be as in Definition 8.6. Suppose* $P \in \mathcal{K}(Q, \nu, M)$ *and* $P' \in \mathcal{K}(Q', \nu', M)$.

(B) *Suppose* $Q \in \Lambda^\sharp$ *and* $Q' \in \Lambda_0 \setminus (\Lambda^\sharp \cup \Lambda_{special})$. *Suppose* $P \in \mathcal{K}(Q, \nu, M)$ *(Definition 8.6) for some* $\nu \in \{1, \cdots, \nu(Q)\}$ *(Definition 8.5). Suppose* $P' \in \Gamma_+(x^\sharp_{\mu(Q')}, S_{KS}(\mu(Q')), M)$*, with* $S_{KS}(\mu(Q'))$ *as in* (8.6).

(C) *Suppose* $Q \in \Lambda^\sharp$ *and* $Q' \in \Lambda_{special} \setminus \Lambda^\sharp$. *Suppose* $P \in \mathcal{K}(Q, \nu, M)$ *(Definition 8.6) for some* $\nu \in \{1, \cdots, \nu(Q)\}$ *(Definition 8.5). Suppose* $P' \in \Gamma_+(x^\sharp_{Q'}, S_{special}(Q'), M)$*, with* $S_{special}(Q')$ *as in* (8.7).

(D) *Suppose* $Q, Q' \in \Lambda_0 \setminus (\Lambda^\sharp \cup \Lambda_{special})$. *Suppose* $P \in \Gamma_+(x^\sharp_{\mu(Q)}, S_{KS}(\mu(Q)), M)$ *and suppose* $P' \in \Gamma_+(x^\sharp_{\mu(Q')}, S_{KS}(\mu(Q')), M)$*, with* $S_{KS}(\mu(Q))$ *and* $S_{KS}(\mu(Q'))$ *as in* (8.6).

(E) *Suppose* $Q \in \Lambda_0 \setminus (\Lambda^\sharp \cup \Lambda_{special})$ *and* $Q' \in \Lambda_{special} \setminus \Lambda^\sharp$. *Suppose* $P \in \Gamma_+(x^\sharp_{\mu(Q)}, S_{KS}(\mu(Q)), M)$*, with* $S_{KS}(\mu(Q))$ *as in* (8.6). *Suppose* $P' \in \Gamma_+(x^\sharp_{Q'}, S_{special}(Q'), M)$*, with* $S_{special}(Q')$ *as in* (8.7).

(F) *Suppose* $Q, Q' \in \Lambda_{special} \setminus \Lambda^\sharp$. *Suppose* $P \in \Gamma_+(x^\sharp_Q, S_{special}(Q), M)$ *and* $P' \in \Gamma_+(x^\sharp_{Q'}, S_{special}(Q'), M)$*, with* $S_{special}(Q)$ *and* $S_{special}(Q')$ *as in* (8.7).

Then

$$|\partial^\alpha(P - P')(x^\sharp_Q)|,\ |\partial^\alpha(P - P')(x^\sharp_{Q'})| \le CM\delta_Q^{2-|\alpha|} \text{ for } |\alpha| \le 2.$$

Proof. Thanks to Lemma 5.1 and Lemma 8.2, we know that

$$|x^\sharp_Q - x^\sharp_{Q'}|,\ |x^\sharp_Q - x^\sharp_{\mu(Q)}|,\ |x^\sharp_{\mu(Q)} - x^\sharp_{\mu(Q')}| \le C\delta_Q \text{ for } Q, Q' \in \Lambda_0 \text{ with } Q \leftrightarrow Q'$$

Therefore, by Lemma 8.3 and Taylor's theorem, it suffices to show that in (A)-(F), the sets S, S' in $\Gamma_+(x^\sharp_\star, S, M) \ni P$, $\Gamma_+(x^\sharp_{\star'}, S', M) \ni P'$ satisfy

$$S(x^\sharp_\star) \subset S \cap S', \text{ for some } x^\sharp_\star \in \left\{ x^\sharp_Q, x^\sharp_{Q'}, x^\sharp_{\mu(Q)}, x^\sharp_{\mu(Q')} \right\}. \tag{8.22}$$

We analyze each scenario.

(A) Recall from (8.18) that $\mathcal{K}(Q, \nu, M) = \Gamma_+(x^\sharp_Q, S(Q, \nu), M)$ and $\mathcal{K}(Q', \nu', M) = \Gamma_+(x^\sharp_{Q'}, S(Q', \nu'), M)$. In this scenario, $S = S(Q, \nu)$ and $S' = S(Q', \nu')$. We let $x^\sharp_\star = x^\sharp_Q$.
We see from (8.15) that $S(x^\sharp_Q) \subset S(Q, \nu)$ and $S(x^\sharp_Q) \subset S(Q', \nu')$, since $Q \leftrightarrow Q'$. Therefore, $S(x^\sharp_Q) \subset S(Q, \nu) \cap S(Q', \nu')$. (8.22) follows.

(B) Recall from (8.18) that $\mathcal{K}(Q, \nu, M) = \Gamma_+(x^\sharp_Q, S(Q, \nu), M)$. In this scenario, $S = S(Q, \nu)$ and $S' = S_{KS}(\mu(Q'))$. We let $x^\sharp_\star = x^\sharp_{\mu(Q')}$.
We see from (8.15) that $S(x^\sharp_{\mu(Q')}) \subset S(Q, \nu)$, since $Q \leftrightarrow Q'$. Recall from (8.6) that $S_{KS}(\mu(Q')) = S(x^\sharp_{\mu(Q')})$. (8.22) follows.

(C) Recall from (8.18) that $\mathcal{K}(Q, \nu, M) = \Gamma_+(x^\sharp_Q, S(Q, \nu), M)$. Thus, in this scenario, $S = S(Q, \nu)$ and $S' = S_{special}(Q')$. We let $x^\sharp_\star = x^\sharp_{\mu(Q)}$.

We see from (8.15) that $S(x^\sharp_{\mu(Q)}) \subset S(Q,\nu)$, since $Q \leftrightarrow Q$ by definition. We see from (8.7) that $S(x^\sharp_{\mu(Q)}) \subset S_{\text{special}}(Q')$ since $Q \leftrightarrow Q'$. (8.22) follows.

(D) In the current scenario, $S = S_{KS}(\mu(Q))$ and $S' = S_{KS}(\mu(Q'))$.
By Lemma 8.2, we have $\mu(Q) = \mu(Q')$. Hence, $S = S'$. Taking $x^\sharp_\star = x^\sharp_{\mu(Q)}$, we see from (8.6) that $S(x^\sharp_{\mu(Q)}) = S \cap S'$. (8.22) follows.

(E) In the current scenario, $S = S_{KS}(\mu(Q))$ and $S' = S_{\text{special}}(Q')$. Let $x^\sharp_\star = x^\sharp_{\mu(Q)}$.
Recall from (8.6) that $S(x^\sharp_{\mu(Q)}) = S_{KS}(\mu(Q))$. From (8.7), we see that $S(x^\sharp_{\mu(Q)}) \subset S_{\text{special}}(Q')$, since $Q' \leftrightarrow Q$. (8.22) follows.

(F) In this scenario, $S = S_{\text{special}}(Q)$ and $S' = S_{\text{special}}(Q')$. We let $x^\sharp_\star = x^\sharp_{\mu(Q)}$.
By (8.7), $S(x^\sharp_{\mu(Q)}) \subset S_{\text{special}}(Q)$, since $Q \leftrightarrow Q$ by definition. By (8.7) again, $S(x^\sharp_{\mu(Q)}) \subset S_{\text{special}}(Q')$, since $Q' \leftrightarrow Q$. (8.22) follows.

We have exhausted all the cases. This concludes the proof of the corollary. □

8.3. Local extension problem

The next lemma states that on the correct local scale, the two-dimensional trace norm behaves in a similar way as the one-dimensional trace norm.

Lemma 8.4. *Let $Q \in \Lambda^\sharp$ and let ϕ be as in Lemma 5.4. Let $\overline{S} \subset E \cap Q^*$. Recall the definition of I_Q in (8.11). There exists a universal constant C such that the following hold.*

(A) Let $f : \overline{S} \to [0,\infty)$. Suppose there exists $F \in C^2_+(100Q)$ with $F = f$ on $\overline{S}$, and $|\partial^\alpha F| \le M\delta_Q^{2-|\alpha|}$ on $100Q$ for $|\alpha| \le 2$. Then there exists $\overline{F}_\nu \in C^2_+(I_Q)$ with $\overline{F}_\nu(s) = f(s,\phi(s))$ for each $(s,\phi(s)) \in \overline{S}$, and $|\partial_s^k \overline{F}_\nu| \le CM\delta_Q^{2-k}$ on I_Q for $k \le 2$.

(B) Let $g : \overline{S} \to \mathbb{R}$. Suppose there exists $G \in C^2(100Q)$ with $G = g$ on $\overline{S}$, and $|\partial^\alpha G| \le M\delta_Q^{2-|\alpha|}$ on $100Q$ for $|\alpha| \le 2$. Then there exists $\overline{G} \in C^2(I_Q)$ with $\overline{G}(s) = g(s,\phi(s))$ for each $(s,\phi(s)) \in \overline{S}$, and $|\partial_s^k \overline{G}| \le CM\delta_Q^{2-|\alpha|}$ on I_Q for $k \le 2$.

Proof. We only prove (A) here. The proof for (B) is identical.

Let Φ be as in Lemma 5.5, and let $\Psi = (\Psi_1, \Psi_2) := \Phi^{-1}$. Let F be as in the hypothesis. Consider the function

$$\overline{F}(s) := F \circ \Psi(s, 0)\,.$$

Since $F \ge 0$, we have $\overline{F} \ge 0$. By Lemma 5.5, we have $\overline{F}(s) = f(s,\phi(s))$ for each $(s,\phi(s)) \in \overline{S}$. It remains to estimate the derivatives for $\overline{F}$. Setting $\partial_1 = \partial_s$ and $\partial_2 = \partial_t$, we have

$$\partial_i(F\circ\Psi) = \sum_{k=1}^{2} \partial_i\Psi_k \cdot \partial_k F\circ\Psi, \quad \text{and}$$

$$\partial_{ij}(F\circ\Psi) = \sum_{k,l=1}^{2} C_{kl}\partial_i\Psi_k\cdot\partial_j\Psi_l + \partial_{kl}F\circ\Psi + \sum_{k=1}^{2}\partial_{ij}\Psi_k\cdot\partial_k F\circ\Psi.$$

Therefore, thanks to Lemma 5.5 and the hypothesis $|\partial^\alpha F| \le M\delta_Q^{2-|\alpha|}$, we can conclude that $|\partial_s^k \overline{F}| \le CM\delta_Q^{2-k}$ on I_Q for $k \le 2$. This concludes the proof of the lemma. □

We can think of the next lemma as a re-scaled local finiteness principle (without a prescribed jet). It is essentially a consequence of Theorem 1.A.

Lemma 8.5. *Let $Q \in \Lambda^\sharp$. For each $\nu = 1, \cdots, \nu(Q)$, let $\overline{S}(Q,\nu)$ be as in Definition 8.4.*

(A) Let $f : E\cap Q^* \to [0,\infty)$. Suppose for each ν, there exists $F_\nu \in C^2_+(100Q)$ such that $F_\nu = f$ on $\overline{S}(Q,\nu)$, and $|\partial^\alpha F_\nu| \le M\delta_Q^{2-|\alpha|}$. Then there exist a universal constant C and a function $\widehat{F}_Q \in C^2_+(\mathbb{R}^2)$ such that

(i) $\widehat{F}_Q\big|_{E\cap Q^*} = f$, and
(ii) $|\partial^\alpha \widehat{F}_Q| \le CM\delta_Q^{2-|\alpha|}$ on $100Q$, $|\alpha| \le 2$.

(B) Let $g : E\cap Q^* \to \mathbb{R}$. Suppose for each ν, there exists $G_\nu \in C^2(100Q)$ such that $G_\nu = g$ on $\overline{S}(Q,\nu)$, and $|\partial^\alpha G_\nu| \le M\delta_Q^{2-|\alpha|}$. Then there exist a universal constant C and a function $\widehat{G}_Q \in C^2(\mathbb{R}^2)$ such that

(i) $\widehat{G}_Q\big|_{E\cap Q^*} = g$, and
(ii) $|\partial^\alpha \widehat{G}_Q| \le CM\delta_Q^{2-|\alpha|}$ on $100Q$, $|\alpha| \le 2$.

Proof. We only prove (A) here. The proof for (B) is identical.

If $\#(E\cap Q^*) \le 3$, then $\nu(Q) = 1$ and $\overline{S}(Q,\nu(Q)) = E\cap Q^*$, and the conclusions follow directly from the definition of $\|f\|_{C^2_+(\overline{S}(Q,\nu(Q)))}$. For the rest of the proof, we assume $\#(E\cap Q^*) > 3$.

Up to a rotation, we know that $E\cap Q^* \subset \{(s,\phi(s)) : s\in\mathbb{R}\}$, where ϕ is as in Lemma 5.4. Enumerate $E\cap Q^*$ as in (8.10). For $\nu = 1,\cdots,N(Q)-2$, we set

$$I_\nu := [s_\nu, s_{\nu+2}]. \tag{8.23}$$

We also set

$$I_0 := (-\infty, s_2] \quad\text{and}\quad I_{N(Q)-1} := [s_{N(Q)-1}, \infty). \tag{8.24}$$

Let $\{\overline{\theta}_\nu\}_{\nu=1}^{N(Q)-1}$ be a partition of unity subordinate to the cover $\{I_\nu\}_{\nu=1}^{N(Q)-1}$, such that

$$|\partial_s^k \overline{\theta}_\nu(s)| \le \begin{cases} C|s_\nu - s_{\nu-1}|^{-k} & \text{if } s \in [s_{\nu-1}, s_\nu] \\ C|s_{\nu+1} - s_\nu|^{-k} & \text{if } s \in [s_\nu, s_{\nu+1}] \end{cases} \quad \text{for } k = 0, 1, 2\,. \tag{8.25}$$

Here it is convenient to use $s_0 := -\infty$, $s_{N(Q)+1} = \infty$, and $\infty^0 = 1$. We set

$$\theta_\nu(s,t) := \overline{\theta}_\nu(s) \quad \text{for } \nu = 0, 1, \cdots, N(Q) - 1\,. \tag{8.26}$$

Let $\overline{F}_\nu$ be as in Lemma 8.4 with $\overline{S} = \overline{S}(Q,\nu)$ for $\nu = 1, \cdots, N(Q) - 2$. By Rolle's Theorem, we have

$$|\partial_s^k(\overline{F}_\nu - \overline{F}_{\nu+1})| \le CM\delta_Q^{2-k}|s_{\nu+1} - s_\nu|^{2-k} \tag{8.27}$$

for $\nu = 1, \cdots, N(Q) - 2$, $s \in I_\nu \cap I_{\nu+1}$, and $k \le 2$.

We also set $\overline{F}_0 := \overline{F}_1$ and $\overline{F}_{N(Q)-1} := \overline{F}_{N(Q)-2}$.

Define

$$F_\nu(s,t) := \overline{F}_\nu(s) \quad \text{for } \nu = 0, 1, \cdots, N(Q) - 1\,.$$

Finally, we set

$$\widehat{F}_Q(x) := \sum_{\nu=0}^{N(Q)-1} \theta_\nu(x) \cdot F_\nu(x)\,.$$

It is clear that $\widehat{F}_Q \ge 0$ on $\mathbb{R}^2$ and $\widehat{F}_Q = f$ on $E \cap Q^*$. By construction, $\partial_t\theta_\nu = \partial_t F_\nu \equiv 0$ for each $\nu = 0, \cdots, N(Q) - 1$. Then, using estimates (8.25) and (8.27), we can conclude that $|\partial^\alpha \widehat{F}_Q| \le CM\delta_Q^{2-|\alpha|}$ on $100Q$ for $|\alpha| \le 2$. □

Repeating the proof of Lemma 7.5 and using Lemma 8.5 (A), we have the following result tailored for the matter at hand.

Lemma 8.6. *For each $B_{\min} > 0$ sufficiently large, we can find $B_{\max}$, depending only on $B_{\min}$, such that the following holds. Let $Q \in \Lambda^\sharp$, and let $\mathcal{K}(Q,\nu,M)$ be as in Definition 8.6. Suppose for each $\nu = 1, \cdots, \nu(Q)$, $\mathcal{K}(Q,\nu,M) \ne \varnothing$. Then at least one of the following holds.*

(A) $f(x) \ge B_{\min} M\delta_Q^2$ *for all* $x \in E \cap Q^*$.
(B) $f(x) \le B_{\max} M\delta_Q^2$ *for all* $x \in E \cap Q^*$.

Proof. Suppose (A) holds. There is nothing to prove.

Suppose (A) fails. We write $B_0 = B_{\min}$ and we fix the number B_0 throughout.

By assumption, there exists $\widehat{x} \in E \cap Q^*$ with $f(\widehat{x}) < B_{\min}\delta_Q^2$. There exists $\widehat{\nu} \in \{1, \cdots, N(Q) - 2\}$ such that $\widehat{x} \in \overline{S}(Q,\widehat{\nu}) \subset S(Q,\widehat{\nu})$. By assumption, $K(Q,\widehat{\nu},M) \ne \varnothing$, so there exists $\widehat{F} \in C^2_+(\mathbb{R}^2)$ with $\widehat{F} = f$ on $S(Q, \widehat{\nu})$, $\|\widehat{F}\|_{C^2(\mathbb{R}^2)} \le M$, and $\mathcal{J}_{x^\sharp_Q}\widehat{F} \in \mathcal{K}(Q, \widehat{\nu}, M)$. By Lemma 7.2, we have

$$|\nabla F(\widehat{x})| \le C\sqrt{MF(\widehat{x})} \le CM\sqrt{B_0}\delta_Q\,.$$

Then Taylor's theorem implies

$$F(x^\sharp_Q) \le CM\left(B_0\delta_Q^2 + \sqrt{B_0}\,|x^\sharp_Q - \widehat{x}|\delta_Q\right) \le CM(\sqrt{B_0}+1)^2\delta_Q^2\,.$$

Let $x_0 \in E \cap Q^*$. Then there exists $\nu(x_0) \in \{1, \cdots N(Q)\}$ such that $x' \in S(Q, \nu(x_0))$.

By assumption, $\mathcal{K}(Q, \nu(x_0), M) \neq \varnothing$. Pick $P \in \mathcal{K}(Q, \nu(x_0), M)$. By Corollary 8.1, we see that

$$|\mathcal{J}_{x^\sharp_Q}\widehat{F} - P(x^\sharp_Q)| \le CM\delta_Q^2\,.$$

Therefore, we have $P(x^\sharp_Q) \le CM(\sqrt{B_0}+1)^2\delta_Q^2$. By the definition of $\mathcal{K}(Q, \nu(x_0), M)$, there exists $F \in C^2_+(\mathbb{R}^2)$ with $F(x_0) = f(x_0)$ and $\mathcal{J}_{x^\sharp_Q}F = P$. In particular, by Lemma 7.2 and Taylor's Theorem, we have

$$|\nabla F(x)| \le CM(\sqrt{B_0}+1)\delta_Q \text{ for all } x \in Q^*\,.$$

By Taylor's theorem again, we have

$$F(x_0) \le CM(\sqrt{B_0}+1)\delta_Q^2\,.$$

Since $x_0 \in E \cap Q^*$ was chosen arbitrarily, (B) follows. □

The next lemma mirrors Lemma 7.3. It says the following. When the local data is big, $\mathcal{K}$ can be viewed as a translate of $\sigma^\sharp$. When the local data is small, $\mathcal{K}$ contains not much more information than the zero jet.

Lemma 8.7. *Let $Q \in \Lambda^\sharp$. Let $\mathcal{K}(Q, \nu, M)$ be as in Definition 8.6. Suppose $\mathcal{K}(Q, \nu, M) \neq \varnothing$ for each $\nu = 1, \cdots, \nu(Q)$.*

(A) There exists a number $B > 0$ exceeding a universal constant such that the following holds. Suppose $f(x) \ge BM\delta_Q^2$ for all $x \in E \cap Q^*$. Then $\mathcal{K}(Q, \nu, M) + M \cdot \sigma^\sharp(x^\sharp_Q, 4k) \subset \mathcal{K}(Q, \nu, CM)$ for each $\nu = 1, \cdots, \nu(Q)$. Here, C is a universal constant.

(B) Let $A > 0$. Suppose $f(x) \le AM\delta_Q^2$ for all $x \in E \cap Q^*$. Then $0 \in \mathcal{K}(Q, \nu, A'M)$ for each $\nu \in \{1, \cdots, \nu(Q)\}$. Here the number A' depends only on A.

Proof. We adapt the proof of Lemma 7.3 with $\mathcal{K}$ in place of $\Gamma^\sharp_+$, and use the fact that $S(Q, \nu)$ contains $\overline{S}(Q, \nu) \subset E \cap Q^*$ (see Definition 8.5). We include the relevant steps here for completeness.

Fix $\nu \in \{1, \cdots, \nu(Q)\}$.

We begin with (A). Let $B > 0$ be a sufficiently large number.

By (8.3), we have $\overline{S}(Q, \nu(Q)) \subset S(Q, \nu)$. Let $P \in \mathcal{K}(Q, \nu, M)$. Repeating the proof of Claim 7.1 in Lemma 7.3, we see that $P(x_Q^\sharp) \geq C(\sqrt{B} - 1/2)^2 M\delta_Q^2$.

By (4.1), Lemma 4.8, and Definition 8.6, there exists a Whitney field

$$\vec{P} = (P, (P^x)_{x\in S(Q,\nu)}) \in W_+^2(\{x_Q^\sharp\} \cup S(Q, \nu))$$

such that $P^x = f(x)$ for all $x \in S(Q, \nu)$, and $\|\vec{P}\|_{W_+^2(\{x_Q^\sharp\}\cup S(Q,\nu))} \leq CM$.

Let $\tilde{P} \in M \cdot \sigma^\sharp(x_Q^\sharp, 4k)$. By Lemma 5.7, $\tilde{P} \in CM \cdot \mathcal{B}(x, \delta_Q)$.

Consider the Whitney field

$$\vec{P}' := (P + \tilde{P}, (P^x)_{x\in S(Q,\nu)}) \,.$$

By the same argument as in the proof of Lemma 7.3, we can verify that

$$\vec{P}' \in W_+^2(\{x_Q^\sharp\} \cup S(Q, \nu)) \quad \text{and } \|\vec{P}'\|_{W_+^2(\{x_Q^\sharp\}\cup S(Q,\nu))} \leq CM \,.$$

Part (A) then follows from Lemma 4.9.

Now we turn to (B).

Let $P \in \mathcal{K}(Q, \nu, M)$. Repeating the argument for Claim 7.2, we have $P(x_Q^\sharp) \leq C(\sqrt{A}+1)^2 M\delta_Q^2$. By the definitions of Γ_+ and $\mathcal{K}$ (see (4.1) and (8.18)), there exists $F \in C_+^2(\mathbb{R}^2)$ with $F(x) = f(x)$ for all $x \in S(Q, \nu)$, $\|F\|_{C_+^2(\mathbb{R}^2)} \leq CM$, and $\mathcal{J}_{x_Q^\sharp} F = P$. By Lemma 7.2 and Taylor's theorem, we have

$$|\partial^\alpha F(x)| \leq CA'' M\delta_Q^{2-|\alpha|} \text{ for all } x \in Q^* \text{ and } |\alpha| \leq 2 \,. \tag{8.28}$$

Here, A'' depends only on A.

Let $\psi \in C_+^2(\mathbb{R}^2)$ be a cutoff function such that $\psi \equiv 1$ near $x_Q^\sharp$, $\psi \equiv 0$ outside of $B(x_Q^\sharp, \frac{c_{\text{rep}}\delta_Q}{100})$, and $|\partial^\alpha \psi| \leq C\delta_Q^{-|\alpha|}$.

We set

$$\tilde{F} := (1 - \psi) \cdot F \,.$$

It is clear that $F \geq 0$ on $\mathbb{R}^2$, $\tilde{F} = f$ on $S(Q, \nu)$, and $\mathcal{J}_{x_Q^\sharp} \tilde{F} \equiv 0$. Using (8.28), we see that $\|\tilde{F}\|_{C^2(\mathbb{R}^2)} \leq A' M$. This proves part (B) and concludes the proof of Lemma 8.7. □

The next lemma mirrors Lemma 7.1. It solves the local interpolation with a prescribed jet in $\mathcal{K}$, so that they can be patched together by a partition of unity.

Lemma 8.8. *Let $Q \in \Lambda^\sharp$. Let $\mathcal{K}(Q, \nu, M)$ be as in Definition 8.6. Suppose $\mathcal{K}(Q, \nu, M) \neq \varnothing$ for each $\nu = 1, \cdots, \nu(Q)$. Then there exist a universal constant C and a function $F_Q \in C_+^2(100Q)$ such that*

(A) $F_Q\big|_{E\cap Q^*} = f$,
(B) $\|F_Q\|_{C^2(\mathbb{R}^2)} \le CM$, *and*
(C) $\mathcal{J}_{x_Q^\sharp} F_Q \in \mathcal{K}(Q, \nu(Q), CM)$.

Proof. We adapt the proof of Lemma 7.1 with the following modifications:

- We use Lemma 8.7 in place of Lemma 7.3.
- We use Lemma 8.6 in place of Lemme 7.5.
- We use $\mathcal{K}$ in place of $\Gamma_+^\sharp$, and the condition $\Gamma_+^\sharp(x_Q^\sharp, 4k, M) \neq \varnothing$ is replaced by $\mathcal{K}(Q, \nu, M) \neq \varnothing$ for each $\nu = 1, \cdots, \nu(Q)$. See Lemma 8.5 and Lemma 8.8.

Here we present the relevant steps for completeness.

Fix $Q \in \Lambda^\sharp$.

Suppose $\#(E \cap Q^*) \le 3$. Recall Definitions 4.1, 8.4, 8.5, and 8.6. By assumption, $\mathcal{K}(Q, \nu(Q), M) = \Gamma_+(x_Q^\sharp, S(Q, \nu(Q)), M) \neq \varnothing$. Pick $P \in \Gamma_+(x_Q^\sharp, S(Q, \nu(Q)), M)$. By the definition of Γ_+, there exists $F_Q \in C_+^2(\mathbb{R}^2)$ such that $F_Q\big|_{S(Q,\nu(Q))} = f$, $\|F_Q\|_{C^2(\mathbb{R}^2)} \le M$, and $\mathcal{J}_{x_Q^\sharp} F_Q \in \Gamma_+(x_Q^\sharp, S(Q, \nu(Q)), M)$. Since $S(Q, \nu) \supset \overline{S}(Q, \nu(Q))$ and $\overline{S}(Q, \nu(Q)) = \overline{S}(Q, 1) = E \cap Q^*$ in this case, the conclusions follow.

From now on, we assume $\#(E \cap Q^*) > 3$.

Let $B_{\min} > 0$ be sufficiently large, and in particular, $B_{\min} > B$, where B is as in Lemma 8.7. Let $B_{\max}$ be given as in Lemma 7.5 with such $B_{\min}$.

Thanks to Lemma 8.6, each $Q \in \Lambda^\sharp$ falls into at least one of the following cases.

(i) $f(x) \ge B_{\min} M \delta_Q^2$ for all $x \in E \cap Q^*$.
(ii) $f(x) \le B_{\max} M \delta_Q^2$ for all $x \in E \cap Q^*$.

We treat (i) first.

Since $\#(E \cap Q^*) \ge 3$, we may select distinct $x_1, x_2 \in E \cap S(Q, \nu(Q)) \cap Q^*$. Pick $P \in \mathcal{K}(Q, \nu(Q), M)$. Let $P^\sharp$ be the unique affine polynomial that interpolates the points $(x_1, f(x_1))$, $(x_2, f(x_2))$, and $(x_Q^\sharp, P(x_Q^\sharp))$. We may repeat the argument for Claim 7.3 and use Lemma 8.7 to show that

$$P^\sharp \in \mathcal{K}(Q, \nu(Q), CM) \quad \text{and} \quad P^\sharp(x_Q^\sharp) \ge C B_{\min} M \delta_Q^2 \,.$$

This, together with Lemma 7.2, implies that

$$\text{dist}\left(x_Q^\sharp, \left\{P^\sharp = 0\right\}\right) \ge C\sqrt{B_{\min}}\delta_Q \,.$$

Therefore, we have

$$P^\sharp(x) \ge C B_{\min} M \delta_Q^2 \text{ for all } x \in 100Q \,. \tag{8.29}$$

Let $g(x) := f(x) - P^\sharp(x)$ for each $x \in E \cap Q^*$. Note that g is not necessarily nonnegative. Since $P^\sharp \in \mathcal{K}(Q, \nu(Q), CM)$, there exists a function $F \in C^2_+(\mathbb{R}^2)$ such that $F|_{S(Q,\nu)} = f$, $\|F\|_{C^2(\mathbb{R}^2)} \le CM$, and $\mathcal{J}_{x_Q^\sharp} F = P$. This, together with the assumption $\mathcal{K}(Q, \nu, M) \neq \varnothing$ and Rolle's theorem, implies that for each $\nu \in 1, \cdots, \nu(Q)$, there exists $G_\nu \in C^2(\mathbb{R}^2)$ such that

$$G_\nu = g \text{ on } \overline{S}(Q,\nu) \quad \text{and} \quad |\partial^\alpha G(x)| \le CM\delta_Q^{2-|\alpha|} \quad \text{for all } x \in 100Q\,,\ |\alpha| \le 2\,. \tag{8.30}$$

By Lemma 8.5(B), there exists $G \in C^2(100Q)$ such that

$$G|_{E\cap Q^*} = g \quad \text{and} \quad |\partial^\alpha G(x)| \le CM\delta_Q^{2-m} \quad \text{for all } x \in 100Q\,,\ |\alpha| \le 2\,. \tag{8.31}$$

Let $\psi \in C^2_+(\mathbb{R}^2)$ be a cutoff function such that

$$\psi \equiv 1 \text{ near } x_Q^\sharp,\ \psi \equiv 0 \text{ outside of } B(x_Q^\sharp, \frac{c_{\rm rep}\delta_Q}{100}), \text{ and } |\partial^\alpha \psi| \le C\delta_Q^{2-|\alpha|}. \tag{8.32}$$

Consider the function

$$F_Q := P^\sharp + (1-\psi)\cdot G\,.$$

- By (8.29) and (8.31), we have $F_Q \ge 0$ on $100Q$.
- Since supp (ψ) is disjoint from $E \cap Q^*$, we have $F_Q(x) = P^\sharp(x) + g(x) = f(x)$ for each $x \in E \cap Q^*$. (A) is satisfied.
- Since $P^\sharp \in \mathcal{K}(Q, \nu(Q), CM)$, we have

  $$\|P^\sharp\|_{C^2(100Q)} \le CM\,.$$

  By (8.31) and (8.32), we have

  $$\|(1-\psi)\cdot G\|_{C^2(100Q)} \le CM\,.$$

  Conclusion (B) then follows from the triangle inequality.
- Since $\psi \equiv 1$ near $x_Q^\sharp$, we have $\mathcal{J}_{x_Q^\sharp} F_Q = \mathcal{J}_{x_Q^\sharp} P^\sharp + 0 = P^\sharp \in \mathcal{K}(Q, \nu(Q), CM)$. (C) is satisfied.

This proves case (i).

Now we turn to case (ii).

Recall the hypothesis $\|f\|_{C^2_+(\overline{S}(Q_\nu))} \le M$ for each ν. By definition, for each $\nu = 1, \cdots, N(Q)$, there exists $F_\nu \in C^2_+(\mathbb{R}^2)$ such that $F_\nu = f$ on $\overline{S}(Q,\nu)$ and $\|F_\nu\|_{C^2(\mathbb{R}^2)} \le CM$. Since $f(x) \le B_{\max}\delta_Q^2$ for all $x \in E \cap Q^*$, by Lemma 7.2, we have

$$|\partial^\alpha F_\nu(x)| \le CM\delta_Q^{2-|\alpha|} \quad \text{for all } x \in 100Q\,.$$

Therefore, the hypotheses of Lemma 8.5(A) are satisfied, and there exists $F \in C^2_+(\mathbb{R}^2)$ such that $F|_{E\cap Q^*} = f$ and

$$|\partial^\alpha F(x)| \le CM\delta_Q^{2-|\alpha|} \quad \text{for all } x \in 100Q\,. \tag{8.33}$$

Let ψ satisfy (8.32). Consider the function

$$F_Q := (1-\psi)\cdot F\,.$$

- Since $F \ge 0$ and $0 \le \psi \le 1$, we have $F_Q \ge 0$ on $100Q$.
- Since $\operatorname{supp}(\psi)$ is away from $E \cap Q^*$, we have $F_Q = f$ on $E \cap Q^*$. (A) is satisfied.
- Thanks to (8.32) and (8.33), we have $\|F_Q\|_{C^2(100Q)} \le CM$. (B) is satisfied.
- Since $\psi \equiv 1$ near $x_Q^\sharp$, we have $\mathcal{J}_{x_Q^\sharp} F_Q \equiv 0 \in \mathcal{K}(Q, \nu(Q), CM)$, thanks to Lemma 8.7. (C) is satisfied.

This concludes the treatment for case (ii) and the proof of Lemma 8.8. □

8.4. Proof of Theorem 5

Now we define the S_ℓ's in the statement of Theorem 5.

Definition 8.7. Recall the definitions of $\Lambda_0, \Lambda^\sharp, \Lambda_{KS}, \Lambda_{special}$ in Definition 8.1 and Lemma 8.2. We set

$$\mathcal{S}^\sharp := \mathcal{S}_1^\sharp \cup \mathcal{S}_2^\sharp \cup \mathcal{S}_3^\sharp, \tag{8.34}$$

where

- $\mathcal{S}_1^\sharp := \left\{S(Q,1),\cdots,S(Q,\nu(Q)) \;:\; Q \in \Lambda^\sharp\right\}$ with $S(Q,\nu)$ as in (8.15),
- $\mathcal{S}_2^\sharp := \{S_{KS}(Q) : Q \in \Lambda_{KS}\}$ with $S_{KS}(Q)$ as in (8.6), and
- $\mathcal{S}_3^\sharp := \{S_{special}(Q) : Q \in \Lambda_{special}\}$ with $S_{special}(Q)$ as in (8.7).

Proof of Theorem 5. Let $\mathcal{S}^\sharp$ be as in (8.34). We enumerate

$$\mathcal{S}^\sharp := \{S_\ell : \ell = 1,\cdots,L\}\,.$$

We claim that the list $S_1,\cdots,S_L$ satisfies the conclusions of Theorem 5.

We examine (A):

- Thanks to (8.16), we have $\#(S_\ell) \le C$ for $S_\ell \in \mathcal{S}_1^\sharp$.
- Thanks to (8.8), we have $\#(S_\ell) \le C$ for $S_\ell \in \mathcal{S}_2^\sharp$.
- Thanks to (8.9), we have $\#(S_\ell) \le C$ for $S_\ell \in \mathcal{S}_3^\sharp$.

Therefore, conclusion (A) holds.

Now we examine (B):

- Thanks to (8.17), $\#(\mathcal{S}_1^\sharp) \leq C \cdot \#(E)$.
- Thanks to Lemma 8.1, $\#(\mathcal{S}_2^\sharp) \leq C \cdot \#(E)$.
- Thanks to Lemma 8.2, $\#(\mathcal{S}_3^\sharp) \leq C \cdot \#(E)$.

Therefore, conclusion (B) holds.

We now turn to conclusion (C). Set

$$M := \max_{\ell=1,\cdots,L} \|f\|_{C^2_+(S_\ell)}$$

It suffices to show that there exists $F \in C^2_+(\mathbb{R}^2)$ such that $F|_E = f$ and $\|F\|_{C^2(\mathbb{R}^2)} \leq CM$.

By the definition of M, we have $\|f\|_{C^2_+(S_\ell)} \leq M$ for all $\ell = 1, \cdots, L$. This implies the following.

- Recall Definitions 8.4, 8.5, and 8.6. For each $Q \in \Lambda^\sharp$, we have

$$\mathcal{K}(Q, \nu, CM) \neq \varnothing \quad \text{and} \quad \|f\|_{C^2_+(\overline{S}(Q,\nu))} \leq M \text{ for } \nu = 1, \cdots, \nu(Q)\,.$$

  This follows from the fact that $S(Q,\nu) \in \mathcal{S}_1^\sharp \subset \mathcal{S}^\sharp$ for $\nu = 1, \cdots, \nu(Q)$ and $\mathcal{K}(Q, \nu, CM) = \Gamma_+(x_Q^\sharp, S(Q,\nu), CM)$ (see Definition 8.6). Therefore, the hypotheses of Lemma 8.8 are satisfied.
- For $Q \in \Lambda_{KS}$ and $x_Q^\sharp$ as in Lemma 5.6, $\Gamma_+(x_Q^\sharp, S_{KS}(Q), CM) \neq \varnothing$. This follows from the fact that $S_{KS}(Q) \in \mathcal{S}_2^\sharp \subset \mathcal{S}^\sharp$ for $Q \in \Lambda_{KS}$ (see Definition 8.7).
- For $Q \in \Lambda_{special}$ and $x_Q^\sharp$ as in Lemma 5.6, $\Gamma_+(x_Q^\sharp, S_{\text{special}}(Q), CM) \neq \varnothing$. This follows from the fact that $S_{\text{special}}(Q) \in \mathcal{S}_3^\sharp \subset \mathcal{S}^\sharp$ for $Q \in \Lambda_{special}$ (see Definition 8.7).

We distinguish three types of squares $Q \in \Lambda_0$.

Type 1 Suppose $E \cap Q^* \neq \varnothing$, that is, $Q \in \Lambda^\sharp$. We set $F_Q^\sharp := F_Q$, where F_Q is as in Lemma 8.8. In particular, we have

$$P_Q^\sharp := \mathcal{J}_{x_Q^\sharp} F_Q^\sharp \in \mathcal{K}(Q, \nu(Q), CM) = \Gamma_+(x_Q^\sharp, S(Q, \nu(Q)), CM), \tag{8.35}$$

with $x_Q^\sharp$ as in Lemma 5.6 and $S(Q, \nu(Q))$ as in (8.15).

Type 2 Suppose $E \cap Q^* = \varnothing$ but $\delta_Q < 1$. Let $\Lambda_{special}, \mu$ be as in Lemma 8.2.

- Suppose $Q \notin \Lambda_{special}$. Pick

$$P_Q^\sharp \in \Gamma_+(x_{\mu(Q)}^\sharp, S_{KS}(\mu(Q)), CM), \tag{8.36}$$

with $x^\sharp_{\mu(Q)}$ as in Lemma 5.6 and $S_{KS}(\mu(Q))$ as in (8.6). We set $F^\sharp_Q := \mathcal{W}^{x^\sharp_{\mu(Q)}}(P^\sharp_Q)$, where $\mathcal{W}^{x^\sharp_{\mu(Q)}}$ is as in Lemma 4.9.

- Suppose $Q \in \Lambda_{special}$. Pick

$$P^\sharp_Q \in \Gamma_+(x^\sharp_Q, S_{\text{special}}(Q), CM), \tag{8.37}$$

with $x^\sharp_Q$ as in Lemma 5.6 and $S_{\text{special}}(Q)$ as in (8.7). We set $F^\sharp_Q := \mathcal{W}^{x^\sharp_Q}(P^\sharp_Q)$, where $\mathcal{W}^{x^\sharp_Q}$ is as in Lemma 4.9.

Type 3 Suppose $E \cap Q^* = \varnothing$ and $\delta_Q = 1$. We set $F^\sharp_Q :\equiv 0$.

To wit, we associate Type 1 squares with clusters in $\mathcal{S}^\sharp_1$, Type 2 non-special squares with clusters in $\mathcal{S}^\sharp_2$, and Type 2 special squares with clusters in $\mathcal{S}^\sharp_3$.

Let $\{\theta_Q : Q \in \Lambda_0\}$ be a C^2 partition of unity that is CZ compatible with Λ_0.

We set

$$F(x) := \sum_{Q \in \Lambda_0} \theta_Q(x) \cdot F^\sharp_Q(x)\,.$$

By construction, $F^\sharp_Q \geq 0$ on $100Q$ and $F^\sharp_Q\big|_{E\cap Q^*} = f$ for each $Q \in \Lambda_0$. Therefore, $F(x) \geq 0$ and $F = f$ on E.

Now we estimate the derivatives of F.

Let $x \in \mathbb{R}^2$. Then there exists $Q \in \Lambda_0$ such that $Q \ni x$. We have

$$\begin{aligned}\partial^\alpha F(x) &= \sum_{Q' \leftrightarrow Q} \partial^\alpha F^\sharp_{Q'}(x) \cdot \theta_{Q'}(x) \\ &\quad + \sum_{Q' \leftrightarrow Q} \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}(F^\sharp_{Q'} - F^\sharp_Q)(x) \cdot \partial^\beta \theta_{Q'}(x)\,.\end{aligned} \tag{8.38}$$

Claim 8.1. *Fix $x \in \mathbb{R}^2$. Let $Q \ni x$, and let $Q' \in \Lambda_0$ with $Q' \leftrightarrow Q$. Then*

$$|\partial^\alpha(F^\sharp_Q - F^\sharp_{Q'})(x)| \leq CM\delta_Q^{2-|\alpha|} \text{ for } |\alpha| \leq 2. \tag{8.39}$$

Suppose the claim is true. Then applying Lemma 4.9, Lemma 8.8, and (8.39) to estimate (8.38), we can conclude that $\|F\|_{C^2(\mathbb{R}^2)} \leq CM$.

Therefore, it suffices to prove Claim 8.1.

Proof of Claim 8.1. By the triangle inequality, we can write

$$\begin{aligned}|\partial^\alpha(F^\sharp_Q - F^\sharp_{Q'})(x)| &\leq |\partial^\alpha(F^\sharp_Q - \mathcal{J}_{x^\sharp_Q} F^\sharp_Q)(x)| + |\partial^\alpha(F^\sharp_{Q'} - \mathcal{J}_{x^\sharp_{Q'}} F^\sharp_{Q'})(x)| \\ &\quad + |\partial^\alpha(\mathcal{J}_{x^\sharp_Q} F^\sharp_Q - \mathcal{J}_{x^\sharp_{Q'}} F^\sharp_{Q'})(x)| \\ &=: \eta_1 + \eta_2 + \eta_3.\end{aligned} \tag{8.40}$$

By Lemma 4.9, Lemma 8.8, and Taylor's theorem, we have

$$\eta_1 + \eta_2 \le CM\delta_Q^{2-|\alpha|}. \tag{8.41}$$

We want to show that

$$\eta_3 = |\partial^\alpha(\mathcal{J}_{x_Q^\sharp} F_Q^\sharp - \mathcal{J}_{x_{Q'}^\sharp} F_{Q'}^\sharp)(x)| = |\partial^\alpha(P_Q^\sharp - P_{Q'}^\sharp)(x)| \le CM\delta_Q^{2-|\alpha|}. \tag{8.42}$$

We consider the following cases.

Case 1 Suppose either Q or Q' is of Type 3. Then (8.42) follows from Lemma 4.9, Lemma 8.8, and Taylor's theorem.

Case 2 Suppose both Q and Q' are of Type 1, that is, $Q, Q' \in \Lambda^\sharp$. Then (8.42) follows from (8.35), scenario (A) of Corollary 8.1, and Taylor's theorem.

Case 3 Suppose one of Q, Q' is of Type 1 and the other is of Type 2. Without loss of generality, we may assume $Q \in \Lambda^\sharp$ and $Q' \in \Lambda_0$. Recall $\Lambda_{\mathsf{special}}$ from Lemma 8.2.

- Case 3-a Suppose $Q' \notin \Lambda_{\mathsf{special}}$. Then (8.42) follows from (8.35), (8.36), scenario (B) of Corollary 8.1, and Taylor's theorem.
- Case 3-b Suppose $Q' \in \Lambda_{\mathsf{special}}$. Then (8.42) follows from (8.35), (8.37), scenario (C) of Corollary 8.1, and Taylor's theorem.

Case 4 Suppose both Q, Q' are of Type 2.

- Case 4-a Suppose $Q, Q' \notin \Lambda_{\mathsf{special}}$. Then (8.42) follows from (8.36), scenario (D) of Corollary 8.1, and Taylor's theorem.
- Case 4-b Suppose $Q \in \Lambda_{\mathsf{special}}$ and $Q' \notin \Lambda_{\mathsf{special}}$. Then (8.42) follows from (8.36), (8.37), scenario (E) of Corollary 8.1, and Taylor's theorem.
- Case 4-c Suppose $Q, Q' \in \Lambda_{\mathsf{special}}$. Then (8.42) follows from (8.37), scenario (F) of Corollary 8.1, and Taylor's theorem.

This proves Claim 8.1. □

The proof of Theorem 5 is now complete. □

References

[1] Y. Brudnyi, P. Shvartsman, The traces of differentiable functions to subsets of $\mathbb{R}^n$, in: Linear and Complex Analysis. Problem Book 3. Part II, in: Lecture Notes in Mathematics, vol. 1574, Springer-Verlag, Berlin, 1994, pp. 279–281.

[2] Y. Brudnyi, P. Shvartsman, Generalizations of Whitney's extension theorem, Int. Math. Res. Not. 1994 (3) (1994) 129–139, 11 pp. (electronic).

[3] Y. Brudnyi, P. Shvartsman, Whitney's extension problem for multivariate $C^{1,\omega}$-functions, Trans. Am. Math. Soc. 353 (6) (2001) 2487–2512 (electronic).

[4] J. Carruth, A. Frei-Pearson, A. Israel, B. Klartag, A coordinate-free proof of the finiteness principle for Whitney's extension problem, Rev. Mat. Iberoam. (2020), online.
[5] C. Fefferman, A sharp form of Whitney's extension theorem, Ann. Math. (2) 161 (1) (2005) 509–577.
[6] C. Fefferman, Interpolation and extrapolation of smooth functions by linear operators, Rev. Mat. Iberoam. 21 (1) (2005) 313–348.
[7] C. Fefferman, Fitting a C^m-smooth function to data. III, Ann. Math. (2) 170 (1) (2009) 427–441.
[8] C. Fefferman, Nearly optimal interpolation of data in $C^2(\mathbb{R}^2)$. Part I, Rev. Mat. Iberoam. 28 (2) (2012) 415–533.
[9] C. Fefferman, A. Israel, Fitting Smooth Functions to Data, CBMS Regional Conference Series in Mathematics, American Mathematical Society, 2020, in press.
[10] C. Fefferman, A. Israel, G.K. Luli, Sobolev extension by linear operators, J. Am. Math. Soc. 27 (1) (2014) 69–145.
[11] C. Fefferman, A. Israel, G.K. Luli, Finiteness principles for smooth selections, Geom. Funct. Anal. 26 (2) (2016) 422–477.
[12] C. Fefferman, A. Israel, G.K. Luli, Interpolation of data by smooth non-negative functions, Rev. Mat. Iberoam. 33 (1) (2016) 305–324.
[13] C. Fefferman, B. Klartag, Fitting a C^m-smooth function to data. I, Ann. Math. (2) 169 (1) (2009) 315–346.
[14] C. Fefferman, B. Klartag, Fitting a C^m-smooth function to data. II, Rev. Mat. Iberoam. 25 (1) (2009) 49–273.
[15] W. Hassler, Analytic extensions of differentiable functions defined in closed sets, Trans. Am. Math. Soc. 36 (1) (1934) 63–89.
[16] W. Hassler, Differentiable functions defined in closed sets. I, Trans. Am. Math. Soc. 36 (2) (1934) 369–387.
[17] W. Hassler, Functions differentiable on the boundaries of regions, Ann. Math. (2) 35 (3) (1934) 482–485.
[18] A. Israel, A bounded linear extension operator for $L^{2,p}(\mathbb{R}^2)$, Ann. Math. 178 (1) (2013) 183–230.
[19] F. Jiang, G.K. Luli, $C^2(\mathbb{R}^2)$ nonnegative extension by bounded-depth operators, in press.
[20] F. Jiang, G.K. Luli, Algorithms for nonnegative $C^2(\mathbb{R}^2)$ interpolation, in press.
[21] P. Shvartsman, The traces of functions of two variables satisfying to the Zygmund condition, in: Studies in the Theory of Functions of Several Real Variables, Yaroslav. Gos. Univ., Yaroslavl, 1982, pp. 145–168 (in Russian).
[22] P. Shvartsman, Traces of functions of Zygmund class, Sib. Mat. Zh. (5) (1987) 203–215 (in Russian). English transl. in Sib. Math. J. 28 (1987) 853–863.
[23] P. Shvartsman, The Whitney extension problem and Lipschitz selections of set-valued mappings in jet-spaces, Trans. Am. Math. Soc. 360 (10) (2008) 5529–5550.
[24] P. Shvartsman, Extension criteria for homogeneous Sobolev spaces of functions of one variable, Rev. Mat. Iberoam. (2020), in press.
[25] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Monographs in Harmonic Analysis, Princeton University Press, Princeton, NJ, 1970.
[26] R. Tyrrell Rockafellar, Convex Analysis. Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks.

Advances in Mathematics 385 (2021) 107756

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Algorithms for nonnegative $C^2(\mathbb{R}^2)$ interpolation

Fushuai Jiang, Garving K. Luli *

ARTICLE INFO

Article history:
Received 11 February 2021
Accepted 28 March 2021
Available online 28 April 2021
Communicated by C. Fefferman

Keywords:
Nonnegative data interpolation
Shape preservation
Interpolation algorithm

ABSTRACT

Let $E \subset \mathbb{R}^2$ be a finite set, and let $f : E \to [0, \infty)$. In this paper, we address the algorithmic aspects of nonnegative C^2 interpolation in the plane. Specifically, we provide an efficient algorithm to compute a nonnegative $C^2(\mathbb{R}^2)$ extension of f with norm within a universal constant factor of the least possible. We also provide an efficient algorithm to approximate the trace norm.

1. Introduction

For integers $m \geq 0, n \geq 1$, we write $C^m(\mathbb{R}^n)$ to denote the Banach space of m-times continuously differentiable real-valued functions such that the following norm is finite

$$\|F\|_{C^m(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} \max_{|\alpha| \leq m} |\partial^\alpha F(x)|.$$

We write $C^m_+(\mathbb{R}^n)$ to denote the collection of nonnegative functions in $C^m(\mathbb{R}^n)$. Let $E \subset \mathbb{R}^n$ be finite. We write $C^m_+(E)$ to denote the collection of functions $f : E \to [0, \infty)$. If S is a finite set, we write $\#(S)$ to denote the number of elements in S. We use C to denote constants that depend only on m and n.

* Corresponding author.
E-mail addresses: fsjiang@math.ucdavis.edu (F. Jiang), kluli@math.ucdavis.edu (G.K. Luli).

https://doi.org/10.1016/j.aim.2021.107756

In this paper, we provide algorithmic solutions to the following problems for $m = n = 2$. These algorithms were announced in [12,13].

Problem 1. *Let* $E \subset \mathbb{R}^n$ *be a finite set. Let* $f : E \to [0, \infty)$. *Compute the order of magnitude of*

$$\|f\|_{C^m_+(E)} := \inf \left\{ \|F\|_{C^m(\mathbb{R}^n)} : F|_E = f \text{ and } F \geq 0 \right\} . \tag{1.1}$$

Problem 2. *Let* $E \subset \mathbb{R}^n$ *be a finite set. Let* $f : E \to [0, \infty)$. *Compute a nonnegative function* $F \in C^m(\mathbb{R}^n)$ *such that* $F|_E = f$ *and* $\|F\|_{C^m(\mathbb{R}^n)} \leq C\|f\|_{C^m_+(E)}$.

By "order of magnitude" we mean the following: Two quantities M and $\tilde{M}$ determined by E, f, m, n are said to have the same order of magnitude provided that $C^{-1}M \leq \tilde{M} \leq CM$, with C depending only on m and n. To compute the order of magnitude of $\tilde{M}$ is to compute a number M such that M and $\tilde{M}$ have the same order of magnitude.

By "computing a function F" from (E, f), we mean the following: After processing the input (E, f), we are able to accept queries consisting of a point $x \in \mathbb{R}^n$, and produce a list of numbers $(f_\alpha(x) : |\alpha| \leq m)$. The algorithm "computes the function F" if for each $x \in \mathbb{R}^n$, we have $\partial^\alpha F(x) = f_\alpha(x)$ for $|\alpha| \leq m$.

Problem 2 is an open problem posed in [7], and Problem 1 is closely related to Problem 2. The theoretical aspects of the problems for $m = n = 2$ were addressed in [12,13]. We refer the readers to [11–13] for a more thorough discussion on the problems.

In this paper, we content ourselves with an idealized computer with standard von Neumann architecture that is able to process exact real numbers. We refer the readers to [10] for discussion on finite-precision computing.

In [12], we proved the following.

Theorem 1. *Let* $E \subset \mathbb{R}^2$ *be a finite set. There exist (universal) constants* C, D, *and a map* $\mathcal{E} : C^2_+(E) \times [0, \infty) \to C^2_+(\mathbb{R}^2)$ *such that the following hold.*

(A) Let $M \geq 0$. *Then for all* $f \in C^2_+(E)$ *with* $\|f\|_{C^2_+(E)} \leq M$, *we have* $\mathcal{E}(f, M) = f$ *on* E *and* $\|\mathcal{E}(f, M)\|_{C^2(\mathbb{R}^2)} \leq CM$.

(B) For each $x \in \mathbb{R}^2$, *there exists a set* $S(x) \subset E$ *with* $\#(S(x)) \leq D$ *such that for all* $M \geq 0$ *and* $f, g \in C^2_+(E)$ *with* $\|f\|_{C^2_+(E)}, \|g\|_{C^2_+(E)} \leq M$ *and* $f|_{S(x)} = g|_{S(x)}$, *we have*

$$\partial^\alpha \mathcal{E}(f, M)(x) = \partial^\alpha \mathcal{E}(g, M)(x) \text{ for } |\alpha| \leq 2 .$$

A few remarks on Theorem 1 are in order. First of all, in [13], we showed that the extension operator $\mathcal{E}$ cannot be linear in general. The constant D appearing in Theorem 1 is called the depth of the extension operator $\mathcal{E}$. This generalizes the notion of the depth of a *linear* extension operator first studied by C. Fefferman in [4,5] (for further discussion on the depth of linear extension operators see also G.K. Luli [14]). The depth of an extension operator (both linear and nonlinear) measures the computational complexity of the extension. The existence of a linear extension operator of bounded depth is one of the main ingredients for the Fefferman-Klartag [9,10] and Fefferman [6] algorithms for solving the interpolation problems without the nonnegative constraints; the algorithms in [6,9,10] are likely essentially the best possible.

In this paper, we will provide another proof of Theorem 1 but with algorithmic complexity in mind. This is the content of Theorem 2.

We start with a definition.

Definition 1.1. Let $\bar{N} \geq 1$ be an integer. Let $\mathcal{B} = \{\xi_1, \cdots, \xi_{\bar{N}}\}$ be a basis of $\mathbb{R}^{\bar{N}}$. Let $\Omega \subset \mathbb{R}^{\bar{N}}$ be a subset. Let X be a set. Let $\Xi : \Omega \to X$ be a map.

- We say Ξ has depth D, if there exists a D-dimensional subspace $V = \mathrm{span}(\xi_{i_1}, \cdots, \xi_{i_D})$, $\xi_{i_1}, \cdots, \xi_{i_D} \in \mathcal{B}$, such that for all $z_1, z_2 \in \Omega$ with $\pi_V(z_1) = \pi_V(z_2)$, we have $\Xi(z_1) = \Xi(z_2)$. Here, $\pi_V : \mathbb{R}^{\bar{N}} \to V$ is the natural projection.
- Suppose Ξ has depth D. Let $V = \mathrm{span}(\xi_{i_1}, \cdots, \xi_{i_D})$ be as above. By an efficient representation of Ξ, we mean a specification of the index set $\{i_1, \cdots, i_D\} \subset \{1, \cdots, \bar{N}\}$ and an algorithm to compute a map $\tilde{\Xi} : \Omega \cap V \to X$ in C_D operations, i.e., given an input $\omega \in \Omega \cap V$, we can compute $\tilde{\Xi}(\omega)$ in C_D operations. Here, the map $\tilde{\Xi}$ agrees with Ξ on $\Omega \cap V$, and C_D is a constant depending only on D.

Note that in general, the set Ω may have complicated geometry. For the purpose of this paper, we will only be considering when Ω is some Euclidean space or the first quadrant of some Euclidean space.

Remark 1.1. Suppose $\Xi : \mathbb{R}^{\bar{N}} \to \mathbb{R}$ is a linear functional. Recall from [10] that a "compact representation" of a linear functional $\Xi : \mathbb{R}^{\bar{N}} \to \mathbb{R}$ consists of a list of indices $\{i_1, \cdots, i_D\} \subset \{1, \cdots, \bar{N}\}$ and a list of coefficients $\chi_{i_1}, \cdots, \chi_{i_D}$, so that the action of Ξ is characterized by

$$\Xi : (\xi_1, \cdots, \xi_{\bar{N}}) \mapsto \sum_{\Delta=1}^{D} \chi_{i_\Delta} \cdot \xi_{i_\Delta}.$$

Therefore, given $v \in \mathrm{span}(\xi_{i_1}, \cdots, \xi_{i_D})$, we can compute $\Xi(v)$ by computing the dot product of two vectors of length D, which requires C_D operations. The present notion of "efficient representation" is a natural generalization adapted to the nonlinear nature of nonnegative interpolation (see [12,13]). Since a nonlinear map in general does not admit a simple representation, we emphasize the complexity of an extension operator rather than its structure.

We think of $C^2_+(E) \cong [0, \infty)^N$. We use the standard orthonormal frame $\mathbb{R}^N$ as a basis for the purpose of defining finite depth. We write $\mathcal{P}^+$ to denote the vector space of polynomials with degree no greater than two, and we write $\mathcal{J}^+_x F$ to denote the two-jet of F at x.

The main theorem of the paper is the following.

Theorem 2. *Suppose we are given a finite set* $E \subset \mathbb{R}^2$ *with* $\#(E) = N$*. Then there exists a collection of maps* $\{\Xi_x : x \in \mathbb{R}^2\}$*, where* $\Xi_x : C^2_+(E) \times [0, \infty) \to \mathcal{P}^+$ *for each* $x \in \mathbb{R}^2$*, such that the following hold.*

(A) *There exists a universal constant* D *such that for each* $x \in \mathbb{R}^2$*, the map* $\Xi_x(\cdot, \cdot) : C^2_+(E) \times [0, \infty) \to \mathcal{P}^+$ *is of depth* D*.*
(B) *Suppose we are given* $(f, M) \in C^2_+(E) \times [0, \infty)$ *with* $\|f\|_{C^2_+(E)} \leq M$*. Then there exists a function* $F \in C^2_+(\mathbb{R}^2)$ *such that*

$$\mathcal{J}^+_x F = \Xi_x(f, M) \text{ for all } x \in \mathbb{R}^2, \ \|F\|_{C^2(\mathbb{R}^2)} \leq CM, \text{ and } F(x) = f(x) \text{ for } x \in E.$$

(C) *There is an algorithm, that takes the given data set* E*, performs one-time work, and then responds to queries.*

A query consists of a point $x \in \mathbb{R}^2$*, and the response to the query is the depth-*D *map* Ξ_x*, given in its efficient representation (see Definition 1.1).*

The one-time work takes $CN \log N$ *operations and* CN *storage. The work to answer a query is* $C \log N$*.*

Remark 1.2. Theorem 2(C) implies that for each $x \in \mathbb{R}^2$, there exists a set $S(x) \subset E$ with $\#(S(x)) \leq D$ such that for all $(f, M), (g, M) \in C^2_+(E) \times [0, \infty)$ with $\|f\|_{C^2_+(E)}, \|g\|_{C^2_+(E)} \leq M$ and $f|_{S(x)} = g|_{S(x)}$, we have $\Xi_x(f, M) = \Xi_x(g, M)$. Moreover, after one-time work using at most $CN \log N$ operations and CN storage, we can perform the following task: Given $x \in \mathbb{R}^2$, we can produce the set $S(x)$ using no more than $C \log N$ operations.

Using Theorem 2, we obtain an algorithmic version of the Sharp Finiteness Principle (see Theorem 5 in [13]):

Theorem 3 *(Algorithmic Sharp Finiteness Principle). Let* $E \subset \mathbb{R}^2$ *with* $\#(E) = N < \infty$*. Then there exist universal constants* C_1, C_2, C_3, C_4, C_5 *and a list of subsets* $S_1, S_2, \cdots, S_L \subset E$ *satisfying the following.*

(A) *We can compute the list* $\{S_\ell : \ell = 1, \cdots, L\}$ *from* E*, using one-time work of at most* $C_1 N \log N$ *operations, and using storage at most* $C_2 N$*.*
(B) $\#(S_\ell) \leq C_3$ *for each* $\ell = 1, \cdots, L$*.*
(C) $L \leq C_4 N$*.*
(D) *Given any* $f : E \to [0, \infty)$*, we have*

$$\max_{\ell=1,\cdots,L} \|f\|_{C^2_+(S_\ell)} \leq \|f\|_{C^2_+(E)} \leq C_5 \max_{\ell=1,\cdots,L} \|f\|_{C^2_+(S_\ell)}.$$

Theorem 3 without condition (A) is the same as Theorem 5 in [13].

In this paper, we will prove Theorem 3 via Theorem 2. Our approach yields an alternate proof of Theorem 5 in [13]. The list of subsets $\{S_\ell : \ell = 1, \cdots, L\}$ that arises in this paper may be different from that in Theorem 5 of [13]. It will be interesting to understand the relationship between them.

Using Theorem 3, we can produce Algorithm 1, solving Problem 1.

Algorithm 1 Nonnegative $C^2(\mathbb{R}^2)$ Interpolation Algorithm - Trace Norm.

DATA: $E \subset \mathbb{R}^2$ finite with $\#(E) = N$.

QUERY: $f : E \to [0, \infty)$.

RESULT: The order of magnitude of $\|f\|_{C^2_+(E)}$. More precisely, the algorithm outputs a number $M \geq 0$ such that both of the following hold.

- We guarantee the existence of a function $F \in C^2_+(\mathbb{R}^2)$ such that $F|_E = f$ and $\|F\|_{C^2(\mathbb{R}^2)} \leq CM$.
- We guarantee there exists no $F \in C^2_+(\mathbb{R}^2)$ with norm at most M satisfying $F|_E = f$.

COMPLEXITY:

- Preprocessing E: at most $CN \log N$ operations and CN storage.
- Answer query: at most CN operations.

Using Theorem 2, we can produce Algorithm 2, solving Problem 2.

Algorithm 2 Nonnegative $C^2(\mathbb{R}^2)$ Interpolation Algorithm - Interpolant.

DATA: $E \subset \mathbb{R}^2$ finite with $\#(E) = N$. $f : E \to [0, \infty)$. $M \geq 0$.

ORACLE: $\|f\|_{C^2_+(E)} \leq M$.

RESULT: A query function that accepts a point $x \in \mathbb{R}^2$ and produces a list of numbers $(f_\alpha(x) : |\alpha| \leq 2)$ that guarantees the following: There exists a function $F \in C^2_+(\mathbb{R}^2)$ with $\|F\|_{C^2(\mathbb{R}^2)} \leq CM$ and $F|_E = f$, such that $\partial^\alpha F(x) = f_\alpha(x)$ for $|\alpha| \leq 2$. The function F is independent of the query point x, and is uniquely determined by (E, f, M).

COMPLEXITY:

- Preprocessing E: at most $CN \log N$ operations and CN storage.
- Answer query: at most $C \log N$ operations.

Theorem 2 also yields Algorithm 3 for computing the representative sets S_ℓ in Theorem 3.

Algorithm 3 Nonnegative $C^2(\mathbb{R}^2)$ Interpolation Algorithm - Representative Sets.

DATA: $E \subset \mathbb{R}^2$ finite with $\#(E) = N$.

RESULT: A query (set-valued) function that accepts a point $x \in \mathbb{R}^2$ and produces a subset $S(x) \subset E$, where $S(x)$ agrees with that in Remark 1.2.

COMPLEXITY:

– Preprocessing E: at most $CN \log N$ operations and CN storage.
– Answer query: at most $C \log N$ operations.

To see how to produce Algorithm 3 from Theorem 2, we simply note that each map Ξ_x in Theorem 2 is stored in its efficient representation (see Definition 1.1). Thus, the set $S(x)$ is given by the corresponding set of indices in the efficient representation of Ξ_x.

Acknowledgment. We are indebted to Jesús A. De Loera, Charles Fefferman, Kevin O'Neill, Naoki Saito, and Pavel Shvartsman, for their valuable comments. We also thank all the participants in the 11th Whitney workshop for fruitful discussions, and Trinity College Dublin for hosting the workshop.

This project is supported by NSF Grant DMS-1554733. The first author is supported by the UC Davis Summer Graduate Student Researcher Award and the Alice Leung Scholarship in Mathematics. The second author is supported by the UC Davis Chancellor's Fellowship.

2. Preliminaries

We use c_*, C_*, C', etc., to denote universal constants. They may be different quantities in different occurrences. We will label them to avoid confusion when necessary.

We assume that we are given an ordered orthogonal coordinate system on $\mathbb{R}^2$, specified by a pair of unit vectors $[e_1, e_2]$. We use $|\cdot|$ to denote Euclidean distance. We use $B(x, r)$ to denote the disk of radius r centered at x. For $X, Y \subset \mathbb{R}^2$, we write $\text{dist}(X, Y) := \inf_{x \in X, y \in Y} |x - y|$.

We use $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2$, etc., to denote multi-indices. We write ∂^α to denote $\partial_{e_1}^{\alpha_1} \partial_{e_2}^{\alpha_2}$. We adopt the partial ordering $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i$ for $i = 1, 2$.

By a square, we mean a set of the form $Q = [a, a+\delta) \times [b, b+\delta)$ for some $a, b \in \mathbb{R}$ and $\delta > 0$. If Q is a square, we write δ_Q to denote the sidelength of the square. For $\lambda > 0$, we use λQ to denote the square whose center is that of Q and whose sidelength is $\lambda\delta_Q$. Given two squares Q, Q', we write $Q \leftrightarrow Q'$ if $\mathtt{closure}(Q) \cap \mathtt{closure}(Q') \neq \varnothing$.

A dyadic square is a square of the form $Q = [2^k \cdot i, 2^k \cdot (i+1)) \times [2^k \cdot j, 2^k \cdot (j+1))$ for some $i, j, k \in \mathbb{Z}$. Each dyadic square Q is contained in a unique dyadic square with sidelength $2\delta_Q$, denoted by Q^+.

Let $\Omega \subset \mathbb{R}^n$ be a set with nonempty interior Ω_0 such that $\Omega \subset \overline{\Omega_0}$. For nonnegative integers m, n, we use $C^m(\Omega)$ to denote the vector space of m-times continuously differentiable real-valued functions up to the closure of Ω, whose derivatives up to order m are bounded. For $F \in C^m(\Omega)$, we define

$$\|F\|_{C^m(\Omega)} := \sup_{x\in\Omega_0} \max_{|\alpha|\le m} |\partial^\alpha F(x)|\,.$$

We write $C^m_+(\Omega)$ to denote the collection of functions $F \in C^m(\Omega)$ such that $F \ge 0$ on Ω.

Let $E \subset \mathbb{R}^n$ be finite. We define the following.

$$C^m(E) := \{f : E \to \mathbb{R}\} \cong \mathbb{R}^{\#(E)} \quad \text{and} \quad \|f\|_{C^m(E)} := \inf\left\{\|F\|_{C^m(\mathbb{R}^n)} : F|_E = f\right\};$$

$$C^m_+(E) := \{f : E \to [0,\infty)\} \cong [0,\infty)^{\#(E)} \quad \text{and}$$

$$\|f\|_{C^m_+(E)} := \inf\left\{\|F\|_{C^m(\mathbb{R}^n)} : F|_E = f \text{ and } F \ge 0\right\}.$$

2.1. Polynomials and Whitney fields

We write $\mathcal{P}$ to denote the vector space of affine polynomials on $\mathbb{R}^2$. It is a three-dimensional vector space. We use $\mathcal{P}^+$ to denote the vector space of polynomials in $\mathbb{R}^2$ with degree no greater than two. It is a six-dimensional vector space.

For $x \in \mathbb{R}^2$ and a function F twice continuously differentiable at x, we write $\mathcal{J}_x F$, $\mathcal{J}^+_x F$ to denote the one-jet, two-jet of F at x, respectively, which we identify with the degree-one, degree-two Taylor polynomials, respectively,

$$\begin{aligned} \mathcal{J}_x F(y) &:= \sum_{|\alpha|\le 1} \frac{\partial^\alpha F(x)}{\alpha!}(y-x)^\alpha, \text{ and} \\ \mathcal{J}^+_x F(y) &:= \sum_{|\alpha|\le 2} \frac{\partial^\alpha F(x)}{\alpha!}(y-x)^\alpha. \end{aligned} \tag{2.1}$$

We use $\mathcal{R}_x$, $\mathcal{R}^+_x$ to denote the rings of one-jets, two-jets at x, respectively. The multiplications on $\mathcal{R}_x$ and $\mathcal{R}^+_x$ are defined in the following way:

$$P \odot_x R := \mathcal{J}_x(PR) \text{ and } P^+ \odot^+_x R^+ := \mathcal{J}^+_x(P^+R^+),$$

for $P, R \in \mathcal{R}_x$ and $P^+, R^+ \in \mathcal{R}^+_x$.

Let $S \subset \mathbb{R}^n$ be a nonempty finite set. A Whitney field on S is an array of polynomials

$$\vec{P} := (P^x)_{x\in S}, \quad \text{where} \quad P^x \in \mathcal{R}_x \text{ for each } x \in S\,.$$

Given $\vec{P} = (P^x)_{x\in S}$, we sometimes use the notation

$$(\vec{P}, x) := P^x \text{ for } x \in S\,.$$

We write $W^2(S)$ to denote the vector space of all Whitney fields on S. For $\vec{P} = (P^x)_{s\in S} \in W^2(S)$, we define

$$\|\vec{P}\|_{W^2(S)} := \max_{x\in S, |\alpha|\le 1} |\partial^\alpha P^x(x)| + \max_{x,y\in S,\, x\neq y,\, |\alpha|\le 1} \frac{|\partial^\alpha (P^x - P^y)(x)|}{|x-y|^{2-|\alpha|}}.$$

We note that $\|\cdot\|_{W^2(S)}$ is a norm on $W^2(S)$.

We write $W^2_+(S)$ to denote a subcollection of $W^2(S)$, such that $\vec{P} \in W^2_+(S)$ if and only if for each $x \in S$, there exists some $M_x \ge 0$ such that

$$(\vec{P}, x)(y) + M_x |y-x|^2 \ge 0 \text{ for all } y \in \mathbb{R}^2. \tag{2.2}$$

For $\vec{P} \in W^2_+(S)$, we define

$$\|\vec{P}\|_{W^2_+(S)} := \|\vec{P}\|_{W^2(S)} + \max_{x\in S}\left(\inf\left\{M_x \ge 0 : (\vec{P},x)(y) + M_x|y-x|^2 \ge 0 \text{ for all } y \in \mathbb{R}^2\right\}\right).$$

The next lemma is a Taylor-Whitney correspondence for $C^2_+(\mathbb{R}^2)$. (A) is simply Taylor's theorem. See [8,13] for a proof of (B).

Lemma 2.1. *There exists a universal constant* C_w *such that the following holds.*
Let $E \subset \mathbb{R}^2$ *be a finite set.*

(A) Let $F \in C^2_+(\mathbb{R}^2)$. *Let* $\vec{P} := (\mathcal{J}_x F)_{x\in E}$. *Then* $\vec{P} \in W^2_+(E)$ *and* $\|\vec{P}\|_{W^2_+(E)} \le C_w \|F\|_{C^2(\mathbb{R}^2)}$.
(B) There exists a map $T^E_w : W^2_+(E) \to C^2_+(\mathbb{R}^2)$ *such that* $\|T^E_w(\vec{P})\|_{C^2(\mathbb{R}^2)} \le C_w \|\vec{P}\|_{W^2_+(E)}$ *and* $\mathcal{J}_x T^E_w(\vec{P}) = (\vec{P}, x)$ *for each* $x \in E$.

2.2. *Trace norm on small subsets*

Let $S \subset \mathbb{R}^2$ be a finite set. We define the following two functions.

$$\begin{aligned} &\mathcal{Q} = \mathcal{Q}_S : W^2(S) \to [0,\infty) \\ &\vec{P} = (P^x)_{x\in S} \mapsto \sum_{\substack{x\in S\\ |\alpha|\le 1}} |\partial^\alpha P^x(x)| + \sum_{\substack{x,y\in S\\ x\neq y\\ |\alpha|\le 1}} \frac{|\partial^\alpha (P^x-P^y)(x)|}{|x-y|^{2-|\alpha|}}, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\mathcal{M} = \mathcal{M}_S : W^2(S) \to [0,\infty] \\ &\vec{P} = (P^x)_{x\in S} \mapsto \begin{cases} \sum_{x\in S} \frac{|\nabla P^x|^2}{P^x(x)} & \text{if } P^x(x) \ge 0 \text{ for each } x \in S \\ \infty & \text{if there exists } x \in S \text{ such that } P^x(x) < 0 \end{cases}. \end{aligned} \tag{2.4}$$

In (2.4), we use the conventions that $\frac{0}{0} = 0$ and $\frac{a}{0} = \infty$ for $a > 0$.

Lemma 2.2. *Let* $S \subset \mathbb{R}^2$ *be a finite set with* $\#(S) \le N_0$ *for some universal constant* N_0*. Let* $\mathcal{Q}$ *and* $\mathcal{M}$ *be as in* (2.3) *and* (2.4)*. Then there exists a universal constant* C *such that*

$$C^{-1}\|\vec{P}\|_{W^2_+(S)} \le (\mathcal{Q}+\mathcal{M})(\vec{P}) \le C\|\vec{P}\|_{W^2_+(S)} \text{ for all } \vec{P} \in W^2_+(S). \tag{2.5}$$

Moreover, $\vec{P} \in W^2(S) \setminus W^2_+(S)$ *if and only if* $\mathcal{M}(\vec{P}) = \infty$*.*

Proof. We write C, C', etc., to denote universal constants.

Fix $\vec{P} = (P^x)_{x\in S} \in W^2_+(S)$.

Suppose $(\mathcal{Q}+\mathcal{M})(\vec{P}) \le M$. We want to show that

$$\|\vec{P}\|_{W^2_+(S)} \le CM. \tag{2.6}$$

Since each summand in the definition of $\mathcal{Q}$ in (2.3) is nonnegative, we have

$$\max_{x\in S, |\alpha|\le 1} |\partial^\alpha P^x(x)| \le CM, \text{ and } \max_{x,y\in S, x\ne y, |\alpha|\le 1} \frac{|\partial^\alpha (P^x - P^y)(x)|}{|x-y|^{2-|\alpha|}} \le CM. \tag{2.7}$$

Since $\mathcal{M}(\vec{P}) \le M$, we have

$$|\nabla P^x|^2 \le M P^x(x) \text{ for } x \in S. \tag{2.8}$$

Therefore, we have

$$P^x(y) + \frac{M}{4}|y-x|^2 \ge 0 \text{ for all } y \in \mathbb{R}^2, x \in S. \tag{2.9}$$

By the definition of $\|\cdot\|_{W^2_+(S)}$, we see that (2.6) follows from (2.7) and (2.9).

Suppose $\|\vec{P}\|_{W^2_+(S)} \le M$. We want to show that

$$(\mathcal{Q}+\mathcal{M})(\vec{P}) \le CM. \tag{2.10}$$

By the definition of $\|\cdot\|_{W^2_+(S)}$, we know that

$$\max_{x\in S, |\alpha|\le 1} |\partial^\alpha P^x(x)| \le M, \quad \max_{x,y\in S, x\ne y, |\alpha|\le 1} \frac{|\partial^\alpha (P^x - P^y)(x)|}{|x-y|^{2-|\alpha|}} \le M, \text{ and} \tag{2.11}$$

$$P^x(y) + M|y-x|^2 \ge 0 \text{ for all } y \in \mathbb{R}^2, x \in S. \tag{2.12}$$

It follows from (2.11) that

$$\mathcal{Q}(\vec{P}) \le CM^2. \tag{2.13}$$

For each $x \in S$, restricting P^x to each line in $\mathbb{R}^2$ passing through x and computing the discriminant, we can conclude from (2.12) that

$$|\nabla P^x|^2 \le CMP^x(x) \text{ for } x \in S. \tag{2.14}$$

It follows from (2.14) that

$$\mathcal{M}(\vec{P}) \le CM. \tag{2.15}$$

(Recall that we use the convention $\frac{0}{0} = 0$). (2.10) then follows from (2.13) and (2.15).

This proves (2.5).

Now we turn to the second statement.

Suppose $\vec{P} \in W^2(S)$ is such that $\mathcal{M}(\vec{P}) = \infty$. Then at least one of the following holds:

- $P^x(x) < 0$ for some $x \in S$, in which case condition (2.2) fails for such P^x, so $\vec{P} \notin W^2_+(S)$.
- There exists $x \in S$ such that $P^x(x) = 0$ but $\nabla P^x \neq 0$, in which case condition (2.2) fails for such P^x, so $\vec{P} \notin W^2_+(S)$.

In conclusion, we have $\vec{P} \notin W^2_+(S)$.

Conversely, suppose $\vec{P} \notin W^2_+(S)$. Then there exists $x \in S$ such that condition (2.2) fails for P^x. This means that either $P^x(x) < 0$, or $P^x(x) = 0$ but $\nabla P^x \neq 0$. In either case, we have $\mathcal{M}(\vec{P}) = \infty$.

Lemma 2.2 is proved. □

For the rest of the subsection, we fix a finite set $S \subset \mathbb{R}^2$ with $\#(S) \le N_0$, where N_0 is a universal constant. We also fix a function $f : S \to [0, \infty)$. We explain how to compute the order of magnitude of $\|f\|_{C^2_+(S)}$.

We adopt the following notation: For $A, B \ge 0$, we write $A \approx B$ if there exists a universal constant C such that $C^{-1}A \le B \le CA$.

We define an affine subspace $\mathbb{A}_f \subset W^2(S)$ by

$$\begin{aligned}\mathbb{A}_f &:= \left\{\vec{P} = (P^x)_{x\in S} \in W^2(S) : P^x(x) = f(x) \text{ and } f(x) = 0 \Rightarrow \nabla P^x = 0 \text{ for } x \in S\right\}\\ &= \left\{\vec{P} = (P^x)_{x\in S} \in W^2_+(S) : P^x(x) = f(x) \text{ for } x \in S\right\}.\end{aligned}$$

Note that $\mathbb{A}_f$ has dimension $2 \cdot (\#(S) - \#(f^{-1}(0))$.

Let $\mathcal{Q}$ and $\mathcal{M}$ be as in (2.3) and (2.4). By Lemma 2.1 and Lemma 2.2,

$$\|f\|_{C^2_+(S)} \approx \inf\left\{(\mathcal{Q} + \mathcal{M})(\vec{P}) : \vec{P} \in \mathbb{A}_f\right\}. \tag{2.16}$$

Let $d := \dim W^2(S) = \#(S) \cdot \dim \mathcal{P} = 3\#(S)$. We identify $W^2(S) \cong \mathbb{R}^d$ via $(P^x)_{x\in S} \mapsto (P^x(x), \partial_{e_1}P^x, \partial_{e_2}P^x)_{x\in S}$. We define the ℓ^1 and ℓ^2-norms, respectively, on $\mathbb{R}^d$ by the formulae

$$\|v\|_{\ell^1} := \sum_{i=1}^{d} |v_i| \text{ and } \|v\|_{\ell^2} := \left(\sum_{i=1}^{d} |v_i|^2\right)^{1/2} \quad \text{for} \quad v = (v_1, \cdots, v_d) \in \mathbb{R}^d.$$

Consider the following objects.

- Let $L_w : W^2(S) \to \mathbb{R}^d$ be a linear isomorphism that maps $\vec{P} \in W^2(S)$ to the vector in $\mathbb{R}^d$ with components

$$\frac{\partial^\alpha(P^y - P^z)(y)}{|y-z|^{2-|\alpha|}} \ , \ \partial^\alpha P^{x_S}(x_S) \quad , \quad |\alpha| \le 1$$

  for suitable $x_S, y, z \in S$ in some order, so that

$$\|L_w(\vec{P})\|_{\ell^1(\mathbb{R}^{d_S})} \approx \mathcal{Q}(\vec{P}) \quad \text{for } \vec{P} \in W^2(S) \tag{2.17}$$

  The construction of such L_w is based on the technique of "clustering" introduced in [1]. See Remark 3.3 of [1]. Since $\#(S)$ is universally bounded, we can compute L_w from S using at most C operations.
- Let $V_f \subset W^2(S)$ be a subspace defined by

$$V_f := \left\{(P^x)_{x\in S} : P^x(x) = 0 \text{ for } x \in S \setminus f^{-1}(0) \text{ and } P^x \equiv 0 \text{ for } x \in f^{-1}(0)\right\}.$$

  Let $\Pi_f = (\Pi_f^x)_{x\in S} : W^2(S) \to V_f$ be the natural projection defined by

$$\Pi_f^x(P^x) = (0, \partial_{e_1}P^x, \partial_{e_2}P^x).$$

  Let $\vec{P}_f \in W^2(S)$ denote the vector

$$\vec{P}_f := (f(x), 0, 0)_{x\in S}.$$

  It is clear that $\mathbb{A}_f = \vec{P}_f + V_f$.
- Let $L_f = (L_f^x)_{x\in S} : W^2(S) \to W^2(S)$ be a linear endomorphism defined by $L_f^x(P^x) = \frac{P^x}{\sqrt{f(x)}}$ for $x \in S \setminus f^{-1}(0)$ and $L_f^x \equiv 0$ for $x \in f^{-1}(0)$.

We see that

$$\mathcal{M}(\vec{P}) \approx \|L_f \Pi_f(\vec{P})\|^2_{\ell^2(\mathbb{R}^d)} \quad \text{for } \vec{P} \in \mathbb{A}_f. \tag{2.18}$$

Combining (2.17) and (2.18), we see that

$$(\mathcal{Q} + \mathcal{M})(\vec{P}) \approx \|L_f \Pi_f(\vec{P})\|^2_{\ell^2(\mathbb{R}^d)} + \|L_w(\vec{P})\|_{\ell^1(\mathbb{R}^d)} \quad \text{for } \vec{P} \in \mathbb{A}_f = \vec{P}_f + V_f. \tag{2.19}$$

Let $\beta := L_w(\vec{P})$ and $A := (L_f \Pi_f)^T (L_f \Pi_f)$. We see from (2.16) and (2.19) that computing the order of magnitude of $\|f\|_{C^2_+(S)}$ amounts to solving the following optimization problem:

$$\text{Minimize } \beta^t A \beta + \|\beta\|_{\ell^1(\mathbb{R}^d)} \quad \text{subject to } L_w^{-1}\beta \in \vec{P}_f + V_f. \tag{2.20}$$

We note that (2.20) is a convex quadratic programming problem with affine constraint. We can find the exact solution to (2.20) by solving for the associated Karush-Kuhn-Tucker conditions, which consist of a bounded system of linear equalities and inequalities [2]. Thus, we can compute the order of magnitude of $\|f\|_{C^2_+(S)}$ in C operations. See Appendix A for details

2.3. Essential convex sets

Definition 2.1. Let $E \subset \mathbb{R}^2$ be a finite set.

- For $x \in \mathbb{R}^2$, $S \subset E$, and $k \geq 0$, we define

$$\begin{aligned} \sigma(x, S) &:= \left\{ \mathcal{J}_x \varphi : \varphi \in C^2(\mathbb{R}^2),\ \ \varphi|_S = 0, \text{ and } \|\varphi\|_{C^2(\mathbb{R}^2)} \leq 1 \right\}, \text{ and} \\ \sigma^\sharp(x, k) &:= \bigcap_{S \subset E, \#(S) \leq k} \sigma(x, S). \end{aligned} \tag{2.21}$$

- Let $f : E \to [0, \infty)$ be given. For $x \in \mathbb{R}^2$, $S \subset E$, $k \geq 0$, and $M \geq 0$, we define

$$\begin{aligned} \Gamma_+(x, S, M, f) &:= \left\{ \mathcal{J}_x F : F \in C^2_+(\mathbb{R}^2),\ \ F|_S = f, \text{ and } \|F\|_{C^2(\mathbb{R}^2)} \leq M \right\}, \text{ and} \\ \Gamma_+^\sharp(x, k, M, f) &:= \bigcap_{S \subset E, \#(S) \leq k} \Gamma_+(x, S, M, f). \end{aligned} \tag{2.22}$$

Adapting the proof of the Finiteness Principle for nonnegative $C^2(\mathbb{R}^2)$ interpolation (Theorem 4 of [13]), we have the following.

Lemma 2.3. *There exists a universal constant C such that the following holds. Let $E \subset \mathbb{R}^2$ be a finite set. Let σ and $\sigma^\sharp$ be as in Definition 2.1. Then for any $x \in \mathbb{R}^2$,*

$$C^{-1} \cdot \sigma^\sharp(x, 16) \subset \sigma(x, E) \subset C \cdot \sigma(x, 16).$$

2.4. Callahan-Kosaraju decomposition

We will use the data structure introduced by Callahan and Kosaraju [3].

Lemma 2.4 *(Callahan-Kosaraju decomposition). Let $E \subset \mathbb{R}^n$ with $\#(E) = N < \infty$. Let $\kappa > 0$. We can partition $E \times E \setminus \mathtt{diagonal}(E)$ into subsets $E_1' \times E_1'', \cdots, E_L' \times E_L''$ satisfying the following.*

(A) $L \leq C(\kappa, n)N$.
(B) *For each* $\ell = 1, \cdots, L$, *we have*

$$\operatorname{diam} E'_\ell, \operatorname{diam} E''_\ell \leq \kappa \cdot \operatorname{dist}(E'_\ell, E''_\ell) .$$

(C) *Moreover, we may pick* $x'_\ell \in E'_\ell$ *and* $x''_\ell \in E''_\ell$ *for each* $\ell = 1, \cdots, L$, *such that the* x'_ℓ, x''_ℓ *for* $\ell = 1, \cdots, L$ *can all be computed using at most* $C(\kappa, n)N \log N$ *operations and* $C(\kappa, n)N$ *storage.*

Here, $C(\kappa, n)$ *is a constant that depends only on* κ *and* n.

3. Algorithm 1: Estimation of trace norm

3.1. Proof of Theorem 3

In this section, we prove Theorem 3 by assuming Theorem 2, whose proof will appear in Section 5.7.

With a slight tweak, the argument in the proof of Lemma 3.1 in [6] yields the following.

Lemma 3.1. *Let* $E \subset \mathbb{R}^2$ *be a finite set. Let* $\kappa_0 > 0$ *be a constant that is sufficiently small. Let* E'_ℓ, E''_ℓ *be as in Lemma 2.4 with* $\kappa = \kappa_0$. *Suppose* $\vec{P} = (P^x)_{x \in E} \in W^2_+(E)$ *satisfies the following.*

(A) $P^x \in \Gamma_+(x, \varnothing, M, f)$ *for each* $x \in E$, *with* Γ_+ *as in* (2.22).
(B) $\left|\partial^\alpha (P^{x'_\ell} - P^{x''_\ell})(x''_\ell)\right| \leq M |x'_\ell - x''_\ell|^{2-|\alpha|}$ *for* $|\alpha| \leq 1$, $\ell = 1, \cdots, L$.

Then $\|\vec{P}\|_{W^2_+(E)} \leq CM$.

Recall Lemma 3.2 of [6].

Lemma 3.2. *Let* $E \subset \mathbb{R}^2$ *be a finite set. Let* E'_ℓ *and* E''_ℓ *be as in Lemma 2.4 with* $\ell = 1, \cdots, L$. *Then every* $x \in E$ *arises as an* x'_ℓ *for some* $\ell \in \{1, \cdots, L\}$.

We now have all the ingredients for the proof of Theorem 3.

Proof of Theorem 3 Assuming Theorem 2. Let $E \subset \mathbb{R}^2$ be a finite set. Let $\{\Xi_x, x \in \mathbb{R}^2\}$ be as in Theorem 2. For each $x \in E$, let $S(x)$ be as in Remark 1.2.

Let κ_0 be as in Lemma 3.1. Let $(x'_\ell, x''_\ell) \in E \times E$, $\ell = 1, \cdots, L$, be as in Lemma 2.4 with $\kappa = \kappa_0$.

We set

$$S_\ell := \{x'_\ell, x''_\ell\} \cup S(x'_\ell) \cup S(x''_\ell) \ , \ \ \ell = 1, \cdots, L. \tag{3.1}$$

Conclusion (A) follows from Theorem 2(C), Remark 1.2, and Lemma 2.4.
Conclusion (B) follows from Theorem 2(C) and Remark 1.2.
Conclusion (C) follows from Lemma 2.4(C).
Now we verify conclusion (D). We modify the argument in [6].
Fix $f : E \to [0, \infty)$. Set

$$M := \max_{\ell=1,\cdots,L} \|f\|_{C^2_+(S_\ell)}. \tag{3.2}$$

Thanks to (3.2), we see that $\|f\|_{C^2_+(S_\ell)} \le M$ for $\ell = 1, \cdots, L$. Thus, for each $\ell = 1, \cdots, L$, there exists $F_\ell \in C^2_+(\mathbb{R}^2)$ such that

$$\|F_\ell\|_{C^2(\mathbb{R}^2)} \le 2M \text{ and } F_\ell(x) = f(x) \text{ for } x \in S_\ell. \tag{3.3}$$

Fix such F_ℓ. For $\ell = 1, \cdots, L$, we define

$$f_\ell : E \to [0, \infty) \text{ by } f_\ell(x) := F_\ell(x) \text{ for } x \in E. \tag{3.4}$$

From (3.3) and (3.4), we see that

$$\|f_\ell\|_{C^2_+(E)} \le 2M \text{ for } \ell = 1, \cdots, L. \tag{3.5}$$

For each $\ell = 1, \cdots, L$, we define

$$P'_\ell := \mathcal{J}_{x'_\ell}\left(\Xi_{x'_\ell}(f_\ell, 2M)\right) \text{ and } P''_\ell := \mathcal{J}_{x''_\ell}\left(\Xi_{x''_\ell}(f_\ell, 2M)\right). \tag{3.6}$$

We will show that the assignment (3.6) unambiguously defines a Whitney field over E.

Claim 3.1. *Let $\ell_1, \ell_2 \in \{1, \cdots, L\}$.*

(a) Suppose $x'_{\ell_1} = x'_{\ell_2}$. Then $P'_{\ell_1} = P'_{\ell_2}$.
(b) Suppose $x''_{\ell_1} = x''_{\ell_2}$. Then $P''_{\ell_1} = P''_{\ell_2}$.
(c) Suppose $x'_{\ell_1} = x''_{\ell_2}$. Then $P'_{\ell_1} = P''_{\ell_2}$.

Proof of Claim 3.1. We prove (a). The proofs for (b) and (c) are similar.
Suppose $x'_{\ell_1} = x'_{\ell_2} =: x_0$. Let $S(x_0)$ be as in Remark 1.2. By (3.1), we see that

$$S(x_0) \subset S_{\ell_1} \cap S_{\ell_2}.$$

Therefore, we have

$$f_{\ell_1}(x) = f_{\ell_2}(x) \text{ for } x \in S(x_0).$$

Thanks to Theorem 2(A), Remark 1.2, and (3.5), we see that

$$\Xi_{x_0}(f_{\ell_1}, 2M) = \Xi_{x_0}(f_{\ell_2}, 2M).$$

By (3.6), we see that $P_{\ell_1} = P_{\ell_2}$. This proves (a). □

By Lemma 3.2, there exists a pair of maps:

$$\begin{aligned}&\text{A surjection } \pi : \{1, \cdots, L\} \to E \text{ such that } \pi(\ell) = x'_\ell \text{ for } \ell = 1, \cdots, L, \text{ and}\\&\text{An injection } \rho : E \to \{1, \cdots, L\} \text{ such that } x'_{\rho(x)} = x \text{ for } x \in E, \text{ i.e., } \pi \circ \rho = \mathrm{id}_E.\end{aligned} \tag{3.7}$$

The surjection π is determined by the Callahan-Kosaraju decomposition (Lemma 2.4), but the choice of ρ is not necessarily unique.

Thanks to Claim 3.1 and the fact that $E'_\ell \times E''_\ell \subset E \times E \setminus \mathtt{diagonal}(E)$, assignment (3.6) produces for each $x \in E$ a uniquely defined polynomial

$$P^x = \mathcal{J}_x\left(\Xi_x(f_{\rho(x)}, 2M)\right), \tag{3.8}$$

with Ξ_x as in Theorem 2 and $\rho(x)$ as in (3.7). Note that, as shown in Claim 3.1, the polynomial P^x in (3.8) is independent of the choice of ρ as a right-inverse of π in (3.7).

Thanks to Theorem 2(B) and (3.5)–(3.8), for each $\ell = 1, \cdots, L$, there exists a function $\tilde{F}_\ell \in C^2(\mathbb{R}^2)$ such that

$$\|\tilde{F}_\ell\|_{C^2(\mathbb{R}^2)} \le CM \text{ and } \tilde{F}_\ell \ge 0 \text{ on } \mathbb{R}^2; \tag{3.9}$$

$$\tilde{F}_\ell = f_\ell(x) = f(x) \text{ for } x \in S_\ell; \text{ and} \tag{3.10}$$

$$\mathcal{J}_{x'_\ell}\tilde{F}_\ell = P^{x'_\ell} = \mathcal{J}_{x'_\ell}\left(\Xi_{x'_\ell}(f_\ell, 2M)\right), \text{ and } \mathcal{J}_{x''_\ell}\tilde{F}_\ell = P^{x''_\ell} = \mathcal{J}_{x''_\ell}\left(\Xi_{x''_\ell}(f_\ell, 2M)\right). \tag{3.11}$$

Thanks to (3.9) and (3.10), we have

$$P^{x'_\ell} \in \Gamma_+(x'_\ell, \{x'_\ell\}, CM, f) \text{ for } \ell = 1, \cdots, L. \tag{3.12}$$

Thanks to (3.9) and (3.11), we have

$$\left|\partial^\alpha(P^{x'_\ell} - P^{x''_\ell})(x''_\ell)\right| \le CM\,|x'_\ell - x''_\ell|^{2-|\alpha|} \text{ for } |\alpha| \le 1, \ell = 1, \cdots, L. \tag{3.13}$$

Therefore, by Lemma 3.1, (3.12), and (3.13), the Whitney field $\vec{P} = (P^x)_{x\in E}$, with P^x as in (3.8), satisfies

$$\vec{P} \in W^2_+(E), P^x(x) = f(x) \text{ for } x \in E, \text{ and } \|\vec{P}\|_{W^2_+(E)} \le CM.$$

By Lemma 2.1(B), there exists a function $F \in C^2_+(\mathbb{R}^2)$ such that $\|F\|_{C^2(\mathbb{R}^2)} \le CM$ and $\mathcal{J}_x F = P^x$ for each $x \in E$. In particular, $F(x) = P^x(x) = f(x)$ for each $x \in E$. Thus, $\|f\|_{C^2_+(E)} \le CM$. This proves conclusion (D).

Theorem 3 is proved. □

3.2. Explanation of Algorithm 1

Below are the steps of Algorithm 1.

Step 1. Compute $S_1, \cdots, S_L$ from E as in Theorem 3.
Step 2. Read $f : E \to [0, \infty)$.
Step 3. For $\ell = 1, \cdots, L$, compute a number M_ℓ such that M_ℓ has the same order of magnitude as $\|f\|_{C^2_+(S_\ell)}$.
Step 4. Return $M := \max\{M_\ell : \ell = 1, \cdots, L\}$.

The number M produced in Step 4 has the same order of magnitude as $\|f\|_{C^2_+(E)}$, thanks to Theorem 3 and Lemma 2.1. Therefore, Algorithm 1 accomplishes what we claim to do.

We now analyze the complexity of Algorithm 1.

By Theorem 3, Step 1 requires no more than $CN \log N$ operations and CN storage.

Step 3 requires no more than CN operations. Indeed, on one hand, computing each M_ℓ requires no more than C operations, thanks to the discussion in Section 2.2; on the other hand, we need to carry out L computations, with $L \le CN$.

Finally, Step 4 requires no more than CN operations.

This concludes our discussion of Algorithm 1.

4. Approximation of $\sigma^\sharp$

This and the next sections will be devoted to the proof of Theorem 2. To prepare the way, in this section, we introduce the relevant objects and show how they can be computed efficiently.

We begin by reviewing some key objects introduced in [9,10], which we will use to effectively approximate the shapes of $\sigma^\sharp(x, 16)$ for $x \in E$.

We will be working with $C^2(\mathbb{R}^2)$ functions instead of $C^2_+(\mathbb{R}^2)$ functions.

4.1. Parameterized approximate linear algebra problems (PALP)

Let $\bar{N} \ge 1$. Let $\{\xi_1, \cdots, \xi_{\bar{N}}\}$ be the standard basis for $\mathbb{R}^{\bar{N}}$. We recall the following definition in Section 6 of [10].

Definition 4.1. A parameterized approximate linear algebra problem (PALP for short) is an object of the form:

$$\underline{\mathcal{A}} = \left[(\underline{\lambda}_1, \cdots, \underline{\lambda}_{i_{\max}}), (\underline{b}_1, \cdots, \underline{b}_{i_{\max}}), (\epsilon_1, \cdots, \epsilon_{i_{\max}})\right], \tag{4.1}$$

where

- Each $\underline{\lambda}_i$ is a linear functional on $\mathcal{P}$, which we will refer to as a "linear functional";

- Each $\underline{b}_i$ is a linear functional on $C^2(E)$, which we will refer to as a "target functional"; and
- Each $\epsilon_i \in [0, \infty)$, which we will refer to as a "tolerance".

Given a PALP $\underline{\mathcal{A}}$ in the form (4.1), we introduce the following terminologies:

- We call i_{max} the length of $\underline{\mathcal{A}}$;
- We say $\underline{\mathcal{A}}$ has depth D if each of the linear functionals $\underline{b}_i$ on $\mathbb{R}^{\bar{N}}$ has depth D with respect to the basis $\{\xi_1, \cdots, \xi_{\bar{N}}\}$ (see Definition 1.1).

Recall Definition 1.1. We assume that every PALP is "efficiently stored", namely, each of the target functionals are stored in its efficient representation. In particular, given a PALP $\underline{\mathcal{A}}$ of the form (4.1) and a target $\underline{b}_i$ of $\underline{\mathcal{A}}$, we have access to a set of indices $\{i_1, \cdots, i_D\} \subset \{1, \cdots, N\}$, such that $\underline{b}_i$ is completely determined by its action on $\{\xi_{i_1}, \cdots, \xi_{i_D}\} \subset \{\xi_1, \cdots, \xi_N\}$. Here $i_D = \text{depth}(\underline{b}_i)$. We define

$$S(\underline{b}_i) := \{x_{i_1}, \cdots, x_{i_D}\} \subset E. \tag{4.2}$$

Given a PALP of the form (4.1), we define

$$S(\underline{\mathcal{A}}) := \bigcup_{i=1}^{i_{max}} S(\underline{b}_i) \subset E \tag{4.3}$$

with $S(\underline{b}_i)$ as in (4.2).

4.2. Blobs and PALPs

Definition 4.2. A blob in $\mathcal{P}$ is a family $\vec{\mathcal{K}} = (\mathcal{K}_M)_{M \geq 0}$ of (possibly empty) convex subsets $\mathcal{K}_M \subset V$ parameterized by $M \in [0, \infty)$, such that $M < M'$ implies $\mathcal{K}_M \subseteq \mathcal{K}_{M'}$. We say two blobs $\vec{\mathcal{K}} = (\mathcal{K}_M)_{M \geq 0}$ and $\vec{\mathcal{K}}' = (\mathcal{K}'_M)_{M \geq 0}$ are C-equivalent if $\mathcal{K}_{C^{-1}M} \subset \mathcal{K}'_M \subset \mathcal{K}_{CM}$ for each $M \in [0, \infty)$.

Let $\underline{\mathcal{A}}$ be a PALP of the form (4.1). For each $\varphi \in C^2(E) \cong \mathbb{R}^{\#(E)}$, we have a blob defined by

$$\begin{gathered} \vec{\mathcal{K}}_\varphi(\underline{\mathcal{A}}) = (\mathcal{K}_\varphi(\underline{\mathcal{A}}, M))_{M \geq 0}\,, \text{ where} \\ \mathcal{K}_\varphi(\underline{\mathcal{A}}, M) := \{P \in \mathcal{P} : |\underline{\lambda}_i(P) - \underline{b}_i(\varphi)| \leq M\epsilon_i \ \text{ for } \ i = 1, \cdots, i_{max}\} \subset V. \end{gathered} \tag{4.4}$$

In this paper, we will be mostly interested in the centrally symmetric (called "homogeneous" in [10]) polytope defined by setting $\varphi \equiv 0$:

$$\sigma(\underline{\mathcal{A}}) := \mathcal{K}_0(\underline{\mathcal{A}}, 1). \tag{4.5}$$

Note that $\sigma(\underline{\mathcal{A}})$ is never empty, since it contains the zero polynomial.

4.3. Essential PALPs and blobs

Let $E \subset \mathbb{R}^2$ be a finite set with $\#(E) = N$. We assume that E is labeled: $E = \{x_1, \cdots, x_N\}$. We identify $C^2(E) \cong \mathbb{R}^N$ with respect to the standard basis $\{\xi_1, \cdots, \xi_N\}$ for $\mathbb{R}^N$.

Definition 4.3. For each $x \in \mathbb{R}^2$ and $\varphi \in C^2(E)$, we define a blob

$$\vec{\Sigma}_\varphi(x) = (\Sigma_\varphi(x, M))_{M \geq 0} \text{ where}$$
$$\Sigma_\varphi(x, M) := \left\{ P \in \mathcal{P} : \begin{matrix} \text{There exists } G \in C^2(\mathbb{R}^2) \text{ with} \\ \|G\|_{C^2(\mathbb{R}^2)} \leq M, G|_E = \varphi, \text{ and } \mathcal{J}_x G = P. \end{matrix} \right\} \tag{4.6}$$

It is clear from the definition of σ in (2.21) that

$$\sigma(x, E) = \Sigma_0(x, 1).$$

Therefore, thanks to Lemma 2.3, we have

$$C^{-1} \cdot \sigma^\sharp(x, 16) \subset \Sigma_0(x, 1) \subset C \cdot \sigma^\sharp(x, 16), \ \ x \in E \tag{4.7}$$

for some universal constant C.

We summarize some relevant results from [10].

Lemma 4.1. *Let $E \subset \mathbb{R}^2$ be finite. Using at most $CN \log N$ operations and CN storage, we can compute a list of PALPs $\{\underline{\mathcal{A}}(x) : x \in E\}$ such that the following hold.*

(A) There exists a universal constant D_0 such that for each $x \in E$, $\underline{\mathcal{A}}(x)$ has length no greater than $3 = \dim \mathcal{P}$ and has depth D_0.
(B) For each given $x \in \mathbb{R}^2$ and $\varphi \in C^2(E)$, the blobs $\vec{\mathcal{K}}_\varphi(\underline{\mathcal{A}}(x))$ as in (4.4) and $\vec{\Sigma}_\varphi(x)$ as in (4.6) are C-equivalent.

See Section 11 of [10] for Lemma 4.1(A), and Sections 10, 11, and Lemma 34.3 of [10] for Lemma 4.1(B).

The main lemma of this section is the following.

Lemma 4.2. *Let $E \subset \mathbb{R}^2$ be given. Let $\{\underline{\mathcal{A}}(x) : x \in E\}$ be as in Lemma 4.1. Recall the definitions of σ and $S(\underline{\mathcal{A}}(x))$ as in (2.21) and (4.3). Then there exists a universal constant C such that, for each $x \in E$,*

$$C^{-1} \cdot \sigma(x, S(\underline{\mathcal{A}}(x))) \subset \sigma^\sharp(x, 16) \subset C \cdot \sigma(x, S(\underline{\mathcal{A}}(x))).$$

Proof. For centrally symmetric $\sigma, \sigma' \subset \mathcal{P}$, we write $\sigma \approx \sigma'$ if there exists a universal constant C such that $C^{-1} \cdot \sigma \subset \sigma' \subset C \cdot \sigma$. Thus, we need to show $\sigma(x, \underline{\mathcal{A}}(x)) \approx \sigma^\sharp(x, 16)$ for $x \in E$.

Thanks to Lemma 2.3, Lemma 4.1(B) (applied to $\varphi \equiv 0$), (4.5), and (4.7), we have

$$\sigma^\sharp(x, 16) \approx \sigma^\sharp(x, E) \approx \mathcal{K}_0(\underline{\mathcal{A}}(x), 1) = \sigma(\underline{\mathcal{A}}(x)) \text{ for } x \in E. \tag{4.8}$$

Therefore, it suffices to show that

$$\sigma(x, S(\underline{\mathcal{A}}(x))) \approx \sigma(\underline{\mathcal{A}}(x)) \text{ for } x \in E.$$

From (4.8) and the definition of σ in (2.21), we see that

$$\sigma(\underline{\mathcal{A}}(x)) \subset C \cdot \sigma(x, E) \subset C \cdot \sigma(x, S(\underline{\mathcal{A}}(x))).$$

It remains to show that

$$\sigma(x, S(\underline{\mathcal{A}}(x))) \subset C \cdot \sigma(\underline{\mathcal{A}}(x)).$$

Let $x \in E$ and let $P \in \sigma(x, S(\underline{\mathcal{A}}(x)))$. Then there exists $\varphi \in C^2(\mathbb{R}^2)$ such that $\|\varphi\|_{C^2(\mathbb{R}^2)} \leq 1$, $\varphi(x) = 0$ for all $x \in S(\underline{\mathcal{A}}(x))$, and $\mathcal{J}_x(\varphi) = P$. Note that $\varphi|_E \in C^2(E)$. We abuse notation and write φ in place of $\varphi|_E$ when there is no possibility of confusion.

It is clear from the definition of $\Sigma_\varphi(x, M)$ in (4.6) that

$$P \in \Sigma_\varphi(x, 1).$$

By Lemma 4.1(B), we have

$$P \in \mathcal{K}_\varphi(\underline{\mathcal{A}}(x), C)$$

with $\mathcal{K}_\varphi(\underline{\mathcal{A}}(x), C)$ as in (4.4). In particular, we have

$$|\underline{\lambda}_i(P) - \underline{b}_i(\varphi)| \leq C\epsilon_i \text{ for } i = 1, \cdots, L = \text{length}(\underline{\mathcal{A}}(x)). \tag{4.9}$$

Here, the $\underline{\lambda}_1, \cdots, \underline{\lambda}_L$, $\underline{b}_1, \cdots, \underline{b}_L$, and $\epsilon_1, \cdots, \epsilon_L$, respectively, are the linear functionals, target functionals, and the thresholds of $\underline{\mathcal{A}}(x)$. However, by the definition of $S(\underline{\mathcal{A}}(x))$ in (4.3) and the fact that $\varphi \equiv 0$ on $S(\underline{\mathcal{A}}(x))$, we see that (4.9) simplifies to

$$|\underline{\lambda}_i(P)| \leq C\epsilon_i \text{ for } i = 1, \cdots, L = \text{length}(\underline{\mathcal{A}}(x)).$$

This is equivalent to the statement

$$P \in \mathcal{K}_0(\underline{\mathcal{A}}(x), C) = C \cdot \sigma(\underline{\mathcal{A}}(x)).$$

Lemma 4.2 is proved. □

5. Algorithm 2: Computing a C-optimal interpolant

Let $E \subset \mathbb{R}^2$ be a finite set. We fix E throughout the rest of the paper.

5.1. Calderón-Zygmund squares

Let $\tilde{\sigma} \subset \mathbb{R}^2$ be a symmetric convex set. We define

$$\operatorname{diam} \tilde{\sigma} := 2 \cdot \sup_{u \in \mathbb{R}^2, |u|=1} p_{\tilde{\sigma}}(u), \tag{5.1}$$

where $p_{\tilde{\sigma}}(u)$ is a gauge function given by

$$p_{\tilde{\sigma}}(u) := \sup\{r \geq 0 : ru \subset \tilde{\sigma}\}. \tag{5.2}$$

Let $\{\underline{\mathcal{A}}(x) : x \in E\}$ be as in Lemma 4.1, and let $\sigma(\underline{\mathcal{A}}(x)) \subset \mathcal{P}$ be as in (4.5). Note that for each $x \in E$, $\sigma(\underline{\mathcal{A}}(x)) \subset \mathcal{P}$ two-dimensional. Indeed, thanks to Lemma 4.1(B) (with $\varphi \equiv 0$), any $P \in \sigma(\underline{\mathcal{A}}(x))$, $x \in E$, must have $P(x) = 0$. Thus, for each $x \in E$, we can identify $\sigma(\underline{\mathcal{A}}(x))$ as a subset of $\mathbb{R}^2$ via the map

$$\sigma(\underline{\mathcal{A}}(x)) \ni P \mapsto (\nabla P \cdot e_1, \nabla P \cdot e_2), \tag{5.3}$$

where $\{e_1, e_2\}$ is the chosen orthonormal system.

Let $A_1, A_2 > 0$ be sufficiently large dyadic numbers. Let $\{\underline{\mathcal{A}}(x) : x \in E\}$ be as in Lemma 4.1. We say a dyadic square $\underline{Q \text{ is OK}}$ if the following hold.

- Either $\#(E \cap 5Q) \leq 1$, or $\operatorname{diam} \sigma(\underline{\mathcal{A}}(x)) \geq A_1 \delta_Q$ for all $x \in E \cap 5Q$. Here and below, the $\operatorname{diam}(\sigma(\underline{\mathcal{A}}(x)))$ is defined using the formula (5.1) via the identification (5.3).
- $\delta_Q \leq A_2^{-1}$.

Definition 5.1. We write Λ_0 to denote the collection of dyadic squares Q such that both of the following hold.

(A) Q is OK (see above).
(B) Suppose $\delta_Q < A_2^{-1}$, then Q^+ is not OK.

Remark 5.1. Note that there are two differences in the definition of Λ_0 than those in [12,13].

- We use $5Q$ in the definition of Λ_0 instead of using $2Q$. This has the advantage that $5Q^+ \subset 5^2 Q = 25Q$.
- We do not require $\operatorname{diam} \sigma(\underline{\mathcal{A}}(x)) \geq A_1 \delta_Q$ for $x \in E \cap 5Q$ when $\#(E \cap 5Q) = 1$.

We will provide explanation when these differences change the structure of the analysis. Otherwise, we will simply add the word "variant" to our reference to results in [12,13].

Lemma 5.1. *Λ_0 enjoys the following properties.*

(A) Λ_0 forms a cover of $\mathbb{R}^2$ with good geometry:

(A1) $\mathbb{R}^2 = \bigcup_{Q\in\Lambda_0} Q$;
(A2) If $Q, Q' \in \Lambda_0$ with $(1+2c_G)Q \cap (1+2c_G)Q' \neq \varnothing$, then

$$C^{-1}\delta_Q \leq \delta_{Q'} \leq C\delta_Q;$$

and as a consequence, for each $Q \in \Lambda_0$,

$$\#\{Q' \in \Lambda_0 : (1+c_G)Q' \cap (1+c_G)Q \neq \varnothing\} \leq C'.$$

Here, C, C' are universal constants, and c_G is a sufficiently small constant, say $1/32$.

(B) Let $Q \in \Lambda_0$. Then there exists $\varphi \in C^2(\mathbb{R})$ such that

$$\rho(E \cap 5Q) \subset \{(t, \varphi(t)) : t \in \mathbb{R}\}, \tag{5.4}$$

where ρ is some rotation about the origin depending only on Q. Moreover, φ satisfies the estimates

$$\left|\frac{d^m}{dt^m}\varphi(t)\right| \leq CA_1^{-1}\delta_Q^{1-m} \text{ for } m = 1, 2, \tag{5.5}$$

with A_1 as in Definition 5.1. Furthermore, suppose for some $x_0 \in E \cap 5Q$ and a unit vector u_0, we have

$$\operatorname{diam} \sigma(\underline{\mathcal{A}}(x_0)) = p_{\sigma(\underline{\mathcal{A}}(x_0))}(u_0)$$

with $\operatorname{diam} \sigma(\underline{\mathcal{A}}(x_0))$ and $p_{\sigma(\underline{\mathcal{A}}(x_0))}(u_0)$ as in (5.1) and (5.2). Then we can take φ to satisfy the following property:

(B1) We can take ρ in (5.4) to be the rotation specified by $u_0 \mapsto e_2$;
(B2) We can take $x_0 = (0, \varphi(0))$.

As a consequence, there exists a C^2-diffeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\Phi \circ \rho(t_1, t_2) = (t_1, t_2 - \varphi(t_1)) \text{ where } \rho \text{ is the rotation as in (5.4)},$$

such that $\Phi(E \cap 5Q) \subset \mathbb{R} \times \{t_2 = 0\}$ *and* $|\nabla^m \Phi|, |\nabla^m \Phi^{-1}| \leq CA_1^{-1}\delta_Q^{1-m}$ *for* $m = 1, 2$*, with* A_1 *as in Definition 5.1.*

Remark 5.2. Lemma 5.1(A) can be found in Section 21 of [10]. See also Lemma 5.1 of [13]. Lemma 5.1(B) follows from the proofs of Lemma 5.4 and 5.5 with a minor modification: For $Q \in \Lambda_0$ with $\#(E \cap 5Q) \leq 1$, we can simply take φ to be a suitable constant function on $\mathbb{R}$.

We recall the following results from [10].

Lemma 5.2. *After one-time work using at most* $CN \log N$ *operations and* CN *storage, we can perform each of the following tasks using at most* $C \log N$ *operations.*

- *(A)* (Section 26 of [10]) *Given a point* $x \in \mathbb{R}^2$*, we compute a list* $\Lambda(x) := \{Q \in \Lambda_0 : (1 + c_G)Q \ni x\}$*.*
- *(B)* (Section 27 of [10]) *Given a dyadic square* $Q \subset \mathbb{R}^2$*, we can compute* $\mathsf{Empty}(Q)$*, with* $\mathsf{Empty}(Q) = \mathsf{True}$ *if* $E \cap 25Q = \varnothing$*, and* $\mathsf{Empty}(Q) = \mathsf{False}$ *if* $E \cap 25Q \neq \varnothing$*.*
- *(C)* (Section 27 of [10]) *Given a dyadic square* $Q \subset \mathbb{R}^2$ *with* $E \cap 25Q \neq \varnothing$*, we can compute* $\mathsf{Rep}(Q) \in E \cap 25Q$*, with the property that* $\mathsf{Rep}(Q) \in E \cap 5Q$ *if* $E \cap 5Q \neq \varnothing$*.*

Definition 5.2. We define the following subcollections of Λ_0:

(5.6) $\Lambda^{\sharp\sharp} := \left\{Q \in \Lambda^\sharp : E \cap (1 + c_G)Q \neq \varnothing\right\};$

(5.7) $\Lambda^\sharp := \{Q \in \Lambda_0 : E \cap 5Q \neq \varnothing\};$

(5.8) $\Lambda_{\text{empty}} := \left\{Q \in \Lambda_0 \setminus \Lambda^\sharp : \delta_Q < A_2^{-1}\right\}$ with A_2 as in Definition 5.1.

We can think of $\Lambda^{\sharp\sharp}$ as the collection of squares with the most "concentrated" information, $\Lambda^\sharp$ as the largest collection of squares that contain information while still having good local geometry, and Λ_{empty} as the collection of squares that do not contain information in their five-time dilation, but are sufficiently small to detect nearby accumulation of points in E.

We begin with the analysis of Λ_{empty} and $\Lambda^\sharp$.

Lemma 5.3. *After one-time work using at most* $CN \log N$ *operations and* CN *storage, we can perform the following task using at most* $C \log N$ *operations: Given* $Q \in \Lambda_0$*, we can decide if* $Q \in \Lambda^\sharp$*,* $Q \in \Lambda_{\text{empty}}$*, or* $Q \in \Lambda_0 \setminus (\Lambda^\sharp \cup \Lambda_{\text{empty}})$*.*

Proof. This is a direct application of Lemma 5.2(B,C) to Q. $\square$

The next lemma tells us how to relay information to squares in Λ_{empty}.

Lemma 5.4. *We can compute a map*

$$\mu : \Lambda_{\mathrm{empty}} \to \Lambda^{\sharp} \tag{5.9}$$

that satisfies

$$(1 + c_G)\mu(Q) \cap 25Q \neq \varnothing \text{ for } Q \in \Lambda_{\mathrm{empty}}. \tag{5.10}$$

The one-time work uses at most $CN \log N$ *operations and* CN *storage. After that, we can answer queries using at most* $C \log N$ *operations. A query consists of a square* $Q \in \Lambda_{\mathrm{empty}}$*, and the response to the query is another square* $\mu(Q)$ *that satisfies* (5.10).

Proof. Suppose $Q \in \Lambda_{\mathrm{empty}}$. Then we have $E \cap 5Q^{+} \neq \varnothing$. By the geometry of Λ_0, we have $5Q^{+} \subset 25Q$. Hence, $E \cap 25Q \neq \varnothing$. Therefore, the map Rep in Lemma 5.2(C) is defined for Q.

We set

$$x := \mathsf{Rep}(Q) \subset E \cap 25Q, \tag{5.11}$$

with Rep as in Lemma 5.2. Note that $x \notin 5Q$, since $Q \in \Lambda_{\mathrm{empty}}$.

Let $\Lambda(x) \subset \Lambda_0$ be as in Lemma 5.2(A). Let $Q' \in \Lambda(x)$. By the defining property of $\Lambda(x)$ and the fact that $x \in E$, we have $Q' \in \Lambda^{\sharp}$. Set

$$\mu(Q) := Q' \in \Lambda^{\sharp}.$$

By the previous comment, we have

$$(1 + c_G)\mu(Q) \ni x. \tag{5.12}$$

Combining (5.11) and (5.12), we see that $(1 + c_G)\mu(Q) \cap 25Q \neq \varnothing$. (5.10) is satisfied.

By Lemma 5.2(A,C), the tasks $\Lambda(\cdot)$ and $\mathsf{Rep}(\cdot)$ require at most $C \log N$ operations, after one-time work using at most $CN \log N$ operations and CN storage. Therefore, computing $\mu(Q)$ requires at most $C \log N$ operations, after one-time work using at most $CN \log N$ operations and CN storage.

This proves Lemma 5.4. □

Lemma 5.5. *After one-time work using at most* $CN \log N$ *operations and* CN *storage, we can perform the following task using at most* $C \log N$ *operations: Given* $Q \in \Lambda^{\sharp}$*, compute a pair of unit vectors* $u_Q, u_Q^{\perp} \in \mathbb{R}^2$*, such that the following hold.*

(A) u_Q *is orthogonal to* $u_Q^{\perp}$*, and the orthogonal system* $[u_Q^{\perp}, u_Q]$ *has the same orientation as* $[e_1, e_2]$.

(B) Let ρ *be the rotation about the origin specified by* $u_Q \mapsto e_2$*, then there exists a function* $\varphi \in C^2(\mathbb{R})$ *that satisfies* (5.4) *and* (5.5) *with this particular* ρ.

Proof. Fix $Q \in \Lambda^\sharp$. This means that $E \cap 5Q \neq \varnothing$. In particular, $\mathsf{Rep}(Q)$ is defined, and by Lemma 5.2(C),

$$x_0 := \mathsf{Rep}(Q) \in E \cap 5Q.$$

Computing x_0 requires at most $C \log N$ operations, after one-time work using at most $CN \log N$ operations and CN storage.

Let $\underline{\mathcal{A}}(x_0)$ be as in Lemma 4.1, and let $\sigma(\underline{\mathcal{A}}(x_0))$ be as in (4.5). By Lemma 4.1(B) (with $\varphi \equiv 0$), any $P \in \sigma(\underline{\mathcal{A}}(x_0))$ must satisfy $P(x_0) = 0$. by Lemma 4.1(A) and definitions (4.4), (4.5) of $\sigma(\underline{\mathcal{A}}(x_0))$, we see that $\sigma(\underline{\mathcal{A}}(x_0))$ is a two-dimensional parallelogram in $\mathcal{P}$ centered at the zero polynomial. Therefore, we have

$$\operatorname{diam} \sigma(\underline{\mathcal{A}}(x_0)) = \mathsf{length}(\Delta_0),$$

where diam is defied in (5.1) and Δ_0 is the longer diagonal of $\sigma(\underline{\mathcal{A}}(x_0))$.

Set u_Q to be a unit vector parallel to Δ_0. Lemma 5.5(B) then follows from Lemma 5.1(B).

We compute another vector $u_Q^\perp$ such that $\{u_Q, u_Q^\perp\}$ satisfies Lemma 5.5(A). Computing $\{u_Q, u_Q^\perp\}$ from $\sigma(\underline{\mathcal{A}}(x_0))$ uses elementary linear algebra, and requires at most C operations.

Lemma 5.5 is proved. □

Lemma 5.6. *After one-time work using at most* $CN \log N$ *operations and* CN *storage, we can perform the following task using at most* $C \log N$ *operations: Given* $Q \in \Lambda_0$*, we can compute a point* $x_Q^\sharp \in Q$ *such that*

$$\operatorname{dist}\left(x_Q^\sharp, E\right) \geq c_0 \delta_Q \tag{5.13}$$

for some universal constant $c_0 \geq 0$.

Proof. Let $Q \in \Lambda_0$ be given.

Suppose $\mathsf{Empty}(Q) = \mathsf{True}$, with $\mathsf{Empty}(\cdot)$ as in Lemma 5.2(B). We set

$$x_Q^\sharp := \mathsf{center}(Q).$$

It is clear that $x_Q^\sharp \in Q$ and (5.13) holds.

Suppose $\mathsf{Empty}(Q) = \mathsf{False}$. Let $x_0 := \mathsf{Rep}(Q) \in E \cap 25Q$.

Suppose $x_0 \notin 5Q$, then $E \cap 5Q = \varnothing$ by Lemma 5.2(C). Again, we set

$$x_Q^\sharp := \mathsf{center}(Q).$$

It is clear that $x_Q^\sharp \in Q$ and (5.13) holds.

Suppose $x_0 \in 5Q$. This means that $Q \in \Lambda^\sharp$ with $\Lambda^\sharp$ as in (5.7). Let u_Q be as in Lemma 5.5.

By Lemma 5.1(B), we have $E \cap 5Q \subset \{(t, \varphi(t)) : t \in \mathbb{R}\}$ up to the rotation $u_Q \mapsto e_2$, and the function φ satisfies $\left|\frac{d^m}{dt^m}\varphi(t)\right| \leq CA_1^{-1}\delta_Q^{1-m}$ for $m = 1, 2$, with A_1 as in Definition 5.1. Therefore, by the defining property of u_Q in Lemma 5.5, we have

$$E \cap 5Q \subset \left\{y \in \mathbb{R}^2 : |(y - x_0) \cdot u_Q| \leq CA_1^{-1}|y - x_0|\right\} =: Z(x_0).$$

Suppose $\operatorname{dist}(\mathrm{center}(Q), Z(x_0)) \geq \delta_Q/1024$. We set

$$x_Q^\sharp := \mathrm{center}(Q).$$

In this case, it is clear that $x_Q^\sharp \in Q$ and (5.13) holds.

Suppose $\operatorname{dist}(\mathrm{center}(Q), Z(x_0)) < \delta_Q/1024$. We set

$$x_Q^\sharp := \mathrm{center}(Q) + \frac{\delta_Q}{4} \cdot u_Q.$$

It is clear that $x_Q^\sharp \in Q$. For sufficiently large A_1, we also have $\operatorname{dist}\left(x_Q^\sharp, Z(x_0)\right) \geq c\delta_Q$ for some constant c depending only on A_1. Thus, (5.13) holds.

After one-time work using at most $CN\log N$ operations and CN storage, the procedure $\mathrm{Empty}(Q)$ requires at most $C\log N$ operations by Lemma 5.2(B); the procedure $\mathrm{Rep}(Q)$ requires at most $C\log N$ operations by Lemma 5.2(C); computing the vector u_Q requires at most $C\log N$ operations; and computing the distance between $\mathrm{center}(Q)$ and $Z(x_0)$ is a routine linear algebra problem, and requires at most C operations.

Lemma 5.6 is proved. □

We now turn our attention to $\Lambda^{\sharp\sharp}$ as in (5.6).

Lemma 5.7. *Using at most* $CN\log N$ *operations and* CN *storage, we can compute the list* $\Lambda^{\sharp\sharp}$ *as in* (5.6).

Proof. This is a direct application of Lemma 5.2(A) to each $x \in E$. □

The next lemma states that we can efficiently sort the data contained in squares in $\Lambda^{\sharp\sharp}$.

Lemma 5.8. *Using at most* $CN\log N$ *operations and* CN *storage, we can compute the following.*

For each $Q \in \Lambda^{\sharp\sharp}$ *with* $\Lambda^{\sharp\sharp}$ *as in* (5.6), *we can compute a sorted list of numbers*

$$\mathrm{Proj}_{u_Q^\perp}(E \cap (1 + c_G)Q - \mathrm{Rep}(Q)) \subset \mathbb{R},$$

where $u_Q^\perp$ *is as in Lemma 5.5,* $\mathsf{Proj}_{u_Q^\perp}$ *is the orthogonal projection onto* $\mathbb{R}u_Q^\perp$*, and* $\mathsf{Rep}(Q)$ *is as in Lemma 5.2(C).*

Proof. By the bounded intersection property in Lemma 5.1(A), we have

$$\#(\Lambda^{\sharp\sharp}) \le CN. \tag{5.14}$$

From the definitions of $\Lambda^{\sharp\sharp}$ and $\Lambda^\sharp$ in (5.6) and (5.7), we see that $\Lambda^{\sharp\sharp} \subset \Lambda^\sharp$. Therefore, we can compute $\mathsf{Rep}(Q)$ and $u_Q^\perp$ for each $Q \in \Lambda^{\sharp\sharp}$ using at most $C \log N$ operations, by Lemma 5.2(B) and Lemma 5.5.

Recall from Lemma 5.7 that we can compute the list $\Lambda^{\sharp\sharp}$ by computing each $\Lambda(x)$ for $x \in E$, with $\Lambda(x)$ as in Lemma 5.2(A). During this procedure, we can store the information $(1+c_G)Q \ni x$ for $Q \in \Lambda(x)$.

By the bounded intersection property in Lemma 5.1(A), we have

$$\sum_{Q\in\Lambda^{\sharp\sharp}} \#(E \cap (1+c_G)Q) \le CN. \tag{5.15}$$

By Lemma 5.2(A) and (5.15), we can compute the list

$$\left\{E \cap (1+c_G)Q : Q \in \Lambda^{\sharp\sharp}\right\}$$

using at most $CN \log N$ operations and CN storage. Then, by Lemma 5.2(C), Lemma 5.5, and (5.14), we can compute the *unsorted* list

$$\mathsf{Proj}_{u_Q^\perp}(E \cap (1+c_G)Q - \mathsf{Rep}(Q)) \tag{5.16}$$

for each $Q \in \Lambda^{\sharp\sharp}$ using at most $CN \log N$ operations and CN storage.

For each $Q \in \Lambda^{\sharp\sharp}$, we can sort the list $\mathsf{Proj}_{u_Q^\perp}(E \cap (1+c_G)Q - \mathsf{Rep}(Q))$ using at most $CN_Q \log N_Q$ operations, where $N_Q := \#(E \cap (1+c_G)Q)$. By (5.15), we can sort the all the lists of the form (5.16) associated with each $Q \in \Lambda^{\sharp\sharp}$ using at most $CN \log N$ operations.

Lemma 5.8 is proved. □

5.2. Local clusters

The next lemma shows how to relay local information to the point $x_Q^\sharp$.

Lemma 5.9. *Let* $Q \in \Lambda^\sharp$. *Let* $x_Q^\sharp$ *be as in Lemma 5.6. Let* $x \in E \cap 5Q$. *Let* $\underline{\mathcal{A}}(x)$ *be as in Lemma 4.1. Let* $S(\underline{\mathcal{A}}(x))$ *be as in* (4.3). *Then*

$$\sigma(x_Q^\sharp, S(\underline{\mathcal{A}}(x))) \subset C \cdot \sigma^\sharp(x_Q^\sharp, 16). \tag{5.17}$$

Proof. Fix x as in the hypothesis. By our choice of $x_Q^\sharp$ in Lemma 5.6, we have

$$\left|x_Q^\sharp - x\right| \geq C\delta_Q. \tag{5.18}$$

Let $P_0 \in \sigma(x_Q^\sharp, S(\underline{\mathcal{A}}(x)))$. By the definition of σ, there exists $\varphi \in C^2(\mathbb{R}^2)$ with $\|\varphi\|_{C^2(\mathbb{R}^2)} \leq 1$, $\varphi|_{S(\underline{\mathcal{A}}(x))} = 0$, and $\mathcal{J}_{x_Q^\sharp}\varphi = P_0$. Set $P := \mathcal{J}_x\varphi$. Then

$$P \in \sigma(x, S(\underline{\mathcal{A}}(x))).$$

Since $x \in E$, by Lemma 4.2, we have

$$P \in \sigma^\sharp(x, 16).$$

Let $S \subset E$ with $\#(S) \leq 16$. By the definition of $\sigma^\sharp$ in (2.21) and Taylor's theorem, there exists a Whitney field $\vec{P} = (P, (P^y)_{y\in S}) \in W^2(S \cup \{x\})$, with $\|\vec{P}\|_{W^2(S\cup\{x\})} \leq C$ and $P^y(y) = 0$ for $y \in S$.

Consider another Whitney field $\vec{P}_0 = (P_0, (P^y)_{y\in S}) \in W^2(S \cup \{x_Q^\sharp\})$ defined by replacing P by P_0 in $\vec{P}$. By the classical Whitney Extension Theorem for finite sets, it suffices to show that $\vec{P}_0$ satisfies

$$P^y(y) = 0 \text{ for } y \in S, \text{ and} \tag{5.19}$$

$$\|\vec{P}_0\|_{W^2(S\cup\{x_Q^\sharp\})} \leq C. \tag{5.20}$$

Note that (5.19) is obvious by construction.

We turn to (5.20).

Since $P_0 = \mathcal{J}_{x_Q^\sharp}\varphi$ and $P = \mathcal{J}_x\varphi$, Taylor's theorem implies

$$\left|\partial^\alpha(P - P_0)(x_Q^\sharp)\right|, |\partial^\alpha(P - P_0)(x)| \leq C\left|x - x_Q^\sharp\right|^{2-|\alpha|} \text{ for } |\alpha| \leq 1. \tag{5.21}$$

Since the Whitney field $\vec{P} = (P, (P^y)_{y\in S})$ satisfies $\|\vec{P}\|_{W^2(S\cup\{x\})} \leq C$, we have

$$\|(P^y)_{y\in S}\|_{W^2(S)} \leq C, \tag{5.22}$$

and

$$|\partial^\alpha(P - P^y)(x)|, |\partial^\alpha(P - P^y)(y)| \leq C|x - y|^{2-|\alpha|} \text{ for } |\alpha| \leq 2, y \in S. \tag{5.23}$$

Applying the triangle inequality to (5.21) and (5.23), and using (5.18), we see that

$$\left|\partial^\alpha(P_0 - P^y)(x_Q^\sharp)\right|, |\partial^\alpha(P_0 - P^y)(y)| \leq C\left|x_Q^\sharp - y\right|^{2-|\alpha|} \text{ for } |\alpha| \leq 1. \tag{5.24}$$

Moreover, since $P_0 \in \sigma(x_Q^\sharp, S(\underline{\mathcal{A}}(x)))$, we have

$$\left|\partial^\alpha P_0(x_Q^\sharp)\right| \leq 1 \text{ for } |\alpha| \leq 1. \tag{5.25}$$

Then, (5.20) follows from (5.22), (5.24), and (5.25).

Lemma 5.9 is proved. □

Let $Q \in \Lambda^\sharp$ with $\Lambda^\sharp$ as in (5.7). Let $\underline{\mathcal{A}}(x), x \in E$ be as in Lemma 4.1. Let $S(\underline{\mathcal{A}}(x))$ be as in (4.3). Let $\mathrm{Rep}(Q)$ be as in Lemma 5.2(C). Let $x_Q^\sharp$ be as in Lemma 5.6. We set

$$S^\sharp(Q) := S(\underline{\mathcal{A}}(\mathrm{Rep}(Q))) \cup \{\mathrm{Rep}(Q)\} \cup \{x_Q^\sharp\}. \tag{5.26}$$

Note that $x_Q^\sharp$ is not a point in E.

5.3. Transition jets

In this section, we want construct a map $T_Q : C_+^2(E) \times [0, \infty) \to \mathcal{P}$ of bounded depth, such that $T_Q(f, M) \in \Gamma_+^\sharp(x_Q^\sharp, 16, CM, f)$ for all $(f, M) \in C_+^2(E) \times [0, \infty)$ with $\|f\|_{C_+^2(E)} \leq M$. We will explain the importance of $\Gamma_+^\sharp(x_Q^\sharp, 16, CM, f)$ in Remark 5.5 towards the end of the section.

Let $S \subset E$. As in (2.3) and (2.4), we consider the following functions, depending on the choice of S:

$$\begin{aligned} &\mathcal{Q}^\sharp : W^2(S) \to [0, \infty) \\ &\vec{P} = (P^x)_{x \in S} \mapsto \sum_{x \in S} |\partial^\alpha P^x(x)| + \sum_{\substack{x,y \in S^\sharp(Q) \\ x \neq y \\ |\alpha| \leq 1}} \frac{|\partial^\alpha (P^x - P^y)(x)|}{|x-y|^{2-|\alpha|}}, \end{aligned} \tag{5.27}$$

and

$$\begin{aligned} &\mathcal{M}^\sharp : W_+^2(S) \to [0, \infty] \\ &(P^x)_{x \in S} \mapsto \begin{cases} \sum_{x \in S} \frac{|\nabla P^x|^2}{P^x(x)} & \text{if } P^x(x) \geq 0 \text{ for each } x \in S \\ \infty & \text{if there exists } x \in S^\sharp(Q) \text{ such that } P^x(x) < 0 \end{cases}. \end{aligned} \tag{5.28}$$

We adopt the conventions that $\frac{0}{0} = 0$ and $\frac{a}{0} = \infty$ for $a > 0$.

For the rest of the section, we fix $Q \in \Lambda^\sharp$, with $\Lambda^\sharp$ as in (5.7). Let $x_Q^\sharp$ be as in Lemma 5.6. Let $S^\sharp(Q)$ be as in (5.26). Recall from (5.26) that $\mathrm{Rep}(Q) \in S^\sharp(Q)$, with Rep as in Lemma 5.2(C).

Let $f \in C^2_+(E)$ be given. We define

$$\mathbb{A}^0_f := \left\{ \vec{P} \in W^2_+(S^\sharp(Q)) : \begin{array}{l} (\vec{P}, x^\sharp_Q) \equiv 0 \text{ and} \\ (\vec{P}, x)(x) = f(x) \text{ for } x \in S^\sharp(Q) \cap E \end{array} \right\}, \text{ and}$$
$$\mathbb{A}^1_f := \left\{ \vec{P} \in W^2_+(S^\sharp(Q) \cap E) : (\vec{P}, x)(x) = f(x) \text{ for } x \in S^\sharp(Q) \cap E \right\}. \tag{5.29}$$

We note that $\mathbb{A}^0_f$ and $\mathbb{A}^1_f$ are affine subspaces of $W^2(S^\sharp(Q))$ and $W^2(S^\sharp(Q) \cap E)$, respectively. They depend only on $f|_{S^\sharp(Q)\cap E}$.

Consider the following minimization problems.

(M0) Let $S = S^\sharp(Q)$ in (5.27) and (5.28). Minimize $\mathcal{Q}^\sharp + \mathcal{M}^\sharp$ over $\mathbb{A}^0_f$.
(M1) Let $S = S^\sharp(Q) \cap E$ in (5.27) and (5.28). Minimize $\mathcal{Q}^\sharp + \mathcal{M}^\sharp$ over $\mathbb{A}^1_f$.

For $\star = 0, 1$, we say a Whitney field $\vec{P} \in \mathbb{A}^\star_f$ is an approximate minimizer of (M$\star$) if

- $(\mathcal{Q}^\sharp + \mathcal{M}^\sharp)(\vec{P}) \le C \cdot \inf \left\{ (\mathcal{Q}^\sharp + \mathcal{M}^\sharp)(\vec{P}') : \vec{P}' \in \mathbb{A}^\star_f \right\}$ for some universal constant C.

Remark 5.3. Recall from Section 2.2 that both (M0) and (M1) can be reformulated as convex quadratic programming problems with affine constraint, and are efficiently solvable [2]. Thus, we can solve for an approximate minimizer of (M$\star$), $\star = 0, 1$, using at most C operations, since $\#(S^\sharp(Q))$ is universally bounded. We call the approximate minimizers for (M0) and (M1) obtained this way $\vec{P}^\sharp_0$ and $\vec{P}^\sharp_1$. Note that $\vec{P}^\sharp_0$ and $\vec{P}^\sharp_1$, respectively, are uniquely determined by $\mathbb{A}^0_f$ and $\mathbb{A}^1_f$.

Lemma 5.10. *Let $Q \in \Lambda^\sharp$. Let $x^\sharp_Q$ be as in Lemma 5.6. Let $(f, M) \in C^2_+(E) \times [0, \infty)$ with $\|f\|_{C^2_+(E)} \le M$. Let $\vec{P} = (P^x)_{x \in S^\sharp(Q)\cap E}$ be an approximate minimizer of (M1) above. Let $P^{\mathsf{Rep}(Q)}$ be the polynomial associated with the point $\mathsf{Rep}(Q)$, i.e., $P^{\mathsf{Rep}(Q)} = (\vec{P}, \mathsf{Rep}(Q))$, with Rep as in Lemma 5.2(C). Let $T^{\mathsf{Rep}(Q)}_w$ be the Whitney extension operator associated with the singleton $\{\mathsf{Rep}(Q)\}$ as in Lemma 2.1(B). Then*

$$\mathcal{J}_{x^\sharp_Q} \circ T^{\mathsf{Rep}(Q)}_w (P^{\mathsf{Rep}(Q)}) \in \Gamma_+(x^\sharp_Q, S^\sharp(Q) \cap E, CM, f).$$

Proof. Let $\vec{P}$ be as in the hypothesis. Let $P_1 := \mathcal{J}_{x^\sharp_Q} \circ T^{\mathsf{Rep}(Q)}_w (P^{\mathsf{Rep}(Q)})$. We adjoin P_1 to $\vec{P}$ to form

$$\vec{P}_1 := \left(P_1, (P^x)_{x \in S^\sharp(Q)\cap E}\right) \in W^2(S^\sharp(Q)).$$

Thanks to Lemma 2.1, it suffices to show that $\vec{P}_1 \in W^2_+(S^\sharp(Q))$ and $\|\vec{P}_1\|_{W^2_+(S^\sharp(Q))} \le CM$.

By Lemma 2.1(B), we see that $T^{\mathsf{Rep}(Q)}_w (P^{\mathsf{Rep}(Q)}) \in C^2_+(\mathbb{R}^2)$ with norm $\|T^{\mathsf{Rep}(Q)}_w (P^{\mathsf{Rep}(Q)})\|_{C^2(\mathbb{R}^2)} \le CM$. Therefore,

$$\left|\partial^\alpha P_1(x_Q^\sharp)\right| \leq CM \text{ for } |\alpha| \leq 1, \text{ and } |\nabla P_1| \leq \sqrt{CMP_1(x_Q^\sharp)}. \tag{5.30}$$

Thus, $\vec{P}_1 \in W^2_+(S^\sharp(Q))$.

Since $\vec{P}$ is an approximate minimizer of (M1) and $\|f\|_{C^2_+(E)} \leq M$, we have

$$\|\vec{P}\|_{W^2_+(S^\sharp(Q)\cap E)} \leq CM. \tag{5.31}$$

For $x \in S^\sharp(Q) \cap E$, we have

$$\begin{aligned}|\partial^\alpha(P^x - P_1)(x)| \leq & \left|\partial^\alpha(P^x - P^{\mathrm{Rep}(Q)})(x)\right| \\ & + \left|\partial^\alpha(P^{\mathrm{Rep}(Q)} - \mathcal{J}_{x_Q^\sharp} \circ T_w^{\mathrm{Rep}(Q)}(P^{\mathrm{Rep}(Q)}))(x)\right|.\end{aligned}$$

Using (5.31) to estimate the first term and Taylor's theorem to estimate the second, we have

$$|\partial^\alpha(P^x - P_1)(x)| \leq CM\left(|x - \mathrm{Rep}(Q)| + \left|x_Q^\sharp - \mathrm{Rep}(Q)\right|\right)^{2-|\alpha|} \leq CM\left|x - x_Q^\sharp\right|^{2-|\alpha|}. \tag{5.32}$$

For the last inequality, we use the fact that dist $\left(x_Q^\sharp, E\right) \geq c\delta_Q$, thanks to Lemma 5.6.

Applying Taylor's theorem to (5.32), we have

$$\left|\partial^\alpha(P^x - P_1)(x_Q^\sharp)\right| \leq CM\left|x - x_Q^\sharp\right|^{2-|\alpha|}. \tag{5.33}$$

Combining (5.30)–(5.33), we see that $\|\vec{P}_1\|_{W^2_+(S^\sharp(Q))} \leq CM$. Lemma 5.10 is proved. □

Definition 5.3. Let $Q \in \Lambda^\sharp$. Let $x_Q^\sharp$ be as in Lemma 5.6. We define

$$T_Q : C^2_+(E) \times [0, \infty) \to \mathcal{P}$$

by the following rule. Let $(f, M) \in C^2_+(E) \times [0, \infty)$ be given, and let (M0) and (M1) be as above. Let $\vec{P}_0^\sharp$ and $\vec{P}_1^\sharp$ be as in Remark 5.3.

(TQ-0) Suppose $\vec{P}_0^\sharp$ satisfies $(\mathcal{Q}^\sharp + \mathcal{M}^\sharp)(\vec{P}_0^\sharp) \leq C_T M$, for some large universal constant C_T. Then we set $T_Q(f, M) \equiv 0$.

(TQ-1) Otherwise, we set $T_Q(f, M) := \mathcal{J}_{x_Q^\sharp} \circ T_w^{\mathrm{Rep}(Q)}(P_1)$. Here, P_1 is the polynomial in $\vec{P}_1^\sharp$ associated with the point $\mathrm{Rep}(Q)$, i.e., $P_1 := (\vec{P}_1^\sharp, \mathrm{Rep}(Q))$; and $T_w^{\mathrm{Rep}(Q)}$ is the Whitney extension operator associated with the singleton $\{\mathrm{Rep}(Q)\}$ as in Lemma 2.1(B).

It is clear that T_Q has bounded depth, since $\vec{P}_0^\sharp$ and $\vec{P}_1^\sharp$ depend only on $f|_{S^\sharp(Q)\cap E}$.

Remark 5.4. Given $Q \in \Lambda^\sharp$ with $\Lambda^\sharp$ as in (5.7), $x_Q^\sharp$ as in Lemma 5.6, $S^\sharp(Q)$ as in (5.26), and $(f, M) \in C^2_+(E) \times [0, \infty)$ with $\|f\|_{C^2_+(E)} \leq M$, computing $T_Q(f, M)$ from the data above amounts to solving for approximate minimizers of (M0) and (M1). Thus, by Remark 5.3, we can compute $T_Q(f, M)$ from the data above using at most C operations.

Recall the following perturbation lemma from [13].

Lemma 5.11 *(variant of Lemmas 5.7 and 7.3 of [13]). Let $E \subset \mathbb{R}^2$ be finite. Let $Q \in \Lambda^\sharp$. Let $x_Q^\sharp$ be as in Lemma 5.6. Let $f \in C^2_+(E)$ be given. Suppose $\Gamma_+^\sharp(x_Q^\sharp, 16, M, f) \neq \varnothing$. The following are true.*

(A) There exists a number $B_0 > 0$ exceeding a large universal constant such that the following holds. Suppose $f(x) \geq B_0 M \delta_Q^2$ for each $x \in E \cap 5Q$. Then

$$\Gamma_+^\sharp(x_Q^\sharp, 16, M, f) + M \cdot \sigma^\sharp(x_Q^\sharp, 16) \subset \Gamma_+^\sharp(x_Q^\sharp, 16, CM, f),$$

for some universal constant C.

(B) Let $A > 0$. Suppose $f(x) \leq AM\delta_Q^2$ for some $x \in E \cap 5Q$. Then

$$0 \in \Gamma_+^\sharp(x_Q^\sharp, 16, A'M, f).$$

Here, A' depends only on A.

The main lemma of this section is the following.

Lemma 5.12. *Let $Q \in \Lambda^\sharp$ with $\Lambda^\sharp$ as in (5.7). Let $x_Q^\sharp$ be as in Lemma 5.6. Let T_Q be as in Definition 5.3. Let $(f, M) \in C^2_+(E) \times [0, \infty)$ with $\|f\|_{C^2_+(E)} \leq M$. Then*

$$T_Q(f, M) \in \Gamma_+^\sharp(x_Q^\sharp, 16, CM, f).$$

Proof. Since $\|f\|_{C^2_+(E)} \leq M$, we have $\Gamma_+^\sharp(x_Q^\sharp, 16, CM, f) \neq \varnothing$. Therefore, the hypotheses of Lemma 5.11 are satisfied.

Recall Definition 5.3.

Suppose $T_Q(f, M)$ is defined in terms of (TQ-0).

By Lemma 2.1, there exists $F \in C^2_+(\mathbb{R}^2)$ with $\|F\|_{C^2(\mathbb{R}^2)} \leq CM$, $F|_{S^\sharp(Q) \cap E} = f$, and $\mathcal{J}_{x_Q^\sharp} F \equiv 0$. Recall from Lemma 5.2(C) and (5.26) that $\mathrm{Rep}(Q) \in S^\sharp(Q) \cap 5Q$. Therefore, by Taylor's theorem, we have

$$f(\mathrm{Rep}(Q)) = F(\mathrm{Rep}(Q)) \leq CM\delta_Q^2.$$

By Lemma 5.11(B), we have $T_Q(f, M) \equiv 0 \in \Gamma_+^\sharp(x_Q^\sharp, 16, CM, f)$.

Suppose $T_Q(f, M)$ is defined in terms of (TQ-1).

For sufficiently large C_T, Taylor's theorem implies, with B_0 as in Lemma 5.11,

$$f(x) \geq B_0 M \delta_Q^2 \text{ for } x \in E \cap 5Q.$$

Thus, the hypothesis of Lemma 5.11(A) is satisfied.

Since $\|f\|_{C^2_+(E)} \leq M$, there exists

$$\hat{F} \in C^2_+(\mathbb{R}^2) \text{ with } \|\hat{F}\|_{C^2(\mathbb{R}^2)} \leq CM,\ \hat{F}|_E = f, \text{ and } \mathcal{J}_{x^\sharp_Q} \hat{F} \in \Gamma_+(x^\sharp_Q, E, CM, f).$$

By Lemma 5.10, we have

$$T_Q(f, M) \in \Gamma_+(x^\sharp_Q, S^\sharp(Q) \cap E, CM, f).$$

Therefore, by Lemma 5.9, the definition of $S^\sharp(Q)$ in (5.26), and the definition of σ in (2.21), we have

$$\mathcal{J}_{x^\sharp_Q} \hat{F} - T_Q(f, M) \in CM \cdot \sigma(x^\sharp_Q, S^\sharp(Q) \cap E) \subset C'M \cdot \sigma^\sharp(x^\sharp_Q, 16).$$

Thus, by Lemma 5.11(A) and the trivial inclusion $\Gamma_+(x^\sharp_Q, E, M, f) \subset \Gamma^\sharp_+(x^\sharp_Q, 16, M, f)$, we have

$$\begin{aligned} T_Q(f, M) &\in \mathcal{J}_{x^\sharp_Q} \hat{F} + CM \cdot \sigma^\sharp(x^\sharp_Q, 16) \\ &\subset \Gamma^\sharp_+(x^\sharp_Q, 16, CM, f) + CM\sigma^\sharp(x^\sharp_Q, 16) \\ &\subset \Gamma^\sharp_+(x^\sharp_Q, 16, C'M, f). \end{aligned}$$

Lemma 5.12 is proved. □

Remark 5.5. We will not use Lemma 5.12 explicitly in this paper. However, jets in $\Gamma^\sharp_+(x^\sharp_Q, 16, M, f)$ are crucial for the following reason:

(5.34) (Lemma 5.3 of [13]) Suppose $Q, Q' \in \Lambda_0$, $x^\sharp_Q$ and $x^\sharp_{Q'}$ as in Lemma 5.6, $P \in \Gamma^\sharp_+(x^\sharp_Q, 16, M, f)$ and $P' \in \Gamma^\sharp_+(x^\sharp_{Q'}, 16, M, f)$, then

$$\begin{aligned} &\left|\partial^\alpha(P - P')(x^\sharp_Q)\right|, \left|\partial^\alpha(P - P')(x^\sharp_{Q'})\right| \\ &\quad \leq CM\left(\delta_Q + \delta_{Q'} + \left|x^\sharp_Q - x^\sharp_{Q'}\right|\right)^{2-|\alpha|} \text{ for } |\alpha| \leq 1. \end{aligned}$$

We can then use (5.34) to control the derivatives when we patch together local extensions. See the proof of Theorem 1 in [12].

5.4. One-dimensional algorithms

We write $\overline{\mathcal{P}}, \overline{\mathcal{P}}^+$, respectively, to denote the collections of single-variable polynomials of degree no greater than one, two. We write $\overline{\mathcal{J}}_t, \overline{\mathcal{J}}_t^+$, respectively, to denote the one-jet, two-jet, of a single variable function at $t \in \mathbb{R}$.

We recall the following results proven in [13].

Theorem 4.A. *Let $\overline{E}_0 \subset \mathbb{R}$ be a finite set with $\#(\overline{E}_0) = N_0$. We think of $C^2_+(\overline{E}_0) \approx [0,\infty)^{N_0}$. Then there exists a collection of maps $\left\{\overline{\Xi}^t_+ : t \in \mathbb{R}\right\}$, where $\overline{\Xi}^t_+ : C^2_+(E) \to \overline{\mathcal{P}}^+$ for each $t \in \mathbb{R}$, such that the following hold.*

(A) There exists a universal constant D_0 such that for each $t \in \mathbb{R}$, the map $\overline{\Xi}^t_+ : C^2_+(\overline{E}_0) \to \overline{\mathcal{P}}^+$ is of depth D_0.
(B) Let $f \in C^2_+(\overline{E}_0)$ be given. Then there exists a function $F \in C^2_+(\mathbb{R})$ such that

$$\overline{\mathcal{J}}_t^+ F = \overline{\Xi}^t_+(f) \text{ for all } t \in \mathbb{R}, \;\; \|F\|_{C^2(\mathbb{R})} \leq C\|f\|_{C^2_+(\overline{E}_0)}, \text{ and } F(t) = f(t) \text{ for } t \in E.$$

(C) There is an algorithm, that takes the given data, performs one-time work, and then responds to queries.

A query consists of a point $t \in \mathbb{R}$, and the response to the query is the depth-D_0 map $\overline{\Xi}^t_+$, given in its efficient representation.

The one-time work takes $CN \log N$ operations and CN storage. The time to answer a query is $C \log N$.

Theorem 4.B. *Let $\overline{E}_0 \subset \mathbb{R}$ be a finite set with $\#(\overline{E}_0) = N_0$. We think of $C^2(\overline{E}_0) \approx \mathbb{R}^{N_0}$. Then there exists a collection of maps $\left\{\overline{\Xi}^t_\pm : t \in \mathbb{R}\right\}$, where $\overline{\Xi}^t_\pm : C^2(E) \to \overline{\mathcal{P}}^+$ for each $t \in \mathbb{R}$, such that the following hold.*

(A) There exists a universal constant D_0 such that for each $t \in \mathbb{R}$, the map $\overline{\Xi}^t_\pm : C^2(\overline{E}_0) \to \overline{\mathcal{P}}^+$ is linear and of depth D_0.
(B) Let $f \in C^2(\overline{E}_0)$ be given. Then there exists a function $F \in C^2(\mathbb{R})$ such that

$$\overline{\mathcal{J}}_t^+ F = \overline{\Xi}^t_\pm(f) \text{ for all } t \in \mathbb{R}, \;\; \|F\|_{C^2(\mathbb{R})} \leq C\|f\|_{C^2(\overline{E}_0)}, \text{ and } F(t) = f(t) \text{ for } t \in E.$$

(C) There is an algorithm, that takes the given data, performs one-time work, and then responds to queries.

A query consists of a point $t \in \mathbb{R}$, and the response to the query is the depth-D_0 map $\overline{\Xi}^t_\pm$, given its efficient representation.

The one-time work takes $CN \log N$ operations and CN storage. The time to answer a query is $C \log N$.

The explanation for Theorems 4.A and 4.B without the complexity statements was given in [13]. We repeat the explanations for completeness, and further elaborate on the complexity.

Using at most $CN_0 \log N_0$ operations and CN_0 storage, we can sort

$$\overline{E}_0 = \{t_1, \cdots, t_{N_0}\} \text{ with } t_1 < \cdots < t_{N_0}.$$

Let us begin with Theorem 4.A.

Suppose $\#(\overline{E}_0) \leq 3$. Let $\overline{\mathcal{Q}}$ and $\overline{\mathcal{M}}$ be as in (2.3) and (2.4), but with $\overline{\mathcal{P}}$ instead of $\mathcal{P}$. Let $\overline{f} : \overline{E}_0 \to [0, \infty)$. Let $\vec{P}_0$ be a section of $\overline{E}_0 \times \overline{\mathcal{P}}$ (i.e., a Whitney field in one-dimension) that minimizes $(\overline{\mathcal{Q}} + \overline{M})$ subject to the constraint $(\vec{P}_0, t)(t) = f(t)$ for $t \in \overline{E}_0$ (see Section 2.2). Let $\overline{T}_W$ be the one-dimensional counterpart of the operator in Lemma 2.1(B). Then $\overline{F} := \overline{T}_W(\vec{P}_0) \in C^2(\mathbb{R})$ with $\overline{F}(t) = f(t)$ and $\overline{F}(t) \geq 0$ on $\mathbb{R}$, thanks to Lemma 2.1(B). By the one-dimensional counterpart of Lemma 2.2, we have $\|\overline{F}\|_{C^2(\mathbb{R})} \leq C\|\overline{f}\|_{C^2_+(\overline{E}_0)}$. Thus, we have constructed a bounded nonnegative extension operator $\overline{\mathcal{E}} : C^2_+(\overline{E}_0) \to C^2_+(\mathbb{R})$ if $\#(\overline{E}_0) \leq 3$. We can simply take the map $\overline{\Xi}_t(\cdot)$ in Theorem 4.A(B) to be $\overline{\mathcal{J}}^+_t \circ \overline{\mathcal{E}}(\cdot)$.

We have shown in Theorem 2.A of [13] that there exists a bounded nonnegative extension operator $\overline{\mathcal{E}} : C^2_+(\overline{E}_0) \to C^2_+(\mathbb{R})$ of bounded-depth in the form

$$\overline{\mathcal{E}}(f)(t) = \sum_{i=1}^{N_0-2} \theta_i(t) \cdot \overline{\mathcal{E}}_i(f)(t), \tag{5.35}$$

where

- $\overline{E}_0^{(i)} = \{t_i, t_{i+1}, t_{i+2}\}$,
- $\overline{\mathcal{E}}^i(\cdot) : C^2_+(\overline{E}_0^{(i)}) \to C^2_+(\mathbb{R})$ is the bounded nonnegative extension operator constructed in the previous step, and
- $\theta_1, \theta_2, \cdots, \theta_{N_0-3}, \theta_{N_0-2}$ form a nonnegative C^2 partition of unity subordinate to the cover $(-\infty, t_3), (t_2, t_4), \cdots, (t_{N_0-3}, t_{N_0-1}), (t_{N_0-2}, \infty)$, such that

$$\left|\frac{d^m}{dt^m}\theta_i(\hat{t})\right| \leq \begin{cases} C\,|t_{i+1} - t_i|^{-m} & \text{if } \hat{t} \in (t_i, t_{i+1}) \\ C\,|t_{i+2} - t_{i+1}|^{-m} & \text{if } \hat{t} \in (t_{i+1}, t_{i+2}) \end{cases}, \quad \text{for } i = 1, \cdots, N_0 - 2.$$

Given $t \in \mathbb{R}$ and $i \in \{1, \cdots, N_0 - 2\}$, we can compute $\overline{\mathcal{J}}^+_t \theta_i$ using at most $C \log N_0$ operations.

Let $t \in \mathbb{R}$ be given. Note that t is supported by at most two of the θ_i's. In $C \log N_0$ operations, we can find all $i', i'' \in \{1, \cdots, N_0 - 2\}$ (possibly $i' = i''$) such that $t \in \operatorname{supp}(\theta_{i'}) \cup \operatorname{supp}(\theta_{i''})$. It is a standard search algorithm and requires at most $C \log N_0$ operations, since $\overline{E}_0$ has been sorted. Finally, we simply set

$$\overline{\Xi}_t(\cdot) := \overline{\mathcal{J}}_t^+ \circ \left(\sum_{i \in \{i', i''\}} \theta_i \cdot \overline{\mathcal{E}}_i(\cdot) \right).$$

It is clear from construction that $\Xi_+^t(\cdot)$ depends only on $f|_{\overline{S}(t)}$, where

$$\overline{S}(t) := \begin{cases} \overline{E}_0 & \text{if } \#(\overline{E}_0) \le 3 \\ \text{three closest points in } \overline{E}_0 \text{ closest to } t & \text{if } \#(\overline{E}_0) > 3 \text{ and } t \notin [t_1, t_{N_0}] \\ \{t_1, t_2, t_3\} & \text{if } t \in [t_1, t_2] \\ \{t_{N_0-2}, t_{N_0-1}, t_{N_0}\} & \text{if } t \in [t_{N_0-1}, t_{N_0}] \\ \{t_1', t_2', t_3', t_4'\} \subset \overline{E}_0 \text{ with } t_1' < t_2' \le t \le t_3' < t_4' & \text{otherwise} \end{cases} \tag{5.36}$$

Theorem 4.A(A) then follows.

Theorem 4.A(B) follows from the fact that the operator $\overline{\mathcal{E}}$ in (5.35) is a bounded nonnegative extension operator on $C_+^2(\overline{E}_0)$.

Theorem 4.A(C) follows from the discussions above on complexity.

We have finished explaining Theorem 4.A.

The explanation for Theorem 4.B is almost identical with some simplification, which we explain below.

When constructing a bounded extension operator for $C^2(\overline{E}_0)$ with $\#(\overline{E}_0) \le 3$, we use

- the natural quadratic form associated with $W^2(\overline{E}_0)$ instead of $(\overline{\mathcal{Q}} + \overline{M})$; and
- the classical Whitney extension operator instead of T_w in Lemma 2.1(B).

See [4,9,10] for details and further discussion on linear extension operators without the nonnegative constraint.

This concludes the explanation for Theorem 4.B.

5.5. Local extension problem

The main lemma of the section is the following.

Lemma 5.13. *Let* $Q \in \Lambda^{\sharp\sharp}$ *with* $\Lambda^{\sharp\sharp}$ *as in* (5.6). *There exists a collection of maps* $\{\Xi_{x,Q} : x \in (1 + c_G)Q\}$ *where* $\Xi_{x,Q} : C_+^2(E) \times [0, \infty) \to \mathcal{P}^+$ *for each* $x \in (1 + c_G)Q$, *such that the following hold.*

(A) There exists a universal constant D *such that for each* $x \in (1 + c_G)Q$, *the map* $\Xi_{x,Q}(\cdot, \cdot) : C_+^2(E) \to \mathcal{P}^+$ *is of depth* D.

(B) Suppose we are given $(f, M) \in C_+^2(E) \times [0, \infty)$ *with* $\|f\|_{C_+^2(E)} \le M$. *Then there exists a function* $F_Q \in C_+^2((1 + c_G)Q)$ *such that*

(B1) $\mathcal{J}_x^+ F_Q = \Xi_{x,Q}(f, M)$ *for all* $x \in (1 + c_G)Q$;

(B2) $\|F_Q\|_{C^2((1+c_G)Q)} \le CM$;
(B3) $F_Q(x) = f(x)$ *for* $x \in E \cap (1+c_G)Q$; *and*
(B4) $\mathcal{J}_{x_Q^\sharp} F_Q \in \Gamma_+^\sharp(x_Q^\sharp, 16, CM, f)$, *with* $x_Q^\sharp$ *as in Lemma 5.6 and* $\Gamma_+^\sharp$ *as in (2.22).*

(C) There is an algorithm, that takes (E, f, M, Q) *as input, performs one-time work, and then responds to queries.*

A query consists of a point $x \in (1+c_G)Q$, *and the response to the query is the depth-*D *map* $\Xi_{x,Q}$, *given its efficient representation.*

The one-time work takes $CN \log N$ *operations and* CN *storage. The time to answer a query is* $C \log N$.

Proof. Repeating the argument of Lemma 3.8 of [12], we can show that there exists a map

$$\mathcal{E}_Q : C_+^2(E) \times [0,\infty) \to C_+^2((1+c_G)Q)$$

such that the following hold.

(5.37) Given $(f, M) \in C_+^2(E) \times [0,\infty)$ with $\|f\|_{C_+^2(E)} \le M$, we have

(a) $\mathcal{E}_Q(f, M) \ge 0$ on $(1+c_G)Q$;
(b) $\mathcal{E}_Q(f, M)(x) = f(x)$ for $x \in E \cap (1+c_G)Q$;
(c) $\|\mathcal{E}_Q(f, M)\|_{C^2((1+c_G)Q)} \le CM$; and
(d) $\mathcal{J}_{x_Q^\sharp} \mathcal{E}_Q(f, M) = T_Q(f, M)$, with $x_Q^\sharp$ as in Lemma 5.6, T_Q as in Definition 5.3.

(5.38) For each $x \in (1+c_G)Q$, there exists a set $S_Q(x) \subset E$ with $\#(S_Q(x)) \le D_0$ for some universal constant D_0, such that the following holds: Given $(f, M), (g, M) \in C_+^2(E) \times [0,\infty)$ with $\|f\|_{C_+^2(E)}, \|g\|_{C_+^2(E)} \le M$ and $f|_{S_Q(x)} = g|_{S_Q(x)}$, we have $\mathcal{J}_x^+ \mathcal{E}_Q(f, M) = \mathcal{J}_x^+ \mathcal{E}_Q(g, M)$.

To prove Lemma 5.13, we need to dissect the operator $\mathcal{E}_Q$ and analyze its complexity. As in Lemma 3.8 of [12], the operator $\mathcal{E}_Q$ takes the following form:

$$\mathcal{E}_Q(f, M) := T_Q(f, M) + (1-\psi) \cdot \widetilde{\mathcal{E}}_Q(f, M), \quad \text{where}$$

$$\widetilde{\mathcal{E}}_Q(f, M) := \Bigg(\overbrace{V \circ \underbrace{\Bigg[\Big(\Delta_{f,M}^Q \overline{\mathcal{E}}_\pm + (1-\Delta_{f,M}^Q) \overline{\mathcal{E}} \Big) \underbrace{\Big((f - T_Q(f, M)\big|_E) \circ \Phi^{-1}\big|_{\mathbb{R}\times\{0\}} \Big)}_{\text{straightening local data}} \Bigg]}_{\text{one-dimensional extension}}}^{\text{vertical extension}} \Bigg) \circ \Phi \,. \tag{5.39}$$

Here, in the order of appearance in (5.39),

- T_Q is as in Definition 5.3;
- $\psi \in C^2_+(\mathbb{R}^2)$ with $\psi \equiv 1$ near $x^\sharp_Q$ (see Lemma 5.6), $\operatorname{supp}(\psi) \subset B(x^\sharp_Q, \frac{c_0}{2}\delta_Q)$ with c_0 as in Lemma 5.6, and $|\partial^\alpha \psi| \le C\delta_Q^{-|\alpha|}$;
- V is the vertical extension map $V(g)(t_1, t_2) := g(t_1)$, for g defined on a subset of $\mathbb{R}$;
- $\Delta^Q_{f,M}$ is an indicator function defined by

$$\Delta^Q_{f,M} := \begin{cases} 1 & \text{if } T_Q(f,M) \text{ is not the zero polynomial} \\ 0 & \text{otherwise} \end{cases};$$

- $\overline{\mathcal{E}}$ and $\overline{\mathcal{E}}_\pm$, respectively, are the one-dimensional extension operators associated with Theorem 4.A and Theorem 4.B (see also Theorems 2.A and 2.B of [13]);
- Φ is the diffeomorphisms in Lemma 5.1(B).

We bring ourselves back to the setting of Lemma 5.13. Recall the definition of $\mathcal{J}^+_x$ as in (2.1). We want to define the maps $\{\Xi_{x,Q} : x \in (1+c_G)Q\}$ by

$$\Xi_{x,Q} := \mathcal{J}^+_x \circ \mathcal{E}_Q \quad \text{for } x \in (1+c_G)Q. \tag{5.40}$$

Lemma 5.13(A) follows from (5.38). Lemma 5.13(B) follows from (5.37).

It remains to examine Lemma 5.13(C). Suppose we have performed the necessary one-time work using at most $CN\log N$ operations and CN storage.

Let $x \in (1+c_G)Q$ be given.

Step 1. We compute

$$t_x := \mathsf{Proj}_{u_Q^\perp}(x - \mathsf{Rep}(Q)).$$

Here,

- $\mathsf{Proj}_{u_Q^\perp}$ denotes orthogonal projection onto $\mathbb{R}u_Q^\perp$;
- the pair $\{u_Q, u_Q^\perp\}$ is as in Lemma 5.5; and
- $\mathsf{Rep}(Q)$ is as in Lemma 5.2(C).

All the procedures involved in this step require at most $C\log N$ operations, thanks to Lemma 5.2(B) and Lemma 5.5.

Step 2. Let ρ be the rotation about the origin specified by $e_2 \mapsto u_Q$. We can compute $\mathcal{J}^+_x\rho$.

Step 3. Let $u_Q^\perp$ and $\mathsf{Proj}_{u_Q^\perp}$ be as in Step 1. We set

$$\overline{E}_Q := \mathsf{Proj}_{u_Q^\perp}(E \cap (1+c_G)Q - \mathsf{Rep}(Q)) \subset \mathbb{R}.$$

Recall from Lemma 5.8 that we can compute the *sorted* list $\overline{E}_Q$ for each $Q \in \Lambda^{\sharp\sharp}$ using at most $CN\log N$ operations and CN storage.

Let $C^2_+(\overline{E}_Q)$ and $C^2(\overline{E}_Q)$ be the one-dimensional trace spaces. Note that we have sorted $\overline{E}_Q$. Let $\overline{\Xi}^{t_x}_+$ and $\overline{\Xi}^{t_x}_\pm$, respectively be the maps associated with $C^2_+(\overline{E}_Q)$ and $C^2(\overline{E}_Q)$, as in Theorems 4.A and 4.B.

Step 4. Recall from Lemma 5.1(B) that the diffeomorphism Φ is defined in terms of a function φ, satisfying (5.4) and (5.5). We compute $\overline{\mathcal{J}}^+_{t_x}\varphi$, where $\overline{\mathcal{J}}^+_{t_x}$ is the single-variable two-jet at t_x. We can accomplish this by simply setting $\overline{\mathcal{J}}^+_{t_x}\varphi := \overline{\Xi}^{t_x}_\pm(\varphi|_{\overline{E}_Q})$, with $\overline{\Xi}^{t_x}_\pm$ as in Theorem 4.B. Since we have already sorted the set $\overline{E}_Q$ in Step 3, computing $\overline{\Xi}^{t_x}_\pm(\varphi|_{\overline{E}_Q})$ requires at most $C\log N$ operations.

Step 5. Similar to Step 4, the query time for $\overline{\mathcal{J}}^+_{t_x}\circ\overline{\mathcal{E}}(\cdot)$[1] and $\overline{\mathcal{J}}^+_{t_x}\circ\overline{\mathcal{E}}_\pm(\cdot)$ is $C\log N$, since set $\overline{E}_Q$ has been sorted in Step 3.

Step 6. By Lemma 5.1(B), the diffeomorphism $\Phi = (\Phi_1,\Phi_2)$ and its inverse $\Phi^{-1} = (\Psi_1,\Psi_2)$ are given by

$$\Phi\circ\rho(t_1,t_2) = (t_1, t_2 - \varphi(t_1)),\ \text{and}$$
$$\rho^{-1}\circ\Phi^{-1}(t_1',t_2') = (t_1', t_2' + \varphi(t_1')).$$

Therefore, we can compute $\mathcal{J}^+_x\Phi_i$ and $\mathcal{J}^+_x\Psi_i$, $i = 1,2$, from the (single-variable) two-jet of φ.

Step 7. We compute $T_Q(f,M)$, as in Definition 5.3. Computing $S^\sharp(Q)$ as in (5.26) requires at most $C\log N$ operations, by Lemma 5.2(C) and Lemma 5.6. After that, we can compute $T_Q(f,M)$ in C operations. See Remark 5.4.

Combining all the steps above, we see that we can compute the map $\Xi_{x,Q}$ in (5.40) via formula (5.39) using at most $C\log N$ operations. After that, given $(f,M)\in C^2(E)\times[0,\infty)$, we can compute $\Xi_{x,Q}(f,M)$ in C operations.

This proves Lemma 5.13. □

5.6. Partitions of unity

Recall the definition of $\mathcal{J}^+_x$ as in (2.1).

We can construct a partition of unity $\{\theta_Q : Q\in\Lambda_0\}$ that satisfies the following properties:

- $\theta \geq 0$;
- $\sum_{Q\in\Lambda_0}\theta_Q \equiv 1$;
- $\operatorname{supp}(\theta_Q)\subset(1+c_G/2)Q$ for each $Q\in\Lambda_0$;
- For each $Q\in\Lambda_0$, $|\partial^\alpha\theta_Q|\leq C\delta_Q^{2-|\alpha|}$ for $|\alpha|\leq 2$;

[1] Note that $\overline{\mathcal{E}}(\cdot)$ is only defined for $\tilde{f}:\overline{E}_Q\to[0,\infty)$.

- After one-time work using at most $CN \log N$ operations and CN storage, we can answer queries as follows: Given $x \in \mathbb{R}^2$ and $Q \in \Lambda_0$, we return $\mathcal{J}_x^+ \theta_Q$. The time to answer query is $C \log N$.

See Section 28 of [10] for details.

5.7. *Proof of Theorem 2*

Proof of Theorem 2. Slightly modifying the proof of Theorem 1 of [12], we can show that there exists a map

$$\mathcal{E} : C_+^2(E) \times [0, \infty) \to C_+^2(\mathbb{R}^2) \tag{5.41}$$

such that the following hold.

(5.42) Given $(f, M) \in C_+^2(E) \times [0, \infty)$ with $\|f\|_{C_+^2(E)} \le M$, we have

(a) $\mathcal{E}(f, M) \ge 0$ on $\mathbb{R}^2$;
(b) $\mathcal{E}(f, M)(x) = f(x)$ for $x \in E$; and
(c) $\|\mathcal{E}(f, M)\|_{C^2(\mathbb{R}^2)} \le CM$.

(5.43) For each $x \in \mathbb{R}^2$, there exists a set $S(x) \subset E$ with $\#(S(x)) \le D$ for some universal constant D, such that the following holds: Given $(f, M), (g, M) \in C_+^2(E) \times [0, \infty)$ with $\|f\|_{C_+^2(E)}, \|g\|_{C_+^2(E)} \le M$ and $f|_{S(x)} = g|_{S(x)}$, we have $\mathcal{J}_x^+ \mathcal{E}(f, M) = \mathcal{J}_x^+ \mathcal{E}(g, M)$.

Moreover, $\mathcal{E}$ takes the form of

$$\mathcal{E}(f, M)(x) := \sum_{Q \in \Lambda_0} \theta_Q(x) \cdot \mathcal{E}_Q^\sharp(f, M)(x) = \sum_{Q \in \Lambda(x)} \theta_Q(x) \cdot \mathcal{E}_Q^\sharp(f, M)(x), \tag{5.44}$$

where

- $\{\theta_Q : Q \in \Lambda_0\}$ is the partition of unity constructed in Section 5.6;
- $\Lambda(x)$ is the set in Lemma 5.2(A); and
- $\mathcal{E}_Q^\sharp$ is defined by the following rule.
  - Suppose $Q \in \Lambda^{\sharp\sharp}$. Then $\mathcal{E}_Q^\sharp(f, M) := \mathcal{E}_Q(f, M)$ with $\mathcal{E}_Q$ as in Lemma 5.13;
  - Suppose $Q \in \Lambda^\sharp \setminus \Lambda^{\sharp\sharp}$. Then $\mathcal{E}_Q^\sharp := T_w^{x_Q^\sharp} \circ T_Q$, with T_Q as in Definition 5.3, $x_Q^\sharp$ as in Lemma 5.6, and $T_w^{x_Q^\sharp}$ as in Lemma 2.1(B) (associated with the singleton $\{x_Q^\sharp\}$).

– Suppose $Q \in \Lambda_{\text{empty}}$. Then $\mathcal{E}^\sharp_Q := T_w^{x^\sharp_{\mu(Q)}} \circ T_{\mu(Q)}$, with μ as in Lemma 5.4, $x^\sharp_{\mu(Q)}$ as in Lemma 5.6, $T_{\mu(Q)}$ as in Definition 5.3, and $T_w^{x^\sharp_{\mu(Q)}}$ as in Lemma 2.1(B) (associated with the singleton $\{x^\sharp_{\mu(Q)}\}$).
– Suppose $Q \in \Lambda_0 \setminus (\Lambda^\sharp \cup \Lambda_{\text{empty}})$. Then $\mathcal{E}^\sharp_Q :\equiv 0$.

We set

$$\Xi_x(f,M) := \mathcal{J}^+_x \circ \mathcal{E}(f,M) = \sum_{Q \in \Lambda(x)} \mathcal{J}^+_x \theta_Q \odot^+_x \mathcal{J}^+_x \circ \mathcal{E}^\sharp_Q(f,M) \text{ for } x \in \mathbb{R}^2. \tag{5.45}$$

Theorem 2(A) follows from (5.42) and Theorem 2(B) follows from (5.43).

We now turn to Theorem 2(C). Suppose we have performed the necessary one-time work using at most $CN \log N$ operations and CN storage.

By Lemma 5.2(A) and Section 5.6, we can compute $\Lambda(x)$ and $\{\mathcal{J}^+_x \theta_Q : Q \in \Lambda(x)\}$ using at most $C \log N$ operations.

By Lemma 5.13, we can compute

$$\left\{\mathcal{J}^+_x \circ \mathcal{E}_Q(f,M) : Q \in \Lambda^{\sharp\sharp} \cap \Lambda(x)\right\}$$

using at most $C \log N$ operations, after computing $\Lambda(x)$.

By Lemma 5.6 and Remark 5.12, we can compute

$$\left\{\mathcal{J}^+_x \circ T_w^{x^\sharp_Q} \circ T_Q(f,M) : Q \in \Lambda(x) \cap (\Lambda^\sharp \setminus \Lambda^{\sharp\sharp})\right\}$$

using at most $C \log N$ operations, after computing $\Lambda(x)$.

By Lemma 5.4, Lemma 5.6 and Remark 5.4, we can compute

$$\left\{\mathcal{J}^+_x \circ T_w^{x^\sharp_{\mu(Q)}} \circ T_{\mu(Q)}(f,M) : Q \in \Lambda_{\text{empty}} \cap \Lambda(x)\right\}$$

using at most $C \log N$ operations, after computing $\Lambda(x)$.

Therefore, we can compute Ξ_x in (5.45) using at most $C \log N$ operations. Given $(f,M) \in C^2(E) \times [0,\infty)$, we can compute $\Xi_x(f,M)$ in C operations. Theorem 2(C) follows.

This proves Theorem 2. $\square$

Appendix A. Convex quadratic programming problem with affine constraint

Let $d \geq 0$ be an integer bounded by a universal constant. We use the standard dot product on $\mathbb{R}^d$ and $\mathbb{R}^{2d}$. We use bold-faced letters to denote given quantities.

We consider a general form of the minimization problem (2.20):

$$\text{Minimize } \beta^t \mathbf{A}\beta + \sum_{i=1}^{d} |\beta^i| \quad \text{subject to } \mathbf{B}\beta = \mathbf{b}. \tag{A.1}$$

Here, $\beta = (\beta^1, \cdots, \beta^d)^t \in \mathbb{R}^d$ is the optimization variable, $\mathbf{A} \in M_{d\times d}$ is a given positive semidefinite, and $\mathbf{B}$ is a given matrix of full rank, and $\mathbf{b}$ is a given vector.

We will solve (A.1) by first augmenting the system (A.1) to remove the absolute values in the objective function. For the augmented system, which is still convex, the solution can be found by solving for a system of linear equalities and inequalities arising from its associated Karush–Kuhn–Tucker (KKT) conditions [2].

We begin with the augmentation. Decomposing β into its positive and negative parts, $\beta = \beta_+ - \beta_-$, i.e., $\beta_+^i := \frac{1}{2}(\beta^i + |\beta^i|)$ and $\beta_-^i := \beta_+^i - \beta^i$, we arrive at the system:

$$\begin{aligned} &\text{Minimize } \begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}^t \begin{pmatrix} \mathbf{A} & -\mathbf{A} \\ -\mathbf{A} & \mathbf{A} \end{pmatrix} \begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix} + (\mathbf{1}_{2d})^t \begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix} \\ &\text{Subject to } (\mathbf{B} \quad -\mathbf{B}) \begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix} = \mathbf{b}, \text{ and } \begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix} \geq \mathbf{0}_{2d}. \end{aligned} \tag{A.2}$$

Note that in order for (A.1) and (A.2) to be equivalent, we have to include in (A.2) the additional sign constraint

$$\beta_+^i \beta_-^i = 0 \text{ for } i = 1, \cdots, d; \tag{A.3}$$

or equivalently, for some $I \subset \{1, \cdots, d\}$,

$$\mathbf{e}_k^t \beta_+ = 0 \text{ for } k \in I \text{ and } \mathbf{e}_k^t \beta_- = 0 \text{ for } k \in \{1, \cdots, d\} \setminus I. \tag{A.4}$$

Here, $\{\mathbf{e}_k : k = 1, \cdots, d\}$ is the standard basis for $\mathbb{R}^d$.

For convenience, set

$$\hat{\beta} := \begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}, \hat{\mathbf{A}} := \begin{pmatrix} \mathbf{A} & -\mathbf{A} \\ -\mathbf{A} & \mathbf{A} \end{pmatrix}, \text{ and } \hat{\mathbf{B}} := (\mathbf{B} \quad -\mathbf{B}) = \begin{pmatrix} \hat{\mathbf{B}}_1 \\ \vdots \\ \hat{\mathbf{B}}_{j_{\max}} \end{pmatrix}.$$

Let $\{\hat{\mathbf{e}}_i : i = 1, \cdots, 2d\}$ be the standard basis for $\mathbb{R}^{2d}$.

The KKT conditions for (A.2) coupled with (A.4) for a *fixed* $I \subset \{1, \cdots, d\}$ are given by

$$
\begin{aligned}
2\hat{\mathbf{A}}\hat{\beta} - \sum_{i=1}^{2d} \mu_i \hat{\mathbf{e}}_i + \sum_{j=1}^{j_{max}} \lambda_j \hat{\mathbf{B}}_j^t + \sum_{k\in I} \nu_k \hat{\mathbf{e}}_k + \sum_{k\in\{1,\cdots,d\}\setminus I} \nu_k \hat{\mathbf{e}}_{k+d} = \mathbf{0}_{2d}, \\
\hat{\beta} \geq \mathbf{0}_{2d}, \\
\hat{\mathbf{B}}\hat{\beta} - \mathbf{b} = \mathbf{0}_{j_{max}}, \\
\hat{\mathbf{e}}_k^t \hat{\beta} = \mathbf{0} \text{ for } k \in I, \\
\hat{\mathbf{e}}_{k+d}^t \hat{\beta} = \mathbf{0} \text{ for } k \in \{1,\cdots,d\}\setminus I, \\
\mu_i \geq \mathbf{0} \text{ for } i = 1,\cdots,2d. \\
\sum_{i=1}^{2d} \mu_i (\hat{\mathbf{e}}_i^t \hat{\beta}) = \mathbf{0}.
\end{aligned} \tag{A.5}
$$

In the above, $\mu_1, \cdots, \mu_{2d}$, $\lambda_1, \cdots, \lambda_{j_{max}}$, $\nu_1, \cdots, \nu_d$ are multipliers, and $\hat{\beta}$ is the primal optimization variable.

Since the matrix $\hat{\mathbf{A}}$ is positive semidefinite, the primal problem in (A.2) is convex. The KKT conditions are necessary and sufficient for the solutions to be primal and dual optimal [2]. Hence, solving (A.2) coupled with (A.4) for a fixed $I \subset \{1, \cdots, d\}$ amounts to solving a bounded system (A.5) of linear inequalities. The latter can be achieved, for instance by the simplex method or elimination [2]. The number of operations involved is at most (doubly) exponential in system size, which is universally bounded. Therefore, we can solve (A.2) coupled with (A.4) for a fixed $I \subset \{1, \cdots, d\}$ using at most C operations.

Finally, we can solve (A.1) using at most C operations by solving (A.2) coupled with (A.4) for every $I \subset \{1, \cdots, d\}$ and compare the minimizers.

It is very likely that one can solve (A.1) more efficiently with advanced techniques. Here we content ourselves with the elementary exposition above. We refer the readers to [2] for a more detailed discussion on convex optimization.

References

[1] E. Bierstone, P.D. Milman, $\mathcal{C}^m$-norms on finite sets and $\mathcal{C}^m$-extension criteria, Duke Math. J. 137 (1) (2007) 1–18.
[2] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.
[3] P.B. Callahan, S.R. Kosaraju, A decomposition of multidimensional point sets with applications to k-nearest-neighbors and n-body potential fields, J. ACM 42 (1995) 67–90.
[4] C. Fefferman, Interpolation and extrapolation of smooth functions by linear operators, Rev. Mat. Iberoam. 21 (1) (2005) 313–348.
[5] C. Fefferman, The structure of linear extension operators for C^m, Rev. Mat. Iberoam. 23 (1) (2007) 269–280.
[6] C. Fefferman, Fitting a C^m-smooth function to data III, Ann. of Math. (2) 170 (1) (2009) 427–441.
[7] C. Fefferman, A. Israel, Fitting Smooth Functions to Data, CBMS Regional Conference Series in Mathematics, American Mathematical Society, 2020.
[8] C. Fefferman, A. Israel, G.K. Luli, Interpolation of data by smooth non-negative functions, Rev. Mat. Iberoam. 33 (1) (2016) 305–324.
[9] C. Fefferman, B. Klartag, Fitting a C^m-smooth function to data. I, Ann. of Math. (2) 169 (1) (2009) 315–346.
[10] C. Fefferman, B. Klartag, Fitting a C^m-smooth function to data. II, Rev. Mat. Iberoam. 25 (1) (2009) 49–273.

[11] C. Fefferman, G.K. Luli, C^m semialgebraic sections over the plane, arXiv:2101.06334, 2021.
[12] F. Jiang, G.K. Luli, $C^2(\mathbb{R}^2)$ nonnegative interpolation by bounded-depth operators, Adv. Math. 375 (2020) 107391.
[13] F. Jiang, G.K. Luli, Nonnegative $C^2(\mathbb{R}^2)$ interpolation, Adv. Math. 375 (2020) 107364.
[14] G.K. Luli, $C^{m,\omega}$ extension by bounded-depth linear operators, Adv. Math. 224 (2010) 1927–2021.

CHAPTER 4

On the shape fields Finiteness Principle

The following paper "On the Shape Fields Finiteness Principle" is to appear in International Mathematics Research Notices [**21**]. The paper is based on joint work with co-author Garving K. Luli at the Department of Mathematics, University of California - Davis and and Kevin O'Neill at the Applied Mathematics Program, Yale University. The authors were supported by UC Davis Summer Graduate Student Researcher Award (F.J.), the Alice Leung Scholarship in Mathematics (F.J.), National Science Foundation Grant DMS-1554733 (G.K.L.), and the UC Davis Chancellor's Fellowship (G.K.L.).

F. Jiang *et al.* (2021) "On the Shape Fields Finiteness Principle,"
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–24
https://doi.org/10.1093/imrn/rnab242

On the Shape Fields Finiteness Principle

Fushuai Jiang[1,*], Garving K. Luli[1], and Kevin O'Neill[2]
[1]Department of Mathematics, UC Davis, One Shields Ave, Davis, CA 95616, USA and [2]Department of Mathematics, Yale University, New Haven, CT 06520

[*]*Correspondence to be sent to: e-mail: fsjiang@math.ucdavis.edu*

In this paper, we improve the finiteness constant for the finiteness principles for $C^m(\mathbb{R}^n, \mathbb{R}^D)$ and $C^{m-1,1}(\mathbb{R}^n, \mathbb{R}^D)$ selection proven in [19] and extend the more general shape fields finiteness principle to the vector-valued case.

1 Introduction

Suppose we are given integers $m \geq 0$, $n \geq 1$, $D \geq 1$. We write $C^m\left(\mathbb{R}^n, \mathbb{R}^D\right)$ to denote the space of all functions $\vec{F} : \mathbb{R}^n \to \mathbb{R}^D$ whose derivatives up to order m are continuous and bounded on $\mathbb{R}^n$, equipped with the norm

$$\|\vec{F}\|_{C^m(\mathbb{R}^n,\mathbb{R}^D)} := \max_{|\alpha|\leq m} \sup_{x\in\mathbb{R}^n} \|\partial^\alpha \vec{F}(x)\|_\infty = \max_{\substack{|\alpha|\leq m \\ 1\leq j\leq D}} \sup_{x\in\mathbb{R}^n} |\partial^\alpha F_j(x)|.$$

Here and below, we view $\partial^\alpha \vec{F}(x) = (\partial^\alpha F_1(x), \cdots, \partial^\alpha F_D(x))$ as a vector in $\mathbb{R}^D$.

We write $\dot{C}^m(\mathbb{R}^n, \mathbb{R}^D)$ to denote the vector space of m-times continuously differentiable $\mathbb{R}^D$-valued functions whose m-th order derivatives are bounded, equipped with the seminorm

$$\|\vec{F}\|_{\dot{C}^m(\mathbb{R}^n,\mathbb{R}^D)} := \max_{|\alpha|=m} \sup_{x\in\mathbb{R}^n} \|\partial^\alpha \vec{F}(x)\|_\infty.$$

Received November 30, 2020; Revised April 20, 2021; Accepted August 2, 2021

We write $C^{m-1,1}\left(\mathbb{R}^n,\mathbb{R}^D\right)$ to denote the space of all $\vec{F}:\mathbb{R}^n\to\mathbb{R}^D$ whose derivatives up to order $m-1$ are bounded and Lipschitz on $\mathbb{R}^n$. When $D=1$, we write $C^m\left(\mathbb{R}^n\right)$ and $C^{m-1,1}\left(\mathbb{R}^n\right)$ in place of $C^m\left(\mathbb{R}^n,\mathbb{R}^D\right)$ and $C^{m-1,1}\left(\mathbb{R}^n,\mathbb{R}^D\right)$.

We write $\vec{\mathcal{P}}$ to denote the vector space $\bigoplus_{j=1}^{D}\mathcal{P}$, where $\mathcal{P}$ is the space of polynomials on $\mathbb{R}^n$ with degree no greater than $m-1$. Note that $\dim\vec{\mathcal{P}}=D\cdot\binom{n+m-1}{m-1}$. We write $\mathcal{J}_x\vec{F}$ to denote the D-tuple of component-wise Taylor polynomials of $\vec{F}$ at x of degree $m-1$.

Quantities $c\,(m,n)$, $C\,(m,n)$, $k\,(m,n)$, etc., denote constants depending only on m,n; these expressions may denote different constants in different occurrences. Similar conventions apply to constants denoted by $C\,(m,n,D)$, $k\,(m,n,D)$, etc.

If S is any finite set, then $|S|$ denotes the number of elements in S.

Let $E\subset\mathbb{R}^n$ be given. Suppose at each $x\in E$, we are given a convex set $K(x)\subset\mathbb{R}^D$. A selection of $(K(x))_{x\in E}$ is a map $\vec{F}:\mathbb{R}^n\to\mathbb{R}^D$ such that $\vec{F}(x)\in K(x)$ for all $x\in E$.

We are interested in the following selection problem.

Problem 1.1. Let $E\subset\mathbb{R}^n$. For each $x\in E$, suppose we are given a convex $K(x)\subset\mathbb{R}^D$. Given a number $M>0$, how can one decide if there exists a selection $\vec{F}\in C^{m-1,1}(\mathbb{R}^n,\mathbb{R}^D)$ or $\vec{F}\in C^m(\mathbb{R}^n,\mathbb{R}^D)$ with $\|\vec{F}\|_{C^m(\mathbb{R}^n,\mathbb{R}^D)}\le C^\sharp M$ or $\|\vec{F}\|_{C^{m-1,1}(\mathbb{R}^n,\mathbb{R}^D)}\le C^\sharp M$, where $C^\sharp$ depends only on m,n,D?

In [19], the authors addressed Problem 1.1 by proving the following

Theorem 1.2 (Finiteness Principle for Smooth Selection [19]). For large enough $k^\sharp=k\,(m,n,D)$ and $C^\sharp=C\,(m,n,D)$, the following hold.

(A) C^m FLAVOR Let $E\subset\mathbb{R}^n$ be finite. For each $x\in E$, let $K\,(x)\subset\mathbb{R}^D$ be convex. Suppose that for each $S\subset E$ with $|S|\le k^\sharp$, there exists $\vec{F}^S\in C^m\left(\mathbb{R}^n,\mathbb{R}^D\right)$ with norm $\|\vec{F}^S\|_{C^m(\mathbb{R}^n,\mathbb{R}^D)}\le 1$, such that $\vec{F}^S\,(x)\in K\,(x)$ for all $x\in S$. Then there exists $\vec{F}\in C^m\left(\mathbb{R}^n,\mathbb{R}^D\right)$ with norm $\|\vec{F}\|_{C^m(\mathbb{R}^n,\mathbb{R}^D)}\le C^\sharp$, such that $\vec{F}\,(x)\in K\,(x)$ for all $x\in E$.

(B) $C^{m-1,1}$ FLAVOR Let $E\subset\mathbb{R}^n$ be arbitrary. For each $x\in\mathbb{R}^n$, let $K\,(x)\subset\mathbb{R}^D$ be a closed convex set. Suppose that for each $S\subset E$ with $|S|\le k^\sharp$, there exists $\vec{F}^S\in C^{m-1,1}\left(\mathbb{R}^n,\mathbb{R}^D\right)$ with norm $\|\vec{F}^S\|_{C^{m-1,1}(\mathbb{R}^n,\mathbb{R}^D)}\le 1$, such that $\vec{F}^S\,(x)\in K\,(x)$ for all $x\in S$. Then there exists $\vec{F}\in C^{m-1,1}\left(\mathbb{R}^n,\mathbb{R}^D\right)$ with norm $\|\vec{F}\|_{C^{m-1,1}(\mathbb{R}^n,\mathbb{R}^D)}\le C^\sharp$, such that $\vec{F}\,(x)\in K\,(x)$ for all $x\in\mathbb{R}^n$.

Therefore, Theorem 1.2 tells us when there exists a $C^{m-1,1}$ selection $\vec{F}$ of $(K(x))_{x\in E}$ for the case of infinite E and provides estimates for the C^m-norm of a selection for finite E.

Theorem 1.2 for the case $D = 1$ and $K(x)$ being a singleton for each $x \in E$ was conjectured by Y. Brudnyi and P. Shvartsman in [6].

The number $k^\sharp$ in Theorem 1.2 is called the finiteness number. The $k^\sharp$ obtained in [19] is $k^\sharp = 100 + (D+2)^{l_*+100}$, where $l_* = \binom{m+n}{n}$.

Here, we give a sharper bound on $k^\sharp$. Our first result is the following.

Theorem 1.3. The $k^\sharp$ found in Theorem 1.2 may be taken to be $k^\sharp = 2^{\dim \vec{\mathcal{P}}}$, where $\dim \vec{\mathcal{P}} = D \cdot \binom{n+m-1}{m-1}$.

A few remarks on Theorem 1.3 are in order. Independently, E. Bierstone and P. Milman [1] and P. Shvartsman [28] proved Theorem 1.3 for the case $D = 1$ and each $K(x)$ is a singleton, that is, $K(x) = \{f(x)\}$ for some real-valued $f : E \to \mathbb{R}$. This corresponds to the finiteness principle proved by C. Fefferman in [12]. In addition, P. Shvartsman proved a weaker version of Theorem 1.3 where each $K(x) \subset \mathbb{R}^D$ is centrally symmetric and we are allowed to dilate $K(x)$. Our present approach is inspired by [1].

In the case $D = 1$, $m = 2$, and each $K(x)$ being a singleton, Theorem 1.3 gives $k^\sharp = 4 \cdot 2^{n-1}$. This is comparable to the finiteness constant $3 \cdot 2^{n-1}$ given by Shvartsman [26], which he shows to be optimal. See also [9].

To prove Theorem 1.3, we will need the following result.

Theorem 1.4. The following holds for $\mathbb{X} = C^m(\mathbb{R}^n, \mathbb{R}^D)$ and $\mathbb{X} = \dot{C}^m(\mathbb{R}^n, \mathbb{R}^D)$.

Let $S \subset \mathbb{R}^n$ be a finite set of diameter at most 1. For each $x \in S$, let $\vec{G}(x) \subset \vec{\mathcal{P}}$ be convex. Suppose that for every subset $S' \subset S$ with $|S| \le 2^{\dim \vec{\mathcal{P}}}$, there exists $\vec{F}^{S'} \in \mathbb{X}$ such that $\|\vec{F}^{S'}\|_{\mathbb{X}} \le 1$ and $\mathcal{J}_x \vec{F}^{S'} \in \vec{G}(x)$ for all $x \in S'$.

Then, there exists $\vec{F} \in \mathbb{X}$ such that $\|\vec{F}\|_{\mathbb{X}} \le \gamma$ and $\mathcal{J}_x \vec{F} \in \vec{G}(x)$ for all $x \in S$.

Here, γ depends only on m, n, D, and $|S|$.

Because the constant γ depends on the number of points in S, following [28], we will refer to Theorem 1.4 as a "weak finiteness principle."

To conclude the introduction, we give an overview of how we prove Theorems 1.3 and 1.4. The proof of Theorem 1.2 given in [19] is via a more general finiteness principle for shape fields, see Theorem 2.4 below. Using Theorem 1.4, we will show an improved bound for $k^\sharp$ in the finiteness principle for shape fields (i.e., Theorem 2.4); we can then deduce the bound for $k^\sharp$ in Theorem 1.2, obtaining the bound asserted in Theorem 1.3. The heart of the matter therefore lies in Theorem 1.4. To put things in perspective, we would like to point out that one can't directly apply the techniques from [1] because of the nonlinear structure in the selection problem and that the result in [28] is for scalar-valued functions. To prove our main theorem (Theorem 1.4), we will adapt the strategy from [1] with some new ingredients: instead of linear structure, we will handle general convex structure using the duality theorem of linear programming to describe the relevant convex sets.

This paper is part of a literature on extension, interpolation, and selection of functions, going back to H. Whitney's seminal work [30–32], and including fundamental contributions by G. Glaeser [23], Y. Brudnyi and P. Shvartsman [4–9, 26, 27, 29], E. Bierstone, P. Milman, and W. Pawłucki [1–3], and C. Fefferman [10–18, 21, 22].

2 Background and Main Results

2.1 Polynomial and Whitney fields

We write $\mathcal{P}$ to denote the vector space of polynomials on $\mathbb{R}^n$ with degree no greater than $m-1$.

For $x \in \mathbb{R}^n$, let F be $(m-1)$-times differentiable at x. We identify the $(m-1)$-jet of F at x with the $(m-1)^{\mathrm{st}}$-degree Taylor polynomial of F at x:

$$\mathcal{J}_x F(y) := \sum_{|\alpha| \le m-1} \frac{\partial^\alpha F(x)}{\alpha!}(y-x)^\alpha.$$

For $P, Q \in \mathcal{P}$ and $x \in \mathbb{R}^n$, we define

$$P \odot_x Q := \mathcal{J}_x(PQ).$$

The operation $\odot_x$ turns $\mathcal{P}$ into a ring, which we denote by $\mathcal{R}_x$.

We define

$$\vec{\mathcal{P}} := \underbrace{\mathcal{P} \oplus \cdots \oplus \mathcal{P}}_{D\text{copies}}.$$

Thus, every $\vec{P} \in \vec{\mathcal{P}}$ has the form $\vec{P} = (P_1, \cdots, P_D)$, with $P_j \in \mathcal{P}$ for $j = 1, \cdots, D$.

Let $\vec{F} = (F_1, \cdots, F_D)$ be a $\mathbb{R}^D$-valued function $(m-1)$-times differentiable at $x \in \mathbb{R}^n$. We define

$$\mathcal{J}_x\vec{F} := (\mathcal{J}_x F_1, \cdots, \mathcal{J}_x F_D) \in \vec{\mathcal{P}}.$$

We will also use the $\mathcal{R}_x$-module structure on $\vec{\mathcal{P}}$, whose multiplication is given by

$$R \odot_x \vec{P} := (R \odot_x P_1, \cdots, R \odot_x P_D) \in \vec{\mathcal{P}},$$

for $x \in \mathbb{R}^n$, $\vec{P} = (P_1, \cdots, P_D) \in \vec{\mathcal{P}}$, and $R \in \mathcal{R}_x$.

Let $S \subset \mathbb{R}^n$ be a finite set. A Whitney field is an array $(\vec{P}^x)_{x\in S}$ parameterized by points in S, where $\vec{P}^x \in \vec{\mathcal{P}}$ for $x \in S$. We write $W^m(S)$ to denote the space of Whitney fields on S.

Given $(\vec{P}^x)_{x\in S} \in W^m(S)$, we define

$$\|(\vec{P}^x)_{x\in S}\|_{W^m(S)} := \max_{\substack{x\in S \\ |\alpha|\le m-1}} \|\partial^\alpha \vec{P}^x(x)\|_\infty + \max_{\substack{x,y\in S, x\neq y \\ |\alpha|\le m-1}} \frac{\|\partial^\alpha(\vec{P}^x - \vec{P}^y)(x)\|_\infty}{|x-y|^{\,m-|\alpha|}}. \tag{2.1}$$

Note that $\|\cdot\|_{W^m(S)}$ is a norm on $W^m(S)$.

We will also be using the seminorm

$$\|(\vec{P}^x)_{x\in S}\|_{\dot{W}^m(S)} := \max_{\substack{x,y\in S, x\neq y \\ |\alpha|\le m-1}} \frac{\|\partial^\alpha(\vec{P}^x - \vec{P}^y)(x)\|_\infty}{|x-y|^{\,m-|\alpha|}}. \tag{2.2}$$

We use $\vec{\mathcal{P}}^*$ to denote the dual of $\vec{\mathcal{P}}$. We use $W^m(S)^*$ to denote the dual of $W^m(S)$. An element $\xi \in W^m(S)^*$ has the form $\xi = (\xi_x)_{x\in S}$, so that

$$\xi\left[(\vec{P}^x)_{x\in S}\right] = \sum_{x\in S} \xi_x(\vec{P}^x) \text{ for } (\vec{P}^x)_{x\in S} \in W^m(S).$$

Thanks to the classical Whitney Extension Theorem for finite sets (see e.g., [24]), we can rephrase Theorem 1.4 in terms of Whitney fields.

Theorem 2.1. The following holds for $\mathbb{X} = W^m$ and $\mathbb{X} = \dot{W}^m$.

Let $S \subset \mathbb{R}^n$ be a finite set of diameter at most 1. For each $x \in S$, let $\vec{G}(x) \subset \vec{\mathcal{P}}$ be convex. Suppose that for every subset $S' \subset S$ with $|S| \le 2^{\dim \vec{\mathcal{P}}}$, there exists $(\vec{P}^x)_{x\in S'} \in \mathbb{X}(S')$ such that $\|(\vec{P}^x)_{x\in S'}\|_{\mathbb{X}(S)} \le 1$ and $\vec{P}^x \in \vec{G}(x)$ for all $x \in S'$.

Then, there exists $(\vec{P}^x)_{x\in S} \in \mathbb{X}(S)$ such that $\|(\vec{P}^x)_{x\in S}\|_{\mathbb{X}(S)} \le \gamma$ and $\vec{P}^x \in \vec{G}(x)$ for all $x \in S$. Here, γ depends only on m, n, D, and $|S|$.

For the rest of the paper, we will be working with Whitney fields instead of C^m functions.

2.2 Shape fields

In this section, we generalize a key object introduced in [19].

Definition 2.2. Let $S \subset \mathbb{R}^n$ be finite. For each $x \in S$, $0 < M < \infty$, let $\vec{\Gamma}(x,M) \subset \vec{P}$ be a (possibly empty) convex set. We say that $(\vec{\Gamma}(x,M))_{x\in S, M>0}$ is a vector-valued shape field if for all $x \in S$ and $0 < M' \le M < \infty$, we have $\vec{\Gamma}(x,M') \subset \vec{\Gamma}(x,M)$.

When $D = 1$, we write $\Gamma(x,M)$ instead of $\vec{\Gamma}(x,M)$, and we omit the adjective "vector-valued."

Definition 2.3. Let $C_w, \delta_{\max}$ be positive real numbers. We say that a vector-valued shape field $(\vec{\Gamma}(x,M))_{x\in S, M>0}$ is $(C_w, \delta_{\max})$-convex if the following condition holds:

Let $0 < \delta \le \delta_{\max}$, $x \in S$, $0 < M < \infty$, $\vec{P}_1, \vec{P}_2 \in \vec{\mathcal{P}}$, $Q_1, Q_2 \in \mathcal{P}$. Assume that

(1) $\vec{P}_1, \vec{P}_2 \in \vec{\Gamma}(x,M)$;

(2) $\|\partial^\alpha(\vec{P}_1 - \vec{P}_2)(x)\|_\infty \le M\delta^{m-|\alpha|}$ for $|\alpha| \le m-1$;

(3) $\left|\partial^\alpha Q_i(x)\right| \le \delta^{-|\alpha|}$ for $|\alpha| \le m-1$ and $i = 1, 2$;

(4) $Q_1 \odot_x Q_1 + Q_2 \odot_x Q_2 = 1$.

Then

(5) $\vec{P} := \sum_{i=1}^{2} (Q_i \odot_x Q_i) \odot_x \vec{P}_i \in \vec{\Gamma}(x, C_w M)$.

2.3 Main technical results

The main technical results are the following two theorems. The first is the Finiteness Principle for vector-valued shape fields, and the second improves the finiteness constant.

Theorem 2.4. There exists $k^\sharp = k^\sharp(m,n,D)$ such that the following holds.

Let $E \subset \mathbb{R}^n$ be a finite set and let $(\vec{\Gamma}(x,M))_{x\in E, M>0}$ be a $(C_w, \delta_{\max})$-convex vector-valued shape field. Let $Q_0 \subset \mathbb{R}^n$ be a cube of side length $\delta_{Q_0} \le \delta_{\max}$ and $x_0 \in E \cap 5Q_0$ and $M_0 > 0$ be given.

Suppose that for each $S \subset E$ with $|S| \leq k^\sharp$, there exists a Whitney field $(\vec{P}^z)_{z\in S}$ such that

$$\|(\vec{P}^z)_{z\in S}\|_{\dot{W}^m(S)} \leq M_0 \tag{2.3}$$

and

$$\vec{P}^z \in \vec{\Gamma}(z, M_0) \text{ for all } z \in S. \tag{2.4}$$

Then, there exist $\vec{P}^0 \in \vec{\Gamma}(x_0, M_0)$ and $\vec{F} \in C^m(Q_0, \mathbb{R}^D)$ such that

- $J_z\vec{F} \in \vec{\Gamma}(z, CM)$ for all $z \in E \cap 5Q_0$.
- $\|\partial^\alpha(\vec{F} - \vec{P}^0)(x)\|_\infty \leq CM_0\delta_{Q_0}^{m-|\alpha|}$ for all $x \in Q_0, |\alpha| \leq m$.
- In particular, $\|\partial^\alpha\vec{F}(x)\|_\infty \leq CM_0$ for all $x \in Q_0, |\alpha| = m$.

The case of scalar-valued shape fields ($D = 1$) was proved in [19]. In this paper, we will use the $D = 1$ case to prove the more general Theorem 2.4 stated above using a gradient trick, inspired by [19, 20].

Theorem 2.5. One may take $k^\sharp = 2^{\dim \vec{\mathcal{P}}}$ in Theorem 2.4.

Proof of Theorem 2.5 via Theorem 1.4. Take as given the hypotheses for Theorem 2.4, but with $k^\sharp = 2^{\dim \vec{\mathcal{P}}}$. This means that for each $S' \subset E$ with $|S'| = 2^{\dim \vec{P}}$, there exists $(\vec{P}^z)_{z\in S'}$ such that

$$\|(\vec{P}^z)_{z\in S'}\|_{\dot{W}^m(S')} \leq M_0 \tag{2.5}$$

and

$$\vec{P}^z \in \vec{\Gamma}(z, M_0). \tag{2.6}$$

Recall that in the definition of shape field, we require $\Gamma(x, M)$ be convex for all $x \in S$ and $M > 0$.

Let $S \subset E$ with $|S| \leq k^\sharp$, where $k^\sharp$ is as initially stated in Theorem 2.4 (and coming from [19] and our gradient trick for $D \geq 2$). Then, the above holds for all $S' \subset S$ with $|S'| = 2^{\dim \vec{\mathcal{P}}}$, so by the homogeneous version of Theorem 1.4, there exists $\vec{F} \in \dot{C}^m(\mathbb{R}^n, \mathbb{R}^D)$

such that

$$\|\vec{F}\|_{\dot{C}^m(\mathbb{R}^n,\mathbb{R}^D)} \leq \gamma M_0 \tag{2.7}$$

and

$$J_x\vec{F} \in \Gamma(x, M_0) \text{ for all } x \in S. \tag{2.8}$$

By (2.7), we have

$$\|(J_x\vec{F}^x)_{x\in S}\|_{\dot{W}^m(S)} \leq C\gamma M_0. \tag{2.9}$$

Thus, the hypotheses for Theorem 2.4 with the $k^\sharp$ from the initial statement are satisfied. ■

At this point, we have shown that the shape fields finiteness principle holds with an improved value of $k^\sharp$ (Theorem 2.5); the next step is to show that the selection problem of Theorems 1.3 and 1.4 may be described through shape fields.

Proof of Theorem 1.3 via Theorem 2.5. Let

$$\vec{\Gamma}(x, M) := \left\{\vec{P} \in \vec{\mathcal{P}} : \|\partial^\alpha \vec{P}(x)\|_\infty \leq M \text{ for } |\alpha| \leq m-1 \text{ and } \vec{P}(x) \in K(x)\right\}. \tag{2.10}$$

It suffices to observe that $(\vec{\Gamma}(x, M))_{x\in E, M>0}$ is a $(C, 1)$-convex shape field when $K(x)$ is convex for each $x \in E$.

Let $\delta \in (0, 1]$, $x \in E$, $M \in (0, \infty)$, $\vec{P}_1, \vec{P}_2 \in \vec{\mathcal{P}}$, and $Q_1, Q_2 \in \mathcal{P}$ be given, such that

(C1) $\vec{P}_1, \vec{P}_2 \in \vec{\Gamma}(x, M)$ with $\vec{\Gamma}(x, M)$ as in (2.10);

(C2) $\|\partial^\alpha(\vec{P}_1 - \vec{P}_2)(x)\|_\infty \leq M\delta^{m-|\alpha|}$ for $|\alpha| \leq m-1$;

(C3) $\left|\partial^\alpha Q_i(x)\right| \leq \delta^{-|\alpha|}$ for $|\alpha| \leq m-1$, $i = 1, 2$; and

(C4) $Q_1 \odot_x Q_1 + Q_2 \odot_x Q_2 = 1$.

We set

$$\vec{P} := \sum_{i=1,2} Q_i \odot_x Q_i \odot_x \vec{P}_i.$$

We want to show that $\vec{P} \in \vec{\Gamma}(x, CM)$ for some $C = C(m, n, D)$.

It is clear from (2.10), (C1), and (C4) that $\vec{P}(x) \in K(x)$. It remains to show that $\|\partial^\alpha \vec{P}(x)\|_\infty \leq CM$.

By the definition of $\vec{\Gamma}$ in (2.10), we have

$$\|\partial^\alpha \vec{P}_i(x)\|_\infty \leq M \text{ for } |\alpha| \leq m-1, i = 1, 2. \tag{2.11}$$

Using the product rule and (C4), we have, for $|\alpha| \leq m-1$,

$$\partial^\alpha(\vec{P} - \vec{P}_1)(x) = \sum_{i=1,2} \sum_{\beta \leq \alpha} \sum_{\gamma \leq \beta} C_{\alpha,\beta,\gamma} \cdot \partial^\gamma Q_i(x) \cdot \partial^{\beta-\gamma} Q_i(x) \cdot \partial^{\alpha-\beta}(\vec{P}_i - \vec{P}_1)(x).$$

It follows from (C2) and (C3) that

$$\|\partial^\alpha(\vec{P} - \vec{P}_1)(x)\|_\infty \leq CM. \tag{2.12}$$

Finally, we see from (2.11), (2.12), and the triangle inequality that $\|\partial^\alpha \vec{P}(x)\|_\infty \leq CM$ for $|\alpha| \leq m-1$. ■

Thus, it remains to establish Theorem 1.4. This will be done in Section 4

3 Whitney Norm and Dual Norm on Clusters

In this section, we review the data structure in [1] and prove a series of results that allows us to reduce the size of supports for linear functionals on $W^m(S)^*$.

We write $|S|$ to denote the cardinality of a finite set $S \subset \mathbb{R}^n$.

If $X, Y \subset \mathbb{R}^n$, we define

$$\text{diam}(X) := \max_{x,x' \in X} |x - x'| \text{ and}$$

$$\text{dist}(X, Y) := \min_{x \in X, y \in Y} |x - y|.$$

A rooted tree ("tree" for short) is an undirected graph with a distinct node (i.e., the root) in which any two nodes are connected by exactly one path. A leaf of a tree is any non-root node of degree one.

Let $S \subset \mathbb{R}^n$ be a finite set. We consider trees $\mathcal{T}$, each node of which corresponds to a subset of S, that satisfy the following properties.

(T1) The root of $\mathcal{T}$, $R(\mathcal{T}) = S$.

(T2) If V is a node, then V corresponds to a subset of S. The children of any node V form a partition of V (unless V is a leaf).

(T3) The nodes of any given level correspond to a partition of S. The leaves of $\mathcal{T}$ are the singletons $\{x\}$, with $x \in S$.

(T4) The number of nodes of level $\ell = 0, 1, \cdots, height(\mathcal{T})-1$ is a strictly increasing function of ℓ. Here, the height of a tree is the number of levels.

A collection of points

$$\underline{x} = \left\{x_V \in S : V \in \mathcal{T} \setminus leaves(\mathcal{T})\right\}$$

is called a set of reference points for $\mathcal{T}$ if, for each V, $x_V \in V$ and $x_V = x_W$ for some child W of V.

We adopt the convention $x_{\{x\}} := x$ in the last level.

Let $\underline{x}$ be a set of reference points of $\mathcal{T}$. For each $V \in \mathcal{T} \setminus leaves(\mathcal{T})$, define

$$V(\underline{x}) := \left\{x_W : W \text{ is a child of } V\right\} .$$

Suppose $x \in S \setminus \left\{x_S\right\}$. Then there is a unique node V of highest level such that $x \in V \setminus \left\{x_V\right\}$. We set

$$ref(x) := x_V. \tag{3.1}$$

We also set

$$U(x) := \text{the node } U \text{ at the lowest level such that } x = x_U. \tag{3.2}$$

A trunk T of $\mathcal{T}$ denotes a directed path from the root S to the second to second-to-last level. In particular, a trunk includes no leaf. Let T be a trunk of $\mathcal{T}$. We define the set of branch nodes $B(T)$ as the set of nodes of $\mathcal{T}$ that are adjacent to T.

We define the notion of "clustering" as follows.

Definition 3.1. Let $S \subset \mathbb{R}^n$ be finite. Let $\mathcal{T}$ be a tree of subsets of S that satisfies (T1) to (T4). We say that $\mathcal{T}$ is a clustering of S if $\mathcal{T}$ has a set of reference points $\underline{x} = \left\{x_V\right\}$ such that for each $\ell = 0, 1, \cdots, height(\mathcal{T}) - 1$, the set

$$\Pi := \left\{V(\underline{x}) : level(V) = \ell\right\}$$

forms a partition of

$$\left\{x_W : \mathit{level}(W) = \ell + 1\right\}$$

satisfying

$$\begin{aligned} |x - y| &\geq c_{\underline{x}} \cdot \operatorname{diam}(S) \text{ for each } S \in \Pi,\ x \neq y \text{ in } S, \text{ and} \\ \operatorname{dist}(S, S') &\geq c_{\underline{x}} \cdot \operatorname{diam}(S) \text{ for all } S, S' \in \Pi, S \neq S'. \end{aligned} \tag{3.3}$$

Here, $0 < c_{\underline{x}} \leq 1$ is called the clustering constant.

We write $\mathcal{C} = \mathcal{C}(\mathcal{T}, \underline{x})$ to denote a clustering $\mathcal{T}$ of S together with a set of reference points $\underline{x}$.

The following lemma is a quick adaptation of Lemma 2.4 of [1].

Lemma 3.2. Given a finite set $S \subset \mathbb{R}^n$, we can always find a clustering $\mathcal{T}$ of S such that for any set of reference points $\underline{x}$ for $\mathcal{T}$, condition (3.3) of Definition 3.1 is satisfied with some $0 < c_{\underline{x}} \leq 1$, where $c_{\underline{x}}$ depends only on n and $|S|$.

Definition 3.3. Let $\mathcal{C} = \mathcal{C}(\mathcal{T}, \underline{x})$ be a clustering of S with a set of reference points $\underline{x}$. We define the C^m-clustering norm $\|\cdot\|_{\mathcal{C}}$ on $W^m(S)$ to be

$$\|(\vec{P}^x)_{x\in S}\|_{\mathcal{C}} := \max\left\{|||(\vec{P}^x)_{x\in S}|||_{\mathcal{C}}\ ,\ \|\vec{P}^{x_S}\|_{x_S}\right\}\ ,$$

where

$$|||\vec{P}|||_{\mathcal{C}} := \max_{\substack{x\in S\setminus\{x_S\}\\ y=ref(x)\\ |\alpha|\leq m-1}} \frac{\|\partial^\alpha(\vec{P}^x - \vec{P}^y)(x)\|_\infty}{|x-y|^{\,m-|\alpha|}} \quad\text{and}\quad \|\vec{P}^{x_S}\|_{x_S} := \max_{|\alpha|\leq m-1} \|\partial^\alpha \vec{P}^{x_S}(x_S)\|_\infty.$$

Lemma 3.4 (Proposition 3.2 of [1]). Let $S \subset \mathbb{R}^n$ be a finite set, and let $\mathcal{C} = \mathcal{C}(\mathcal{T}, \underline{x})$ be a clustering of S with a set of reference points $\underline{x}$ and clustering constant $c_{\underline{x}}$. Then

$$\|(\vec{P}^x)_{x\in S}\|_{W^m(S)} \leq C\|(\vec{P}^x)_{x\in S}\|_{\mathcal{C}}. \tag{3.4}$$

Here, $C = C(c_{\underline{x}}, m, n, |S|, B)$, where B is an upper bound on $\operatorname{diam}(S)$.

Next we characterize linear functionals on clusters.

Let $S \subset \mathbb{R}^n$ be finite, and let $\xi = (\xi_x)_{x \in S} \in W^m(S)^*$.

Let $\mathcal{C}(\mathcal{T}, \underline{x})$ be a clustering of S. For each node $V \in \mathcal{T}$, we define $\xi_V \in \vec{\mathcal{P}}^*$ by the formula

$$\xi_V := \sum_{x \in V} \xi_x \,. \tag{3.5}$$

Lemma 3.5 (Lemma 5.1 of [1]). Let $S \subset \mathbb{R}^n$ be a finite set, and let $\mathcal{C} = \mathcal{C}(\mathcal{T}, \underline{x})$ be a clustering of S with a set of reference points $\underline{x}$ and clustering constant $c_{\underline{x}}$. Let $\mathit{ref}(x)$ and $U(x)$ be as in (3.1) and (3.2), respectively. The action of $\xi \in W^m(S)^*$ has the form:

$$\xi[(\vec{P}^x)_{x \in S}] = \sum_{x \in S \setminus \{x_S\}} \xi_{U(x)}(\vec{P}^x - \vec{P}^{\mathit{ref}(x)}) + \xi_S(\vec{P}^{x_S}) \,. \tag{3.6}$$

As in Remark 5.2 of [1], we can compute the cluster dual norm using the following formula:

$$\begin{aligned} \|\xi\|_{\mathcal{C}^*} = &\sum_{\substack{x \in S \setminus \{x_S\} \\ |\alpha| \le m-1 \\ 1 \le j \le D}} |x - \mathit{ref}(x)|^{m-|\alpha|} \left| \xi_{U(x)} \left(0, \ldots, 0, \frac{(\cdot - x)^\alpha}{\alpha!}, 0, \ldots, 0 \right) \right| \\ &+ \sum_{\substack{|\alpha| \le m-1 \\ 1 \le j \le D}} \left| \xi_S \left(0, \ldots, 0, \frac{(\cdot - x_S)^\alpha}{\alpha!}, 0, \ldots, 0 \right) \right| \,. \end{aligned} \tag{3.7}$$

In the above, the nontrivial expression in the arguments of ξ_S and $\xi_{U(x)}$ are in the j-th coordinates.

Lemma 3.6. Let $S \subset \mathbb{R}^n$ be a finite set, and let $\Phi : S \times \vec{\mathcal{P}}^* \to \mathbb{R}$ be a function that is positively homogeneous with degree one on the fibers and vanishes along the zero section. We write $\Phi_x(\cdot)$ for $\Phi(x, \cdot)$. Let $\mathcal{T}$ be a clustering of of S. Let $\xi \in W^m(S)^*$. For each $V \in \mathcal{T}$, define ξ_V as in (3.5). Define

$$\Phi(\xi_V) := \sum_{x \in V} \Phi_x(\xi_x) \,,$$

and set $\bar{\xi}_V := (\xi_V, \Phi(\xi_V)) \in \vec{\mathcal{P}}^* \oplus \mathbb{R}$. Let T be a trunk of $\mathcal{T}$, and let $\Xi(T)$ denote the linear span of $\left\{\bar{\xi}_V : V \in B(T)\right\}$ in $\vec{\mathcal{P}}^* \oplus \mathbb{R}$. Assume

$$\dim \Xi(T) < \#(B(T)) \,.$$

Then there exists $\eta \in W^m(S)^*$ such that the following hold.

(1) For all $V \in \mathcal{T} \setminus T$, $\eta_V = \theta_V \xi_V$ for some $0 \le \theta_V \le 2$.

(2) For some $V \in B(T)$, $\eta_x \equiv 0$ for all $x \in V$.

(3) $\sum_{x\in S} \xi_x = \sum_{x\in S} \eta_x$ as elements of $\vec{\mathcal{P}}^*$.

(4) $\sum_{x\in S} \Phi_x(\xi_x) = \sum_{x\in S} \Phi_x(\eta_x)$.

Moreover, for such η, we have

$$\|\eta\|_{\mathcal{C}^*} \le 2\|\xi\|_{\mathcal{C}^*} \tag{3.8}$$

Proof. We modify the proof of Lemma 6.1 of [1].

Since $\dim \Xi(T) \le \#(B(T))$, $\{\bar{\xi}_W, W \in B(T)\}$ is not linearly independent, so we may find $V \in B(T)$ such that

$$\bar{\xi}_V = \sum_{W\in B(T)\setminus V} \lambda_{VW} \cdot \bar{\xi}_W \text{ where all } |\lambda_{VW}| \le 1\,.$$

For each $x \in S$, we set $\eta_x := \theta_x \cdot \xi_x$, where

$$\theta_x := \begin{cases} 0 & \text{if } x \in V\,. \\ 1 + \lambda_{VW} & \text{if } x \in W \text{ and } W \in B(T) \setminus V\,. \\ 1 & \text{otherwise.} \end{cases}$$

Conclusions (1) and (2) then follow by construction.

Now we prove (3) and (4). First we make the following crucial observation. Thanks to our assumption on Φ and the conditions on the λ_{VW}'s, we see that

$$\Phi_x\left((1+\lambda_{VW})\xi_x\right) = \Phi_x(\xi_x) + \lambda_{VW}\Phi_x(\xi_x). \tag{3.9}$$

Therefore,

$$\begin{aligned} \sum_{x\in S} \bar{\eta}_x &= \sum_{x\in S} \bar{\xi}_x - \sum_{x\in V} \bar{\xi}_x + \sum_{W\in B(T)\setminus V} \lambda_{VW} \sum_{x\in W} \bar{\xi}_x \\ &= \sum_{x\in S} \bar{\xi}_x - \left(\bar{\xi}_V - \sum_{W\in B(T)\setminus V} \lambda_{VW}\bar{\xi}_W\right) \\ &= \sum_{x\in S} \bar{\xi}_x\,. \end{aligned}$$

We see that (3) and (4) follow.

Lastly, (3.8) follows from (3.7) and conclusions (1) and (3). ■

Let $S \subset \mathbb{R}^n$ and let $\mathcal{T}$ be a clustering of S. For any subset $S' \subset S$, $\mathcal{T}$ determines a clustering $\mathcal{T}'$ of S' by restriction.

The main result of the section is the following.

Lemma 3.7. Let $k \geq 2$. Under the hypotheses of Lemma 3.6, if $|S| \leq k$, then there exists $S' \subset S$ satisfying the following.

(1) Let $\mathcal{T}'$ be the clustering of S' determined by $\mathcal{T}$. For every trunk T' of $\mathcal{T}'$, let $\Xi(T')$ denote the linear span of $\left\{\bar{\xi}_V : V \in B(T')\right\}$ in $\vec{\mathcal{P}}^* \oplus \mathbb{R}$. Then we have

$$\#\left(B(T')\right) \leq \dim \Xi(T').$$

(2) There exists $\eta \in W^m(S)^*$ such that the following hold.

- (a) η_x is a multiple of ξ_x for each $x \in S$, and $\eta_x = 0$ for $x \in S \setminus S'$.
- (b) $\|\eta\|_{W^m(S)^*} \leq C\|\xi\|_{W^m(S)^*}$, where $C = C(m, n, k, B)$ with B being an upper bound for $\mathrm{diam}(S)$.
- (c) $\sum_{x \in S} \xi_x = \sum_{x \in S} \eta_x$.
- (d) $\sum_{x \in S} \Phi_x(\xi_x) = \sum_{x \in S} \Phi_x(\eta_x)$, with Φ_x as in Lemma 3.6.

Proof. Suppose S itself does not satisfy both of the conclusions. Taking $\eta = \xi$, we see that S satisfies (2). Therefore, S does not satisfy (1). Using conclusion (2) of Lemma 3.6, we may shrink S by one point at a time until conclusions (1), (2a), (2c), and (2d) hold. Meanwhile, (2b) holds, thanks to Lemma 3.2, Lemma 3.4, and (3.8). ■

We will couple Lemma 3.7 with the following result to prove Theorem 1.4.

Lemma 3.8 (Lemma 6.4 of [1]). Let $S \subset \mathbb{R}^n$ be finite with $\#(S) \geq 2$. Let $\mathcal{T}$ be a clustering of S. Suppose that for every trunk T of $\mathcal{T}$, $|(B(T))| \leq N$ for some $N \in \mathbb{N}$. Then $|S| \leq 2^{N-1}$.

4 Proof of the Main Theorem

We begin the proof of Theorem 1.4 by showing that one can approximate the convex sets Γ arbitrarily well by polytopes, which will allow us to use linear programming. While finer levels of approximation to these convex sets will generally require an arbitrarily increasing number of linear constraints to describe, the constants arising in our proof will be independent of this number.

By a polytope in a finite-dimensional normed vector space V, we mean the finite intersection of half-spaces of the form $\{v \in V : \xi(v) \leq c\}$, where $\xi \in V^*$ and $c \in \mathbb{R}$.

Let v, w be two Euclidean vectors. We write $v \geq w$ if each of the entries of $v - w$ is nonnegative.

Lemma 4.1. Let V be a finite-dimensional normed vector space with norm $\| \cdot \|_V$, and let $K \subset V$ be convex. Given $\delta > 0$, there exists a convex polytope K_δ such that $K \subset K_\delta \subset B_\delta(K)$, where $B_\delta(K)$ is the δ-neighborhood of K under the metric determined by $\| \cdot \|_V$.

Proof. We first address the case where $V = \mathbb{R}^d$, where the norm is the ℓ^∞ norm given by $\|(x_1, ..., x_d)\| = \max_{1 \leq j \leq d} |x_j|$.

Let $\mathcal{Q}$ be the set of cubes of the form

$$Q = [k_1\delta, (k_1 + 1)\delta] \times ... \times [k_d\delta, (k_d + 1)\delta],$$

where $k_1, ..., k_d \in \mathbb{Z}$. Define

$$K' = \bigcup_{\substack{Q \in \mathcal{Q} \\ Q \cap K \neq \emptyset}} Q,$$

and let $K'' = Conv(K')$, where $Conv(\cdot)$ is used to denote the convex hull of a set. Thus, K'' is a convex polytope. By definition, $K \subset K'$.

Let $x \in K''$. Then, there exist $y', z' \in K'$ such that x is on the line segment from y' to z'. Since $y', z' \in K'$, there exist $y, z \in K$ such that $\|y - y'\|, \|z - z'\| < \delta$.

Consider the function $f(t) = \|t(y' - y) + (1 - t)(z' - z)\|$. Then $f(0), f(1) < \delta$ and f is a convex, nonnegative function, so $f(t) < \delta$ for all $t \in [0, 1]$. Pick $t_0 \in [0, 1]$ such that $x = t_0 y' + (1 - t_0)z'$. Then, $f(t_0) < \delta$ means that $\|x - [ty + (1 - t)y']\| < \delta$. Since K is convex, $ty + (1 - t)y' \in K$, so x is within distance δ of K. Thus, $K' \subset B_\delta(K)$, completing the proof in the case $V = \mathbb{R}^d$.

Now suppose that V is an arbitrary d-dimensional, normed space. Since any two norms on a finite-dimensional space are equivalent, there exists $M < \infty$ such that $M^{-1}\|v\|_V \leq \|T^{-1}(v)\|_{\ell^\infty(\mathbb{R}^d)} \leq M\|v\|_V$.

Let $T : \mathbb{R}^d \to V$ be a linear isomorphism and let $\tilde{K} \subset \mathbb{R}^n$ be a polytope satisfying $T^{-1}(K) \subset \tilde{K} \subset B_\epsilon(T^{-1}(K))$, where $\epsilon > 0$ is to be determined. It follows that

$K \subset T(\tilde{K}) \subset B_{M\epsilon}(K)$ and that $T(\tilde{K})$ is a polytope in V. (To see the latter, observe that for linear functionals ξ on $\mathbb{R}^d$, $\xi(v) \leq c$ if and only if $\xi \circ T^{-1}(T(v)) \leq c$ and $\xi \circ T^{-1} \in V^*$.)

Thus, choosing $\epsilon = \delta/M$, we see that $T(\tilde{K})$ is the desired polytope. ■

4.1 Theorem 1.4 with $\mathbb{X} = C^m(\mathbb{R}^n, \mathbb{R}^D)$

Proof of Theorem 1.4 with $\mathbb{X} = C^m(\mathbb{R}^n, \mathbb{R}^D)$. Given $E \subset \mathbb{R}^n$ and $K(x) \subset \vec{\mathcal{P}}$ for each $x \in E$, we define

$$\|(K(x))_{x\in E}\|_{W^m(E)} := \inf\{\|(\vec{P}^x)_{x\in E}\|_{W^m(E)} : \vec{P}^x \in K(x) \text{ for all } x \in E\}. \tag{4.1}$$

While not strictly a norm, the above notation allows for a concise description of a quantity which is the main focus of the proof.

Our goal is to show there exists $C = C(m, n, D, B)$ such that for any finite $S \subset \mathbb{R}^n$ satisfying $|S| \leq B$, there exists $S' \subset S$ with $|S'| \leq 2^{\dim \vec{\mathcal{P}}}$ such that

$$C^{-1}\|(G(x))_{x\in S'}\|_{W^m(S')} \leq \|(G(x))_{x\in S}\|_{W^m(S)} \leq C\|(G(x))_{x\in S'}\|_{W^m(S')}.$$

If so, it follows that

$$\|(G(x))_{x\in S'}\|_{W^m(S')} \leq 1 \text{ for all } S' \subset S \text{ satisfying } |S'| = 2^{\dim \vec{\mathcal{P}}}$$

implies

$$\|(G(x))_{x\in S}\|_{W^m(S)} \leq C$$

for all $S \subset E$ satisfying $|S| = k^\sharp$.

We now make the following reduction: it suffices to prove Theorem 1.4 in the case that each $G(x)$ is a polytope.

If not, replace each $G(x)$ with $G(x)_\delta$ for sufficiently small $\delta > 0$, where $G(x)_\delta$ is the polytope guaranteed by Lemma 4.1. By taking $\delta > 0$ small enough, one may approximate both $\|(G(x))_{x\in S}\|_{W^m(S)}$ and $\|(G(x))_{x\in S'}\|_{W^m(S')}$ within a factor of 2, as these norms are continuous with respect to the relevant metrics.

To this end, we replace each $G(x)$ with $G(x)_\delta$, which will now be denoted K_x, as δ is fixed. For each x, we write $K_x = \{\vec{P} : \Omega_x\vec{P} \leq \vec{c}_x\}$ for some linear map $\Omega_x : \vec{\mathcal{P}} \to \mathbb{R}^{m_x}$, where $m_x \in \mathbb{N}$. We will occasionally write $\Omega : W^m(S) \to \prod_x \mathbb{R}^{m_x}$ to denote the mapping which sends $(\vec{P}^x)_{x\in S}$ to $(\Omega_x\vec{P}^x)_{x\in S}$.

We begin by writing $\|(K_x)_{x\in S}\|_{W^m(S)}$ as the solution to a linear programming problem:

$$\|(K_x)_{x\in S}\|_{W^m(S)} = \inf_{(\Omega_x \vec{P}^x \leq \vec{c}_x)_{x\in S}} \|(\vec{P}^x)_{x\in S}\|_{W^m(S)} \tag{4.2}$$

$$= \inf_{(\Omega_x \vec{P}^x \leq \vec{c}_x)_{x\in S}} \sup_{\|(\xi_x)_{x\in S}\|_{W^m(S)^*}\leq 1} (\xi_x)_{x\in S}[(\vec{P}^x)_{x\in S}]. \tag{4.3}$$

By Lemma 4.1, the unit ball in $W^m(S)^*$ may be approximated within a factor of 2 by a polytope, written as $\left\{(\xi_x)_{x\in S} : L(\xi_x)_{x\in S} \leq \mathbf{1}_k\right\}$ for some $k \in \mathbb{N}$ and linear map $L : W^m(S)^* \to \mathbb{R}^k$. Thus, we may rewrite (4.2) as

$$\|(K_x)_{x\in S}\|_{W^m(S)} \approx \inf_{(\Omega_x \vec{P}^x \leq \vec{c}_x)_{x\in S}} \sup_{L(\xi_x)_{x\in S}\leq \mathbf{1}_k} (\xi_x)_{x\in S}[(\vec{P}^x)_{x\in S}] \tag{4.4}$$

for some linear map $L : W^m(S)^* \to \mathbb{R}^k$ and some $k \in \mathbb{N}$.

The advantage of this formulation is that it becomes possible to apply the LP Duality Theorem (Lemma A.2 in Appendix) to the supremum above, giving us

$$\begin{aligned}\|(K_x)_{x\in S}\|_{W^m(S)} &\approx \inf_{(\Omega_x \vec{P}^x \leq \vec{c}_x)_{x\in S}} \inf_{\substack{y\geq 0 \\ L^T y = (\vec{P}^x)_{x\in S}}} \mathbf{1}_k \cdot y \\ &= \inf_{\substack{(\Omega_x \vec{P}^x \leq c_x)_{x\in S} \\ y\geq 0 \\ L^T y = (\vec{P}^x)_{x\in S}}} \mathbf{1}_k \cdot y \\ &= \inf_{\substack{-\Omega L^T y \geq -c \\ y\geq 0}} \mathbf{1}_k \cdot y.\end{aligned}$$

Note the referenced linear programming problem is feasible, as its solution corresponds to finding the smallest norm of a vector in a closed set.

Applying the Duality Theorem again, one obtains

$$\|(K_x)_{x\in S}\|_{W^m(S)} \approx \sup_{\substack{(z_x\geq 0)_{x\in S} \\ L(-\Omega^T)z\leq \mathbf{1}_k}} \sum_x -\vec{c}_x \cdot z_x \tag{4.5}$$

$$= \sup_{\substack{L(\xi_x)_{x\in S}\leq \mathbf{1}_k \\ (\xi_x\in(-\Omega_x^T)\mathbb{R}_+^{m_x})_{x\in S}}} \sup_{\substack{(z_x\geq 0)_{x\in S} \\ (-\Omega_x^T z_x = \xi_x)_{x\in S}}} \sum_x -\vec{c}_x \cdot z_x \tag{4.6}$$

$$\approx \sup_{(\xi_x\in(-\Omega_x^T)\mathbb{R}_+^{m_x})_{x\in S}} \frac{\sum_{x\in S} f_x(\xi_x)}{\|(\xi_x)_{x\in S}\|_{W^m(S)^*}}, \tag{4.7}$$

where
$$f_x(\xi_x) = \sup_{\substack{z\geq 0\\ -\Omega_x^T z=\xi_x}} \sum_x -\vec{c}_x \cdot z.$$

Fix $(\xi_x)_{x\in S}$ such that $\xi_x \in (-\Omega_x^T)\mathbb{R}_+^{m_x}$ for all $x \in S$. We see that f satisfies the hypotheses of Lemmas 3.6 and 3.7. By Lemma 3.8, we may apply Lemma 3.7 repeatedly until the S' in the conclusion satisfies $|S'| \leq 2^{\dim \mathcal{P}}$.

Let $(\eta_x)_{x\in S}$ be as guaranteed in the conclusion of Lemma 3.7 and recall $S' = \{x \in S : \eta_x \neq 0\}$. Thus,

$$\|(\eta_x)_{x\in S}\|_{W^m(S')^*} = \|(\eta_x)_{x\in S}\|_{W^m(S)^*} \lesssim \|(\xi_x)_{x\in S}\|_{W^m(S)^*}$$

and $|S'| \leq 2^{\dim \vec{\mathcal{P}}}$. Note that each η_x is obtained by multiplying some ξ_x by a nonnegative scalar; thus, $\eta_x \in (-\Omega_x^T)\mathbb{R}_+^{m_x}$ for all $x \in S$.

By this reasoning and (4.5) applied both as written above and with S' in place of S,

$$\begin{aligned}\|(K_x)_{x\in S}\|_{W^m(S)} &\approx \sup_{(\xi_x\in\Omega_x^T\mathbb{R}_+^{m_x})_{x\in S}} \frac{\sum_{x\in S} f_x(\xi_x)}{\|(\xi_x)_{x\in S}\|_{W^m(S)^*}}\\ &\lesssim \sup_{(\eta_x\in\Omega_x^T\mathbb{R}_+^{m_x})_{x\in S'}} \frac{\sum_{x\in S} f_x(\eta_x)}{\|(\eta_x)_{x\in S}\|_{W^m(S')^*}}\\ &\approx \|(K_x)_{x\in S'}\|_{W^m(S')}.\end{aligned}$$

■

4.2 Theorem 1.4 with $\mathbb{X} = \dot{C}^m(\mathbb{R}^n, \mathbb{R}^D)$

In this section, we point out the modifications needed in order to prove Theorem 1.4 for the case $\mathbb{X} = \dot{C}^m(\mathbb{R}^n, \mathbb{R}^D)$.

Let $S \subset \mathbb{R}^n$ be a finite set. Recall the definition of $\|\cdot\|_{\dot{W}^m(S)}$ in (2.2). We define

$$H(S) := \operatorname{span}\left\{\xi_{\alpha,y,z,j} : y, z \in S,\, z \neq y,\, |\alpha| \leq m-1,\, 1 \leq j \leq D\right\},$$

where each $\xi_{\alpha,y,z,j} \in W^m(S)^*$ is characterized by the action

$$\xi_{\alpha,y,z,j}[(\vec{P}^x)_{x\in S}] = \frac{\partial^\alpha(P_j^y - P_j^z)(y)}{|y-z|^{\,m-|\alpha|}}.$$

Then the norm $\|\cdot\|_{\dot{W}^m(S)}$ can be computed via the formula

$$\|(\vec{P}^x)_{x\in S}\|_{\dot{W}^m(S)} = \sup_{\substack{\xi\in H(S)\\ \|\xi\|_{W^m(S)^*}\leq 1}} \xi[(\vec{P}^x)_{x\in S}],.$$

Mirroring (4.1), we define the selection "seminorm" to be

$$\|(K_x)_{x\in S}\|_{\dot{W}^m(S)} := \inf\{\|(\vec{P}^x)_{x\in S}\|_{\dot{W}^m(S)} : \vec{P}^x \in K(x) \text{ for all } x \in S\}.$$

We repeat proof of Theorem 1.4 with $\mathbb{X} = C^m(\mathbb{R}^n, \mathbb{R}^D)$ in the previous section, but with the following modifications.

- We use $\|\cdot\|_{\dot{W}^m(S)}$ in place of $\|\cdot\|_{W^m(S)}$ (both the Whitney seminorm and the selection "seminorm").
- All the linear functionals will be chosen from $H(S) \subset W^m(S)^*$.
- The map L in (4.4) will be replaced by a suitable linear map $\tilde{L} : H(S) \to \mathbb{R}^{\tilde{k}}$ for some $\tilde{k} \in \mathbb{N}$.

This concludes all the necessary modifications for the proof of Theorem 1.4 with $\mathbb{X} = \dot{C}^m(\mathbb{R}^n, \mathbb{R}^D)$.

The proof of Theorem 1.4 is complete.

5 Vector-Valued Shape Fields Finiteness Principle

In this section we use what is colloquially known as the "gradient trick" to prove Theorem 2.4 using the $D = 1$ case proven in [19]. (See [19, 20].)

The following proof will require working in both $\mathbb{R}^n$ and $\mathbb{R}^{n+D}$, so we provide a brief introduction to some of the notation.

The variable for $\mathbb{R}^n$ will be x, while $\mathbb{R}^{n+D}$ will be viewed as $\{z = (x, \xi) : x \in \mathbb{R}^n, \xi \in \mathbb{R}^D\}$. The appropriate level of regularity for $\mathbb{R}^{n+D}$ will be C^{m+1}, so let $\mathcal{P}^+$ denote the vector space of $\mathbb{R}$-valued, m-degree polynomials over $\mathbb{R}^{n+D}$. (Recall that $\vec{\mathcal{P}}$ is the vector space of $\mathbb{R}^D$-valued, $(m-1)$-degree polynomials over $\mathbb{R}^n$.)

Proof of Theorem 2.4. Let $E, Q_0 \subset \mathbb{R}^n$, $(\vec{\Gamma}(x, M))_{x\in E, M>0}$, C_w, $0 < \delta_{Q_0} \leq \delta_{\max}$, $x_0 \in E \cap 5Q_0$ as in the hypotheses of Theorem 2.4 be given.

Let $E^+ = \{(x, 0) : x \in E\} \subset \mathbb{R}^{n+D}$. For $(x_0, 0) \in E^+$, define

$$\Gamma((x_0, 0), M) = \{P \in \mathcal{P}^+ : P(x, 0) = 0, \nabla_\xi P(x, 0) \in \vec{\Gamma}(x_0, M)\}. \tag{5.1}$$

We now show that $(\Gamma(z,M))_{z\in E^+}$ satisfies the hypotheses of the $D = 1$ case of Theorem 2.4.

Let $S^+ \subset E^+$ with $|S^+| \leq k^\sharp$. By definition, S^+ is of the form $\{(x,0) : x \in S\}$ for some $S \subset E$ with $|S| \leq k^\sharp$.

By hypothesis of Theorem 2.4, there exist $(\vec{P}^x)_{x\in S}$ such that

$$\|(\vec{P}^x)_{x\in S}\|_{\dot{W}^m(S)} \leq M_0. \tag{5.2}$$

and

$$\vec{P}^x \in \vec{\Gamma}(x, M_0) \text{ for all } x \in S. \tag{5.3}$$

For $z = (x_0, 0) \in E^+$, define

$$P^z(x,\xi) = P^{(x_0,0)}(x,\xi) := \sum_{j=1}^{D} \xi_j P_j(x). \tag{5.4}$$

Clearly, $P^{(x_0,0)}(x_0,0) = 0$ and $\nabla_\xi P^{(x_0,0)} = \vec{P}^{x_0}$, so $P^z \in \Gamma(z, M_0)$ for all $z \in E^+$.

Let $(x_0,0),(y_0,0) \in E^+$. Then,

$$\partial_x^\alpha P^{(x_0,0)}(x,0) = 0 \text{ and } \partial_x^\alpha \partial_\xi^\beta P^{(x_0,0)}(x,0) = 0 \text{ for } |\beta| \geq 2 \tag{5.5}$$

by definition, and for $1 \leq j \leq D$,

$$\left|\partial_x^\alpha \partial_{\xi_j}\left(P^{(x_0,0)} - P^{(y_0,0)}\right)(x_0,0)\right| = \left|\partial_x^\alpha (P_j^{x_0} - P_j^{y_0})(x_0,0)\right| \tag{5.6}$$

$$\leq C|x_0 - y_0|^{m-|\alpha|} \tag{5.7}$$

$$= C|(x_0,0) - (y_0,0)|^{(m+1)-(|\alpha|+1)}. \tag{5.8}$$

Thus, $(P^z)_{z\in S^+}$ satisfy (2.3).

To demonstrate $(C_w, \delta_{\max})$-convexity, let $0 < \delta \leq \delta_{\max}, x \in S^+, M < \infty, P_1, P_2, Q_1, Q_2 \in \mathcal{P}^+$ be as in Definition 2.2. If $P := Q_1 \odot_{(x_0,0)} Q_1 \odot_{(x_0,0)} P_1 + Q_2 \odot_{(x_0,0)} Q_2 \odot_{(x_0,0)} P_2$, then $P(x_0,0) = 0$ and

$$\nabla_\xi P(x,0) = [Q_1 \odot_{(x_0,0)} Q_1 \odot_{(x_0,0)} \nabla_\xi P_1](x,0) + [Q_2 \odot_{(x_0,0)} Q_2 \odot_{(x_0,0)} \nabla_\xi P_2](x,0), \tag{5.9}$$

which lies in $\Gamma(x, C_w M)$ by the $(C_w, \delta_{\max})$-convexity of the $\vec{\Gamma}(x,M)$.

Let Q' be the unit cube in $\mathbb{R}^D$. By the $D = 1$ case of Theorem 2.4 applied to $E^+ \subset \mathbb{R}^{n+D}, (\Gamma(z,M))_{z\in E^+,M>0}, (x_0,0), Q_0 \times Q'$, we have the following. There exist $F \in C^{m+1}(\mathbb{R}^{n+D},\mathbb{R})$ and $P^0 \in \Gamma((x_0,0), CM_0)$ such that

$$J_{(x,0)}F \in \Gamma((x,0), CM) \text{ for all } (x,0) \in E^+; \tag{5.10}$$

$$|\partial_x^\alpha \partial_\xi^\beta (F - P^0)(x,\xi)| \leq CM_0 \text{ for all } (x,\xi) \in Q_0 \times Q', |\alpha| + |\beta| \leq m+1; \tag{5.11}$$

and

$$\text{In particular, } |\partial_x^\alpha \partial_\xi^\beta F(x,\xi)| \leq CM_0 \text{ for } |\alpha| + |\beta| = m+1. \tag{5.12}$$

Define $\vec{G}(x) := \nabla_\xi F(x,0)$ and $\vec{Q}^0(x) = \nabla_\xi P^0(x,0)$. We claim $\vec{G} \in C^m(\mathbb{R}^n,\mathbb{R}^D)$ and $\vec{Q}^0 \in \vec{\Gamma}(x,CM)$ are the desired function and jet, respectively, found in the conclusion of Theorem 2.4.

First, by (5.10),

$$J_x G(y) = \nabla_\xi J_{(y,0)} F(x,0) \in \vec{\Gamma}(x, CM) \tag{5.13}$$

because $J_{(y,0)}F(x,0) \in \vec{\Gamma}((x,0), CM)$.

Next, for any $|\alpha| \leq m$ and $1 \leq j \leq D$,

$$|\partial_x^\alpha (G_j - Q_j^0)(x)| = |\partial_x^\alpha (\partial_{\xi_j} F - \partial_{\xi_j} P^0)(x,0)| \tag{5.14}$$

$$\leq CM_0 \delta_{Q_0}^{(m+1)-(|\alpha|+1)} = CM_0 \delta_{Q_0}^{m-|\alpha|} \tag{5.15}$$

by (5.11).

Lastly, for $|\alpha| = m$,

$$|\partial_x^\alpha G_j(x)| = |\partial_x^\alpha \partial_{\xi_j} F| \leq CM_0 \tag{5.16}$$

via (5.12). ■

Acknowledgments

We are indebted to Pavel Shvartsman and the referee for their valuable comments.

Funding

This work was supported by the UC Davis Summer Graduate Student Researcher Award [to F.J.]; the Alice Leung Scholarship in Mathematics [to F.J.]; the National Science Foundation Grant [DMS-1554733 to G.K.L.]; and the UC Davis Chancellor's Fellowship [to G.K.L.].

A Linear Programming and Duality

Lemma A.1 (LP Duality Theorem). Let p, q be positive integers. Let $c \in \mathbb{R}^p$ and $b \in \mathbb{R}^q$. Let $A : \mathbb{R}^p \to \mathbb{R}^q$ be a linear map. Consider the following two optimization problems.

$$\text{Maximize } c^T \cdot x \text{ subject to } Ax \leq b\,. \tag{A.1}$$

$$\text{Minimize } b^T \cdot y \text{ subject to } A^T y = c \text{ and } y \geq 0\,. \tag{A.2}$$

Suppose one of (A.1) or (A.2) has a feasible solution, then both have feasible and optimal solutions. Moreover, if x_0 optimizes (A.1) and y_0 optimizes (A.2), then $c^T \cdot x_0 = b^T \cdot y_0$, that is, the maximum of (A.1) equals the minimum of (A.2).

The same conclusion holds if we replace "$Ax \leq b$" by "$Ax \leq b$ and $x \geq 0$" in (A.1) and "$A^T y = c$" by "$A^T y \geq c$" in (A.2).

See [25] for a proof.

We generalize the theorem above to finite dimensional normed spaces.

Lemma A.2. Let V be a finite-dimensional normed vector space with norm $\|\cdot\|_V$ and dual V^*. Let $L : V^* \to \mathbb{R}^q$ be a linear map and let $L^* : \mathbb{R}^q \to V$ be the dual operator of L defined by

$$x^T \cdot L(\phi) = \langle \phi, L^* x \rangle \text{ for all } x \in \mathbb{R}^q \text{ and } \phi \in V^*.$$

(Here we identify the dual of any Euclidean space with itself via the dot product.)

Let $b \in \mathbb{R}^q$. Suppose there exists $\phi_0 \in V^*$ such that $L(\phi_0) \leq b$. Then

$$\sup_{L(\phi) \leq b} \langle \phi, v \rangle = \inf_{\substack{y \geq 0 \\ L^* y = v}} b^T \cdot y. \tag{A.3}$$

Proof. Let $p = \dim V < \infty$. There exists a linear isomorphism $J : V \to \mathbb{R}^p$. Let $J^* : \mathbb{R}^p \to V^*$ denote its dual. Note that J^* is also a linear isomorphism. We have the following diagram.

$$\begin{array}{ccccc} \mathbb{R}^p & \xleftarrow[J]{} & V & \xleftarrow[L^*]{} & \mathbb{R}^q \\ \big\downarrow & & & & \big\downarrow \\ \mathbb{R}^p & \xrightarrow{J^*} & V^* & \xrightarrow{L} & \mathbb{R}^q \end{array}$$

For each $v \in V$ and $\phi \in V^*$, there exist unique $c, x \in \mathbb{R}^p$ such that $J^{-1}(p) = v$ and $J^*(x) = \phi$. Thus, thanks to LP Duality Theorem (Lemma A.1), we have

$$\sup_{L(\phi)\le b} \langle \phi, v\rangle = \sup_{L\circ J^*(x)\le b} \langle J^*(x), J^{-1}(c)\rangle$$
$$= \sup_{L\circ J^*(x)\le b} c^T \cdot x$$
$$= \inf_{\substack{(L\circ J^*)^T y = c\\ y\ge 0}} b^T \cdot y.$$

Notice that $(L\circ J^*)^T = J\circ L^*$. Moreover, since J is an isomorphism, the equality $J\circ L^* y = c$ is equivalent to $L^* y = J^{-1}(c) = v$. (A.3) follows. ■

References

[1] Bierstone, E. and P. D. Milman. "C^m-norms on finite sets and C^m-extension criteria." *Duke Math. J.* 137, no. 1 (2007): 1–18.

[2] Bierstone, E., P. D. Milman, and W. Pawłucki. "Differentiable functions defined in closed sets. A problem of Whitney." *Invent. Math.* 151, no. 2 (2003): 329–52.

[3] Bierstone, E., P. D. Milman, and W. Pawłucki. "Higher-order tangents and Fefferman's paper on Whitney's extension problem." *Ann. Math. (2)* 164, no. 1 (2006): 361–70.

[4] Brudnyi, Y. and P. Shvartsman. "A linear extension operator for a space of smooth functions defined on a closed subset in $\mathbb{R}^n$." *Dokl. Akad. Nauk SSSR* 280, no. 2 (1985): 268–72.

[5] Brudnyi, Y. and P. Shvartsman. "Generalizations of Whitney's extension theorem." *Int. Math. Res. Not.* 3, no. 129 (1994).

[6] Brudnyi, Y. and P. Shvartsman. "The Traces of Differentiable Functions to Subsets of $\mathbb{R}^n$." In *Linear and Complex Analysis. Problem Book 3*, vol. 1574 of Lecture Notes in Mathematics, 279–81. Berlin: Springer, 1994.

[7] Brudnyi, Y. and P. Shvartsman. "The Whitney problem of existence of a linear extension operator." *J. Geom. Anal.* 7, no. 4 (1997): 515–74.

[8] Brudnyi, Y. and P. Shvartsman. "The trace of jet space $J^k\Lambda^\omega$ to an arbitrary closed subset of $\mathbb{R}^n$." *Trans. Amer. Math. Soc.* 350, no. 4 (1998): 1519–53.

[9] Brudnyi, Y. and P. Shvartsman. "Whitney's extension problem for multivariate $C^{1,\omega}$-functions." *Trans. Amer. Math. Soc.* 353, no. 6 (2001): 2487–512. (electronic).

[10] Fefferman, C. "A generalized sharp Whitney theorem for jets." *Rev. Mat. Iberoam.* 21, no. 2 (2005): 577–688.

[11] Fefferman, C. "Interpolation and extrapolation of smooth functions by linear operators." *Rev. Mat. Iberoam.* 21, no. 1 (2005): 313–48.

[12] Fefferman, C. "A sharp form of Whitney's extension theorem." *Ann. Math. (2)* 161, no. 1 (2005): 509–77.

[13] Fefferman, C. "Whitney's extension problem for C^m." *Ann. Math. (2)* 164, no. 1 (2006): 313–59.

[14] Fefferman, C. "C^m extension by linear operators." *Ann. Math. (2)* 166, no. 2 (2007): 779–835.

[15] Fefferman, C. "The structure of linear extension operators for C^m." *Rev. Mat. Iberoam.* 23, no. 1 (2007): 269–80.

[16] Fefferman, C. "Fitting a Cm-smooth function to data, III." *Ann. Math. (2)* 170, no. 1 (2009): 427–41.

[17] Fefferman, C. "Whitney's extension problems and interpolation of data." *Bull. Amer. Math. Soc. (N.S.)* 46, no. 2 (2009): 207–20.

[18] Fefferman, C. "The C^m norm of a function with prescribed jets." *Rev. Mat. Iberoam.* 26, no. 3 (2010): 1075–98.

[19] Fefferman, C., A. Israel, and G. K. Luli. "Finiteness principles for smooth selection." *Geom. Funct. Anal.* 26, no. 2 (2016): 422–77.

[20] Fefferman, C. and G. K. Luli. "The Brenner–Hochster–Kollár and Whitney problems for vector-valued functions and jets." *Rev. Mat. Iberoam.* 30, no. 3 (2014): 875–92.

[21] Fefferman, C. and P. Shvartsman. "Sharp finiteness principles for Lipschitz selections." *Geom. Funct. Anal.* 28 (2018): 1641–705.

[22] Fefferman, C. L. and G. K. Luli. "C^m semialgebraic sections over the plane." (2021): preprint https://arxiv.org/abs/2101.06334.

[23] Glaeser, G. "Étude de quelques algèbres tayloriennes." *J. Anal. Math.* 6 (1958): 1–124.

[24] Stein, E. M. *Singular Integrals and Differentiability Properties of Functions*, vol. 2 of Monographs in Harmonic Analysis. Princeton, NJ: Princeton University Press, 1970.

[25] Matousek, J. and B. Gärtner. *Understanding and Using Linear Programming.* Universitext. Springer Berlin Heidelberg, 2007.

[26] Shvartsman, P. "Traces of functions of zygmund class." *Sibirsk. Mat. Zh.* 5 (1987): 203–15.

[27] Shvartsman, P. "Lipschitz selections of set-valued mappings and Helly's theorem." *J. Geom. Anal.* 12, no. 2 (2002): 289–324.

[28] Shvartsman, P. "The Whitney extension problem and Lipschitz selections of set-valued mappings in jet-spaces." *Trans. Amer. Math. Soc.* 360, no. 10 (2008): 5529–50.

[29] Shvartsman, P. "On Lipschitz selections of affine-set valued mappings." *Geom. Funct. Anal.* 11, no. 4 (2001): 840–68.

[30] Whitney, H. "Analytic extensions of differentiable functions defined in closed sets." *Trans. Amer. Math. Soc.* 36, no. 1 (1934): 63–89.

[31] Whitney, H. "Differentiable functions defined in closed sets. I." *Trans. Amer. Math. Soc.* 36, no. 2 (1934): 369–87.

[32] Whitney, H. "Functions differentiable on the boundaries of regions." *Ann. Math. (2)* 35, no. 3 (1934): 482–5.

Bibliography

[1] V. Baramidze and M. Lai, *Nonnegative data interpolation by spherical splines*, Journal of Computational and Applied Mathematics, 342 (2018), pp. 463–477.

[2] E. Bierstone and P. D. Milman, *$\mathcal{C}^m$-norms on finite sets and $\mathcal{C}^m$-extension criteria*, Duke Mathematical Journal, 137 (2007), pp. 1–18.

[3] E. Bierstone, P. D. Milman, and W. Pawłucki, *Differentiable functions defined in closed sets. A problem of Whitney*, Invent. Math., 151 (2003), pp. 329–352.

[4] Y. Brudnyi and P. Shvartsman, *Generalizations of Whitney's extension theorem.*, Internat. Math. Res. Notices, 3 (1994).

[5] ———, *Whitney's extension problem for multivariate $C^{1,\omega}$-functions*, Trans. Amer. Math. Soc., 353 (2001), pp. 2487–2512 (electronic).

[6] C. Fefferman, *A generalized sharp Whitney theorem for jets*, Rev. Mat. Iberoam., 21 (2005), pp. 577–688.

[7] ———, *Interpolation and extrapolation of smooth functions by linear operators*, Rev. Mat. Iberoam., 21 (2005), pp. 313–348.

[8] ———, *A sharp form of Whitney's extension theorem*, Ann. of Math. (2), 161 (2005), pp. 509–577.

[9] ———, *Whitney's extension problem for C^m*, Ann. of Math. (2), 164 (2006), pp. 313–359.

[10] ———, *Fitting a C^m-smooth function to data III*, Ann. of Math. (2), 170 (2009), pp. 427–441.

[11] C. Fefferman, A. Israel, and G. K. Luli, *Finiteness principles for smooth selections*, Geom. Funct. Anal., 26 (2016), pp. 422–477.

[12] ———, *Interpolation of data by smooth non-negative functions*, Rev. Mat. Iberoam., 33 (2016), p. 305—324.

[13] C. Fefferman, F. Jiang, and G. K. Luli, *C^2 interpolation with range restriction*, https://arxiv.org/abs/2107.08272, (2021).

[14] C. Fefferman and B. Klartag, *Fitting a C^m-smooth function to data. I*, Ann. of Math. (2), 169 (2009), pp. 315–346.

[15] ———, *Fitting a C^m-smooth function to data. II*, Rev. Mat. Iberoam., 25 (2009), pp. 49–273.

[16] C. Fefferman and P. Shvartsman, *Sharp finiteness principles for Lipschitz selections*, Geom. Funct. Anal., 28 (2018), pp. 1641–1705.

[17] G. Glaeser, *Étude de quelques algèbres tayloriennes*, J. Analyse Math., 6 (1958), pp. 1–124.

[18] F. Jiang and G. K. Luli, $C^2(\mathbb{R}^2)$ *nonnegative interpolation by bounded-depth operators*, Advances in Math., 375 (2020), p. 107391.

[19] ———, *Nonnegative* $C^2(\mathbb{R}^2)$ *interpolation*, Advances in Math., 375 (2020), p. 107364.

[20] ———, *Algorithms for nonnegative* $C^2(\mathbb{R}^2)$ *interpolation*, Advances in Math., 385 (2021), p. 107756.

[21] F. Jiang, G. K. Luli, and K. O'Neill, *On the shape fields finiteness principle*, Int. Mat. Res. Not., (2021), p. rnab242.

[22] M.-J. Lai and C. Meile, *Scattered data interpolation with nonnegative preservation using bivariate splines and its application*, Computer Aided Geometric Design, 34 (2015), pp. 37–49.

[23] J. Schmidt and W. Hess, *S-convex, monotone, and positive interpolation with rational bicubic splines of* C^2*-continuity*, BIT, 33 (1993), pp. 496–511. cited By 11.

[24] L. L. Schumaker and H. Speleers, *Nonnegativity preserving macro-element interpolation of scattered data*, Computer Aided Geometric Design, 27 (2010), pp. 245–261.

[25] P. Shvartsman, *Traces of functions of two variables, which satisfy Zygmund's condition*, in Research on the Theory of Functions of Several Real Variables (Russian), Yaroslavl State University, 1982, pp. 145–168.

[26] ———, *Lipschitz sections of set-valued mappings and traces of functions from the Zygmund class on an arbitrary compactum*, Dokl. Akad. Nauk SSSR, 276 (1984), pp. 559–562.

[27] ———, *Lipschitz sections of multivalued mappings*, in Studies in the Theory of Functions of Several Real Variables (Russian), Yaroslavl State University, 1986, pp. 121–132.

[28] ———, *Traces of functions of Zygmund class*, Sib. Math. J., 28 (1987), pp. 853–863.

[29] ———, *Lipschitz selections of set-valued mappings and Helly's theorem*, J. Geom. Anal., 12 (2002), pp. 289–324.

[30] ———, *The Whitney extension problem and Lipschitz selections of set-valued mappings in jet-spaces*, Trans. Amer. Math. Soc., 360 (2008), pp. 5529–5550.

[31] ———, *On Lipschitz selections of affine-set valued mappings*, Geom. Funct. Anal., 11 (840-868), p. 2001.

[32] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc., 36 (1934), pp. 63–89.

[33] ———, *Differentiable functions defined in closed sets. I*, Trans. Amer. Math. Soc., 36 (1934), pp. 369–387.

[34] ———, *Functions differentiable on the boundaries of regions*, Ann. of Math. (2), 35 (1934), pp. 482–485.