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A Type of Algebraic Structure Related to Sets of Intervals

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Abstract

F. Wehrung has asked: Given a family $\mathcal{C}$ of subsets of a set Ω, under what conditions will there exist a total ordering on Ω with respect to which every member of $\mathcal{C}$ is convex?¶ We look at the family $\mathcal{P}$ of subsets of Ω generated by $\mathcal{C}$ under certain partial operations which, when Ω is given with a total ordering, preserve convexity; we determine the possible structures of $\mathcal{P}$ under these operations if $\mathcal{C}$, and hence $\mathcal{P}$, is finite, and note a condition on that structure that is necessary and sufficient for there to exist an ordering of Ω of the desired sort. From this we obtain a criterion which works without the finiteness hypothesis on $\mathcal{C}$.¶ Bounds are obtained on the cardinality of the set $\mathcal{P}$ generated under these operations by an n-element set $\mathcal{C}$.¶ We end by noting some other ways of answering Wehrung's question, using results in the literature. The bibliography lists still more related literature.

1 Introduction

If $(\Omega, \leq)$ is a totally ordered set, we shall call a subset A of Ω *convex* if for p, q, $r \in \Omega$ with $p \leq q \leq r$, the conditions $p, r \in A$ imply $q \in A$. (I use the word "interval" in the title, since it concisely suggests "convex subset of a totally ordered set", but in general, I shall write "convex set". The word "interval" will, however, come up in Section 7, in connection with results in the literature.)

This note answers the question:

(1) (F. Wehrung, personal correspondence related to [12]; slightly reworded.) Let Ω be a set (usually finite), and $\mathcal{C}$ a set of subsets of Ω. When does there exist a total order on Ω with respect to which every member of $\mathcal{C}$ is convex?

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In Section 2, we show that given a set $\mathcal{C}$ of subsets of a set Ω, there exists a total ordering $\leq$ on Ω having the desired property for $\mathcal{C}$ if and only if for every finite subset $\mathcal{C}' \subseteq \mathcal{C}$, there exists a total ordering $\leq_{\mathcal{C}'}$ on Ω having that property for $\mathcal{C}'$. Thus, the general problem reduces to the corresponding problem for finite $\mathcal{C}$.

In Section 3 we note some natural partial operations on the set of convex subsets of a totally ordered set Ω, which can be described solely in set-theoretic terms, and we name an arbitrary set $\mathcal{P}$ of subsets of a set Ω which is closed under these partial operations a "patchwork". Thus, a set $\mathcal{C}$ of subsets of Ω has all its members convex under some total ordering $\leq$ on Ω if and only if the patchwork $\mathcal{P}$ that it generates has the same property. Moreover, the patchwork generated by a finite set $\mathcal{C}$ of subsets of Ω is finite; so question (1) for finite sets $\mathcal{C}$ comes down to the same question for finite patchworks $\mathcal{P}$.

In Section 4 we determine the structures of all *finite* patchworks $\mathcal{P}$ of subsets of a set Ω, and note for which of these there exist orderings of Ω making all members of $\mathcal{P}$ convex. In Section 5, we combine this result with that of Section 2 to get a condition on a not necessarily finite patchwork that is necessary and sufficient for the existence of such an ordering.

In Section 6 we examine how many elements the patchwork $\mathcal{P}$ generated by an n-element set $\mathcal{C}$ can have.

After an earlier draft of this note was sent out, it was pointed out to me that there are results in the literature that can be used to answer question (1) in other ways. These are noted in Section 7. So perhaps the main value of this paper is the detailed structure it reveals of the family $\mathcal{P}$ arising from a family $\mathcal{C}$ for which question (1) has a positive answer – and possibly also the corresponding structural information when (1) is not assumed to have a positive answer. (Whether that case has interesting applications I do not know.)

One of the referees has pointed out that the sort of structure I call a patchwork is named a "weakly partitive family" in [3, 4, 9].

We remark that the arguments of Section 4 below that determine the possible structures of a finite patchwork are rather lengthy and intricate; but the reasoning is elementary. In contrast, the argument of Section 2 that reduces the general question to the finite case calls on the Compactness Theorem of model theory, or, alternatively, on ultrafilters; so that proof, though brief, is the one non-elementary piece of reasoning in this note.

2 Reduction to the Case of Finite $\mathcal{C}$

Lemma 1 *Let Ω be a set, and $\mathcal{C}$ a set of subsets of Ω. Then the following conditions are equivalent.*

(i) *There exists a total order on Ω with respect to which every member of $\mathcal{C}$ is convex.*
(ii) *For each finite subset $\mathcal{C}' \subseteq \mathcal{C}$, there exists a total order on Ω with respect to which every member of $\mathcal{C}'$ is convex.*

Proof (i) $\implies$ (ii) is clear, since an ordering that works for $\mathcal{C}$ works for any $\mathcal{C}' \subseteq \mathcal{C}$.

The reader familiar with the Compactness Theorem of model theory [5, Corollary V.5.6] will see that that theorem implies the reverse implication. (One uses a language with a constant for each element of Ω, a unary relation for each member of $\mathcal{C}$, and a binary relation "$\leq$".) I sketch below a variant of this argument that uses ultrafilters [5, §V.5] rather than the Compactness Theorem.

Assume (ii). Let X be the set of all finite subsets $\mathcal{C}' \subseteq \mathcal{C}$, and let us choose for each $\mathcal{C}' \in X$ a total ordering $\leq_{\mathcal{C}'}$ on Ω under which all members of $\mathcal{C}'$ are convex. Since the ordering of X by inclusion is upward directed, the nonempty up-sets of X generate a proper filter $\mathcal{F}$ on X, which can be extended to an ultrafilter $\mathcal{U}$. For every pair of distinct elements $a, b \in \Omega$, either the set $\{\mathcal{C}' \mid a \leq_{\mathcal{C}'} b\}$ or its complement $\{\mathcal{C}' \mid b \leq_{\mathcal{C}'} a\}$ belongs to $\mathcal{U}$; let us write $a \leq b$ in the former case, $b \leq a$ in the latter.

From the fact that each $\leq_{\mathcal{C}'}$ is a total ordering on Ω, it is easily deduced that $\leq$ is a total ordering on Ω. To show that every $A \in \mathcal{C}$ is convex under that ordering, suppose $a \leq b \leq c$, and $a, c \in A$. The definition of $\leq$ and the inequalities $a \leq b \leq c$ show that $\{\mathcal{C}' \in X \mid a \leq_{\mathcal{C}'} b \leq_{\mathcal{C}'} c\} \in \mathcal{U}$, while the definition of $\mathcal{F}$ shows that $\{\mathcal{C}' \in X \mid A \in \mathcal{C}'\} \in \mathcal{F} \subseteq \mathcal{U}$; hence the intersection of these two sets is nonempty. Taking $\mathcal{C}'$ in that intersection, we see that since $A \in \mathcal{C}'$ is convex under $\leq_{\mathcal{C}'}$, we indeed have $b \in A$, as required $\qquad \square$

3 Patchworks

The concept we will define next is motivated by the following easily verified result.

Lemma 2 *If $(\Omega, \leq)$ is a totally ordered set, and A, B are convex subsets of Ω which have nonempty intersection, and neither of which contains the other, then $A \cap B$, $A \cup B$, and $A \setminus B$ are also convex.*

Also, $\emptyset$ and Ω are convex. $\qquad \square$

(Each of the conclusions of the first sentence of Lemma 2 needs less than the full set of assumptions on A and B. But the cases where the unnecessary assumptions fail are trivial, so we lose nothing in using this formulation.)

Definition 3 Let Ω be a set.

We will say that subsets A, $B \subseteq \Omega$ *overlap* if they have nonempty intersection, and neither contains the other.

For any set Ω, a *patchwork* of subsets of Ω (which may be shortened to "a patchwork on Ω", or simply "a patchwork" when the context makes Ω clear) will mean a set $\mathcal{P}$ of subsets of Ω such that for every pair of *overlapping* sets A, $B \in \mathcal{P}$, the set $\mathcal{P}$ also contains $A \cap B$, $A \cup B$, and $A \setminus B$, and such that $\emptyset$ and Ω also belong to $\mathcal{P}$.

If $\mathcal{P}$ is a patchwork on Ω, members of $\mathcal{P}$ will be called $\mathcal{P}$-*sets*.

We note

Lemma 4 *The patchwork $\mathcal{P}$ on a set Ω generated by a finite set $\mathcal{C}$ of subsets of Ω (i.e., the closure of $\mathcal{C}$ under the partial operations of Definition 3) is finite.*

Proof $\mathcal{P}$ is contained in the Boolean ring of subsets of Ω generated by $\mathcal{C}$, and finitely generated Boolean rings are finite. $\qquad \square$

4 The Case of Finite $\mathcal{P}$.

Convention 5 *Throughout this section, Ω will be a fixed set, and $\mathcal{P}$ a finite patchwork on Ω.*

A key to analyzing the structure of $\mathcal{P}$ is to look at those of its members to which its partial operations *cannot* be applied:

Definition 6 A $\mathcal{P}$-set A will be called *autonomous* if it is nonempty, and does not overlap any member of $\mathcal{P}$ (cf. Definition 3). We will generally shorten "autonomous $\mathcal{P}$-set" to "autonomous set".

If $A \in \mathcal{P}$ is autonomous, then the set of maximal autonomous *proper* subsets of A will be called the *cohort* under A. Elements of A that are not in the union of the cohort under A (if any) will be called *non-cohort elements* of A.

Note that

Lemma 7 *Any two autonomous members of $\mathcal{P}$ are either disjoint, or one contains the other.*

Proof By the definition of "autonomous" this is, in fact, true of any two $\mathcal{P}$-sets at least *one* of which is autonomous. □

By Lemma 7 the autonomous members of $\mathcal{P}$ form a tree under inclusion, branching downward from the top element, Ω. It follows from the finiteness of $\mathcal{P}$ that for every proper subset A of Ω, there is a *least* autonomous $\mathcal{P}$-set B properly containing A.

Note that every *minimal* autonomous set A consists entirely of non-cohort elements. Non-minimal autonomous $\mathcal{P}$-sets necessarily have nonempty cohorts under them; we shall see that they may or may not also have non-cohort elements. Each element of Ω is a non-cohort element of a unique autonomous set, namely, the least autonomous set containing it.

Not every union of autonomous sets need be a $\mathcal{P}$-set, but the converse is true:

Lemma 8 *Let A be a $\mathcal{P}$-set. Then*

(i) *A is a union of autonomous sets; in fact it is the disjoint union of the maximal elements of the set of autonomous sets which it contains.*

   Moreover, if $A \neq \Omega$, then letting B be the least autonomous set properly containing A, we have

(ii) *The maximal autonomous sets contained in A (cf. (i)) all belong to the cohort under B.*

(iii) *If A' is a $\mathcal{P}$-set which overlaps A, then the least autonomous set properly containing A' is also B.*

(iv) *If A is not autonomous, then it can be written as a disjoint union of two nonempty $\mathcal{P}$-sets, each of which is the union of a subset of the cohort under B.*

Proof (i): If A is autonomous this is trivial: it is the union of the 1-element family $\{A\}$. If A is not autonomous, assume inductively that all $\mathcal{P}$-sets properly contained in A are unions of autonomous sets. Since A is not autonomous, it overlaps some $\mathcal{P}$-set A'. Then $A \cap A'$ and $A \setminus A'$ are $\mathcal{P}$-sets properly contained in A, so by the inductive hypothesis, they are unions of autonomous sets, hence so is their union, A. That A is the union of the *maximal* autonomous sets that it contains follows, since $\mathcal{P}$ is finite; and by Lemma 7, these sets are disjoint.

(ii): Let C be a maximal autonomous subset of A. Then C is a proper autonomous subset of B; we need to show it is maximal among such subsets. If not, suppose D were an autonomous set with $C \subset D \subset B$. By the maximality of C in A, we have $D \nsubseteq A$; by minimality of B among autonomous sets properly containing A, we have $A \nsubseteq D$; so, as $A \cap D \supseteq C \neq \emptyset$, the sets A and D overlap, contradicting the assumption that D is autonomous. So no such D exists, as desired.

(iii): We shall show that B contains A'. This implies that B contains the least autonomous set containing A'. Moreover, reversing the roles of A and A', it follows that the least autonomous set containing A' contains B; so those two autonomous sets must be the same, as desired.

To see that B contains A', note that $A \cup A'$ is a $\mathcal{P}$-set which is not disjoint from B, so by the autonomy of B, it must either be contained in B or properly contain B. If the latter were true, then A' would have elements outside B, and also contain the nonempty set $B \setminus A$, so it would overlap B, which is impossible since B is autonomous. So $A \cup A'$ must be contained in B; in particular, A' is contained in B, as claimed.

(iv): Since A is nonautonomous, we can choose a $\mathcal{P}$-set C which it overlaps. Thus, by (iii), the least autonomous set containing C is B, so by (i) and (ii), A and C are both unions of members of the cohort under B. Writing A as the union of the disjoint nonempty $\mathcal{P}$-sets $A \cap C$, and $A \setminus C$, we have the desired decomposition. $\square$

The above results suggest the question: Given an autonomous set with nonempty cohort under it, *which* unions of subsets of this cohort can be $\mathcal{P}$-sets? We will use the following concept.

Definition 9 Two $\mathcal{P}$-sets A and B will be called *adjacent* if they are disjoint, and their union is again a $\mathcal{P}$-set.

The *adjacency graph* of a set of $\mathcal{P}$-sets will mean the graph having the members of this set as vertices, and having an edge between a pair of such vertices if and only if they are adjacent as $\mathcal{P}$-sets.

The word "adjacent" is motivated by the case where $\mathcal{P}$ consists of sets convex under a total ordering $\leq$ on Ω, but our definition is not limited to that case; and even in that case, two sets $A, B \in \mathcal{P}$ that are adjacent as convex subsets of Ω need not be adjacent under our present definition, if our patchwork $\mathcal{P}$ doesn't happen to include $A \cup B$.

If we had given the above definition earlier, we could, in Lemma 8(iii), have changed "a $\mathcal{P}$-set which overlaps A" to "a nonempty $\mathcal{P}$-set which overlaps A or is adjacent to A", since the proof uses only the fact that the union $A \cup A'$ is a $\mathcal{P}$-set distinct from both A and A'. Thus we have

Lemma 10 *If A and A' are adjacent nonempty $\mathcal{P}$-sets, then the least autonomous set properly containing A and the least autonomous set properly containing A' are the same.*
In particular, any two adjacent autonomous $\mathcal{P}$-sets belong to the same cohort. $\square$

We can now prove

Lemma 11 *A subset A of Ω is a $\mathcal{P}$-set if and only if it can be written as the union of a family of autonomous sets whose adjacency graph is connected.*

Proof The statement is trivial if A is an autonomous set or $\emptyset$, so assume that neither of these is the case.

To get the "if" direction, suppose A is the union of a family of more than one autonomous sets and has connected adjacency graph. The desired implication is true if the graph in question has exactly 2 vertices, by the definition of adjacency; so assume it has ≥ 3 vertices. Let us also assume inductively that the corresponding implication is true for every set A' that can be written as the union of a family of fewer autonomous sets than we are using for A.

Let us now choose a vertex B in the graph for A such that the subgraph obtained by removing B, i.e., the adjacency graph of the family with union $A \setminus B$, is still connected. (Such a vertex exists in any finite connected graph G with at least two vertices [1, Theorem 3.2.10]. Indeed, choosing any vertex x, a vertex y at maximum distance from x has the desired property, since for any other vertex z, a shortest path from x to z cannot pass through y, whence in $G \setminus \{y\}$, every vertex z lies in the connected component of x.) Thus, by our inductive assumption, $A \setminus B$ is a $\mathcal{P}$-set. Let C be a vertex of the graph for A which is adjacent to B. Since the graph for A has ≥ 3 vertices, it has vertices other than B and C. Hence neither of the $\mathcal{P}$-sets $A \setminus B$ or $B \cup C$ contains the other, moreover, they overlap in C; so their union, A, is a $\mathcal{P}$-set, as desired.

Conversely, suppose A is a $\mathcal{P}$-set which is neither autonomous nor empty, and B the least autonomous set containing A. By Lemma 8(i) and (ii), A is the union of some subset of the cohort under B; we wish to show that the adjacency graph of that subset is connected. Again, this is clear if that graph has ≤ 2 vertices, so suppose it has at least 3, and assume by induction that every proper subset of that set whose union is a $\mathcal{P}$-set does have connected adjacency graph.

Since A is non-autonomous, Lemma 8(iv) gives us a decomposition of our expression for A into two disjoint subfamilies, whose unions are $\mathcal{P}$-sets A_1 and A_2. By our inductive assumption, the graphs corresponding to these $\mathcal{P}$-sets are connected, so to show that the graph corresponding to A is connected, it will suffice to find a connected subgraph thereof whose vertex-set meets both those subgraphs. Again invoking our inductive assumption, we will have this if we can find a proper $\mathcal{P}$-subset $A_3 \subset A$ which is a union of members of the cohort under B, and which meets both A_1 and A_2.

Since the graph associated with A had at least 3 vertices, the subgraph associated with one of A_1, A_2 must have more than one vertex; i.e., one of those two $\mathcal{P}$-subsets, say A_1, must be non-autonomous. Let C be a $\mathcal{P}$-set which overlaps A_1. By Lemma 8(iii), C is also a union of members of the cohort under B. If C also has nonempty intersection with A_2, then we see that $A \cap C$ will be a $\mathcal{P}$-subset A_3 of the sort we need. On the other hand, if $C \cap A_2 = \emptyset$, then the fact that $C \not\subseteq A_1$ implies $C \not\subseteq A$, so $A \setminus C$ is a $\mathcal{P}$-set, and will serve as our A_3. $\qquad\square$

(Remark: In the above result, if A is not autonomous, then it will have a *unique* expression of the indicated sort. If it *is* autonomous, then as noted, its expression as the union of the singleton $\{A\}$ is always of the indicated sort; in this case A may or may not have another such expression. Namely, we will see later that an autonomous set A may or may not be the union of the cohort under it, and that if it is, the adjacency graph for that decomposition may or may not be connected.)

To lead up to our criterion for the existence of an ordering of Ω under which all $\mathcal{P}$-sets are convex, we look next at two conditions which (it is not hard to see) are incompatible with the existence of such an ordering.

Lemma 12 *If an autonomous set A is adjacent to more than two other autonomous sets, then every two autonomous sets adjacent to A are adjacent to one another.*

In fact, when this is true, then in the adjacency graph of the cohort containing A, the connected component containing A has an edge between every two vertices.

Proof Let B be the least autonomous set properly containing A. In view of Lemma 10, all the autonomous sets referred to in the lemma are members of the cohort under B.

To get the first assertion of the lemma, we first consider any two autonomous sets $C \neq C'$ adjacent to A. By the adjacency assumption, $A \cup C$ and $A \cup C'$ are $\mathcal{P}$-sets, and they clearly overlap, so their union $A \cup C \cup C'$ is a $\mathcal{P}$-set. Now by our hypothesis there is at least one other autonomous set C'' adjacent to A, so $A \cup C''$ is a $\mathcal{P}$-set which overlaps $A \cup C \cup C'$; hence $(A \cup C \cup C') \backslash (A \cup C'') = C \cup C'$ is a $\mathcal{P}$-set, i.e., C and C' are adjacent, as claimed.

To get the second assertion, let us first show that in the situation of the above paragraph, if C is adjacent to yet another member D of the cohort under B, then A is also adjacent to D. Indeed, since C is adjacent to the three sets A, C' and D, the first assertion of the lemma shows that these are adjacent to one another, in particular, A and D are adjacent. By induction, all members of the connected component in question are adjacent to A, so by yet another application of the first assertion of the lemma, they are all adjacent to one another. $\square$

Lemma 13 *Suppose $A_1, \ldots, A_n$ $(n \geq 3)$ are distinct autonomous sets forming a "cycle of adjacency", i.e., such that A_i is adjacent to A_{i+1} $(1 \leq i < n)$, and A_n to A_1. Then, again, in the adjacency graph of the cohort containing the A_i, the connected component containing these elements has an edge between every two vertices.*

Proof First suppose $n \geq 4$. Then $A_1 \cup A_2 \cup A_3$ and $A_3 \cup \cdots \cup A_n \cup A_1$ will be overlapping $\mathcal{P}$-sets, hence their intersection, $A_1 \cup A_3$ is a $\mathcal{P}$-set; i.e., A_1 and A_3 are adjacent. Hence A_3 is adjacent to at least the three sets A_1, A_2 and A_4, so the preceding lemma is applicable.

On the other hand, suppose $n = 3$. If the connected component in question is simply $\{A_1, A_2, A_3\}$, we are done. If not, then by connectedness, some other member C of that connected component must be adjacent to one of the A_i. This again makes that A_i adjacent to three distinct members of the component, so again, the preceding lemma is applicable. $\square$

We can now prove

Theorem 14 *Let $\mathcal{P}$ be a finite patchwork, and $B \in \mathcal{P}$ an autonomous set. Then exactly one of the following statements is true.*

(i) *The cohort under B has at least 3 members, every two members of that cohort are adjacent, and the union of the cohort is all of B.*

(ii) *The cohort under B has at least 2 members, the adjacency graph of that cohort is a path (i.e., consists of elements $A_1, \ldots, A_n$ for some $n \geq 2$, with A_i and A_j adjacent if and only if $|i - j| = 1$), and again, the union of the cohort is all of B.*

(iii)   *No two members of the cohort under B are adjacent. (In this case, B may or may not contain non-cohort elements. If it does, the cohort under B may be empty or nonempty.)*

Proof Consider the adjacency graph of the cohort under B. If this has any edges, then the members of that cohort comprising a nontrivial connected component of that graph will have as union a $\mathcal{P}$-subset $A \subseteq B$ by Lemma 11. This A must be autonomous, since if it overlapped another $\mathcal{P}$-set A', then by Lemma 8(iii) $A \cup A'$ would still be a $\mathcal{P}$-subset of B, hence its adjacency graph would again correspond to a subgraph of the cohort of B, which, by Lemma 11, would be a larger connected subgraph of that graph, a contradiction.

So such an A will be an autonomous $\mathcal{P}$-subset of B which properly contains members of the cohort under B; hence A must equal B. This shows that if the adjacency graph of the cohort under B has any edges, then it is connected, and B is the union of that cohort, i.e., has no non-cohort elements.

By Lemmas 12 and 13, if the above adjacency graph has either any "branching" or any cycles, it must have an edge between every pair of points, and since it will also have at least three vertices, it will fall under case (i). On the other hand, a connected graph with at least one edge which has neither branching nor cycles is a path, so if that is so, we are in case (ii). Finally, if the adjacency graph of B has no edges, we are in case (iii).

(As noted earlier, a minimal autonomous set B has empty cohort under it, and so necessarily has non-cohort elements. For an example of an autonomous set having both a nonempty cohort under it and non-cohort elements, the reader can examine the case where Ω is a totally ordered set, and $\mathcal{P}$ is the patchwork generated by an interval B in Ω and one or more pairwise disjoint subintervals of B whose union is not all of B.)   □

We can now answer question (1) for finite patchworks.

Theorem 15 *The following three conditions on a finite patchwork $\mathcal{P}$ are equivalent:*

(a) Ω *admits a total ordering $\leq$ under which all $\mathcal{P}$-sets are convex.*
(b)   *No three nonempty $\mathcal{P}$-sets are pairwise adjacent.*
(c)   *Every autonomous $\mathcal{P}$-set $B \subseteq \Omega$ falls under case* (ii) *or* (iii) *of Theorem 14.*

Proof (a) $\implies$ (b) is intuitively clear. To supply some details: if a convex set $A \subseteq \Omega$ is adjacent to each of a pair of disjoint convex sets A' and A'', then one of A' and A'' must lie above it and the other below it under $\leq$, otherwise A' and A'' could not be disjoint. Since A' and A'' have the nonempty set A between them, their union is not convex, so they are not adjacent.

(b) $\implies$ (c) is immediate: If some B fell under case (i) of Theorem 14, the cohort under it would include three pairwise adjacent sets.

To prove (c) $\implies$ (a) we shall, assuming (c), construct recursively an ordering $\leq$ as in (a).

Given an autonomous set $B \subseteq \Omega$, assume recursively that for each A in the cohort under B, we have constructed a total ordering on A under which all $\mathcal{P}$-sets contained in A are convex. (If B is minimal, the cohort under it is empty, and this assumption is vacuous.)

If B falls under case (ii) of Theorem 14, then writing the members of the cohort under B as $A_1, \ldots, A_n$, as in that theorem, we order their union B so that $A_1, \ldots, A_n$ form *successive* intervals of B (either putting A_i below A_{i+1} for all $i < n$, or putting A_i above

A_{i+1} for all such i), with each A_i ordered internally by the order previously constructed for it.

On the other hand, if B falls under case (iii), we can order it by arranging the members of the cohort under it (if any) in any order, simply making each a convex subset of B, again with the internal ordering previously constructed, and likewise put the non-cohort elements of B (if any) in any order above, below or between those intervals.

We see that for every $\mathcal{P}$-set A, if B is the least autonomous $\mathcal{P}$-set containing it, then A becomes convex under the ordering so constructed on B, and remains so as we extend this ordering to larger autonomous sets. Hence when our construction reaches the top set, Ω, we have established (a). □

Going back to Theorem 14, let us turn that result upside down, and describe, somewhat informally, how to construct all finite patchworks:

Start with any partially ordered set T having the form of a finite downward-branching tree, with greatest element denoted Ω. Choose any graph structure on the elements of T such that edges, if any, occur only among elements of T that lie immediately below a common element, and such that for each element, the resulting graph structure on the set of all elements immediately below it is either a complete graph, a path, or edgeless. Now assign to each element of T such that the graph structure we have given to the set of elements immediately below it is edgeless a set that is to be its set of non-cohort elements, assigning to distinct elements of T disjoint sets, and requiring the set so assigned to every element of T with ≤ 1 elements of T immediately below it to be nonempty. Let each element of T become the name of the union of the sets of non-cohort elements assigned to it *and* to all elements of T anywhere below it, and call the sets so named *autonomous sets*. (Apologies for this informal step of turning abstract elements into names for sets!) In particular, Ω is the name assigned to the set of *all* elements that have been introduced as non-cohort elements at any step. Finally, let the members of $\mathcal{P}$ be the unions of those families of autonomous sets which, under our graph structure on T, form connected subgraphs.

5 The Case of not Necessarily Finite $\mathcal{P}$

Returning to Theorem 15, we can combine it with Lemma 1 and deduce:

Theorem 16 *Let Ω be a set, and $\mathcal{P}$ a (not necessarily finite) patchwork of subsets of Ω. Then (as in Theorem 15), the following conditions are equivalent.*

(a) Ω *admits a total ordering $\leq$ under which all $\mathcal{P}$-sets are convex.*
(b) *No three nonempty $\mathcal{P}$-sets are pairwise adjacent.*

Proof (a) $\implies$ (b) is clear, as in the proof of Theorem 15.

Conversely, assume (b) holds. Then the same condition holds for all finite sub-patchworks $\mathcal{P}' \subseteq \mathcal{P}$, hence by Theorem 15, for each such $\mathcal{P}'$ there exists an ordering $\leq_{\mathcal{P}'}$ making all elements of $\mathcal{P}'$ convex. Since every finite subset $\mathcal{C}' \subseteq \mathcal{P}$ generates a finite sub-patchwork $\mathcal{P}'$, an ordering $\leq_{\mathcal{P}'}$ which makes all members of $\mathcal{P}'$ convex will do the same for $\mathcal{C}'$; so we can now apply Lemma 1 (with $\mathcal{P}$ in the role of $\mathcal{C}$) to get an ordering $\leq$ as in (a), completing the proof. □

We could not bring condition (c) of Theorem 15 into Theorem 16, because as one goes from one finite patchwork $\mathcal{P}_0$ on Ω to a larger one, sets that were autonomous can cease to be so, and this may leave us with no proper autonomous sets in the infinite patchwork $\mathcal{P}$. For two contrasting examples, let Ω be the real line, let $\mathcal{P}_1$ consist of $\emptyset$, Ω, and all the half-open intervals $[a, b)$ $(a < b \in \mathbb{R})$, and let $\mathcal{P}_2$ consist of all finite unions of members of $\mathcal{P}_1$. Both these families are patchworks, and it is easy to see that neither has any autonomous elements other than Ω. The sets in $\mathcal{P}_1$ are convex under the standard ordering on Ω, but there can be no ordering under which the sets in $\mathcal{P}_2$ are convex, since condition (b) fails: *every* two disjoint nonempty members of $\mathcal{P}_2$ are adjacent in $\mathcal{P}_2$, so any three such sets show the failure of (b). So despite having the same (trivial) family of autonomous sets, one of these patchworks does, and the other does not, satisfy the conditions of the above theorem.

It would be interesting to know whether the kind of structures we have called patchworks are useful in other contexts than the study of convex subsets of ordered sets.

We remark that in defining the concept of a patchwork, rather than making it a family of subsets of a set Ω, we could, with slightly greater formal generality, have made it a family of elements of a general Boolean ring. The "set of subsets of a set" version seemed most natural for our purposes, but the more general version might be useful in some other contexts.

6 The Cardinality of a Patchwork Generated by n Elements

We noted in Lemma 4 that every finitely generated patchwork is finite. Let us get some sharper bounds.

The proof of the next result uses an observation that we have not yet stated explicitly: any patchwork $\mathcal{P}$ is closed under pairwise intersections, and hence under all finite intersections. Indeed, by definition $\mathcal{P}$ is closed under intersections of overlapping elements, while if two elements are comparable, their intersection is one of them, while if they are disjoint, their intersection is $\emptyset \in \mathcal{P}$.

Proposition 17 *If n is a nonnegative integer, Ω a set, and $\mathcal{P}$ a patchwork generated by n subsets of Ω, then $\mathcal{P}$ has at most $2^{2^n-1} + 1$ elements. Moreover, for all $n \neq 2$, there exist examples achieving this bound.*

Proof Let $\mathcal{B}(n)$ be the free Boolean ring on n generators [8, Theorem 26, p.260, and Corollary 1]. This can be described as the Boolean ring of all subsets of the set 2^n of subsets of $\{1, \ldots, n\}$; its free generators are the sets $X_1, \ldots, X_n$, where X_i is the set of those subsets containing i. Thus $\mathcal{B}(n)$ has cardinality 2^{2^n}. The *nonunital* subring $\mathcal{B}(n)^{\mathrm{o}}$ of $\mathcal{B}(n)$ generated by the X_i consists of those members of $\mathcal{B}(n)$ which, like $X_1, \ldots, X_n$, do not have as a member the empty subset of $\{1, \ldots, n\}$; so it has cardinality 2^{2^n-1}.

(Some readers might find it more comfortable to think of what I am calling 2^n, not as the set of subsets of $\{1, \ldots, n\}$, but as the set of length-n strings of 0's and 1's; so that elements of $\mathcal{B}(n)$ become, not sets of sets, but sets of words. What I have said above, and will do below, is easily translated into that viewpoint.)

If $\mathcal{P}$ is the patchwork on a set Ω generated by n subsets $A_1, \ldots, A_n$, let us map the free Boolean ring $\mathcal{B}(n)$ homomorphically into the Boolean ring of all subsets of Ω by sending each free generator X_i to A_i. It is not hard to see that the partial binary operations

on subsets of Ω used in Definition 3 carry the image of $\mathcal{B}(n)^{\circ}$ into itself, so the elements of $\mathcal{P}$ obtained from $A_1, \ldots, A_n$ using those operations will lie in that image, as will the element $\emptyset$. The one member of $\mathcal{P}$ that may not be contained in that image is Ω, so counting this, we see that $\mathcal{P}$ must have cardinality $\leq 2^{2^n-1} + 1$.

To get an example where this value is achieved, let us take Ω to be 2^n, meaning, as above, the set of all subsets of $\{1, \ldots, n\}$, and examine the patchwork $\mathcal{P}$ on Ω generated by the X_i.

Since every pair X_i, X_j $(i \neq j)$ overlaps, $\mathcal{P}$ will contain all the sets $X_i \setminus X_j$ $(i \neq j)$. I now claim that for any $a \in 2^n$ which contains some but not all of $1, \ldots, n$, the singleton $\{a\}$ is a $\mathcal{P}$-set. Indeed, it is the intersection of those sets $X_i \setminus X_j$ such that $i \in a$ but $j \notin a$. The singleton whose unique member is the whole set $\{1, \ldots, n\}$ is also in $\mathcal{P}$, being the intersection of the X_i themselves. On the other hand, $\emptyset \in \Omega$ is not contained in any of the X_i, so the singleton $\{\emptyset\}$ is not a $\mathcal{P}$-set.

This gives us $2^n - 1$ singleton autonomous sets; let us now look at adjacency relationships among them. I claim that if two elements $a \neq b \in \Omega$ are nonempty, and differ only in the presence or absence of a single one of $1, \ldots, n$, then $\{a, b\} \in \mathcal{P}$, i.e., $\{a\}$ and $\{b\}$ are adjacent. Indeed, of the $n - 1$ members of $\{1, \ldots, n\}$ at which a and b agree, they must agree in *containing* at least one, otherwise one of a or b would be empty. If they also agree in *not* containing at least one, then, imitating the trick of the preceding paragraph, we can obtain $\{a, b\}$ as an appropriate intersection of sets $X_i \setminus X_j$, while if they contain all $n - 1$ elements at which they agree, then, again using the idea of the preceding paragraph, we find that $\{a, b\}$ will be an intersection of sets X_i. Using these adjacency relations, we can get a chain of adjacency from any nonempty a to the element $\{1, \ldots, n\}$; so the adjacency graph of these $2^n - 1$ singleton autonomous sets is connected. Moreover, the n elements of cardinality $n - 1$ are all adjacent to $\{1, \ldots, n\}$, so if $n \geq 3$, Lemma 12 tells us that every two of these singletons are adjacent; so the union of every subfamily of these singletons is a $\mathcal{P}$-set, giving 2^{2^n-1} elements. Bringing in Ω itself (which is not such a union, because it contains not only the 2^{n-1} nonempty subsets of $\{1, \ldots, n\}$ but also the empty subset), we have the asserted $2^{2^n-1} + 1$ elements.

We assumed in the next-to-last sentence that $n \geq 3$. Looking at lower values, if $n = 1$, so that $\{1, \ldots, n\}$ has a unique nonempty subset $a = \{1\}$, we have $2^{2^n-1} + 1 = 3$, and $\mathcal{P}$ indeed has 3 elements, $\emptyset \subset \{a\} \subset \{\{\emptyset\}, a\} = \Omega$. If $n = 0$, so that $\Omega = \{\{\emptyset\}\}$, the empty set $\mathcal{C}$ of subsets of Ω generates the patchwork $\{\emptyset, \Omega\}$, which has cardinality $2 = 2^{2^n-1} + 1$.

The case $n = 2$, on the other hand will fall under the next result. (See discussion after the proof thereof.) $\qquad\square$

Proposition 18 *If n is a nonnegative integer, Ω a set, and $\mathcal{P}$ a patchwork generated by n subsets of Ω whose members are all convex under some ordering $\leq$ on Ω, then $\mathcal{P}$ has at most $\binom{2n}{2} + 2 = 2n^2 - n + 2$ elements; and for all n there exist examples achieving this bound.*

Proof To prove the stated bound, suppose $(\Omega, \leq)$ is a totally ordered set, and let us define a "cut" χ in Ω to be a pair (χ^ℓ, χ^u) (the superscripts standing for "lower" and "upper") such that Ω is the disjoint union of χ^ℓ and χ^u, and every element of χ^ℓ is less than every element of χ^u. (So such a cut is equivalent to an isotone map from Ω to $\{0, 1\}$. Note that χ^ℓ or χ^u may be empty.) Let us call a cut χ "higher than" a cut χ' if $\chi^\ell \supset \chi'^\ell$. Clearly, the cuts are totally ordered under this relation.

Observe that every nonempty convex subset A of Ω is determined by two cuts, $\chi_{A\uparrow}$ and $\chi_{A\downarrow}$, the former characterized by the property that $\chi_{A\uparrow}^u$ consists of the elements of Ω that are greater than all elements of A (equivalently, that $\chi_{A\uparrow}^\ell$ consists of the elements that are less than or equal to at least one element of A), the latter by the property that $\chi_{A\downarrow}^\ell$ consists of the elements that are less than all elements of A (equivalently, that $\chi_{A\downarrow}^u$ consists of the elements greater than or equal to at least one element of A). Thus, $A = \chi_{A\uparrow}^\ell \cap \chi_{A\downarrow}^u$.

Given overlapping convex sets A and B, it is not hard to see that for each of $A \cup B$, $A \cap B$, $A \setminus B$ and $B \setminus A$, the two cuts determining the set in question are taken from among the four cuts $\chi_{A\downarrow}$, $\chi_{A\uparrow}$, $\chi_{B\downarrow}$, and $\chi_{B\uparrow}$. (Which pair of cuts correspond to each of these depends on whether A extends upward from $A \cap B$ and B downward therefrom, or vice versa. For instance, in the former case, $A \setminus B$ is determined by the two cuts $\chi_{A\uparrow}$ and $\chi_{B\uparrow}$; i.e., $A \setminus B = \chi_{A\uparrow}^\ell \cap \chi_{B\uparrow}^u$.)

Hence if we start with a set $\mathcal{C}$ of n convex sets, those of them that are nonempty determine at most $2n$ distinct cuts, and applying our four partial binary operations recursively, we get a set of nonempty convex sets each of which is determined by one of the $\leq \binom{2n}{2}$ unordered pairs of distinct cuts from this set. Bringing in the sets Ω and $\emptyset$ (which may or may not have belonged to $\mathcal{C}$, or, in the case of Ω, arisen via our partial operations), we see that the patchwork generated by $\mathcal{C}$ can have at most $\binom{2n}{2} + 2$ elements.

Let us now show that for every n, this upper bound is achieved. For integers $i \leq j$, let us here write $[i, j]$ for $\{k \in \mathbb{Z} \mid i \leq k \leq j\}$. Let Ω be the set $[-n, n]$, under the usual ordering of the integers, and let $\mathcal{C}$ consist of the n convex subsets $A_i = [-n + i, i - 1]$ $(i = 1, \ldots, n)$, each of which has n elements. We shall show below that the closure $\mathcal{P}$ of $\mathcal{C}$ under the partial *binary* operations in the definition of a patchwork consists of all the nonempty convex subsets of $[-n + 1, n - 1]$. Note that the latter $2n-1$-element ordered set admits $2n$ cuts (one "below" the least element, $2n - 2$ between successive elements, and one "above" the greatest element), hence it has $\binom{2n}{2}$ nonempty convex subsets. Once we know that our partial binary operations give us all these sets, then bringing in Ω (which, since it contains $-n$ and n, is not in that closure) and $\emptyset$, it will follow that the patchwork generated by $\mathcal{C}$ indeed has $\binom{2n}{2} + 2$ elements.

To show that we get all convex sets $[i, j]$ with $-n + 1 \leq i \leq j \leq n - 1$, it will suffice to show that we get those of the forms $[i, n - 1]$ and $[-n + 1, j]$, since $[i, j] = [i, n - 1] \cap [-n + 1, j]$. To get $[i, n - 1]$, note that if $i < 0$, that set is the union of the overlapping sets $[i, i+n-1] = A_{i+n}$ and $[0, n-1] = A_n$; if $i = 0$, it is $[0, n-1] = A_n$, and if $i > 0$, it is a difference of overlapping sets, $[0, n - 1] \setminus [-n + i, i - 1] = A_n \setminus A_i$. The symmetric argument shows that we can get $[-n + 1, j]$, completing the proof. $\square$

Let us observe that for $n \leq 2$, the members of *every* family $\mathcal{C}$ of n subsets of a set Ω are convex under some ordering of Ω. This is immediate for $n = 0$ and $n = 1$. To see the case $n = 2$, let $\mathcal{C} = \{A_1, A_2\}$. If A_1 and A_2 are disjoint or one contains the other, it is again easy to see how to get such an ordering. If they overlap, then by choosing any ordering in which all elements of $A_1 \setminus A_2$ lie above all elements of $A_1 \cap A_2$, and these in turn lie above all elements of $A_2 \setminus A_1$, while every element not in $A_1 \cup A_2$ lies above or below all elements of $A_1 \cup A_2$, we again get the desired convexity conditions.

For $n = 0, 1, 2$, the upper bounds given by Proposition 17 are 2, 3, 9, while those given by Proposition 18 are 2, 3, 8, so these bounds differ first at $n = 2$. By the orderability result of the preceding paragraph for $n = 2$, the bound given by Proposition 18 holds for that case even in the context of Proposition 17.

A consequence of Proposition 18 is that given a set $\mathcal{C}$ of n subsets of a finite set Ω, one can determine in polynomial time whether there exists an ordering of Ω making all elements of $\mathcal{C}$ convex. The idea is that starting with a list of the n sets in $\mathcal{C}$, together with the sets Ω and $\emptyset$, one searches the list for overlapping pairs A, B, and when one finds such a pair, adds to the list their union, their intersection, and their relative complements, and repeats this process recursively. Within polynomial time, one will know whether or not this recursive process terminates before the cardinality of the list goes above $\binom{2n}{2} + 2$ (i.e., whether we get a family closed under those partial operations without getting above that cardinality). If it doesn't so terminate, then by Proposition 18 there can be no ordering of the desired sort. If it does, then that recursive construction has given us an enumeration of $\mathcal{P}$, and we can work out its adjacency structure and apply Theorem 15 (a) $\Longleftrightarrow$ (b) to determine whether there exists such an ordering. The time needed will be polynomial in n and the number of elements in Ω. (If, rather, Ω is infinite, then whether one can do the same in a time that is polynomial in n will depend on what sort of descriptions we have of Ω and the subsets of Ω that form $\mathcal{C}$. If we assume the operations of the Boolean algebra of subsets of Ω "given", then we have polynomial time in n.)

7 Other Solutions to Question (1), Based on Results in the Literature

Given a set $\mathcal{C}$ of subsets of a set Ω, the graph which has the members of $\mathcal{C}$ as vertices, and an edge between $A, B \in \mathcal{C}$ if and only if $A \cap B \neq \emptyset$, is called the *intersection graph* of $\mathcal{C}$. There has been considerable study of the intersection graphs of finite sets of intervals of the real line. Graphs that can be so represented are called *interval graphs*, and several characterizations of such graphs have been established (see [13], and for some details, [1, Theorem 9.4.4], [7, Chapter 3] and [10]).

The literature varies, inter alia, as to whether intervals are assumed to be closed or open [1, Remark 9.4.2, point 2], but it is not hard to show that a finite graph is an interval graph in either sense if and only if it satisfies the formally weaker condition that for *some* totally ordered set $(\Omega, \leq)$, the graph can be represented as the intersection graph of a set of convex subsets of Ω.

Criteria for a finite graph to be an interval graph do not, themselves, answer the finite-$\mathcal{C}$ case of question (1): If the intersection graph of a set $\mathcal{C}$ of sets is an interval graph, this only tells us that the graph is *isomorphic* to the intersection graph of some set of convex subsets of some totally ordered set. Observe, however, that the complete graph with three vertices is the intersection graph of the family of three sets $\{1\} \subset \{1, 2\} \subset \{1, 2, 3\}$, which are convex under the standard ordering of the integers, but is also the intersection graph of $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, which are not simultaneously convex under any ordering of $\{1, 2, 3\}$.

However, Dave Witte Morris (personal communication) has pointed out that one can get around this difficulty by bringing in the singleton subsets of Ω alongside the elements of $\mathcal{C}$ (assuming for the moment that Ω is finite). We develop the result below. Here we take an *interval* to mean a set $[s, t] = \{x \in \mathbb{R} \mid s \leq x \leq t\}$ where $s < t \in \mathbb{R}$. (Because intervals are by definition nonempty, we shall, in defining the set $\mathcal{C}^+$ below, not only append to $\mathcal{C}$ all singletons, but also remove $\emptyset$ if it belonged to $\mathcal{C}$.)

Proposition 19 (D. W. Morris) *Let Ω be a finite set, $\mathcal{C}$ a set of subsets of Ω, and $\mathcal{C}^+$ the set consisting of all members of $\mathcal{C}$ other than $\emptyset$, and also all singletons $\{a\}$ $(a \in \Omega)$. Then the following two conditions are equivalent:*

(i) *There exists a total ordering $\leq$ of Ω under which all members of $\mathcal{C}$ are convex.*

(ii) *The intersection graph of $\mathcal{C}^+$ is an interval graph (i.e., is isomorphic to the intersection graph of a set of intervals of the real line).*

Thus, applying to $\mathcal{C}^+$ any of the criteria in [13] *for a finite graph to be an interval graph, one gets criteria for the members of $\mathcal{C}$ to be convex under some ordering of Ω.*

Proof First assume we have an ordering $\leq$ as in (i), and list the elements of Ω in order as $a_1 < \cdots < a_n$. Choose real numbers $s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n$. These determine disjoint intervals $I_i = [s_i, t_i] \subseteq \mathbb{R}$, $(i = 1, \ldots, n)$, ordered like the elements of Ω.

To each nonempty convex subset A of Ω, we can now associate an interval $f(A) \subseteq \mathbb{R}$: the convex closure of $\bigcup_{a_i \in A} I_i$. It is straightforward to verify that for nonempty convex subsets A and B of Ω, we have $A \cap B \neq \emptyset$ if and only if $f(A) \cap f(B) \neq \emptyset$. (The forward implication is immediate; conversely, if A and B are disjoint, one can see from their convexity that $f(A)$ and $f(B)$ will be disjoint.) Applying f to the members of $\mathcal{C}^+$, all of which are convex in $(\Omega, \leq)$, we get the desired representation of the intersection graph of $\mathcal{C}^+$ as an interval graph.

Conversely, suppose the intersection graph of $\mathcal{C}^+$ is an interval graph, i.e., that there exists a map f carrying members of $\mathcal{C}^+$ to intervals in the real line, such that

(2) $f(A) \cap f(B) \neq \emptyset$ if and only if $A \cap B \neq \emptyset$.

Since distinct singletons $\{a\}$ and $\{b\}$ $(a, b \in \Omega)$ are disjoint, (2) implies that the $f(\{a\})$ $(a \in \Omega)$ will be disjoint intervals, hence this family of intervals has a natural ordering based on the ordering of $\mathbb{R}$. In other words, there is a way of listing the elements of Ω as $a_1, \ldots, a_n$ so that, writing $f(\{a_i\}) = [s_i, t_i]$, we have $s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n$. Hence let us order Ω by setting $a_1 < \cdots < a_n$.

For each $a \in \Omega$ and nonempty $A \in \mathcal{C}$, note that

(3) $a \in A \iff \{a\} \cap A \neq \emptyset \iff f(\{a\}) \cap f(A) \neq \emptyset$,

where the second equivalence holds by (2). Now the convexity of $f(A)$, together with the fact that the $f(\{a_i\})$ $(i = 1, \ldots, n)$ form an ordered string of disjoint intervals, shows that those $f(\{a_i\})$ that are contained in $f(A)$ form a consecutive substring of this string of intervals, whence by (3) the set of $a \in A$ is a consecutive string under our ordering of Ω; so every nonempty $A \in \mathcal{C}$ is indeed convex under that ordering. $\mathcal{C}$ might also contain $\emptyset$, which we had to exclude from $\mathcal{C}^+$; but $\emptyset$ is vacuously convex, so our proof of (i) is complete. $\square$

Note that since we have been considering finite graphs only, in the above result we had to take not only $\mathcal{C}$, but also Ω to be finite; something I avoided doing in earlier sections so that we could make use of Lemma 1. However, given a not necessarily finite set Ω and a finite set $\mathcal{C}$ of subsets of Ω, we can proceed as follows. For $a, b \in \Omega$, let $a \sim b$ if and only if the set of members of $\mathcal{C}$ containing a is the same as the set of those containing b. This is an equivalence relation on Ω with $\leq 2^{\mathrm{card}(\mathcal{C})}$ equivalence classes. Let $\Omega_0 \subseteq \Omega$ be a set of representatives of these finitely many equivalence classes, noting that for $A \neq B$ in $\mathcal{C}$, we have $A \cap \Omega_0 \neq B \cap \Omega_0$, and let $\mathcal{C}_0 = \{A \cap \Omega_0 \mid A \in \mathcal{C}\}$, a set of subsets of Ω_0. Then, on the one hand, if every member of $\mathcal{C}$ is convex under an ordering of Ω, every member of $\mathcal{C}_0$ is clearly convex under the induced ordering of Ω_0. On the other hand, if there is an ordering $\leq_0$ of Ω_0 under which all members of $\mathcal{C}_0$ are convex, we can construct an ordering $\leq$ of Ω by putting an arbitrary internal total ordering on each equivalence

class under $\sim$, and arranging those classes one above another as their representatives are arranged in Ω_0. Then we see that all members of $\mathcal{C}$ will be convex under the resulting ordering.

Proposition 19 gives criteria for the members of $\mathcal{C}_0$ to be convex under an ordering of Ω_0, and as we have seen, this is equivalent to the existence of an ordering of Ω under which the members of $\mathcal{C}$ are convex.

Finally, we can apply Lemma 1 to get criteria for such an ordering to exist in the case of infinite $\mathcal{C}$.

Two variants of the approach of Proposition 19 were pointed out by Martin Milanič (personal communication):

On the one hand, one may consider a family $\mathcal{C}$ of subsets of a set Ω as determining a *bipartite* graph, where one family of vertices corresponds to the points of Ω and the other to the elements of $\mathcal{C}$, and an edge connects the vertex corresponding to $p \in \Omega$ with the one corresponding to $S \in \mathcal{C}$ if and only if $p \in S$. Results are known on when such a bipartite graph corresponds to a family $\mathcal{C}$ of convex sets under an ordering on Ω [2, § 9.7.2].

On the other hand, a pair $(\Omega, \mathcal{C})$ can be regarded as a *hypergraph*, with Ω the set of vertices, and $\mathcal{C}$ the set of hyperedges. The question of when the hyperedges of a hypergraph are intervals under some ordering of the vertex-set is studied in [11]; see also [2, § 8.7].

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