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FORMULAS FOR STRESS RATIO AND EFFECTIVE FLANGE WIDTH OF SIMPLE AND CONTINUOUS I, T AND BOX BEAMS

by

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Abstract

In this investigation, formulas for calculating the stress ratio of simple and continuous T, I and box beams are presented. The stress ratio is defined as the ratio of the longitudinal stress at a point found by shear lag theory to the stress at the same point found by elementary beam theory. In these formulas, geometric parameters, loading types and section location are taken into account. A wide range of stress ratio problems are solved. Numerous examples and comparisons are given to check the accuracy of the formulas.

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Notation

S	= stress ratio = σ_s / σ_b
σ_s	= shear lag theory longitudinal stress at web-flange junction
σ_b	= beam theory longitudinal stress at same point as σ_s
e	= distance from center of web to middle surface of top flange
M^*, N^*	= total bending moment and total axial force at a section
c	= distance from web center to centroid of overall cross-section, positive upward
A, I	= area and moment of inertia of web within half of total cross-section
r_i	= $2bhe^2/I$
r_a	= $2bh/A$
r_i'	= r_a' = $(r_i + r_a)/2$
h	= thickness of flange
h_w	= thickness of web within half of total cross-section (Fig. 1)
h'	= flange thickness of equivalent I beam, see Eq. (10)
ψ_m	= see Eq. (14)
α	= $n\pi/l$
d	= overall depth of web
b	= half width of flange, measured from edge of web
A_t, I_t	= area and moment of inertia of half of total section
η	= S - 1 = complementary stress ratio
λ	= b_e/b = effective flange width coefficient
β	= 1 - λ = complementary effective flange width coefficient
b_e	= Half effective width of flange, measured from edge of web
S_i	= stress ratio of equivalent I beam

η_i = complementary stress ratio of equivalent I beam

L = total length of beam

l = L/b

a_j = j-th span length of continuous beam

x = relative x-coordinate of cross section

ξ = relative x-coordinate of application point of concentrated load or reaction (Fig. 2)

$c_1 = \begin{cases} .76 & \text{(box beam)} \\ .75 & \text{(I beam)} \end{cases}$

$c_2 = \begin{cases} .60 & \text{(box beam)} \\ .65 & \text{(I beam)} \end{cases}$

$\Delta\beta = \begin{cases} 0 & \text{(for box beam and I beam with } l > 30) \\ .088\text{-}.0455\, l^{\cdot 2} & \text{(for I beam with } l \leq 30) \end{cases}$

η_o = peak complementary stress ratio

M_w, M_j^p, M_j^r = bending moment in a simple beam due to uniform load, j-th concentrated load and j-th interior reaction

$\eta_w, \eta_j^p, \eta_j^r$ = complementary stress ratios in a simple beam due to uniform load, j-th concentrated load and j-th interior reaction

η_j = complementary stress ratio at the j-th interior support section

R_j = reaction of j-th interior support of continuous beam

M^*_j = total bending moment at j-th interior support section

L_1 = $2\xi L$

L_2 = $2(1-\xi)L$

G = $I_t N^* / A_t M^*$

e' = distance from center of web to middle surface of bottom flange

1. Introduction

In wide flange beams, the distribution of longitudinal stresses across the flange is not uniform due to shear deformation in the flange, known as shear lag. An effective flange width or a stress ratio, between which a simple relation can be found (see later, section 2), are used to account for this effect in design. The effective flange width is defined as that width which resists a longitudinal force equal to the actual force in the flange, if the longitudinal stresses across the flange were constant and equal to the actual maximum stress. The stress ratio is defined as the ratio between the longitudinal stress found using shear lag theory and the stress at the same point calculated by elementary beam theory. Only the stress ratio at the web-flange junction is dealt with in this study.

The problem of effective flange width for a T-beam has been studied for a long time. Numerous research results have been published by many authors and various recommendations have been suggested in design codes from various countries. However, among these research results and recommendations, a wide disparity can be found [3, 4]. For I beams and box beams, the study of the related problem seems to be insufficient. Therefore, a more rational approach to the problem is needed.

In a recent study [3], tables for determining the effective flange width of T beams, which are helpful for design work, were presented. In Ref. [3], however, only a geometric parameter b/l and three loading cases were considered. Furthermore, at the most critical sections, the section directly under a concentrated load or the interior support section, the effective widths were not given. A lower estimate for the shear lag effect was obtained in Ref. [3] due to the use of a finite element analysis with a rather coarse mesh (see later, section 4.2).

A rigorous analysis and the correct effective width or stress ratio can be obtained by the computer program given in Ref. [2], provided a sufficiently large number of harmonics is adopted. In Ref. [1], a simplified, but accurate, analysis and a computer program SHLAG have been presented, by which the shear lag effect and the stress ratio or effective flange width can be found with a minimum amount of input data. However, using these programs would be somewhat cumbersome for engineers who want to find quickly only the most important data on shear lag as it affects design problems.

In this study, based on the results calculated by SHLAG of Ref. [1], formulas for determining the stress ratio and effective flange width, in which loading types, section location, l' and r_i (or r_a) are taken into account, are suggested (see Notation for definition of symbols). The computational work for all formulas can be done by ordinary hand calculator and most formulas are simple and easy to use. A variety of shear lag problems for simple and continuous T, I and box beams can be solved by these formulas. Moreover, these formulas can also be used as a tool to study the stress ratio problem since the parameters included appear in an explicit form and the effect of each parameter can be estimated easily.

In section 2 of this study, some basic relations developed in the present study are given. The T section is more complicated than the I or box section with symmetric flanges. An extra parameter, the location of the centroid of the total cross section, must be considered. To simplify the formulation, the calculation of the T beam will be performed through an equivalent I beam. This problem will also be discussed in this section.

In section 3, some simple and essential formulas are presented, by which the most important features of the shear lag effect for a simple or continuous beam can be estimated quickly. These should be useful for design problems usually encountered.

In section 4, more complete formulas are given and a wide range of shear lag problems are solved. These formulas should prove useful even for research as well as design work in this field.

Although all formulas in this study are derived for T beams and I or box beams with symmetric flanges, these formulas can be used approximately for beams with slightly nonsymmetric flanges.

Numerous typical numerical examples are given to show how to use the formulas. Comparisons with the results by SHLAG and other authors have been performed to check the accuracy of the formulas.

2. General Remarks

2.1 Basic Relations

Solutions for the stresses in T, I and box beams (Fig. 1) have been studied in detail in Ref. [1]. For most practical structural materials, the Poisson's ratio ν varies only from .15 to .3, the effect of which upon the stresses is negligible and will be neglected in this study. Cross sections of beams studied in this investigation are shown in Fig. 1.

At the web-flange junction, the longitudinal stress in a beam (Fig. 2) can be expressed as [1] (see Notation for definition of symbols):

$$\sigma_S = \sum_{n=1}^{\infty} A_n F_n(l) \sin n\pi x \tag{1}$$

$F(l)$ can be found from Eqs. (8) and (11) of Ref. [1] by letting $y = b$.

The total bending moment and axial force at any section of the beam can be expressed as a Fourier series:

$$\begin{aligned} M^* &= M^*(x) = \sum_{n=1}^{\infty} m_n \sin n\pi x \\ N^* &= N^*(x) = \sum_{n=1}^{\infty} n_n \sin n\pi x \end{aligned} \tag{2}$$

For a T beam

$$[p_n(l) + \frac{r_i + r_a}{2} q_n(l)]A_n = -\frac{e}{I} m_n + \frac{n_n}{A} + \frac{e}{I} n_n c \tag{3}$$

For an I beam with symmetric flanges,

$$[p_n(l) + r_i q_n(l)]A_n = -e m_n / I$$

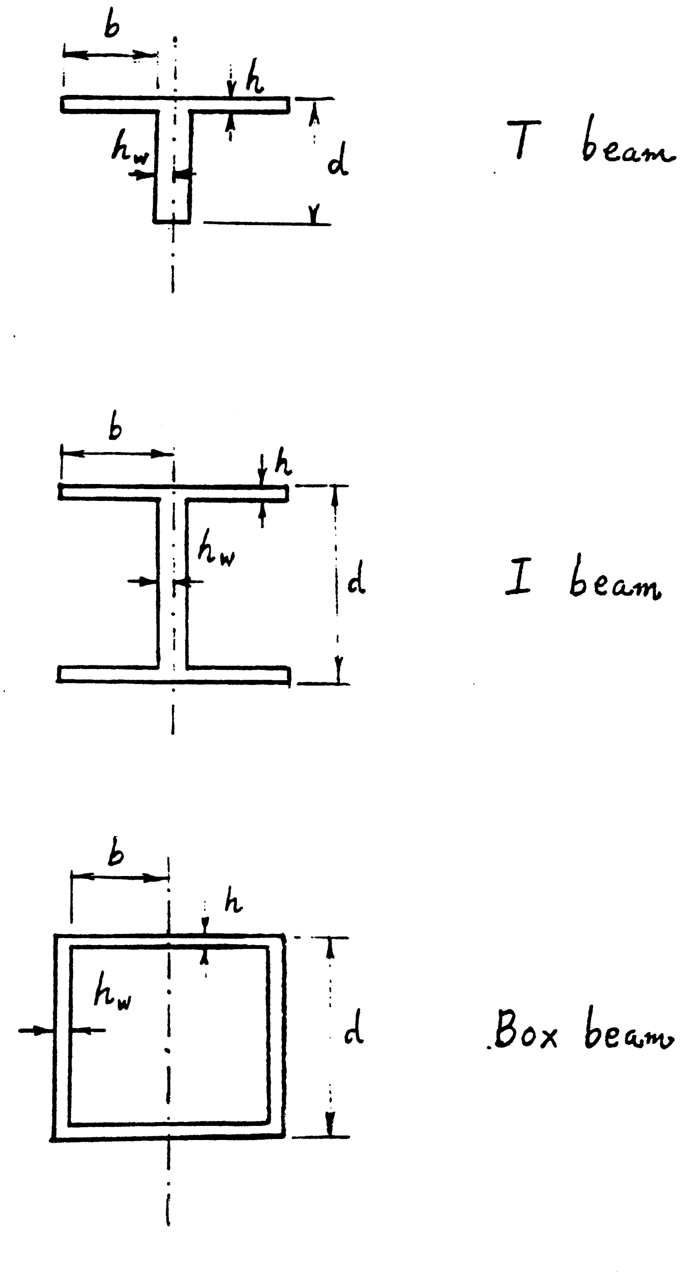


Fig. 1. Cross Sections of Beams

or

$$[p_n(l) + r_a q_n(l)]A_n = n_n / A \tag{4}$$

where $p_n(l)$ and $q_n(l)$ can be found from Eq. (9), (10) of Ref. [1].

For a box beam with symmetric flanges (see Ref. [1]):

$$[2+r_i(1 - \tanh^2\alpha + \frac{\tanh\alpha}{\alpha})]A_n = -\frac{e}{I}m_n$$

or

$$[2 + r_a(1 - \tanh^2\alpha + \frac{\tanh\alpha}{\alpha})]A_n = \frac{n_n}{A} \tag{5}$$

It can be seen from Eqs. (4) and (5) that for an I or box beam, σ_s will be dependent only upon l, r_i (or r_a), e/I or A and is independent of the specific dimensions of the cross section. The beam theory stress can be expressed as*

$$\sigma_b = \frac{N^*}{A_t} \pm \frac{M^*e}{I_t} = \frac{N^*}{(1 + r_a)A} \pm \frac{M^*}{1 + r_i} \cdot \frac{e}{I} \tag{6}$$

Therefore, the stress ratio $S = \sigma_s / \sigma_b$ is further independent of e/I (or A) and thus only loading types, location of section x, l, and r_i (or r_a) are chosen as the parameters in the formulas of this study.

Both η and β are used in the formulas. These two quantities can be simply related by

$$\eta = \beta / [(r_i + 2bh^3/h_w d^3)^{-1} + 1 - \beta] \text{ (for bending)} \tag{7}$$

or

$$\eta = \beta / (r_a^{-1} + 1 - \beta) \text{ (for axial load)} \tag{8}$$

2.2 Stress Ratio of a T Beam

It can be concluded from Eqs. (3) and (4) that under bending the shear lag stress σ_s of a T beam can be determined from an equivalent I beam, if I, b and e of both these beams are the same but r_i of the latter (denoted by r_i') is equal to $(r_i + r_a)/2$ of the former. Therefore, the longitudinal shear lag stress σ_s of the T beam to be calculated is equal to

$$\sigma_s = \frac{M^*e}{I(1 + r_i') + bh'^3/6} S_i \tag{9}$$

where, S_i and h' = stress ratio and flange thickness of the equivalent I beam, and

$$h' = \frac{I}{2be^2} r_i' \tag{10}$$

The beam theory stress of the T beam can be expressed as

$$\sigma_b = \frac{M^*(e - c)}{I_t} = \frac{M^*e(1 - c/e)}{I_t} \tag{11}$$

---

*$I_t \approx (1 + r_i)I$ is used. Accurately, $I_t = (1 + r_i)I + 2bh^3/12$.

we can find

$$\frac{I_t}{1 - c/e} = [1 + r_i' + (1 + \frac{r_a}{2})\frac{bh^3}{12I}]\, I \tag{12}$$

Hence, the stress ratio of a T beam under bending can be calculated by

$$S = \sigma_s/\sigma_b = \psi_m S_i \tag{13}$$

$$\psi_m = [1 + r_i' + (1 + \frac{r_a}{2})\frac{bh^3}{12}]/(1 + r_i' + \frac{bh'^3}{6}) \tag{14}$$

Similarly, when only axial load N is acting ($m_n = 0$), it can be proved that the stress ratio of a T beam is exactly equal to that of its equivalent I beam provided A of both these beams are the same but r_a of the latter (denoted by r_a') is equal to $(r_i + r_a)/2$ of the former.

For a T beam, stress ratio S, and effective flange width coefficient λ can be related as follows.

Under bending, from Eqs. (11) and (12) we can find

$$S = \frac{I_t}{M^*e(1 - c/e)}\,\sigma_s = \frac{1 + r_i' + (1 + \frac{r_a}{2})\frac{bh^3}{12I}}{1 + \lambda r_i' + (1 + \frac{r_a}{2}\lambda)\frac{bh^3}{12I}\,\lambda} \tag{15}$$

For axial load,

$$S = \frac{\sigma_s}{N/A_t} = \frac{2 + r_a}{2 + \lambda r_a} \tag{16}$$

3. Formulas for Basic Cases

All formulas of this study were derived by empirically matching with the calculated results of the computer program SHLAG of Ref. [1] (see Appendix). The accuracies of the formulas are checked by comparison with the results of SHLAG and other authors. Symbols in all formulas are defined in the Notation section.

3.1 Basic Formulas for Simple Beams

Under uniform load, from the results of various comparison calculations, it can be concluded that the effect of r_i upon the effective flange width can be neglected when $l \geq 4$. At any section of a simple beam, the complementary effective width

$$\beta = 3.77\, l^{-1.9}[1 + (3.1 - 99\, l^{-3.0})\,|\,.5 - x\,|^{1.5}] + \Delta\beta \qquad (l \geq 4) \tag{17}$$

Under a concentrated load at any point $x = \xi$ (Fig. 2), at the section directly under the load $x = \xi$,

$$\eta_o = \begin{cases} \dfrac{c_1}{\xi(1-\xi)l}\,(r_i^3 - c_2) & (r_i \geq .6) \\[2ex] \dfrac{c_3}{4\xi(1 - \xi)(r_i^{-1} + 1 - c_3)} & (r_i < .6) \end{cases} \tag{18}$$

$$c_3 = \begin{cases} 1.35/l^{.8} - .02 & (l \leq 140) \\ .006 & (l > 140) \end{cases} \tag{19}$$

If a simple beam is subjected to uniform load and n concentrated loads, S can be obtained by superposition as:

$$S = (M_w S_w + \sum_{j=1}^{n} M_j^p S_j^p)/M^*$$

where

M_w, S_w = bending moment and stress ratio at section under consideration due to uniform load alone;

M_j^p, S_j^p = as M_w, S_w, but due to the j-th concentrated load alone;

M^* = total bending moment at the section under consideration.

3.2. Two-span Continuous Beam under Uniform Load

For a two-span continuous beam (Fig. 3), the stress ratio at any section can be expressed as:

$$S = \frac{M_w S_w + M_1^r S_1^r}{M^*} \tag{20}$$

where

M_w = bending moment due to the uniform load in a simple beam formed by removing the interior support;

M_1^r = as M_w, but due to the reaction R_1 of the interior support;

$M^* = M_w + M_1^r$.

Eq. (20) can be transformed to

$$\eta = S - 1 = \frac{M_w \eta_w + M_1^r \eta_1^r}{M^*} \tag{21}$$

At the interior support section $x = \xi$,

$M_w = \xi(1-\xi)wL^2/2$

$M_1^r = -(1 + \xi - \xi^2)wL^2/8$

Substituting in Eq. (21) and letting $\eta = \eta_1$, we can obtain

$$\eta_1 = \frac{(1 + \xi - \xi^2)\eta_o - 4(\xi - \xi^2)\eta_w}{1 - 3(\xi - \xi^2)} \tag{22}$$

η_w and η_o can be calculated by Eqs. (17), (7), and (18), respectively.

Eq. (22) can be used for any continuous beam with two unequal spans. If the two spans are equal, Eq. (22) can be simplified to:

$$\eta_1 = \frac{4c_1}{l}(r_i^3 - c_2) - \frac{4(3.77l^{-1.9} + \Delta\beta)}{r_i^{-1} + 1 - (3.77l^{-1.9} + \Delta\beta)} \tag{23}$$

Some results calculated for equal and unequal two span beams are shown in Tables 1 and 2. In Table 1, structures A, B, C and D are the box beams in Ref. [2] with nonsymmetric flanges, for which the r_i can be calculated by summation of the

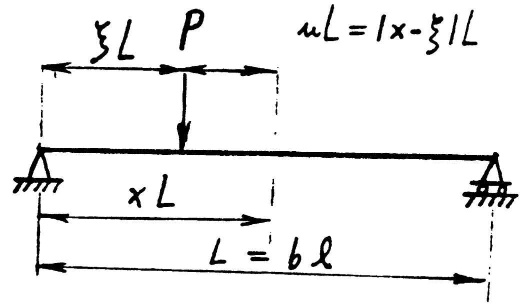


Fig. 2 Simple Beam Subjected to Concentrated Load

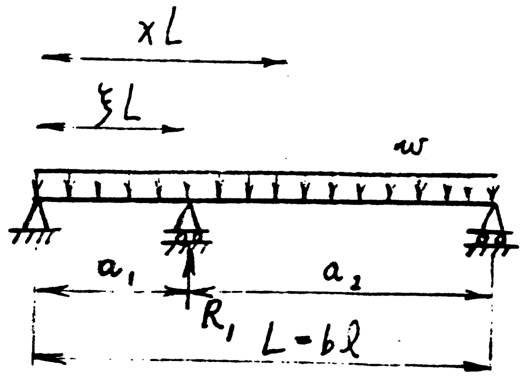


Fig. 3 Two-Span Continuous Beam under Uniform Load

Table 1 Stress Ratios at Center Support Section of Continuous Beams with Two Equal Spans under Uniform Load

beam case	l	d	r_i	stress ratio S		
				Eq. (23)	SHLAG	Ref. [2]
I-1	16		9.72	2.14	2.18	
I-2	30		9.72	1.65	1.66	
B-1	16		24.3	2.83	2.84	
B-2	30		24.3	2.00	2.00	
Stru. A	32	7.5	5.90	1.51		1.41, 1.44*
Stru. B Ref.	32	5	7.76	1.58	1.58, 1.53*	1.44, 1.46*
Stru. C [2]	64	15	3.34	1.19	1.19, 1.17*	1.13, 1.20*
Stru. D	64	10	4.71	1.23		1.13, 1.20*

*Stress ratio for top and bottom flange

Table 2 Stress Ratios at Interior Support Section of Continuous Box Beam with Two Unequal Spans under Uniform Load

ratio of spans	l	r_i	β	η_w	η_o	stress ratio S	
						Eq. (22)	SHLAG
	16	4.86	.0235	.020	.215	1.74	1.74
$\xi = 1/3$	60	4.86	.0019	.002	.058	1.21	1.21
$\frac{a_2}{a_1} = 2$	16	24.3	.0235	.023	.428	2.51	2.53
	30	24.3	.0070	.007	.229	1.82	1.83
$\xi = 1/4$	16	4.86	.027	.023	.256	1.66	1.65
	60	4.86	.002	.002	.068	1.18	1.18
$\frac{a_2}{a_1} = 3$	16	24.3	.027	.027	.509	2.34	2.34
	30	24.3	.008	.008	.272	1.72	1.73

second moments of top and bottom flanges:

$$r_i = \frac{2 \times 9.375 \times .75(d/2 - .375)^2 + 9.37 \times 1(d/2 - .5)^2}{1.25d^3/12}$$

From Tables 1 and 2 it can be seen that the accuracy of Formulas (22) and (23) is excellent for beams with symmetric flanges. For beams with nonsymmetric flanges, the stress ratio is different for top and bottom flange and the results of Eq. (22) and (23) agree well with the larger one.

The accuracy of Eqs. (17) and (18) is much better than Eqs. (22) and (23), since in the latter subtraction of large numbers may occur. Therefore, for a simple beam subjected to uniform load and numerous concentrated loads in the same direction, Eqs. (17) and (18) would be very accurate.

3.3. Multiple Span Continuous Beams under Uniform Load

For a continuous beam with n spans (Fig. 4), similar to Eq. (21), at any section,

$$\eta = (M_w \eta_w + \sum_{j=1}^{n-1} M_j^r \eta_j^r)/M^* \tag{24}$$

where

M_j^r = as in Eq. (20), but due to the reaction R_j of the j-th interior support

M^* = total bending moment at section under consideration.

When calculating the stress ratio at a certain interior supported section of most practical engineering structures, except when the total length, span and r_i of the beam is very small (say, $l < 12$, $a < 4$ and $r_i < .6$), the effect of all other reactions can be neglected. Hence, at the j-th interior supported section, Eq. (24) can be simplified to:

$$\eta_j = (M_w \eta_w + M_j^r \eta_j^r)/M_j^* \tag{25}$$

For a continuous beam with n equal spans (Fig. 5)

$$M_w = j(n - j)wL^2/2n^2$$

$$M_j^r = -j(n - j)R_j L/n^2 \tag{26}$$

Substituting Eqs. (26), (7), and (18) in Eq. (25), we can obtain

$$n_j = [\frac{j(n - j)}{2n^2} \frac{\beta r_i}{1 + (1 - \beta)r_i} - \frac{R_j}{wL} \frac{c_1}{l}(r_i{}^3 - c_2)]/\frac{M^*{}_j}{wL^2} \tag{27}$$

where

β can be calculated by Eq. (17) and

$M^*{}_j$ = total bending moment at the j-th interior supported section.

For the stress ratio at the interior support section of a continuous box beam with three equal spans and $r_i = 4.86$, we can obtain

$$\eta_1 = \eta_2 = \frac{25.26}{l} - \frac{45.7}{1.206l^{1.9} - 4.57} \tag{28}$$

Some results calculated for continuous beams with multiple equal spans are shown in Tables 3 and 4, from which it can be seen that Eq. (27) is sufficiently accurate even for continuous beams with ten equal spans and nonsymmetric flanges.

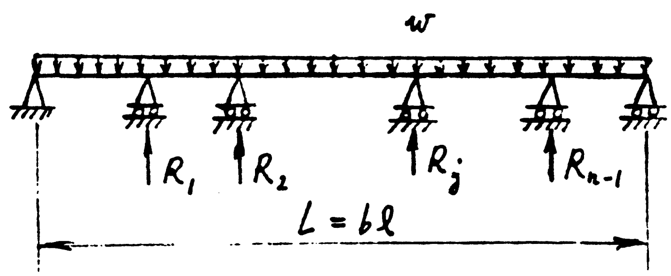


Fig. 4. Multiple-Span Continuous Beam Under Uniform Load

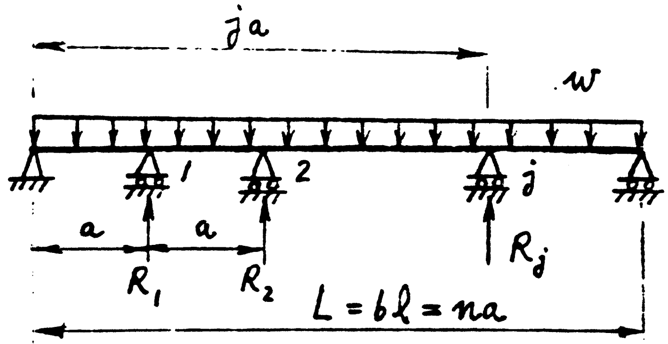
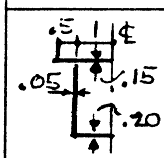


Fig. 5. Multiple Equal Span Continuous Beam
Under Uniform Load

Table 3 Stress Ratio S at Interior Support Sections of Continuous Box Beams with Three Equal Spans under Uniform Load

l =		12	24	45	90
S	Eq. (28)	2.76	1.96	1.53	1.27
S	SHLAG	2.76	1.97	1.54	1.28

Table 4 Stress Ratio S at Interior Support Sections of Continuous Beams with Ten Equal Spans under Uniform Load

beam case		j		1	2	3	4	5
		R_j/wL		.1134	.0964	.1010	.0997	.1002
		$10^3 \times M_j{}^*/wL^2$		-1.058	-.773	-.849	-.827	-.837
box beam	l = 100	S	Eq. (27)	1.62	1.69	1.65	1.66	1.66
	r_i = 3.07		SHLAG	1.59	1.68	1.64	1.65	1.65
I beam	l = 100	S	Eq. (27)	1.57	1.63	1.60	1.61	1.61
	r_i = 3.07		SHLAG	1.56	1.64	1.61	1.62	1.62
	l = 240 r_i = 10.6	S	Eq. (27)	1.48	1.55	1.52	1.53	1.53
	d = 2		SHLAG*	1.47	1.55	1.53	1.53	1.53

*Stress ratio for bottom flange

4. Extension of Basic Formulas and Application

To study the longitudinal distribution of the stress ratio in a simple or continuous beam, it is a fundamental requirement that a formula for the longitudinal variation of the stress ratio under a single concentrated load must be derived.

4.1. Stress Ratio at Any Section of Simple Beam under a Concentrated Load at Midspan

The complementary stress ratio can be expressed empirically as:

$$\eta = \eta_o \exp(-\alpha u) \tag{29}$$

in which, $u = |x - .5|$, η_o can be found from Eq. (18) by letting $\xi = .5$. When $r_i \geq .6$,

$$\eta_o = 4c_1(r_i^3 - c_2)/l$$

$$\alpha = \frac{.8}{\eta_o}\left\{1 + b(\exp[(\frac{3}{l})^{.43}\, ln(\frac{r_i}{.6})] - 1)\right\} \tag{30}$$

where, for box beams,

$$b = \begin{cases} 11.1\,(0.0111\, l^{1.6} + .0114) & (l \leq 6) \\ 11.1\,(l^a - .99) & (l > 6) \end{cases}$$

$$a = \begin{cases} .14 & (l \geq 10) \\ .1 + .01(l - 6) & (l < 10) \end{cases}$$

for I beams,

$$b = \begin{cases} .121\, l^{1.51} & (l \leq 6) \\ 11(\, l^{.12} - 1.076) & (l > 6) \end{cases}$$

when $r_i < .6$

$$\eta_o = c_3/(r_i^{-1} + 1 - c_3)$$

$$\alpha = 1.33 r_i / \eta_o \tag{31}$$

For a continuous beam with two equal spans under uniform load, the stress ratio at any section can be calculated by Eq. (21), in which η_w and η_1^r can be found from Eqs. (17), (7), and (30).

For example, the stress ratio at section $x = .3$ of beam B-1 in Table 1 can be computed as follows. $\eta_w = .025$ can be obtained from Eqs. (17) and (7). From Eq. (30), $\eta_o = .382$ and $\alpha = 59.1$ can be found. Thus, $\eta_1^r = 2.8 \times 10^{-6} \approx 0$. Substituting $M_w = .105wL^2$ and $M_1^r = -.09375wL^2$ in Eq. (21), we can obtain

$$\eta = \frac{.105 \times .025 - 0}{.105 - .09375} = .23$$

$$S = 1.23$$

The stress ratios for various sections of beams in Table 1 can be calculated similarly and are shown in Tables 5 and 6.

Table 5 Longitudinal Variation of Stress Ratio S for Beams in Table 1 under Uniform Load

beam case		x	.2	.3	.4	.45	.475	.5
		u	.3	.2	.1	.05	.025	0
B-1	S	η_1^r	0	0	.001	.020	.087	.382
		Eq. (21)	1.13	1.23	.52	1.02	1.44	2.83
		SHLAG	1.15	1.23	.37	.96	1.43	2.84
B-2	S	η_1^r	0	0	0	.003	.025	.204
		Eq. (21)	1.04	1.07	.86	.98	1.12	2.00
		SHLAG	1.05	1.07	.84	.93	1.08	2.00
I-1	S	η_1^r	0	0	.007	.042		.249
		Eq. (21)	1.16	1.30	.50	1.15		2.14
		SHLAG	1.16	1.25	.58	1.11		2.17
I-2	S	η_1^r	0	0	0	.008		.133
		Eq. (21)	1.04	1.06	.87	1.03		1.65
		SHLAG	1.05	1.08	.84	.98		1.66

Table 6 Longitudinal Variation of Stress Ratio of Structure B and C of Ref. [2] (Table 1)

		x	.25	.4667	.48	.4867	.4933	.5
Stru. B	S	η_1^r	0	.009	.025	.042	.070	.119
		Eq. (21)	1.04	1.03	1.13	1.21	1.35	1.58
		SHLAG*	1.06,1.06	1.01,1.01	1.10,1.09	1.16,1.15	1.36,1.33	1.58,1.53
		Ref. [2]*	1.04,1.07		1.08,1.11	1.13,1.16	1.35,1.37	1.44,1.46
		x	.1875	.25	.49	.4933	.4967	.5
Stru. C	S	η_1^r	0	0	.011	.017	.026	.040
		Eq. (21)	1.01	1.01	1.05	1.08	1.13	1.20
		SHLAG*	1.01,1.01	1.01,1.01	1.05,1.04	1.08,1.07	1.12,1.11	1.19,1.17
		Ref. [2]*	.99,1.01	.99,1.02	1.02,1.02	1.08,1.12	1.13,1.19	1.13,1.20

*Stress ratio for top and bottom flange

4.2. Calculation of T Beam

As mentioned in section 2.2, a T beam can be calculated from its equivalent I beam, for which the above-mentioned formulas can be used.

A family of T beams from Ref. [3] were calculated by the above-mentioned approximate formulas and the computer program SHLAG and are compared with results in Ref. [3]. The geometric dimensions of the beams are shown in Table 7. The stress ratios calculated for simple beams are shown in Table 8. The stress ratios calculated for two equal span continuous beams, i.e., propped cantilever beam in Ref. [3], are shown in Table 9.

From Tables 8 and 9 it can be seen that the approximate formula stress ratios agree well with those of SHLAG, except for the two span continuous beam T-3, in which the relative span a/b = 2.7 is very short and r_i' is large, but Ref. [3] gave a somewhat lower estimate of the stress ratios. This is logical since in Ref. [3] a finite element analysis with a rather coarse mesh was used. Furthermore, it can be expected that at a section directly under the concentrated load or at the interior support section the stress ratios by the analysis of Ref. [3] would be even lower.

4.3. Longitudinal Distribution of Stress Ratio in Simple Beam under A Concentrated Load at Any Point

For the case of a concentrated load acting at any point $x = \xi$, we shall calculate the stress ratio S with the aid of two auxiliary beams, Beam 1 with span $L_1 = 2\xi L$ and Beam 2 with span $L_2 = 2(1-\xi)L$ (Fig. 6). Both Beam 1 and Beam 2 are loaded by a concentrated load at midspan, for which the complementary stress ratios η_1 and η_2 can be found by Eqs. (30) or (31).

It can be easily found from Eq. (18) that the peak complementary stress ratio η_0 of the original beam, Beam o, is equal to the average of those of Beam 1 and Beam 2. At a section A on the left side and far away from the load, the stress ratio S of Beam o will approach the value S_1 at the corresponding section A' of Beam 1 (Fig. 6). Hence, we can assume approximately that at the left support section, S of Beam o is equal to S_1 of beam 1. A similar assumption can also be used on the right side.

To meet the above requirements we can assume for the original beam: when $u = \xi - x \geq 0$,

$$S = \frac{1}{2}[(1 + \frac{u}{\xi})S_1 + (1 - \frac{u}{\xi})S_2]$$

when $u = x - \xi \geq 0$,

$$S = \frac{1}{2}[(1 + \frac{u}{1-\xi})S_2 + (1 - \frac{u}{1-\xi})\,S_1]$$

and

$$\eta = \begin{cases} \frac{1}{2}[(1 + \frac{u}{\xi})\ \eta_1 + (1 - \frac{u}{\xi})\ \eta_2] & (u = \xi - x \geq 0) \\ \frac{1}{2}[(1 - \frac{u}{1-\xi})\ \eta_1 + (1 + \frac{u}{1-\xi})\ \eta_2] & (u = x - \xi \geq 0) \end{cases} \tag{32}$$

in which, η_1 can be found from Eqs. (29), (30), or (31) by using $l_1 = 2\xi l$ for l and

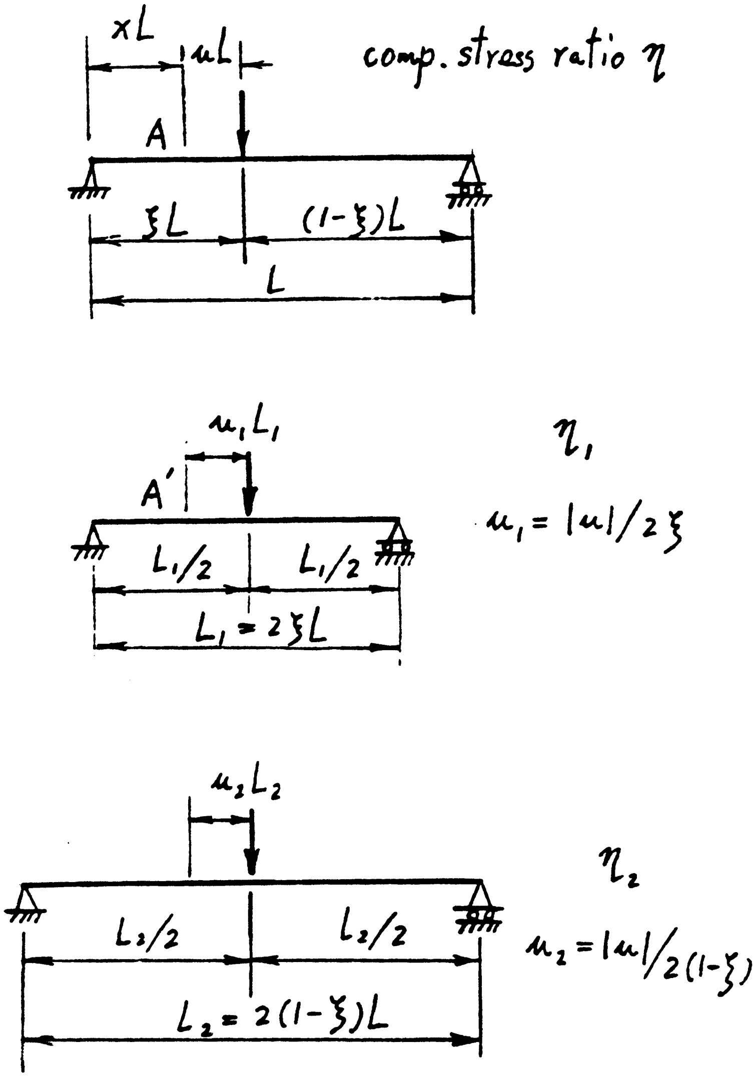


Fig. 6. Simple Beam Subjected to Concentrated Load at Any Point

Table 7 Geometry of Beams in Ref. [3] ($h/b = 2/39$, $h_w/b = 1/13$)*

beam	b'/l'	l	d/b	r_i	r_a	$r'_i = r'_a$	h'/b	ψ_m
T-1	.3	7.18	.359	8.19	3.71	5.98	.0373	1.011
T-2	.5	4.31	.2154	10.78	6.19	8.49	.0403	1.057
T-3	.8	2.69	.1347	11.38	9.90	10.64	.0479	1.243

*b', l',λ' denotes b, l, λ of Ref. [3] respectively, and $l = 2.154l'/b'$,
λ (this study) = $(14\lambda' - 1)/13$.

Table 8 Simple T Beams under Uniform and Concentrated Load*

load	beam			approximate formula			SHLAG	Ref. [3]		
		b'/L	x	β	η_i	S	S	λ'	λ	S
uniform load	T-1	.3	.325	.128	.123	1.14	1.13	.89	.882	1.11
			.475	.111	.105	1.12	1.11	.90	.892	1.10
			.5	.111	.105	1.12	1.12			
	T-2	.5	.325	.294	.357	1.44	1.41	.73	.709	1.38
			.475	.264	.309	1.39	1.38	.76	.742	1.32
			.5	.264	.309	1.39	1.38			
conce. load at midspan	T-1	.3	.325		.025	1.04	1.05	.95	.946	1.05
			.475		.293	1.31	1.30	.76	.742	1.29
			.5		.442	1.46	1.46			
	T-2	.5	.325		.161	1.23	1.27	.81	.795	1.24
			.475		.684	1.78	1.74	.60	.569	1.67
			.5		.870	1.98	1.99			
	T-3	.8	.325		.545	1.94	2.12	.58	.548	1.92
			.475		1.328	2.92	3.00	.43	.386	2.69
			.5		1.540	3.19	3.40			

*λ' is given in Ref. [3], $S = \psi_m(1 + \eta_i)$ in the approximate formula method.

Table 9 Continuous T Beams with Two Equal Spans under Uniform Load*

beam		approximate formula					SHLAG	Ref. [3]	
case	x	β	η_w	η_1^r	η_i	S	S	λ'	S
T-1,	.2375	.044	.039	0	.216	1.23	1.19	.83	1.19
$l=14.36$	.4875	.034	.030	.151	.703	1.72	1.75	.57	1.67
	.5	.034	.030	.221	.985	2.01	2.02		
T-2,	.2375	.106	.105	.002	.575	1.66	1.62	.64	1.57
$l=8.62$	.4875	.081	.079	.333	1.490	2.63	2.65	.39	2.53
	.5	.081	.078	.435	1.863	3.03	3.06		
T-3,	.2375	.229	.267	.018	1.400	2.98	2.95	.44	2.62
$l=5.39$	.4875	.178	.197	.643	2.674	4.57	4.81	.27	4.43
	.5	.178	.196	.770	3.066	5.05	5.44		

*λ', S as in Table 8.

$u_1 = |u|/2\xi$ for u, η_2 can be calculated similarly by using $l_2 = 2(1-\xi)l$ for l and $u_2 = |u|/2(1-\xi)$ for u.

For example, the stress ratio S at section $x = .2$ of the simple beam ($l = 8$, $r_i = 9.72$) shown in Fig. 7 can be calculated as follows. Substituting $l_1 = 2 \times .25 \times 8 = 4$ and $u_1 = .05/2 \times .25 = .1$ in Eqs. (29) and (30) for l and u, we can obtain $\eta_1 = .347$. Similarly, using $l_2 = 2(1 - .25) \times 8 = 12$ and $u_2 = .05/2 \times .75 = .0333$, we can find $\eta_2 = .087$. Substituting $u = .05$ and η_1, η_2 in Eq. (32), we can obtain for section $x = .2$

$$\eta = \frac{1}{2}[(1 + \frac{.05}{.25}) \times .347 + (1 - \frac{.05}{.25}) \times .087] = .243$$

Similarly, at section $x = .3$,

$$\eta = \frac{1}{2}[(1 - \frac{.05}{.75}) \times .347 + (1 + \frac{.05}{.75}) \times .087] = .208$$

Results for some simple box beams by the above approximate formulas are shown in Fig. 7, in which those by SHLAG are also plotted for comparison.

4.4. Longitudinal Distribution of Stress Ratio in Continuous Beams under Uniform Load

For a continuous beam with two unequal spans under uniform load, the stress ratio at any section can be determined by Eq. (21), in which η_1^r can be calculated by Eq. (32) and η_1, η_2 can be obtained from Eq. (29) and (30). The stress ratio calculated for some typical two span continuous box and I beams are shown in Table 10 and 11 with the necessary intermediate results.

For a continuous beam with n spans (Fig. 4) under uniform and m concentrated loads, the stress ratio at any section can be found from Eq. (24) by adding the effect of the concentrated loads:

$$\eta = (M_w \eta_w + \sum_{j=1}^{n-1} M_j^r \eta_j^r + \sum_{i=1}^{m} M_i^p \eta_i^p)/M^* \tag{33}$$

where M_i^p, η_i^p = bending moment and complementary stress ratio in a simple beam due to the i-th concentrated load.

In most practical structures, for calculating the stress ratio at a section in the j-th span, only the effect of uniform load, interior reactions R_{j-1} and R_j and the concentrated loads in this span need to be accounted for. Therefore, if only uniform load is acting, Eq. (33) can be simplified to:

$$\eta = (M_w \eta_w + M_{j-1}^r \eta_{j-1}^r + M_j^r \eta_j^r)/M^* \tag{34}$$

For a continuous box beam with three equal spans, with $l = 12$ and $r_i = 4.86$, subjected to uniform load (Table 12), at section $x = .475$, η can be computed as follows.

For reaction R_1, from Eq. (32), we can obtain

$$\eta_1^r = \frac{1}{2}[(1 - \frac{.475 - 1/3}{1 - 1/3}) \times .0035 + (1 + \frac{.475 - 1/3}{1 - 1/3}) \times .0018] = .0025$$

for reaction R_2, similarly,

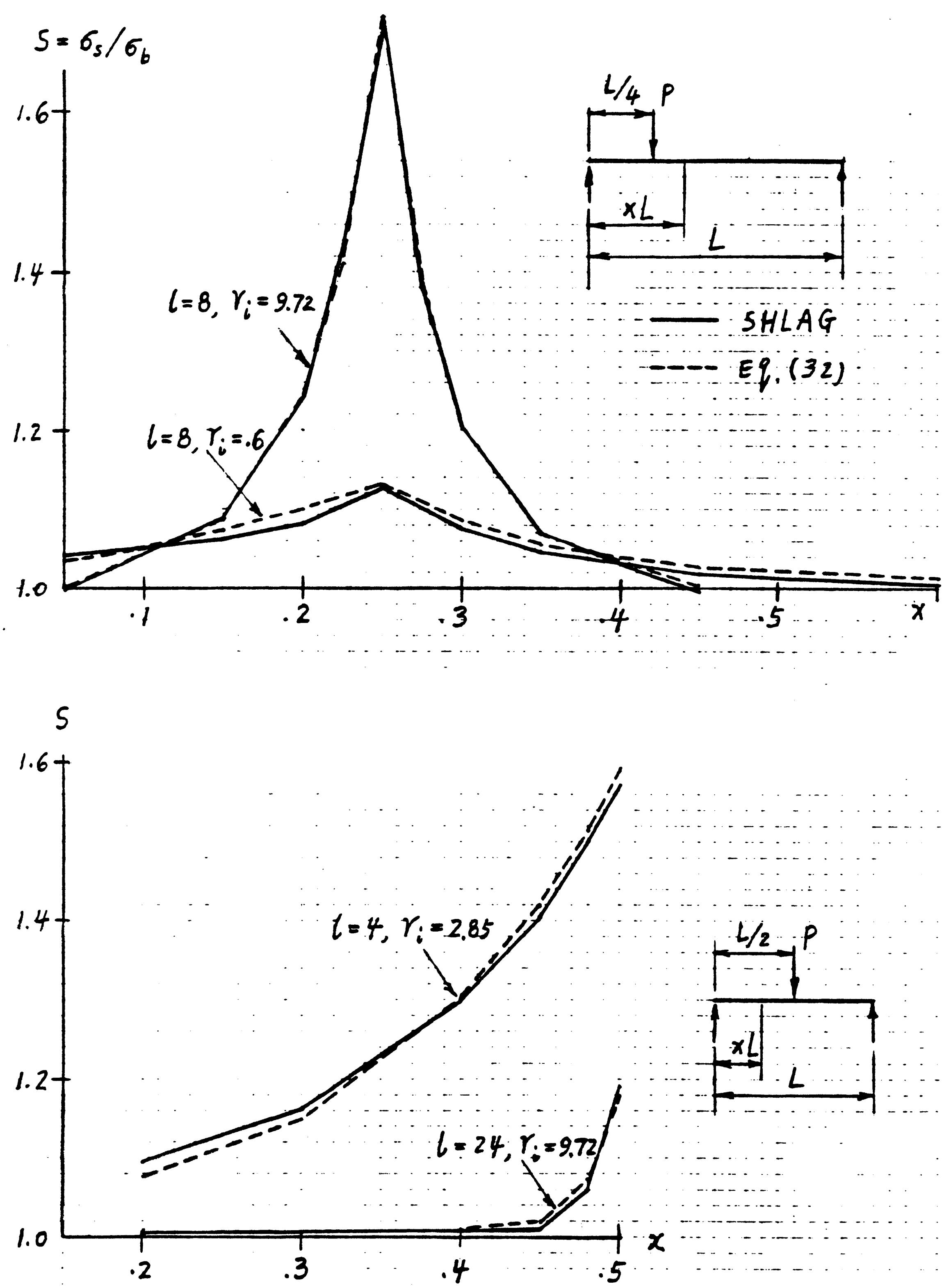


Fig. 7. Longitudinal Distribution of Stress Ratio in Simple Beam under a Concentrated Load

Table 10 Longitudinal Variation of Stress Ratio for Two Span Continuous Beams with ξ = 1/3 and a_2/a_1 = 2 under Uniform Load

beam	l, r_i	x		.0833	.1667	.3	.3333	.3667	.5	.75
box beam	60, 4.86	S	η_1^r	0	0	.001	.058	.001	0	0
			Eq. (21)	∞	.98	1.00	1.21	1.00	1.01	1.01
			SHLAG	-72	1.02	.98	1.21	.98	1.00	1.01
	16, 4.86	S	η_1^r	0	0	.048	.216	.047	0	0
			Eq. (21)	∞	.74	1.14	1.74	1.16	1.19	1.06
			SHLAG	-571	.74	1.16	1.74	1.18	1.19	1.06
	16, 24.3	S	η_1^r	0	0	.065	.428		0	0
			Eq. (21)	∞	.69	1.20	2.51		1.23	1.07
			SHLAG	-744	.77	1.08	2.53		1.18	1.07
I beam	60, 4.86	S	η_1^r	0	0	.002	.054	.002	0	0
			Eq. (21)	∞	.98	1.00	1.19	1.00	1.02	1.01
			SHLAG	-74	1.01	.98	1.20	1.00	1.00	1.01
	16, 4.86	S	η_1^r	0	.001	.070	.202		.001	0
			Eq. (21)	∞	.68	1.21	1.67		1.25	1.08
			SHLAG	-594	.72	1.18	1.69		1.21	1.07

Table 11 Longitudinal Variation of Stress Ratio for Two Span Continuous Beams with ξ = 1/4 and a_2/a_1 = 3 under Uniform Load

beam	l, r_i		x	.10	.22	.24	.25	.26	.29	.5
box beam	60, 4.86		η_1^r	0	.001	.018	.068	.018	0	0
		S	Eq. (21)	.99	1.00	1.05	1.18	1.05	1.00	1.01
			SHLAG	.99	.99	1.08	1.19	1.07	.97	1.01
	16, 4.86		η_1^r	0	.074	.170	.256	.165	.049	0
		S	Eq. (21)	.91	1.17	1.43	1.66	1.44	1.13	1.08
			SHLAG	.84	1.19	1.48	1.65	1.49	1.10	1.09
	16, 24.3		η_1^r	0	.074	.267	.509	.262	.039	0
		S	Eq. (21)	.89	1.16	1.69	2.34	1.71	1.17	1.09
			SHLAG	.80	1.14	1.75	2.34	1.75	.99	1.10
I beam	60, 4.86		η_1^r	0	.003	.022	.064	.023	.001	0
		S	Eq. (21)	1.00	1.01	1.06	1.17	1.07	1.00	1.00
			SHLAG	.99	1.00	1.08	1.18	1.08	.98	1.01
	16, 4.86		η_1^r	.002	.093	.175	.239	.170	.062	0
		S	Eq. (21)	.89	1.21	1.43	1.60	1.44	1.16	1.11
			SHLAG	.85	1.20	1.45	1.61	1.46	1.13	1.10

$$\eta_2' = \frac{1}{2}[(1 + \frac{2/3-.475}{2/3}) \times .0003 + (1-\frac{2/3 - .475}{2/3}) \times .0007] = .0004$$

Substituting in Eq. (34), we obtain

$$\eta = \frac{.12469 \times .0290 - .06417 \times .0025 - .05806 \times .0004}{.12469 - .06417 - .05806} = 1.39$$

Some results are shown in Table 12, from which the following conclusions for calculating the stress ratio of a multiple span continuous beam can be made:

1. In most practical structures, only one reaction nearest to the section under consideration needs to be considered, thus Eq. (34) can be further simplified.
2. For sections near midspan, the effect of interior reactions can be neglected, thus Eq. (34) can be simplified to:

$$\eta = M_w \eta_w / M^* \tag{35}$$

3. To estimate the general features of the shear lag effect in a continuos beam with multiple spans, only the stress ratio at sections near midspan and at interior supports are required and can be calculated by the simple formulas Eqs. (17), (7) and (18).

4.5. Beams under Prestress Load

The following two empirical formulas can be used to calculate the stress ratio for beams under prestress load.

Under a pair of axial loads at two ends of a simple beam, η can be computed for any section by:

for box beams,

$$\eta = 1.11\, r_a^{.96}\, l^{-.08} \exp(-\alpha x)$$

$$\alpha = \begin{cases} 1.27l\,[1 + .189\, l^{-.8}\,(2.8 r_a^{.64} - 1)] & (r_a \geq .2) \\ 1.27l & (r_a < .2) \end{cases} \tag{36}$$

for I beams,

$$\eta = \eta_o \exp(-\alpha x)$$

$$\eta_0 = (.194 + .0009l)\{\exp[(.99 - .0045l)\ln(5r_a)] - 1\} + 1.43$$

$$(\text{if } \eta_o < 0, \text{ let } \eta_o = 0) \tag{37}$$

$$\alpha = (3 + 1.19l - 64/l^4) r_a^{1.15} / \eta_o \quad (l \geq 4)$$

Eq. (37) can be also used for T beams, provided r_a is replaced by $(r_i + r_a)/2$.

Under a partial uniform load symmetric about midspan (Fig. 8), at the midspan section,

$$\eta = \eta_w + (\eta_o - \eta_w) \exp(-\alpha v / l)$$

$$\alpha = c_4 r_i^{.75}\, l^{-.54} / \eta_o \tag{38}$$

where, η_w, η_o can be found from Eq. (17), (7), and (18) by letting $x = \xi = .5$ and

Table 12. Longitudinal Variation of Stress Ratio for Continuous Beams with Three Equal Spans and r_i = 4.86 under Uniform Load

l	x	.1	.2	.3	.3233	.3333	.3433	.4	.5
12	η_w	.052	.044	.037	.035	.035	.034	.032	.029
	η_1^r	0	.004	.099	.209	.288	.205	.034	.0015
	η_2^r	0	0	0	0	0	0	0	.0015
	S Eq. (34)	1.28	1.50	1.67	2.41	2.76	2.34	.42	2.39
	SHLAG	1.48	1.59	1.79	2.51	2.76	2.47	.84	2.33
45	η_w	.0041	.0034	.0029	.0028	.0027	.0027	.0025	.0022
	η_1^r	0	0	.0027	.0281	.0768	.0281	.0001	0
	η_2^r	0	0	0	0	0	0	0	0
	S Eq. (34)	1.02	1.04	.98	1.21	1.54	1.21	.87	1.10
	SHLAG	1.04	1.06	.89	1.27	1.54	1.27	1.06	1.01

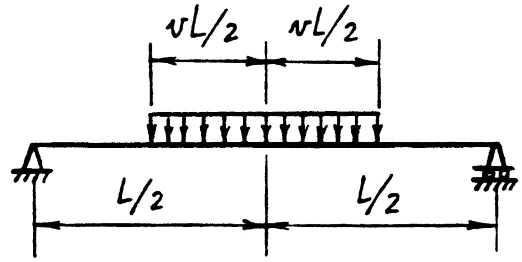


Fig. 8. Simple Beam under Partial Uniform Load Symmetric about Midspan

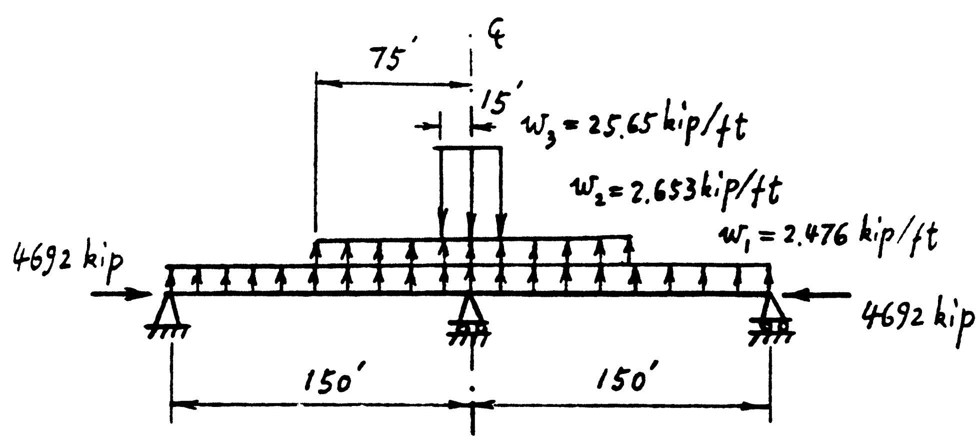


Fig. 9. Prestress Load of Structure B in Ref. [2]

Table 13 Two Equal Span Continuous T Beam under Prestress Load

x	η_m^i	η_n^i	$G/(e-c)$	S_t		σ_s^t (Mp$_a$)*		
				Eq. (39)	SHLAG	Eq. (39)	SHLAG	Ref. [3]
.2375	.485	.041	-2.91	1.18	1.17	-2.56	-2.51	-2.68
.4875	1.500	0	1.733	-1.28	-1.00	.87	.68	.52
.4933	1.663	0	1.627	-1.95	-1.82	1.21	1.12	
.5	1.851	0	1.520	-2.94	-3.06	1.62	1.67	

*Stress in the middle of the top flange.

Table 14 Fundamental Formulas for Simple Beams

Loading Case	No. of Formula
uniform load	(17)
concentrated load at midspan	(29), (30)
concentrated load at any point	(32)
partial uniform load symmetric about midspan	(38)
a pair of axial loads at the two ends	(36), (37)

$$c_4 = \begin{cases} 1.74 \text{ (for box beam)} \\ 1.60 \text{ (for I beam)} \end{cases}$$

The total prestress load from a tendon profile made up of parabolas can be considered as a combination of axial load N and uniform loads w. For I and box beams, under the combination of bending moment M^* and axial force N^*, η is equal to

for the top flange,

$$\eta_t = \frac{-\eta_m + \eta_n G/(e - c)}{-1 + G/(e - c)} \tag{39}$$

for the bottom flange,

$$\eta_b = \frac{\eta_m + \eta_n G/(e' + c)}{1 + G.(e' + c)} \tag{40}$$

where η_m and η_n = complementary stress ratio under bending and axial loads, the latter can be found from Eq. (36) or (37).

For T beams, Eq. (39) should be replaced by

$$S_t = 1 + \eta_t = \frac{-\psi_m(1 + \eta_m^i) + (1 + \eta_n^i)G/(e - c)}{-1 + G.(e - c)} \tag{41}$$

in which η_m^i and η_n^i = complementary stress ratio of the equivalent I beam under bending and axial load, ψ_m can be calculated by Eq. (14).

As the first example, a continuous T beam with two equal spans used as a design example in Ref. [3] is analyzed. The loading and dimensions of the beam are:

Uniform distributed net load $\quad w = 33.72$ KN/m
Prestressing axial force at two ends $\quad N = -4700$ KN
$b = 4.64$ m, $l/b = 8.62$, $h_w/b = .077$, $h/b = .0513$, $d/b = .215$.

The results are shown and compared with Ref. [3] in Table 13.

A box beam, structure B of Ref. [2], can be used as the second example. Under the prestress load (Fig. 9), the η due to each partial uniform load can be found by Eq. (38). Under load w_2, for instance, $v/l = .5$ and $\eta = .0052$ can be obtained. The bending moment in the basic structure (a simple beam), due to w_2, is equal to $-.2488L^2$. Thus, η_m can be found by superposition (at the center support section):

$$\eta_m = \frac{(-.310 \times .0046 - .2488 \times .0052 + .609 \times .0446 + .0442 \times .119)L^2}{(-.310 - .2488 + .609 + .0442)L^2}$$

$$= .310$$

from Eq. (36), $\eta_n \approx 0$. Substituting η_m, η_n and $G = -2.11$ in Eq. (39), we can obtain $\eta_t = .141$ and then $\sigma_t = 1.141 \times (-1.99) = -2.28$ ksi ($\sigma_t = -2.20$ and -2.18 ksi by SHLAG and Ref. [2] respectively.

5. Summary and Conclusion

Empirical formulas for the stress ratio and effective flange width of a simple I or box beam under various loading cases have been derived and the fundamental formulas are listed in Table 14. All formulas in Table 14 can be used for any section of the beam except Eq. (38), which can only be used for the midspan section.

Formulas for continuous beams are obtained from the formulas in Table 14 by superposition. For convenience, numerous formulas for special cases have also been given.

In section 3, some simple and essential formulas are presented. The most important features of the shear lag effect for a simple or continuous beam can be estimated quickly by these formulas. These should be useful for design problems usually encountered.

For a T beam, the shear lag calculation can be done by its equivalent I beam as discussed in Section 2.

Several typical numerical examples have been given to show how to use the formulas and to check the accuracy by comparison with the results by the method of Ref. [1] and other authors. It is shown that the empirical formulas presented in this study are accurate enough for design problems and useful for research studies on the shear lag effect in beams as affected by various basic parameters.

6. References

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Appendix

All formulas of this study are derived empirically by matching with the results calculated by the computer program SHLAG of Ref. [1]. For example, for the case of a concentrated load at midspan of a simple I-beam, the formula for η_o can be derived as follows. If the calculated data by SHLAG for l = 4, 8, 24 and r_i = .6, 1.19, 2.85, 5.70, 9.72, 10.83 are plotted using r_i as the horizontal coordinate, a family of similar $r_i - \eta_o$ curves are formed as shown in Fig. A1. If we use $x = r_i^{0.3}$ as the horizontal coordinate, the above curves can be transformed to a family of straight lines converging to a point (.65, 0), except for the case of l = 2, in which the curve slightly deviates from a straight line. Therefore, for $r_i \geq 0.6$, η_o can be expressed approximately as:

$$\eta_o = 3(r_i^{.3} - .65)$$

All other formulas can be derived similarly, although for some cases the deviation is somewhat more complicated.

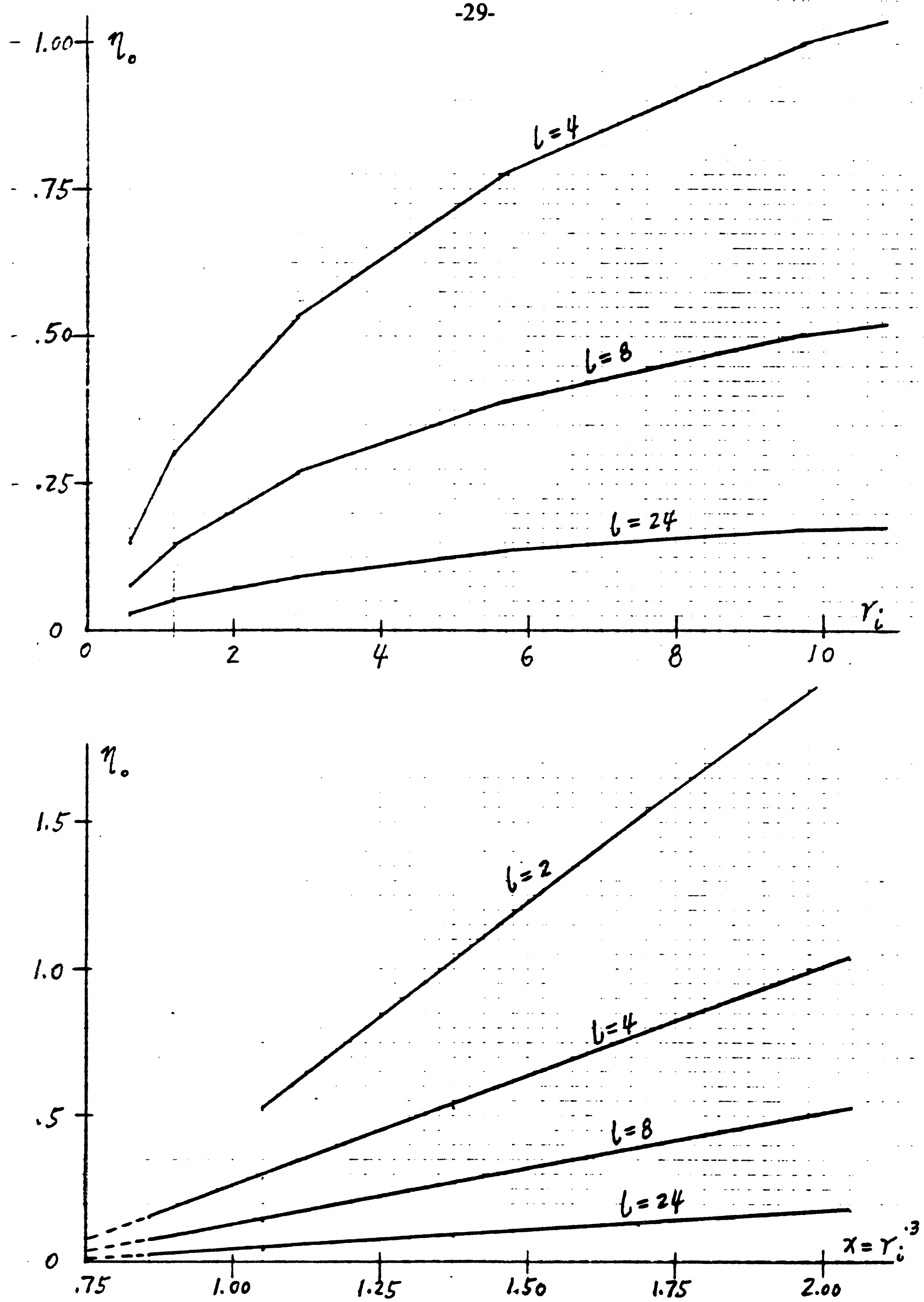

Fig. A1. $\eta_o - r_i$ **Curves for Simple I Beam**