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ABSTRACT OF THE DISSERTATION

Essays on the Economics of Networks

by

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Doctor of Philosophy in Economics

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Professor Simon Adrian Board, Chair

This dissertation contains three chapters on how economic networks affect various market situations. Broadly, they cover contracting and monopoly pricing in the presence of economic networks.

The first chapter considers a principal, many agents contracting problem. Agents sit on a network of complementarities. That is, the effort of one agent affects the value of effort for those with whom he connects. Given this structure on effort, I characterize the first best contract. This contract induces efforts that reflect the agents Bonacich centrality in the symmetrized network. I then consider a variety of bilateral contracts, and compare their values for the principal. First, I consider bilateral forcing contracts. These contracts induce less effort per agent than the first best contract. Agents' effort distortions depends on their bibliographic coupling. I show that it is this novel measure that drives effort down for certain agents. Networks with high total bibliographic coupling have a large profit gap from first to second best forcing contracts. I compare these contracts to bilateral linear contracts, and show that linear contracts outperform the forcing contracts. Finally, I show that base and bonus contracts are profit maximizing for the principal, and implement first best.

The second chapter considers a monopolist who introduces a new durable good to a base of consumers who are connected on a network of communication. Consumers are initially unaware of the product, and must learn about its existence through their neighbors. Each consumer who purchases informs a group of neighbors, and the information flows through consumers as a branching process. The monopolist commits to a dynamic

price path on the infinite horizon. I find that though consumers are fully strategic, the monopolist finds it optimal to serve the entire consumer base infinitely often, which implies a sales structure. I then derive the optimal price path for a simplified model of two agents, and derive comparative statics.

The third chapter considers a monopolist who sells to a consumer base that is largely unaware of the product. The monopolist spreads the information of the product to consumers by the past purchasers. I assume that the monopolist knows the exact network structure on which consumers live, and sets prices based off of consumers positions and the aware set of consumers. I consider three different pricing strategies. First, I consider a setting where the monopolist can price discriminate based on the consumers' network position. In this case I am able to find which consumers are important to the information flow. Consumers who are aware early get a discount, along with agents who are critical to the information flow. If there are consumers who can only be reached through one consumer purchasing, this consumer is offered a discounted price. I see that these ideas follow through to the single priced monopolist case, where prices fluctuate if many critical agents exist. Finally, I consider the optimal mechanism, where the monopolist can price discriminate based off of network position and price. In this case the monopolist can find the optimal flow of information and implement it.

The dissertation of Alexander Graupner is approved.

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2020

To my parents and my bubby

TABLE OF CONTENTS

1 Contracting with Local Complementarities **1**

1.1 Introduction 1

1.2 The Model 6

1.3 First Best 9

1.4 Bilateral Forcing Contracts 12

1.5 Bilateral Linear Contracts 20

1.6 Optimal Bilateral Contracts 26

1.7 Conclusion 28

1.8 Appendix 30

2 Durable Good Pricing with Interdependent Demand **37**

2.1 Introduction 37

2.2 The Model 42

2.3 Optimal Monopoly Pricing 46

2.4 Two Type Example 47

2.5 Conclusion 50

2.6 Proofs 51

3 Good Allocation on Social Networks: A Mechanism Design Approach 58

3.1 Introduction 58

3.1.1 Literature Review 60

3.2 Model 61

3.2.1 The Network of Communication 63

3.2.2 The Solution Strategy 66

3.3 Price Discrimination 67

3.4 Single Price Monopolist 76

3.4.1 The Strategy of Consumers 82

3.4.2 Active Set Updating and Pricing 84

3.5 Optimal Mechanisms 89

3.6 Conclusion 94

References **96**

LIST OF FIGURES

1.1 Three Agent line 4

1.2 4 agents, 3 links. 12

1.3 3 Agent Star Networks 16

1.4 Two Agent line 23

2.1 Rocket League Prices over time 38

3.1 Timing of Word of Mouth 62

3.2 A Network of Communication 64

3.3 The Potential Directions of Communication 64

3.4 Ex-Post Flow When All Purchase 65

3.5 Ex-Post Information Flow When B Doesn't Purchase. 65

3.6 Simple Line 68

3.7 Competing Word of Mouth 69

3.8 Arbitrarily long line 71

3.9 Critical consumers 75

3.10 Individual Word Of Mouth Value 78

3.11 Joint Word of Mouth Effect 78

3.12 Joint Word of Mouth Effect 79

3.13 Critical consumers 84

3.14 Independent Lines 88

3.15 Mechanism Design Example 90

3.16 Overlapping Network Effects 93

LIST OF TABLES

1.1 Efforts 17

1.2 Profits 17

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VITA

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CHAPTER 1

Contracting with Local Complementarities

1.1. Introduction

The endeavors taken by collaborating firms exhibit a complementary aspect in many industries. For example, in R&D industries such as information technology and biotechnology, positive inter-firm externalities are present. In these industries, firms form R&D networks to share information and technologies. If one firm develops a new process or technology, the other firms in their network benefit from access to these innovations. Collaboration is particularly prevalent in the information technology industry, as Hagedoorn and Schakenraad (1990) report over 200 collaborative agreements made every year during the 1980's.

Outside agencies contract with firms in these industries to pursue ventures on behalf of their clientele. In biotechnology, Contract Research Organizations (CROs) develop drugs and vaccines on the behalf of major pharmaceutical companies such as Pfizer. Similarly, the government contracts with information technology companies such as IBM to develop new technologies for use in the military, legislative branches, and for general acquisition. For example, in 2014 the federal government awarded 60 contracts across 33 IT firms to create "The Federal Marketplace at HealthCare.gov" as part of the Affordable Care Act.

This paper studies contracts in a single principal, multiple agent setting where the agents reside on a network of complementarities. In his seminal work, Segal (1999) introduced the idea of externalities on trade on a complete network. In this paper I consider a setting where the externalities between agents are asymmetric. Agents share different connections of varying strengths, which affect the magnitudes of the externalities they exert on each other. This relates to the idea that when a biotechnology firm works to develop a new vaccine, their relations with other firms will vary in importance. A firm working on a vaccine may have a certain agreement with another firm working on a similar project, while having a different relationship with a third firm who has some technology relevant to production. As in Segal, the contract written for one agent affects the payoffs and actions of other agents. For example, a government contract with one vaccine developer will also benefit other firms in the biotechnology space. Firms with whom the government does not contract may also pursue ventures in this area, in particular if they have agreements with other firms who themselves contract with the government. These different levels of interaction are captured by the model.

I show how network externalities affect various types of bilateral contracts. I show the inefficiencies that arise from fixed price (also known as forcing) contracts, which are preferred by the United States government.[1] Importantly, how efforts and profits are distorted from first best depends on the underlying network structure and the spectral properties of the network. In particular, effort is distorted downward, and the network's "bibliographic coupling" plays a vital role in determining which agents have their effort distorted downward. This is a novel feature to the model. This measure also determines from which networks the principal suffers the greatest loss in profits from first best.

I show that the optimal linear contracts, which allow agents to adjust effort, outperform the optimal bilateral forcing contracts. Finally, I construct a set of bilateral contracts that recovers the first best efforts and profits. The value of the contracts to the principal depends on the size of the endogeneous outside options for the agents, and first best can be achieved if the contracts force these outside options to zero.

The model, as outlined in Section 1.2, considers a single principal contracting with N agents who live on a publicly observed network of complementarities. In the first stage, the principal offers contracts to all agents simultaneously, and agents accept or reject their contracts simultaneously. In the second stage, the set of accepted contracts is publicly observed, and then agents choose effort levels simultaneously. I assume that efforts, contracts, and contract acceptance or rejection are all observable. This would correspond closely to government contracting, where information on contracts is public

[1] See the Federal Acquisition Regulation,
https://www.acquisition.gov/sites/default/files/current/far/pdf/FAR.pdf

knowledge.

In Section 1.3, I consider the first-best. This benchmark relates the optimal efforts and profits to the Bonacich centralities of the "symmetrized" network. The symmetrized network takes a network with links reversed and combines it with the original network. This implies that the direction of links between agents is not important. Only the presence of an externality between agents matters to the principal, and the direction of that externality is not important. Given this result, I characterize how optimal effort levels depend on the underlying network structure, and explore which network structures are best for the principal.

In Section 1.4, I study bilateral contracting. Motivated by the fixed price contracts commonly used by the government when contracting with R&D firms, I consider forcing contracts which prescribe an effort level and offer a payment for that effort. Though these contracts are the United States government's preferred structure, they exhibit large inefficiencies due to the high outside options that they induce. When an agent thinks about rejecting her contract, she benefits from the high level of effort of all the other agents who accept their contracts (in equilibrium); this means her outside option is valuable and makes it expensive to persuade her to accept the contract. In order to reduce these rents, the contracted effort is biased down.

For small values of network interaction intensity, the agents who have effort distorted downward the most are those with the highest "bibliographic coupling". These are agents who have an effect on other agents who are themselves affected by many. Intuitively, agents who are affected by many other agents have a large outside option, as they internalize the effort of all these others. The outside options are quadratic, and therefore the principal optimally keeps those highly affected agents from having too high an outside option by lowering the efforts of those connected to them.

I show how the network structure allows for a richness in identifying the important agents to the contracting problem. A novel aspect of the model is that the important agents of this model are not necessarily those most central by traditional measures. Consider the example exhibited in Figure 1.1.

In this line, agent 2 internalizes the efforts of agents 1 and 3. We will find that in the

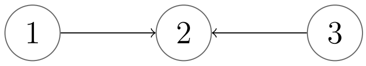

Figure 1.1: Three Agent line

first best, agent 2 will be prescribed the highest effort level. Naturally, agent 2 is central in the network. But when we move to bilateral contracts, we see that agents 1 and 3 are the vital ones. They get their efforts distorted downward by greater magnitude that agent 2, as they have high levels of bibliographic coupling.

Additionally, I relate profits to the eigenvalues of the adjacency matrices that represent the networks. As a result, I am able to derive properties on which networks are better (in terms of profits) for the principal, and which networks cause the greatest loss from restriction to bilateral forcing contracts. These spectral properties are related to the maximal and average degrees in the network of interaction, and the maximal profit that can be achieved from a network is bounded by the maximal degree on that network. Intuitively, when agents have fewer connections, there are less profits available. At the same time, the loss between the first best and the optimal bilateral contract also shrinks.

In Section 1.5, I study another contract common in practice: linear contracts. I characterize these contracts and find they perform better than the optimal bilateral forcing contracts. This is because when an agent rejects his contract, the other agents are free to adjust their effort downward, taking into account that the rejecting agent has lower incentive to put forth effort. This adjustment lowers the outside option of the agent below that of the bilateral forcing contracts, which is in turn better for the principal.

In Section 1.6, I show that the principal can profit maximize and achieve first best in the bilateral contracting case by using “bonus” contracts. These contracts offer a base wage and a bonus amount that is given only if the agent takes their first best effort. Agents are able to punish each other sufficiently for not accepting the contracts, and there is no profitable deviation. Though the government does not prefer these types of contracts over forcing contracts, some contracts written by Pfizer to their CROs have a flavor of this structure through milestone payments, implying that these firms approach the first best outcome.[2]

This works contributes to a variety of literatures. Since the seminal work by Holmstrom (1982), economists have analyzed contracting with teams. Segal (1999) and Segal and Whinston (2003) considered contracting with general externalities. Segal finds that, when there are externalities on "nontraders" who do not sign a contract, inefficiencies arise. I confirm this finding, and show how the size of the inefficiency depends on the contract written and the structure of the underlying network. In Segal's papers, if one player rejects a contract, they get zero units in trade. This is natural in settings where the principal is trading some physical good with the agents. In this paper an agent will still exert effort when they do not accept a contract, as they have a benefit dependant on the other agents' actions. It is the tradeoff between inducing high effort in the contract and the endogeneous outside options which drives the principal's contracting problem. I also contrast with Segal by assuming an explicit network structure that allows me to study which agents are more important in terms of effort and distortions. With this explicit network structure I am able to characterize which agents have their effort distorted downward more, based on differing network measures.

I build on the existing literature on linear quadratic network games (See Jackson and Zenou (2014) for a review). Choices by firms and consumers on networks have been analyzed through the lense of linear quadratic games for much of the last decade. Ballester, Calvo-Armengol, and Zenou (2006) consider key-player analysis in linear quadratic games, relating the key player in the network to their intercentrality. Bloch and Querou (2013) consider pricing the network effects that agents have on each other, finding higher prices for those with higher degree, and for those who have many neighbors with small degree. These papers relate optimal solutions to measures of Bonacich centrality. I find similar results, but to my knowledge this is the first paper that relates the optimal solutions to the bibliographic coupling of the players.

There is a growing literature of contracting when agents live on an explicit network structure. Shi and Xing (2019) and Jadbabaie and Kakhbod (2019) both consider models

[2]Pfizer writes milestone payments into contracts with many of their CROs. See https://www.sec.gov/Archives/edgar/data/1609351/000119312514457223/d776249dex1018.htm for an example with the agreement between Pfizer and Spark Therapeutics, INC.

of screening agents through menus of contracts for different specifications of incomplete information. These papers don't allow for endogenous outside options. In Shi and Xing, there is incomplete information on an agent's position in the network, while in Jadbabaie and Kakhbod there is incomplete information on the level of interaction between the agents. There are other papers on changing the incentives of agents in networks. Belhaj and Deroian (2019) consider a contracting environment with fixed budget where the principal contracts with a subgroup of the network, while Galeotti, Golub, and Goyal (2017) consider changing the marginal benefits of agents to solve different objectives. These papers are interested in which agents to incentivize, while my paper considers contracting with the entire population of agents.

Finally, the literature on R&D networks and their inter-firm collaborations relates to my analysis. A vast literature is reviewed in Goyal and Moraga-Gonzalez (2001), where they also consider how these collaborations come about, in a two-stage game where firms first form links, and then compete. In contrast, I take the network as exogeneous, but consider a third-party (principal) who contracts with these agents.

1.2. The Model

There is a single principal and N agents.[3] The N agents live on a directed, weighted network G, which I additionally use to represent the matrix of interaction between the agents. The network is assumed to arise exogeneously and be public information, with agents' identities on the network being known to each other and the principal. An element of G, $g_{ij} \in [0, 1]$, is equal to the weight of the effect that agent j has on agent i. I take the regular convention that $g_{ii} = 0$ for all i.

Effort e_i taken by each agent i is observable. The principal and agents play a two stage game. In the first stage, the principal publically offers a contract, specified by $t_i(e_i)$ (sometimes abbreviated as t_i), to each agent simultaneously. The agents choose to accept or reject their contracts simultaneously. Then, in the second stage, the set of contracts that were accepted are publicly viewed by all players. Agents choose their effort levels

[3] I slightly abuse notation, letting N represent both the number of agents and the set of agents.

simultaneously, and utilities and profits realize.

I consider three specifications of contracts. First, I consider forcing contracts, for which $t_i(e_i) = t_i$ if e_i is the prescribed level, and $-\infty$ if not. I contrast these contracts with linear contracts which give payment of the form $t_i(e_i) = \alpha_i + \beta_i e_i$. The third and final contract I consider is the base and bonus contract. This contract is of the form $t_i(e_i) = \alpha_i + \gamma(e_i)$, where $\gamma(e_i)$ gives a payment if the prescribed effort level is taken, and is equal to zero if not.

The utility of an agent depends on the contract offered to him, his benefit from the complementarities, and his private cost. I assume that the benefit for the agent comes strictly from the interaction with his neighbors, and is equal to $\theta e_i \sum_{j=1}^{N} g_{ij} e_j$. This term represents a benefit from collaboration with neighbors, and can also be thought of as a cost reduction. Importantly, I abstract away from any benefit the agent may receive strictly from his effort alone. This corresponds to the complementarity benefit acting as a cost reduction from collaboration, but also applies to various applications. For example, in biotechnology, a firm that develops a certain technology to help produce a vaccine may have no value for their development unless another firm with whom they collaborate is working on the underlying biological structure of the vaccine.

The parameter $\theta > 0$ measures the level of intensity of the network interaction, which is commonly known and the same for every agent. To focus on the network effect, I assume a convex, quadratic cost. This assumption places the model in the world of linear quadratic games. The utility of an agent who signs a contract is therefore

$$U_i(e_i) = t_i(e_i) - \frac{e_i^2}{2} + \theta e_i \sum_{j=1}^{N} g_{ij} e_j \tag{1.1}$$

I make the following assumption on the network intensity, which guarantees a finite and positive trade profile in the first and second best cases.

Assumption 1.2.1. *Let λ_1 be the spectral radius (i.e. largest eigenvalue) of the matrix $G + G^T$. Then $\theta < \frac{1}{\lambda_1}$.*

This assumption is similar to one common in the literature on network games. For example, Ballester et al. (2006) assume that θ is less than the inverse of the spectral radius of the network G. This assumption is necessary to guarantee existence of equilibrium in quadratic network games. I instead require that θ be bounded by the symmetrized network $G + G^T$, as it is this network which is important to much of the analysis. Note that this assumption encompasses the one used in Ballester et al.

The principal wishes to maximize the sum of the efforts less the transfers that she must pay. That is, for transfers $\{t_i(e_i)\}$, the principal's profit function is

$$\Pi = \sum_{i=1}^{N} e_i - t_i(e_i) \tag{1.2}$$

As a benchmark, consider a situation where the principal does not offer any contracts to the agents. Then, each agent maximizes his benefit minus his cost[4], so that he solves the following program:

$$\max_{e_i} \quad \theta e_i \sum_{j=1}^{N} g_{ij} e_j - \frac{e_i^2}{2} \tag{1.3}$$

For each agent i, the solution to this problem is

$$e_i = \theta \sum_{j=1}^{N} g_{ij} e_j \tag{1.4}$$

If an agent does not have a contract, he still wishes to exert effort if his neighbors in the network are exerting effort, as he receives positive value out of the effort through the complementarity aspect. I abstract away from any value of effort that the agent may get privately, and only consider the value in the complementarity aspect as to isolate the effect of the network.

Let E be a $N \times 1$ column vector of efforts of the agents. Then, (1.4) stacks to be

$$E = \theta G E \implies (\mathbb{I} - \theta G)E = \mathbf{0} \tag{1.5}$$

The only solution to this system is $E = \mathbf{0}$. Therefore, the utility of each agent in the case of no contracting is also 0. Thus, to induce effort from the agents, the principal must offer transfers for that effort. This can be thought as the agent's failure to internalize the

[4]This is analogous to the agent minimizing a cost function $C(e_i) = \frac{e_i^2}{2} - \theta e_i \sum_{j=1}^{N} g_{ij} e_j$

externalities they enact on others when there are no incentives given by the principal. Without incentives to take effort toward a venture, none of the agents find it profitable to take effort. This equilibrium is the outcome if no contracts are accepted.

1.3. First Best

The first best is derived by maximizing the sum of the principal's profit in Equation 1.2 and the agents' utilities in Equation 1.3. That is, I solve

$$\max_{\{e_i\}_{i\in N}} \quad \Pi + \sum_{i=1}^{N} U_i(e_i) = \sum_{i=1}^{N} e_i - \frac{e_i^2}{2} + \theta e_i \sum_{j=1}^{N} g_{ij} e_j \tag{1.6}$$

Written in matrix form, this is

$$\max_{E} \quad \mathbf{1}^T E - \frac{E^T E}{2} + \theta E^T G E \tag{1.7}$$

Proposition 1.3.1. *The first best effort profile is* $E = (\mathbb{I} - \theta(G + G^T))^{-1}\mathbf{1}$

Proof. Appendix. □

The principal can implement the first best through a multilateral forcing contract. This contract specifies for each agent i an effort level e_i and a transfer t_i which the agent receives only if he takes the prescribed effort level. If the agent takes any other effort level, he gets a payment (and hence utility) of $-\infty$. This specification guarantees agents will not deviate from their prescribed effort when in the contract. The payment of $-\infty$ can be thought of as a value of not following the contract, and the effects that this may lead to. Suppose, for example, that an R&D firm were contracted by the government to develop a new product or implement a new process. If the firm were to break this contract, they would not only lose the payment for completing the task, but the government would also likely not hire them for future projects. Therefore, the $-\infty$ payment can be thought of a dynamic loss from breaking the contract.[5]

The principal's problem is as follows:

[5] In fact, as is see when deriving the contract, a payment of $-\infty$ is not necessary for the contract. A large negative payment $-C$ such that no agent deviates once they sign the contract also implements the first best in the multilateral contract.

$$\begin{aligned}\Pi = \max_{\{(t_i,e_i)\}_{i\in N}} \quad & \sum_{i=1}^{N}(e_i - t_i) \\ \text{s.t.} \quad & t_i - \frac{e_i^2}{2} + \theta e_i \sum_{j=1}^{N} g_{ij}e_j \geq 0, \quad \forall i \in N\end{aligned} \tag{1.8}$$

That is, the principal maximizes the sum of the efforts minus the transfers paid, such that each agents individual rationality (IR) constraint holds. Agents must be willing to accept, rather than reject, the contract. When an agent rejects the contract, all contracts are void, and all efforts (and therefore utilities) are zero. Thus the outside option is zero if an agent rejects a contract. To solve this problem, I note that at the profit maximizing values, the agents' IR constraints should all bind. Substituting out the transfers gives the unconstrained problem:

$$\Pi = \max_{\{e_i\}_{i\in N}} \quad \sum_{i=1}^{N} e_i - \frac{e_i^2}{2} + \theta e_i \sum_{j=1}^{N} g_{ij}e_j \tag{1.9}$$

Or, in matrix form,

$$\Pi = \max_{E} \quad \mathbf{1}^T E - \frac{E^T E}{2} + \theta E^T G E \tag{1.10}$$

Proposition 1.3.2. *The optimal multilateral forcing contract implements first best and specifies an effort profile:* $E_{FB} = (\mathbb{I} - \theta(G + G^T))^{-1}\mathbf{1}$, *Profits:* $\pi_{FB} = \frac{\mathbf{1}^T E_{FB}}{2}$, *and Transfers:* $T_{FB} = \frac{E_{FB}}{2}$

Proof. Appendix. □

Proposition 1.3.2 allows us to analyze how effort and profits depend on the underlying structure of the network. To analyze the proposition, I recall the definition of Bonacich centrality:

Definition 1.3.1. (Bonacich Centrality) *Given a matrix M, parameter* θ*, and vector of ones* $\mathbf{1}$*, Bonacich centrality is defined as*

$$(\mathbb{I} - \theta M)^{-1}\mathbf{1} =: [b_1, b_2, ..., b_n]^T$$

where b_i *is the Bonacich Centrality of agent i.*

The first-best effort is the Bonacich Centrality of the *symmetrized* network $G + G^T$. The symmetrized network "undirects" the underlying network G. The effort levels dependence on this Bonacich centrality comes from the linear quadratic aspect of the game. The linear best responses of the agents give a clean characterization of the optimal actions. Since the first best effort allocations depend on the symmetrized network, it does not matter if there is a link from agent i to agent j, or from agent j to agent i; all that matters is the existence of such a link. This is in contrast to linear quadratic games where agents do not internalize their effect on others in the network. When the principal, acting as an outside entity, views the network, she fully internalizes all the links, and therefore prescribes efforts based off of the symmetrized network, for each link has a value to the group as a whole.[6]

A clear implication is that the network that maximizes the sum of the Bonacich centralities in the symmetrized network is profit maximizing for the principal. This is is because the principal splits the profits with the agents equally, leaving a profit of $\pi_{FB} = \frac{\mathbf{1}^T E_{FB}}{2}$ for themselves. This is another feature of the linear quadratic nature of the game. This implication is illustrated in the following example.

Example 1.3.1. *4 Agent 3 Link Networks*

Consider 4 agents, with 3 links between the agents. There are 4 networks (up to permutation on nodes and links) that can be formed between these agents.They are found in Figure 1.2. Bonacich Centralities of each agent in the symmetrized network are given next to the nodes.

The profit (and therefore welfare, as agent utilities are zero) for network A is $\pi_A = \frac{2+3\theta}{1-3\theta^2}$*, for network B is* $\pi_B = \frac{\theta+2}{1-\theta-\theta^2}$*, for network C is* $\pi_C = \frac{2+3\theta-3\theta^2}{1-5\theta^2}$ *and for network D is* $\pi_D = \frac{2-3\theta}{1-3\theta+2\theta^2}$*. The profit from the star network A is higher than that of the line network B, while the highest profit comes from network C, which exhibits a partially hierarchical structure. As we look across these different networks, it is generally better to have an asymmetry in the centrality of the agents. The principal generally prefers links to be*

[6] Results based off of this measure are not uncommon in the literature, as Ballester et al. (2006) and Bloch and Querou (2013) also consider how an outside entity interacts with a network, finding results based on $G + G^T$ as well.

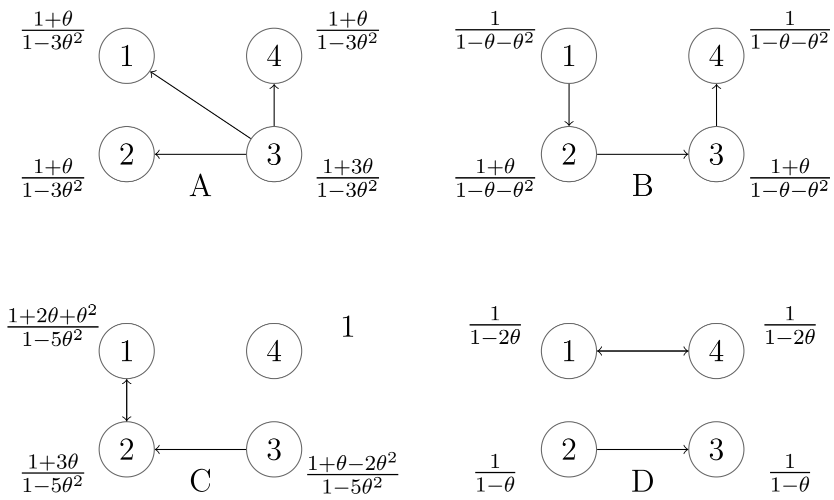

Figure 1.2: 4 agents, 3 links.

focused on agents to enhance the complementarity, such as agent 3 in network A, and agent 2 in network C. While the direction of the influence does not matter, it is valuable to have agents affect each other in both directions, as exhibited in network C.

Multilateral contracts, while optimal to the principal, are not realistic in a variety of settings. When Pfizer contracts with multiple CROs, they can not condition one agreement on the agreements written with other CROs. This could lead to anticompetitive behavior between the CROs. These firms could collude against the Pfizer for better terms. Similarly, when the government contracts with information technology firms, these firms are often competitors, meaning that they can not condition a contract on the other contracts, for this may induce agreements across firms that are active in the same industry, arising antitrust issues. In both of these cases the principal would write the contracts for each firm
independently.

1.4. Bilateral Forcing Contracts

I assume that the principal is restricted to contracting bilaterally through forcing contracts. These contracts are a favorite of the federal government, and were the focus of the

Obama era overhaul on government contracting practices. When the federal government contracted to create "healthcare.gov", the healthcare website under the ACA, they awarded these fixed price forcing contracts to several separate contractors. They award these contracts to firms and require them to succeed in various tasks, such as setting up the Federal Exchange. Though I show that these contracts are suboptimal, their vast use by the United States government makes them of substantial interest.

A forcing contract is defined as a payment t_i and a prescribed effort level e_i. The agent receives the transfer t_i if and only if he takes effort level e_i and receives $-\infty$ otherwise, being effectively tied to this effort level if he accepts his contract. Therefore, the deviations of interest are whether an agent rejects his contract. An agent will only truly make a decision when he decides to sign the contract or not, as the effort decision is trivial once he signs.[7] Agents implicitly make their effort choice at the same time that they decide to accept the contract.

Let the effort vector prescribed to the agents in the optimal bilateral forcing contracts be $E^* = [e_1^*, e_2^*, ..., e_N^*]^T$. If an agent decides to reject the contract, given the others all accept their contracts, they best respond to this effort vector, playing:

$$e_i = \theta \sum_{j=1}^{N} g_{ij} e_j^* \tag{1.11}$$

This best response leads to a utility of

$$u_{i0} = \frac{\theta^2 (\sum_{j=1}^{N} g_{ij} e_j^*)^2}{2} \tag{1.12}$$

This is the outside option for each agent if they plan to reject their contract. Importantly, this outside option is now endogeneous to the contracts that are written. When inducing a higher effort from certain agents, the principal also makes the outside options of others more attractive. This trade off drives the principal's optimal effort levels.

The principal's problem is

[7] This setting is identical to a setting where all agents sign their contracts and the agents get zero payment if they deviate within contract. In that case, the deviation comes only within the contract, instead of when deciding to accept or reject the contract. Both cases lead to the same optimal bilateral forcing contract payments in equilibrium, so I consider with the type of contract that is consistent with the multilateral case.

$$\max_{\{(t_i,e_i)\}_{i\in N}} \quad \sum_{i=1}^{N}(e_i - t_i)$$
$$\text{s.t.} \quad t_i - \frac{e_i^2}{2} + \theta e_i \sum_{j=1}^{N} g_{ij}e_j \geq \frac{\theta^2(\sum_{j=1}^{N} g_{ij}e_j)^2}{2}, \quad (\forall i \in N) \tag{1.13}$$

There is a tension in choosing how much effort to prescribe each agent. If an agent i affects many other agents, then on one hand they should be prescribed a high effort level in order to make the value of effort for other agents higher. On the other hand, increasing the effort of this agent makes the outside option of others more enticing, and so the principal may desire to lower the effort of this agent.

To solve for the optimal bilateral forcing contracts, I note that the individual rationality constraint must hold at equality for each agent. Substituting yields the unconstrained problem,

$$\max_{\{e_i\}_{i\in N}} \quad \sum_{i=1}^{N}(e_i - \frac{e_i^2}{2} + \theta e_i \sum_{j=1}^{N} g_{ij}e_j - \frac{\theta^2(\sum_{j=1}^{N} g_{ij}e_j)^2}{2}) \tag{1.14}$$

I write this equation in matrix form:

$$\max_{E} \quad \mathbf{1}^T E - \frac{1}{2}E^T E + \theta E^T G E - \frac{\theta^2}{2}E^T G^T G E \tag{1.15}$$

I solve for the optimal vector E^* and state the optimal effort levels, transfers, and profits in the following theorem on the optimal bilateral forcing contracts.

Theorem 1.4.1. *The optimal bilateral forcing contract specifies an effort profile:*

$$E_{SB} = ((\mathbb{I} - \theta G^T)(\mathbb{I} - \theta G))^{-1}\mathbf{1} \tag{1.16}$$

Profits: $\pi_{SB} = \frac{\mathbf{1}^T E_{SB}}{2}$, *and Transfers:* $T_{SB} = \frac{E_{SB}}{2}$. *Efforts are distorted downward from the first best:* $E_{SB} \leq E_{FB}$, *and* $E_{SB} < E_{FB}$ *if the network is weakly connected.*[8]

Proof. Appendix. □

It is clear that the efforts prescribed by the principal in this contract are all less than the first-best efforts. While the complementarity has stayed the same, the efforts of the

[8] *A weakly connected network is one such that if all directed links were made undirected, the network would be connected.*

agents now factor into the outside option of others who are affected by these agents. Therefore, any agent who affects the outside option of others will have his effort distorted downward. Then, any agent who has a complementarity in their effort with one of these agents will also have his effort distorted downward, as their benefit has diminished. This effect diffuses throughout the network, and as long as the network is weakly connected, the effort for all agents is distorted downward from the multilateral to bilateral forcing contracts. If an agent contributes to the outside option of someone with a high outside option, the marginal contribution is relatively high due to the quadratic nature of the outside utility. The principal optimally wants to lower the efforts of these agents who contribute to the outside options of those who internalize efforts of many agents.

This intuition comes through with the equation for the effort in the optimal bilateral forcing contracts. The effort levels from these contracts relate to the Bonacich centrality of a modified network. Taking Equation 1.16 and multiplying through gives the following function of the effort vector:

$$E_{SB} = (\mathbb{I} - \theta(G + G^T - \theta G^T G))^{-1}\mathbf{1} \tag{1.17}$$

Efforts are the Bonacich centrality of the modified network $G + G^T - \theta G^T G$. The term that makes the difference from the first best is $\theta^2 G^T G$. This term can be thought as the influence an agent shares with a second agent over a third. The adjacency matrix G has tells us all the walks of length one in the network. If I square the matrix, the element ij of G^2 will tell us the number of walks of length 2 from agent i to agent j, and so on. If I consider the matrix $G^T G$, this gives the walks that go one step in the direction of the directed links in the graph, and one step in the opposite direction of the links in the graph, which would be the direction of the links in the transpose network G^T. This measure $G^T G$ is known as the *bibliographic coupling* of a network. This term comes from the literature on citations, and refers to when two papers cite a common third paper. Two papers' bibliographic coupling will tell how many papers they both cite. Similarly to the citation interpretation, the ijth entry of $G^T G$ in this setting tells how many agents agent i and j both affect. If the ith row of $G^T G$ sums to a large number, this means that agent i has a high total bibliographic coupling, such that he affects agents who are highly affected by others. This implies that agent i should have his effort distorted downward to a greater magnitude. Intuitively, if an agent affects another who is highly affected, then the initial agent will have their effort distorted downward more, as the agent who is affected is receiving a large outside option, that grows in a quadratic fashion. This implies that in a network of CROs, when a firm has agreements across many other firms, Pfizer would wish to incentivize the firms on the peripheries less, and focus more of their incentives into the firm who internalizes much of the effort of others.

These ideas are captured in the following example.

Example 1.4.1. *Effort Distortion*

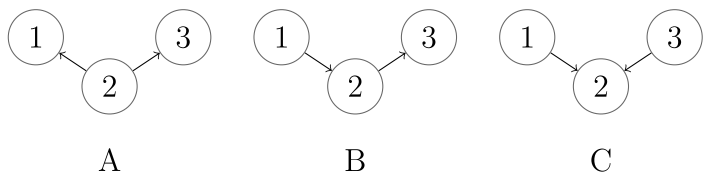

Figure 1.3: 3 Agent Star Networks

Consider a setting of 3 agents situated in a star network structure. I consider the 3 different networks as shown in Figure 1.3. Each network has two links, with a single directed link between agent 2 and each of the other agents. Each network is identical when symmetrized, and therefore the first-best efforts and profits are identical across networks. I am interested in which agents get their efforts distorted downward more within and across each network. Consider agent 2. As we look from network A to B to C, agent 2's total bibliographic coupling (the sum of coupling across all agents, including himself) goes from 2 to 1 to 0. As this occurs, agents 2's effort increases. This is because agent 2 goes from contributing to both of the outside options of the other agents, to one of theirs, to none. Similarly, agent 1's effort monotonically decreases as his total bibliographic coupling increases from 0 to 1 to 2. Agent 3's effort increases at first, due to a compounding effect of the complementarity of the effort, and then decreases as his coupling increases. These results of this example are summarized in Table 1.1.

It is a different notion of the centrality that decides the efforts across each network. Consider network A. In the first best, agent 2 would be prescribed the highest effort level of any of the agents, as he is the most central in the symmetrized network. In comparison,

Network	Effort 1	Effort 2	Effort 3
A	$1+\theta+2\theta^2$	$1+2\theta$	$1+\theta+2\theta^2$
B	$1+\theta+\theta^2$	$1+2\theta+\theta^2+\theta^3$	$1+\theta+2\theta^2+\theta^3+\theta^4$
C	$1+\theta$	$1+2\theta+2\theta^2$	$1+\theta$
First Best	$\frac{1+\theta}{1-2\theta^2}$	$\frac{1+2\theta}{1-2\theta^2}$	$\frac{1+\theta}{1-2\theta^2}$

Table 1.1: Efforts

Network	Profit
A	$1.5+2\theta+2\theta^2$
B	$1.5+2\theta+2\theta^2+\theta^3+0.5\theta^4$
C	$1.5+2\theta+\theta^2$
First Best	$\frac{1.5+2\theta}{1-2\theta^2}$

Table 1.2: Profits

in the second best, agent 2 will be prescribed they lowest effort level for high enough θ. For example, if $\theta = 0.6$, then agent 2 is prescribed effort 2.2, while agents 1 and 3 are prescribed effort 2.32. Agent 2 has their effort distorted further down than the other agents because he has a high bibliographic coupling.

Profits act similarly to effort. The networks with lower bibliographic coupling lead to higher profits, while the network with the high bibliographic coupling has smaller profits, as summarized in Table 1.2. It is noteworthy that Network B gives higher profits than Network A, even though they have the same level of bibliographic coupling. This is because in Network B there is a compounding effect of the complementarity of effort. There are two competing effects of the externality: a negative effect through the bibliographic coupling, and a positive effect through the initial complementarity.

The example shows that the bibliographic coupling of the network plays a role in the distortion of effort from first best throughout the network. When the level of interaction θ is "small" we see that this intuition is exact, and it is uniquely this measure which affects which agents have their effort distorted down, and who has his effort distorted downward the most.

Proposition 1.4.1. *For small θ, effort is distorted further downward for agents with higher bibliographic coupling. Across different networks, profits are distorted further downward based on the aggregate bibliographic coupling across all agents.*

Proof. When θ is small, I take a second order Taylor approximation around zero to see that, for the effort profile in the first best,

$$E_{FB} = (\mathbb{I} - \theta(G + G^T))^{-1}\mathbf{1} = [\mathbb{I} + \theta(G + G^T) + \theta^2(G + G^T)^2]\mathbf{1} \tag{1.18}$$

Similarly, for the second best effort profile I have that

$$E_{SB} = ((\mathbb{I} - \theta G^T)(\mathbb{I} - \theta G))^{-1}\mathbf{1} = [\mathbb{I} + \theta(G + G^T) + \theta^2(G + G^T)^2 - \theta^2 G^T G]\mathbf{1} \tag{1.19}$$

I simply combine the two equations to see that $E_{FB} - E_{SB} = \theta^2 G^T G\mathbf{1}$.

For the profits, I note that

$$\pi_{FB} - \pi_{SB} = \frac{\mathbf{1}^T(E_{FB} - E_{SB})}{2} = \frac{\mathbf{1}^T\theta^2 G^T G\mathbf{1}}{2} \tag{1.20}$$

□

Therefore, when θ is small, agents with higher total bibliographic coupling across all other agents have their effort distorted downward more than those with low total coupling. Agents with no bibliographic coupling i.e. those with no effect on other agents, don't have any distortion down from their first best effort. This is a property of a small θ. When θ is large, the drop in effort from those with nonzero bibliographic coupling will be felt throughout the network, shading efforts downward. These effects are on higher order, and not felt with small θ.

Profits also depend on the coupling in the network. Optimal bilateral forcing contracts achieve lower profits when networks have higher bibliographic coupling. More links will lead to a bigger gap in profits between the multilateral and bilateral forcing contracts, as a link adds to the bibliographic coupling. If there are agents who are affected by many others, the network has a high coupling, and there is a large gap in profits. This is because those agents have very large outside options, and any additional effort prescribed to those that affect them will raise the outside option much higher. Thus to keep payments to these agents down, all of those who affect him will have their effort distorted downward. This effect eventually dominates any direct effect from the benefit of the complementarity, highly affected agents put a downward pressure on the efforts of many others, and profits are lower, as exhibited in network C of Example 1.4.1.

I can also analyze the principal's profits across the first and second best forcing contracts for more general θ by restricting to special networks. Assume that the network is undirected and regular.[9] This is a network such that every agent has the same degree. This restriction allows me to make comparisons across networks between the optimal forcing contracts in the multilateral and bilateral cases. When a network is undirected, the matrix G is diagonalizable. That is, if Λ is a diagonal matrix with the eigenvalues λ_i of G on the diagonal, then there exists an orthonormal matrix V such that

$$V^{-1}GV = \Lambda \tag{1.21}$$

Since V is invertible, I can write $G = V\Lambda V^{-1}$. Through this diagonalization I am able to characterize the effort vectors in the multilateral and bilateral forcing contracts. Let Λ_{FB} be a diagonal matrix with terms $\frac{1}{1-2\theta\lambda_i}$ on the diagonal, and Λ_{SB} be a diagonal matrix with terms $\frac{1}{(1-\theta\lambda_i)^2}$ on the diagonal. Then, in the multilateral case:

$$E_{FB} = V\Lambda_{FB}V^{-1} \tag{1.22}$$

In the bilateral case:

$$E_{SB} = V\Lambda_{SB}V^{-1} \tag{1.23}$$

Therefore the optimal effort vectors are related to the spectral properties of the matrix representation G. Since the profits to the principal are equal to half the sum of the efforts in both multilateral and bilateral contracting, this representation allows for exact comparisons across networks based on the spectral properties of the network.

Proposition 1.4.2. *Let G be an undirected, regular network, with agent degree d. Then* $\Pi_{FB} = \frac{N}{2(1-2\theta d)}$, $\Pi_{SB} = \frac{N}{2(1-\theta d)^2}$ *and the proportional loss,* $\frac{\Pi_{FB}-\Pi_{SB}}{\Pi_{FB}} = \frac{\theta^2 d^2}{(1-\theta d)^2}$

Proof. Appendix □

[9]The appendix extends the following analysis to general undirected networks.

The profits to the principal are increasing in the number of agents and the degree of agents in both the multilateral and bilateral forcing contracts, but the gap in profits also increases with the degree. Therefore there are competing effects at play. Though the principal has higher profits in both cases as the degree of agents increases, the gap between the multilateral and bilateral cases is also increasing. As the network becomes more densely connected, the principal loses more through the restriction to bilateral contracts, as outside options increase.

When contracting with a large group of firms, the government will find it more profitable if they have a higher level of connectivity, but at the same time would be more willing to create multilateral contracts which would close the large gap from first best which extends as the connectivity increases. In general, high connectivity is profitable for the principal but the gap from the first best also increases.

1.5. Bilateral Linear Contracts

The forcing contracts of the previous section tie an agent to his effort level once he has accepted the contract. While this guarantees a certain level of effort for the principal, these contracts leave the agent with a high outside option. When an agent rejects the contract, given all the other agents are tied to their effort levels, there is no way to punish this agent for rejecting. He has a high outside option, which the principal finds detrimental.

To address this issue, I consider a type of contract that is also very common in practice, the linear contract. Linear contracts specify a fixed "base" payment and an incentive portion, which is often based off a fraction of the output. These contracts allow the agent to best respond continuously; to adjust his effort depending on the efforts of others. I derive the optimal linear contracts in the network setting, and show that by allowing agents to adjust their efforts, the principal attains higher profits.

I assume that the principal observes each of the agent's efforts individually, and therefore can condition each agent's contract on the effort level of that agent. That is, the principal announces a base payment α_i and incentive payment β_i such that the entire wage payed to the agent is $\alpha_i + \beta_i e_i$. The timing of the game is the same as the previous sections. The principal proposes wages, the agents accept or reject, then they view all the accepted contracts and choose their optimal action. Let the optimal action chosen after agent i rejects a contract be $\tilde{e}_j$ for each j.

The principal solves the following problem

$$\max_{\{(\beta_i,\alpha_i,e_i)\}_{i\in N}} \quad \sum_{i=1}^{N}(e_i - \beta_i e_i - \alpha_i)$$

$$\text{s.t.}$$

$$\text{(IR)} \quad \beta_i e_i + \alpha_i - \frac{e_i^2}{2} + \theta e_i \sum_{j=1}^{N} g_{ij} e_j \geq \frac{\theta^2(\sum_{j=1}^{N} g_{ij}\tilde{e}_j)^2}{2}, \quad (\forall i \in N) \tag{1.24}$$

$$\text{(IC)} \quad e_i \in argmax_{e_i} \quad \beta_i e_i + \alpha_i - \frac{e_i^2}{2} + \theta e_i \sum_{j=1}^{N} g_{ij} e_j, \quad (\forall i \in N)$$

The principal now faces two different types of constraints for each agent i: an individual rationality constraint and an incentive compatibility constraint. When writing forcing contracts, the principal effectively collapsed both constraints into one. Now, the incentive compatibility constraint ensures the agents choose the optimal action, as they now receive payment for a continuum of action choices. This highlights how the agent can adjust his effort in the contract in response to the actions of others. The individual rationality constraint contains the modified effort term $\tilde{e}_j$. This represents the efforts taken by others when they see that an agent has rejected his contract. They adjust their effort level to acknowledge that the agent who rejected his contract will be best responding to the others who have signed their contracts. It is in this way that agents punish those who do not sign their contract. Agents will lower their effort levels when they see someone has not signed a contract, giving an equilibrium that makes the outside option less attractive.

I first characterize the optimal linear contract. By the incentive compatibility constraint, for each i,

$$\beta_i - e_i + \theta \sum_{j=1}^{N} g_{ij} e_j = 0 \tag{1.25}$$

Or, in matrix notation,

$$E = (\mathbb{I} - \theta G)^{-1}\beta \tag{1.26}$$

This equation holds true when every agent accepts his contract. If an agent rejects his contract, then he plays his best response, given by

$$e_i = \theta \sum_{j=1}^{N} g_{ij} e_j \tag{1.27}$$

The other agents see that agent i has rejected his contract, and readjust their efforts accordingly. Given these other agents accept their contracts, their best responses are

$$e_j = \beta_j + \theta \sum_{k=1}^{N} g_{jk} e_k \tag{1.28}$$

Let E_{-i} be the effort vector when agent i has rejected his contract. Let $\mathbb{I}_{-i}$ be the identity matrix, with the iith component equal to zero instead of 1. Then, stacking Equations 1.27 and 1.28,

$$E_{-i} = (\mathbb{I} - \theta G)^{-1} \mathbb{I}_{-i} \beta \tag{1.29}$$

Equipped with these effort vectors and the individual rationality constraint, I write the principal's unconstrained problem

$$max_E \quad \pi(E) = \mathbf{1}^T E - \frac{E^T E}{2} + \theta E^T G E - \frac{1}{2} \sum_{i=1}^{N} E_{-i}^T u_i u_i^T E_{-i} \tag{1.30}$$

u_i is a column vector with 1 in the ith spot, and 0 elsewhere. The solution to this problem is summarized in the proposition.

Proposition 1.5.1. *The optimal linear contracts specify, for each i,*

$$\beta = [(\mathbb{I} - \theta(G + G^T))(\mathbb{I} - \theta G)^{-1} + \sum_{i=1}^{N} (\mathbb{I} - \theta G^T)\mathbb{I}_{-i}(\mathbb{I} - \theta G^T)^{-1} u_i u_i^T (\mathbb{I} - \theta G)^{-1} \mathbb{I}_{-i}]^{-1} \mathbf{1}$$

$$\alpha_i = -\beta_i e_i + \frac{e_i^2}{2} - \theta e_i \sum_{j=1}^{N} g_{ij} e_j + \frac{\theta^2 (\sum_j g_{ij} \tilde{e}_j)^2}{2}, \quad (\forall i \in N)$$

and induces effort

$$E = [\mathbb{I} - \theta(G + G^T) + \sum_i (\mathbb{I} - \theta G^T)\mathbb{I}_{-i}(\mathbb{I} - \theta G^T)^{-1} u_i u_i^T (\mathbb{I} - \theta G)^{-1} \mathbb{I}_{-i}(\mathbb{I} - \theta G)]^{-1} \mathbf{1}$$

Proof. Appendix. □

These linear contracts always perform at least as well as the optimal bilateral forcing contracts, and strictly outperform them under very weak connectivity assumptions on the network. By allowing the agents to adjust their efforts, the principal induces a punishment for not signing the contract, which lowers the outside options of the agents. In accomplishing this, the principal maintains higher profits. To see how this works, we consider the following example.

Example 1.5.1. *Consider a network with two agents who affect each other, as in Figure 1.4.*

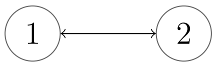

Figure 1.4: Two Agent line

I consider the optimal linear contract. The problem for the principal is as follows.

$$
\begin{aligned}
&\max_{\{\beta_i,\alpha_i,e_i\}} \quad e_1 + e_2 - (\beta_1 e_1 + \alpha_1) - (\beta_2 e_2 + \alpha_2) \\
&\qquad s.t. \\
&\text{(IR)} \quad \beta_1 e_1 + \alpha_1 - \frac{e_1^2}{2} + \theta e_1 e_2 \geq \frac{\theta^2 \tilde{e}_2^2}{2}, \quad \beta_2 e_2 + \alpha_2 - \frac{e_2^2}{2} + \theta e_2 e_1 \geq \frac{\theta^2 \tilde{e}_1^2}{2} \\
&\text{(IC)} \quad e_1 \in argmax_{e_1} \quad \beta_1 e_1 + \alpha_1 - \frac{e_1^2}{2} + \theta e_1 e_2, \quad e_2 \in argmax_{e_2} \quad \beta_2 e_2 + \alpha_2 - \frac{e_2^2}{2} + \theta e_2 e_1
\end{aligned}
$$

By the (IC) *condition,* $e_1 = \beta_1 + \theta e_2$*, and* $e_2 = \beta_2 + \theta e_1$*, implying that* $e_1 = \frac{\beta_1+\theta\beta_2}{1-\theta^2}$ *and* $e_2 = \frac{\beta_2+\theta\beta_1}{1-\theta^2}$*. If agent 1 rejects his contract, then agent 2 responds with* $\tilde{e}_2 = \frac{\beta_2}{1-\theta^2}$*. This effort is shaded down from the one within the contract, leading to a smaller outside option for agent 1. Similarly,* $\tilde{e}_1 = \frac{\beta_1}{1-\theta^2}$*. Importantly, the agents have lowered their effort levels in response to the other rejecting their contract. Given these values of effort, we plug the* (IC) *and* (IR) *constraints into the principal's objective to obtain*

$$
\begin{aligned}
\max_{\beta_1,\beta_2} \quad & \frac{\beta_1+\theta\beta_2}{1-\theta^2} + \frac{\beta_2+\theta\beta_1}{1-\theta^2} - \frac{1}{2}\left(\frac{\beta_1+\theta\beta_2}{1-\theta^2}\right)^2 - \frac{1}{2}\left(\frac{\beta_2+\theta\beta_1}{1-\theta^2}\right)^2 \\
& + 2\theta\left(\frac{\beta_1+\theta\beta_2}{1-\theta^2}\right)\left(\frac{\beta_2+\theta\beta_1}{1-\theta^2}\right) - \frac{\theta^2}{2}\left(\frac{\beta_2}{1-\theta^2}\right)^2 - \frac{\theta^2}{2}\left(\frac{\beta_1}{1-\theta^2}\right)^2
\end{aligned}
\tag{1.31}
$$

Solving for the optimal β*, we get* $\beta_1 = \beta_2 = \frac{(1-\theta)(1+\theta)^2}{1-2\theta^2-2\theta^3}$*. This leads to effort levels* $e_1 = e_2 = \frac{(1+\theta)^2}{1-2\theta^2-2\theta^3}$*. Profits are also* $\frac{(1+\theta)^2}{1-2\theta^2-2\theta^3}$*. This is in comparison to the optimal* *bilateral forcing contracts, which give efforts and profits equal to* $\frac{1}{(1-\theta)^2}$. *Efforts and profits are higher in the linear contracting case.*

In the example we see that the linear contract outperforms the bilateral forcing contract. This intuition is formalized in the theorem.

Theorem 1.5.1. *The principal's profits under linear contracting are at least as large as under bilateral forcing contracts. Profits are strictly higher if there exists a strongly connected component in the network.*[10]

Proof. I construct a linear contract that incentivizes the same effort as the bilateral forcing contracts, and show that this contract is both feasible and gives at least as much profit as the second best forcing contracts. Then, since the optimal linear contract performs at least as well as this constructed linear contract, I conclude that linear contracts outperform bilateral forcing contracts.

The linear contract I construct must induce effort level in Equation 1.16. I set β equal to $(\mathbb{I} - \theta G^T)^{-1}\mathbf{1}$, inducing the following effort

$$E = (\mathbb{I} - \theta G)^{-1}\beta = (\mathbb{I} - \theta G)^{-1}(\mathbb{I} - \theta G^T)^{-1}\mathbf{1} = ((\mathbb{I} - \theta G^T)(\mathbb{I} - \theta G))^{-1}\mathbf{1} \tag{1.32}$$

Thus the effort from the optimal bilateral forcing contract is feasible through linear contracts. Clearly β_i is positive for each agent, through our assumptions on θ. The effort levels taken in equilibrium are

$$E = \sum_{k=0}^{\infty} \theta^k G^k \beta \implies e_i = \sum_{j=1}^{N} \sum_{k=0}^{\infty} \theta^k g_{ij}^k \beta_j \tag{1.33}$$

Where g_{ij}^k is the ijth element of G^k The derivative of this value with respect to β_l is

$$\frac{\partial e_i}{\partial \beta_l} = \sum_{k=0}^{\infty} \theta^k g_{il}^k \geq 0 \tag{1.34}$$

The inequality is strict as long as there is a path from l to i. Therefore, decreasing β_l (to zero) decreases the efforts of all agents on a path from agent l. Since the outside

[10] *A strongly connected component is a subset of a directed network such that each agent in the subset can reach every other agent in the subset. Note that being in a strongly connected component is not a strong assumption, for example, two agents who affect each other form a strongly connected component.*

option of agent l is $u_{lo} = \frac{\theta^2}{2}(\sum_{j=1}^{N} g_{lj}e_j)^2$, and depends on the efforts taken when he doesn't sign a contract, the outside utility is lower than in the optimal bilateral forcing contract case, as long as agent l is in some strongly connected component. If he is in a strongly connected component, this means that his rejection causes at least one agent who affects him to lower their effort, lowering the his outside option. If he is not in a strongly connected component, then the outside utility is the same as in the bilateral forcing contract. Therefore, the aggregate level of outside options will be less than in the forcing contracts if there is at least one strongly connected component, and equal otherwise. Therefore, since the profit of the principal is just equal to the total surplus of the contract minus the outside options of the agents, we see that this constructed contract gives at least as large profits to the principal as the second best forcing contract, and strictly larger if there is a strongly connected component. Therefore, the optimal linear contract gives weakly higher profits as well, and strictly higher profits when there is a strongly connected component.

□

Therefore, linear contracts outperform bilateral forcing contracts. Since agents reoptimize based on the contracts signed, they effectively make deviation from the contract less enticing. By allowing the agents to adjust their effort, the principal effectively has the agents punish each other for rejecting a contract, holding each other accountable to their effort levels.

Therefore there is theoretical justification for linear contracts when facing a group of agents who have complementarities in their efforts, as formalized in Theorem 1.5.1. When contracting with firms that have network externalities in their effort, the government prefers to sign "fixed price" contracts, which gives a fixed price for a product or services, while not taking into account costs of effort or the time taken by the agent. The proposition above implies that these contracts are suboptimal whenever at least two agents internalize each others efforts, even indirectly. The government could do better by offering an incentive scheme in the form of a linear contract that allows the agents to adjust the effort necessary depending on what contracts exist in the industry.

In fact, some private companies such as Pfizer now implement contracts that give additional payments to firms that reach certain benchmarks. In biotechnology, this is implemented by giving incentives for discoveries and production. This type of contract is a spin on the classic linear contract, as it incentivizes certain benchmarks rather than effort continuously. As we will see in the next section, this type of contract can implement the first best, and is the optimal bilateral contract for the principal. Some private companies tend to lean towards this type of contract, while agencies including the U.S. government prefer contracts that, while simplistic in nature, are clearly far from optimal.

1.6. Optimal Bilateral Contracts

Sections 1.4 and 1.5 make it clear that the principal finds it profitable to allow agents the freedom to adjust their efforts to the contracts accepted in their network. The optimal bilateral contract takes this intuition further, incentivizing agents to punish those who do not accept the contract in the harshest way possible. The principal is able to implement the first best effort, with the threat that the agent will obtain zero utility if they do not accept the contract. By achieving first best and leaving no utility to the agents, the principal maximizes his profits.

The contract that obtains these goals has a "base and bonus" structure. Agents receive a "base" salary α_i which they receive no matter their choice of effort. Then, if they take their first best effort level, they receive a "bonus" γ_i on top of the base amount. The base and bonus are set up in a way that is both incentive compatible and individually rational, along with a third consideration, which states that when someone deviates from their contract (either by rejecting the contract or not taking first best effort), then each agent is willing to deviate from his first best effort as well. I show that for all agents in the network, if a subset of the agents linked to them deviate from the contract, they would also like to deviate from the first best and respond to the others. Therefore when one agent rejects a contract, there is an equilibrium where all agents ignore their contracts and play their best responses as illustrated by Equation (1.5). In this equilibrium the agent who rejected his contract obtains zero utility. Thus the outside option for all agents is the lowest possible, leading to profit maximization for the principal and zero rents for the agents. The theorem outlines these contracts.

Theorem 1.6.1. *The optimal bilateral contract implements the first-best effort profile* $E_{FB} = [e_1^*, e_2^*, ..., e_N^*]^T$*, and specifies for each* i *a base payment* $\alpha_i = -\frac{\theta^2(\sum_{j=1}^N g_{ij}e_j^*)^2}{2}$ *and a bonus payment* $\gamma_i(e_i^*) = \frac{e_i^{*2}}{2} - \theta e_i^* \sum_{j=1}^N g_{ij}e_j^* + \frac{\theta^2(\sum_{j=1}^N g_{ij}e_j^*)^2}{2}$*, which is received only if the agent takes his first best effort level.*

Proof. Appendix. □

The base payment is weakly negative, and is zero only if the agent is not affected by any of the other agents. This can be thought as a "sell the firm" contract as in Holmstrom (1979). Each agent "buys" their outside option, and then is paid it back along with the rest of their payment in terms of a bonus. In the proof, I outline the constraints that must hold and show that they hold with these values of the contract. There are three constraints that must hold for each agent: the incentive compatibility and individual rationality constraints of which we are familiar, and a "punishment constraint" which states that agents must be willing to ignore their contracts and best respond to the actions of others if someone rejects their contract. The proof shows that given the contract above, all the constraints hold for all agents. When an agent rejects their contract, this leads to an equilibrium in the second stage where all agents are willing to play an effort of zero and get zero utility. In the subgame after all agents accept their contract, they all find it incentive compatible to play their first best effort. Thus the principal achieves first best, while leaving zero utility to the agents, implying this is the best that the principal can hope to achieve.

The optimal contracts put the agents on a knife's edge. If they take any action other than their prescribed action they get paid nothing, and lose money through the α_i term. These contracts, while economically optimal, are not practical for a variety of reasons. In particular, the contract does not fully implement first best. When an agent rejects his contract, it is possible that some of those who affect him are indifferent between taking the first best effort level and taking zero effort. Also, even if all agents sign their contract, there exists an equilibrium where all agents play zero effort, and so in this contract the multiple equilibria problem arises. Thus we only have partial implementation of first best. That being said, some private firms that contract for R&D, such as Pfizer, use a type of contract that reflects these optimal contracts. Pfizer will offer "benchmark" contracts which give lump payments for achieving certain goals. This has the flavor of these optimal contracts, implying that Pfizer may be closer to reaching the first best through contracting than the federal government.

1.7. Conclusion

When agents live on a network of complementarities, the principal must take into account the externalities present when contracting. In particular, they need to balance the opposing forces of complementarity in effort costs and outside options endogeneous to the contracts. In this paper I discussed contracts that are used commonly in government spending on R&D industries: forcing contracts. These contracts, while not optimal, are implemented by the U.S. government in a variety of industries. The loss of profits from first best to the optimal bilateral forcing contracts are characterized through the spectral properties of the underlying network, and I find that efforts and profits decrease more when agents have high bibliographic coupling.

I also look at bilateral linear contracts. Surprisingly, these contracts perform better than bilateral forcing contracts. This is because the linear contracts allow agents to adjust their effort based on the contracts that are observed, which brings down the outside option of an agent who rejects their contract. As long as there exists at least one strongly connected component of any size, this type of contract strictly outperforms the forcing contracts. This implies that in situations where agents collaborate in mutually beneficial relationships, the linear contracts perform better than the forcing contracts. In network structures without this two way collaboration, such as in directed tree networks, we could expect to see forcing contracts, as they perform just as well as the linear contracts. The optimal bilateral contracts continue the logic of allowing adjust their efforts when someone deviates, and incentivizes agents to play zero effort when someone has rejected a contract.

This analysis leads to policy implications for the government and other entities that contract for R&D. Fixed price contracts, though considered simple and easy to implement, are suboptimal when network externalities are present. The results find that allowing for some leeway in the efforts of the agents would give higher profits to the principal and would outperform these fixed contracts.

There are various avenues for future research based off of this work. As hinted at in Example 1.3.1, there are efficient networks in this game which rely on the Bonacich centrality of the symmetrized network. Which networks maximize this measure (conditional on a fixed number of nodes and links) is an open question. If the principal could create the network at a cost per link, the optimal network would rely on this measure of centraity. These network design questions are still open, and are left for future research. I also assume that everything is observable in the model. Allowing for unobservable contracts and efforts could potentially lead to other insights on contracting in the presence of these network externalities.

1.8. Appendix

PROOF OF PROPOSITION 1.3.1 The first order condition of (1.7) with respect to E is

$$\mathbf{1}^T - E^T + \theta E^T(G+G^T) = 0 \implies \mathbf{1} = (\mathbb{I} - \theta(G+G^T))E \implies E = (\mathbb{I} - \theta(G+G^T))^{-1}\mathbf{1} \tag{1.35}$$

PROOF OF PROPOSITION 1.3.2

The first order condition of (1.9) with respect to E is

$$\mathbf{1}^T - E^T + \theta E^T(G+G^T) = 0 \implies \mathbf{1} = (\mathbb{I} - \theta(G+G^T))E \implies E = (\mathbb{I} - \theta(G+G^T))^{-1}\mathbf{1} \tag{1.36}$$

To prove the third equation, I need the following lemma

Lemma 1.8.1. *Let X be a $N \times 1$ vector, $A_1 = \mathbb{I} - A$, and $A_2 = \mathbb{I} - .5A - .5A^T$. Then, $X^T A_1 X = X^T A_2 X$.*

Proof. This is simple arithmetic. Since $X^T AX$ is a scalar, $X^T AX = X^T A^T X$. Then,

$$\begin{aligned} X^T A_2 X = X^T(\mathbb{I} - .5A - .5A^T)X &= X^T X - .5X^T AX - .5X^T A^T X \\ &= X^T X - .5X^T AX - .5X^T AX \\ &= X^T X - X^T AX = X^T A_1 X \end{aligned}$$

□

With the lemma, I can now prove the third equation. Transfers that the principal must pay are

$$\sum_{i=1}^{N} t_i = \frac{E^T E}{2} - \theta E^T G E = \frac{1}{2}E^T(\mathbb{I} - 2\theta G)E \tag{1.37}$$

Plugging in the first best effort allocation gives

$$\sum_{i=1}^{N} t_i = \frac{1}{2}\mathbf{1}^T(\mathbb{I} - \theta(G+G^T))^{-1}(\mathbb{I} - 2\theta G)(\mathbb{I} - \theta(G+G^T))^{-1}\mathbf{1} \tag{1.38}$$

I apply the lemma and simplify to get

$$\frac{1}{2}\mathbf{1}^T(\mathbb{I}-\theta(G+G^T))^{-1}(\mathbb{I}-\theta G-\theta G^T)(\mathbb{I}-\theta(G+G^T))^{-1}\mathbf{1} = \frac{1}{2}\mathbf{1}^T(\mathbb{I}-\theta(G+G^T))^{-1}\mathbf{1} \tag{1.39}$$

So that $\sum_{i=1}^{N} t_i = \frac{1}{2}\mathbf{1}^T E_{FB}$. It follows immediately that $\pi_{FB} = \frac{\mathbf{1}^T E_{FB}}{2}$.

PROOF OF THEOREM 1.4.1

Taking a first order condition with respect to the effort vector E gives

$$\mathbf{1}^T - E^T + \theta E^T(G + G^T) - .5\theta^2 E^T(G^T G + (G^T G)^T) = 0 \tag{1.40}$$

Which implies that

$$E = (\mathbb{I} - \theta G - \theta G^T + \theta^2 G^T G)^{-1}\mathbf{1} = ((\mathbb{I} - \theta G^T)(\mathbb{I} - \theta G))^{-1}\mathbf{1} \tag{1.41}$$

To prove the third equation, I note that the sum of the transfers paid by the principal is

$$\sum_{i=1}^{N} t_i = \frac{1}{2}E^T E - \theta E^T G E + \frac{\theta^2}{2} E^T G^T G E = \frac{1}{2}E^T(\mathbb{I} - 2\theta G + \theta^2 G^T G)E \tag{1.42}$$

Applying Lemma 1.8.1, I find that

$$\sum_{i=1}^{N} t_i = \frac{1}{2}E^T(\mathbb{I} - \theta G - \theta G^T + \theta^2 G^T G)E = \frac{1}{2}E^T(\mathbb{I} - \theta G^T)(\mathbb{I} - \theta G)E \tag{1.43}$$

Applying the values of E, I get $\sum_{i=1}^{N} t_i = \frac{1}{2}\mathbf{1}^T E_{SB}$. Then it follows immediately that $\pi_{SB} = \frac{\mathbf{1}^T E_{SB}}{2}$.

PROOF OF PROPOSITION 1.4.2

I first prove a more general statement for undirected networks, and then show the result holds for regular networks.

The first best effort profile is

$$E_{FB} = (\mathbb{I} - 2\theta G)^{-1}\mathbf{1} \tag{1.44}$$

Writing E_{FB} as an infinite sum gives the form

$$E_{FB} = (\sum_{i=1}^{\infty}(2\theta G)^k)\mathbf{1} = (\sum_{i=1}^{\infty}(2\theta)^k(V\Lambda V^{-1})^k)\mathbf{1} = V(\sum_{i=1}^{\infty}(2\theta)^k(\Lambda)^k)V^{-1}\mathbf{1} = V\Lambda_{FB}V^{-1}\mathbf{1} \tag{1.45}$$

Where Λ_{FB} is a diagonal matrix with zeros off the diagonal and at each element ii, the term $\frac{1}{1-2\theta\lambda_i}$ where λ_i is the ith eigenvalue of the matrix G.

Solving for the second best allocation in terms of the diagonalized matrix is as follows. When the matrix is symmetric, the second best allocation is of the form

$$E_{SB} = ((\mathbb{I} - \theta G)^{-1})^2 \mathbf{1} \tag{1.46}$$

Which I can write as an infinite sum, as

$$E_{SB} = (\sum_{i=1}^{\infty} (\theta G)^k)^2 \mathbf{1} \tag{1.47}$$

Plugging in the diagonalized form for G gives the following formulation

$$E_{SB} = (\sum_{i=1}^{\infty} (\theta)^k (V \Lambda V^{-1})^k)^2 \mathbf{1} = V (\sum_{i=1}^{\infty} (\theta \Lambda)^k)^2 V^{-1} \mathbf{1} = V \Lambda_{SB} V^{-1} \mathbf{1} \tag{1.48}$$

Where Λ_{FB} is a diagonal matrix with zeros off the diagonal and at each element ii, the term $\frac{1}{(1-\theta\lambda_i)^2}$ where λ_i is the ith eigenvalue of the matrix G.

I use the following result, stated without proof

Lemma 1.8.2. *Courant-Fischer-Weyl Min-Max Principle*

Let M be a square matrix with n eigenvalues. Then

$$\lambda_{min} \leq \frac{x^T M x}{x^T x} \leq \lambda_{max} \tag{1.49}$$

I apply this lemma to the profits in both the first and second best. Take the first best effort profile, $E_{FB} = (\mathbb{I} - 2\theta G)^{-1} \mathbf{1}$. The profit from the first best is $\frac{\mathbf{1}^T E_{FB}}{2} = \frac{\mathbf{1}^T (\mathbb{I} - 2\theta G)^{-1} \mathbf{1}}{2}$. Therefore, I apply Courant-Fischer-Weyl to see that

$$\frac{\tilde{\lambda}_{min}}{2} \leq \frac{\mathbf{1}^T (\mathbb{I} - 2\theta G)^{-1} \mathbf{1}}{2N} \leq \frac{\tilde{\lambda}_{max}}{2} \tag{1.50}$$

Where $\tilde{\lambda}$ are the eigenvalues of $(\mathbb{I} - 2\theta G)^{-1}$. From the above discussion, we see that the eigenvalues of this matrix are $\frac{1}{1-2\theta\lambda_i}$. This is increasing in λ in the relevant region, so that $\tilde{\lambda}_{max} = \frac{1}{1-2\theta\lambda_{max}}$. Applying this to (1.50), we see that

$$\frac{N}{2(1 - 2\theta\lambda_{min})} \leq \Pi_{FB} \leq \frac{N}{2(1 - 2\theta\lambda_{max})} \tag{1.51}$$

Now, $d_{ave} \leq \lambda_{max} \leq d_{max}$, and if and only if G is regular, $d_{ave} = d_{max}$, and so $\lambda_{max} = d_{ave} = d_{max}$. Now,

$$\lambda_{max} = \max_{\mathbf{x}} \quad \frac{\mathbf{x}^T G \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \geq \frac{\mathbf{1}^T G \mathbf{1}}{\mathbf{1}^T \mathbf{1}} = \frac{\sum_{ij} G_{ij}}{N} = \frac{\sum_i d_i}{N} = d_{ave} \tag{1.52}$$

Therefore the inequality holds with equality. Similarly, I can show the same for the matrices $(\mathbb{I} - 2\theta G)^{-1}$ and $(\mathbb{I} - \theta G)^{-2}$. From Courant-Fischer-Weyl, we have that

$$\frac{1}{1 - 2\theta\lambda_{max}} = \max_{\mathbf{x}} \quad \frac{\mathbf{x}^T (\mathbb{I} - 2\theta G)^{-1} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \geq \frac{\mathbf{1}^T (\mathbb{I} - 2\theta G)^{-1} \mathbf{1}}{\mathbf{1}^T \mathbf{1}} = \frac{\sum_i b_i}{N} = b_{ave} \tag{1.53}$$

Where b_{ave} is the average Bonacich centrality, and b_i is the Bonacich centrality of agent i. I can bound the maximal eigenvalue of $(\mathbb{I} - 2\theta G)^{-1}$ above as well, using the definition of eigenvalue. Let x be the eigenvector corresponding to the largest eigenvalue, and x_i be the maximal coordinate of this eigenvector. Then

$$\frac{1}{1 - 2\theta\lambda_{max}} |x_i| = |((\mathbb{I} - 2\theta G)^{-1} x)|_i = |\sum_j b_j x_j| \leq \sum_j b_j |x_i| \leq b_{max} |x_i| \tag{1.54}$$

So that $\frac{1}{1-2\theta\lambda_{max}} \leq b_{max}$. I therefore can bound the maximal eigenvalue between the maximal and average Bonacich Centralities. If a network is regular, $b_{max} = b_{ave}$, and thus (1.53) shows that the profit bound holds with equality. A proof for the second best is almost identical.

So, as $\lambda_{max} = d$ for the regular matrices, we obtain profits $\Pi_{FB} = \frac{N}{2(1-2\theta d)}$ and $\Pi_{SB} = \frac{N}{(1-\theta d)^2}$ respectively. The proportional loss equation follows.

PROOF OF PROPOSITION 1.5.1

I can plug in equations (1.26) and (1.29) into (1.30) to get the profit equation as a function of just the incentive vector β.

$$\begin{aligned} \pi = \mathbf{1}^T (\mathbb{I} - \theta G)^{-1} \beta - \frac{\beta^T (\mathbb{I} - \theta G^T)^{-1} (\mathbb{I} - \theta G)^{-1} \beta}{2} \\ + \theta \beta^T (\mathbb{I} - \theta G^T)^{-1} G (\mathbb{I} - \theta G)^{-1} \beta - \frac{1}{2} \sum_{i=1}^{N} \beta^T \mathbb{I}_{-i} (\mathbb{I} - \theta G^T)^{-1} u_i u_i^T (\mathbb{I} - \theta G)^{-1} \mathbb{I}_{-i} \beta \end{aligned} \tag{1.55}$$

Taking a first order condition with respect to the vector β gives the first order conditions

$$\mathbf{1}^T(\mathbb{I}-\theta G)^{-1} - \beta^T(\mathbb{I}-\theta G^T)^{-1}(\mathbb{I}-\theta G)^{-1} \\ + \theta\beta^T(\mathbb{I}-\theta G^T)^{-1}(G^T+G)(\mathbb{I}-\theta G)^{-1} \\ - \sum_i \beta^T \mathbb{I}_{-i}(\mathbb{I}-\theta G^T)^{-1}u_i u_i^T(\mathbb{I}-\theta G)^{-1}\mathbb{I}_{-i} = 0 \quad (1.56)$$

Solving this for β, and noting that the values α_i come from the individual rationality constraint in (1.24) leads to the optimal linear contract.

PROOF OF THEOREM 1.5.1

I show that the contracts of the form $\{\alpha, \gamma\}$ satisfy all incentive constraints necessary for the agents, implement first-best, and leave the agents with zero utility, implying that these contracts are optimal for the principal.

I first consider every agent has accepted their contract. The incentive compatibility constraint implies that

$$\alpha_i + \gamma_i - \frac{e_{i_{FB}}^2}{2} + \theta e_{i_{FB}} \sum_{j=1}^N g_{ij} e_{j_{FB}} \geq \alpha_i + \frac{\theta^2 (\sum_{j=1}^N g_{ij} e_{j_{FB}})^2}{2} \quad (1.57)$$

That is, every agent would rather take the contract's recommendation and receive their payoff of the base and bonus than best respond to everyone else taking their first best payoff and just receive their base payment plus the value of their best response.

Next, I require that every agent would rather deviate from their contract and play their best response if one or more of their neighbors are deviating from their contract. This constraint causes a cascade through the network such that every agent would like to deviate from their contracts and play best responses, leading to zero effort throughout.

For an agent to want to best respond given one neighbor deviates (rejects or ignores their contract) the constraint is:

$$\alpha_i + \gamma_i - \frac{e_{i_{FB}}^2}{2} + \theta e_{i_{FB}} (\sum_{j \neq k} g_{ij} e_{j_{FB}} + g_{ik} e_{k_{BR}}) \leq \alpha_i + \frac{\theta^2 (\sum_{j \neq k} g_{ij} e_{j_{FB}} + g_{ik} e_{k_{BR}})^2}{2} \quad (1.58)$$

This punishment condition implies that the outside option of rejecting a contract is zero, and so the individual rationality constraint is

$$\alpha_i + \gamma_i - \frac{e_{i_{FB}}^2}{2} + \theta e_{i_{FB}} \sum_{j=1}^{N} g_{ij} e_{j_{FB}} \geq 0 \tag{1.59}$$

These conditions must hold for the contract to be implementable. We see that at the optimum for the principal, no rents should be left to the agents, and so equations (1.57) and (1.59) should hold with equality. Therefore, equation (1.57) pins down γ_i so that

$$\gamma_i = \frac{e_{i_{FB}}^2}{2} - \theta e_{i_{FB}} \sum_{j=1}^{N} g_{ij} e_{j_{FB}} + \frac{\theta^2 (\sum_{j=1}^{N} g_{ij} e_{j_{FB}})^2}{2} \tag{1.60}$$

Then, I bring this equation to equation (1.59), and see that

$$\alpha_i = -\frac{\theta^2 (\sum_{j=1}^{N} g_{ij} e_{j_{FB}})^2}{2} \tag{1.61}$$

Therefore we have the parameter values as stated in the theorem. What is left to show is that if an agent rejects their contract, all other agents would like to ignore their contracts and play their best responses. That is, equation (1.58) must hold. For this to hold, we have

$$\alpha_i + \frac{\theta^2 (\sum_{j \neq k} g_{ij} e_{j_{FB}} + g_{ik} e_{k_{BR}})^2}{2} \geq \alpha_i + \gamma_i - \frac{e_{i_{FB}}^2}{2} + \theta e_{i_{FB}} (\sum_{j \neq k} g_{ij} e_{j_{FB}} + g_{ik} e_{k_{BR}}) \tag{1.62}$$

Cancelling the α_i and plugging in the γ_i gives

$$\frac{\theta^2 (\sum_{j \neq k} g_{ij} e_{j_{FB}} + g_{ik} e_{k_{BR}})^2}{2} \geq \frac{\theta^2 (\sum_{j=1}^{N} g_{ij} e_{j_{FB}})^2}{2} - \theta e_{i_{FB}} g_{ik} e_{k_{FB}} + \theta e_{i_{FB}} g_{ik} e_{k_{BR}} \tag{1.63}$$

I group like terms and apply the square to get

$$\theta e_{i_{FB}} g_{ik} (e_{k_{FB}} - e_{k_{BR}}) \geq \frac{\theta^2}{2} (2 g_{ik} (e_{k_{FB}} - e_{k_{BR}}) \sum_{j \neq k} g_{ij} e_{j_{FB}} + g_{ik}^2 (e_{k_{FB}} + e_{k_{BR}})(e_{k_{FB}} - e_{k_{BR}})) \tag{1.64}$$

If $g_{ik} = 0$ this is trivial. If it is 1, we can cancel, noting that

$$e_{k_{FB}} = 1 + \theta \sum_{j=1}^{N} g_{kj} e_{j_{FB}} + \theta \sum_{j=1}^{N} g_{jk} e_{j_{FB}} > \theta \sum_{j=1}^{N} g_{kj} e_{j_{FB}} = e_{k_{BR}} \tag{1.65}$$

So that

$$e_{i_{FB}} \geq \frac{\theta}{2}(2\sum_{j\neq k} g_{ij}e_{j_{FB}} + g_{ik}(e_{k_{FB}} + e_{k_{BR}})) \tag{1.66}$$

Plugging in for $e_{i_{FB}}$ and simplifying gives

$$1 + \frac{\theta}{2}g_{ik}e_{k_{FB}} + \theta\sum_{j=1}^{N} g_{ji}e_{j_{FB}} \geq \frac{\theta}{2}g_{ik}e_{k_{BR}} \tag{1.67}$$

Which follows immediately from $e_{k_{FB}} > e_{k_{BR}}$, and so the inequality must hold. Therefore agents who are affected by an agent who deviates will rather best respond, and play $e_i = \theta\sum_{i=1}^{N} g_{ij}e_j$. Therefore, if an agent rejects a contract, all of those who are affected by him will deviate from their contracts. Continuing by this logic, any agent affected by these secondary agents would also deviate from the contract and best respond. This effect diffuses throughout the network. This implies that if one agent deviates from the contract, then all who can be reached from him will also best respond, ignoring their contracts.

The final part to note is that some agents can not be reached by a deviator, and therefore may not want to deviate from the contract. Equation (A30) shows that this does not matter. Agents who are not affected by the deviator are willing to play effort zero, as the contract makes them indifferent between playing the first best effort, or playing an effort of zero. Therefore, when someone deviates, everyone either weakly or strictly wants to play effort of zero, leading to zero outside utility.

Therefore, agents are all willing to accept their contracts, play the first best effort, and get zero utility, implying this contract maximizes profits for the principal.

CHAPTER 2

Durable Good Pricing with Interdependent Demand

2.1. Introduction

With the emergence of the internet, and specifically social media, consumers are more connected than ever before. They are able to share their product experiences to a vast network of peers, sharing information to those with whom they interact. This connectivity enables consumers to become informed about new products and opportunities. Firms need to take this consumer communication into account when making strategy decisions. When introducing a new product to market, firms can rely on consumer communication as a mechanism to expand awareness. Consumer word of mouth acts as a substitute to advertising, and is especially important early in the product's lifespan. The potential for the product to become "viral" is vital for many new products to be successful in a variety of markets; for example it is particularly prevalent in the online video game industry.

This paper is interested in a monopolist's optimal price path when consumers engage in communication across generations. I consider the classic durable good problem, with incoming demand that depends on those that purchase in previous periods. Except for a small subset of the population, consumers are initially unaware of the product and learn about its existence when a neighbor purchases the good. Consumers are strategic in their purchasing decision; choosing when to buy given the optimal price path. This paper analyzes how consumers' communication affects the monopolist's optimal price path, when this
communication affects the future demand of the product.

The classic results in durable good pricing with strategic buyers date back to Stokey (1979). In this seminal work, she finds that when consumers are strategic in their

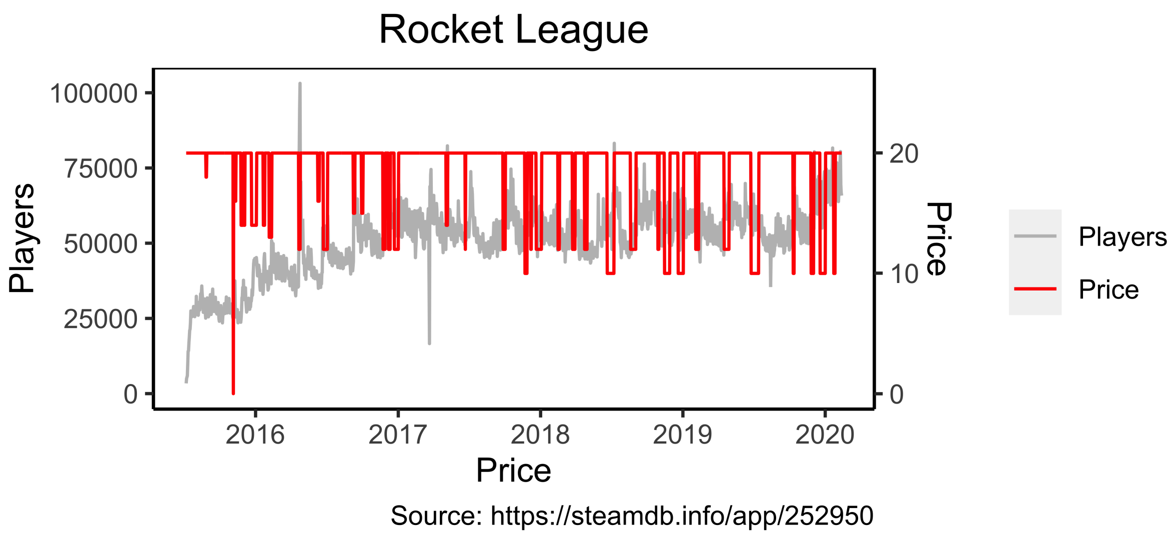

Figure 2.1: Rocket League Prices over time

purchasing decisions, the monopolist finds it optimal to keep a fixed price. Consumer's ability to delay implies that too many wait to purchase at a lower price, making price discrimination over time unprofitable. In contrast to this result, in many industries sales do exist. For example, in the online video game industry, the price paths of many games exhibit a sales structure. Consider the game "Rocket League". As seen in Figure 2.1,[1] the price stays relatively fixed, dropping to very quick, random sales. I theorize that the reason for these sales is to harness a consumer communication effect. For this reason the monopolist wishes to lower the price so that low valued informed consumers purchase and inform their peers about the good. By the time their peers have become aware of the good, the price is high once again, and the firm makes high revenues off of these later purchasers.[2] The desire to harness this consumer word of mouth is so strong that the price is pushed all the way to zero at some points.

In the video game industry, there are many price trackers that keep consumers aware of the current prices, and any sales that may arise. These price tracking softwares, such as the website "www.isthereanydeal.com", keep aware consumers in this industry in tune

[1]Price and usage data comes from the Steam database: steamdb.info.

[2]An AR(1) regression shows a significant increase in users the period after the price increases, implying that more consumers use the product even after the price has increased.

with prices. Even with these tools for consumers to monitor prices, which would allow consumers to be strategic in their purchasing decision, frequent sales are still seen for many online games.

The main result of the paper shows that, when consumers engage in word of mouth communication with each other, the price path will exhibit a sales structure as long as the monopolist is patient enough. Even though consumers are strategic, the monopolist finds it optimal to sell to the entire consumer base over time, holding sales in order to get consumers involved in the information diffusion. By serving all of the current consumer base the product, the monopolist is able to reach as many consumers as possible, whereas excluding consumers by setting a high price may cause demand to die out over time. The result is relevant as the monopolist and the consumers both become very patient. This means that sales are optimal even when consumers' incentive to delay increases.

This paper contributes to a variety of different literatures. First, it contributes to the works on dynamic pricing with strategic consumers. Classic results show that when a monopolist can commit to a price path, prices stay fixed over time. In contrast to this result, we see sales in a variety of settings. Garrett (2016) considers the changing valuations of consumers as a potential explanation for fluctuating prices. I offer another explanation: the externality of consumer communication. Other papers that consider dynamic pricing include Conlisk, Gerstner, and Sobel (1984), and Board (2008). These papers consider incoming demand over time, where the demand that enters in a period is independent of the current consumer base. In contrast, this paper considers when the incoming demand is interlinked with current consumers. Consumption today affects the incoming demand tomorrow, and the monopolist must take this into account when setting prices.

That sales are optimal to aid the diffusion of a product is not obvious. There is a classic literature on diffusion on networks, dating back to Bass (1969), that find monotone or single peaked price paths. In his seminal work, Bass considers a diffusion process, abstracting away from any network or strategic assumptions. Adoption of a good diffuses through random meetings of different consumers who find out about the good. Bass, Krishnan, and Jain (1994) introduce price into the diffusion model, where they find the common “S-shaped” diffusion curve. They find that the optimal price path increases until a certain point and then decreases thereafter. Importantly, this model has no network or strategic considerations. Several diffusion models introduce price as a decision variable, including Doland and Jeuland (1981), Robinson and Lakhani (1975) and Kalish (1983). In particular, Kalish assumes that consumers are myopic, but the monopolist chooses prices to maximize profits dynamically. Kalish finds that prices are monotone or single peaked. This is in stark contrast to the price path I find on a fixed network with consumer and monopolist strategies. This is because these papers consider a mean field approximation of diffusion, whereas in this paper I consider consumers on an explicit network structure. Mean diffusion models have been popular in the literature for a while. Jackson and Yariv (2005, 2007) have a few papers that use mean field approximations to analyze diffusion on networks. The mean field methods have been a popular approximation in diffusion models, but as seen in this paper, assuming a complex network gives vastly different results.

There has been a recent literature on diffusion on explicit networks. Sadler (2020) lays the framework for diffusion on networks using branching process theory. This paper uses a similar setting. Sadler notes how assuming an explicit network structure can give very different results than mean field models. I confirm this, showing that in contrast to the mean field models of the famed Bass curve models, I get an optimal price path that exhibits frequent sales. When a consumer does not purchase the good, they halt information spread throughout the network. As enough of these consumers build up, they create a bottleneck, which drives the price down in order to create future demand.

There is a large literature on static pricing in networks, going back to Campbell (2013), who considers a static pricing game when word of mouth communication is necessary for the consumers to become aware of the good. He finds that there is a set of prices that induce a “giant component” of purchasers, and it is one of these prices that the monopolist should set. In this paper I allow the price to change over time, and thus finding the giant component is not longer necessary. The monopolist can do better by lowering the price at certain periods, and keeping it high in others. Many other papers on static pricing on networks are interested about goods with consumption externalities. Examples of these are Fainmesser and Galeotti (2015), and Bloch and Querou (2013). In these papers there is a externalities in consumption between consumers. In contrast, in this paper the externality comes through in communication, and is enjoyed only by the monopolist.

There is also a large literature in marketing on consumer word of mouth communication, and in particular, how this word of mouth communication can be controlled by the firm. Campbell (2017) considers a model of "buzz" where the firm is interested in spreading information of its product as far as possible. Carroni, Pin, and Righi (2020) and Kamada and Ory (2020) consider firms who can reward referrals. They endogenize word of mouth communication and do not consider dynamic pricing. Zhong et al. (2018) consider a discount pricing strategy for a firm that takes into account viral marketing. They consider word of mouth on a variety of network structures and how price discounts affect information flow and profits. Shen, Duenvas, and Kapuscinski (2014), Li et al. (2018), Ajorlou et al. (2018) and Campbell (2015) consider dynamic pricing models when consumers engage in word of mouth communication. These papers find fluctuations in prices under various communication assumptions, but do not consider demand cannibalization that occurs when consumers are strategic. My model of consumer communication is most related to Campbell's, who assumes that consumers engage in word of mouth via a branching process. He considers the sales of a nondurable good to a myopic consumer base. In contrast I consider a durable good, which induces stronger consumer incentives to delay, and find not only do prices fluctuate, but all consumers are served the good over time. Lehmann and Esteban-Bravo (2006) consider when consumers should be given a good for free to jump-start a diffusion process. They find conditions when consumers should be given a product at the beginning of the products lifespan. In contrast, this paper shows that some consumers should also be given the product for free (or at a large discount) at intermediate points in its lifespan. A common theme across the word of mouth communication literature is offering discounts and finding price fluctuations. These papers largely ignore any strategic considerations of the consumer base. In contrast, this paper considers fully strategic buyers on an explicit network, and shows that sales will still occur in the face of demand cannibalization that comes from consumers' strategic delay.

The rest of the paper proceeds as follows. Section 2.2 sets up the model of consumer communication in a branching process framework. Section 2.3 gives the main result. Section 2.4 gives an example with two types of consumers. Section 2.5 concludes. Section 2.6 provides the proofs of the results.

2.2. The Model

Time is discrete and infinite, $t \in \{0, 1, 2...\}$. A monopolist introduces a new durable good to a consumer base at zero cost. Consumers and the monopolist share a common discount factor $\delta \in [0, 1)$. Consumers have an exogeneous taste for the good, distributed discretely as $\theta \in \{\theta_0, \theta_1, ...\theta_n\}$ with cumulative distribution function $F(\theta)$. The monopolist commits to a publicly known price path p_t at time $t = 0$. At each time t, consumers must be aware of the good in order to make a purchasing decision. An aware consumer with value θ who purchases at time t receives utility

$$\delta^t(\theta - p_t) \tag{2.1}$$

The monopolist is assumed to have zero cost, and thus commits to a price path at time zero that maximizes her expected total revenues. Let D_t be the size of the set of consumers who purchase at time t. Given a price path p_t, the monopolist's profit is

$$\Pi = \sum_{t=0}^{\infty} \delta^t p_t D_t \tag{2.2}$$

Timing is as follows. At $t = 0$, the monopolist publicly commits to a price path $\{p_t\}$. At the beginning of each period, a price is posted. Aware consumers who have not purchased decide whether or not to purchase. Then these purchasers engage in word of mouth to their neighbors. The next period these neighbors are aware and able to make a purchasing decision, after the new price is posted.

I assume a unit mass of consumers are aware of the good at time zero, and other consumers become aware through word of mouth when those before them purchase the good. Consumers are strategic in their purchasing decision. Given a price path $\{p_t\}$, once a consumer i becomes aware at time t_i, he chooses some time τ in which to purchase, such that

$$\tau \in argmax_{t \geq t_i} \quad \delta^t(\theta - p_t) \tag{2.3}$$

When a period begins, the price is set, and aware consumers who have not yet purchased decide whether or not to purchase the good. Consumers become aware through word of mouth, which I model using a homogeneous branching process. I assume that a unit mass of consumers are initially aware of the product, and each consumer who purchases the good informs N new consumers a single time. I allow N to be a positive integer, so that communication has an impact on the problem.[3] The branching process continues *ad infinum*.

There are three assumptions on the word of mouth process. First, the consumers only engage in word of mouth communication when they purchase the good. This implies that the price has an actual effect on the word of mouth process. If all consumers engaged in word of mouth whether or not they purchased, the prices would be irrelevant to the diffusion process and papers have shown that a fixed price is optimal.[4] Second, each consumer tells N new consumers deterministically the period that they have purchased, and there is no strategic consideration into the word of mouth decision. Consumers are strategic in their purchasing decision, but not in their contribution to the word of mouth process. Finally, each consumer tells N others, and these other consumers can be reached uniquely by the initial consumer, as I abstract away from cases where consumers can be reached in multiple ways.[5]

As an illustrative benchmark, I consider a single price monopolist to exhibit how the branching process is controlled by the price. In this case, demand grows as follows

$$E[D_t|D_{t-1}] = N(1 - F(p))D_{t-1} \tag{2.4}$$

Noting that $D_0 = (1 - F(p))$, we can write $E[D_t]$ in time zero terms as

$$E[D_t] = N^t(1 - F(p))^{t+1} \tag{2.5}$$

[3] If $N = 0$ then we have the classic problem of Stokey (1979).

[4] For example, see Board (2008)

[5] At the beginning of a diffusion process, the network of information flow resembles a tree network, and so this assumption can be thought of as relevant for the earlier stages of the products lifespan.

Assuming a unit mass of consumers are initially aware, the expected profits of a fixed price are therefore

$$\Pi = p(1-F(p))+\delta Np(1-F(p))^2+\delta^2N^2(1-F(p))^3+... = p(1-F(p))\sum_{t=0}^{\infty}(\delta N(1-F(p)))^t \tag{2.6}$$

First, to guarantee a convergence of profits, I assume that $\delta N \leq 1$. This can be thought as the monopolist having a low patience level, so that she only cares about a few periods ahead. This relates to the early time periods of a new product, when the whole population has not yet become aware of the good.

Under this assumption, the profits converge to

$$\Pi = \frac{p(1-F(p))}{1-\delta N(1-F(p))} \tag{2.7}$$

If, for simplicity, I assume that the number of types is large and uniformly distributed, so that we can approximate probabilities with the cumulative distribution function $F(\theta) = \theta$, this equation becomes

$$\Pi = \frac{p(1-p)}{1-\delta N(1-p)} \tag{2.8}$$

This is maximized at $p = \frac{\sqrt{1-\delta N}-(1-\delta N)}{\delta N}$. As the monopolist becomes more patient, we see that the price approaches zero. The value of future consumers becomes higher to the monopolist as she becomes more patient, and so there is a downward pressure on the price. Similarly if N becomes large so that $\delta N \to 1$, the price becomes zero as there are a large amount of potential buyers reached behind each aware consumer. As the monopolist becomes impatient, or N decreases, the price approaches the static monopoly price, which is $\frac{1}{2}$ in this simplified case.

There is a clear intuition and solution when the profits converge for all prices. When $\delta N > 1$, there is a critical price such that the branching process dies with some probability less than one. In these cases, prices which allow the branching process to continue indefinitely would be chosen, and profits would diverge.

To see this, consider the profit function for the single priced monopolist

$$p(1-F(p))\sum_{k=0}^{\infty}(\delta N)^k(1-F(p))^k \tag{2.9}$$

Since $\delta N > 1$, there exists a p^* so that $\delta N(1 - F(p^*)) > 1$. Setting the price at p^* forever will therefore lead to infinite profits, as long as $p^* > 0$. For some distributions $p^* = \theta_0$, but for all other distributions we have infinite profits with a price fixed above the lowest type valuation.

There is not profit maximizing price in these cases, but any positive price low enough so that $\delta N(1-F(p)) > 1$ will lead to infinite expected profits. The monopolist's problem is effectively to find the price at which the value of the branching process is expected to expand indefinitely, and set this price or lower to keep the information flow going forever, which in turns gives revenues forever. The demand effectively expands at a rate faster than the discounting, which causes profits to diverge.

For any fixed price p, the flow of information and purchasers will either (probabilistically) continue forever or will die out with probability 1, depending on the value of N. The probability of incoming demand halting corresponds the mean offspring amount for a fixed price being greater or less than 1. If the mean offspring size is greater than 1 then the information flow may continue indefinitely, while if the mean offspring size is less than 1 it will die out in finite time for certain.

In general there are 3 cases. First, in the case that $N = 1$ the branching process will die out for all prices except the one that serves all consumers. Second, the branching process may continue probabilistically for all prices below the static monopoly price, which the monopolist will clearly never set a price above. Third, the branching process may die out for certain at some prices below the static monopoly price, and continue probabilistically for some other prices. These cases depend on the mean offspring size based on the number of consumers each purchaser informs.

This idea translates into the profits of the monopolist. The offspring distribution of consumers will either explode towards infinity, or will eventually die out. The monopolist's choice of price will influence which of these happen, and the monopolist would prefer consumers to continue to show up. If the number of new consumers slows, the monopolist may find it optimal to serve more of the aware set, in order to bring in a larger consumer base. The monopolist gains marginal revenue from a consumer along two channels. She receives a private marginal revenue from the purchase of the consumer, but she also receives a continuation marginal revenue by her ability to reach, and therefore sell to, more consumers in the future.

2.3. Optimal Monopoly Pricing

With the tools and intuition of the branching process in hand I now consider the optimal price path of the monopolist. To maintain an interesting solution, I assume that $\delta N \leq 1$. As I allow $\delta \to 1$, the assumption boils down to $N = 1$. When this condition doesn't hold the monopolist can obtain infinite profits with a variety of pricing strategies, as discussed in the single price case.

Even though consumers are strategic, I find that it is optimal for the monopolist to serve all consumers, setting a price equal to the low end of the distribution of types infinitely often.

Theorem 2.3.1. *Let $N = 1$ and $\delta \to 1$. On the optimal price path, a price of $p = \theta_0$ is set infinitely often.*

The idea of the proof is as follows. When the monopolist sells the good to a consumer they receive two benefits. First, they receive revenue from the customer who purchases. Second, they reach those that this customer informs about the product, and have the ability to now sell to them as well. If the price is always above θ_0, then the number of new consumers decreases each period. There is a build up of consumers who do not purchase the good, and therefore there is a build up of unreached consumers. Since the network is fixed, there are potential consumers who can not be reached unless those that do not purchase the good eventually do buy the product. If δ is high enough, the monopolist will eventually wish to break through the blockage that these low valuation consumers cause, by offering a low price so that all consumers will purchase.

This force is strong enough that it overcomes the strategic actions of the consumers, and the loss from consumer delay is small with respect to the gain from reaching new consumers. The amount of cannibalization of demand is small compared to the previously unavailable demand base that the monopolist reaches. Even as consumers become very patient, the value from selling to low valued consumers outweighs all other considerations, leading to an optimal price path with sales.

The theorem gives a valuable insight into the price path of newly introduced products. When the monopolist wishes to build a consumer base, a potential strategy is to lower the price frequently. This causes many of those aware of the product to purchase it, and when these consumers engage in word of mouth, information spreads to new consumers. Since new consumers come in after the monopolist drops the price, there is a larger consumption base from which to extract high revenues in future periods. By allowing the information to spread to previously unaware people in the branching process, the monopolist opens herself up to a new consumer base from which she can charge high prices and gain previously unavailable revenues.

The result is in a simplified case, and relies on the assumption that $\delta N \leq 1$. As seen in the single price monopolist case, if $\delta N > 1$ (i.e. $N > 1$ and $\delta \to 1$) the monopolist can get infinite expected profits by setting a fixed price forever. In many cases, this fixed price will be below the classic monopoly price, confirming the result in Campbell (2013).

Corollary 2.3.1. *Let $N > 1$. Then, as $\delta \to \frac{1}{N}$, the optimal price path must go below any price p such that $(1 - F(p))N < 1$ infinitely often.*

2.4. Two Type Example

When I restrict to two types of consumers, I am able to derive the entire price path for the monopolist. I assume a simplified taste distribution for consumers, where consumers have a binary taste for the good $\theta \in \{\theta_L, \theta_H\}$. With two types, there are three types of price paths that could potentially be optimal for the monopolist. First, the monopolist could set the low price forever, selling to both types of consumers in the market. Second, the monopolist could sell to high type consumers θ_H only. Finally, the monopolist could choose to sell to both types, selling to only the high type in some periods and selling to both in others. I characterize when each type is optimal, showing that setting the high price forever is never optimal.

I assume that each consumer can inform N others about the existence of the good after they purchase, where N is no longer restricted to be one. Each consumer has high valuation for the good θ_H with probability $q < 1$. For an interesting solution, I assume the profits of setting the high price forever or the low price forever converge. There are three different price paths considered. First, if the monopolist sets the high price forever, profits are

$$\Pi_H = \theta_H q + \theta_H q^2 \delta N + \theta_H q^3 \delta^2 N^2 ... = \theta_H q \sum_{k=0}^{\infty} (q\delta N)^k = \frac{\theta_H q}{1 - q\delta N} \tag{2.10}$$

If the monopolist sets the low price forever profits are

$$\Pi_L = \theta_L + \theta_L \delta N + \theta_L \delta^2 N^2 ... = \theta_L \sum_{k=0}^{\infty} (\delta N)^k \tag{2.11}$$

Before deriving the sales price path, we see that if equation (2.11) is greater than equation (2.10), Theorem 2.3.1 still holds and we have the low price being set infinitely often. Depending on the parameters, the price path of low forever may be optimal. Even though pricing dynamically may not be optimal, it is never profit maximizing to set the high price forever.

I now consider a sales price path. Consider a price path that serves just the high types for the first $\tau - 1$ periods, and then all types in period τ. After the low price is set, the game resets as all agents aware purchase in time τ. Then the price path repeats, and there is a cyclical sale structure.

An optimal price path can be thought of as a dynamic allocation rule over time. That is, the monopolist chooses to allocate the good to only high types in some periods, and all types in others. A sales price path is one such that the monopolist chooses to allocate the good to the high type consumers only in periods $\{0, 1, .., \tau - 1\}$, and then to all types in period τ.

Many different prices will implement this allocation. The optimal prices set before the sale price are done so in such a way that no high type consumer ever strategically delays. We can work backwards from time τ. At time τ, the price is $p_\tau = \theta_L$. As we iterate backwards to time zero, the price increases so that the high type consumer is indifferent between purchasing today and waiting one more period. That is, in time $\tau - 1$ the price is found by the equation

$$\theta_H - p_{\tau-1} = \delta(\theta_H - \theta_L) \tag{2.12}$$

Or $p_\tau = \theta_H - \delta(\theta_H - \theta_L)$. In general, the price at time $\tau - k$ can be found from the recursive equation

$$\theta_H - p_k = \delta(\theta_H - p_{k+1}) \tag{2.13}$$

Or, iterating all the way to the last period, $p_k = \theta_H - \delta^{\tau-k}(\theta_H - \theta_L)$. Given these prices and the cyclical nature of the price path, we get the following profit, derived in full in Section 2.6.

$$\Pi_S = \frac{1-qN}{1-qN-\delta^{\tau+1}N(1-q+q^{\tau+1}N^\tau - q^{\tau+1}N^{\tau+1})} * \left(\frac{q\theta_H(1-(q\delta N)^\tau)}{1-q\delta N} - \frac{q(\theta_H-\theta_L)\delta^\tau(1-(qN)^\tau)}{1-qN} + \frac{\delta^\tau\theta_L(1-q+q^{\tau+1}N^\tau - q^{\tau+1}N^{\tau+1})}{1-qN}\right) \tag{2.14}$$

Note that Π_L and Π_H come from this profit equation by setting $\tau = 0$ and $\tau = \infty$ respectively.

Therefore, finding the optimal price path boils down to finding the τ that maximizes Equation (2.14). In general this is difficult to do analytically. That said, with some assumptions we can derive some comparative statics, as stated in the proposition. For simplicity, I assume that $\theta_L = 0$, so that setting the low price forever can never be optimal.

Proposition 2.4.1. *Let $q = \frac{1}{N}$, and $\theta_L = 0$. There are two cases depending on δN. If $\delta N \leq 1$, then the optimal sales time is decreasing in N. The optimal sales time is increasing in δ for all optimal $\tau \leq \tau^*$, and is decreasing in δ for all optimal $\tau \geq \tau^*$, for some time τ^*. If $\delta N > 1$, then the optimal sales time is increasing in N. The optimal sales time is decreasing in δ.*

The proof is in Section 2.6. The proposition shows how the monopolist reacts to changes in the number of consumers that a buyer informs, and the discount factor. Clearly if q becomes larger, the sales time increases. This makes the first comparative static unsurprising. Consider that $\delta N \leq 1$. If N increases (and q decreases keeping Nq constant), the time between sales goes down. There is a larger amount of consumers who do not purchase at each time, and they lead to a larger amount of untapped demand. Surprisingly, this static flips when $N\delta > 1$. In this case, larger N increases the time between sales. This is because the monopolist is happy to continue selling to the high types as demand is expanding at a higher rate, implying that the monopolist does not need to reach untapped parts of the network as quickly.

The static on δ is a bit more sensitive. In the case that $\delta N > 1$, the monopolist increases sales time in δ. That is because as the monopolist and consumers become more patient, the monopolist wants to put more time between sales, so that she gets higher revenue from the high types, who are now more willing to delay. As demand is expanding relatively quickly, sales are less important to the monopolist to continue the information diffusion. When $\delta N \leq 1$, the sales time varies depending on where it currently sits. If the time between sales is small, then it is decreasing in δ. If the time between sales is large, it is increasing in δ. Therefore, as consumers become more patient, if the time between sales was already small, the monopolist wishes to shorten it further, approaching just selling the low price forever. If the time between sales was large enough, the monopolist wishes to extend it, so that she can get higher revenues off the high type for longer.

2.5. Conclusion

Consumer communication is an important aspect of dynamic interactions between firms and consumers. In a connected world, interactions between a consumer and his peers can lead to differences in purchasing decisions and general awareness of products. In this paper we see that when new products are introduced to market, a firm can harness this communication to spread awareness as well as maximize profits. Importantly, a sales structure arises in markets where firms rely on this communication, even when consumers are strategic in their purchasing decisions.

Sales work as a substitute to other methods of information exposure, such as advertising. Instead of spending money on advertisements, the monopolist may find it a better strategy to lower the price, and entice the consumer base to effectively do their advertising for them. Even though dropping the price acts as a cost on the monopolist through lost revenues, the monopolist is able to harness the consumer base to spread knowledge of the good itself, without spending on advertising.

This methodology may be more effective in industries such as the video game industry for a variety of reasons. Consumers in this industry have a vast amount of products to choose between, and would likely be swayed to pick a product that their peers use, rather than one they see advertised towards them in the traditional sense. Consumers trust their peers more than a company, and therefore would likely listen to their peers over a firm when being told about a product. Though this model does not capture this aspect of consumer communication, this psychological phenomenon would make harnessing consumer communication more enticing.

2.6. Proofs

Proof of Theorem 2.3.1 To arrive at a contradiction, assume that the optimal price path does not reach θ_0 infinitely often. Let $T = 0$ be the time period after the last time θ_0 is the price. Then, there exists a price $p_t = \theta_i$ reached infinitely often such that the price path stays above this price for all times after some time τ_0. Let $\tau > \tau_0$ be such that $p_{\tau-1} = \theta_i$. We consider a new price path that is the same as the assumed optimal price path, with $p_\tau = \theta_0$, and $p = \theta_j$ for some j, for all prices at time $t > \tau$. That is, the monopolist continues the same price path with prices changed after time τ to a sale price and then a fixed price afterwards. I show that this price path gives higher profits than the assumed optimal path, implying that there can never be a "final" sale.

In the assumed optimal price path, the amount of buyers from time τ onward are at most

$$(1 - F(\theta_i))^\tau + (1 - F(\theta_i))^{\tau+1} + (1 - F(\theta_i))^{\tau+2} ... = \frac{(1 - F(\theta_i))^\tau}{F(\theta_i)} \tag{2.15}$$

Therefore, I can bound above the profits from time τ onward by

$$\delta^\tau (1 - F(\theta_i))^\tau \sum_{k=0}^{\infty} \delta^k (1 - F(\theta_i))^k = \frac{p_m \delta^\tau (1 - F(\theta_i))^\tau}{1 - \delta(1 - F(\theta_i))} \tag{2.16}$$

Where p_m is the static monopoly price, the highest price the monopolist would ever set.

Now, given this price path, there is an amount of consumers who would not purchase the good at any time. The size of these consumers is

$$F(\theta_i)+(1-F(\theta_i))F(\theta_i)+(1-F(\theta_i))^2F(\theta_i)+...+(1-F(\theta_i))^{\tau-2}F(\theta_i)+(1-F(\theta_i))^{\tau-1} \quad (2.17)$$

This equation sums to 1. On the new price path, the monopolist sells to all these consumers for a price of θ_0, and then sets a price for all future consumers of θ_j Therefore, the profits from time τ onward of the new price path is

$$\delta^\tau\theta_0 + \delta^{\tau+1}\theta_j(1-F(\theta_j)) + \delta^{\tau+2}\theta_j(1-F(\theta_j))^2... = \delta^\tau(\theta_0 + \frac{\delta\theta_j(1-F(\theta_j))}{1-\delta(1-F(\theta_j))}) \quad (2.18)$$

Note that after the sale no consumers ever delay as the price is fixed at θ_j.

Setting the price at the lower price θ_0 at time τ leads to cannibalization of sales back to time 0. I upperbound this cannibalization. At time k, a consumer will delay to time τ if

$$\delta^{\tau-k-1}(\theta-\theta_i) < \theta - p_k < \delta^{\tau-k}(\theta-\theta_0) \quad (2.19)$$

That is, the consumer was not willing to wait until period $\tau-1$ to purchase, but would now wait further than that to time τ to purchase. Many consumers may have this equality hold but also not wish to purchase at a new time on the modified price path, so this is an upperbound on the types who would wait. I upper bound this interval even more by replacing the leftmost term with $\delta^{\tau-k}(\theta-\theta_i)$, so that $\delta^{\tau-k}(\theta-\theta_i) < \theta-p_k < \delta^{\tau-k}(\theta-\theta_0)$. Rearranging, we see that this inequality is

$$\frac{p_k-\delta^{\tau-k}\theta_i}{1-\delta^{\tau-k}} < \theta < \frac{p_k-\delta^{\tau-k}\theta_0}{1-\delta^{\tau-k}} \quad (2.20)$$

So that within a period k, the fraction of the newly aware set that would delay is

$$F(\frac{p_k-\delta^{\tau-k}\theta_0}{1-\delta^{\tau-k}}) - F(\frac{p_k-\delta^{\tau-k}\theta_i}{1-\delta^{\tau-k}}) \quad (2.21)$$

Using the uniform approximation, we get that

$$\frac{p_k-\delta^{\tau-k}\theta_0}{1-\delta^{\tau-k}} - \frac{p_k-\delta^{\tau-k}\theta_i}{1-\delta^{\tau-k}} = \frac{\delta^{\tau-k}(\theta_i-\theta_0)}{1-\delta^{\tau-k}} \quad (2.22)$$

The loss can then be upperbounded by

$$\sum_{k=0}^{\tau-1}\frac{p_m\delta^k\delta^{\tau-k}(\theta_i-\theta_0)(1-F(\theta_i))^k}{1-\delta^{\tau-k}} = p_m\delta^\tau(\theta_i-\theta_0)\sum_{k=0}^{\tau-1}\frac{(1-F(\theta_i))^k}{1-\delta^{\tau-k}} \quad (2.23)$$

Therefore, the profits from the newly constructed price path are higher if the continuation profits dominate the remaining profits of the optimal price path plus the loss from waiting consumers. That is, if

$$\delta^\tau(\theta_0 + \frac{\delta\theta_j(1-F(\theta_j))}{1-\delta(1-F(\theta_j))}) > \frac{p_m\delta^\tau(1-F(\theta_i))^\tau}{1-\delta(1-F(\theta_i))} + p_m\delta^\tau(\theta_i - \theta_0)\sum_{k=0}^{\tau-1}\frac{(1-F(\theta_i))^k}{1-\delta^{\tau-k}} \quad (2.24)$$

Cancelling the δ terms and letting $\tau \to \infty$ gives

$$\theta_0 + \frac{\delta\theta_j(1-F(\theta_j))}{1-\delta(1-F(\theta_j))} > p_m(\theta_i - \theta_0)(\frac{1}{F(\theta_i)}) \quad (2.25)$$

Letting $\delta \to 1$, this is

$$\theta_0 + \frac{\theta_j(1-F(\theta_j))}{F(\theta_j)} > p_m(\theta_i - \theta_0)(\frac{1}{F(\theta_i)}) \quad (2.26)$$

Letting θ_j be chosen to maximize $\frac{\theta_j(1-F(\theta_j))}{F(\theta_j)}$ will solve this equation. For example, if $F(\theta) = \theta$, then $\theta_j = 0$, and so we have $1 > \frac{1}{2}\theta_i(\frac{1}{\theta_i}) = \frac{1}{2}$.

Therefore the assumed optimal price path is not optimal, implying there is no "final" sales period.

Proof of Corollary 2.3.1

The proof is almost identical to that of Theorem 1. To arrive at a contradiction, assume that the optimal price path does not reach θ_0 infinitely often. Let $T = 0$ be the time period after the last time θ_0 is the price. Then, there exists a price $p_t = \theta_i$ reached infinitely often such that the price path stays above this price for all times after some time τ_0. Let $\tau > \tau_0$ be such that $p_{\tau-1} = \theta_i$. We consider a new price path that is the same as the assumed optimal price path, with $p_\tau = \theta_0$, and $p = \theta_j$ for some j, for all prices at time $t > \tau$. That is, the monopolist continues the same price path with prices changed after time τ to a sale price and then a fixed price afterwards. I show that this price path gives higher profits than the assumed optimal path, implying that there can never be a "final" sale.

In the assumed optimal price path, the amount of buyers from time τ onward are at most

$$N^\tau(1-F(\theta_i))^\tau + N^{\tau+1}(1-F(\theta_i))^{\tau+1} + N^{\tau+2}(1-F(\theta_i))^{\tau+2}... = \frac{N^\tau(1-F(\theta_i))^\tau}{1-N(1-F(\theta_i))} \quad (2.27)$$

Therefore, I can bound above the profits from time τ onward by

$$\delta^\tau N^\tau (1-F(\theta_i))^\tau \sum_{k=0}^{\infty} \delta^k N^k (1-F(\theta_i))^k = \frac{p_m \delta^\tau N^\tau (1-F(\theta_i))^\tau}{1-\delta N(1-F(\theta_i))} \tag{2.28}$$

Where p_m is the static monopoly price, the highest price the monopolist would ever set.

Now, given this price path, there is an amount of consumers who would not purchase the good at any time. The size of these consumers is

$$F(\theta_i) + N(1-F(\theta_i))F(\theta_i) + N^2(1-F(\theta_i))^2 F(\theta_i) + ... + N^{\tau-1}(1-F(\theta_i))^{\tau-1} \tag{2.29}$$

This sums to $\frac{F(\theta_i)-N^{\tau-1}(1-F(\theta_i))^\tau(N-1)}{1-N(1-F(\theta_i))}$ On the new price path, the monopolist sells to all these consumers for a price of θ_0, and then sets a price for all future consumers of θ_j Therefore, the profits from time τ onward of the new price path is

$$\delta^\tau \left(\frac{F(\theta_i) - N^{\tau-1}(1-F(\theta_i))^\tau (N-1)}{1-N(1-F(\theta_i))} \right) (\theta_0 + \delta^{\tau+1} N \theta_j (1-F(\theta_j)) + \delta^{\tau+2} N^2 \theta_j (1-F(\theta_j))^2 ...)$$
$$= \delta^\tau \left(\frac{F(\theta_i) - N^{\tau-1}(1-F(\theta_i))^\tau (N-1)}{1-N(1-F(\theta_i))} \right) (\theta_0 + \frac{\delta N \theta_j (1-F(\theta_j))}{1-\delta N(1-F(\theta_j))}) \tag{2.30}$$

Note that after the sale no consumers ever delay as the price is fixed at θ_j.

Setting the price at the lower price θ_0 at time τ leads to cannibalization of sales back to time 0. I upperbound this cannibalization. At time k, a consumer will delay to time τ if

$$\delta^{\tau-k-1}(\theta-\theta_i) < \theta - p_k < \delta^{\tau-k}(\theta-\theta_0) \tag{2.31}$$

That is, the consumer was not willing to wait until period $\tau-1$ to purchase, but would now wait further than that to time τ to purchase. Many consumers may have this equality hold but also not wish to purchase at a new time on the modified price path, so this is an upperbound on the types who would wait. I upper bound this interval even more by replacing the leftmost term with $\delta^{\tau-k}(\theta-\theta_i)$, so that $\delta^{\tau-k}(\theta-\theta_i) < \theta - p_k < \delta^{\tau-k}(\theta-\theta_0)$. Rearranging, we see that this inequality is

$$\frac{p_k - \delta^{\tau-k}\theta_i}{1-\delta^{\tau-k}} < \theta < \frac{p_k - \delta^{\tau-k}\theta_0}{1-\delta^{\tau-k}} \tag{2.32}$$

So that within a period k, the fraction of the newly aware set that would delay is

$$F(\frac{p_k - \delta^{\tau-k}\theta_0}{1-\delta^{\tau-k}}) - F(\frac{p_k - \delta^{\tau-k}\theta_i}{1-\delta^{\tau-k}}) \tag{2.33}$$

Using the uniform approximation, we get that

$$\frac{p_k - \delta^{\tau-k}\theta_0}{1-\delta^{\tau-k}} - \frac{p_k - \delta^{\tau-k}\theta_i}{1-\delta^{\tau-k}} = \frac{\delta^{\tau-k}(\theta_i - \theta_0)}{1-\delta^{\tau-k}} \tag{2.34}$$

The loss can then be upperbounded by

$$\sum_{k=0}^{\tau-1} \frac{p_m \delta^k \delta^{\tau-k}(\theta_i - \theta_0) N^k (1-F(\theta_i))^k}{1-\delta^{\tau-k}} = p_m \delta^\tau (\theta_i - \theta_0) \sum_{k=0}^{\tau-1} \frac{N^k(1-F(\theta_i))^k}{1-\delta^{\tau-k}} \tag{2.35}$$

Therefore, the profits from the newly constructed price path are higher if the continuation profits dominate the remaining profits of the optimal price path plus the loss from waiting consumers. That is, if

$$\delta^\tau \left(\frac{F(\theta_i) - N^{\tau-1}(1-F(\theta_i))^\tau (N-1)}{1-N(1-F(\theta_i))} \right) (\theta_0 + \frac{\delta N \theta_j (1-F(\theta_j))}{1-\delta N(1-F(\theta_j))})$$
$$> \frac{p_m \delta^\tau N^\tau (1-F(\theta_i))^\tau}{1-\delta N(1-F(\theta_i))} + p_m \delta^\tau (\theta_i - \theta_0) \sum_{k=0}^{\tau-1} \frac{N^k (1-F(\theta_i))^k}{1-\delta^{\tau-k}} \tag{2.36}$$

Assume that $N(1-F(\theta_i)) < 1$. Cancelling the δ terms and letting $\tau \to \infty$ gives

$$(\frac{F(\theta_i)}{1-N(1-F(\theta_i))})(\theta_0 + \frac{\delta N \theta_j (1-F(\theta_j))}{1-\delta N(1-F(\theta_j)}) > p_m(\theta_i - \theta_0)(\frac{1}{1-N(1-F(\theta_i))}) \tag{2.37}$$

Letting $\delta \to \frac{1}{N}$, this is

$$\tag{2.38}$$

Which is the same condition as in Theorem 1. Letting θ_j be chosen to maximize $\frac{\theta_j(1-F(\theta_j))}{F(\theta_j)}$ will solve this equation. For example, if $F(\theta) = \theta$, then $\theta_j = 0$, and so we have $1 > \frac{1}{2}\theta_i(\frac{1}{\theta_i}) = \frac{1}{2}$. Therefore the corollary holds.

Proof of Proposition 2.4.1

With the prices as pinned down in Equation 20, we can write the profit within a sales cycle as

$$\pi = p_0 q + p_1 q^2 \delta N + ... + p_{\tau-1} q^\tau \delta^{\tau-1} N^{\tau-1} + ((1-q)\sum_{k=0}^{\tau-1}(qN)^k + (qN)^\tau)\delta^\tau \theta_L$$
$$= q \sum_{k=0}^{\tau-1} p_k q^k \delta^k N^k + ((1-q)\sum_{k=0}^{\tau-1}(qN)^k + (qN)^\tau)\delta^\tau \theta_L \tag{2.39}$$

Substituting in for the prices gives

$$\pi = q\sum_{k=0}^{\tau-1}\theta_H(q\delta N)^k - \delta^\tau q(\theta_H-\theta_L)\sum_{k=0}^{\tau-1}(qN)^k + ((1-q)\sum_{k=0}^{\tau-1}(qN)^k + (qN)^\tau)\delta^\tau\theta_L \quad (2.40)$$

Applying the sums gives

$$\pi = \frac{q\theta_H(1-(q\delta N)^\tau)}{1-qN\delta} - \frac{\delta^\tau q(\theta_H-\theta_L)(1-(qN)^\tau)}{1-qN} + \frac{\delta^\tau\theta_L(1-q+q^{\tau+1}N^\tau - q^{\tau+1}N^{\tau+1})}{1-qN} \quad (2.41)$$

This profit repeats on a growing consumer base every $\tau+1$ periods. Total profit Π is

$$\Pi_S = \pi + \delta^{\tau+1}N\frac{(1-q+q^{\tau+1}N^\tau - q^{\tau+1}N^{\tau+1})}{1-qN}\pi + \delta^{(\tau+1)^2}N^2\left(\frac{(1-q+q^{\tau+1}N^\tau - q^{\tau+1}N^{\tau+1})}{1-qN}\right)^2\pi... \quad (2.42)$$

The total profit from the sales price path is

$$\Pi_S = \frac{1-qN}{1-qN-\delta^{\tau+1}N(1-q+q^{\tau+1}N^\tau - q^{\tau+1}N^{\tau+1})} * \left(\frac{q\theta_H(1-(q\delta N)^\tau)}{1-q\delta N} - \frac{q(\theta_H-\theta_L)\delta^\tau(1-(qN)^\tau)}{1-qN} + \frac{\delta^\tau\theta_L(1-q+q^{\tau+1}N^\tau - q^{\tau+1}N^{\tau+1})}{1-qN}\right) \quad (2.43)$$

When $\theta_L = 0$ and $q = \frac{1}{N}$, this profit becomes

$$\Pi_S = \frac{1}{1-\delta^{\tau+1}(N\tau-\tau+N)}(\frac{1}{N}\sum_{k=0}^{\tau-1}\theta_H(\delta)^k - \delta^\tau\frac{1}{N}(\theta_H)\sum_{k=0}^{\tau-1}1) \quad (2.44)$$

Which equals

$$\frac{\theta_H}{N(1-\delta^{\tau+1}(N*\tau-\tau+N))}\left(\frac{1-\delta^\tau}{1-\delta} - \delta^\tau * \tau\right) \quad (2.45)$$

Taking a first order condition with respect to τ gives

$$\frac{\theta_H\delta^\tau(1-\delta N)(1-\delta^{\tau+1}+(\tau+1)ln(\delta))}{-N(1-\delta)(1+\tau\delta^{\tau+1}-N(\tau+1)\delta^{\tau+1})^2} = 0 \quad (2.46)$$

Which implies that

$$\theta_H\delta^\tau(1-\delta N)(1-\delta^{\tau+1}+(\tau+1)ln(\delta)) = 0 \quad (2.47)$$

I apply the implicit function theorem to this equation. I obtain the following three derivatives:

$$\frac{\partial}{\partial \tau} = \theta_H \delta^{\tau}(1-\delta N)Ln(\delta)(2(1-\delta^{\tau+1}) + (\tau+1)Ln(\delta) \tag{2.48}$$

$$\frac{\partial}{\partial N} = -\theta_H \delta^{\tau+1}(1-\delta^{\tau+1} + (\tau+1)ln(\delta) \tag{2.49}$$

$$\frac{\partial}{\partial \delta} = \delta^{\tau-1}(\theta_H(1-\delta^{\tau+1})(1-2\delta N+2\tau(1-\delta N)) - H(\tau+1)(\delta N-\tau(1-\delta N))Ln(\delta)) \tag{2.50}$$

First we consider the change in τ when N changes.

$$\frac{\partial \tau}{\partial N} = \frac{\delta(1-\delta^{\tau+1} + (\tau+1)Ln(\delta))}{(1-\delta N)Ln(\delta)(2(1-\delta^{\tau+1}) + (\tau+1)Ln(\delta))} \tag{2.51}$$

If $\delta N < 1$, this derivative is always negative. If $\delta N > 1$, this derivative is always positive.

Next, we consider the change in τ when δ changes. This is

$$\frac{\partial \tau}{\partial \delta} = -\frac{(1-\delta^{\tau+1})(1-2\delta N+2\tau(1-\delta N)) - (\tau+1)(\delta N-\tau(1-\delta N))Ln(\delta)}{\delta(1-\delta N)Ln(\delta)(2(1-\delta^{\tau+1}) + (\tau+1)Ln(\delta))} \tag{2.52}$$

By the previous comparative static, when $\delta N \leq 1$, the denominator is positive, and so the sign of this derivative is the sign of the numerator. In this case, the numerator is less than zero for small τ and greater than zero for large τ. Therefore the monopolist increases the time between sales if the time is already large enough, and decreases it if the time is small enough.

CHAPTER 3

Good Allocation on Social Networks: A Mechanism Design Approach

3.1. Introduction

With the emergence of the internet, companies are able to collect information on consumers in greater detail than ever. Particularly, firms are able to see consumers' relations with each other on a large scale, viewing the underlying social network of a consumer base. Companies take advantage of this knowledge in many ways; they may offer discounts for informing friends about a good, or offer an influential person a product for free, assuming that this person will inform many new potential consumers about the good. In particular, when a firm is introducing a new product to an unaware consumer base, they may wish to set prices in such a way to harness this consumer word of mouth.

This paper is interested in a monopolist's optimal price path when facing a network of consumers who are initially unaware of the product. Consumers can only become aware of the product when one of their neighbors purchases the product themselves. I assume that the monopolist has complete information over the network of consumers and knows who is currently aware. I show that with this information, the optimal price path for the monopolist depends on both the private and "network" values of the consumers. The monopolist wishes to offer discounts to both consumers who are aware early in the game, and those who are critical to the information flow.

Firms often take into consideration the close networks of their consumers. Influencer marketing, for example, harnesses consumers who are highly connected in a network. These "influencers" are used to spread the knowledge of a products existence to the rest of a consumer base. When a firm offers a product to an influencer they are likely thinking of who the influencer can inform, and potentially who those informed can in turn make aware of the product. It is unlikely that the firm thinks too far ahead in the diffusion process. Also, it is found empirically that average path lengths and diameters in networks are often very short. For example, the famed Milgrom letter experiment (1967), which aimed to send a letter from one random person to another, found path lengths with a median of 5.

For this reason I consider relatively small networks. This is to reflect a forward looking monopolist who doesn't look too far into the future, and to match the empirical findings on the diameter and path lengths of networks. Nodes can also be thought of as collective groups of consumers. To reach a new group, the monopolist must serve others through the network, who then engage in word of mouth, moving the information of the product throughout the network.

I consider a variety of pricing mechanisms. I consider price discrimination based on network position, no price discrimination, and price discrimination based on both network position and private valuation. Price discrimination based on network position, considered in Section 3, allows me to isolate the network communication value. Since the monopolist can set prices for each consumer individually, she can internalize the value of the communication. I find that the optimal pricing strategy sets prices based on the "private" marginal revenue of the consumer and the "public" value, which is the value of future consumers, discounted by the probability that this potential consumer is reached uniquely because of the current consumer.

In Section 4, I consider a single price monopoly. In this case the intuition from Section 3 holds. As the monopolist accumulates critical buyers, there is a downward pressure on the price. I show that even with infinitely patient buyers, the monopolist still offers sales in certain networks. I provide some intuitive comparative statics on the number of aware consumers and their network effects.

Section 5 characterizes the fully optimal mechanism for a variety of networks. The monopolist is able to discriminate on both network position and individual taste. Because the monopolist can fully control the information flow, she creates the optimal network of communication flow based on consumer types and the initial aware set.

3.1.1. Literature Review

This paper contribute to a variety of different strands of literature. First, it contributes to the vast literature on word of mouth communication of durable goods. Papers in this literature usually rely on mean field assumptions to derive price paths. The famed "Bass curve" predicts that as the good diffuses over a network of consumers, prices will start low, and raise monotonically until they reach a peak, and then they decrease over time. Bass (1969) pioneered this area, and has many papers showing this price path. Importantly, in this literature the price is often taken as an exogeneous variable, and consumers are assumed to interact probabilistically with any others in the population. In contrast to this, I assume that the price is a decision of the monopolist, and that consumers can only inform their neighbors. This leads to different predictions than in the mean field models.

Given the small network feature, I am able to derive an exact optimal price path without making any sort of approximation on the network. Most papers on diffusion and price paths on networks make the mean field assumptions or branching process assumptions on the networks (see Bass Krishnan, and Jain (1994) and Sadler (2020) for both types of assumptions respectively). In contrast I do not allow for mean field or branching structure assumptions on the network. I consider smaller networks, in the class of multilayer graphs. Since the networks are small, this can be thought of as a finite period game, with relatively few stages. By allowing the monopolist to see the entire network and the aware consumers, I am able to directly calculate the network value of each consumer exactly.

There is a vast literature on dynamic pricing of durable goods, starting with the classic durable good problem posed by Stokey (1979), which finds that the monopolist does best to not price dynamically when consumers are strategic. Conlisk, Gerstner, and Sobel (1984) and Board (2008) consider the dynamic monopoly problem with incoming demand. In this paper incoming demand is endogeneous to the current purchasers of the good. New consumers tomorrow are made aware by those who purchase today. In this way the future consumer base depends on today's purchasers. This endogeneous incoming demand makes it so that the monopolist does wish to price dynamically, taking network values of the consumers into account.

There is a large literature on pricing in networks, going back to Campbell (2013), who considers a static pricing game when word of mouth communication is necessary for the consumers to become aware of the good. He find that there is a price that induces a "giant component" of purchasers, and it is this price that the monopolist should set. In this paper I consider a directed word of mouth process on a small network, and so finding a giant component is not necessary. Instead, I consider the total network value of consumers and price based off of these values. Campbell (2015) also considers a monopolist selling a nondurable good to an initially unaware consumer base. In this paper the good is durable, giving consumers a stronger incentive to delay. In much of the paper I assume that consumers are infinitely patient, and still find that sales can be optimal. Many other papers on pricing on networks are interested in goods with complementarities in consumptions. Examples of these are Fainmesser and Galeotti (2015), and Bloch and Querou (2013). In these papers there is a complementarity in consumption for consumers. In contrast, in this paper the complementarity comes through in communication, and mainly enjoyed only by the monopolist. Consumers enjoy the complementarity in communication through the pricing discounts they receive.

The rest of the paper proceeds as follows. Section 3.2 presents the model. Section 3.3 gives the price discrimination solution. Section 3.4 discusses a single priced monopolist. Section 3.5 gives the optimal mechanism. Section 3.6 concludes.

3.2. Model

A monopolist introduces a durable good to a group of consumers who live on a complex network structure G, which is commonly known by both the monopolist and the consumers. A subset of consumers are initially aware of the existence of the product, but a majority of the consumer base is unaware of the product at the beginning of the game. Consumers become aware of the good if one of their neighbors purchases the good. In particular, the timing of the game is as in Figure 3.1. At the beginning of each period, the monopolist sets the price (or prices in the price discrimination case). Aware consumers who have

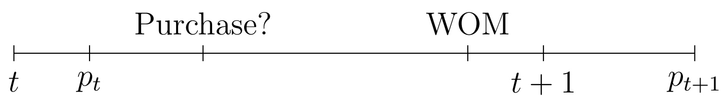

Figure 3.1: Timing of Word of Mouth

yet to purchase the good see the price, and decide whether or not to purchase the good. Then, after making a purchasing decision, those who purchase the good engage in word of mouth communication to all of their unaware neighbors. Then, the next period, after the new price is set, the newly aware consumers can make a purchasing decision of their own.

The monopolist and the consumers share a common discount factor $\delta \in [0, 1]$. The monopolist is assumed to have zero costs and therefore maximizes her expected revenues. The monopolist commits to a price path conditional on who purchases the good each period. Given this price path, aware consumers choose an optimal time at which to purchase in order to maximize their utility. Consumer utility is made up of a private valuation for the good $\theta \in [\underline{\theta}, \overline{\theta}]$ which is distributed by a distribution function $F(\theta)$. Consumers' utility from purchasing at time t is

$$U(\theta, p_t) = \delta^t(\theta - p_t) \tag{3.1}$$

Thus, strategic consumers choose an optimal time τ at which to purchase the good, and for a consumer who is made aware at time $t-1$, we can characterize $\tau(\theta, t)$ as

$$\tau(\theta, t) \in argmax_{s \geq t}\ \delta^s(\theta - p_s) \tag{3.2}$$

Note that $\tau(\theta, t)$ can possibly be infinite, implying that the consumer never purchases the good given the price path. Given the consumers optimal decisions, the monopolist maximizes her total expected revenue

$$\Pi = \sum_{t=1}^{T} \int_{\underline{\theta}}^{\overline{\theta}} \delta^{\tau(\theta,t)} p_{\tau(\theta,t)} dF(\theta) \tag{3.3}$$

The monopolist commits to a price path $\{p_t\}$ to maximize these profits. We consider a variety of pricing capabilities of the monopolist in the following sections, restricting to a single price, allowing third degree price discrimination, and finally finding the optimal mechanism.

3.2.1. The Network of Communication

I consider networks with N consumers. I allow the networks to be of arbitrary size, but consider relatively small networks for exposition. This implies that the monopolists pricing strategy is also of finite length. The existence of a final pricing period is vital to the problem at hand, for it allows us to solve the monopolist's problem through backwards induction. We can find the prices in the final period, given an absence of word of mouth value in this period, and work backwards.

The word of mouth process could be quite complicated, given the vast number of networks and options of the flow of information across networks. With this in mind, we consider a subgroup of networks which match the ex-post information flow of word of mouth. These are networks known as "multilayer" networks. A directed network G is multilayer if it is an acyclic network, with the additional property that for all triplets (i, j, j'), if $(ij, ij') \in G \times G$, then there is no directed path from j to j' or j' to j. Thus these are networks with no cycles, and also if there is a common link towards two nodes, then they can not be connected to each other.

These multilayer networks are a natural setting for the word of mouth process. In fact, after the game has ended, the realized network of word of mouth communication will be a multilayer network. For this reason, we focus on these networks, as justified by the following lemma.

Lemma 3.2.1. *The realized word of mouth network on a general undirected graph G is a multilayer graph.*

The lemma is trivial to show. Consider a network G, and an consumer $i \in G$. Consider this consumer becomes aware. He can tell consumers who are connected to him about the good after he is aware, and they in turn can tell others about the good. But, since consumer i initially becomes aware, it is impossible for him to become aware again, and therefore he can not have a path of communication back to himself, implying he can not be involved in a cycle. Therefore the network must be acyclical.

Next, to show the second property of a multicyclical graph holds, consider three consumers (i, j, j') such that i is connected directly to j and j'. Consider that consumer i becomes aware of the good and informs consumer j and j'. Then it is impossible to consumer j to reach consumer j' through a path of communication and visa-versa since as they are already aware they can again not be made aware a second time. Therefore the ex-post graph of communication is a multilayer graph. To visualize how this works, consider the network shown in Figure 3.2.

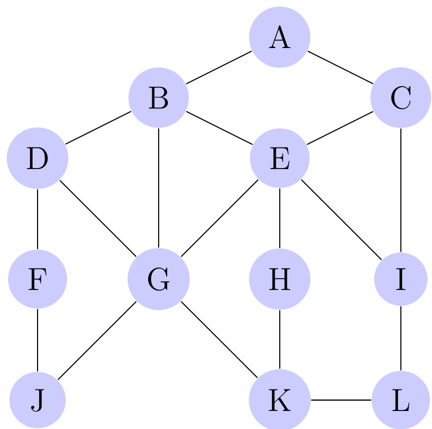

Figure 3.2: A Network of Communication

We assume that the only aware consumer is consumer A. Then from this initial seeding we can put all the possible directions of word of mouth, assuming that consumers tell all of their connections if they engage in word of mouth. This is shown in Figure 3.3.

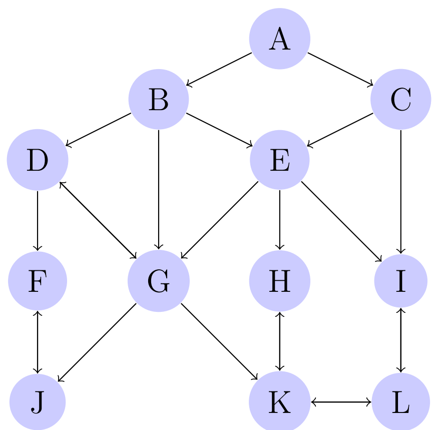

Figure 3.3: The Potential Directions of Communication

From this graph there are assumptions to be made over who engages in the word of mouth and who does not. We assume that those who purchase the good engage in the word of mouth. For illustrative purposes, here we will assume that everyone purchases the good when they become aware, as if the product were free. Therefore, the ex-post information flow looks as in Figure 3.4.

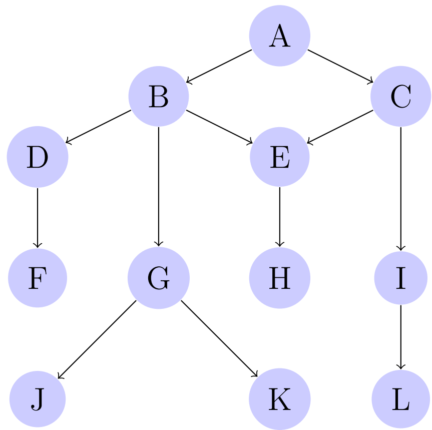

Figure 3.4: Ex-Post Flow When All Purchase

This network is a multilayer graph, and would be a tree if not for the links from B to E and C to E. Now, consider that everyone purchases when made aware except for consumer B. Then the ex-post network is as in Figure 3.5.

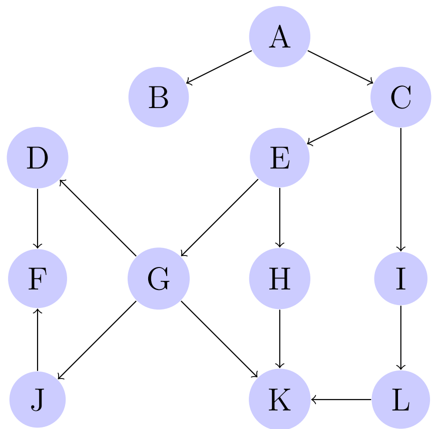

Figure 3.5: Ex-Post Information Flow When B Doesn't Purchase.

This much more complicated network is also a multilayer graph. Simple multilayer graphs, such as trees, can give a simple exposition to how the word of mouth process could act, but the multilayer aspect also allows for some more interesting networks for the information flow. Consider for example, consumer F in the above network. This consumer can be informed by either consumer D or consumer J. Since both of these consumers can only be made aware by consumer G, if the underlying network is as above, we see that either of consumers D or J can make consumer F informed after they purchase. The monopolist only needs one of them to purchase to make consumer F aware, and so has a trade off of wanting one of these consumers to purchase to allow for a purchase from consumer F, while also trying to maximize profits from the currently aware consumers. It is this trade-off that is at the heart of the analysis. The monopolist wishes to gain high revenues off of the consumers who are currently aware of the good, but at the same time want a lot of consumers to purchase the good in order to spread the information on the existence of the good throughout the network.

3.2.2. The Solution Strategy

To solve for the optimal prices for the monopolist, we follow a general solution approach akin to Board (2008). Since consumers are acting optimally, the envelope theorem applies, and we see that the utility for an consumer at time t is

$$u_t(\theta) = \int_{\underline{\theta}}^{\theta} \delta^{\tau(x,t)} dx + u_t(\underline{\theta}) \tag{3.4}$$

Utility for the lowest type is zero. The usual integration by parts argument gives

$$\int_{\underline{\theta}}^{\overline{\theta}} u_t(\theta) dF(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} \delta^{\tau(\theta,t)} (1 - F(\theta)) d\theta \tag{3.5}$$

The welfare generated by an consumer at time t is

$$\int_{\underline{\theta}}^{\overline{\theta}} \delta^{\tau(\theta,t)} (\theta + \tilde{V}_{i,\tau(\theta,t)}) dF(\theta) \tag{3.6}$$

Where $\tilde{V}_{i,\tau(\theta,t)}$ is the expected Word of Mouth value generated by the consumer purchasing the good and sharing the information of the good with his neighbors. This $\tilde{V}_{i,\tau(\theta,t)}$ is vital to the monopolists problem. Given that profits for the monopolist are equal to the total welfare minus the utility of a consumer, we see that the profit gained by the monopolist from a single consumer i at time t is

$$\Pi_{i,t} = \int_{\underline{\theta}}^{\overline{\theta}} \delta^{\tau(\theta,t)}(\theta + V_{i,\tau(\theta,t)} - \frac{(1-F(\theta))}{f(\theta)})f(\theta)d\theta \tag{3.7}$$

$V_{i,\tau(\theta,t)}$ is the monopolist's value derived from the future welfare. Thus the monopolist would like to serve a consumer whenever their "adjusted marginal revenue" is positive. In classic mechanism design literature, we have the marginal revenue values of $\theta - \frac{(1-F(\theta))}{f(\theta)}$. In this model we have that value as the private marginal revenue, and then we have the externality value $V_{i,\tau(\theta,t)}$ which is gained by the monopolist if and only if the consumer is served the good. Therefore, based on the private valuation θ and the externality value V, the monopolist would like to serve consumer i at time t if and only if

$$MR_i(\theta, V) = \theta + V_{i,\tau(\theta,t)} - \frac{(1-F(\theta))}{f(\theta)} \geq 0 \tag{3.8}$$

The value of $V_{i,\tau(\theta,t)}$ is independent of θ, and so there is a cut off rule based on θ of which consumers should be served. To make sure this cut off rule is well defined, we make the following assumption.

Assumption 3.2.1. $\theta - \frac{(1-F(\theta))}{f(\theta)}$ *is increasing in* θ.

This assumption allows us to calculate the solution as an optimal cutoff rule, as formalized in the following lemma.

Lemma 3.2.2. *At each time* t*, and for each consumer with value* $V_{i,\tau(\theta,t)}$ *there is a* θ^* *such that the monopolist wishes to serve all consumers with type* $\theta \geq \theta^*$*, and does not wish to serve consumers with types* $\theta < \theta^*$

The existence of a cutoff rule means that I just need to find the cutoffs at each time (and for each consumer in the price discrimination case) in order to find the optimal strategy of the monopolist. I first start with the case of price discrimination over network position. This case gives insight into the value of network positioning, and is helpful in finding the optimal pricing strategies in single price as well.

3.3. Price Discrimination

We begin the analysis by considering the case where the monopolist can see the positions of the consumers on the network, but can not see their private taste valuations. The

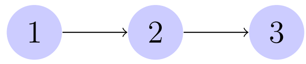

Figure 3.6: Simple Line

monopolist can set a price for each consumer who is actively aware of the good. Allowing the monopolist to price discriminate based on the network position allows us to isolate the individual contribution to the network and see how, based on network location, different consumers have different value to the monopolist. With this in mind, we work one of the simplest examples to get insight into how the network value works. In this example, there is only ever a single consumer who is aware and has not purchased, and therefore the difference in pricing strategies is not important.

Example 3.3.1. *Consider a line of consumers, with private values distributed uniformly on the unit interval. Consumer 1 is initially aware, while the others must be informed in order to purchase. Let* $\delta = 1$*.*

This example is illustrated in Figure 3.6. We consider network value of each consumer in reverse. If consumer 3 becomes aware, then the whole network is aware and his network value is 0. Since $F(\theta) = \theta$*, the private value is* $\theta - \frac{1-F(\theta)}{f(\theta)} = 2\theta - 1$*, and so consumer 3 is served if his type is above the cutoff of* $\theta = \frac{1}{2}$*. To implement this cutoff, consumer 3 is offered a price* $p = \frac{1}{2}$ *and purchases with probability* $\frac{1}{2}$*. Now, we consider consumer 2. If consumer 2 is served, he engages in word of mouth communication with consumer 3, and this is the only way that consumer 3 can be informed of the good. Then, after consumer 3 is made aware, he purchases with a* $\frac{1}{2}$ *probability and the monopolist makes* $\frac{1}{2}$ *off of him. Therefore, the network value of consumer 2 is* $\frac{1}{4}$*. So, the monopolist would like to serve consumer 2 if* $2\theta - 1 + \frac{1}{4} \geq 0$*, or* $\theta \geq \frac{3}{8}$*.*

Finally, we consider the first consumer. If they purchase the good, then they share word of mouth with consumer two, who purchases with probability $\frac{5}{8}$*. So, with probability* $\frac{5}{8}$*, the monopolist would get the* $\frac{3}{8}$ *from consumer 2, plus the value* $\frac{1}{4}$ *from consumer 3, leading to a network value of* $\frac{25}{64}$*. So, the monopolist wishes to sell the good to consumer 1 if* $2\theta - 1 + \frac{25}{64} \geq 0$*, or if* $\theta \geq \frac{39}{128}$*. Prices are increasing over time conditional on consumers purchasing, or stay constant if a consumer does not purchase the good. Therefore, the optimal price path for the monopolist is* $p_1 = \frac{39}{128}$*,* $p_2 = \frac{3}{8}$*,* $p_3 = \frac{1}{2}$*. The price only increases*

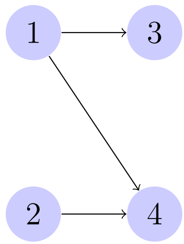

Figure 3.7: Competing Word of Mouth

if the current consumer purchases the good.

We see that as we move through a path, in this case along the tree, we have that prices are increasing as consumers purchase. As we move along a path, the continuation value of that path decreases for the monopolist and thus the price offered to the current consumer approaches the monopoly price. The cutoff for each consumer depends on their network value, which depends on the consumers that get reached strictly because of the current consumer being served the good. To understand how the current consumer must be "vital" in reaching the other consumers, consider the following example, illustrated in Figure 3.7.

Example 3.3.2. *Competing Word of Mouth*

In this network, we assume that consumers 1 and 2 are at first aware of the good, while consumers 3 and 4 need to be made aware. Now, consumer 3 can only be made aware of the good if consumer 1 purchases the good himself, as consumer 3 can only be reached on the unique path from consumer 1. This means, of course, that the probability of being reached uniquely by consumer 1's purchasing is 1.

Now, consumer 4 can be reached by each of consumer 1 and consumer 2. So, the network value that each consumer has at the time they are offered the price is dependent on the other consumers type, and whether the other consumer is served. We continue to let $F(\theta) = \theta$ and $\delta = 1$. Let θ_1^* and θ_2^* be the cut offs of consumer 1 and 2 respectively at the time that they are both initially aware. Then, the marginal value of consumer 1 is

$$M(\theta_1, V_1) = 2\theta_1 - 1 + \frac{1}{4} + \frac{1}{4}\theta_2^* \tag{3.9}$$

That is, consumer 1 has a private marginal value, then the value of reaching consumer 3 is $\frac{1}{4}$, and the value of reaching consumer 4 is $\frac{1}{4}$. The latter is reached uniquely by *consumer 1 with probability equal to θ_2^*, since that is the probability that consumer 2 does not get served the product. Similarly, for consumer 2, we have their marginal value as*

$$M(\theta_2, V_2) = 2\theta_2 - 1 + \frac{1}{4}\theta_1^* \tag{3.10}$$

We want to serve either of these consumers when each of these equations are weakly greater than 0. So, we set equations (9) and (10) equal to 0, and find the cutoffs to be $\theta_1^* = \frac{20}{63}$ and $\theta_2^* = \frac{29}{63}$. The cutoffs come from the probabilities of the other consumers being served and form a fixed point. The network value of consumer 2 in this example is zero if consumer 1 also purchases, and the network value of consumer 1 is the $\frac{1}{4}$, the value from consumer 3, if consumer 2 has also purchased. So, the value to the network comes from the probability that a consumer is the unique way to reach other consumers.

Continuing to consider this example, we can see what happens in the second round if one of consumers 1 and 2 were to not purchase in the first round. A consumer will not purchase in the first round if they either have too low of a value, or if they are being strategic and expect a lower price in the future. Consider that consumer 1 purchases and consumer 2 does not purchase. Then, in the next period, consumer 2 now has no network value, as consumer 4 is now aware, and therefore there is no one left for consumer 2 to inform. Thus the new cutoff, and therefore price, for consumer 2 is $\frac{1}{2}$. This is greater than the original cut off, and so consumer 2 would not want to delay their purchase if they believe the other consumer is purchase the good. A similar observation exists for consumer 1. If consumer 2 purchases, consumers 1's network value now only comes through his connection to consumer 3. So, consumer 1 would be served with a cutoff, and therefore price, of $\frac{3}{8}$. Therefore consumer 1 would also not wish to delay their purchases. If neither consumer purchases the good then the monopolist can commit to keeping the same prices forever, again leading the consumers to not wish to delay.

From the example, note there is no need to distinguish between myopic and strategic consumers in small networks with price discrimination. The network value of each consumer is weakly decreasing and therefore prices offered to the consumer would be increasing weakly. This allows us to isolate the effect of the network on prices, and so we

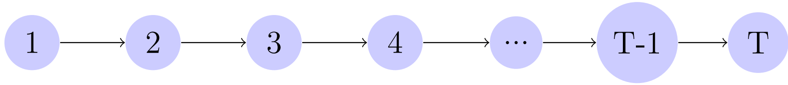

Figure 3.8: Arbitrarily long line

can examine how network structure alone affects the prices set by the monopolist.

We see that the main issue on calculating the network value of the monopolist come from finding the probabilities that consumers are reached uniquely because of the currently aware consumer. We can find the probabilities that these consumers are made aware because of our current aware consumer, and then find the value we would get off of these consumers in a recursive manner. Let λ_{ijt} be the probability that consumer j is reached in the word of mouth process because of consumer i being served at time t. Then, the total network value for consumer i at time t is

$$V_{i,\tau(\theta,t)} = \sum_{j \in G} \lambda_{ijt} \delta^{t_j} \int_{\theta^*_{t_j}}^{\bar{\theta}} MR(\theta_j, 0) dF \tag{3.11}$$

The monopolist values the network value of an consumer who purchases at their optimal time as the sum over all the probabilities of reaching other consumers, times their discounted marginal revenues, integrated up from the cutoffs. This network value is consistently changing as consumers in the network purchase the good and themselves engage in word of mouth communication. The monopolist can figure out the values by seeing how information would flow through the network if all consumers purchased and engaged in word of mouth from the initially aware consumers onward, and then working backwards from the last consumers to be made aware.

The probabilities can be calculated by finding, based on the cutoffs, how likely consumers are to be reached by the consumer of interest, and only the consumer of interest. To see how these probabilities work, we consider the following example.

Example 3.3.3. *Consider a line of arbitrary length, and seen in Figure 3.8. We assume that consumer 1 is the initially aware consumer. Now, assume that consumer i just becomes aware in the period $t-1$. Then in period t, we can find the probabilities of another consumer becoming aware when consumer i is served the good. If $j \leq i$, then $\lambda_{ijt} = 0$ as these consumers are already aware and have purchased the good. If $j = i+1$,* *then the probability of them becoming aware is 1, i.e.* $\lambda_{ijt} = 1$. *If* $j > i + 1$, *then the probability of becoming aware is* $\lambda_{ijt} = \prod_{k\in(i,j)}(1 - F(\theta_k^*))$. *That is, it is the probability that every consumer between i and j is also served so that consumer j becomes aware.*

Now, we can see that $i < j$ *implies* $V_{i,t} > V_{j,t}$ *so that*

$$\theta_i^* = MR_i^{-1}(0, V_{i,t}) < MR_j^{-1}(0, V_{j,t}) = \theta_j^* \tag{3.12}$$

.

Prices, just as cutoffs, are increasing over time as network values decrease from consumer to consumer.

In fact, for many distributions, as the monopolist becomes very patient, prices on the line of consumers will fall to zero as the number of consumers increases. Consider, for example, that types are distributed uniformly on the unit interval, so that $F(\theta) = \theta$, *and assume a discount factor* $\delta \leq 1$. *Then we can write the profit as*

$$\pi = p_1(1 - p_1) + \delta(1 - p_1)p_2(1 - p_2) + ... = \sum_{i=1}^{N} p_i\delta^{i-1}\prod_{k=1}^{i}(1 - p_k) \tag{3.13}$$

Consider the optimal choice of p_1. *Taking a first order condition, we get*

$$p_1 = \frac{1}{2} - \frac{1}{2}\sum_{i=2}^{N} p_i\delta^{i-1}\prod_{k=2}^{i}(1 - p_k) \tag{3.14}$$

Considering the last term, we note that

$$\sum_{i=2}^{N} p_i\delta^{i-1}\prod_{k=2}^{i}(1-p_k) \geq max_p \quad \sum_{i=2}^{N} \delta^{i-1}p(1-p)^{i-1} = max_p \quad \frac{p(1-p)\delta(1-(\delta(1-p))^{N-1})}{1-\delta(1-p)} \tag{3.15}$$

In this limit, this implies that

$$\sum_{i=2}^{\infty} p_i\delta^{i-1}\prod_{k=2}^{i}(1 - p_k) \geq max_p \quad \frac{p(1-p)\delta}{1-\delta(1-p)} \tag{3.16}$$

The right hand side is maximized when $p = \frac{\sqrt{1-\delta}-(1-\delta)}{\delta}$. *Equation 16 becomes*

$$\sum_{i=2}^{\infty} p_i\delta^{i-1}\prod_{k=2}^{i}(1 - p_k) \geq \frac{(1-\sqrt{1-\delta})(\sqrt{1-\delta}-(1-\delta))}{\delta\sqrt{1-\delta}} \tag{3.17}$$

Taking this value to the first order condition on the first price, we get

$$p_1 \leq \frac{1}{2} - \frac{1}{2}\left(\frac{(1-\sqrt{1-\delta})(\sqrt{1-\delta}-(1-\delta))}{\delta\sqrt{1-\delta}}\right) \tag{3.18}$$

As $\delta \to 1$, we see that this implies that $p_1 \leq 0$. Since p_1 would never be negative, this implies that as the number of consumers on the line increases, and as the monopolist is patient enough to care about all consumers almost equally, the price approaches zero.

The examples show an algorithmic way to find the optimal prices in the price discrimination case at each time period. The algorithm is mapped out in the following theorem.

Theorem 3.3.1. *Pricing Algorithm*

When the monopolist can discriminate based on network position, he serves customers if and only if

$$\theta_i - \tfrac{1-F(\theta_i)}{f(\theta_i)} + V_{i,t} \geq 0$$

Let $\theta^*_{i,t}$ be the threshold for to consumer i to be served at time t. Then at each t, prices can be found by the following system of equations

$$\begin{bmatrix} \theta^*_1 - \frac{(1-F(\theta^*_1)}{f(\theta^*_1)} + \sum_{j\in G}\lambda_{1jt}\delta^{t_j}\int_{\theta^*_\tau}^{\overline{\theta}} MR(\theta,0)dF \\ \theta^*_2 - \frac{(1-F(\theta^*_2)}{f(\theta^*_2)} + \sum_{j\in G}\lambda_{2jt}\delta^{t_j}\int_{\theta^*_\tau}^{\overline{\theta}} MR(\theta,0)dF \\ \ldots \\ \theta^*_N - \frac{(1-F(\theta^*_N)}{f(\theta^*_N)} + \sum_{j\in G}\lambda_{Njt}\delta^{t_j}\int_{\theta^*_\tau}^{\overline{\theta}} MR(\theta,0)dF \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix}$$

Such that $\theta^*_i \geq \underline{\theta}$. The inequality binds if $\theta^*_i > \underline{\theta}$. Setting $p_{j,t} = \theta^*_{j,t}$ maximizes profits.

The result follows from the exposition above. Since the network value is weakly positive, the monopolist wishes to serve a greater fraction of types θ than the monopoly amount without any network effect. At each time period, the monopolist must update its probabilities that consumers are reached by serving those in the aware set. It solves the system of equations at each time period to maximize its profits, until the information flow stops. This result follows for general multilayer networks, but it is computationally easier for the monopolist if they are considering smaller networks. It quickly becomes computationally difficult to calculate optimal price paths over large networks.

The inequality in the theorem stems from the possibility of consumers being offered the good at the lowest end of the taste distribution. It is possible that the desire to reach consumers later in the network is so strong that the monopolist wishes to give the good to a consumer no matter what. This is considered in the following example.

Example 3.3.4. *Large Network Values*

Let $\delta = 1$ and $F(\theta) = \theta$. Consider two consumers who are initially aware, and that both of them are connected to the same N additional consumers. The values for the initial consumers one and two are

$$2\theta_1 - 1 + \frac{N\theta_2}{4} \tag{3.19}$$

and

$$2\theta_2 - 1 + \frac{N\theta_1}{4} \tag{3.20}$$

The system of equations that must be solved for the optimal cutoffs are

$$2\theta_1 - 1 + \frac{N\theta_2}{4} \geq 0 \tag{3.21}$$

and

$$2\theta_2 - 1 + \frac{N\theta_1}{4} \geq 0 \tag{3.22}$$

If $N \leq 8$, then the optimal cutoffs are $\theta_1 = \theta_2 = \frac{4}{N+8}$. If $N > 8$, we have corner solutions. The monopolist would like to reach the untapped consumer base with certainty, and therefore offers it for free (a price of zero) to one of the consumers. Assume that the cutoff for consumer 1 is $\theta_1^* = 0$. Then, by Equation 3.22, $\theta_2^* = \frac{1}{2}$. Given this, we see that Equation 3.11 holds as an inequality, as $\frac{N}{8} > 1$ when $N > 8$. The profit for the monopolist when setting the same price to both consumers is $\frac{(N+4)^2}{4(N+8)}$ and when giving the good to one for free is $\frac{N+1}{4}$. The latter overtakes the former as N increases past 8. Therefore, if the number of consumers to be reached is large and there is a large enough continuation value for the monopolist, she will want to offer the good for free to one of the consumers, while implementing the monopoly cutoff with the monopoly price for the other consumer. Thus it is possible that the monopolist offers the good for free to some,

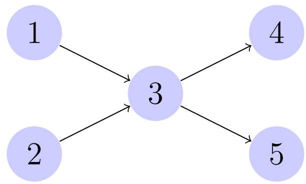

Figure 3.9: Critical consumers

while charging a price to others that does not take into account the network that could be reach by the agent who is offered the good for free.

The pricing strategy may seem at first to be strictly increasing over time, as in all of the examples we have seen thus far. Of course, as more consumers have purchased the good, there is less value over the network for which the monopolist to gain, implying that he would wish to raise the price over time as the good diffuses over the network. We have observed that this is true when the good is diffusing down a line, or along unique paths. But this is not true in general, as seen in the next example.

Example 3.3.5. *Consider the following network, shown in Figure 3.9. We assume that the leftmost consumers, consumer 1 and 2, are initially aware of the product. We wish to find the optimal price path for the monopolist. As in the previous examples, we assume that $F(\theta) = \theta$ and $\delta = 1$.*

In this network, either consumer 1 or 2 (or both) can inform consumer 3, who then can inform both consumers 4 and 5. So, starting backwards at consumers 4 and 5, we see that the cutoff for each consumer will be $\theta_4^* = \theta_5^* = \frac{1}{2}$. Then, for consumer 3, we see that his marginal value is

$$M(\theta_3, V_3) = 2\theta_3 - 1 + \frac{1}{2} \tag{3.23}$$

This is greater than or equal to 0 when $\theta_3 \geq \frac{1}{4}$ so that the cutoff (and therefore price) is $\theta_3^* = p_3 = \frac{1}{4}$. Finally, we consider the first stage, where consumers 1 and 2 are the only aware consumers in the network. The marginal value of consumer 1 is

$$M(\theta_1, V) = 2\theta_1 - 1 + \theta_2^*(\frac{3}{4} * \frac{1}{4} + \frac{3}{4} * \frac{1}{2}) \tag{3.24}$$

For consumer 2, the marginal value is similar

$$M(\theta_2, V) = 2\theta_2 - 1 + \theta_1^*(\frac{3}{4} * \frac{1}{4} + \frac{3}{4} * \frac{1}{2}) \quad (3.25)$$

Setting both marginal values equal to zero gives cut off values of $\theta_1^* = \theta_2^* = \frac{16}{41}$*. These are also the prices charged by the monopolist. So, we can see that* $\theta_1^* > \theta_3^*$*. Even though consumer 1 and 2 are aware before consumer 3, they are offered higher prices than consumer 3 is. This is because neither consumer 1 or 2 are uniquely critical to the word of mouth process. Consumer 3 can be reached by either (or both) of the initially aware consumers. This means that neither of the two are critical to the flow of information, in the way that consumer 3 is critical. Consumer 3 gets a bigger "discount" for his consumption because he is uniquely necessary to the continuation of the word of mouth process.*

This idea of critical consumers is important to understanding why the price would drop. Even though the total value of the network is decreasing as consumers purchase, as there are less consumers available to purchase the good, when the diffusion process runs into these "blocking" consumers, there is a downward pressure on the price for these consumers, since there is no other way to reach parts of the network through the word of mouth process. As we will see in the next section, it is these consumers who force "sales" i.e. periods of price drops when the monopolist is restricted to setting a single price.

3.4. Single Price Monopolist

We turn to a monopolist restricted to setting a single price for all consumers. We assume still that the monopolist knows the exact network structure, and knows exactly who in the network is aware of the good and has not purchased. Thus the information is the same as in the previous subsection, but now the monopolist must choose a single price to set.

We have seen in a few examples that being able to price discriminate across consumers does not have bite. For example, on a line of consumers, there is only one active consumer each period and therefore no need to price discriminate across consumers. In other situations where there are many paths to consumers and paths are no longer necessarily unique, being bound to a single price is restrictive to the monopolist, who must balance gaining current revenues from consumers and expanding the potential consumer base.

With a single price monopolist, the strategy of consumers becomes nontrivial. In the previous section, a consumer had no incentive to delay their purchase. The monopolist would always offer them a price that would only increase over time. Therefore a consumer could never do better than purchasing the good when they are first offered a price or never purchasing. This is not the case when the monopolist offers a fixed price. Consumers can try and wait and get joined with an important, "critical" cohort and receive a lower price as the monopolist feels downward pressure on the price. The optimal strategy of the monopolist now has to balance the private marginal revenues of the aware group of consumers with their combined network value. Aware consumers who have no network effect will drive the price up, keeping the price higher even when critical consumers are reached.

For the monopolist to find the optimal price path, we have to aggregate the marginal revenues of each consumer to find the aggregate marginal revenue. Using the same methodology in the previous section we know that the marginal revenue of an individual consumer i is

$$MR_i(\theta, V) = \theta + V_{i,\tau(\theta,t)} - \frac{(1 - F(\theta))}{f(\theta)} \tag{3.26}$$

When the monopolist sets a single price, he needs to consider the cumulative marginal revenue based off of the active set of consumers. Let $A(t)$ be the set of active consumers who are aware but have not purchased at time t. Then, at each time t, the cumulative marginal revenue is

$$\sum_{i \in A(t)} \theta + V_{i,\tau(\theta,t)} - \frac{(1 - F(\theta))}{f(\theta)} \tag{3.27}$$

The private value is the same for every consumer, so we can isolate the network values. The cumulative network value is

$$\sum_{i \in A(t)} V_{i,t} = \sum_j \sum_i \lambda_{ijt} \delta^{\tau(\theta_j,t)} \int_{\theta^*_\tau}^{\overline{\theta}} MR(\theta, 0) dF = \sum_j \lambda_{Aj\tau} \delta^{\tau(\theta_j,t)} \int_{\theta^*_\tau}^{\overline{\theta}} MR(\theta, 0) dF \tag{3.28}$$

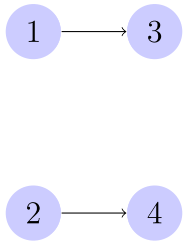

Figure 3.10: Individual Word Of Mouth Value

$\lambda_{Aj\tau}$ is the probability that an consumer is reached uniquely from one of the consumers in the aware set. This is the aggregate probability that an unaware consumer is made aware by a single other consumer.

To gain intuition on the aggregate probabilities and how they relate to the marginal value of the active set, we consider a network on four consumers, where consumers 1 and 2 are initially aware, and consumers 3 and 4 are unaware. For simplicity we again assume that $F(\theta) = \theta$, and $\delta = 1$. First, we consider the network configuration as in Figure 3.10.

In this network, the network effects are independent, and so the problem is similar to the price discrimination case. The marginal revenue from each aware consumer is $M_i(\theta, V) = 2\theta - 1 + \frac{1}{4}$, implying that the aggregate network effect is

$$M_1(\theta, V) + M_2(\theta, V) = 2\theta - 1 + \frac{1}{4} + 2\theta - 1 + \frac{1}{4} = 4\theta - 2 + \frac{1}{2} \tag{3.29}$$

We see that the cutoff here is $\theta^* = \frac{3}{8}$.

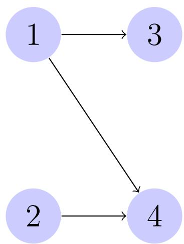

Figure 3.11: Joint Word of Mouth Effect

Now, we consider adding a link between consumer 1 and consumer 4, as seen in Figure 3.11. Now the consumer 1 has some individual word of mouth effect that does not depend on consumer 2, but consumer 2's word of mouth effect depends on consumer 1.

Given only one cut off is set for the two of them, the marginal value of consumer 1 is $M_1(\theta, V) = 2\theta - 1 + \frac{1}{4} + \frac{\theta}{4}$. The marginal value of consumer 2 is $M_2(\theta, V) = 2\theta - 1 + \frac{\theta}{4}$. The word of mouth value of consumer 2 depends on the probability that consumer 1 is not served. With probability θ, consumer 2 is the sole reason that consumer 4 becomes aware of the good. Similarly, with the same probability consumer 1 is the sole reason consumer 4 becomes aware. Consumer 1 also can make consumer 3 aware, and he is the only way that consumer 3 can become aware. Taking these two marginal values, we can find the cumulative marginal revenue of the aware set, which is

$$M_A(\theta, V) = 2\theta - 1 + \frac{1}{4} + \frac{\theta}{4} + 2\theta - 1 + \frac{\theta}{4} = \frac{9}{2}\theta - \frac{7}{4} \tag{3.30}$$

This gives a cutoff of $\theta^* = \frac{7}{18}$. In the previous section we analyzed this network in price discrimination terms. We found the cutoffs of $\theta_1^* = \frac{20}{63}$ and $\theta_2^* = \frac{29}{63}$. Therefore, consumer 1 is harmed and consumer 2 benefits when they must be pooled together, which is the classic result we get when consumers the monopolist is forced to offer a single price to heterogeneous consumers.

If we also add a link from consumer 2 to consumer 3 as seen in Figure 3.12, then consumers 1 and 2 are identical, with marginal revenues of $M(\theta, V) = 2\theta - 1 + \frac{\theta}{2}$. The total marginal revenue is

$$M(\theta, V) = 4\theta - 2 + \theta \tag{3.31}$$

So that the cutoff is $\theta^* = \frac{2}{5}$.

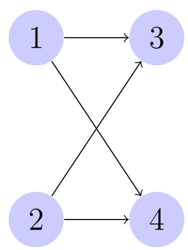

Figure 3.12: Joint Word of Mouth Effect

Considering Figures 3.10, 3.11 and 3.12, we first see that no consumer has any incentive to delay, as once another consumer purchases, their network value weakly decreases and most all consumers become aware. The expected profits the monopolist gains from from these three networks are $\frac{25}{32}$, $\frac{121}{144}$, and $\frac{9}{10}$ respectively. The monopolist's profit is increasing in the number of links between consumers. In particular, it is increasing in the number of links from the active set to the unaware consumers. Adding a link between aware consumers or towards a consumer who has already purchased is clearly superfluous. But, adding a link to the unaware consumers increases the probability that this consumer is reached, while at the same time making each individual consumer less important. This raises the cut off, which, as the cutoff is always below the static monopoly cutoff, raises the revenues.

Another observation is that the cutoffs also generally raise with the number of agents that are aware. To exhibit this property, we consider the complete network of N agents, with M of them initially aware. Staying with our assumptions on $F(\theta)$ and δ, we get the following marginal revenue for an individual aware consumer

$$M_i(\theta, V) = 2\theta - 1 + \theta^{M-1} * \frac{N-M}{4} \tag{3.32}$$

That is, with probability θ^{M-1}, this individual agent is responsible for the rest of the network ($N - M$ consumers) to become aware, and then be offered a price of $\frac{1}{2}$. The total value of the aware set is therefore

$$M_A(\theta, V) = M(2\theta - 1) + M\theta^{M-1} * \frac{N-M}{4} \tag{3.33}$$

The cutoff is so that this marginal revenue is equal to zero. A closed form solution for θ is not easily found, but we can use the implicit function theorem to see that

$$\frac{\partial\theta}{\partial M} = -\frac{\theta(4\theta - 8\theta^2 + 2M\theta^M - N\theta^M + M^2\theta^M ln[\theta] - MN\theta^M ln[\theta])}{M(-8\theta^2 - M\theta^M + M^2\theta^M + N\theta^M - MN\theta^M)} \tag{3.34}$$

This term is strictly greater than zero, implying that the cutoff used by the monopolist is increasing in the number of agents aware. This is intuitive, as each individual agent becomes less important to the information flow, the cutoff raises as there are more chances to find a high valued consumer who will purchase and make the rest of the network aware of the good. More aware consumers makes the individual consumer less important to the information flow, which raises the cutoff.

Similarly, we use the implicit function theorem to see that

$$\frac{\partial \theta}{\partial N} = -\frac{\theta^{M-1}}{8 + (M-1)(N-M)\theta^{M-2}} \tag{3.35}$$

This term is clearly less than zero, implying that, while keeping the number of aware fixed, the cutoff decreases with the size of the potential consumers in the second period. This is intuitive. Since there is a larger potential base to reach in the second period, the monopolist finds it important to have at least one purchaser, and therefore lowers the cutoff to increase that probability.

Of course, this logic does not follow when agents have individual network effects. Consider a network of N individual agents aware, who are each connected to one other unaware consumer. Then, each aware consumer's marginal revenue does not rely on the probability of others purchasing, and the cutoff stays constant at the level as if there was only a single aware consumer. Profits are obviously higher, but the same fraction of consumers are served no matter how many consumers there are.

So we see that the number of both links and aware consumers both can put an upward pressure on the price, as it enhances the word of mouth process. In general though, this is not true, as counter examples exist to show that more links or more aware consumers may not necessarily raise the cutoffs. In general, there are not clean comparative statics to be made.

Thus we see that the probabilities of being reach through the active set purchasing affects on aggregate the cutoff levels. If we look in general at the marginal value of a set $A(t)$, we get the following aggregate value

$$MR(\theta, V_{A,t}) = |A(t)|(\theta - \frac{1-F(\theta)}{f(\theta)}) + \sum_j \lambda_{Ajt}\delta^{\tau} \int_{\theta^*_{\tau}}^{\overline{\theta}} MR(\theta, 0) dF \tag{3.36}$$

That is, for each θ, the monopolist receives the private marginal revenue from each consumer, multiplied by the number in the active set, plus the aggregate word of mouth value. The monopolist wishes to serve consumers with types θ such that $MR(\theta, V_{A,t}) \geq 0$, or, rearranging (3.36), consumer types such that

$$\frac{\sum_j \lambda_{Ajt}\delta^{\tau} \int_{\theta^*_{\tau}}^{\overline{\theta}} MR(\theta, 0) dF}{|A|} \geq -(\theta - \frac{1-F(\theta)}{f(\theta)}) \tag{3.37}$$

This equation shows that the monopolist would like to serve types θ such that the average word of mouth value is at least as great as the loss of revenue from the current actively aware consumers. The right hand side of this inequality is positive for all viable cutoffs except for the static monopoly cut off, where it is zero. The monopolist has a trade off, between serving active consumers for high prices, and offering low prices to get more of the market involved.

Equation (37) gives insight into how the marginal revenue of the active set changes over time. If the monopolist were to accumulate buyers with no network value, this would force the cut off closer to the static monopoly cut off, as increasing the size of $A(t)$ makes the first term of the marginal revenue more consequential, and since this term is negative, it pushes the value of θ up. On the other hand, if agents who are "critical" are in the active group, for example agents such that $\lambda_{ijt} = 1$ for some j. These consumers cause a downward pressure on the cutoffs and prices, as they necessarily need to consume in order for others to become aware of the good. As these consumers accumulate, their existence puts a downward pressure on the price, until the monopolist lowers the price enough for them to consume, if at all. We will discuss how this updating occurs in the the upcoming section, but first a discussion on consumer strategy is necessary.

3.4.1. The Strategy of Consumers

When the monopolist is restricted to setting a single price, consumers have an incentive to delay their purchases, in order to get a lower price in the future. We have seen in the price discrimination case that an agent does not wish to delay, as their network value can only decrease, leading to a higher cutoff, and therefore price, offered by the monopolist. This is no longer the case with a single price monopolist, as consumers can get lumped into groups which change in value over time. A consumer may find it valuable to delay their purchase in order to get a better price in the future. To take this into account, we consider what the optimal strategy of the consumer would be. As we have seen in the above, the price of the monopolist each period depends on the "active" set of consumers. Therefore, consumers choose to wait based on their expectation of future prices, which depend on their beliefs of the future active sets of agents. For simplicity, and to make the consumers as optimistic as possible, I assume that consumers believe that all those in the active set will purchase the good in the given time period in which they are making a decision to purchase or to wait. This means that each period the consumer decides whether to purchase now, or wait for the other active members to purchase so that he is grouped in with what he believes would be a completely new consumer base. I adopt this assumption for the following reason. If all consumers delay their purchase, then the monopolist sets the same price as in the previous period. So, if all consumers delay, then the price is constant forever, through the monopolist's commitment strategy. So, if a consumer believes that no one else is buying, then waiting is never optimal. So, I consider the opposite, that the consumer believes that the diffusion moves fluidly without him. Under these beliefs the consumer chooses a time to purchase such that

$$\tau(\theta, t) \in argmax_{s \geq t} \ E[\delta^s(\theta - p_s)] \tag{3.38}$$

Where the expectation is over their belief that all purchase the good. We will assume that the monopolist holds beliefs that are agnostic to which consumers had not purchased. The monopolist will just look at the active set each period, and make no believe of the type distribution, except that it is distributed by $F(\theta)$ which we allow to be unit uniform. We do not allow the monopolist to update any belief on the consumers type if they do not purchase.

This strategy implies that the monopolist can infer an individual consumers willingness to wait based off of the current active set and who they would communication with in the next period. This raises some difficulty in the monopolist's pricing strategy, as different consumers have different incentives to wait. But, there are only specific cases where the consumer would have incentive to wait. In many cases, as we have seen, a consumer can only expect the aggregate network value to go down, implying that they would never wait as prices are increasing. But we will see in examples that there are situations in which agents have an incentive to wait. The monopolist would like these waiting agents to purchase as early as possible, and therefore must set the price so that agents will not wait. That means, after choosing the cut off θ_t^*, the monopolist must choose a price such that

$$\theta_t^* - p_t \geq \delta(\theta_t^* - p_{t+1}) \tag{3.39}$$

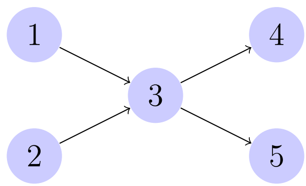

Figure 3.13: Critical consumers

Where the pricing changes come from the price that makes the consumer with the largest incentive to deviate indifferent. The monopolist can start at the end of the game, with a price of $\frac{1}{2}$ and move iteratively backwards to set the price path. This strategy only works for small enough networks, for as networks become large it becomes computationally hard to work backwards as there are too many possible networks that can arise through the word of mouth process.

3.4.2. Active Set Updating and Pricing

We go back to the monopolist's problem. Given a diffusion of the good through the network, the monopolist wishes to make the cut offs based off of the size of the active set, the probabilities of agents being reached from the active set, and the private marginal revenues from each agent. When consumers are added to the active set with no network value, they raise the cutoff. When they have network value, they may raise or lower the cutoff depending on the probability that other consumers are reached through them. If the probability that another consumer than them becomes active raises highly enough, then the cutoff may be lowered. For example, if there are consumers that can only be reached by a unique active consumer, this consumer puts a strong downward pressure on the price. Intuitively, if the monopolist accumulates many of these types of consumers, then eventually the price will fall to a "sales" price.

Since the networks and flow of the active set can get very cumbersome very quickly, we consider a variety of examples to work through the ideas and forces that are driving the monopoly pricing problem. Consider the following example, shown in Figure 3.13, which we considered in the price discrimination case as well.

In this example we assume again that agent 1 and 2 are initially aware. We work backwards again from the case where all consumers are aware. Types are uniform on the unit interval and the discount factor is 1. If everyone is aware, then the monopolist sets a price such that the private marginal revenues are zero, which is a price of $\frac{1}{2}$. A price of $\frac{1}{2}$ will implement this cutoff. Now, given this we move backwards through the diffusion of information. Assume that consumers 1, 2, and 3 are aware of the product. For this to be possible, either agent 1 or agent 2, or both agents, must have purchased the good already. Assuming that both consumers had purchased the good, we are left with the problem we have already solved in the price discrimination case. Consumer 3 should be served if

$$M(\theta, V_A) = 2\theta - 1 + \frac{1}{2} \geq 0 \tag{3.40}$$

So that the cut off is $\theta^* = \frac{1}{4}$. A price of $\frac{1}{4}$ will implement this cutoff.

Now, if only one of the initially aware consumers had purchased the good, we would be in a situation where again consumers 1, 2, and 3 are aware. Say that agent 1 purchased the good, but agent 2 did not purchase the good. The active set is therefore agents 2 and 3. The agent 2 now has no network value, while agent 3 has the network value of $\frac{1}{2}$. We combine these network values to find the marginal revenue of the active set to be

$$M(\theta, V_A) = 2\theta - 1 + 2\theta - 1 + \frac{1}{2} \geq 0 \tag{3.41}$$

We have just added the network values of the two agents together to get the network value of the entire active set. This gives the monopolist a cutoff of $\theta^* = \frac{3}{8}$. The cutoff is raised from when agent 2 had already purchased. This consumer puts a negative externality on the agent who still has a network value. The existence of an agent who has no network externality doesn't allow the price to fall too low. The price that can implement this value is $\frac{3}{8}$ as well, as in the next period the price will either go up or stay constant.

Now, we consider the first stage, where only agents 1 and 2 are aware. The value of each of them is

$$M(\theta, V_1) = M(\theta, V_2) = 2\theta - 1 + \theta^*(\frac{3}{4} * \frac{1}{4} + \frac{3}{4} * \frac{1}{2}) \tag{3.42}$$

Therefore, the total marginal revenue across the two consumers is

$$M(\theta, V_A) = 2(2\theta - 1) + \frac{9}{8}\theta \tag{3.43}$$

This gives a cut off of $\theta^* = \frac{16}{41}$. Now, since $\frac{16}{41} > \frac{3}{8}$, the agents have incentive to wait if the price were equal to $\frac{16}{41}$. If one agent were to wait and the other purchased, then they could be offered a lower price. We can see here that the incentive to wait depends on the beliefs of one consumer on the type of the other. This is where the assumption on the consumer beliefs comes into play. By assuming that the other consumer purchases for sure, the strategic consumer has the strongest incentive to wait, harming the monopolist the most. It is in this way that this assumption on the consumer beliefs harms the monopolist the most.

The monopolist must choose a price such that the cutoff type, and therefore all above him, will like to purchase the good in the first round. That is, the monopolist chooses a price such that

$$\frac{16}{41} - p \geq \frac{16}{41} - \frac{3}{8} \implies p = \frac{3}{8} \tag{3.44}$$

Of course, if consumers are infinitely patient, they will be willing to wait given they believe the other consumer will purchase. The only way to get them to purchase would be to offer the same price that they would offer in the next period. If we assume that either the consumer has a discount factor $\delta \leq 1$, or alternatively they have a probabilistic belief on the other consumer purchasing, we see that they price is such that

$$\frac{16}{41} - p \geq \delta(\frac{16}{41} - \frac{3}{8}) \implies p = \frac{128 - 5\delta}{328} \tag{3.45}$$

The price is increasing as the discount rate decreases. We see in this example two separate and important observations when the monopolist can only set a single price. First, we see that consumers may wish to delay their purchases, something that was clearly not the case when the monopolist can price discriminate. We also see that *even though the consumer can wait*, the monopolist would sometimes like to drop the price. This is because of the aggregate network effect throughout the network.

3.4.2.1. General Pricing Strategies

The ideas from the previous example follow through to ideas on the general network pricing game. Recall the decision of the monopolist, where the cutoffs are decided by the following equation

$$\frac{\sum_j \lambda_{Ajt}\delta^{\tau} \int_{\theta^*_{\tau}}^{\overline{\theta}} MR(\theta,0)dF}{|A|} = -(\theta - \frac{1-F(\theta)}{f(\theta)}) \tag{3.46}$$

There are some general comparative statics that can be found from this equation. We know that this equation says that the monopolist would like to serve the consumer base if the average marginal revenue is equal to the loss from the private value.

If there is no network value, then the left hand side is zero, and so the monopolist just serves those with positive private marginal revenue, which corresponds to the static monopoly price.

As we increase the amount of agents aware, without a change in the network value, we see that the cutoff raises. This is because as we add agents without a network effect, the loss from these agents being overserved outweighs any value we can get, raising the value up to the monopoly cutoff. Of course, when an agent becomes aware, there are other effects going into play. The probability that this agent becomes reached disappears from the numerator of the left hand side, lowering the left had side. At the same time, the probabilities of other agents being reach will raise if the newly aware agent is on a path to these consumers. Which effect dominates will affect whether the cutoff increases or decreases.

It is not possible to say in general which effect dominates, but we saw as in the previous case that it can move either direction. If a consumer is “critical”, that is if their purchase is necessary for a subgroup of consumers to purchase, then they may likely be the type of consumer for which the cutoff drops when they become aware.

If a critical agent exists, there are parts of the network which will not become aware until this agent purchases this good. As these agents accumulate, they have a downward pressure on the price. When there are many ways to reach an agent, the price can remain

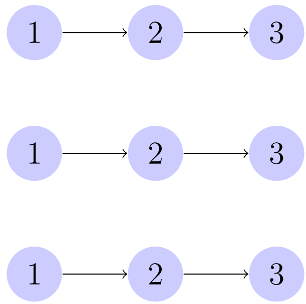

Figure 3.14: Independent Lines

high and there is a high likelihood that the agent will be reached. But, after the number of paths falls, the individual agent becomes more important, which puts downward pressure on the price.

The existence of critical agents alone does not guarantee that prices go down. There must be some interdependence between agents to reach these critical agents. To see this consider a setting of N independent finite lines of 3 agents, where the one agent on the periphery of each line is independent, as in Figure 3.14.

If the first agent is aware in each network, then each agent has the following network value

$$M(\theta, V_i) = 2\theta - 1 + \frac{25}{64} \tag{3.47}$$

Therefore the total network effect is

$$M(\theta, V_A) = N(2\theta - 1) + N * \frac{25}{64} \tag{3.48}$$

This gives a $\theta^* = \frac{39}{128}$. Now, if a subset M of these agents purchase the good, then the marginal revenue from each newly aware agent is

$$M(\theta, V) = 2\theta - 1 + \frac{1}{4} \tag{3.49}$$

Combining these marginal revenues with that of previous group, we have the following aggregate marginal revenue.

$$M(2\theta - 1 + \frac{1}{4}) + (N - M)(2\theta - 1 + \frac{25}{64}) \tag{3.50}$$

This leads to a cutoff of $\theta^* = \frac{39}{129} + \frac{9M}{128N}$. This is always a higher cutoff than the previous part, implying that being critical alone is not quite enough. Other effects must be at play, such as going from an unimportant group of consumers to a critical consumer, such as in Figure 3.1.3.

Setting a single price gives this trade off between revenues now and later, and dealing with the ability to wait of the consumers. The monopolist must weigh these issues in a way that they didn't have to when they could discriminate based on network position. This issue is due to arise in the many settings where the monopolist can see the consumers positioned and aware of their good, but can not price differently between them in the current time period. With many firms gaining information on consumers and their networks through social media sites and other technologies, this situation arises often.

3.5. Optimal Mechanisms

We now consider a case where the monopolist can discriminate not only over network position but also over the private types of consumers. We are interested in finding the optimal mechanism at each time period in a network of communication. We assume the following mechanism. The monopolist views the network and the active set. He then collects a report of types from each member of the active set. Then, based on these reports, and the network externalities available, the monopolist allocates the good across consumers. Then, a new group of consumers become aware, and the mechanism continues. We look to find the mechanism that maximizes profits, and which truth telling is incentive compatible and individually rational.

This mechanism differs from the price discrimination case for now the monopolist can decide whether to allocate an agent after viewing the types of all agents. This allows the monopolist to choose only one agent to get the good when we have a block in information flow. The optimal mechanism will again be very cumbersome in a variety of networks, so we exhibit the forces behind the network in our example that has been used throughout the paper, as seen in Figure 3.15.

We begin by considering the optimal mechanism when there is complete information

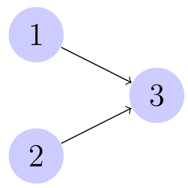

Figure 3.15: Mechanism Design Example

as a benchmark. Assume that the aware set includes just agent 3 aware. The optimal mechanism is simple. The agent reports his type. If the type is greater than or equal to $\frac{1}{2}$ he is sold the good for a price the same, and if not he receives nothing. The expected revenues from this formulation are, as we have seen many times, equal to $\frac{1}{4}$. This is just the monopoly profit gained off of an individual consumer.

Now, we consider the time period in which the consumers 1 and 2 are aware, and consumer 3 needs to be made aware to purchase the good. What is important here is that the monopolist only needs one of the consumers to purchase the good in order for the word of mouth effect to take hold. That is, there is some region of the type space $\Theta = \theta \times \theta \in [0,1] \times [0,1]$ such that the monopolist may choose to serve one or the other of the agents.

We first note that in the complete information case there are two cut offs of note. First, we have the "monopoly price" cutoff of $\frac{1}{2}$. If a type is above this threshold then the monopolist would like to serve the good to them no matter what. The other cutoff is $\frac{3}{8}$. Below this cutoff, the continuation value of $\frac{1}{4}$ is not enough for the loss the monopolist would take on selling to the agent, and so agents with types below $\frac{3}{8}$ would never be served the good.

So we deal with these cutoffs to find the optimal allocation rule for the agents. Consider the perspective of consumer 1's report. We have the following optimal allocation rule based on the type report.

- If consumer 1 reports $\theta \in [\frac{1}{2}, 1]$, he is served the good.
- If consumer 1 reports $\theta \in [0, \frac{3}{8})$, he is not served the good.
- If consumer 1 reports $\theta \in [\frac{3}{8}, \frac{1}{2})$ his allocation depends on the report of consumer 2.

– If consumer 2 reports $\theta \in [0, \frac{3}{8})$, consumer 1 is served the good.

– If consumer 2 reports $\theta \in [\frac{3}{8}, \frac{1}{2})$, the consumer with the highest type report is served the good.

– If consumer 2 reports $\theta \in [\frac{1}{2}, 1]$, then consumer 1 is not served the good.

We therefore see what the optimal allocation would be for the monopolist. The monopolist only needs one of the two agents to purchase the good, and so when the agents have low enough types, the monopolist chooses to sell to only one of them, to gain the expected revenues from the future agent, while only taking a loss in marginal revenue on one of the agents. We can implement this with prices of depending on the region the types fall in. If one type reports $\theta \in [\frac{1}{2}, 1]$, then the price is $\frac{1}{2}$. If one type reports $\theta \in [0, \frac{3}{8})$ and the other reports type $\theta \in [\frac{3}{8}, \frac{1}{2})$, then the price set is $\frac{3}{8}$ to implement the sale to just one of the consumers. If both report $\theta \in [0, \frac{3}{8})$ then a sale does not occur. Finally, if both report $\theta \in [\frac{3}{8}, \frac{1}{2})$, then a price of $max(\theta_1, \theta_2)$ will implement the allocation. This leads to profits of

$$(\frac{1}{4} * \frac{1}{2} + \frac{1}{4} * \frac{1}{2} + \frac{1}{4} * (2 * \frac{1}{2}) + \frac{3}{64} * \frac{3}{8} + \frac{3}{64} * \frac{3}{8} + \frac{1}{64} * \frac{11}{24}) + (1 - \frac{9}{64})\frac{1}{4} = \frac{1163}{1536} \quad (3.51)$$

The terms break down as follows. The first three terms represent when one or both of the consumers report a type at least $\frac{1}{2}$. The next two terms represent when one consumer reports a type in $[\frac{3}{8}, \frac{1}{2}]$ and the other reports a type less than $\frac{3}{8}$. The next term represents the expectation of the maximum type revelation in that relevant interval. Finally, the $(1 - \frac{9}{64})\frac{1}{4}$ term represents the expected value of reaching consumer 3 times the probability of reaching him.

We now take the same situation and see how the monopolist would implement such a mechanism when there is not full information. We now assume the same network as in Figure 3.15, with the types of consumers now being private information. We first see that for agent 3, reporting truthfully is optimal, for he will be charged a price of $\frac{1}{2}$ no matter what. Thus we do not need to worry about the incentive compatibility or the individual rationality of this consumer.

Now, for consumers 1 and 2, when they are aware of the good, they have incentive to act strategically in the mechanism. We consider the incentive to deviate of each agent for each type. We look for an incentive compatible and individually rational mechanism. In doing so we need to find what prices offered would induce the optimal allocation in the network. We first consider those with types in $[0, \frac{3}{8}]$ If they announce a type that is above $\frac{3}{8}$, they will possibly receive the good for a price higher than their valuation. If they announce a type below $\frac{3}{8}$, they will not receive the good. For this reason it is incentive compatible for the agent to reveal their true type.

Now, we consider a consumer with type in $[\frac{3}{8}, \frac{1}{2}]$. If they announce their true type, they will either get the good for $\frac{3}{8}$, not get the good, or if the other agent announces a type that is in the same region, they will potentially get the good. We implement a second price auction when both agents announce types in $[\frac{3}{8}, \frac{1}{2}]$. The second price auction is IC, so a consumer would not deviate from their type to another announcement in this region. They would also not deviate below this region. To see if they would deviate above this region, we need to see what prices are offered to those who announce $\frac{1}{2}$. Consider the IC constraint of the high type. We let p_1 be the price offered to those announcing above $\frac{1}{2}$, p_2 be the price offered to those with value between $\frac{3}{8}$ and $\frac{1}{2}$. Then, the IC constraint is

$$\theta - p_1 \geq \frac{3}{8}(\theta - p_2) + \frac{1}{8}(\theta - \frac{7}{16}) \tag{3.52}$$

We note that the best deviation for this consumer is to announce a type just below $\frac{1}{2}$. This gives him the good for sure in the second price auction. We notice that $\frac{3}{8}$ will be the price for p_2 because low types will never deviate upwards and high types can't deviate below this to gain utility. So, we solve the above equation with equality when $\theta = \frac{1}{2}$, to get $p_1 = \frac{57}{128}$. At this price those with a type above $\frac{1}{2}$ will not deviate from truth telling.

The last thing to do is to check that those in the middle of the type distribution do not wish to deviate. If they deviate upward, they get

$$\theta - \frac{57}{128} \tag{3.53}$$

If they do not deviate, they get

$$\frac{3}{8}(\theta - \frac{3}{8}) + \frac{1}{8}(8\theta - 3)(\theta - \frac{1}{16}(8\theta - 3)) \tag{3.54}$$

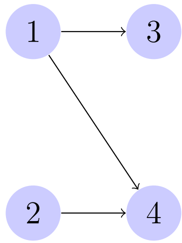

Figure 3.16: Overlapping Network Effects

They do not wish to deviate up as long as

$$\frac{3}{8}(\theta - \frac{3}{8}) + \frac{1}{8}(8\theta - 3)(\theta - \frac{1}{16}(8\theta - 3)) \geq \theta - \frac{57}{128} \implies 15 - 40\theta + 32\theta^2 \geq 0 \tag{3.55}$$

This holds for all $\theta \in [\frac{3}{8}, \frac{1}{2}]$. Therefore this mechanism is incentive compatible and individually rational. The expected profits from this mechanism are $\frac{283}{384}$, which gives higher profits than in the price discrimination and the single price monopolist case, which are, in both cases, $\frac{25}{36}$. Clearly being able to moderate the information flow in a more sophisticated manner is valuable to the monopolist.

The general property of the mechanism is that it allows the monopolist to decide who to serve in terms of their network positions and types and allows them to keep information flowing while losing the smallest amount possible in current marginal revenues. When dealing with a general network, the monopolist can collect type profiles from the active set, and then decide who to serve dependent on the network effect and the types given.

To see how the competing network values work, we consider another network which we have seen before, recreated in Figure 3.16.

Here, we see consumer 1 has network effects independent of consumer 2. This will imply that the monopolist wishes to serve consumer 1 no matter what for types lower than $\frac{1}{2}$. We see that the monopolist would like to serve consumer 2 if he has a type higher than $\frac{1}{2}$, or for a type higher than $\frac{3}{8}$ only if consumer 1 reports to have a type lower than $\frac{1}{4}$.

To summarize the allocation rule, we see that for consumer 1

- If $\theta \in [0, \frac{1}{4}]$, he is not served
- If $\theta \in [\frac{1}{4}, \frac{3}{8}]$ being served depends on consumer 2.

  - If consumer 2 reports a type less than $\frac{1}{2}$, then consumer 1 is served
  - If consumer 2 reports a type greater than $\frac{1}{2}$, the consumer 1 is not served

- If $\theta \in [\frac{3}{8}, 1]$, he is served.

We have a similar rule for consumer 2:

- If $\theta \in [0, \frac{3}{8}]$, then he is not served
- If $\theta \in [\frac{3}{8}, \frac{1}{2}]$, being served depends on consumer 1
  - If consumer 1 reports a type less than $\frac{1}{4}$, then consumer 2 is served
  - If consumer 1 reports a type greater than $\frac{1}{4}$, then consumer 2 is not served

To induce this allocation we set prices so that types do not wish to deviate downward, as in the previous example. Starting with consumer 2, we set price cutoffs of $\frac{3}{8}$ and $\frac{15}{32}$ for types in $[\frac{3}{8}, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ respectively. The price is shaded downward for the higher types so that they do not wish to deviate to get a lower price.

For consumer 1, the prices are $\frac{1}{4}$ and $\frac{5}{16}$ for types in $[\frac{1}{4}, \frac{3}{8}]$ and $[\frac{3}{8}, 1]$ respectively. We see that consumer 1 is more important to the information flow than consumer 2, and consumer 1 is rewarded for this. Consumer 2 is rarely vital to the information flow, and so he is rarely awarded the good for his network value.

Consumer 1 has a network value that is not affected by the existence of consumer 2. For this reason, the monopolist is more incline to serve consumer 1. This puts a negative externality on consumer 2, causing him to be serves much less often than if he were the only one who could reach consumer 4. Since consumer 1 has a unique path to agent 3, his network value can not diminish so far, leading to a larger set of types of consumer 1 that are served.

3.6. Conclusion

When a monopolist has information over consumer networks of communication, she can use this information to optimize when pricing dynamically. In this paper we see how the monopolist can calculate the network value of different consumers based on the information spread that they engage in. The monopolist offers lower prices to consumers who are aware early, and those that are critical to the information flow. When the monopolist is restricted to setting a single price, these critical agents put a downward pressure on the price. Finally, we consider the optimal mechanism when consumers engage in word of mouth. In the optimal mechanism the monopolist is able to harness the word of mouth effect perfectly, picking exactly the consumers who should be served the good to maximize profits.

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