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The Varieties of Multidimensional Necessity

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

in Philosophy

by

Fabio Dal Conti Lampert

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2020

DEDICATION

To my wife, Camila Aimbiré, with love and gratitude.

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ABSTRACT OF THE DISSERTATION

The Varieties of Multidimensional Necessity

By

Fabio Dal Conti Lampert

Doctor of Philosophy in Philosophy

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Dean's Professor Kai Wehmeier, Chair

Two-dimensional semantics is one of the main theories of meaning in contemporary analytic philosophy. Much of what makes it interesting is the unified manner it has of providing a semantic analysis of statements involving the metaphysical notions of necessity, possibility, and actuality, and the epistemic notions of a priori and a posteriori knowledge or knowability. This unified account allows for the development of modal logics endowed with semantic structures which are inspired by two-dimensional theories of meaning, in which the formal languages contain operators corresponding to the aforementioned metaphysical and epistemic notions.

In this dissertation, I develop first-order two-dimensional modal logics for different modal languages and present sound and complete semantic tableaux for all of them. Additionally, I argue that the first-order extensions call for new actuality operators so that certain statements in natural language can be formalized. I argue that this brings new issues for two-dimensional modal logics, such as the problem of omniscience with respect to actual truths. Finally, from the perspective of mathematical logic, I show that extensions of certain propositional two-dimensional modal logics to the n-dimensional case are finitely axiomatizable. Decidability of n-dimensional modal logics is proved by systematic constructions of semantic tableaux.

Chapter 1

Introduction

Since at the very least the early work of Kripke (1959, 1963) in modal semantics, the guiding idea behind the model-theoretic analysis of statements of the form "it is necessary that p" and "it is possible that p" is to analyse the former as p being true at every possible world (in a model), and the latter as p being true at some possible world (in a model). A possible world in this case could be taken (intuitively) as a maximally specific way things could have been. For example, there is a possible world where "Soccer is the favorite sport amongst the citizens of the United States" is true, but there is no possible world according to which "$2 + 2 = 5$" is true. Consequently, the former statement is possibly true, but the latter is not. A formal language for necessity and possibility is then given by the language of the propositional calculus with the addition of two unary operators, $\Box$ for necessity, and $\Diamond$ for possibility. A model for such language is usually defined as a triple, $\mathcal{M} = (W, R, V)$, where W is a set of points intuitively identified with possible worlds, R is a binary 'accessibility' relation between worlds, and for each propositional symbol p in the language, $V(p) \subseteq W$ is the set of worlds at which p is true. Then for p to be necessarily true at a possible world $w \in W$ in $\mathcal{M}$ is just for $\Box p$ to be true at w in $\mathcal{M}$, which in turn means that p is true at every world v such that wRv, that is, every world v that is R-accessible from w; similarly, for p to be possible at a world w in $\mathcal{M}$ is just for $\Diamond p$ to be true at w in $\mathcal{M}$, which in turn means that there is a world v such that wRv and p is true at v.

Later on in the development of modal semantics, especially with the works of Hazen (1976) and Crossley and Humberstone (1977), a logical analysis of statements of the form "it is actually the case that p" was proposed as p being true in the 'actual world' (of the model in question). Relative to a model $\mathcal{M}$, one can simply take a world of W, say, w^*, to represent (intuitively) the actual world, and add an actuality operator @ to the formal language, where $@p$ is true at a world w in the model $\mathcal{M}$ just in case p is true at w^*. The addition of an actuality operator to the modal language allows one to evaluate a formula as true or false at a possible world while keeping an eye, as it were, on what is necessary or possible at the actual world, and so one in effect keeps track of what is true or false with respect to a *pair* of worlds. In a sense, this is the beginning of what is called 'two-dimensional semantics'.

Two-dimensional semantics has played an important role in contemporary analytic philosophy as an overarching theory, generally speaking, designed to explain epistemic notions such as a priori and a posteriori knowledge as well as the metaphysical modalities of possibility, necessity, and actuality. In general, two-dimensional semanticists defend an extension of the formal apparatus of possible worlds semantics in modal logic in order to analyze the meaning of linguistic expressions. Even though there is a wide variety of theses and somewhat distinct approaches under the umbrella of two-dimensionalism, it is constitutive of the theory that statements be evaluated concerning their truth values at a pair of indices regarded as possible worlds, states, contexts, epistemic scenarios, or combinations thereof.

With respect to a logical analysis of statements concerning what is a priori knowable, the standard two-dimensional analysis of "it is a priori that p" is that p is true at every diagonal pair of possible worlds or, more generally, indices, i.e. pairs whose coordinates are identical. Passing over a number of distinctions and subtleties involving the many variations of two-dimensional semantics, this is the analysis appearing in the works of theorists such as Stalnaker (1978), Evans (1979), Davies and Humberstone (1980), Kaplan (1989), Jackson (1998), Chalmers (1996, 2004), and others. One way of motivating such an analysis is by means of the plausible claim that a statement p is knowable a priori if and only if no matter which world is the actual one, p is true at that world. For in that case, no empirical knowledge about the world one is in is required in order to know that it is a p-world. By extending the evaluation of formulas in a modal language with respect to pairs (w, v) of possible worlds in a model $\mathcal{M}$, we can then declare that, where A is the a priori operator, Ap is true at (w, v) if and only if for every world z in $\mathcal{M}$, p is true at (z, z).

The main objectives of this dissertation involve taking the tradition of two-dimensional semantics as a fruitful semantic theory and investigating several of its aspects that remain somewhat neglected in recent scholarship in the philosophy of language, metaphysics, and modal logic. Although some axiomatic systems have been presented for certain logics endowed with two-dimensional semantics, other proof systems such as semantic tableaux have not been previously investigated. Furthermore, most of the logical development in the literature has focused on propositional two-dimensional modal languages, leaving first-order languages and the philosophical issues that come along with quantification rather unexplored. From a logical point of view, it is also interesting to ask how two-dimensional modal logic might be generalized to n-dimensional modal logics, and whether known completeness and decidability results for certain axiomatizations of two-dimensional modal logics extend to the general, n-dimensional case. Are n-dimensional modal logics complete? Are they decidable? Issues like these are investigated in the three chapters below. In what follows an overview of each chapter is presented.

Chapter 2: Actuality, Tableaux, and Two-dimensional Modal Logics. (Published in *Erkenntnis*, 83(3): 403-443, 2018.)

In this chapter I develop semantic tableau methods for a variety of two-dimensional modal logics. Most of the existing development of logics for a priori knowability is purely axiomatic, and the purpose of this chapter is to provide alternative approaches to proof-theoretical investigations of such logics. Additionally, because most of the literature deals with propositional languages only, the proof systems in this chapter are developed for first-order languages with identity. All formulas in the language are doubly-indexed in the proof systems, with the upper indices intuitively representing actual worlds or epistemic scenarios, and the lower indices representing worlds of evaluation – or first and second dimensions.

Many philosophical questions have arisen from two-dimensional semantics concerning the right account of validity for modal languages. Should we define validity as truth in all worlds in every model, or as truth in the actual world of every model? In this chapter I remain neutral regarding such questions. I discuss several philosophical implications of each definition, but more importantly, the tableaux I propose modulate over these different notions of validity, and I show how to adapt each proof system to the various validity notions.

I also motivate the introduction of a new actuality operator into two-dimensional languages and explore some of the philosophical questions raised by it concerning the relations between actuality, necessity, and the a priori, that seem to undermine traditional intuitive readings of two-dimensional operators.

Chapter 3: Actuality and The A Priori. (Published in *Philosophical Studies*, 175(3): 809-830, 2018.)

One of the main motivations for adding an actuality operator to the basic modal language is to augment the expressive power of this language. Ordinary modal statements such as "it is possible for everyone who is actually rich to be poor" have readings whose intended truth conditions cannot be formalized in a first-order modal language without an actuality operator. So the actuality operator fulfills an important role when added to the modal language as it enhances its expressive capabilities. In this chapter I consider natural-language sentences that cannot be formally represented in a first-order language for epistemic two-dimensional semantics. This case resembles the examples of sentences not expressible in first order one-dimensional modal languages, and I take this to be a strong motivation for adding a new kind of actuality operator to the two-dimensional language. It turns out, however, that the most natural ways to repair the expressive inadequacy of the first-order language for epistemic two-dimensional semantics render moot the original philosophical motivation of formalizing a priori knowability as necessity along the diagonal.

Chapter 4: The Logic of Sequence Frames (Published in *The Review of Symbolic Logic*, 1-44, doi:10.1017/S1755020320000325, 2020.)

Logicians have studied a variety of multidimensional modal logics. Several results about these logics are well known, including which logics are complete or decidable relative to certain classes of structures. But no investigations have been carried out concerning generalizations to arbitrary finite dimensions of two-dimensional modal logics as used in the philosophical literature about a priori knowability. This chapter investigates and develops generalizations of two-dimensional modal logics of this kind to any finite dimension n. These logics are natural extensions of some multidimensional systems already known from the literature on two-dimensional modal logics, or logics for a priori knowledge. I prove a variety of results the most important of which include a completeness theorem for an axiomatic system relative to the class of what I call 'sequence frames' for n-dimensional modal logics, and I define semantic tableaux generalizing the prefixed tableaux for one-dimensional modal logics found in Fitting and Mendelsohn (1998) by means of which a decidability result is demonstrated via a systematic procedure for constructing tableaux.

Chapter 2

Actuality, Tableaux, and Two-Dimensional Modal Logics

2.1 Introduction

In this chapter we present semantic tableaux for two-dimensional modal logics. First we devise tableaux for a system we call $\mathbf{S5}_{2D}$. The logic is somewhat based on the systems presented in both Crossley and Humberstone (1977) and Davies and Hum- berstone (1980), but it extends these significantly. The language contains the usual Boolean connectives, modal operators, first-order quantifiers, identity, an actuality operator, a Ref operator, a fixedly operator, and an extra actuality operator that we call "distinguishedly". All the formulas in the language are doubly-indexed, with the upper indices intuitively representing the actual or reference worlds, and the lower indices representing worlds of evaluation — first and second dimensions, respectively. Additionally, we show the tableaux to be modular in the sense that they can be adjusted for different notions of validity: local, general, or diagonal. We show the tableau methods for $\mathbf{S5}_{2D}$ to be general enough for different logics such as the

S5 system with an actuality operator presented in both Crossley and Humberstone (1977) and Hazen (1978), the logic for the fixedly operator developed in Davies and Humberstone (1980), as well as variations of a different system containing the so-called diagonal or apriority operators as primitive, which is presented in both Restall (2012) and Fritz (2013, 2014).

Models for such logics are well known, as there are several axiomatic systems for different two-dimensional logics.[1] By contrast, different proof systems have been rather unexplored. Besides the axiomatic developments we note only the hypersequent system introduced by Restall (2012) containing apriority operators.[2] Furthermore, the great majority of axiomatic systems — and Restall's hypersequent system — are presented exclusively for the propositional case.[3] This is because the main innovations are engendered by the two-dimensional operators themselves, whence not much appears to be gained by adding first-order quantifiers, especially given the notorious complications they bring into modal contexts. However, the motivation for the introduction of two-dimensional operators is usually drawn from examples in a first-order language, as illustrated by (1) [and (3)] below, and, moreover, there are several interesting questions concerning names, identity, and rigidity, to name a few, once we take quantification into account in logics containing both modal and epistemic operators.[4] We shall explore some of these questions in due course.

Moreover, no tableau system seems to have been developed thus far for two-dimensional logics.[5] One advantage of this approach — and, as argued in Ray and Lampert (unpublished

[1]Axiomatizations for different two-dimensional logics can be found in, for example, Segerberg (1973), Davies and Humberstone (1980), Kaplan (1989), and Fritz (2013).

[2]Hazen (1978) develops a Fitch-style natural deduction system, except that it only contains an actuality operator alongside the modal operators.

[3]First-order axiomatic systems with an actuality operator have been investigated in Hodes (1984b) and Stephanou (2005).

[4]Although, as we shall see in due course, only some two-dimensional operators will be said to be explicitly epistemic. In Sect. 5 we shall explore just some of those questions concerning quantification in order to set up a system intended to represent the discourse involving a priori knowledge following Chalmer's version of two-dimensional semantics. In Ray and Lampert (unpublished manuscript), we develop a Lemmon-style natural deduction system for quantified two-dimensional modal logic involving actuality and fixedly operators, although two-dimensional developments of systems in the style of Fitch and Gentzen are also yet to be explored.

[5]During the time the paper on which this chapter is based was being reviewed Gilbert (2016) was

manuscript), also natural deduction systems — for modal logics is that tableaux containing numeric indices can be construed in a quite natural and elegant manner, where the indices refer intuitively to possible worlds. This provides a certain semantic transparency to the proof system that becomes manifest in the rules, for they basically display the semantics for each operator.[6] Another advantage is, or course, its practicality. It is very simple to prove theorems in it, whereas the same is not true for axiomatic systems. And, in effect, in order to develop two-dimensional tableaux we can assume much of what is already familiar regarding tableau techniques for modal logic, which is more of a virtue than anything else; as two-dimensional modal logic tends to be at least as complicated than its one-dimensional counterpart, a simple proof system for it is readily motivated as a useful tool for philosophers. Lastly, on philosophical grounds, the modularity of 2D-tableaux over different notions of validity presents itself as a very attractive property, for it is far from obvious how to achieve the same modularity with axiomatic proof systems.

The chapter is divided in the following way: in Sect. 2 we provide a quick overview of the rationale behind the introduction of actuality, fixedly, distinguishedly, and *Ref* operators in two-dimensional modal languages; in Sect. 3 we present sound and complete semantic tableaux for $\mathbf{S5}2\mathcal{D}$ encompassing the operators listed above; in Sect. 4 we show how to modulate the 2D-tableaux for different notions of validity, namely, local, general, and diagonal validity; in Sect. 5 we show how tableaux for different systems — including, especially, a logic for epistemic two-dimensional semantics — are yielded based on the methods herein presented; finally, in Sect. 6 we discuss further the problem of expressive incompleteness in multidimensional languages, which has motivated the addition of some

published, developing two-dimensional tableaux independently from the present chapter. The system considered by Gilbert is basically David and Humberstone's propositional $\mathbf{S5}\mathcal{AF}$ with the addition of a *Ref* operator. In addition, Gilbert presents some decidability results for different two-dimensional logics, which of course will not hold for the first-order systems herein considered. Many thanks to Shawn Standefer and an anonymous reviewer for pointing me to Gilbert's paper.

[6]Not everyone is happy with this. Poggiolesi and Restall (2012) accuse labelled systems of exploiting explicit semantic notions in the proof system, which is unacceptable, so they claim, on the basis that the latter should employ purely syntactic tools. We do not engage in this debate here, although a response against the charge of "semantic pollution" in labelled systems can be found in Read (2015).

of the operators already mentioned, by focusing on several philosophical consequences and questions raised by introducing the distinguishedly operator in two-dimensional modal languages, some of which bear directly upon how two-dimensional operators, in particular, diagonal necessity operators, should be intuitively understood. Proofs of soundness and completeness for the several systems discussed here can be found in Appendices 1, 2, and 3.

2.2 Two-Dimensional Modal Logic

2.2.1 Actually and Fixedly

Crossley and Humberstone defined an **S5** modal logic with an actuality operator, $\mathcal{A}$,[7] whose motivation was the untranslatability in a modal language of sentences such as

(1) It is possible for everything that is actually red to be shiny.

As suggested by Crossley and Humberstone, once we enrich our modal language with $\mathcal{A}$, we can translate (1) as follows:

(2) $\Diamond(\forall x)(\mathcal{A}(red(x)) \supset (shiny(x)))$

The resulting logic is a conservative extension of **S5** called **S5**$\mathcal{A}$, where a Kripke model is a triple, $\mathcal{M} = \langle W, w*, V \rangle$, consisting of a non-empty set of 'possible worlds', W, a distinguished member of W, $w*$, called the 'actual world', and a valuation function, V, from propositional variables and members of W to truth-values. The semantics for the modal operators is as usual, and we say that for any formula, φ, $\mathcal{A}\varphi$ is true at a world just in case φ is true at the actual world, i.e. $\mathcal{M}, w \vDash \mathcal{A}\varphi$ if and only if $\mathcal{M}, w* \vDash \varphi$.

[7]We avoid using quotation marks for the sake of presentation, but use-mention distinctions should be clear given the context.

Besides an actuality operator, Crossley and Humberstone also introduced the fixedly operator, $\mathcal{F}$, which works as a universal quantifier over worlds taken as actual. Its original semantics is rather complex. A relation of variance between models is defined such that a model $\mathcal{M}'$ is a variant of $\mathcal{M}$ (written $\mathcal{M}' \approx \mathcal{M}$) if and only if $\mathcal{M}'$ is like $\mathcal{M}$ except possibly with respect to which world is actual. Thus, $\mathcal{M}, w \vDash \mathcal{F}\varphi$ if and only if for all $\mathcal{M}' \approx \mathcal{M}$, $\mathcal{M}', w \vDash \varphi$. The introduction of $\mathcal{F}$ was motivated by the fact that $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$ comes out as an axiom in **S5**$\mathcal{A}$, despite of its "intuitive invalidity."[8] For the axiom says that whatever is actually true is necessarily actually true, but even though grass is actually green, this is not necessarily so. An alternative to $\Box$ as representing necessity, then, would be the concatenation $\mathcal{FA}$, which seems to deliver a much stronger sense of necessity indeed: if $\mathcal{FA}\varphi$ holds, then φ is true at every world taken as actual. Moreover, $\mathcal{A}\varphi \supset \mathcal{FAA}\varphi$ is not valid in **S5**$\mathcal{AF}$, i.e. **S5**$\mathcal{A}$ with the addition of $\mathcal{F}$.[9] Thus, although whatever is actually true is also $\Box$-necessarily true, it is not $\mathcal{FA}$-necessarily true, in which case we might take $\mathcal{FA}$ as a more faithful representation of the necessity we have in mind when we deny that actual truths are necessary.

Subsequently, Davies and Humberstone offered a doubly-indexed semantics for the logic **S5**$\mathcal{AF}$ with formulas being evaluated with respect to a pair of worlds, whereby it becomes unnecessary to define variance between models for the semantics of the fixedly operator. A new and elegant semantic clause for $\mathcal{F}$ is given as follows: $\mathcal{M}^v_w \vDash \mathcal{F}\varphi$ if and only if for all $z \in W$, $\mathcal{M}^z_w \vDash \varphi$, where the upper world is the actual world under consideration (the first dimension), and the lower world is the world of evaluation (the second dimension).

This doubly-indexed semantics accounts for the two-dimensional flavour of the system since now we can evaluate formulas as being true at a world relative to a certain world considered as actual. A difference in the models for **S5**$\mathcal{AF}$ is that Davies and Humberstone define

[8] Humberstone (2004, p.21).

[9] We should point out that, originally, Crossley and Humberstone use the formula $\mathcal{A}\varphi \supset \mathcal{FA}\varphi$, which is fine since the repeated actuality operator can be deleted in this case without loss.

them as pairs, $\mathcal{M} = \langle W, V \rangle$, rather than the triples with a distinguished element as in **S5**$\mathcal{A}$. The actual world is deposed from the models because in **S5**$\mathcal{AF}$ it is the upper world in the evaluation of formulas that represents which world is actual. The result are 'floating' actual worlds in the same model, rather than a unique and distinguished one. Finally, there is also a need to adjust the semantic clause for $\mathcal{A}$ in **S5**$\mathcal{AF}$: $\mathcal{M}^v_w \vDash \mathcal{A}\varphi$ if and only if $\mathcal{M}^v_v \vDash \varphi$. The actuality operator now has a 'copy down' function: it copies the upper world to the lower one. Hence, rather than pointing at a distinguished element identified in the models — the actually-actual world, as it were — A points to the upper world, whatever it is, with respect to which the relevant formula is being evaluated.

2.2.2 Introducing *Distinguishedly*

As mentioned above, Davies and Humberstone argue that in **S5**$\mathcal{AF}$ we can represent an interesting and stronger sense of necessity, according to which a formula can be true at a world whichever world is taken as actual, by simply concatenating $\mathcal{F}$ and $\mathcal{A}$. This results in the well-known 'truth on the diagonal' that has been widely used in discussions concerning the a priori.[10] According to Davies and Humberstone (1980, p. 3), $\mathcal{FA}$ provides us with "a formal rendering of a distinction invoked by Gareth Evans between deep and superficial necessity," which was used by Evans to account for cases where a statement was contingently true but knowable a priori — namely, the well-known examples in Kripke (1980). Davies and Humberstone suggested that Evans' distinction could be formally represented in the sense that a formula, φ, is superficially necessary just in case $\Box\varphi$ holds, and deeply necessary just in case $\mathcal{FA}\varphi$ holds. The idea is that a formula would be a priori knowable just in case it is deeply necessary, i.e. if and only if its $\mathcal{FA}$-modalization holds. Thus, assuming that it is a contingent matter which world turns out to be actual, the formula $\mathcal{A}\varphi \equiv \varphi$ for instance, where φ is any contingent truth, is both contingent and a priori, for its $\mathcal{FA}$-modalization

[10] In particular, see Stalnaker (1978), Jackson (1998a), and Chalmers (1996, 2004) for different usages of this notion.

is valid even though its $\Box$-modalization is not.[11] Although there is nothing epistemic built into the formal system since the modalities are alethic, arguably, the resulting logic can be used in this way to represent some important a priori truths, thereby shedding some light on the contingent a priori.

Now, **S5**$\mathcal{AF}$ is very impressive when it comes to its expressiveness. In addition to necessity, possibility, and actuality, we also have deep necessity, which, as alluded above, has been used to represent apriority.[12] Moreover, we can also define the dual of $\mathcal{F}$, which we call "shiftably", by the following semantic clause: $\mathcal{M}_w^v \vDash \mathcal{S}\varphi$ if and only if for some $z \in W$, $\mathcal{M}_w^z \vDash \varphi$.[13] Since a formula is deeply necessary just in case its $\mathcal{FA}$-modalization holds, we suggest that a formula is deeply possible just in case its $\mathcal{SA}$-modalization holds.[14] However, now it is possible to generate sentences such as the following:

(3) It is deeply possible for everything that is actually red to be shiny.

We should clearly be able to formalize (3) in the logic of deep necessity. But how should we formalize it? As it happens, (3) is a two-dimensional analogue of (1), which motivated the introduction of $\mathcal{A}$ in a basic modal language. And, similarly, if we bring back a distinguished element in the models for **S5**$\mathcal{AF}$, then there will be no way to formalize (3) if we take "actually" as referring to the actually-actual world, or the distinguished element of the model, as we take (3) to be suggesting. For instance, the most intuitive strategy to formalize (3) would be to translate "actually" by $\mathcal{A}$, thereby leading to (4):

(4) $\mathcal{SA}(\forall x)(\mathcal{A}(red(x)) \supset (shiny(x)))$

[11] Analogously, one could get necessary truths that are only a posteriori knowable, for instance, by substituting any empirical truth for φ in $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$.

[12] In Restall (2012) and Fritz (2013), a priori operators behave exactly as $\mathcal{FA}$. Thus, in Sect. 5 we develop tableau systems taking $\mathcal{FA}$ as a single operator.

[13] The shiftably operator was previously defined in Ray and Lampert (unpublished manuscript).

[14] Hanson (2006, p. 452), defends that, to a certain extent, deep necessity and possibility seem to capture our intuitions about necessity and possibility in general even better than superficial necessity and possibility.

Nevertheless, $\mathcal{A}$ now points to the world introduced by $\mathcal{SA}$, and not to the actually-actual world, as we claim it should. The actuality operator is relativized, as it were, in **S5**$\mathcal{AF}$: again, it is just a copy down operator. Since A becomes sensitive to embedding in $\mathcal{F}$ and $\mathcal{S}$ contexts, this leaves us unable to say in the language things we want to be able to say, such as that something is actually so, and not just actually so relative to a world taken as actual.

In **S5**$\mathcal{A}$ we have the means to refer back to the distinguished world whenever we want, but this is not generally the case for **S5**$\mathcal{AF}$ even if we add a distinguished world to the models. In order for this to be possible we suggest adding — besides a distinguished world to the models — a new operator to the language, called "distinguishedly", which takes any world in the upper position to the distinguished one — hence, distinguishedly works as an $\mathcal{F}$- or $\mathcal{S}$-inhibitor.[15] Finally, if we denote distinguishedly by $\odot$, and let $w*$ be the distinguished element of the models, we can present its semantics as follows:

$$\mathcal{M}_w^v \models \odot\varphi \text{ if and only if } \mathcal{M}_w^{w*} \models \varphi.$$

The idea is that we can now translate (3) into our language as the following:

(5) $\mathcal{SA}(\forall x)(\odot\mathcal{A}(red(x)) \supset (shiny(x)))$

Thus we have both a way of denoting any floating actual world by using $\mathcal{F}$, $\mathcal{S}$, and $\mathcal{A}$, as well as the distinguished element in the models by using $\odot$. For obvious reasons, we sometimes refer to the latter as a rigid actuality operator, thereby leaving to $\mathcal{A}$ the title of non-rigid actuality operator. There are several philosophical issues to be discussed concerning these, and we address some of them in Sect. 6.

[15]In Humberstone (1982, p. 452), $\mathcal{A}$ is called an inhibitor (in a logic without $\mathcal{F}$), for it "protects what is in its scope from the influence of an outlying modal operator."

2.2.3 *Ref*

It would be interesting enough to have a proof system for all the operators discussed above. However, we can increase the expressive power of our two-dimensional logic even more if we add what Cresswell (1990) calls the "Ref" operator. This is very similar to Vlach's (1973) "then" operator for temporal logics, which is usually studied in the company of Prior's (1968) "now" operator, the tense analogue of $\mathcal{A}$.[16] The need for such an operator comes from an attempt to formalize sentences such as

(6) It might have been that, if everyone then rich might have been poor, then someone is happy.

What this example is purported to show is the following.[17] The term "then", similarly to "actual", points to a certain world. Except that where "actual" points to the distinguished element in the models (or the upper world), "then" points to the possible world introduced in the beginning of (6). Thus, we cannot formalize (6) as

(7) $\Diamond(\Diamond(\forall x)(\mathcal{A}(rich(x)) \supset (poor(x))) \supset (\exists x)(happy(x)))$

What (7) says, in effect, is that there is a possible world, w, such that, for some possible world z, if everyone who is rich at the actual world, $w*$, is poor at z, then someone is happy at w.[18] However, what we want is to quantify over whoever is rich at w, and not $w*$. In order to do that we need to somehow mark the first possible world introduced, w, such that w will now be taken as the reference world. We do this by using *Ref*, which we symbolize

[16]This operator can also be found in Stalnaker (1978), being symbolized by †. On p. 320, Stalnaker offers $\Box\dagger$ as an apriority operator, which is equivalent to $\mathcal{FA}$. More on this in Sect. 4.1.

[17]This example can be found in Sider (2010, p. 225).

[18]Assuming, of course, a rigid actuality operator, where $\mathcal{A}$ always takes us to the actual world. In a two-dimensional logic, $\mathcal{A}$ copies down the upper world, whatever it is, whereby it is enough for the purposes of (7) that the upper world is not w.

henceforth as $\otimes$:[19]

$\mathcal{M}_w^v \models \otimes\varphi$ if and only if $\mathcal{M}_w^w \models \varphi$.

Now we can formalize (6) as follows:

(8) $\Diamond \otimes (\Diamond(\forall x)(\mathcal{A}(rich(x)) \supset (poor(x))) \supset (\exists x)(happy(x)))$

It can be easily checked that (8) gives us the correct formal rendering of (6). Just as $\mathcal{A}$ is a copy down operator, $\otimes$ can be seen as a copy up operator, for it copies up the world of evaluation to the upper position, the actual world under consideration. In the next section we present tableau methods for first-order $\mathbf{S5}_{2D}$, consisting of the two- dimensional operators alluded thus far alongside the modal operators for necessity and possibility.

2.3 Tableaux for $\mathbf{S5}_{2D}$

2.3.1 Syntax and Semantics for $\mathbf{S5}_{2D}$

Definition 2.3.1 (First-order language) For the language $\mathcal{L}_{2D}$, let $\{c_1, c_2, ...\}$ be a set of *constant symbols*, $\{x_1, x_2, ...\}$ a set of *individual variables*, and $\{P_1^n, P_2^n, ...\}$ a set of n-place *predicate symbols* for each $n \in \mathbb{N}$. The terms t and formulas φ are recursively generated by the following grammar ($i, n \in \mathbb{N}$):

$t ::= c_i \mid x_i$

$\varphi ::= P_i^n(t_1, ..., t_n) \mid t = t' \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \mathcal{A}\varphi \mid \odot\varphi \mid \otimes\varphi \mid \mathcal{F}\varphi \mid \exists x_i\varphi$

[19]Sider formalizes *Ref* by using $\times$. But since we use $\times$ as a syntactic mark for when a branch of tableau closes, we choose a slightly different symbol instead.

The other Boolean connectives and the universal quantifier $\forall x_i \varphi$ are defined as usual. Moreover, we define $\Diamond$ as $\neg\Box\neg$, and $\mathcal{S}$ as $\neg\mathcal{F}\neg$.[20] In addition, we define the set of *basic formulas* of $\mathcal{L}_{2D}$. Where AT is the set of atomic formulas of $\mathcal{L}_{2D}$, the basic formulas, ψ, of $\mathcal{L}_{2D}$, are generated as follows:

$$\psi ::= AT \mid \neg AT$$

With respect to the models, we have already mentioned different ways to define them for the two-dimensional case. We can either define *centered* models, with a distinguished element, whereby $\mathcal{F}$ changes the models to variant ones, or we can evaluate formulas with respect to pairs of worlds with $\mathcal{F}$ quantifying universally over the first coordinate of every pair. In the latter case, there was originally no need for a distinguished point in the models, since the first coordinate of every pair plays the role of the actual world under consideration anyway. But since we have a distinguishedly operator in our language, we definitely want a distinguished element fixed in the models. Our strategy, then, will be to evaluate formulas with respect to pairs of worlds, since this will provide us with a more uniform treatment for different two-dimensional systems, in which case the set of possible worlds in the models is defined accordingly in order to contain ordered pairs rather than single worlds. This can be accomplished by simply taking the usual set W of possible worlds and letting it be $Z \times Z$, for a non-empty set Z, with $w*$ being its distinguished element.[21] Consequently, $\odot\mathcal{A}$ will have the function of pointing to the distinguished element of two-dimensional models. We also define accessibility relations for both $\Box$ and $\mathcal{F}$ formulas, which we take to be equivalence relations, but which might be restricted to different properties as usual, generating two-dimensional versions of different modal logics like **T**, **B**, and so on.

Definition 2.3.2 (Constant domain 2D-centered model) A *constant domain 2D- centered model* is a tuple, $\mathcal{M} = \langle W, w*, \mathcal{R}_\Box, \mathcal{R}_\mathcal{F}, \mathcal{D}, V \rangle$, such that

[20] Also, in what follows we use $a, b, c, ...$ for constant symbols and $x, y, z, ...$ for individual variables.
[21] Since $w* \in W$ is the distinguished element of W. More on this below.

- $W = Z \times Z$ for some set Z,
- $w*$ is the distinguished element of Z,
- $\mathcal{R}_\Box \subseteq W \times W$, the $\Box$-accessibility relation, is the least relation such that for every $v, w, z \in Z$, $\langle v, w\rangle \mathcal{R}_\Box \langle v, z\rangle$,
- $\mathcal{R}_\mathcal{F} \subseteq W \times W$, the $\mathcal{F}$-accessibility relation, is the least relation such that for every $v, w, z \in Z$, $\langle v, w\rangle \mathcal{R}_\mathcal{F} \langle z, w\rangle$,
- $\mathcal{D}$ is a non-empty domain of quantification, and
- V is a function assigning to each constant c_i of $\mathcal{L}_{2D}$ and $\langle v, w\rangle \in W$, an object $V(c_i, \langle v, w\rangle) \in \mathcal{D}$, and to each n-place predicate symbol P_i^n and $\langle v, w\rangle \in W$, a set $V(P_i^n, \langle v, w\rangle) \in \mathcal{D}^n$.

Once we have a language, we extend the language to a new language, say, $\mathcal{L}(\mathcal{M})$, by adding a constant symbol c for each object in the domain. $\mathcal{M}$ interprets each of the new constants c as the object c from which it gets its name. For the moment we want our constants to be *rigid designators*, that is, with a world-invariant designation, not only with respect to $\mathcal{R}_\Box$ but also with respect to $\mathcal{R}_\mathcal{F}$-accessible worlds. Because we have two indices to keep track of in our semantics, there are two ways constants can be said to denote rigidly: a constant is $\mathcal{R}_\Box$-*rigid* if it denotes the same objects with respect to $\mathcal{R}_\Box$-accessible pairs of worlds, and a constant is $\mathcal{R}_\mathcal{F}$-*rigid* if it denotes the same objects with respect to $\mathcal{R}_\mathcal{F}$-accessible pairs of worlds. Since we want constants to be rigid in both senses at least for now, we assume the following constraints on models:[22] where c_i is any constant symbol in $\mathcal{L}_{2D}$,

($\mathcal{R}_\Box$-rigidity) For every $v, w, z \in Z$, if $\langle v, w\rangle \mathcal{R}_\Box \langle v, z\rangle$, then $V(c_i, \langle v, w\rangle) = V(c_i, \langle v, z\rangle)$,

[22] Those are almost the same as the ones defined by Holliday and Perry (2014). In Sect. 5 we define a different system with non-rigid constants for a new kind of accessibility relation.

($\mathcal{R}_{\mathcal{F}}$-rigidity) For every $v, w, z \in Z$, if $\langle v, w\rangle \mathcal{R}_{\mathcal{F}} \langle z, w\rangle$, then $V(c_i, \langle v, w\rangle) = V(c_i, \langle z, w\rangle)$.

Such conditions, of course, could be lifted in favour of non-rigid constants. But, for simplicity, we start with rigid constants, moving to non-rigid designation in Sect. 5. We also impose a third condition on models corresponding to an important feature of the semantics presented by Davies and Humberstone:

(Neutrality) If φ is a basic formula, then for every $v, w, z \in Z$, $\mathcal{M}^v_w \vDash \varphi$ if and only if $\mathcal{M}^z_w \vDash \varphi$

This constraint is intended to make the truth of basic formulas sensitive only to the second coordinate in a pair, in which case the actual world under consideration turns out to be a free parameter for atoms (and their negations) of the language. With this assumption at hand, our models will validate $\mathcal{F}\varphi \equiv \varphi$ for every basic formula φ, corresponding thereby to Davies and Humberstone's models.[23]

Definition 2.3.3 (Truth) We define 'φ is true at w relative to v in $\mathcal{M}$', written $\mathcal{M}^v_w \vDash \varphi$,

[23] In Sect. 5 we define another system in which this constraint is abandoned, and we discuss some philosophical issues related to it.

by recursion on φ. For a pair $\langle v, w\rangle \in W$,

$$\begin{aligned}
\mathcal{M}^v_w \vDash P^n_i(t_1, ..., t_n) &\iff \langle V(t_1, \langle v, w\rangle), ..., V(t_n, \langle v, w\rangle)\rangle \in V(P^n_i, \langle v, w\rangle);\\
\mathcal{M}^v_w \vDash t = t' &\iff V(t, \langle v, w\rangle) = V(t', \langle v, w\rangle);\\
\mathcal{M}^v_w \vDash \neg\varphi &\iff \mathcal{M}^v_w \nvDash \varphi;\\
\mathcal{M}^v_w \vDash \varphi \wedge \psi &\iff \mathcal{M}^v_w \vDash \varphi \text{ and } \mathcal{M}^v_w \vDash \psi;\\
\mathcal{M}^v_w \vDash \Box\varphi &\iff \text{for every } z \in Z, \text{ if } \langle v, w\rangle \mathcal{R}_\Box \langle v, z\rangle, \text{ then } \mathcal{M}^v_z \vDash \varphi;\\
\mathcal{M}^v_w \vDash \Diamond\varphi &\iff \text{for some } z \in Z, \langle v, w\rangle \mathcal{R}_\Box \langle v, z\rangle \text{ and } \mathcal{M}^v_z \vDash \varphi;\\
\mathcal{M}^v_w \vDash \mathcal{A}\varphi &\iff \mathcal{M}^v_v \vDash \varphi;\\
\mathcal{M}^v_w \vDash \odot\varphi &\iff \mathcal{M}^{w*}_w \vDash \varphi;\\
\mathcal{M}^v_w \vDash \otimes\varphi &\iff \mathcal{M}^w_w \vDash \varphi;\\
\mathcal{M}^v_w \vDash \mathcal{F}\varphi &\iff \text{for every } z \in Z, \text{ if } \langle v, w\rangle \mathcal{R}_\mathcal{F} \langle z, w\rangle, \text{ then } \mathcal{M}^z_w \vDash \varphi;\\
\mathcal{M}^v_w \vDash \mathcal{S}\varphi &\iff \text{for some } z \in Z, \langle v, w\rangle \mathcal{R}_\mathcal{F} \langle z, w\rangle \text{ and } \mathcal{M}^z_w \vDash \varphi;\\
\mathcal{M}^v_w \vDash \forall x\varphi &\iff \text{for each constant symbol } c \text{ of } \mathcal{L}(\mathcal{M}), \mathcal{M}^v_w \vDash \varphi[c/x];\\
\mathcal{M}^v_w \vDash \exists x\varphi &\iff \text{for some constant symbol } c \text{ of } \mathcal{L}(\mathcal{M}), \mathcal{M}^v_w \vDash \varphi[c/x];
\end{aligned}$$

Lastly, a formula φ is *false* at w relative to v in $\mathcal{M}$ if and only if it is not true at w relative to v in $\mathcal{M}$.

Definition 2.3.4 (Logical properties) A formula, φ, is *true simpliciter* under $\mathcal{M}$, written $\mathcal{M} \vDash \varphi$, if and only if φ is true at $w*$ relative to $w*$ in $\mathcal{M}$ (i.e. if and only if $\mathcal{M}^{w*}_{w*} \vDash \varphi$); a formula φ is *valid* if and only if it is true *simpliciter* under every $\mathcal{M}$; and a formula φ is a *logical consequence* of a set of formulas Γ if and only if for every $\mathcal{M}$, if $\mathcal{M} \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \vDash \varphi$.

Notice that we depart from Crossley and Humberstone's notion of general validity — truth at all worlds in all models — by adopting what they call 'real-world validity', which is just the definition of validity given in Kripke (1963a, p. 69). In Sect. 4, however, we show how to adapt the tableaux for general validity as well.

2.3.2 2S-Tableaux

Tableaux for two-dimensional modal logic are similar to the modal tableaux with numeric indices presented in both Fitting and Mendelsohn (1998) and Priest (2008). In both of these cases, every node of a tableau is an indexed formula, where the index is a numeral denoting (intuitively) a possible world according to which the relevant formula is evaluated.[24] In the 2D case, however, we need double indexing in order to denote (intuitively) the actual world under consideration — the upper index — as well as the usual possible world — the lower index. Let $n, m, ...$ denote lower indices, and $i, j, ...$ denote upper indices.[25] For the numeric index '0', we fix its interpretation to name $w*$, in which case the pair of indices $\langle 0, 0\rangle$ denotes the distinguished element in the respective model. Moreover, a doubly-indexed formula is an expression, $[\varphi]^i_n$ where i and n constitute a pair of indices, $\langle i, n\rangle$, and φ is a formula. All doubly-indexed formulas are enclosed within brackets.

The root of a 2D-tableau always contains the negation of the formula we are attempting to prove doubly-indexed by the pair $\langle 0, 0\rangle$. So, for any formula, φ, a 2D-tableau proof of φ begins with $[\neg\varphi]^0_0$. This says, intuitively, that $\neg\varphi$ is true at the possible world denoted by '0' relative to '0' taken as actual. On the other hand, if we attempt to prove φ from a non-empty set of premises, we add $[\psi]^0_0$, for every premise ψ, to the first lines of the 2D-tableau.

Now we present the rules for 2D-tableaux. The rules for the Boolean connectives are just

[24]In Priest's (2008) case, some nodes may be expressions such as 'nrm', where n and m are numerals such that the possible world denoted by m is r-accessible from n. Fitting and Mendelsohn (1998), on the other hand, do not make use of single nodes reflecting the accessibility relations of the system in question. Rather, formulas may be prefixed by sequences of indices such as $\sigma.n$, which means intuitively that n is a world accessible from σ; whereby there is no need to add nodes with the single purpose of denoting an accessibility relation.

[25]We avoid using the letter 'o' to denote lower indices since it may be confused with the number 0.

the ones found in Smullyan (1995) grouped into conjunctive and disjunctive rules, except that they now receive double indexing. Moreover, for the Boolean connectives, the double indexing of a formula is preserved to its immediate descendants.

Definition 2.3.5 (Conjunctive Rules)

$$\text{For any pair of indices } \langle i, n \rangle,$$
$$\begin{array}{cccc} [\varphi \wedge \psi]^i_n & [\neg(\varphi \vee \psi)]^i_n & [\neg(\varphi \supset \psi)]^i_n & [\varphi \equiv \psi]^i_n \\ \downarrow & \downarrow & \downarrow & \downarrow \\ [\varphi]^i_n & [\neg\varphi]^i_n & [\varphi]^i_n & [\varphi \supset \psi]^i_n \\ [\psi]^i_n & [\neg\psi]^i_n & [\neg\psi]^i_n & [\psi \supset \varphi]^i_n \end{array}$$

Definition 2.3.6 (Disjunctive Rules)

$$\text{For any pair of indices } \langle i, n \rangle,$$
$$\begin{array}{cccc} [\varphi \vee \psi]^i_n & [\neg(\varphi \wedge \psi)]^i_n & [\varphi \supset \psi]^i_n & [\neg(\varphi \equiv \psi)]^i_n \\ \swarrow \quad \searrow & \swarrow \quad \searrow & \swarrow \quad \searrow & \swarrow \quad \searrow \\ [\psi]^i_n \quad [\varphi]^i_n & [\neg\varphi]^i_n \quad [\neg\psi]^i_n & [\neg\varphi]^i_n \quad [\psi]^i_n & [\neg(\varphi \supset \psi)]^i_n \quad [\neg(\psi \supset \varphi)]^i_n \end{array}$$

Moreover, we define a double negation rule as well:

Definition 2.3.7 (Double Negation Rule)

$$\text{For any pair of indices } \langle i, n \rangle,$$
$$\begin{array}{c} [\neg\neg\varphi]^i_n \\ \downarrow \\ [\varphi]^i_n \end{array}$$

Next we present the rules for modal and two-dimensional operators. These are grouped as follows. The actuality rules contain the rules for $\mathcal{A}$, $\odot$, and $\otimes$; the possibility rules contain the rules for $\Diamond$ and $\mathcal{S}$; and the necessity rules comprise the rules for $\Box$ and $\mathcal{F}$. For the possibility rules there is a constraint on the indices added to the branch where the rule is applied, namely, they have to be new to the branch. This is just the usual set up for possibility rules in modal tableaux.[26] But since the formulas in 2D-tableaux are doubly-indexed, this ought to be generalized for two indices. Next we state the rules for modal and two-dimensional operators:

Definition 2.3.8 (Actuality Rules)

For any pair of indices $\langle i, n \rangle$,

$$\begin{array}{cccccc} [\mathcal{A}\varphi]^i_n & [\neg\mathcal{A}\varphi]^i_n & [\odot\varphi]^i_n & [\neg\odot\varphi]^i_n & [\otimes\varphi]^i_n & [\neg\otimes\varphi]^i_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ [\varphi]^i_i & [\neg\varphi]^i_i & [\varphi]^0_n & [\neg\varphi]^0_n & [\varphi]^n_n & [\neg\varphi]^n_n \end{array}$$

Definition 2.3.9 (Possibility Rules)

If the index z is new to the branch,

$$\begin{array}{cccc} [\Diamond\varphi]^i_n & [\neg\Box\varphi]^i_n & [\mathcal{S}\varphi]^i_n & [\neg\mathcal{F}\varphi]^i_n \\ \downarrow & \downarrow & \downarrow & \downarrow \\ [\varphi]^i_z & [\neg\varphi]^i_z & [\varphi]^z_n & [\neg\varphi]^z_n \end{array}$$

Definition 2.3.10 (Necessity Rules)

For every index z occurring on the branch,

$$\begin{array}{cccc} [\Box\varphi]^i_n & [\neg\Diamond\varphi]^i_n & [\mathcal{F}\varphi]^i_n & [\neg\mathcal{S}\varphi]^i_n \\ \downarrow & \downarrow & \downarrow & \downarrow \\ [\varphi]^i_z & [\neg\varphi]^i_z & [\varphi]^z_n & [\neg\varphi]^z_n \end{array}$$

In addition, we define the following rule corresponding to (Neutrality):

Definition 2.3.11 (Upper exchange Rule)

If φ is a basic formula, then for any index j on the branch,

$$\begin{array}{c} [\varphi]^i_n \\ \downarrow \\ [\varphi]^j_n \end{array}$$

[26] See, for instance, Fitting and Mendelsohn (1998, p. 49).

Finally, we present the rules for identity and quantifiers. The rules for the quantifiers are just the usual ones found in tableaux for constant domain modal logic, so there is nothing particularly new to them.[27] We adopt constant domain here because of its convenience, but it should be clear how the rules for variable domains can be similarly adapted. In the case of identity, one of the rules allows us some freedom with respect to the lower index, just as in the basic modal case for identity, while keeping the upper index unmoved.

Definition 2.3.12 (Identity Rules)

For any constant c and pair of indices $\langle i, n \rangle$ already occurring on the branch,

$$[c = c]^i_n$$

If $[c = d]^i_n$ and $[\varphi(c)]^i_m$ already occur on the branch,

$$\begin{array}{c} [c = d]^i_n \\ [\varphi(c)]^i_m \\ \downarrow \\ [\varphi(d)]^i_m \end{array}$$

Definition 2.3.13 (Universal Rules)

For any constant c and pair of indices $\langle i, n \rangle$,

$$\begin{array}{cc} [(\forall x)\varphi]^i_n & [\neg(\exists x)\varphi]^i_n \\ \downarrow & \downarrow \\ [\varphi[c/x]]^i_n & [\neg\varphi[c/x]]^i_n \end{array}$$

Definition 2.3.14 (Existential Rules)

For any constant c not occurring on the branch, and pair of indices $\langle i, n \rangle$,

$$\begin{array}{cc} [(\exists x)\varphi]^i_n & [\neg(\forall x)\varphi]^i_n \\ \downarrow & \downarrow \\ [\varphi[c/x]]^i_n & [\neg\varphi[c/x]]^i_n \end{array}$$

[27]See, for instance, Priest (2008, p. 310).

We say that a branch $\mathfrak{b}$ of a 2D-tableau is *closed* just in case for some formula, φ, and a pair of indices $\langle i, n \rangle$, both $[\varphi]^i_n$ and $[\neg\varphi]^i_n$ occur on $\mathfrak{b}$. Otherwise, we say the 2D-tableau is *open*. A 2D-tableau is *closed* just in case all of its branches are closed. Finally, we define what it is to be a 2D-tableau proof:

Definition 2.3.15 (2D-tableau proof) A *2D-tableau proof* of a formula, φ, is a closed 2D-tableau for $[\neg\varphi]^0_0$. Moreover, for a (possibly empty) set of formulas, Γ, and a formula, φ, if φ is provable from Γ in $\mathbf{S5}_{2D}$ (i.e. if there is a closed tableau for $[\neg\varphi]^0_0$ with the elements of Γ listed in its origin) we write $\Gamma \vdash \varphi$. As usual, φ is a *theorem* in $\mathbf{S5}_{2D}$ just in case it is provable from an empty set of premises.

Soundness and completeness proofs for $\mathbf{S5}_{2D}$ can be found in appendix A. It is illustrative to see how 2D-tableaux work, so in what follows we prove some theorems of $\mathbf{S5}_{2D}$. We begin with the propositional portion of $\mathbf{S5}_{2D}$, and then we exhibit a quantified case:

$\vdash \mathcal{F}\mathcal{A}p \equiv \Box p$

$$1.\ [\neg(\mathcal{F}\mathcal{A}p \equiv \Box p)]^0_0$$

$2.\ [\neg(\mathcal{F}\mathcal{A}p \supset \Box p)]^0_0$	$3.\ [\neg(\Box p \supset \mathcal{F}\mathcal{A}p)]^0_0$
$4.\ [\mathcal{F}\mathcal{A}p]^0_0$	$12.\ [\Box p]^0_0$
$5.\ [\neg\Box p]^0_0$	$13.\ [\neg\mathcal{F}\mathcal{A}p]^0_0$
$6.\ [\neg p]^0_1$	$14.\ [\neg\mathcal{A}p]^1_0$
$7.\ [\mathcal{A}p]^0_0$	$15.\ [\neg p]^1_1$
$8.\ [\mathcal{A}p]^1_0$	$16.\ [p]^0_0$
$9.\ [p]^0_0$	$17.\ [p]^0_1$
$10.\ [p]^1_1$	$18.\ [p]^1_1$
$11.\ [p]^0_1$	$\times$
$\times$	

Items 2, 3, 4, 5, 12, and 13 result from truth-functional rules, while 6, 16, and 17 result from the possibility and necessity rules for $\neg\Box$ and $\Box$, respectively. 7, 8, and 14 result from the necessity and possibility rules for $\mathcal{F}$ and $\neg\mathcal{F}$, respectively, while 9, 10, and 15 are just applications of the rules for $\mathcal{A}$. Finally, the upper exchange rule was used in 11 and 18. Another example, this time involving a failed tableau proof attempt, is the following:

$\nvdash \mathcal{S}\Diamond p \supset \Box\mathcal{F}\Box p$

1. $[\neg(\mathcal{S}\Diamond p \supset \Box\mathcal{F}\Box p)]_0^0$

2. $[\mathcal{S}\Diamond p]_0^0$

3. $[\neg\Box\mathcal{F}\Box p]_0^0$

4. $[\neg\mathcal{F}\Box p]_1^0$

5. $[\neg\Box p]_1^2$

6. $[\neg p]_3^2$

7. $[\Diamond p]_0^4$

8. $[p]_5^4$

The only branch of the tableau remains open. Since this case only involves the propositional portion of $\mathbf{S5}_{2D}$, a propositional model, $\mathcal{M} = \langle W, w*, \mathcal{R}_{\Box}, \mathcal{R}_{\mathcal{D}}, V\rangle$, can be constructed from the open branch above in the following way. Where k is any single index, $\mathfrak{b}$ is the open branch above, and $Z = \{k \mid k \in \mathfrak{b}\}$, set $W = Z \times Z$, where 0 is the distinguished element of Z. Moreover, for every $\langle i, n\rangle, \langle i, m\rangle \in W$, set $\langle i, n\rangle\mathcal{R}_{\Box}\langle i, m\rangle$; similarly, for every $\langle i, n\rangle, \langle j, n\rangle \in W$, set $\langle i, n\rangle\mathcal{R}_{\mathcal{F}}\langle j, n\rangle$. Since we are assuming that both accessibility relations are equivalence relations, we drop any mention of them in what follows. Finally, for any propositional letter, say, q, if $[q]_n^i$ occurs on $\mathfrak{b}$, then $\mathcal{M}_n^i \vDash q$; if, on the other hand, $[\neg q]_n^i$ occurs on $\mathfrak{b}$, then $\mathcal{M}_n^i \nvDash q$; if neither, set either $\mathcal{M}_n^i \vDash q$ or $\mathcal{M}_n^i \nvDash q$. Thus, a countermodel for the open branch in the tableau above will have $Z = \{0, 1, 2, 3, 4, 5\}$, $W = Z \times Z$, $\mathcal{M}_3^2 \nvDash p$, and $\mathcal{M}_5^4 \vDash p$.

The set W contains a long list of pairs generated from Z, but we can represent a countermodel using a 2D-matrix where the vertical axis designates reference or actual worlds and the horizontal axis designates possible worlds:

$$\begin{pmatrix} & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & \times & \times & \times & \times & \times & \times \\ 1 & \times & \times & \times & \times & \times & \times \\ 2 & \times & \times & \times & \neg p & \times & \times \\ 3 & \times & \times & \times & \times & \times & \times \\ 4 & \times & \times & \times & \times & \times & p \\ 5 & \times & \times & \times & \times & \times & \times \end{pmatrix}$$

Since $\mathcal{M}_5^4 \vDash p$, it follows that $\mathcal{M}_0^4 \vDash \Diamond p$, and hence $\mathcal{M}_0^0 \vDash \mathcal{S}\Diamond p$. On the other hand, $\mathcal{M}_3^2 \nvDash p$, whence $\mathcal{M}_1^2 \nvDash \Box p$ and $\mathcal{M}_1^0 \nvDash \mathcal{F}\Box p$. Therefore, $\mathcal{M}_0^0 \nvDash \Box\mathcal{F}\Box p$. Finally, in what follows we show how a countermodel can be read-off from a failed proof with quantifiers:

$\nvdash (\exists x)Px \supset (\forall x)\mathcal{F}\mathcal{A}\Diamond \otimes Px$

$$\begin{gathered}
1.\ [\neg((\exists x)Px \supset (\forall x)\mathcal{F}\mathcal{A}\Diamond \otimes Px)]_0^0 \\
2.\ [(\exists x)Px]_0^0 \\
3.\ [\neg(\forall x)\mathcal{F}\mathcal{A}\Diamond \otimes Px]_0^0 \\
4.\ [Pa]_0^0 \\
5.\ [\neg\mathcal{F}\mathcal{A}\Diamond \otimes Pb]_0^0 \\
6.\ [\neg\mathcal{A}\Diamond \otimes Pb]_0^1 \\
7.\ [\neg\Diamond \otimes Pb]_1^1 \\
8.\ [\neg \otimes Pb]_0^1 \\
9.\ [\neg \otimes Pb]_1^1 \\
10.\ [\neg Pb]_0^0 \\
11.\ [\neg Pb]_1^1
\end{gathered}$$

Again, set $W = Z \times Z$, where Z is the set containing every (single) index occurring on the open branch above, $\mathfrak{b}$, and let 0 be the distinguished element of Z. Now we define C as the set of all constants, c, occurring on $\mathfrak{b}$, and $\mathscr{D} = \{c \mid c \in C\}$. Moreover, set $V(c, \langle i, n \rangle) = c$ to each $c \in C$ and $\langle i, n \rangle \in W$, and to each n-place predicate symbol, R, on $\mathfrak{b}$, let $V(R, \langle i, n \rangle) = \{\langle c_1, ..., c_n \rangle \mid R(c_1, ..., c_n)^i_n \text{ occurs on } \mathfrak{b}\}$. Given both rigidity conditions, for any $\langle i, n \rangle, \langle j, m \rangle \in W$ and $c \in C$, set $V(c, \langle i, n \rangle) = V(c, \langle j, m \rangle)$. The following is a countermodel for the open branch above: $Z = \{0, 1\}$, in which case we have $W = \{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle\}$, $V(P, \langle 0, 0 \rangle) = \{a\}$. We can represent the model by the following 2D-matrix:

$$\begin{pmatrix} & 0 & 1 \\ 0 & Pa & \times \\ 1 & \times & \times \end{pmatrix}$$

We prove that $\mathcal{M}^0_0 \nvDash (\exists x)Px \supset (\forall x)\mathcal{FA}\Diamond \otimes Px$. Consider $\mathcal{M}^0_0 \vDash Pa$. Thus, $\mathcal{M}^0_0 \vDash (\exists x)Px$. However, for some constant b, $\mathcal{M}^0_0 \nvDash Pb$ and $\mathcal{M}^1_1 \nvDash Pb$, in which case we have both $\mathcal{M}^1_0 \nvDash \otimes Pb$ and $\mathcal{M}^1_1 \nvDash \otimes Pb$. So, $\mathcal{M}^1_1 \nvDash \Diamond\otimes Pb$ and $\mathcal{M}^1_0 \nvDash \mathcal{A}\Diamond\otimes Pb$, whence $\mathcal{M}^0_0 \nvDash \mathcal{FA}\Diamond\otimes Pb$. Therefore, $\mathcal{M}^0_0 \nvDash (\forall x)\mathcal{FA}\Diamond \otimes Px$.

2.4 Local, General, and Diagonal Validity

In the semantics for $\mathbf{S5}_{2D}$ we defined validity as truth at the distinguished element of every model. This is just what you would expect as an extension of the canonical notion of validity found in Kripke's early work in modal semantics.[28] However, this has not become the standard definition of validity for modal logics since, after Kripke's early papers, validity is usually defined as *truth at all worlds in every model*. In Crossley and Humberstone (1977) those are called *real world* and *general* validity, respectively.[29] A small terminological difference is that here we use *local* validity for the former.

In basic modal logics without actuality operators both notions of validity are equivalent, but this does not hold in general for languages containing $\mathcal{A}$. For example, the formula $\mathcal{A}p \supset p$ is locally but not generally valid. This is easy to check, for in a basic modal semantics we can just set $w* \in V(p)$ and $w \notin V(p)$, where w is any world other than $w*$, which makes $\mathcal{A}p$ true at w and p false at w.[30] Since a formula is generally valid just in case it holds at every world of every model, the above suffices for a model falsifying $\mathcal{A}p \supset p$. On the other hand, its $\mathcal{A}$-modalization, $\mathcal{A}(\mathcal{A}p \supset p)$, is in effect generally valid. This reflects the fact that for any locally valid formula, ψ, its $\mathcal{A}$-modalization, $\mathcal{A}\psi$, is generally valid.

The set of valid formulas in modal logics containing an actuality operator is conditional on our choice between local and general validity, which brings about important philosophical consequences for the resulting logic. Zalta (1988), for instance, argues that local validity is the best alternative for modal logics since it is defined in terms of truth *simpliciter*, or truth in a model. General validity, on the other hand, bypasses the notion of truth in a model, being thus defined in terms of truth at all worlds. Hence, while local validity clearly characterizes the canonical Tarskian notion of logical truth, general validity does not. Besides, truth at all worlds is best seen as characterizing necessity, and there seems to be no principled reason to identify validity with necessary truth, at least not without an argument.[31] The upshot, according to Zalta, is that modal logics with an actuality operator, when characterized by

[28]See Kripke (1958) and (1963a).

[29]Zalta (1988) provides a list with many influential textbooks in modal logic in which the authors define validity as general rather than real-world. More recently, we could also add Fitting and Mendelsohn (1998) and Priest(2008) to the list.

[30]In our case, this has to be adjusted to pairs of worlds.

[31]For more on this, see Nelson and Zalta (2012), and Hanson (2006, 2014) for a response.

the right notion of validity, generate contingent logical truths, whose paradigmatic example is $\mathcal{A}p \supset p$. It is straightforward, moreover, to see that this move entails a failure of the rule of necessitation, for even though $\mathcal{A}p \supset p$ is locally valid, its necessitation, $\Box(\mathcal{A}p \supset p)$, is not.[32] Therefore, in logics containing an actuality operator, the rule of necessitation must be restricted accordingly.

2.4.1 Modular Tableaux

There are several questions to be asked concerning the best notion of validity in modal logics with actuality operators. Should we count, after all, $\mathcal{A}p \supset p$ as a logical truth? Do we need to have an account of validity directly in terms of truth in a model? Perhaps modal semantics requires logical truth to be delineated differently. Perhaps it does not. Be that as it may, our present task does not involve defending a particular notion of validity, for it is a clear advantage of 2D-tableaux that they afford us with a simple way to modulate between local and general validity, which is a palliative for those uncomfortable with contingent logical truths.[33] With respect to local validity, 2D-tableaux are exactly as they were set up above, that is, the root of a 2D-tableau is the negation of the formula we want to prove, doubly-indexed by $\langle 0, 0\rangle$, i.e. $[\neg\varphi]_0^0$. (We remind the reader that the index '0' is fixed to denote $w*$). Thus, given the soundness and completeness theorems in appendix A, a formula, φ, is locally valid just in case there is a proof of φ (from zero premises) indexed by $\langle 0, 0\rangle$.

In order to make a distinction between tableaux for different kinds of validity, we call tableaux for local validity *local 2D-tableaux*. Next we show how to set up *general 2D-tableaux* corresponding to general validity.

[32]This causes the rule of necessitation to fail in $\mathbf{S5}_{2D}$ as well, unless one defines general rather than local validity. In that case, $\mathcal{A}p \supset p$ would not count as a theorem of $\mathbf{S5}_{2D}$.

[33]This was previously shown in Ray and Lampert (manuscript) for natural deduction systems with numeric indices. Since the tableaux presented here work in a similar way, this result could be easily adapted. I am grateful to Greg Ray for helpful suggestions concerning this section.

Tableaux for general validity

In the case of general validity, the tableaux are set up as follows:

Definition 2.4.1 (General 2D-tableaux) A *general 2D-tableau* is defined just like local 2D-tableaux, except that the root of a general 2D-tableau is the negation of the formula we want to prove, doubly-indexed by $\langle n, m\rangle$, i.e. $[\neg\varphi]^n_m$, where n and m denote any indices different from '0', and $n \neq m$.

Thus, if φ is provable from zero premises where its negation was doubly-indexed by $\langle n, m\rangle$ on the root of the tableau, the soundness theorem in the appendix guarantees that for every 2D-centered model, $\mathcal{M} = \langle W, w*, \mathcal{R}_{\Box}, \mathcal{R}_{\mathcal{F}}, \mathscr{D}, V\rangle$, φ is true at $\langle n, m\rangle$ under $\mathcal{M}$. Since there are no constraints imposed on neither n nor m besides their being different and distinct from '0', they can pick out any two worlds in any model,[34] whence φ will be generally valid. Now the definition of a 2D-tableau proof ought to be restated, too, for general validity:

Definition 2.4.2 A *general 2D-tableau proof* of a formula, φ, is a closed general 2D-tableau for $[\neg\varphi]^n_m$, where n and m denote any indices different from '0', and $n \neq m$.

Furthermore, the generalized notion of consequence corresponding to general validity says that a formula, φ, is a *general consequence* of a set of formulas Γ if and only if for every model $\mathcal{M}$ and every pair $\langle v, w\rangle \in W$, if $\mathcal{M}^v_w \vDash \gamma$, for all $\gamma \in \Gamma$, then $\mathcal{M}^v_w \vDash \varphi$.

Now we give an example of a general 2D-tableau proof attempt of a formula that is locally but not generally valid. We use the indices '1','2' for n and m, respectively, at the root of

[34] For languages lacking a distinguishedly operator, one of the indices does not need to be different from '0'. Thus, general 2D-tableaux for $\odot$-free languages might begin with the pair $\langle 0, n\rangle$, where $0 \neq n$. However, things are trickier in languages containing $\odot$. For example, by starting a general 2D-tableau with $\langle 0, 1\rangle$ we would be able to prove $\Box p \supset \odot\mathcal{A}p$, which is valid when (Neutrality) is assumed, but not otherwise. Since in §5 we investigate two-dimensional logics where the models are not constrained by (Neutrality), had we defined general 2D-tableaux without requiring of both indices in the tableau's root that they must be different from '0', we would have to change our definition and incorporate this restriction. Because of this, the requirement that both indices should be different from '0' gives us a more general treatment to deal with general 2D-tableaux.

the tableau.

$\nvDash \mathcal{A}p \supset p$

$$
\begin{array}{r l}
1. & [\neg(\mathcal{A}p \supset p)]^1_2 \\
2. & [\mathcal{A}p]^1_2 \\
3. & [\neg p]^1_2 \\
4. & [p]^1_1
\end{array}
$$

The general 2D-tableau remains open, and a countermodel can be read-off from the open branch where $W = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$, $\mathcal{M}^1_1 \vDash p$, and $\mathcal{M}^1_2 \nvDash p$. The following 2D-matrix illustrates the countermodel:

$$
\begin{pmatrix}
 & 1 & 2 \\
1 & p & \neg p \\
2 & \times & \times
\end{pmatrix}
$$

Since $\mathcal{M}^1_1 \vDash p$, it follows that $\mathcal{M}^1_1 \vDash \mathcal{A}p$, whence $\mathcal{M}^1_1 \vDash \mathcal{A}p \supset p$. On the other hand, since $\mathcal{M}^1_1 \vDash p$, it also follows that $\mathcal{M}^1_2 \vDash \mathcal{A}p$. But since $\mathcal{M}^1_2 \nvDash p$, we have $\mathcal{M}^1_2 \nvDash \mathcal{A}p \supset p$, whence $\mathcal{A}p \supset p$ is not generally valid.

Tableaux for diagonal validity

There is yet another notion of validity that is somewhat natural for two-dimensional modal logics, namely, diagonal validity. A formula, φ, is *diagonally valid* if and only if φ holds at every pair $\langle w, w \rangle$ in every model.[35] As remarked in §2, when Davies and Humberstone offered a two-dimensional semantics, evaluating formulas with respect to pairs of worlds, they dropped a distinguished element from the models. This is because the intuitive function of representing the actual world was taken up by the upper world in the evaluation of formulas. So, rather than a single distinguished element defined in the models, we now have a large set of actual worlds, i.e. the distinguished element of each variant of the original model, or every upper index in every model. In fact, the novelty with respect to the fixedly operator is that it changes the model in question to a variant one, where a possibly different world becomes the actual one.

Although the two-dimensional notation of Davies and Humberstone simplifies this in a very elegant manner, it nevertheless leaves out an important aspect of the original account: once the fixedly operator takes effect, we are evaluating formulas relative to variants of the original model, where the distinguished element of the latter is supposed to represent intuitively the (only) real world, the other actual worlds in the variant models being alternatives with respect to it. In this sense, we can say that there is at most one actual world, even though different worlds might be considered as actual. Informally and intuitively, one of the insights that a distinguishedly operator brings back to two-dimensional modal logic is the following: that there is a real world in the space of two-dimensional possibilities to which we can always refer, even within contexts where we are far removed from it by embedding under modal or fixedly operators. *This is just what the actuality operator does in one-dimensional modal logic.* Another way to think of it is that distinguishedly marks the original model, bringing us back from any variant thereof.

In the absence of a distinguishedly operator, diagonal validity tells us that a formula is valid just in case it holds at every world considered as actual. The formula $\mathcal{A}\varphi \supset \varphi$, for example, happens to be both locally and diagonally valid. Also, there is a strong sense of

[35] Humberstone (2008, p. 259–264), also discusses diagonal validity. Moreover, Kocurek (2016) adopts diagonal validity for a system containing actually and fixedly operators.

necessity featuring in diagonal validity that was lacking in local validity. Since a formula is diagonally valid just in case it holds at every pair along the diagonal, it holds at every possible world taken as actual. Admittedly, this is not the same as the intuitive notion of metaphysical necessity, which is represented in our 2D-matrices by truth along rows, and not truth along the diagonal. But it is a necessity nonetheless. In fact, diagonal validity brings to light precisely the notion of necessity defended by Davies and Humberstone, namely, $\mathcal{FA}$-necessity, whereas $\Box$-necessity usually corresponds to what is generally valid. Hence, even though we could not have $\mathcal{A}\varphi \supset \varphi$ to come out valid when validity was taken to correspond to $\Box$-necessary truth (general validity), it is valid when a different kind of necessity is in play.

Now, diagonal validity, or truth at every world taken as actual, can also be represented by diagonal tableaux as follows:

Definition 2.4.3 (Diagonal 2D-tableaux) A *diagonal 2D-tableau* is defined just like local 2D-tableaux, except that the root of the 2D-tableau is now the negation of the formula we want to prove, doubly-indexed by $\langle n, n\rangle$, where n is any index other than '0', i.e. $[\neg\varphi]_n^n$.

Diagonal 2D-tableaux, then, begin with any pair of identical indices, except that they are different from '0', corresponding thereby to an arbitrary point on the diagonal. We also define the notion of a 2D-tableau proof for the diagonal case as follows:

Definition 2.4.4 (Diagonal 2D-tableau proof) A *diagonal 2D-tableau proof* of a formula, φ, is a closed diagonal 2D-tableau for $[\neg\varphi]_n^n$, where n is any index other than '0'.

Lastly, diagonal validity corresponds to a limiting case of diagonal consequence, where a formula, φ, is a *diagonal consequence* of a set of formulas Γ if and only if for every model $\mathcal{M}$ and every pair $\langle w, w\rangle \in W$, if $\mathcal{M}_w^w \vDash \gamma$, for all $\gamma \in \Gamma$, then $\mathcal{M}_w^w \vDash \varphi$.

Now it can be verified by a tableau proof that $\mathcal{A}p \supset p$ is diagonally valid. We use the index

‘1’ for n at the root of the tableau.

$\vdash \mathcal{A}p \supset p$

$$1.\ [\neg(\mathcal{A}p \supset p)]_1^1$$
$$2.\ [\mathcal{A}p]_1^1$$
$$3.\ [\neg p]_1^1$$
$$4.\ [p]_1^1$$
$$\times$$

The proof above suggests that local and diagonal validity coincide. However, the formula $\odot\mathcal{A}p \supset p$ is locally, but not diagonally valid. In fact, this formula also fails to be generally valid. Next we provide a failed tableau proof attempt of this formula using diagonal 2D-tableaux:

$\nvdash \odot\mathcal{A}p \supset p$

$$1.\ [\neg(\odot\mathcal{A}p \supset p)]_1^1$$
$$2.\ [\odot\mathcal{A}p]_1^1$$
$$3.\ [\neg p]_1^1$$
$$4.\ [\mathcal{A}p]_1^0$$
$$5.\ [p]_0^0$$

The diagonal 2D-tableau remains open, and a countermodel can be read-off from the open branch where $W = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$, $\mathcal{M}_0^0 \vDash p$, and $\mathcal{M}_1^1 \nvDash p$. The following 2D-matrix illustrates the countermodel:

$$\begin{pmatrix} & 0 & 1 \\ 0 & p & \times \\ 1 & \times & \neg p \end{pmatrix}$$

Since $\mathcal{M}_0^0 \vDash p$, it follows that $\mathcal{M}_0^0 \vDash \mathcal{A}p$, and also $\mathcal{M}_0^0 \vDash \odot\mathcal{A}p$. Therefore, $\mathcal{M}_0^0 \vDash \odot\mathcal{A}p \supset p$. On the other hand, since $\mathcal{M}_0^0 \vDash p$, it also follows that $\mathcal{M}_1^0 \vDash \mathcal{A}p$, and $\mathcal{M}_1^1 \vDash \odot\mathcal{A}p$. But since $\mathcal{M}_1^1 \nvDash p$, we have $\mathcal{M}_1^1 \nvDash \odot\mathcal{A}p \supset p$, whence $\mathcal{A}p \supset p$ fails to hold at some point on the diagonal, and consequently it is not diagonally valid.

It is simple to check that local and diagonal validity coincide for distinguishedly-free formulas, for in the absence of $\odot$, there is no operator marking $w*$ in the language. Thus, if a $\odot$-free formula fails to be locally valid, it fails on some point on the diagonal in a model. On the other hand, if a $\odot$-free formula fails on some diagonal point in a model, if fails to be locally valid in the model generated by making that point the distinguished one.

2.5 Tableaux for Different Two-Dimensional Systems

The system $\mathbf{S5}_{2D}$ generalizes providing sound and complete tableaux for a variety of two-dimensional modal logics. The *locus classicus* of the logic $\mathbf{S5}\mathcal{A}$ is Crossley and Humberstone's seminal paper where a sound and complete axiomatization using general validity is offered for the propositional case. A Fitch-style natural deduction system for the same logic can be found in Hazen (1978), the only difference being that Hazen uses local rather than general validity. Now, by modulating between local and general tableaux one gets sound and complete systems for the logics defined by Hazen as well as Crossley and Humberstone, for $\mathbf{S5}_{2D}$ is clearly a conservative extension of $\mathbf{S5}\mathcal{A}$ with local validity, and our general tableaux are sound and complete too for $\mathbf{S5}\mathcal{A}$ with general validity.[36] Obviously, the same can be said for $\mathbf{S5}\mathcal{AF}$, the logic of the fixedly operator defined by Davies and Humberstone.

2.5.1 Tableaux for Epistemic Two-Dimensional Semantics

In this section we define a formal system based on the ideas of epistemic two-dimensional semantics as developed by Chalmers (1996, 2004). For this purpose, the system will be very similar to the two-dimensional logics for a priori knowledge considered in both Restall (2012) and Fritz (2013, 2014). The languages of these systems comprise truth-functional connectives, the modal operators $\Box$ and $\Diamond$, an actuality operator $\mathcal{A}$, and a primitive apriority operator, which we symbolize as $\mathcal{D}$, behaving similarly to $\mathcal{FA}$, i.e. it quantifies over every point on the diagonal. Fritz's logic is directly based on the epistemic two-dimensional semantics proposed by Chalmers, and so he suggests we should also define the dual of the a priori operator, which may be read intuitively as a conceivability operator as motivated by Chalmers[37] — even though Fritz himself does not commit to it being a formal rendering of conceivability.[38] In what follows we define the formal language and semantics of our system, which we call $\mathbf{S5E}_{2D}$. The main difference between $\mathbf{S5E}_{2D}$ and the logics of both Fritz and Restall[39] is that $\mathbf{S5E}_{2D}$ is extended to the first-order case, in which case we can introduce distinctions concerning rigidity.

Our language will be defined similarly to Definition 2.3.1, except that we make changes concerning the choice of two-dimensional operators:

Definition 2.5.1 (First-order Language) Let $\{c_1, c_2, ...\}$ be a set of *constant symbols*, $\{x_1, x_2, ...\}$ a set of *individual variables*, and $\{P_1^n, P_2^n, ...\}$ a set of n-place *predicate symbols*

[36]In appendix C we show some results concerning how $\mathbf{S5}_{2D}$ extends those systems.
[37]See Chalmers (2004), p. 219.
[38]See Fritz (2014), p. 386. Although Chalmers (2004, p. 219), motivates *conceivability* as the dual of *apriority*, see Chalmers (2011b) for a criticism.
[39]These are equivalent. See Fritz (2014).

for each $n \in \mathbb{N}$. The terms t and formulas φ of $\mathcal{L}_{E2D}$ are recursively generated by the following grammar ($i, n \in \mathbb{N}$):

$$t ::= c_i \mid x_i$$

$$\varphi ::= P_i^n(t_1, ..., t_n) \mid t = t' \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \mathcal{A}\varphi \mid \mathcal{D}\varphi \mid \exists x_i \varphi$$

The basic formulas of the language comprise, again, atoms and their negations. Besides $\forall x_i \varphi$, $\Diamond$, and the other Boolean connectives, which are defined as usual, we also define $\mathcal{C}$, the dual of $\mathcal{D}$, as $\neg\mathcal{D}\neg$. Our models will be just as in Definition 2.3.2, except for having a $\mathcal{D}$-accessibility relation rather than an $\mathcal{F}$-accessibility relation:

Definition 2.5.2 (Constant domain 2D-centered models) A *constant domain 2D-centered model* is a tuple, $\mathcal{M} = \langle W, w*, \mathcal{R}_\Box, \mathcal{R}_\mathcal{D}, \mathscr{D}, V \rangle$, such that

- $\mathcal{R}_\mathcal{D} \subseteq W \times W$, the $\mathcal{D}$-accessibility relation, is the least relation such that for every $v, w, z \in Z$, $\langle v, w \rangle \mathcal{R}_\mathcal{D} \langle z, z \rangle$.

Again, this accessibility relation might be restricted in order to generate different systems. Moreover, this time we only impose the $\mathcal{R}_\Box$-rigidity condition on models. Even though a similar $\mathcal{R}_\mathcal{D}$-rigidity condition might be applied, the logic is intended to capture traditional features of epistemic two-dimensional semantics in which a formula can be necessarily true without being a priori knowable. As we shall see below, having an $\mathcal{R}_\Box$-rigidity condition while lacking a similar $\mathcal{R}_\mathcal{D}$-rigidity condition gives us just what we want. Moreover, we also abandon (Neutrality), which gave us freedom with respect to which world was taken to be the actual one for basic formulas. We come back to this topic below.

We notice, nonetheless, that the logic in question does not contain explicit epistemic elements such as epistemic scenarios, for the two indices of evaluation are members of the same set of possible worlds. This, however, is just a simplification, and it is an assumption that both Restall and Fritz share in their systems. One may refer, then, to the upper index position intuitively as the *epistemic dimension*, or *scenarios*, and to the lower one as the *metaphysical dimension*, or possible worlds, in the spirit of Chalmers' two-dimensional semantics. Also, the distinguished element specified in the models may be dropped in the absence of a distinguishedly operator, in which case either diagonal or local validity might be naturally adopted — diagonal and local validity coincide for $\mathcal{L}_{E2D}$. The truth clauses in the semantics are very similar to what we already have in Definition 2.3.3, but we shall state those explicitly since $\mathbf{S5E}_{2D}$ involves, after all, a different language.

Definition 2.5.3 (Truth) For a pair $\langle v, w\rangle \in W$,

$$
\begin{array}{lcl}
\mathcal{M}^v_w \vDash P^n_i(t_1, ..., t_n) & \iff & \langle V(t_1, \langle v, w\rangle), ..., V(t_n, \langle v, w\rangle)\rangle \in V(P^n_i, \langle v, w\rangle); \\
\mathcal{M}^v_w \vDash t = t' & \iff & V(t, \langle v, w\rangle) = V(t', \langle v, w\rangle); \\
\mathcal{M}^v_w \vDash \neg\varphi & \iff & \mathcal{M}^v_w \nvDash \varphi \\
\mathcal{M}^v_w \vDash \varphi \wedge \psi & \iff & \mathcal{M}^v_w \vDash \varphi \text{ and } \mathcal{M}^v_w \vDash \psi; \\
\mathcal{M}^v_w \vDash \Box\varphi & \iff & \text{for every } z \in Z, \text{ if } \langle v, w\rangle \mathcal{R}_\Box \langle v, z\rangle, \text{ then } \mathcal{M}^v_z \vDash \varphi; \\
\mathcal{M}^v_w \vDash \Diamond\varphi & \iff & \text{for some } z \in Z, \langle v, w\rangle \mathcal{R}_\Box \langle v, z\rangle \text{ and } \mathcal{M}^v_z \vDash \varphi; \\
\mathcal{M}^v_w \vDash \mathcal{A}\varphi & \iff & \mathcal{M}^v_v \vDash \varphi; \\
\mathcal{M}^v_w \vDash \mathcal{D}\varphi & \iff & \text{for every } z \in Z, \text{ if } \langle v, w\rangle \mathcal{R}_\mathcal{D} \langle z, z\rangle, \text{ then } \mathcal{M}^z_z \vDash \varphi; \\
\mathcal{M}^v_w \vDash \mathcal{C}\varphi & \iff & \text{for some } z \in Z, \langle v, w\rangle \mathcal{R}_\mathcal{D} \langle z, z\rangle \text{ and } \mathcal{M}^z_z \vDash \varphi; \\
\mathcal{M}^v_w \vDash \forall x\varphi & \iff & \text{for each constant symbol } c \text{ of } \mathcal{L}(\mathcal{M}), \mathcal{M}^v_w \vDash \varphi[c/x]; \\
\mathcal{M}^v_w \vDash \exists x\varphi & \iff & \text{for some constant symbol } c \text{ of } \mathcal{L}(\mathcal{M}), \mathcal{M}^v_w \vDash \varphi[c/x];
\end{array}
$$

The notions of validity and consequence are defined as before.[40] Now, the tableau rules for $\mathbf{S5E}_{2D}$ are basically the ones we already have for the relevant portion of the language of $\mathbf{S5}_{2D}$. Notice that the second identity rule, the one involving substitution, can be assumed

[40] One can find the same semantics defined in chapter 3.

without any change in $\mathbf{S5E}_{2D}$. In fact, we could have presented a stronger rule for identity substitution in $\mathbf{S5}_{2D}$, involving change in both upper and lower indices, given the rigidity conditions imposed on the models, that is, if $[c = d]^i_n$ and $[\varphi(c)]^j_m$ already occur on the branch, then $[\varphi(d)]^j_m$ can be added to the same branch. This was not needed, however, given our (Neutrality) constraint and the upper exchange rule, in which case this variation of the identity substitution rule can be shown to be a derived rule:

1. $[c = d]^i_n$
2. $[\varphi(c)]^j_m$
3. $[\varphi(c)]^i_m$
4. $[\varphi(d)]^i_m$
5. $[\varphi(d)]^j_m$

3 results from 2 by upper exchange, 4 follows from 1 and 3 by identity substitution, while 5 follows from 4 by upper exchange. Thus, our identity rules were fit to assume rigidity exclusively for $\mathcal{R}_\Box$-accessible worlds, whereupon they do not need to be modified for $\mathbf{S5E}_{2D}$. The only rule that needs, in effect, to be abandoned is the upper exchange rule, since we do not have (Neutrality) constraining the models for $\mathbf{S5E}_{2D}$. In the following we define the new rules for $\mathcal{D}$ and $\mathcal{C}$, where the former is classified as a necessity rule, and the latter as a possibility rule:

Definition 2.5.4 (Possibility Rules for $\mathcal{C}$)

If the index z is new to the branch,

$$\begin{array}{cc} [\mathcal{C}\varphi]^i_n & [\neg\mathcal{D}\varphi]^i_n \\ \downarrow & \downarrow \\ [\varphi]^z_z & [\neg\varphi]^z_z \end{array}$$

Definition 2.5.5 (Necessity Rule for $\mathcal{D}$)

For every index z occurring on the branch,

$$\begin{array}{cc} [\mathcal{D}\varphi]^i_n & [\neg\mathcal{C}\varphi]^i_n \\ \downarrow & \downarrow \\ [\varphi]^z_z & [\neg\varphi]^z_z \end{array}$$

The purpose of dropping (Neutrality) and the $\mathcal{R}_{\mathcal{D}}$-rigidity condition is that, in $\mathbf{S5E}_{2D}$, we are now able to find countermodels for $\Box(x = y) \supset \mathcal{D}(x = y)$. This is desirable in order to capture the view usually held by two-dimensionalists and exemplified by the Hesperus/Phosphorus case: although Hesperus is identical to Phosphorus, and necessarily so, this identity is not a priori knowable. We can take the first coordinate of a pair of worlds as determining the reference of a name just like a Fregean sense does. Because of this, it is desirable for names not to designate rigidly under the first dimension. The name 'Hesperus', for example, can be said to pick out Venus relative to a scenario where this is the brightest object in the evening sky. By contrast, if we consider a different scenario where the brightest object in the evening sky is Neptune, then 'Hesperus' picks out Neptune relative to this scenario.[41] Although 'Hesperus is Phosphorus' is necessarily true, it is not a priori since 'Hesperus' and 'Phosphorus' can pick out distinct entities relative to different scenarios — or first coordinates of members of W. This can be illustrated, formally, by the following tableau proof attempt:

$\nvdash \Box(a = b) \supset \mathcal{D}(a = b)$

[41] A more detailed exposition can be found in Chalmers (2004).

$$1.\ [\neg(\Box(a=b) \supset \mathcal{D}(a=b))]^0_0$$

$$2.\ [\Box(a=b)]^0_0$$

$$3.\ [\neg\mathcal{D}(a=b)]^0_0$$

$$4.\ [\neg a=b]^1_1$$

$$5.\ [a=b]^0_0$$

$$6.\ [a=b]^0_1$$

The tableau remains open, and a countermodel can be read-off from the open branch where $Z = \{0,1\}$, and thus $W = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$, $V(a,\langle 0,0\rangle) = V(b,\langle 0,0\rangle)$, and $V(a,\langle 0,1\rangle) = V(b,\langle 0,1\rangle)$. (Notice that this respects the $\mathcal{R}_\Box$-rigidity condition.) The following 2D-matrix illustrates the countermodel:

$$\begin{pmatrix} & 0 & 1 \\ 0 & a=b & a=b \\ 1 & \times & \times \end{pmatrix}$$

Since $\mathcal{M}^0_1 \vDash a=b$ and $\mathcal{M}^0_0 \vDash a=b$, $\mathcal{M}^0_0 \vDash \Box(a=b)$. However, since $\mathcal{M}^1_1 \nvDash a=b$, we have $\mathcal{M}^0_0 \nvDash \mathcal{D}(a=b)$, in which case $\nvDash \Box(a=b) \supset \mathcal{D}(a=b)$. Note, however, that if we assume (Neutrality) and upper index rigidity, we validate $\Box(a=b) \supset \mathcal{D}(a=b)$, for the upper exchange rule becomes valid as well, in which case the tableau above can be closed by an application of upper exchange to item 6, whereby we have 7. $[a=b]^1_1$, contradicting 4.

2.5.2 Semantic Neutrality and the Conceivability/Possibility Link

Now, suppose for a moment that we were to assume (Neutrality) in the semantics. This move validates the conceivability/possibility link, as illustrated below by a proof in the propositional portion of the language:

$\vdash \mathcal{C}p \equiv \Diamond p$

$$1.\ [\neg(\mathcal{C}p \equiv \Diamond p)]^0_0$$

2. $[\neg(\mathcal{C}p \supset \Diamond p)]^0_0$	3. $[\neg(\Diamond p \supset \mathcal{C}p)]^0_0$
4. $[\mathcal{C}p]^0_0$	10. $[\Diamond p]^0_0$
5. $[\neg\Diamond p]^0_0$	11. $[\neg\mathcal{C}p]^0_0$
6. $[p]^1_1$	12. $[p]^0_1$
7. $[\neg p]^0_0$	13. $[\neg p]^0_0$
8. $[\neg p]^0_1$	14. $[\neg p]^1_1$
9. $[p]^0_1$	15. $[p]^1_1$
$\times$	$\times$

The use of upper exchange in the above (items 9 and 15) is licensed by (Neutrality). It also indicates something interesting about what is being proved in the logic, namely, that the truth of p is independent of the upper index. As we know, this feature is distinctive of Davies and Humberstone's semantics, and it was adopted in the system $\mathbf{S5}_{2D}$ accordingly.[42] By and large, the link between $\mathcal{C}$ and $\Diamond$ is due to this, which fits nicely with the claim in Chalmers (2006, §3.5) that where φ is a *semantically neutral formula*, φ is conceivable just in case it is also possible. According to Chalmers, "a semantically neutral expression is one whose extension in counterfactual worlds does not depend on how the actual world turns out." (p. 192) Some examples of semantically neutral expressions provided by Chalmers include 'and', 'consciousness', 'causal', 'philosopher', etc.. One can think of expressions of this kind as ones which are not Twin-Earthable, in contrast to most proper names, natural

[42]Although, of course, it might be dropped, generating thereby a different set of validities.

kind terms, and indexicals, for instance. A more formal account of semantic neutrality is also given by Fritz (2014, p. 410), where a formula is semantically neutral in a model just in case "its truth is independent of the epistemic index", which translates to us as truth independent of the upper index. Thus, for semantically neutral formulas the upper exchange rule may be used freely, whereby a tight link between $\mathcal{C}$ and $\Diamond$ formulas is attained (analogously for $\mathcal{D}$ and $\Box$ formulas).[43]

A system for semantically neutral terms can be defined along the following lines. Let $\mathbf{S5EN}_{2D}$ be the system resulting from $\mathbf{S5E}_{2D}$ by constraining the models with (Neutrality) and adding the upper exchange rule to the stock of rules already in $\mathbf{S5E}_{2D}$. All the constants in the language will be semantically neutral. Additionally, we want these to involve a certain notion of *epistemic rigidity*, that is, rigidity with respect to the upper index, or first dimension. Although semantically neutral expressions are true or false independently of the upper index, we need to make sure the constants still denote the same entities whenever there is variation with respect to the upper index. For the system $\mathbf{S5}_{2D}$ this was accomplished under the guise of rigidity for $\mathcal{R}_{\mathcal{F}}$-accessible worlds. In $\mathbf{S5E}_{2D}$, however, we lack an accessibility relation that is exclusive for the first coordinates of the members of W. And even though we could still define rigidity for semantically neutral constants with respect to every first coordinate of every pair by stipulation, we can also make use of a more general notion of *super-rigidity*, which is also employed by Chalmers (2012, p. 370), besides being closely related to semantic neutrality.[44] The notion of super-rigidity can be defined formally by stipulating that for any constant symbol c_i in the language of $\mathbf{S5EN}_{2D}$,

$$\text{(Super-rigidity) For every } u, v, w, z \in Z,\ V(c_i, \langle u, v\rangle) = V(c_i, \langle w, z\rangle).$$

[43]It is a simple matter to check that for any $\mathcal{A}$-free formula φ, we have both $\Diamond\varphi \equiv \mathcal{C}\varphi$ and $\Box\varphi \equiv \mathcal{D}\varphi$, the latter corresponding to axiom $\mathcal{F}6$ in Davies and Humberstone's $\mathbf{S5}\mathcal{AF}$. See appendix C.

[44]Chalmers thinks all super-rigid expressions are semantically rigid. Although there are some wrinkles in the converse claim, if a semantically neutral expression is not equivalent to a super-rigid expression, it is at least equivalent to a compound of super-rigid expressions. For more details, see Chalmers (2012, p. 370).

Super-rigidity, therefore, makes semantically neutral constants rigid with respect to both upper and lower indices of evaluation.[45] It is also a simple generalization of both $\mathcal{R}_{\Box}$ and $\mathcal{R}_{\mathcal{F}}$ rigidities. Such constants will denote the same entities at any point in a two-dimensional model, whereupon they will not be susceptible of Twin-Earthability. $\mathbf{S5EN}_{2D}$ is, consequently, a logic for semantically neutral terms. It is also a subsystem of $\mathbf{S5}_{2D}$, since $\mathcal{D}$ can be defined as $\mathcal{FA}$, whence $\mathbf{S5EN}_{2D}$ is sound and complete as well. Thus, in effect, we already had in $\mathbf{S5}_{2D}$ a comprehensive system for semantically neutrality, except for lacking primitive apriority operators. The system herein presented, therefore, delivers what one would expect when it comes to a logic for conceivability/possibility based on a semantics such as the one originally proposed by Chalmers, as long as one imposes (Neutrality) on the models, substantiated by appropriate rigidity conditions and the upper exchange rule. Finally, the same modularity can be achieved regarding validity in the tableaux for both $\mathbf{S5E}_{2D}$ and $\mathbf{S5EN}_{2D}$.

2.6 Coda: Inexpressibility, Necessity, and Multidimensionality

Before closing, let us come back to the topic of expressive incompleteness in modal languages, which was briefly addressed by examples in the beginning of this chapter, as well as to the relations between actuality, necessity, and the a priori. Many of the issues in this section should be taken as exploratory, with several open questions for future research, both formal and philosophical. Nonetheless, we must acknowledge some of the pressing philosophical questions that arise concerning different actuality operators, to wit, rigid and non-rigid, in the context of two-dimensional modal logics, and how they relate to the two notions of

[45] In Chalmers' terms, this kind of expression has a constant two-dimensional intension. See Chalmers (2012, p. 370).

necessity ($\Box$ and $\mathcal{FA}/\mathcal{D}$) at work in the systems defined above. In particular, we focus on the system $\mathbf{S5}_{2D}$, although similar issues apply, *mutatis mutandis*, to $\mathbf{S5E}_{2D}$.

The introduction of a significant number of operators in the language of $\mathbf{S5}_{2D}$ was motivated by the claim that basic modal languages, whose modal operators include only $\Box$ and $\Diamond$, are expressively incomplete, that is, they are not sufficiently powerful in order to fully represent modal discourse. In particular, this was the reason for introducing $\mathcal{A}$, $\odot$, and $\otimes$. We have started with Crossley and Humberstone's sentence (1), the inexpressibility of which has motivated the introduction of $\mathcal{A}$ into a first-order modal language, and then we offered a two-dimensional generalization of (1), namely, (3), which was in turn claimed to be inexpressible in a first-order two-dimensional language, thereby motivating the addition of $\odot$ into our stock of operators. Similar considerations have underpinned the adoption of $\otimes$ as well.

At this point, however, one cannot but question whether there would not be similar inexpressibility issues in even more powerful languages, such as the very same claimed to be sufficiently rich in order to formalize (3), for example. After all, do we have reason to believe our current suite of modal operators to be appropriate in order to represent the kind of modal discourse we are interested here?

As we shall now see, any adequate answer to this bears upon what is meant by the appropriateness of a formal language, or a logical system itself. While there is certainly a question regarding the expressive completeness of a language such as $\mathcal{L}_{2D}$, this is the kind of result deserving a different chapter. By contrast, one might also ask about whether the formal language or system at hand is appropriate in the sense of being ultimately philosophically adequate. This notion of adequacy, in turn, can be clarified with a couple of examples. The modal logic $\mathbf{T}$ is characterized by the axiom $\Box\varphi \supset \varphi$, whence it seems to be inadequate as a logic for obligation: it is simply not the case that whenever p ought to be the case, then p. On the other hand, it is usually assumed that knowledge is factive, that is, if one knows that p, then p, whence **T** is sometimes claimed to be adequate as a logic for knowledge.[46] Another, familiar cases, whose details we skip, are Salmon's (1989) argument based on the essentiality of origin to the effect that modal logics containing the **S4** axiom $\Box\varphi \supset \Box\Box\varphi$ are inadequate in order to represent our intuitive notion of (metaphysical) necessity, as well as Dummett's argument that metaphysical modality does not obey the **B** axiom $\varphi \supset \Box\Diamond\varphi$. The former denies that metaphysical necessity is transitive, while the latter denies it is symmetric. Hence, both reject **S5** as an adequate logic of necessity.[47] Likewise, we saw in §2.1 that $\mathcal{F}$ was introduced into modal languages with an actuality operator because of the intuitive invalidity of $\mathcal{A}\varphi \supset \Box\mathcal{A}$, for, as Humberstone (2004, p. 21) points out, "to many there seems to be a sense in which what is actually the case need not be necessarily actually the case," whence the alternative notion of necessity delivered by $\mathcal{FA}$. While a basic modal language for **S5** is expressively deficient, one containing $\mathcal{A}$ might be claimed to be, in a sense, philosophically impaired. The question we face, therefore, is about the philosophical adequacy of the systems herein defined.

Regarding this matter we now observe that the introduction of a distinguishedly operator in a language for two-dimensional modal logic comes not without a cost. Despite being needed on pain of expressive weakness, the addition of $\odot$ results in the formula $\odot\mathcal{A}\varphi \supset \mathcal{FA}\odot\mathcal{A}\varphi$ being valid in $\mathbf{S5}_{2D}$.[48] This formula is, in effect, a two-dimensional analogue of the already much discussed formula $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$. Consequently, if something is distinguishedly actually true, it is deeply necessary that it is so, in which case the distinguishedly operator seems to subvert the very motivation for moving to a two-dimensional modal logic! Furthermore, notice that the role of $\odot\mathcal{A}$ in this case is analogous to the one of a rigid $\mathcal{A}$ in one-dimensional modal logic. Notwithstanding its non-rigidity in a two-dimensional framework, $\mathcal{A}$ plays the role of a distinguished actuality operator in $\mathbf{S5}\mathcal{A}$. Therefore, the intuition that it is a contingent

[46] Williamson (2013b), for example, argues that **T** is the correct system for a logic of knowledge.

[47] Ultimately, these questions amount to the kind of accessibility relation one wants for a basic modal system.

[48] This holds regardless of the chosen notion of validity — local, general, or diagonal.

matter which world is the actual one should be understood in terms of rigid actuality: despite being superficially necessary, it is deeply contingent which world is the distinguished actual world, whence the thought that $\odot\mathcal{A}\varphi \supset \mathcal{F}\mathcal{A} \odot \mathcal{A}\varphi$ is, too, intuitively invalid. In the midst of repairing the expressive deficiency of two-dimensional languages, it looks as though we throw the baby out with the bathwater.

Not surprisingly, this problem may be solved by the addition of a third index of evaluation and a new necessity operator, herein symbolized by $\mathcal{N}$:

$$\mathcal{M}^{\substack{u\\v}}_{w} \vDash \mathcal{N}\varphi \text{ if and only if for every } z \in Z,\ \mathcal{M}^{\substack{z\\v}}_{w} \vDash \varphi.$$

We can also define the dual of $\mathcal{N}$, symbolized by $\mathcal{P}$, as $\neg\mathcal{N}\neg$. Models for propositional three-dimensional modal logic can be defined by letting W be $Z \times Z \times Z$, for some set Z, where $w*$ is a distinguished element of Z, $\mathcal{R}_{\Box}$, $\mathcal{R}_{\mathcal{F}}$, and $\mathcal{R}_{\mathcal{N}}$ are accessibility relations corresponding to $\Box$, $\mathcal{F}$, and $\mathcal{N}$ formulas, and V is a function from propositional letters to members of W, which are now ordered triples $\langle u, v, w\rangle$. To be more explicit, the semantic clauses can be defined as follows:

$$\mathcal{M}^{u}_{\substack{v\\w}} \models p \iff \langle u, v, w\rangle \in V(p);$$

$$\mathcal{M}^{u}_{\substack{v\\w}} \models \neg\varphi \iff \mathcal{M}^{u}_{\substack{v\\w}} \not\models \varphi$$

$$\mathcal{M}^{u}_{\substack{v\\w}} \models \varphi \wedge \psi \iff \mathcal{M}^{u}_{\substack{v\\w}} \models \varphi \text{ and } \mathcal{M}^{u}_{\substack{v\\w}} \models \psi;$$

$$\mathcal{M}^{u}_{\substack{v\\w}} \models \Box\varphi \iff \text{for every } z \in Z, \text{ if } \langle u, v, w\rangle \mathcal{R}_{\Box} \langle u, v, z\rangle, \text{ then } \mathcal{M}^{u}_{\substack{v\\z}} \models \varphi;$$

$$\mathcal{M}^{u}_{\substack{v\\w}} \models \Diamond\varphi \iff \text{for some } z \in Z, \langle u, v, w\rangle \mathcal{R}_{\Box} \langle u, v, z\rangle \text{ and } \mathcal{M}^{u}_{\substack{v\\z}} \models \varphi;$$

$$\mathcal{M}^{u}_{\substack{v\\w}} \models \mathcal{A}\varphi \iff \mathcal{M}^{u}_{\substack{v\\v}} \models \varphi;$$

$$\mathcal{M}^{u}_{\substack{v\\w}} \models \odot\varphi \iff \mathcal{M}^{u}_{\substack{u\\u}} \models \varphi;$$

$$\mathcal{M}^{u}_{\substack{v\\w}} \models \otimes\varphi \iff \mathcal{M}^{u}_{\substack{w\\w}} \models \varphi;$$

$$\mathcal{M}^{u}_{\substack{v\\w}} \models \mathcal{F}\varphi \iff \text{for every } z \in Z, \text{ if } \langle u, v, w\rangle \mathcal{R}_{\mathcal{F}} \langle u, z, w\rangle, \text{ then } \mathcal{M}^{u}_{\substack{z\\w}} \models \varphi;$$

$$\mathcal{M}^{u}_{\substack{v\\w}} \models \mathcal{S}\varphi \iff \text{for some } z \in Z, \langle u, v, w\rangle \mathcal{R}_{\mathcal{F}} \langle u, z, w\rangle \text{ and } \mathcal{M}^{u}_{\substack{z\\w}} \models \varphi;$$

$$\mathcal{M}^{u}_{\substack{v\\w}} \models \mathcal{N}\varphi \iff \text{for every } z \in Z, \text{ if } \langle u, v, w\rangle \mathcal{R}_{\mathcal{N}} \langle z, v, w\rangle, \text{ then } \mathcal{M}^{z}_{\substack{v\\w}} \models \varphi;$$

$$\mathcal{M}^{u}_{\substack{v\\w}} \models \mathcal{P}\varphi \iff \text{for some } z \in Z, \langle u, v, w\rangle \mathcal{R}_{\mathcal{N}} \langle z, v, w\rangle \text{ and } \mathcal{M}^{z}_{\substack{v\\w}} \models \varphi.$$

The notions of truth, consequence, and validity, can be appropriately defined by generalizing the corresponding notions in the two-dimensional case. For example, we can define local validity as truth at $\langle w*, w*, w*\rangle$ in every model, general validity as truth at every pair $\langle u, v, w\rangle$ in every model, or even, say, *three-dimensional validity*, as truth at every triple $\langle w, w, w\rangle$ in every model. Notice that given the semantics just outlined, although we still have the validity of $\odot\mathcal{A}\varphi \supset \mathcal{F}\mathcal{A} \odot \mathcal{A}\varphi$, the formula $\odot\mathcal{A}\varphi \supset \mathcal{N} \odot \odot\mathcal{A}\varphi$ is not valid, as desired.[49]

Now there are several questions about the philosophical prospects of a three-dimensional modal logic based on the semantics above, which directly affect its status as a plausible solution to the problem under consideration.[50] Does $\mathcal{N}$ in itself define any interesting notion of necessity?[51] And what about the compound $\mathcal{N}\odot$ delivering truth at every triple $\langle z, z, z\rangle$? Does that in turn correspond to any intuitive or useful notion of necessity? Since distinguishedly actual truths, despite being deeply necessary, are not necessary in the sense of $\mathcal{N}\odot$, this invites the question about whether it is the latter, then, instead of $\mathcal{FA}$, that best represents the intuitive notion of necessity like the one that have motivated the introduction of $\mathcal{F}$.

In spite of all this, the attentive reader will notice that $\odot$ is not rigid in this three-dimensional language anymore: it is a copy down operator, just like $\mathcal{A}$. This 'de-rigidification' serves to illustrate that in a three-dimensional language lacking a rigid actuality operator there will be no formal representation of a sentence equivalent to the following truth conditions: where Σ is an existential quantifier over members of W,

(9) $\Sigma\langle w, w, w\rangle\forall x(Rx\text{-}at\text{-}\langle w*, w*, w*\rangle \supset Sx\text{-}at\text{-}\langle w, w, w\rangle)$.

The best we could do here would be to formalize (9) as (10):

[49] In both cases, "validity" means either local, general, or three-dimensional.

[50] Tableaux for three-dimensional modal logic can be easily defined by extending the two-dimensional case with another numeric index. Note that we can also define a three-dimensional version of $\otimes$, whereby the third index of evaluation is copied up to the first one. If we symbolize this as $\oplus$, its semantics can be defined as $\mathcal{M}^{u}_{\substack{v\\w}} \vDash \oplus\varphi$ if and only if $\mathcal{M}^{w}_{\substack{w\\w}} \vDash \varphi$. More generally, for any dimension n, we can define new actuality and *Ref* operators accordingly. The same is true, of course, for the other two-dimensional operators mentioned here.

[51] The answer should be negative in case a condition similar to (Neutrality) is assumed:

(3D-Neutrality) If φ is a basic formula, then for every $t, u, v, w, z \in Z$, $\mathcal{M}^{u}_{\substack{v\\w}} \vDash \varphi$ if and only if $\mathcal{M}^{t}_{\substack{z\\w}} \vDash \varphi$.

Once (3D-Neutrality) is assumed, the models validate $\varphi \equiv \mathcal{N}\varphi$ for every basic formula φ, whence $\mathcal{N}$ does not represent any notion of necessity in itself.

(10) $\mathcal{P} \odot \forall x(\odot Rx \supset Sx)$.

However, $\odot$ will not take us to the triple $\langle w*, w*, w* \rangle$ as desired. This problem is already familiar, whence we can solve it by likewise familiar methods: a new actuality operator, $\mathcal{B}$, can be added to the language, pointing invariably to $w*$ in Z:

$$\mathcal{M}^{u}_{\substack{v\\w}} \models \mathcal{B}\varphi \text{ if and only if } \mathcal{M}^{w*}_{\substack{v\\w}} \models \varphi,$$

in which case (9) can now be formalized as

(11) $\mathcal{P} \odot \forall x(\mathcal{B} \odot Rx \supset Sx)$.

Is there an English reading available for (11)? Unless there are natural readings for three-dimensional operators, we do not have any expectations of finding one. Besides, there is an element of artificiality in three-dimensional operators like the ones just defined: apart from the fact that they do not seem to have any natural-language readings, there does not seem to be philosophical or intuitive interpretations for them as well unless one takes $\mathcal{N}\odot$, for instance, as a deep necessity operator, which leaves $\mathcal{FA}$ unexplained. Maybe some natural interpretation will appear that makes multidimensional necessities more intuitive. At the moment, however, they strike us as rather puzzling. This might cause one to raise an eyebrow and suspect that, even if we could make intuitive sense of such a thing like a three-dimensional modal logic, it seems to be much more the product of logical curiosity than anything else.

Nevertheless, a three-dimensional system helps to illustrate just how the problems motivating the introduction of a new dimension to modal logics, as well as new actuality operators, are more general than we once thought them to be, besides being interestingly related: on pain of expressive deficiency, we need rigid actuality operators in modal languages; yet, because they are rigid, they will inhibit whatever necessity operators we have, shielding, as it were, the formulas within their scope, thereby validating that rigidly actual truths are necessarily so. If, on the one hand, we accept the latter, the philosophical motivation to introduce a fixedly operator seems to fall flat. On the other hand, by adding more dimensions and operators there is no longer a principled reason, so it seems, to endorse that deep necessity is represented by $\mathcal{FA}$; as we have just shown, we might as well take $\mathcal{N}\odot$ to be our deep necessity operator, since rigidly actual truths turn out to be $\mathcal{FA}$-necessary despite the intuition that they are (deeply) contingent. Once, however, we have the new (rigid) actuality operator $\mathcal{B}$ in our three-dimensional language, it is an easy matter to see that $\mathcal{B} \odot \varphi \supset \mathcal{N} \odot \mathcal{B} \odot \varphi$ also becomes valid, which we might take in turn as reason to define a four-dimensional modal logic, and so on. Now, of course, we are back to square one, and the problem appears to recur in any dimension whatsoever.

On the two-dimensionalist's behalf, it might be claimed that some of these issues are purely formal, especially the ones involving three-dimensional logics or more. However, this does not change the fact that an explanation for the validity of $\odot\mathcal{A}\varphi \supset \mathcal{FA} \odot \mathcal{A}\varphi$ is needed if we are supposed to identify $\mathcal{FA}$ as a deep necessity operator. After all, *there seems to be a sense in which what is distinguishedly actually the case need not be deeply necessarily distinguishedly actually the case.* In any event, it seems clear enough that $\mathcal{FA}$ defines a notion of necessity, but far from clear it is a *deep* one; whatever it is, it just might not be what Davies and Humberstone, let alone Evans, had in mind.[52]

2.7 Conclusion

We have shown tableau calculi for first-order two-dimensional modal logics to be sound and complete. The methods in use here present themselves as simple generalizations of familiar tools for first-order modal logics, and there seem to be no special difficulties in generalizing

[52]In chapter 3 we point out similar issues involving the apriority operator in a semantics based on epistemic two-dimensional semantics, as developed by Chalmers.

them for more dimensions as well. Furthermore, we have motivated a new operator, called "distinguishedly", that brings to light a formal distinction between rigid and non-rigid actuality operators in two-dimensional languages. Not only that, but we have also observed that it might give rise to unpleasant philosophical consequences for two-dimensional modal logic itself. In a nutshell: despite being well-motivated, the philosophical interpretation of two-dimensional operators is far from settled, especially the necessity attributed to $\mathcal{FA}$ that has long been part of philosophical lore. For the moment we cannot avoid but closing this chapter with several open questions:

- Call a Kripke model 'deterministic' if it validates formulas of the kind $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$. Can Kripke models for languages that are not expressively deficient in the sense above (i.e. containing actuality operators) be defined in non-deterministic ways?

- What should we make of the claim that deep necessity and/or a priori knowledge corresponds to truth along the diagonal once our models validate formulas such as $\odot\mathcal{A}\varphi \supset \mathcal{FA} \odot \mathcal{A}\varphi$? Are the issues raised here a problem for different brands of two-dimensionalism such as the versions developed by Kaplan, Stalnaker, Jackson, and Chalmers?

- There are interesting questions regarding the epistemic profile of rigid and non-rigid actual truths. While $\mathcal{A}\varphi \equiv \varphi$ is $\mathcal{FA}$-necessary and usually taken to be a priori knowable (Evans (1979, p. 200), Davies (2004, p. 100)), $\odot\mathcal{A}\varphi \equiv \varphi$ is not $\mathcal{FA}$-necessary. In this sense, the thought that grass is actually green seems radically different from the corresponding thought that grass is distinguishedly actually green, although the formulas just mentioned only differ when occurring embedded in $\mathcal{F}$ contexts.

- The formula $\odot\mathcal{A}\varphi \supset \varphi$ is locally, but it is neither diagonally nor generally valid. If a local account of validity is preferred, though, this means that some valid formulas do not hold along the diagonal. This brings complications to the traditional view that diagonal necessity corresponds to apriority. Is this a reason for rejecting local validity for two-dimensional modal logics, or diagonal necessity as corresponding to apriority?

- The formula $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$ holds in normal systems of modal logic from **K** up to **S5** endowed with an actuality operator. Thus, it also holds in systems where the accessibility relation is restricted at most by seriality, such as the system **D**, standardly assumed for deontic modality. Moreover, there are also reasons for adding actuality operators to modal languages for deontic logic (Humberstone (1982)). If so, is $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$ appropriate once we interpret $\Box$ in a deontic fashion?

- Humberstone (1982) distinguishes between actuality and subjunctive operators on the grounds that the former inhibits while the latter activates modal operators. This distinction becomes obscured once we consider multidimensional languages. $\mathcal{A}$ inhibits only the basic modal operators $\Box$ and $\Diamond$, but it is binded by $\mathcal{F}$ and $\mathcal{D}$. By contrast, $\odot\mathcal{A}$ inhibits all of these. Are there similar issues with subjunctive operators? Moreover, how does the issues discussed here concerning different actuality operators affect modal systems with subjunctive markers, such as the ones defined by Wehmeier (2004, 2005), in which an actuality operator is used as a tool for mood distinctions?

- Concerning proof systems, indexed tableaux seem to generalize nicely for multidimensional modal logics including several kinds of operators. There appears to be no impediment, at least in principle, to generalize indexed tableaux for n-dimensional modal logics. What about different proof systems such as hypersequents or natural deduction? For instance, could Fitch-style natural deduction systems be defined for multidimensional modal logics?

Appendix

2.A Soundness and Completeness for $\mathbf{S5}_{2D}$

The soundness theorem for $\mathbf{S5}_{2D}$ is an extension of corresponding proofs for modal logic presented, for instance, in Fitting and Mendelsohn (1998) and Priest (2008).

Definition 2.A.1 (Satisfiability) Let S be a set of doubly-indexed formulas. We say S is *satisfiable* in $\mathcal{M} = \langle W, w*, \mathcal{R}_\Box, \mathcal{R}_\mathcal{F}, \mathscr{D}, V \rangle$ just in case there is a function, f, assigning to each (single) index n occurring in S a possible world $f(n) \in Z$, where $f(0) = w*$, $W = Z \times Z$, such that,

- If $[\varphi]^i_n \in S$, then φ is true at $f(n)$ relative to $f(i)$, i.e. $\mathcal{M}^{f(i)}_{f(n)} \vDash \varphi$.
- If the pairs of indices $\langle i, n \rangle$ and $\langle i, m \rangle$ are in S, then $\langle f(i), f(n) \rangle \mathcal{R}_\Box \langle f(i), f(m) \rangle$.
- If the pairs of indices $\langle i, n \rangle$ and $\langle j, n \rangle$ are in S, then $\langle f(i), f(n) \rangle \mathcal{R}_\mathcal{F} \langle f(j), f(n) \rangle$.

Definition 2.A.2 A tableau branch $\mathfrak{b}$ is *satisfiable* just in case the set of doubly-indexed formulas on it is satisfiable in some model, and a tableau is *satisfiable* just in case some branch of it is satisfiable.

Lemma 2.A.1 *A closed tableau is not satisfiable.*

Proof. Suppose that $\mathcal{T}$ is a closed tableau that is also satisfiable. By Definition 2.A.2, some branch $\mathfrak{b}$ of $\mathcal{T}$ is satisfiable, whence there is a model, $\mathcal{M} = \langle W, w*, \mathcal{R}_\Box, \mathcal{R}_\mathcal{F}, \mathscr{D}, V\rangle$, satisfying the set S of doubly-indexed formulas on $\mathfrak{b}$ by way of the function f, by Definition 2.A.1. But since, by assumption, $\mathcal{T}$ is also closed, for some formula φ and pair of indices $\langle i, n\rangle$, $[\varphi]^i_n \in S$ and $[\neg\varphi]^i_n \in S$. Hence, by Definition 2.A.1, $\mathcal{M}^{f(i)}_{f(n)} \vDash \varphi$ and $\mathcal{M}^{f(i)}_{f(n)} \nvDash \varphi$, which is impossible. □

Lemma 2.A.2 *If one of the rules in* $\boldsymbol{S5}_{2D}$ *is applied to a satisfiable 2D-tableau, it results in another satisfiable 2D-tableau.*

Proof. Suppose $\mathfrak{b}$ is a branch of a satisfiable 2D-tableau $\mathcal{T}$ and that we apply one of the rules in $\mathbf{S5}_{2D}$ to a doubly-indexed formula occurring on $\mathfrak{b}$. It is easily seen that the result is another satisfiable tableau. Since, by assumption, $\mathcal{T}$ is satisfiable, by Definition 2.A.2 at least one of its branches, say, $\mathfrak{b}*$, is satisfiable. Now, if $\mathfrak{b}* \neq \mathfrak{b}$, then $\mathfrak{b}*$ remains satisfiable after we apply a rule on $\mathfrak{b}$, whence we have a satisfiable 2D-tableau by Definition 2.A.2. On the other hand, if $\mathfrak{b}* = \mathfrak{b}$, we need to consider several cases. We omit the arguments for the Booleans, modal operators, quantifiers, and identity, since these are routine.[53] In order to carry on such arguments in the 2D framework one just needs to add a pair of indices where the first coordinate does not do any work.

Suppose $[\mathcal{A}\varphi]^i_n$ occurs on $\mathfrak{b}$, and we apply the rule for $\mathcal{A}$. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \mathcal{A}\varphi$, by Definition 2.A.1. Thus, $\mathcal{M}^{f(i)}_{f(i)} \vDash \varphi$, by the truth clause for $\mathcal{A}$. Therefore, applying the rules for $\mathcal{A}$ to a satisfiable 2D-tableau results in another satisfiable 2D-tableau. The argument for $\neg\mathcal{A}$ is analogous.

Suppose $[\odot\varphi]^i_n$ occurs on $\mathfrak{b}$, and we apply the rule for $\odot$. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \odot\varphi$, by Definition 2.A.1. Thus, $\mathcal{M}^{w*}_{f(n)} \vDash \varphi$, by the truth clause for $\odot$, in which case $\mathcal{M}^{f(0)}_{f(n)} \vDash \varphi$, since '0' is fixed to $w*$. Therefore, applying the rules for $\odot$ to a satisfiable 2D-tableau results in another satisfiable 2D-tableau. The argument for $\neg\odot$ is analogous.

[53]See, for instance, Fitting and Mendelsohn (1998, pp. 58–9, 122–123), and Priest (2008, pp. 31–2, 322–323).

Suppose $[\otimes\varphi]^i_n$ occurs on $\mathfrak{b}$, and we apply the rule for $\otimes$. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \otimes\varphi$, by Definition 2.A.1. Thus, $\mathcal{M}^{f(n)}_{f(n)} \vDash \varphi$, by the truth clause for $\otimes$. Therefore, applying the rules for $\otimes$ to a satisfiable 2D-tableau results in another satisfiable 2D-tableau. The argument for $\neg\otimes$ is analogous.

Suppose $[\mathcal{F}\varphi]^i_n$ occurs on $\mathfrak{b}$, and we apply the rule for $\mathcal{F}$. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \mathcal{F}\varphi$, by Definition 2.A.1. Also, for every pair of indices $\langle i, n\rangle$ and $\langle j, n\rangle$ occurring on $\mathfrak{b}$, $\langle f(i), f(n)\rangle \mathcal{R}_{\mathcal{F}} \langle f(j), f(n)\rangle$. Thus, $\mathcal{M}^{f(j)}_{f(n)} \vDash \varphi$, by the truth clause for $\mathcal{F}$. On the other hand, suppose $[\neg\mathcal{S}\varphi]^i_n$ occurs on $\mathfrak{b}$. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \neg\mathcal{S}\varphi$, by Definition 2.A.1, and so $\mathcal{M}^{f(i)}_{f(n)} \nvDash \mathcal{S}\varphi$. Hence, for every $\langle f(i), f(n)\rangle \mathcal{R}_{\mathcal{F}} \langle f(j), f(n)\rangle$, $\mathcal{M}^{f(j)}_{f(n)} \nvDash \varphi$, by the truth clause for $\mathcal{S}$, and so $\mathcal{M}^{f(j)}_{f(n)} \vDash \neg\varphi$. Therefore, applying the rules for $\mathcal{F}$ to a satisfiable 2D-tableau results in another satisfiable 2D-tableau.

Suppose $[\mathcal{S}\varphi]^i_n$ occurs on $\mathfrak{b}$, and we apply the rule for $\mathcal{S}$. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \mathcal{S}\varphi$, by Definition 2.A.1. Thus, for some $z \in Z$, $\langle f(i), f(n)\rangle \mathcal{R}_{\mathcal{F}} \langle z, f(n)\rangle$ and $\mathcal{M}^{z}_{f(n)} \vDash \varphi$, by the truth clause for $\mathcal{S}$. Let g be like f except that $g(j) = z$, where j is a new index occurring on $\mathfrak{b}$. Since g and f agree except for j, the tableau extended by g is satisfiable. Moreover, by definition of g, $\langle g(i), g(n)\rangle \mathcal{R}_{\mathcal{F}} \langle g(j), g(n)\rangle$ and $\mathcal{M}^{g(j)}_{g(n)} \vDash \varphi$. By choice of g, it follows that $\mathcal{M}^{f(j)}_{f(n)} \vDash \varphi$. The argument for $\neg\mathcal{F}$ is analogous. Therefore, applying the rules for $\mathcal{S}$ to a satisfiable 2D-tableau results in another satisfiable 2D-tableau.

Finally, let φ be a basic formula and suppose $[\varphi]^i_n$ occurs on $\mathfrak{b}$, and that we apply the upper exchange rule. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \varphi$, by Definition 2.A.1. Moreover, since φ is a basic formula, $\mathcal{M}^{f(j)}_{f(n)} \vDash \varphi$, for any first coordinate $f(j)$ of any pair of worlds, given that the models are constrained by (Neutrality). Therefore, applying the upper exchange rule to a satisfiable 2D-tableau results in another satisfiable 2D-tableau. $\square$

Theorem 2.A.1 (Soundness) *If φ has a 2D-tableau proof, then φ is valid.*

Proof. Suppose φ has a 2D-tableau proof, in which case there is a closed 2D-tableau, $\mathcal{T}$, beginning with $[\neg\varphi]^0_0$. For a contradiction, assume that φ is not valid. Thus, there is a 2D-centered model, $\mathcal{M} = \langle W, w*, \mathcal{R}_\Box, \mathcal{R}_\mathcal{F}, \mathscr{D}, V\rangle$, such that $\mathcal{M}^{w*}_{w*} \nvDash \varphi$. Let f be a function such that $f(0) = w*$. By Definition 2.A.1, $\{[\neg\varphi]^0_0\}$ is satisfiable. Moreover, since its only branch is satisfiable, $\mathcal{T}$ is also satisfiable, and so is any 2D-tableau extending it by way of the rules in $\mathbf{S5}_{2D}$, by Lemma 2.A.2. Therefore, $\mathcal{T}$ is both closed and satisfiable, contradicting Lemma 2.A.1, whence $\mathcal{M}^{w*}_{w*} \vDash \varphi$. □

Infinite Tableaux and Systematic Procedure. Some 2D-tableaux may run infinitely, and by König's Lemma these will have at least one infinite branch. In what follows we give an example of a failed proof attempt in $\mathbf{S5}_{2D}$ generating an infinite 2D-tableau:

1. $[\neg\mathcal{S}(p\supset\Box p)]^0_0$
2. $[\neg(p\supset\Box p)]^0_0$
3. $[p]^0_0$
4. $[\neg\Box p]^0_0$
5. $[\neg p]^0_1$
6. $[\neg(p\supset\Box p)]^1_0$
7. $[p]^1_0$
8. $[\neg\Box p]^1_0$
9. $[\neg p]^1_2$

$\vdots$

A 2D-matrix for the above looks like this:

$$\begin{pmatrix} & 0 & 1 & 2 & 3 & \dots \\ 0 & p & \neg p & \times & \times & \\ 1 & p & \times & \neg p & \times & \\ 2 & p & \times & \times & \neg p & \\ \vdots & \vdots & & & & \ddots \end{pmatrix}$$

Using the same procedure as before to devise a countermodel we have the following:

Since $\mathcal{M}_0^0 \vDash p$ and $\mathcal{M}_1^0 \nvDash p$, it follows that $\mathcal{M}_0^0 \nvDash \Box p$. Hence, $\mathcal{M}_0^0 \nvDash p \supset \Box p$.

Since $\mathcal{M}_0^1 \vDash p$ and $\mathcal{M}_2^1 \nvDash p$, it follows that $\mathcal{M}_1^1 \nvDash \Box p$. Hence, $\mathcal{M}_0^1 \nVdash p \supset \Box p$.

Since $\mathcal{M}_0^2 \vDash p$ and $\mathcal{M}_3^2 \nvDash p$, it follows that $\mathcal{M}_2^2 \nvDash \Box p$. Hence, $\mathcal{M}_0^2 \nvDash p \supset \Box p$.

$\dots$ Hence, $\mathcal{M}_0^0 \nvDash \mathcal{S}(p \supset \Box p)$.

Even though a finite countermodel can be easily constructed to invalidate $\mathcal{S}(p \supset \Box p)$,[54] we need a *systematic procedure* for constructing 2D-tableaux in order to prove that any tableau so constructed is such that, if it has an open branch, finite or not, there is a countermodel for it.

Now, several systematic procedures of this kind can be found in the literature, even for the modal cases. In particular, we direct the reader again to Fitting and Mendelsohn (1998, pp. 126–7,)[55] since our strategy will be based on it. Thus, let $[\varphi]_n^i$ be any doubly-indexed formula of which we want to know whether or not it is satisfiable. For stage $n = 1$, introduce $[\neg\varphi]_n^i$ to the 2D-tableau as line 1. Next, suppose n stages have been completed. If the 2D-tableau is already closed, then we have produced a proof of $[\varphi]_n^i$. On the other hand,

[54]We leave this case to the reader.
[55]Smullyan (1995, pp. 58–9), presents one for first-order logic.

if the 2D-tableau remains open, then we proceed to stage $n+1$ in which we take an open branch $\mathfrak{b}$ of the 2D-tableau such that $\mathfrak{b}$ is the leftmost highest point on it, and for each doubly-indexed formula $[\psi]^j_m$[56] such that it occurs on $\mathfrak{b}$ we extend the 2D-tableau as follows:

I If $[\psi]^j_m$ is a basic formula, say, $[\chi]^k_p$, add $[\chi]^l_p$ to the end of $\mathfrak{b}$, where l is an upper index added by either $\mathcal{S}$ or $\mathcal{F}$ (if it does not already contain it).

II If a constant, say, c, is on $\mathfrak{b}$, add $[(c=c)]^k_p$ to the end of $\mathfrak{b}$ for all pairs of indices $\langle k,p\rangle$ occurring on $\mathfrak{b}$.

III If the pair of indices $\langle k,p\rangle$ is on $\mathfrak{b}$, add $[(c=c)]^k_p$ to the end of $\mathfrak{b}$ for all constants c on $\mathfrak{b}$.

IV If $[\psi]^j_m$ is $[(c=d)]^k_q$, for any constants c and d, and $[\varphi(c)]^k_p$ occurs on $\mathfrak{b}$, add $[\varphi(d)]^k_p$ to the end of $\mathfrak{b}$ (if it does not already contain it).

V If $[\psi]^j_m$ is $[\neg\neg\chi]^k_p$, add $[\chi]^k_p$ to the end of $\mathfrak{b}$ (if it does not already contain it).

VI If $[\psi]^j_m$ is $[(\zeta\wedge\chi)]^k_p$, add both $[\zeta]^k_p$ and $[\chi]^k_p$ to the end of $\mathfrak{b}$ (if it does not already contain them — analogously for the other conjunctive cases).

VII If $[\psi]^j_m$ is $[(\zeta\vee\chi)]^k_p$, split the end of $\mathfrak{b}$ into $\mathfrak{b}_1$ and $\mathfrak{b}_2$, adding $[\zeta]^k_p$ and $[\chi]^k_p$ to the end of $\mathfrak{b}_1$ and $\mathfrak{b}_2$, respectively (if it does not already contain them — analogously for the other disjunctive cases).

VIII If $[\psi]^j_m$ is $[\Diamond\chi]^k_p$, take the first lower numeric index, q, which is new to $\mathfrak{b}$, and add $[\chi]^k_q$ to the end of $\mathfrak{b}$ (if it does not already contain it — analogously for $\neg\Box$).

IX If $[\psi]^j_m$ is $[\Box\chi]^k_p$, add every doubly-indexed formula $[\chi]^k_q$ to the end of $\mathfrak{b}$ for every index q occurring on the branch (if it does not already contain it — analogously for $\neg\Diamond$).

[56] Where the indices i and n may be identical to j and m, respectively.

X If $[\psi]^j_m$ is $[\odot\chi]^k_p$, add $[\chi]^0_p$ to the end of $\mathfrak{b}$ (if it does not already contain it — analogously for $\neg\odot$).

XI If $[\psi]^j_m$ is $[\otimes\chi]^k_p$, add $[\chi]^p_p$ to the end of $\mathfrak{b}$ (if it does not already contain it — analogously for $\neg\otimes$).

XII If $[\psi]^j_m$ is $[\mathcal{A}\chi]^k_p$, add $[\chi]^k_k$ to the end of $\mathfrak{b}$ (if it does not already contain it — analogously for $\neg\mathcal{A}$).

XIII If $[\psi]^j_m$ is $[\mathcal{S}\chi]^k_p$, take the first upper numeral index, l, which is new to $\mathfrak{b}$, and add $[\chi]^l_p$ to the end of $\mathfrak{b}$ (if it does not already contain it — analogously for $\neg\mathcal{F}$).

XIV If $[\psi]^j_m$ is $[\mathcal{F}\chi]^k_p$, add every doubly-indexed formula $[\chi]^l_p$ to the end of $\mathfrak{b}$ for every index l occurring on the branch (if it does not already contain it — analogously for $\neg\mathcal{S}$).

XV If $[\psi]^j_m$ is $[(\exists x)\chi]^k_p$, take the first constant, c, not appearing on $\mathfrak{b}$, and add $[\chi[c/x]]^k_p$ to the end of $\mathfrak{b}$ (if it does not already contain it — analogously for $(\neg\forall x)$).

XVI If $[\psi]^j_m$ is $[(\forall x)\chi]^k_p$, add every doubly-indexed formula $[\chi[c/x]]^k_p$ to the end of $\mathfrak{b}$ such that c is among the first countably many individual constants available (if it does not already contain it — analogously for $(\neg\exists x)$).

Once this procedure is completed for the leftmost highest point on the 2D-tableau, repeat it for the next highest point closest from it until the rightmost highest point. This is where stage $n+1$ is concluded. Now there are three possible cases to consider. Either the systematic procedure resulted in a closed 2D-tableau, thereby producing a proof of $[\varphi]^i_n$; the procedure terminated, producing an open branch; the procedure does not terminate, producing a possibly infinite open branch.

Completeness for $\mathbf{S5}_{2D}$. We show how to construct a countermodel such that, if $\mathfrak{b}$ is any complete open branch of a 2D-tableau, finite or not, then $\mathfrak{b}$ is satisfiable, where $\mathfrak{b}$ is a *complete* open branch of a 2D-tableau $\mathcal{T}$ just in case every rule that can possibly be applied to it has been applied. In order to construct a countermodel we use the same procedure as in the counterexamples above.

Definition 2.A.3 (Equivalence class) Let $\approx$ be an equivalence relation over the set of constants in $\mathcal{L}_{2D}$ such that $c_0 \approx c_1$ iff '$[c_0 = c_1]^i_n$' occurs on $\mathfrak{b}$ for any pair of indices $\langle i, n\rangle$. Moreover, for any c, let $[c]$ denote the equivalence class of c relative to $\approx$.

Definition 2.A.4 (Model) For the countermodel, if k is any single index occurring on $\mathfrak{b}$, let $Z = \{k \mid k \in \mathfrak{b}\}$, and $W = Z \times Z$. Let 0 be the distinguished element of Z. For every $\langle i, n\rangle, \langle i, m\rangle \in W$, set $\langle i, n\rangle \mathcal{R}_{\Box} \langle i, m\rangle$, and for every $\langle i, n\rangle, \langle j, n\rangle \in W$, set $\langle i, n\rangle \mathcal{R}_{\mathcal{F}} \langle j, n\rangle$. Next we define C as the set of all constants occurring on $\mathfrak{b}$, and $\mathscr{D} = \{[c] \mid c \in C\}$. Furthermore, set $V(c, \langle i, n\rangle) = [c]$ to each $c \in C$, and to each n-place predicate symbol, R, on $\mathfrak{b}$, let $V(R, \langle i, n\rangle) = \{\langle [c_0], ..., [c_n]\rangle : R(c_0, ..., c_n)^i_n$ occurs on $\mathfrak{b}\}$. Given both rigidity conditions, for every constant symbol c and $\langle i, n\rangle, \langle j, m\rangle \in W$ we have $V(c, \langle i, n\rangle) = V(c, \langle j, m\rangle)$. In case φ is atomic, if $[\varphi]^i_n$ occurs on $\mathfrak{b}$ then $\mathcal{M}^i_n \vDash \varphi$, otherwise set $\mathcal{M}^i_n \nvDash \varphi$.[57] This is enough to define a constant domain 2D-centered model $\mathcal{M} = \langle W, 0, \mathcal{R}_{\Box}, \mathcal{R}_{\mathcal{F}}, \mathscr{D}, V\rangle$.

Lemma 2.A.3 (Truth lemma) *Let $\mathfrak{b}$ be a complete open branch of a 2D-tableau, and $\mathcal{M}$ be a constant domain 2D-centered model, $\mathcal{M} = \langle \mathrm{W}, 0, \mathcal{R}_{\Box}, \mathcal{R}_{\mathcal{F}}, \mathscr{D}, V\rangle$. For every doubly-indexed formula, $[\varphi]^i_n$,*

$$[\varphi]^i_n \text{ occurs on } \mathfrak{b} \Leftrightarrow \mathcal{M}^i_n \vDash \varphi$$

Proof. By induction on φ. We only consider the relevant two-dimensional operators.

Let φ be $\mathcal{A}\psi$. If $[\mathcal{A}\psi]^i_n$ occurs on $\mathfrak{b}$, then $[\psi]^i_i$ is also on $\mathfrak{b}$. By I.H., $\mathcal{M}^i_i \vDash \psi$, whence $\mathcal{M}^i_n \vDash \mathcal{A}\psi$, by the truth clause for $\mathcal{A}$ (the argument for $\neg\mathcal{A}\psi$ is analogous).

Let φ be $\odot\psi$. If $[\odot\psi]^i_n$ occurs on $\mathfrak{b}$, then $[\psi]^0_n$ is also on $\mathfrak{b}$. By I.H., $\mathcal{M}^0_n \vDash \psi$, whence

[57] Of course, we assume the models are constrained by (Neutrality).

$\mathcal{M}_n^i \vDash \odot\psi$, by the truth clause for $\odot$ (analogously for $\neg \odot \psi$).

Let φ be $\otimes\psi$. If $[\otimes\psi]_n^i$ occurs on $\mathfrak{b}$, then $[\psi]_n^n$ is also on $\mathfrak{b}$. By I.H., $\mathcal{M}_n^n \vDash \psi$, whence $\mathcal{M}_n^i \vDash \otimes\psi$, by the truth clause for $\otimes$ (analogously for $\neg \otimes \psi$).

Let φ be $\mathcal{F}\psi$. If $[\mathcal{F}\psi]_n^i$ occurs on $\mathfrak{b}$, then for every j on $\mathfrak{b}$, $[\psi]_n^j$ is also on $\mathfrak{b}$. By I.H. and construction, $\mathcal{M}_n^j \vDash \psi$ for every $\langle i, n\rangle \mathcal{R}_{\mathcal{F}} \langle j, n\rangle$, whence $\mathcal{M}_n^i \vDash \mathcal{F}\psi$, by the truth clause for $\mathcal{F}$. On the other hand, let φ be $\neg\mathcal{S}\psi$. If $[\neg\mathcal{S}\psi]_n^i$ occurs on $\mathfrak{b}$, then for every j on $\mathfrak{b}$, $[\neg\psi]_n^j$ also occurs on $\mathfrak{b}$. By I.H. and construction, $\mathcal{M}_n^j \nvDash \psi$ for every $\langle i, n\rangle \mathcal{R}_{\mathcal{F}} \langle j, n\rangle$, whence $\mathcal{M}_n^i \nvDash \mathcal{S}\psi$, by the truth clause for $\mathcal{S}$.

Let φ be $\mathcal{S}\psi$. If $[\mathcal{S}\psi]_n^i$ occurs on $\mathfrak{b}$, then for some j on $\mathfrak{b}$, $[\psi]_n^j$ is also on $\mathfrak{b}$. By I.H. and construction, $\mathcal{M}_n^j \vDash \psi$ for some $\langle i, n\rangle \mathcal{R}_{\mathcal{F}} \langle j, n\rangle$, whence $\mathcal{M}_n^i \vDash \mathcal{S}\psi$, by the truth clause for $\mathcal{S}$. On the other hand, let φ be $\neg\mathcal{F}\psi$. If $[\neg\mathcal{F}\psi]_n^i$ occurs on $\mathfrak{b}$, then for some j on $\mathfrak{b}$, $[\neg\psi]_n^j$ is also on $\mathfrak{b}$. By I.H., $\mathcal{M}_n^j \nvDash \psi$ for some $\langle i, n\rangle \mathcal{R}_{\mathcal{F}} \langle j, n\rangle$, whence $\mathcal{M}_n^i \nvDash \mathcal{F}\psi$, by the truth clause for $\mathcal{F}$. □

Theorem 2.A.2 (Completeness) *If φ is valid, then φ has a 2D-tableau proof.*

Proof. We prove the contrapositive. Suppose φ does not have a 2D-tableau proof, in which case any 2D-tableau $\mathcal{T}$ beginning with $[\neg\varphi]_0^0$ remains open. Let $\mathfrak{b}$ be a complete open branch of $\mathcal{T}$ such that $[\neg\varphi]_0^0$ occurs on $\mathfrak{b}$. By Lemma 2.A.3, there is a constant domain 2D-centered model, $\mathcal{M} = \langle W, 0, \mathcal{R}_{\Box}, \mathcal{R}_{\mathcal{F}}, \mathscr{D}, V\rangle$, such that $\mathcal{M}_0^0 \nvDash \varphi$. Consequently, φ is not valid. □

2.B Soundness and Completeness Theorems for General and Diagonal Tableaux

It might be useful to indicate how the soundness and completeness theorems for local 2D-tableaux can be modified for general and diagonal 2D-tableaux. The only differences concern the main theorems for soundness and completeness, namely, theorems 2.A.1 and 2.A.2. In what follows we show how they are adjusted for general 2D-tableaux.

Theorem 2.B.1 (Soundness for general 2D-tableaux) *If φ has a general 2D-tableau proof, then φ is generally valid.*

Proof. Suppose φ has a general 2D-tableau proof, in which case there is a closed general 2D-tableau, $\mathcal{T}$, which begins with $[\neg\varphi]^n_m$, where $n \neq m$, and both n and m are different from '0'. For a contradiction, assume that φ is not generally valid. Thus, there is a 2D-centered model, $\mathcal{M} = \langle W, w*, \mathcal{R}_\Box, \mathcal{R}_\mathcal{F}, \mathscr{D}, V\rangle$, and a pair $\langle v, w\rangle \in W$, such that $\mathcal{M}^v_w \nvDash \varphi$. Let f be a function such that $f(n) = v$ and $f(m) = w$, in which case $\mathcal{M}^{f(n)}_{f(m)} \nvDash \varphi$. By Definition 2.A.1, $\{[\neg\varphi]^n_m\}$ is satisfiable. Moreover, since its only branch is satisfiable, $\mathcal{T}$ is also satisfiable, and so is any general 2D-tableau extending it by way of the $\mathbf{S5}_{2D}$ rules, by Lemma 2.A.2. Therefore, $\mathcal{T}$ is both closed and satisfiable, contradicting Lemma 2.A.1, whence $\mathcal{M}^v_w \vDash \varphi$. □

Theorem 2.B.2 (Completeness for general 2D-tableaux) *If φ is generally valid, then φ has a general 2D-tableau proof.*

Proof. We prove the contrapositive. Suppose φ does not have a general 2D-tableau proof, in which case any general 2D-tableau $\mathcal{T}$ beginning with $[\neg\varphi]^n_m$, where $n \neq m$, both n and m are different from '0', and that remains open. Let $\mathfrak{b}$ be a complete open branch of $\mathcal{T}$ such that $[\neg\varphi]^n_m$ occurs on $\mathfrak{b}$. By Lemma 2.A.3, there is a constant domain 2D-centered model, $\mathcal{M} = \langle W, 0, \mathcal{R}_\Box, \mathcal{R}_\mathcal{F}, \mathscr{D}, V\rangle$, such that $\mathcal{M}^n_m \nvDash \varphi$. Consequently, φ is not generally valid. □

The arguments for diagonal 2D-tableaux are very similar, we just have to consider a formula, φ, indexed by any pair, $\langle n, n\rangle$, where $n \neq 0$. Besides this, the arguments are essentially the same.

2.C Different systems

We show that sound and complete general 2D-tableau can be obtained for different systems extended by $\mathbf{S5}_{2D}$.

2.C.1 S5$\mathcal{A}$

Crossley and Humberstone offer the following axioms for **S5**$\mathcal{A}$ with general validity, besides the usual **S5** axioms:[58]

(A1) $\mathcal{A}(\varphi \supset \psi) \supset (\mathcal{A}\varphi \supset \mathcal{A}\psi)$

(A2) $\mathcal{A} \equiv \neg\mathcal{A}\neg\varphi$

(A3) $\Box\varphi \supset \mathcal{A}\varphi$

(A4) $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$.

Lemma 2.C.1 *$\mathbf{S5}_{2D}$ with general validity extends $\mathbf{S5}\mathcal{A}$ with general validity.*

Proof. General 2D-tableau proof schemata can be given to prove axioms (A1)-(A4). We show (A3) and (A4), and leave (A1) and (A2) for the reader.

[58] In their original paper, Crossley and Humberstone use axioms with a rule of uniform substitution. In Davies and Humberstone (1980), axiom-schemata are offered, and a redundant axiom from Crossley and Humberstone (1977) is omitted. We follow Davies and Humberstone's presentation here.

1. $[\neg(\Box\varphi \supset \mathcal{A}\varphi)]^0_1$
2. $[\Box\varphi]^0_1$
3. $[\neg\mathcal{A}\varphi]^0_1$
4. $[\neg\varphi]^0_0$
5. $[\varphi]^0_0$

$\times$

1. $[\neg(\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi)]^0_1$
2. $[\mathcal{A}\varphi]^0_1$
3. $[\neg\Box\mathcal{A}\varphi]^0_1$
4. $[\neg\mathcal{A}\varphi]^0_2$
5. $[\neg\varphi]^0_0$
6. $[\varphi]^0_0$

$\times$

□

Corollary 2.C.1 *Soundness and Completeness for* ***S5***$\mathcal{A}$ *with general validity.*

Proof. By Lemma 2.C.1 and soundness and completeness for $\mathbf{S5}_{2D}$ general 2D-tableaux. □

Moreover, it can be shown that $\mathbf{S5}_{2D}$ extends $\mathbf{S5}\mathcal{A}$ with local validity as well by proving the axiom schema $\mathcal{A}\varphi \supset \varphi$, which is straightforwardly done by local 2D-tableaux.

2.C.2 S5$\mathcal{AF}$

In what follows we exhibit Davies and Humberstone's axiomatization of $\mathbf{S5}\mathcal{AF}$. Besides the basis provided by $\mathbf{S5}\mathcal{A}$, we have:

($\mathcal{F}$1) $\mathcal{F}(\varphi \supset \psi) \supset (\mathcal{F}\varphi \supset \mathcal{F}\psi)$

($\mathcal{F}$2) $\mathcal{F}\varphi \supset \varphi$

($\mathcal{F}$3) $\mathcal{F}\varphi \supset \mathcal{F}\mathcal{F}\varphi$

($\mathcal{F}4$) $\varphi \supset \mathcal{F}\neg\mathcal{F}\neg\varphi$

($\mathcal{F}5$) $\varphi \supset \mathcal{F}\varphi$, for any $\mathcal{A}$-free formula φ.

($\mathcal{F}6$) $\mathcal{F}\mathcal{A}\varphi \equiv \Box\varphi$, for any $\mathcal{A}$-free formula φ.

Lemma 2.C.2 *$\mathbf{S5}_{2D}$ with general validity extends $\mathbf{S5}\mathcal{A}\mathcal{F}$ with general validity.*

Proof. Again, general 2D-tableau proof schemata can be given for axioms ($\mathcal{F}1$)-($\mathcal{F}6$). $\Box$

Corollary 2.C.2 *Soundness and Completeness for $\mathbf{S5}AF$ with general validity.*

Proof. By Lemma 2.C.2 and soundness and completeness for $\mathbf{S5}_{2D}$ general 2D-tableaux. $\Box$

Theorem 2.C.1 *Soundness for $\mathbf{S5E}_{2D}$.*

Proof. The arguments are very similar to the corresponding proofs for $\mathbf{S5}_{2D}$. We only need the following adjustments. The definition of satisfiability needs to be modified by taking into account $\mathcal{R}_{\mathcal{D}}$ rather than $\mathcal{R}_{\mathcal{F}}$, in which case for a constant domain 2D-centered model, $\mathcal{M} = \langle W, w*, \mathcal{R}_{\Box}, \mathcal{R}_{\mathcal{D}}, \mathscr{D}, V\rangle$, a set S of doubly-indexed formulas, and a function f from each individual index n in S to a possible world in Z, if the pairs of indices $\langle i, n\rangle$ and $\langle z, z\rangle$ are in S, then $\langle f(i), f(n)\rangle \mathcal{R}_{\mathcal{D}} \langle f(z), f(z)\rangle$. The rest of the definition remains the same. The soundness lemma corresponding to Lemma 2.A.2 varies with respect to the latter in minor details: it does not need arguments for $\otimes$ and $\odot$, it contains clauses for $\mathcal{D}$ and $\mathcal{C}$ rather than $\mathcal{F}$ and $\mathcal{S}$, which will be very similar, besides not containing arguments for upper exchange. We leave the details for the reader. $\Box$

Theorem 2.C.2 *Completeness for $\mathbf{S5E}_{2D}$.*

Proof. The arguments are, again, very similar to the ones for $\mathbf{S5}_{2D}$. However, for the countermodel corresponding to Definition 2.A.4, since we are not assuming rigidity with respect to the upper index, there needs to be some adjustments in our definitions. First, we need to modify Definition 2.A.3 of equivalent classes. Let $\approx_{i,n}$ be an equivalence relation over the set of constants in $\mathcal{L}_{E2D}$ such that $c_0 \approx_{i,n} c_1$ iff '$[c_0 = c_1]^i_n$' occurs on $\mathfrak{b}$ for any constants c_0, c_1, and pair of indices $\langle i, n\rangle$. Moreover, for any c, let $[c]_{i,n}$ denote the equivalence class of c relative to $\approx_{i,n}$. Now, if k is any index occurring on $\mathfrak{b}$, let $Z = \{k \mid k \in \mathfrak{b}\}$, and $W = Z \times Z$. Let 0 be the distinguished element of Z. For every $\langle i, n\rangle, \langle i, m\rangle \in W$, set $\langle i, n\rangle \mathcal{R}_{\Box} \langle i, m\rangle$, and for every $\langle i, n\rangle, \langle z, z\rangle \in W$, set $\langle i, n\rangle \mathcal{R}_{\mathcal{D}} \langle z, z\rangle$. Let $\mathscr{D} = \{[c]_{i,n} \mid c, \langle i, n\rangle \text{ on } \mathfrak{b}\}$. Concerning rigidity, this time we only assume that for any constant symbol c and $\langle i, n\rangle, \langle i, m\rangle \in W$, $V(c, \langle i, n\rangle) = V(c, \langle i, m\rangle)$, where $V(c, \langle i, n\rangle) = [c]_{i,n}$ and to each n-place predicate symbol, R, on $\mathfrak{b}$, let $V(R, \langle i, n\rangle) = \{\langle [c_0]_{i,n}, ..., [c_n]_{i,n}\rangle : R(c_0, ..., c_n)^i_n \text{ occurs on } \mathfrak{b}\}$. We do not assume (Neutrality) anymore. The rest of the definition remains the same as in Definition 2.A.4. Furthermore, the truth lemma is basically the same except for, again, having clauses for $\mathcal{D}$ and $\mathcal{C}$ rather than $\mathcal{F}$ and $\mathcal{S}$, which are straightforward. □

Chapter 3

Actuality and The A Priori

In this chapter we investigate some questions concerning the expressive power of a first-order modal language with two-dimensional operators. In particular, a language endowed with a two-dimensional semantics intended to provide a logical analysis of the discourse involving a priori knowledge. We consider a natural-language sentence that cannot be formally represented in such a language. This was firstly conjectured in chapter 2, but here we present a proof. It turns out, however, that the most natural ways to repair this expressive inadequacy render moot the original philosophical motivation of formalizing a priori knowability as necessity along the diagonal.

In what follows, first we present the basic principles involved in a quantified semantics containing the operators for necessity, actuality, and apriority. In the second part of the chapter we argue that any attempt to formalize the relevant natural-language sentence fails, and a proof that there is actually no sentence of the language with equivalent truth conditions can be found in the appendix. Finally, we show how adding a new operator or plural quantifiers to the language are not available options for proponents of this kind of two-dimensional semantics, for in both cases the models validate intuitively false principles concerning the a priori.

3.1 Quantified Two-Dimensional Semantics

By and large, two-dimensional semantics involve evaluating sentences with respect to a pair of possible worlds, states, points, or whatever.[1] In terms of possible worlds, for example, this means that a formula will be considered true at a world *relative to* or *with respect to* another possible world, although not necessarily a different one. Two-dimensional semantics have been developed in order to investigate the semantics of "now" and "then" in tense logics (Kamp (1971), Vlach (1973)), as well as the semantics of "actually" in modal logics (Crossley and Humberstone (1977), Davies and Humberstone (1980), Cresswell (1990)).[2] In particular, it has also been illuminating in investigations concerning a priori knowledge (Davies and Humberstone (1980), Kaplan (1989), Davies (2004), Chalmers (1996, 2004)). More recently, formal systems have been defined by Restall (2012) and Fritz (2013, 2014) containing primitive a priori knowability operators; in particular, the logic defined by Fritz is explicitly motivated by Chalmers' epistemic two-dimensional semantics.

There are several important differences regarding not only the motivations involved in such proposals, but also with respect to many formal details. Since our concern, however, involves the notion of a priori knowability, we shall pass over those and concentrate on what is common to the logical systems defined by Davies and Humberstone, Restall, Fritz, as well as Chalmers' version of two-dimensional semantics.[3] Broadly speaking, what is common amongst them is a semantics involving the so-called metaphysical modality expressed in

[1]For more on this, and different characterizations of two-dimensionality, see Humberstone (2004).

[2]Such a framework proved to be very useful also for the purposes of investigating context-dependent terms (Montague (1968), Lewis (1970a), Kaplan (1989)), the pragmatics of assertion (Stalnaker (1978)), as well as conceptual analysis and reductive explanation (Jackson (1998), Chalmers and Jackson (2001)).

[3]Perhaps the main point can be generalized for other cases as well. What is important, as we shall make explicit, is a semantics considering necessity and actuality interpreted by the corresponding modal operators, as well as apriority taken as truth on the diagonal.

terms of $\Box$-modalization, a priori knowability, which is taken as necessity along the diagonal, as well as an indexical interpretation of actuality formalized by the actuality operator. The pertinent question then is how should we expand this kind of semantics to the first-order case with identity? Quantified modal semantics already brings up too many philosophical problems. But, in effect, we can naturally extend propositional two-dimensional modal semantics with quantifiers in a way that is very similar to the one-dimensional case.

In the usual Kripke semantics, first-order quantifiers can be defined as ranging over the domains of each world of evaluation or even the entire domain of all possible worlds. The former gives us variable domains, and the latter gives us a constant domain semantics for the quantifiers. In each case, formulas can also be interpreted in a two-dimensional manner by adding an extra world parameter to the semantic evaluation clauses, namely, the distinguished or actual world of the models.[4] Since the basic modal semantics make no explicit use of such world, except, maybe, in defining logical properties such as validity, this simple addition will not cause any difference in the resulting logics. Basic modal semantics can be taken without harm as containing such a hidden parameter for actuality. By making it explicit, we get the simplest two-dimensional modal logic. Thus, a quantified two-dimensional semantics may consider the range of the quantifiers to be similar to the one-dimensional case, with the exception that the actual world now varies, whereupon the extensions of the individual constants and predicate symbols of the language will be determined by each pair of possible worlds.

In what follows we give a more formal treatment of a quantified two-dimensional semantics along these lines. We use a constant domain semantics for the quantifiers, although a variable domain semantics can also be easily defined.[5] We discuss the semantics for identity before closing this section.

[4]Where $\mathcal{M}$ denotes a Kripke model and $w*$ is the distinguished element of a set W, this can be done by rewriting $\mathcal{M}, w \vDash p$ as $\mathcal{M}, \langle w*, w \rangle \vDash p$.

[5]We purposefully avoid single quotation marks in order to not clutter the presentation. The context should make use-mention distinctions clear.

Definition 3.1.1 (First-order Language) For our language, $\mathcal{L}$, let $\{c_1, c_2, ...\}$ be a set of *constant symbols*, $\{x_1, x_2, ...\}$ a set of *individual variables*, and $\{P_1^n, P_2^n, ...\}$ a set of n-place *predicate symbols* for each $n \in \mathbb{N}$. The terms t and formulas φ are recursively generated by the following grammar ($i, k \in \mathbb{N}$):

$$t ::= c_i \mid x_i$$

$$\varphi ::= P_i^n(t_1, ..., t_k) \mid t = t' \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \mathcal{A}\varphi \mid \mathcal{D}\varphi \mid \exists x_i \varphi$$

We define $\forall x_i \varphi$, $\Diamond$, and the other Boolean connectives as usual. Moreover, we define $\mathcal{C}$, the dual of $\mathcal{D}$, as $\neg\mathcal{D}\neg$. In what follows we occasionally drop the subscripts involved in terms, writing $a, b, c, ...$ for constant symbols and $x, y, z, ...$ for individual variables. Next we define models equipped with accessibility relations for both $\Box$ and $\mathcal{D}$ formulas.[6] Once we have a language, we extend the language to a new language, say, $\mathcal{L}(\mathcal{M})$, by adding a constant symbol c for each object in the domain. $\mathcal{M}$ interprets each of the new constants c as the object c from which it gets its name. Since our set W of possible worlds will only contain pairs, its distinguished element will be a pair as well :

Definition 3.1.2 (2D-centered models) A *constant domain 2D-centered model* is a tuple, $\mathcal{M} = \langle W, w*, \mathcal{R}_\Box, \mathcal{R}_\mathcal{D}, \mathscr{D}, V \rangle$, such that

- $W = Z \times Z$ for some set Z,
- $w*$ is a distinguished element of Z,
- $\mathcal{R}_\Box \subseteq W \times W$, the $\Box$-accessibility relation, is the least relation such that for every $v, w, z \in Z$, $\langle v, w \rangle \mathcal{R}_\Box \langle v, z \rangle$,
- $\mathcal{R}_\mathcal{D} \subseteq W \times W$, the $\mathcal{D}$-accessibility relation, is the least relation such that for every $v, w, z \in Z$, $\langle v, w \rangle \mathcal{R}_\mathcal{D} \langle z, z \rangle$,

[6] The accessibility relations $\mathcal{R}_\Box$ and $\mathcal{R}_\mathcal{D}$ are just the ones in Fritz (2013, p. 1758), except that he also adds an accessibility relation for $\mathcal{A}$ formulas, which would be useful for us if we were investigating properties of the logic corresponding to how the frames are defined.

- $\mathscr{D}$ is a non-empty domain of quantification, and
- V is a function assigning to each constant c_i of $\mathcal{L}$ and $\langle v, w\rangle \in W$, an object $V(c_i, \langle v, w\rangle) \in \mathscr{D}$, and to each n-place predicate symbol P_i^n and $\langle v, w\rangle \in W$, a set $V(P_i^n, \langle v, w\rangle) \subseteq \mathscr{D}^n$.

Moreover, for any constant symbol c_i and $v, w, z \in Z$, let $V(c_i, \langle v, w\rangle) = V(c_i, \langle v, z\rangle)$. We call this the *rigidity condition*.[7]

Definition 3.1.3 (Truth) We define 'φ is true at w relative to v in $\mathcal{M}$', written $\mathcal{M}_w^v \vDash \varphi$,[8] by recursion on φ. For a pair $\langle v, w\rangle \in W$, and a valuation V in $\mathcal{M}$,

$\mathcal{M}_w^v \vDash P_i^n(t_1, ..., t_n)$	iff	$\langle V(t_1, \langle v, w\rangle), ..., V(t_n, \langle v, w\rangle)\rangle \in V(P_i^n, \langle v, w\rangle)$;
$\mathcal{M}_w^v \vDash t = t'$	iff	$V(t, \langle v, w\rangle) = V(t', \langle v, w\rangle)$;
$\mathcal{M}_w^v \vDash \neg\varphi$	iff	$\mathcal{M}_w^v \nvDash \varphi$
$\mathcal{M}_w^v \vDash \varphi \wedge \psi$	iff	$\mathcal{M}_w^v \vDash \varphi$ and $\mathcal{M}_w^v \vDash \psi$;
$\mathcal{M}_w^v \vDash \Box\varphi$	iff	for every $\langle v, z\rangle \in W$ such that $\langle v, w\rangle \mathcal{R}_\Box \langle v, z\rangle$, $\mathcal{M}_z^v \vDash \varphi$;
$\mathcal{M}_w^v \vDash \mathcal{A}\varphi$	iff	$\mathcal{M}_v^v \vDash \varphi$;
$\mathcal{M}_w^v \vDash \mathcal{D}\varphi$	iff	for every $\langle z, z\rangle \in W$ such that $\langle v, w\rangle \mathcal{R}_\mathcal{D} \langle z, z\rangle$, $\mathcal{M}_z^z \vDash \varphi$;
$\mathcal{M}_w^v \vDash \exists x\varphi$	$\Longleftrightarrow$	for some constant symbol c of $\mathcal{L}(\mathcal{M})$, $\mathcal{M}_w^v \vDash \varphi[c/x]$;

We say that a formula φ is *true* in $\mathcal{M}$, written $\mathcal{M} \vDash \varphi$, if and only if φ is true at $\langle w*, w*\rangle$ in $\mathcal{M}$ (i.e. $\mathcal{M}_{w*}^{w*} \vDash \varphi$), and a formula φ is a *logical consequence* of a set of formulas Γ if and only if for every $\mathcal{M}$, if $\mathcal{M} \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \vDash \varphi$. The notion of validity generalizes logical consequence as usual, and we can define *local validity* for a formula as truth in every

[7]Here we follow Holliday and Perry (2014) in their quantified two-dimensional semantics. Even though their semantics is defined to deal with the Hintikka-Kripke problem in the context of an epistemic logic, their rigidity condition is useful for our purposes. More on this below.

[8]About notation: Davies and Humberstone (1980, p. 4) write $\mathcal{M} \vDash_w^v \varphi$, appending the superscripts and subscripts to the right side of the turnstile. We simply prefer having the world variables on the left side, which, in its present form, is intended as a two-dimensional version of $\mathcal{M}, \langle v, w\rangle \vDash \varphi$.

model, *general validity* as truth at every pair $\langle v, w \rangle$ in every model, and *diagonal validity* as truth at every coincident pair $\langle w, w \rangle$ in every model.

The notion of validity for logics containing actuality operators is notoriously controversial, and a satisfactory philosophical defense of either local or general validity is not our main objective.[9] Nonetheless, there is something to be said about the notion of validity in logics containing diagonal operators. Amongst the ones mentioned above, the ones in vogue comprise either general or diagonal validity, and not local validity as we have defined it. Davies and Humberstone's logic of fixedly actually, or $\mathbf{S5}\mathcal{AF}$, is defined with respect to general validity, and the two-dimensional logics developed by Restall and Fritz both use diagonal validity.[10] In particular, both define a set of distinguished elements rather than fixing a specific point in the models as we did. This set in turn comprises every diagonal point in a frame, which makes sense vis-à-vis the fact that the first coordinate of each pair is intuitively an actual world. Logical truth, consequently, turns out to be truth at every diagonal point in every model. And this is plausible on philosophical grounds since it counts $\mathcal{D}p \supset p$ as valid — also, $\mathcal{R}_{\mathcal{D}}$ can be immediately seen to be reflexive on the diagonal points of our models. Presumably, if it is a priori that p, then p is true, since apriority is usually taken to be factive.

However, by inspection of the semantics defined above, it is obvious that a formula is diagonally valid if and only if it is locally valid.[11] Suppose that a formula, say, φ, is diagonally valid. Then φ holds at every diagonal point of every model, in which case it is obviously locally valid as well. On the other hand, suppose φ is not diagonally valid, in which case it fails to hold at some diagonal point in some model. Since there is a model in which that

[9] See, for example, Crossley and Humberstone (1977, p. 14–15) and Hanson (2006, 2014) for a defense of general validity, and Zalta (1988), Nelson and Zalta (2012) for arguments in favour of local validity.

[10] This is also the case in Kocurek (forthcoming).

[11] Local validity and general validity coincide for basic one-dimensional modal logics without actuality operators since there is no use for a distinguished element in the models. Similarly, there is no use for a distinguished point on the diagonal of a 2D frame since the actuality operator is not fixed to any particular point but to the first coordinate of every pair of possible worlds.

diagonal point is the distinguished one, φ also fails to be locally valid. Consequently, by adopting local validity we have not changed the set of validities in comparison to the logics of both Restall and Fritz insofar as the propositional portion is concerned.[12]

There are philosophical motivations, however, which we take to speak in favour of local validity. Informally, every actual world is alternative to *this* one, and this idea is better captured by fixing a single point in the models as distinguished. By the same token, Restall calls every diagonal point an indicative alternative, in contrast to the subjunctive alternatives delivered by horizontal, or $\mathcal{R}_{\Box}$-related points.[13] But not having a single distinguished point in the models makes it unclear in relation to what those points are ultimately supposed to be alternatives to. All of this is less plausible, on the other hand, if we take the first coordinates of every pair to be epistemic scenarios, for there is no apparent reason to have a single privileged scenario in a class of models. Since, however, the resulting logic is the same whether we single out a point or every diagonal point as distinguished, at least momentarily, such remarks matter only to the extent of a more informal level of philosophical motivation. It is only in §3.2 that we make an important use of a distinguished point given the new actuality operator therein defined, and so we have found it useful to define our models in this manner from the outset.

Informally, we can read the semantic clauses for $\Box$, $\mathcal{A}$, and $\mathcal{D}$, respectively, as saying that 'Necessarily φ' is true at $\langle v, w\rangle$ if and only if for every $\mathcal{R}_{\Box}$-related pair of possible worlds $\langle v, z\rangle$, φ is true at $\langle v, z\rangle$; 'Actually φ' is true at $\langle v, w\rangle$ if and only if φ is true at $\langle v, v\rangle$; and 'It is a priori that φ' is true at $\langle v, w\rangle$ if and only if for every $\mathcal{R}_{\mathcal{D}}$-related pair of possible worlds $\langle z, z\rangle$, φ is true at $\langle z, z\rangle$. Hence, the apriority operator, $\mathcal{D}$, delivers truth along the diagonal. This is easy to see by displaying a 2D matrix, where the worlds arranged on the

[12] Some terminological remarks are in order. It is usual to say 'real-world' instead of 'local' validity, in accordance with the terminology used in Crossley and Humberstone (1977). We find it odd, however, to call real-world validity truth at the distinguished *pair* in every model, otherwise we have no reason to prefer a different nomenclature.

[13] See p. 1611.

vertical represent scenarios or actual worlds, and the worlds on the horizontal are the worlds of evaluation:

$$\begin{pmatrix} & v & w \\ v & - & - \\ w & - & - \end{pmatrix}$$

If a formula is a priori knowable, or $\mathcal{D}$-true, it receives the value T in the following manner:

$$\begin{pmatrix} & v & w \\ v & T & - \\ w & - & T \end{pmatrix}$$

Apart from several notational variances, the semantics just defined conservatively extends the semantics presented by both Restall and Fritz. With respect to Davies and Humberstone's case, there are more significant differences besides the fact that **S5**$\mathcal{AF}$ is defined only as a propositional logic. First, they define a fixedly operator, $\mathcal{F}$, rather than a primitive apriority operator, where $\mathcal{F}\varphi$ is true at $\langle v, w\rangle$ if and only if for every pair of possible worlds $\langle z, w\rangle \in W$, φ is true at $\langle z, w\rangle$.[14] Thus, it is the concatenation $\mathcal{FA}$ that defines diagonal necessity in **S5**$\mathcal{AF}$.[15] Davies and Humberstone claim that $\mathcal{FA}$ corresponds to Evans' (1979) notion of deep necessity, identified by the latter with what is a priori knowable. Such a notion

[14] We have adapted, of course, the semantic clause of $\mathcal{F}$ for pairs of worlds.

[15] Davies (2004, p. 89) makes it clear that they did not intend to formalize anything like an epistemic logic, although the resulting system does give rise to a priori truths. Moreover, in response to Evans' criticisms to the fixedly operator, he also considered the possibility of adding a primitive operator $\mathcal{D}$ for diagonal necessity (cf. p. 92).

was defined in contrast to superficial necessity, which is just the necessity delivered by the modal operator $\Box$, and it was used by Evans to argue that Kripke's (1980) examples of the contingent a priori involved cases of superficial contingency but deep necessity, where a sentence is said to be superficially contingent just in case its $\Box$-modalization and the $\Box$-modalization of its negation are both false. Davies and Humberstone claim they have not found counterexamples in the language of $\mathbf{S5}\mathcal{AF}$ of sentences that are a priori but not deeply necessary, or necessary along the diagonal, even though the identification of diagonal necessity with the a priori is made with some reservations (p. 10). Finally, the truth of propositional letters in $\mathbf{S5}\mathcal{AF}$ is sensitive only to the world of evaluation, and not to a pair of worlds. This feature transforms the actual world in the evaluation of an atomic formula into a free parameter, resulting in an equivalence for $\Box$ and $\mathcal{D}$-formulas not containing $\mathcal{A}$.[16] Yet, neither $\Box p \supset \mathcal{D}p$ nor $\mathcal{D}p \supset \Box p$ are valid in the semantics defined here, in sharp contrast to Davies and Humberstone's $\mathbf{S5}\mathcal{AF}$.[17]

Now it is not difficult to see that our semantics is also very much in agreement with the semantics developed by Chalmers, where expressions are evaluated with respect to pairs of scenarios and possible worlds. Chalmers take scenarios to be epistemically possible worlds, i.e. "ways things might be that cannot be ruled out a priori." (2004, p. 211) For the purposes of a more formal treatment, however, he allows those to be considered just as possible worlds. (idem) The same simplification is made by Fritz (2013, p. 1757) in his formal system. Thus, we might just as well take the Cartesian product $Z \times Z$ in a 2D-centered frame as intuitively representing scenarios and possible worlds, respectively. Although this is a somewhat implausible assumption as scenarios give rise to distinct, epistemic modalities, it does not affect our main points in any substantial way.[18] Furthermore, despite the fact that

[16] Axiom $\mathcal{F}6$, for example, of $\mathbf{S5}\mathcal{AF}$, reads $\Box\varphi \leftrightarrow \mathcal{FA}\varphi$ for $\mathcal{A}$-free φ. See Davies and Humberstone (1980, p. 4).

[17] For more on this, see §3. Fritz (2013, p. 1761) makes similar observations.

[18] Both Restall and Fritz constrain their frames in such a way that there are at least as many horizontal as vertical worlds. See Restall (2012, p. 1618), and Fritz (2013, p. 1761). Without loss of generality, we can define 2D frames in a similar manner.

Chalmers usually takes semantic values in terms of primary, secondary, and two-dimensional intensions in order to ascribe meanings to expressions, his semantics for necessity, actuality, and apriority correspond exactly to our operators $\Box$, $\mathcal{A}$, and $\mathcal{D}$, respectively.[19]

The more interesting aspects of our quantified two-dimensional semantics, however, involve identity and proper names. In modal logics, the former is usually defined irrespective of the world of evaluation, consequently validating the *necessity of identity*, viz., the claim that if two things are identical, they are necessarily identical. In contrast, an unrestricted semantics for identity in a two-dimensional framework would result in the models validating that it is a priori that two things are the same whenever two things are the same.[20] But this is certainly not intended by Chalmers' version of two-dimensional semantics, or any formal treatment designed to capture what is intuitively taken to be a priori knowable. For even though 'Hesperus is Phosphorus' is a true identity claim, and hence necessary, it is cognitively significant, whence two-dimensionalists take it to be only a posteriori knowable.

Following Kripke (1980), we take proper names to be rigid designators, where a proper name designates rigidly just in case it has the same extension in every possible world. In a two-dimensional framework this means that a proper name has the same extension in every possible world w relative to a world v taken as actual. Following Chalmers' terminology, the secondary or counterfactual intensions of both 'Hesperus' and 'Phosphorus' are the same, which accounts for the necessity of 'Hesperus is Phosphorus.' This is captured by the rigidity condition assumed in 2D-centered models, which in turn gives us $V(Hesperus, \langle v, w \rangle) = V(Phosphorus, \langle v, w \rangle)$ for all $w \in Z$, whence this identity is $\mathcal{R}_\Box$ necessary.[21] In contrast, we want the primary or actual intensions of both 'Hesperus'

[19]See, for example, Chalmers (2014, p. 212), although in the same paper he recognizes difficulties for the semantics of the apriority operator caused by the nesting problem.

[20]Proof: Suppose $\mathcal{M} \vDash x = y$ for an assignment V. By the unrestricted semantics of identity terms, $V(x) = V(y)$. Let $\langle w, w \rangle$ be any pair of possible worlds in W such that $\langle w*, w* \rangle \mathcal{R}_\mathcal{D} \langle w, w \rangle$. Then, given $V(x) = V(y)$, it follows that $\mathcal{M}^w_w \vDash x = y$, whence $\mathcal{M} \vDash \mathcal{D}x = y$, therefore $\mathcal{M} \vDash x = y \supset \mathcal{D}x = y$. An analogous result is available for Davis and Humberstone's **S5**$\mathcal{AF}$, the only difference being that it involves the compound operator $\mathcal{FA}$ rather than $\mathcal{D}$.

[21]We thereby assimilate proper names to individual constants in $\mathcal{L}$. This follows closely the presentation

and 'Phosphorus' to not designate rigidly, for they pick out something more related to a Fregean descriptive sense like the last bright object in the morning sky and the first bright object in the evening sky, respectively. And in some scenarios the first bright object in the evening sky is not Venus, whence 'Hesperus is Phosphorus' is only a posteriori knowable.[22] If we let this scenario be some $z \in Z$, then we can have for some $\langle v, w\rangle \mathcal{R}_{\mathcal{D}} \langle z, z\rangle$, $V(Hesperus, \langle z, z\rangle) \neq V(Phosphorus, \langle z, z\rangle)$. In fact, the 2D matrix below illustrates a 2D model in which 'Hesperus is Phosphorus' is necessary and yet a posteriori:

$$\begin{pmatrix} & v & w \\ v & H = P & H = P \\ w & H \neq P & H \neq P \end{pmatrix}$$

Since, 'Hesperus is Phosphorus' is not true along the diagonal, it is not a priori as desired. There are, of course, more subtleties involved in the relationship between names, designation, and a two-dimensional framework, but this is enough for the present purposes.

3.2 Expressive Incompleteness

It is well-known that there are sentences that cannot be formalized in a language for quantified modal logic **S5**. A classic example from Crossley and Humberstone (1977) is

(1) It is possible for everything that is actually red to be shiny.

The problem is that (1) intends to quantify over the actual red things, except that the

in Holliday and Perry (2014, §4.5).

[22]Cf. Chalmers (2004, pp. 158–162, 2014, p. 212).

quantifier occurs within the scope of a possibility operator, whereby it can only quantify over the entities at that possible world. As a solution, Crossley and Humberstone suggest adding an actuality operator to the language designed to block the effect of any occurrences of modal operators to its left.[23] This can be seen in the semantics defined above for $\mathcal{A}$, except that Crossley and Humberstone's logic, called **S5**$\mathcal{A}$, only extends **S5** with $\mathcal{A}$, and not with apriority or diagonal operators, in which case there is no variation in the actual world of the models.[24] Thus, in **S5**$\mathcal{A}$ the actuality operator rigidifies the evaluation of formulas in its scope to the distinguished element of the models. (1) can then be formalized as follows:

(2) $\Diamond\forall x(\mathcal{A}Rx \supset Sx)$

It might be thought, however, that a language for a two-dimensional modal logic enriched with actuality and diagonal operators is expressive enough so that similar problems do not arise. But consider the following:

(3) It is not a priori knowable that something that is actually red is shiny.

There is a reading of this sentence that resists formalization in the language $\mathcal{L}$ for quantified two-dimensional semantics, namely:

(3a) $\neg\forall w\exists x(Rx\text{-at-}\langle w*, w*\rangle \wedge Sx\text{-at-}\langle w, w\rangle)$.

Or, equivalently,

(3b) $\exists w\forall x(Rx\text{-at-}\langle w*, w*\rangle \supset \neg Sx\text{-at-}\langle w, w\rangle)$.

Notice that in Davies and Humberstone's **S5**$\mathcal{AF}$, (3) can also be stated as

[23] The same strategy was employed in tense logics by Kamp (1971) for the "now" operator.
[24] In fact, in **S5**$\mathcal{A}$ there is no need at all of evaluating formulas with respect to a pair of worlds.

(3c) It is not deeply necessary that something that is actually red is shiny.[25]

Moreover, if one takes *conceivability* as the dual of apriority, another rendition of (3) might be

(3d) It is conceivable that everything that is actually red fails to be shiny.[26]

The most natural attempt to formalize such sentences would be as follows:

(4) $\neg\mathcal{D}\exists x(\mathcal{A}Rx \wedge Sx)$

This will not do, however, for it says that for some $\langle z, z\rangle \in W$ such that $\langle w*, w*\rangle \mathcal{R}_{\mathcal{D}} \langle z, z\rangle$, something that is red at $\langle z, z\rangle$ is shiny at $\langle z, z\rangle$. In contrast, the reading of (3) we are interested in says that something at $\langle z, z\rangle$ that is *in fact* red, i.e. red at $\langle w*, w*\rangle$, is shiny at $\langle z, z\rangle$. And this cannot be formalized by using the actuality operator as suggested in (4). The problem occurs because the actuality operator is designed to inhibit any outlying occurrences of the modal operators $\Box$ and $\Diamond$, but it does not scope out of apriority operators. When occurring within the scope of $\mathcal{D}$, the actuality operator merely reproduces, as it were, the actual world (or scenario) already introduced by $\mathcal{D}$. Since, however, (3) clearly belongs to the discourse intended to be captured by the language in question, $\mathcal{L}$ is expressively incomplete. Moreover, the expressive deficit in this case is exhibited by an analogue of the quantified **S5** case without an actuality operator, indicating iterations of the same problem as we add different operators to the language.[27] In the appendix we show that there is in

[25]This is similar to how the sentence appears in chapter 2. The only difference being that (3*) was formulated in terms of *deep possibility*, the dual of deep necessity. Admittedly, not much of a natural-language statement, since the notion of deep necessity seems to be even more philosophically loaded in comparison to apriority. Yet, in a language intended to formalize it, this is exactly the kind of sentence expected to be expressible.

[26]Chalmers (2004, p. 219) motivates conceivability as the dual of a priori knowability. However, in Chalmers (2011a) he argues that this is problematic.

[27]Notice that a rendering similar to (4) is available for Davies and Humberstone's **S5**$\mathcal{AF}$, the only difference being that $\mathcal{FA}$ occupies the place of $\mathcal{D}$.

fact no sentence of $\mathcal{L}$ with the same truth conditions as (3).[28]

3.3 The *Distinguished Actuality* Operator

The first and most natural approach to the expressive incompleteness of $\mathcal{L}$ would be to revise it by adding yet another actuality operator, which in turn will be designed to protect any formula in its scope from modal or apriority operators affixed to its left. In chapter 2 we suggested adding an operator called *distinguishedly*, symbolized by $\odot$, with the following semantics:

$$\mathcal{M}_w^v \models \odot\varphi \quad \text{iff} \quad \mathcal{M}_w^{w*} \models \varphi$$

The distinguishedly operator was designed to protect the formulas in its scope from occurrences of the fixedly operator in a logic resembling Davies and Humberstone's **S5**$\mathcal{AF}$.[29] If we affix $\odot$ to the immediate left of $\mathcal{A}$ in (4), we get in effect the desired truth conditions. However, we can also just define a new operator, @, pronounced *distinguished actuality*, which takes any pair of worlds to the distinguished element of a 2D-centered model:

$$\mathcal{M}_w^v \models @\varphi \quad \text{iff} \quad \mathcal{M}_{w*}^{w*} \models \varphi$$

This suffices in order to correctly formalize (3), for we will just need to replace the actuality operator in (4) with @, whereby we have the following:

(5) $\neg\mathcal{D}\exists x(@Rx \land Sx)$

[28] Hodes (1984a, p. 25, Theorem 15) proved that a formula resembling (1), but restricted to a single predicate letter, is not representable in **S5**, although similar inexpressibility results were previously conjectured by Hazen (1976). Wehmeier (2001) offers an elegant simplification of Hodes' argument. More recently, Kocurek (forthcoming) presents a thorough investigation of several inexpressibility results using bisimulations.

[29] See chapter 2.

It should be clear why we call @ distinguished actuality, for this is in contrast with the behavior of $\mathcal{A}$ in a two-dimensional modal semantics. Rather than evaluating the formulas in its scope relative to the distinguished element of the models, $\mathcal{A}$ just 'copies down' whatever possible world is momentarily taken as actual. @, on the other hand, behaves similarly to $\mathcal{A}$ in a standard one-dimensional modal language. Thus, we might just as well call $\mathcal{A}$, in a two-dimensional framework, *relative actuality.*

The actual truth conditions for any sentences φ, $\mathcal{A}\varphi$, and $@\varphi$ are the same, but they differ under both subjunctive and indicative alternatives. In a one-dimensional modal language it makes no difference if we add either $\mathcal{A}$ or @, for $\mathcal{A}\varphi$ and $@\varphi$ hold at a possible world w just in case φ holds at the actual world. It is only in a two-dimensional framework that the two come apart — and, in effect, that @ is needed. This also suggests that $\mathcal{A}$ does not correspond to the English adverb 'actually' after all. Hazen (1976, p. 40), for instance, claims that $\mathcal{A}$ is "quite well attested in ordinary English." And his suggestion seems to be that just like $\Box$ and $\Diamond$ correspond to the English adverbs 'necessarily' and 'possibly,' $\mathcal{A}$ stands for 'actually.' However, as Wehmeier (2004) also observes, an occurrence of 'actually' in a sentence seems to be neither necessary nor sufficient for its logical form to contain $\mathcal{A}$.[30] As a matter of fact, it is plausible that both (1) and (3) above could have been written without the word 'actually' while retaining the intended reading where $\mathcal{A}$ and @ were needed. For 'actually' seems to have an emphatic role in both (1) and (3), and we could have used different locutions like 'in fact' while preserving the same meaning.[31] On the other hand, Wehmeier uses an example from Kripke (1980, p. 124) as evidence that occurrences of 'actually' are also not sufficient for a sentence to contain an actuality operator.

> Consider a counterfactual situation in which, let us say, fool's gold or iron pyrites was actually found in various mountains in the United States, or in areas of South

[30] This is also acknowledged by Humberstone. See his (1982, fn. 16).

[31] This contrasts with a purely rhetorical use of 'actually', as pointed out by Crossley and Humberstone (1977, p. 11).

Africa and the Soviet Union.

The above does not look like a case where "found" is evaluated with respect to the actual world, but rather to a counterfactual one. Another example comes from Wehmeier (2005, fn. 6):

(6) Under certain circumstances, no-one would believe in aliens, although there would actually be aliens.

Again, the claim is that 'actually' does not force evaluation with respect to the actual world. But despite the fact that in those cases 'actually' does not force evaluation with respect to *the* actual world, it is also plausible that it forces evaluation relative to an alternative actual world. If this is true, both examples involve not counterfactual possibilities, but rather diagonal alternatives, in which case an actuality operator, although unnecessary, could be used in their formalization. In those cases, a two-dimensional semantics can be illuminating. Of course, this is not a decisive argument for the claim that those examples involve diagonal possibilities, but it is plausible that they do. In any event, that 'actually' is not sufficient for the logical form of a sentence to contain $\mathcal{A}$ is shown in two-dimensional contexts where the English adverb requires a distinguishedly actually operator.

Now, this solution to the expressive inadequacy of $\mathcal{L}$ seems simple and well motivated, since it comes down to a simple generalization of the analogous fix in **S5**. Moreover, it makes sense to have @ contraposing apriority operators in the same way $\mathcal{A}$ relates to the modal operators $\Box$ and $\Diamond$. One consequence, however, of adding $\mathcal{A}$ to **S5** is that the semantics validates the following:

(7) $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$

Even though (7) is valid in **S5**$\mathcal{A}$, there is a certain intuitive sense in which it seems false.

For take any sentence that is actually true such as 'grass is green'. Then, according to (7), 'grass is actually green' is a necessary truth, even though it is contingent whether grass is actually green. This feature of $\mathcal{A}$ is, in effect, what motivated Crossley and Humberstone to add a fixedly operator to **S5**$\mathcal{A}$, which in turn has the effect of varying the actual world of the models, thereby making it a contingent matter which world turns out to be actual.

a formula resembling (7) is obtained when we add @ to $\mathcal{L}$, for the following now turns out to be valid:

(8) $@\varphi \supset \mathcal{D}@\varphi$

Notwithstanding its validity, (8) is intuitively false. For it says of anything that is distinguishedly actually true, or *d*-actual, that it is a priori that it is *d*-actual. Thus, given that water is *d*-actually H_20, it follows that it is a priori knowable that water is *d*-actually H_20. However, this is clearly not the kind of thing we can know a priori. Furthermore, since the converse of (8) is obviously valid, the semantics ends up licensing the following material equivalence:

(9) $@\varphi \leftrightarrow \mathcal{D}@\varphi$.

This, moreover, is also relevant to Davies and Humberstone's claim about not having noticed "any examples of truths expressed in terms of '$\Box$', '$\mathcal{F}$', and '$\mathcal{A}$' which are a priori knowable and also not deeply necessary." (p. 10) But adding @ to $\mathcal{L}$ immediately results in one:

(10) $@\varphi \supset \varphi$

Despite the fact that 'water is *d*-actually H_20' is not a priori knowable, the conditional 'if water is *d*-actually H_20, then water is H_20' seems to be a priori knowable. But in order to see that this does not hold along the diagonal, just let $\langle w*, w*\rangle \in V(p)$ and $\langle v, v\rangle \notin V(p)$, in which case $\mathcal{M}_v^v \vDash @p$ but $\mathcal{M}_v^v \nvDash p$, resulting in $\mathcal{M}_v^v \nvDash @p \supset p$. True, this is not expressible just in terms of $\Box$, $\mathcal{F}$, and $\mathcal{A}$, but neither is (3*). Hence, this leaves us with a dilemma: either one ignores the expressive deficit of $\mathcal{L}$, or one enriches it with a distinguished actuality operator. In any case, the language turns out to be inadequate as a formal rendering of the a priori.

Another interesting issue about (10) concerns its status as a logical truth. As we have already pointed out, there are philosophical motivations to endorse a local account of validity. In modal semantics, such is, in fact, the orthodox view, for this is the definition of validity appearing in Kripke's (1963b) development of modal semantics. Even though there is no important distinction to be made between local and general accounts of validity with respect to basic modal languages, since those turn out to be equivalent, it is well-known that once we add an actuality operator this equivalence is severed. Zalta (1988) argues that there are contingent logical truths, since $\mathcal{A}\varphi \supset \varphi$ is locally but not generally valid, and he favours a local account of validity. We do not find contingent logical truths particularly appealing. However, by admitting local validity, we have contingent logical truths, viz. (10), that are not a priori knowable, since $\mathcal{D}(@\varphi \supset \varphi)$ is not locally valid. Of course, one could take logic to be an a posteriori enterprise altogether, but we confess being unsure whether this is an esteemed company for two-dimensional semantics. In any case, we shall refrain from announcing the discovery of a posteriori logical truths.

3.4 Plural Quantification

Is there any other way to formalize (3) without the need of a distinguished actuality operator? Bricker (1989) suggested using plural quantification in order to formalize sentences such as (1) in first-order **S5**.[32] This seems to be the best approach so far avoiding the addition of

[32] A similar proposal can be found in Forbes (1989, pp. 93–102). Nonetheless, the first time a proposal like this one appeared in the literature seems to be in Humberstone (1982, p. 2), with the only difference

new operators into basic modal languages, whence it is pertinent to investigate whether it can be generalized to a two-dimensional modal language. A two-dimensional modal language with plural quantifiers, $\mathcal{L}_{pl}$, will be just like $\mathcal{L}$, except that it is two-sorted, for now we add *plural variables* $\{x_1x_1, x_2x_2, ...\}$ and *plural constant symbols* $\{c_1c_1, c_2c_2, ...\}$, both of which we call *plural terms*, denoted by tt, as well as a two-place predicate, $\prec$, relating a single term to a plural one. Quantification over plural variables, $\exists x_ix_i\varphi$, is read as *there are some things such that...*,[33] and $y \prec xx$ is read as *y is one of xx's.*

The model theory should also be modified appropriately. A constant domain 2D-centered model for plural quantification is a tuple, $\mathcal{M} = \langle W, w*, \mathcal{R}_{\Box}, \mathcal{R}_{\mathcal{D}}, \mathscr{D}, \mathscr{D}*, V\rangle$ where everything is just as in Definition 3.1.2 except that we add another domain, $\mathscr{D}*$, for the plural variables, where $\mathscr{D}*$ is a set of non-empty subsets of $\mathscr{D}$. Moreover, V also assigns pluralities to each plural constant cc of $\mathcal{L}_{pl}$, $V(cc) \in \mathscr{D}*$. Regarding the semantics, we just need to consider the new atomic formula $y \prec xx$ and the plural quantifiers:

$$\mathcal{M}^v_w \vDash t \prec tt' \quad \text{iff} \quad V(t) \in V(tt');$$

$$\mathcal{M}^v_w \vDash \exists xx\varphi \quad \text{iff} \quad \text{for some } cc \text{ of } \mathcal{L}(\mathcal{M}), \mathcal{M}^v_w \vDash \varphi[cc/xx];$$

Finally, we define $\forall x_ix_i\varphi$ as $\neg\exists\neg x_ix_i\varphi$. A treatment of plural quantification along these lines added to a basic modal language allows us to regiment (1) as follows:

(11) $\exists xx(\forall y(y \prec xx \leftrightarrow Ry) \wedge \Diamond\forall y(y \prec xx \supset Sy))$

Accordingly, the plural quantifier $\exists xx$ in (11) should be taken as ranging "neither over sets, nor classes, nor properties; it ranges in an irreducibly plural way over the [red things] themselves." (Bricker (1989, p. 389)) And it is true that (11) delivers the correct truth conditions for (1).[34] Now, it is straightforward to generalize this proposal for the two-

that he used explicit quantification over sets.

[33] This interpretation comes from Boolos (1984).

[34] We should point out that Bricker's formalization depends on his assumption that $y \prec xx$ may hold at

dimensional language, for the apparatus of plural quantification seems to give the correct result for (3) as well:

(12) $\exists xx(\forall y(y \prec xx \leftrightarrow Ry) \wedge \neg\mathcal{D}\exists y(y \prec xx \wedge Sy))$

Since the existential plural quantifier does not occur within the scope of $\mathcal{D}$ or any modal operator, it is evaluated with respect to the distinguished element of the model. Consequently, any witness of the single quantifier $\exists y$ is an object that is red at $\langle w*, w*\rangle$, which is exactly what we would get had we use @ rather than plural quantification. This indicates just how powerful plural quantification can be when dealing with scope issues in a modal language. Also, it suggests a certain elegance to the original proposal of adding plural quantifiers to modal languages. After all, (3) can be seen as mere iteration in a two-dimensional framework of the expressibility problem illustrated by (1) in **S5**, while the very same solution is available for the two cases.

The first difficulty with this proposal, however, is the move from seemingly unproblematic *de dicto* attributions of a priori knowledge to *de re* attributions. For (12) in effect says that

(13) There are red things such that it is not a priori that one of them is shiny,

and so the apriority operator occurs within the scope of a plural quantifier, whereby the plural version of (3) is *de re*, in contrast with the *de dicto* occurrence of $\mathcal{D}$ in (3). Thus, although (12) might get the desired truth conditions, it might nevertheless be unsatisfying as a formalization of (3) given its commitment to an apparent *de re* ascription of a priori knowledge. On the face of it, however, both (13) and (3) involve no attributions of a priori knowledge at all, but the lack thereof, whence this objection can be dismissed as a

world at which the objects assigned to the plural variables do not exist. (See p. 387) We run along with Bricker on this, but only to illustrate how plural quantifiers do not solve the problem for the two-dimensional case even on such controversial grounds.

non-starter.[35] Nevertheless, a constant domain for the single quantifiers validates a two-dimensional version of the Barcan Formula and its converse, which does involve *de re* a priori knowledge ascriptions. In particular, the *2D Converse Barcan Formula* is as follows:

(14) $\mathcal{D}\forall x\varphi \supset \forall x\mathcal{D}\varphi$

It is easy to check that any instance of (14) is valid in 2D-centered models.[36] However, cases of *de re* knowledge involve, presumably, a relation of acquaintance between the knower and the object of knowledge. But it is far from clear how this can be made consistent with cases of a priori knowledge. Nonetheless, (14) shows that *de re* a priori knowledge ascriptions arise independently of plural quantification. The natural solution in order to avoid this — at least in this case — would be to assume a variable domain semantics for the quantifiers, but this will also cause problems for sentences like (12), for varying the domain of the quantifiers might incur loss of objects while changing from world to world. In (12), we need the objects at the pair of worlds introduced by the apriority operator to be red at the distinguished point of the model. But there will be no guarantee that there will be objects in the domains of both the distinguished point and the relevant pair of possible worlds in a varying domain semantics. This is a reason for why a constant domain semantics was always preferable in order to formalize sentences such as (1) and (3).[37]

A word is in order, too, with respect to the nature of the semantics specified for the plural quantifiers. As defined above, the truth of an atomic formula $y \prec xx$ relative to

[35]The move from *de dicto* to *de re* knowledge ascriptions has been defended by Soames (2004) on the basis of the following exportation principle — understood relative to an assignment function and pair of possible worlds:

(E) For any name n and predicate F, if 'a knows/believes that n is F' is true, then 'a knows/believes that $[x/n]$ is F' is true.

Soames takes the above principle to be "intuitively compelling," (p. 261) which seems to be the only support offered in favour of (E). Yet, some two-dimensionalists — Chalmers, in particular — deny that (E) has any plausibility whatsoever. See Chalmers (2011, pp. 630–633) for counterexamples to (E) based on knowledge attributions. A more recent defense of *de re* a priori knowledge can nevertheless be found in Dorr (2011).

[36]Ditto for the corresponding *2D-Barcan formula*.

[37]For example, see Humberstone (1982, p. 13).

an assignment is given independently of pairs of possible worlds. This is just the usual model theory for plural quantification in the modal case,[38] whereby plural terms are given a purely extensional interpretation.[39] In **S5** endowed with plural quantification, this will validate the following two principles:

$$(\Box \prec)\ \forall y \forall xx(y \prec xx \leftrightarrow \Box y \prec xx)$$

$$(\Box \not\prec)\ \forall y \forall xx(\neg y \prec xx \leftrightarrow \Box \neg y \prec xx)$$

For any objects, if something is one of those, then it is necessarily one of those. And since our two-dimensional modal logic extends **S5**, such principles are also valid in the two-dimensional case. Their validity is easy to check given the extensional character of plural terms — they are interpreted rigidly in the same manner as set membership, and so the truth of $y \prec xx$ becomes "world-invariant."[40] This yields the non-contingency of $\prec$:[41]

$$(\text{NC}{\prec})\ \forall y \forall xx(\Box y \prec xx \vee \Box \neg y \prec xx),$$[42]

For any objects, something is either necessarily one of them or necessarily not one of them. Now, whether the principles $(\Box \prec)$ and $(\Box \not\prec)$ should be validated by one's favourite modal logic is something which we shall not discuss, although this has been defended on philosophical grounds.[43] However, in a two-dimensional framework, the following two principles are also valid:

$$(\mathcal{D} \prec)\ \forall y \forall xx(y \prec xx \leftrightarrow \mathcal{D} y \prec xx)$$

$$(\mathcal{D} \not\prec)\ \forall y \forall xx(\neg y \prec xx \leftrightarrow \mathcal{D} \neg y \prec xx)$$

[38]See, for example, Uzquiano (2011, p. 225) and Williamson (2003, pp. 456–7).
[39]It would be the same if we were quantifying over sets rather than plurally.
[40]See Uzquiano (2011, p. 225).
[41]This is also defended by Bricker (1989, p. 387) and Forbes (1985, p. 109).
[42]See also Williamson (2013a, p. 248) for similar remarks.
[43]For recent discussion on this, see Uzquiano (2011), Williamson (2003, 2010, 2013a, pp. 241–254), and Linnebo (2016).

In our two-dimensional modal language, the evaluation of the open formula $y \prec xx$ is not just world-invariant, it is also pairwise world-invariant, whereby ($\mathcal{D} \prec$) and ($\mathcal{D} \not\prec$) are valid in the corresponding models. If $y \prec xx$ holds at any point in a 2D-centered model, then it holds on the diagonal as well.[44] But even though we might want both ($\Box \prec$) and ($\Box \not\prec$), ($\mathcal{D} \prec$) and ($\mathcal{D} \not\prec$) seem to deliver intuitively false principles. First of all, those are clear cases involving *de re* a priori knowledge ascriptions. Moreover, ($\mathcal{D} \prec$) says of any objects that if something is one of those, it is a priori that it is one of those. But why should we think that? George is one of the Cheerios lovers, but this is not a priori knowable even on a *de dicto* reading. Additionally, ($\mathcal{D} \prec$) and ($\mathcal{D} \not\prec$) also give us the a priori analogue of (NC$\prec$), whence the relation denoted by $\prec$ is not *a posteriori*:

(NP$\prec$) $\forall y \forall xx(\mathcal{D}y \prec xx \vee \mathcal{D}\neg y \prec xx)$

But it is not a priori of Frege that he is one of the philosophers, and it is not a priori of him that he is not. Alternatively, one could read (NP$\prec$) as attributing *de re* a priori knowledge of the plurality, but it is also not a priori of the philosophers that Frege is (or is not) one of them. A solution might be to develop an alternative model theory in which the truth of the atomic formula $y \prec xx$ is relativized to a pair of possible worlds just like the other atomic cases in the semantics. Consequently, we would be able to invalidate principles such as ($\mathcal{D} \prec$) and ($\mathcal{D} \not\prec$). The only semantic clause we would need to reconsider is the following:

$$\mathcal{M}^v_w \vDash t \prec tt' \quad \text{iff} \quad V(t, \langle v, w\rangle) \in V(tt', \langle v, w\rangle)$$

It makes sense to restrict, too, the valuation of plural constants to pairs of possible worlds, whereupon it imitates the valuation of the single case: $V(cc, \langle v, w\rangle) \in \mathscr{D}*$. This is sufficient to invalidate ($\mathcal{D} \prec$) and ($\mathcal{D} \not\prec$), since a counter-model for the former can be constructed by

[44] The proof is similar to the necessity of identity. For suppose that $\mathcal{M} \vDash y \prec xx$ for an assignment V. Where $\langle w, w\rangle$ is any pair of possible worlds in W such that $\langle w*, w*\rangle \mathcal{R}_{\mathcal{D}} \langle w, w\rangle$, since $V(y) \in V(xx)$, it follows that $\mathcal{M}^w_w \vDash y \prec xx$, whence $\mathcal{M} \vDash \mathcal{D}y \prec xx$. The other direction is obvious, and the argument for ($\mathcal{D} \not\prec$) is analogous.

letting $V(c, \langle w*, w*\rangle) \in V(cc, \langle w*, w*\rangle)$ and $V(c, \langle w, w\rangle) \notin V(cc, \langle w, w\rangle)$, for every single and plural constants c and cc, respectively, where $\langle w, w\rangle$ is any pair on the diagonal different from the distinguished one — a similar argument is available to falsify $(\mathcal{D} \not\prec)$. This model can be illustrated by the 2D matrix below:

$$\begin{pmatrix} & w* & w \\ w* & c \prec cc & c \prec cc \\ w & c \not\prec cc & c \not\prec cc \end{pmatrix}$$

It should be noted that we still have the correct truth conditions for (12) in this restricted semantics, as well as both $(\Box \prec)$ and $(\Box \not\prec)$ given the rigidity condition for $\mathcal{R}_\Box$-related worlds.

Would this be a good move for the two-dimensionalist? Even though it invalidates the problematic principles $(\mathcal{D} \prec)$ and $(\mathcal{D} \not\prec)$, restricting the truth conditions of $\prec$ formulas to a pair of worlds does nothing in order to avoid explicit *de re* a priori knowledge ascriptions delivered by the single quantifiers.[45] Yet, plural quantification brings with it even more of those cases. Consider, for instance, the uncontroversial principle that for any objects, something is one of them:

(15) $\forall xx \exists y\, y \prec xx$

One can check that (15) is valid in any constant domain 2D-centered model with plural quantifiers since the domains $\mathscr{D}$ and $\mathscr{D}*$ are non-empty. However, if (15) is locally valid, i.e.

[45] Another possible move is to use full quantification over sets, rather than pluralities. This is the strategy adopted by Meyer (2013) in order to eliminate the actuality operator in quantified modal logic. However, the same issues arise concerning what is validated by the models, regardless of whether we assume sets or pluralities in the semantics.

true at the distinguished point of every model, then it is diagonally valid as well, since local and diagonal validity are equivalent for $\mathcal{L}$ and $\mathcal{L}_{pl}$. Therefore, the following is also valid:

(16) $\mathcal{D}\forall xx \exists y\, y \prec xx$

This is an instance of a two-dimensional analogue of the rule of *necessitation* in basic modal logics:

$$\text{Diagonalization} \quad \frac{\varphi}{\mathcal{D}\varphi}$$

If a formula, φ, is valid, then $\mathcal{D}\varphi$ is valid too.[46] But given the plural version of the 2D Converse Barcan Formula, which is valid notwithstanding the restriction on the relation $\prec$, one can deduce (17) by truth-functional reasoning:

(17) $\forall xx \mathcal{D}\exists y\, y \prec xx$

However, (17) seems to be quite objectionable. For it is not a priori of someone that he or she is amongst the Cheerios lovers, or of the Cheerios lovers that someone is one of them. We take this to be a high price for two-dimensionalists to pay. Unless they can show such principles to be reasonable, plural quantifiers do not seem to offer an adequate solution for the expressive deficit of the language. Quantification seems to be, again, the Achilles' heel of modal semantics.

[46] Again, this inference rule is valid for any sentence unless @ is in the language.

Appendix

3.A Proof of the Expressive Incompleteness of $\mathcal{L}$

Since bisimulation implies model equivalence, we show that there is a bisimulation between two constant domain 2D-centered models differing with respect to sentence (3). But first some notation is in order. Let $\vec{c}$ denote a sequence of objects or terms of $\mathcal{L}$, where its length is denoted by $\|\vec{c}\|$. The nth-member of $\vec{c}$ is denoted by $\vec{c}_n$. Furthermore, let $\varphi|\vec{x}|$ be a formula whose free variables are all in $\vec{x}$, and whenever $\|\vec{c}\| = \|\vec{x}\|$, let $\varphi|\vec{c}|$ be the result of appropriately substituting constants in $\vec{c}$ for variables in $\vec{x}$. Also, we use $\langle v, w\rangle$ for members of W_1 and $\langle y, z\rangle$ for members of W_2, as well as c for members of $\mathscr{D}_1$ and d for members of $\mathscr{D}_2$. We omit any mention of $\mathcal{M}$ in the evaluation clauses by writing ${}^{v}_{w} \vDash_1 \varphi$ whenever φ holds at $\langle v, w\rangle$ in $\mathcal{M}_1$ — similarly for $\mathcal{M}_2$.

Next we define the general notion of a world-object bisimulation for constant domain 2D-centered models:

Definition 3.A.1 (World-Object Bisimulation) Let $\mathcal{N}_1 = \langle W_1, w*, \mathcal{R}_{\Box 1}, \mathcal{R}_{\mathcal{D}1}, \mathscr{D}_1, V_1\rangle$ and $\mathcal{N}_2 = \langle W_2, v*, \mathcal{R}_{\Box 2}, \mathcal{R}_{\mathcal{D}2}, \mathscr{D}_2, V_2\rangle$ be two constant domain 2D-centered models. A *world-object bisimulation* between $\mathcal{N}_1$ and $\mathcal{N}_2$ is a non-empty relation $\cong \subseteq (W_1 \times \mathscr{D}_1) \times (W_2 \times \mathscr{D}_2)$ such that $\langle w*, w*\rangle\vec{c} \cong \langle w*, w*\rangle\vec{d}$, satisfying the following conditions:

1. $(\langle v,w\rangle\vec{c}\cong\langle y,z\rangle\vec{d})\Rightarrow({}^{v}_{w}\vDash_1 P^n_i\vec{c}\Leftrightarrow{}^{y}_{z}\vDash_2 P^n_i\vec{d})$, for every predicate symbol P^n_i (the *atomic* condition);

2. $(\langle v,w\rangle\vec{c}\cong\langle y,z\rangle\vec{d})\Rightarrow\forall n,m\leq\|\vec{c}\|({}^{v}_{w}\vDash_1 \vec{c}_n=\vec{c}_m\Leftrightarrow{}^{y}_{z}\vDash_2 \vec{d}_n=\vec{d}_m)$ (the *identity* condition);

3. $((\langle v,w\rangle\vec{c}\cong\langle y,z\rangle\vec{d})\wedge\langle y,z\rangle\mathcal{R}_{\Box 2}\langle y,z'\rangle)\Rightarrow(\exists\langle v,w'\rangle\in W_1:\langle v,w\rangle\mathcal{R}_{\Box 1}\langle v,w'\rangle\wedge(\langle v,w'\rangle\vec{c}\cong\langle y,z'\rangle\vec{d}))$ (the $\mathcal{R}_\Box$-forth condition);

4. $((\langle v,w\rangle\vec{c}\cong\langle y,z\rangle\vec{d})\wedge\langle v,w\rangle\mathcal{R}_{\Box 1}\langle v,w'\rangle)\Rightarrow(\exists\langle y,z'\rangle\in W_2:\langle y,z\rangle\mathcal{R}_{\Box 2}\langle y,z'\rangle\wedge(\langle v,w'\rangle\vec{c}\cong\langle y,z'\rangle\vec{d}))$ (the $\mathcal{R}_\Box$-back condition);

5. $(\langle v,w\rangle\vec{c}\cong\langle y,z\rangle\vec{d})\Rightarrow(\langle v,v\rangle\vec{c}\cong\langle y,y\rangle\vec{d})$ (the *actuality* condition);

6. $((\langle v,w\rangle\vec{c}\cong\langle y,z\rangle\vec{d})\wedge\langle y,z\rangle\mathcal{R}_{\mathcal{D}2}\langle y',y'\rangle)\Rightarrow(\exists\langle v',v'\rangle\in W_1:\langle v,w\rangle\mathcal{R}_{\mathcal{D}1}\langle v',v'\rangle\wedge(\langle v',v'\rangle\vec{c}\cong\langle y',y'\rangle\vec{d}))$ (the $\mathcal{R}_\mathcal{D}$-forth condition);

7. $((\langle v,w\rangle\vec{c}\cong\langle y,z\rangle\vec{d})\wedge\langle v,w\rangle\mathcal{R}_{\mathcal{D}1}\langle v',v'\rangle)\Rightarrow(\exists\langle y',y'\rangle\in W_2:\langle y,z\rangle\mathcal{R}_{\mathcal{D}2}\langle y,y'\rangle\wedge(\langle v',v'\rangle\vec{c}\cong\langle y',y'\rangle\vec{d}))$ (the $\mathcal{R}_\mathcal{D}$-back condition);

8. $((\langle v,w\rangle\vec{c}\cong\langle y,z\rangle\vec{d})\wedge d\in\mathscr{D}_2)\Rightarrow(\exists c\in\mathscr{D}_1:\langle v,w\rangle\vec{c},c\cong\langle y,z\rangle\vec{d},d)$ (the *quantifier*-forth condition);

9. $((\langle v,w\rangle\vec{c}\cong\langle y,z\rangle\vec{d})\wedge c\in\mathscr{D}_1)\Rightarrow(\exists d\in\mathscr{D}_2:\langle v,w\rangle\vec{c},c\cong\langle y,z\rangle\vec{d},d)$ (the *quantifier*-back condition);

Conditions 1, 3, and 4 are standard for both atomic and modal formulas, although they appear here under a two-dimensional framework. Similarly, 8 and 9 are usual for constant domain quantifiers.[47] One can find conditions 2 and 5 for (unrestricted) identity and actuality in Kocurek (2018), which also presents conditions for the fixedly operator. Our conditions 6 and 7 for $\mathcal{R}_\mathcal{D}$ are defined in order to handle the $\mathcal{D}$ operator.

[47]See, for instance, van Benthem (2010) and, more recently, Urquhart (2016).

Lemma 3.A.1 (Invariance) *For any constant domain 2D-centered models* $\mathcal{N}_1$ *and* $\mathcal{N}_2$*, if* $\cong$ *is a world-object bisimulation between* $\mathcal{N}_1$ *and* $\mathcal{N}_2$*,* $\langle v, w\rangle \in W_1$*,* $\langle y, z\rangle \in W_2$*, and* $\varphi|\vec{x}|$ *is a formula,*

$$(\langle v, w\rangle\vec{c} \cong \langle y, z\rangle\vec{d}) \Rightarrow ({}^{v}_{w}\vDash_1 \varphi|\vec{c}| \Leftrightarrow {}^{y}_{z} \vDash_2 \varphi|\vec{d}|).$$

Proof. By induction on $\varphi|\vec{x}|$. The atomic cases including identity hold by construction given conditions 1 and 2, while the truth-functional cases are straightforward.

Let $\varphi|\vec{x}|$ be $\Box\varphi|\vec{x}|$. Suppose that $\langle v, w\rangle\vec{c} \cong \langle y, z\rangle\vec{d}$ and ${}^{v}_{w} \vDash_1 \Box\varphi|\vec{c}|$. If $\langle y, z\rangle\mathcal{R}_{\Box 2}\langle y, z'\rangle$, then $\exists\langle v, w'\rangle \in W_1$ such that $\langle v, w\rangle\mathcal{R}_{\Box 1}\langle v, w'\rangle$ and $\langle v, w'\rangle\vec{c} \cong \langle y, z'\rangle\vec{d}$, by condition 3. Thus, ${}^{v}_{w'} \vDash_1 \varphi|\vec{c}|$. By the induction hypothesis, ${}^{y}_{z'} \vDash_2 \varphi|\vec{d}|$. Therefore, ${}^{y}_{z} \vDash_2 \Box\varphi|\vec{d}|$. The other direction is analogous given condition 6.

Let $\varphi|\vec{x}|$ be $\mathcal{A}\varphi|\vec{x}|$. Suppose that $\langle v, w\rangle\vec{c} \cong \langle y, z\rangle\vec{d}$ and ${}^{v}_{w} \vDash_1 \mathcal{A}\varphi|\vec{c}|$. Thus, ${}^{v}_{v} \vDash_1 \varphi|\vec{c}|$, by the semantics of $\mathcal{A}$. By condition 5, $\langle v, v\rangle\vec{c} \cong \langle y, y\rangle\vec{d}$, whence ${}^{y}_{y} \vDash_2 \varphi|\vec{d}|$, by the induction hypothesis. Therefore, ${}^{y}_{z} \vDash_2 \mathcal{A}\varphi|\vec{d}|$, by the semantics of $\mathcal{A}$. The other direction is analogous.

Let $\varphi|\vec{x}|$ be $\mathcal{D}\varphi|\vec{x}|$. Suppose that $\langle v, w\rangle\vec{c} \cong \langle y, z\rangle\vec{d}$ and ${}^{v}_{w} \vDash_1 \mathcal{D}\varphi|\vec{c}|$. If $\langle y, z\rangle\mathcal{R}_{\mathcal{D}2}\langle y', y'\rangle$, then $\exists\langle v', v'\rangle \in W_1$ such that $\langle v, w\rangle\mathcal{R}_{\mathcal{D}1}\langle v', v'\rangle$ and $\langle v', v'\rangle\vec{c} \cong \langle y', y'\rangle\vec{d}$, by condition 6. Thus, ${}^{v'}_{v'} \vDash_1 \varphi|\vec{c}|$. By the induction hypothesis, ${}^{y'}_{y'} \vDash_2 \varphi|\vec{d}|$. Therefore, ${}^{y}_{z} \vDash_2 \mathcal{D}\varphi|\vec{d}|$. The other direction is analogous given condition 7.

Let $\varphi|\vec{x}|$ be $\exists y\varphi|\vec{x}, y|$. Suppose that $\langle v, w\rangle\vec{c} \cong \langle y, z\rangle\vec{d}$ and ${}^{v}_{w} \vDash_1 \exists y\varphi|\vec{c}, y|$. Thus, $\exists c \in \mathscr{D}_1$ such that ${}^{v}_{w} \vDash_1 \varphi|\vec{c}, c|$. By condition 9, $\exists d \in \mathscr{D}_2$ such that $\langle v, w\rangle\vec{c}, c \cong \langle y, z\rangle\vec{d}, d$. By the induction hypothesis, ${}^{y}_{z} \vDash_2 \varphi|\vec{d}, d|$, whence ${}^{y}_{z} \vDash_2 \exists y\varphi|\vec{d}, y|$. The other direction is analogous given condition 8. □

We define our two models as follows.[48] Let $\mathcal{M}_1 = \langle W_1, w*, \mathcal{R}_{\Box 1}, \mathcal{R}_{\mathcal{D}1}, \mathscr{D}_1, V_1\rangle$ be such that

[48]The models are constructed resembling the ones in Wehmeier (2001), where he proves a similar result

W_1 is the set of all pairs of subsets of $\mathbb{N}$ that are both infinite and coinfinite containing at least one odd number, $w*$ is the set $2\mathbb{N}+1$ of odd numbers, $\mathscr{D}_1$ is $\mathbb{N}$, for each $\langle i, j\rangle \in W_1$ let $V_1(R, \langle i, j\rangle) = V_1(S, \langle i, j\rangle) = i \cup j$, and let the accessibility relations be defined just as in Definition 3.1.2. The extension of every other predicate symbol is empty. On the other hand, our second model, $\mathcal{M}_2 = \langle W_2, w*, \mathcal{R}_{\Box 2}, \mathcal{R}_{\mathcal{D}2}, \mathscr{D}_2, V_2\rangle$, is just like $\mathcal{M}_1$ except for $W_2 = W_1 \cup \{\langle 2\mathbb{N}, 2\mathbb{N}\rangle\}$, where $2\mathbb{N}$ is the set of even numbers, and $V_2(R, \langle 2\mathbb{N}, 2\mathbb{N}\rangle) = V_2(S, \langle 2\mathbb{N}, 2\mathbb{N}\rangle) = 2\mathbb{N} \cup 2\mathbb{N}$. No other predicate has an extension defined on $\langle 2\mathbb{N}, 2\mathbb{N}\rangle$.

In order to prove that $\mathcal{M}_1$ and $\mathcal{M}_2$ are bisimilar, let $\langle v, w\rangle \in W_1$ and $\langle v', w'\rangle \in W_2$ be corresponding pairs of worlds in the models $\mathcal{M}_1$ and $\mathcal{M}_2$. For any $\langle v, w\rangle \in W_1$ and $\langle v', w'\rangle \in W_2 - \{\langle 2\mathbb{N}, 2\mathbb{N}\rangle\}$, say that the mapping ρ from W_1 onto $W_2 - \{\langle 2\mathbb{N}, 2\mathbb{N}\rangle\}$ is an *isomorphism* between $\mathcal{M}_1$ and the submodel of $\mathcal{M}_2$ defined on $W_2 - \{\langle 2\mathbb{N}, 2\mathbb{N}\rangle\}$ such that $\rho(c) = d$, in which case for $\langle v, w\rangle \in W_1$ and $\langle v', w'\rangle \in W_2$ other than $\langle 2\mathbb{N}, 2\mathbb{N}\rangle$, and for any non-empty predicate symbol P_i^n, we have

$$(c \in V_1(P_i^n, \langle v, w\rangle)) \Leftrightarrow (\rho(c) \in V_2(P_i^n, \langle v', w'\rangle)).$$

Moreover, we define the following relations between $\mathcal{M}_1$ and $\mathcal{M}_2$. For any $\langle v, w\rangle \in W_1$ and $\langle v', w'\rangle \in W_2$ other than $\langle 2\mathbb{N}, 2\mathbb{N}\rangle$, set

$$(\langle v, w\rangle \vec{c} \cong \langle v', w'\rangle \vec{d}) \Leftrightarrow (\rho(\vec{c}) = \vec{d}).$$

For the actuality condition, let

$$(\langle v, w\rangle \vec{c} \cong \langle v', w'\rangle \vec{d}) \Leftrightarrow (\langle v, v\rangle \vec{c} \cong \langle v', v'\rangle \rho(\vec{c})).$$

In the case of the extra pair of worlds $\langle 2\mathbb{N}, 2\mathbb{N}\rangle \in W_2$, we define

for a first-order modal language.

$$(\langle v,v\rangle\vec{c}\cong\langle 2\mathbb{N},2\mathbb{N}\rangle\vec{d})\Leftrightarrow\forall n\leq\|\vec{c}\|(\vec{c}_n\in(V_1(S,\langle v,v\rangle))\;\Leftrightarrow\;(\vec{d}_n\in V_2(S,\langle 2\mathbb{N},2\mathbb{N}\rangle))),$$

and

$$(\langle v,v\rangle\vec{c}\cong\langle 2\mathbb{N},2\mathbb{N}\rangle\vec{d})\Leftrightarrow\forall n\leq\|\vec{c}\|(\vec{c}_n\in(V_1(R,\langle v,v\rangle))\;\Leftrightarrow\;(\vec{d}_n\in V_2(R,\langle 2\mathbb{N},2\mathbb{N}\rangle))).$$

Finally, set

$$(\langle v,w\rangle\vec{c}\cong\langle 2\mathbb{N},2\mathbb{N}\rangle\vec{d})\Leftrightarrow(\langle v,v\rangle\vec{c}\cong\langle 2\mathbb{N},2\mathbb{N}\rangle\vec{d}).$$

Lemma 3.A.2 $\cong$ *is a bisimulation between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$.

Proof. Conditions 1, 2, and 5 hold by construction. Since $\langle 2\mathbb{N},2\mathbb{N}\rangle$ is not $\mathcal{R}_\Box$-accessible with respect to any pair of worlds, conditions 3 and 4 are easily seen to be met as well.

For conditions 6 and 7, the only cases we need to consider involve $\langle 2\mathbb{N},2\mathbb{N}\rangle$. Suppose $\langle v,w\rangle\vec{c}\cong\langle v',w'\rangle\vec{d}$ and that $\langle v',w'\rangle\mathcal{R}_{\mathcal{D}2}\langle 2\mathbb{N},2\mathbb{N}\rangle$. Since $\vec{c}$ contains a single element, we have only two cases. If $\vec{c}$ is even, choose a pair $\langle v,w\rangle\mathcal{R}_{\mathcal{D}1}\langle y,y\rangle$ such that $\vec{c}\in V_1(S,\langle y,y\rangle)$. By the definition of $\mathcal{M}_1$, we know that there will be such a pair, in which case we have $\langle y,y\rangle\vec{c}\cong\langle 2\mathbb{N},2\mathbb{N}\rangle\vec{d}$, by construction. If $\vec{c}$ is odd, choose a pair $\langle v,w\rangle\mathcal{R}_{\mathcal{D}1}\langle z,z\rangle$ such that $\vec{c}\notin V_1(S,\langle z,z\rangle)$, whence $\vec{d}\notin V_2(S,\langle 2\mathbb{N},2\mathbb{N}\rangle)$. Therefore, $\langle z,z\rangle\vec{c}\cong\langle 2\mathbb{N},2\mathbb{N}\rangle\vec{d}$. Condition 7 is analogous.

With respect to conditions 8 and 9, again, we only need to check the cases involving the extra pair $\langle 2\mathbb{N},2\mathbb{N}\rangle$. Suppose that $\langle v,w\rangle\vec{c}\cong\langle 2\mathbb{N},2\mathbb{N}\rangle\vec{d}$ and that $c\in\mathscr{D}_1$. We only have cases involving the predicates S and R, since all the other predicates are empty. If $c\in V_1(S,\langle v,v\rangle)$, then $\langle v,w\rangle\vec{c},c\cong\langle 2\mathbb{N},2\mathbb{N}\rangle\vec{d},d$, by construction, since $V_2(S,\langle 2\mathbb{N},2\mathbb{N}\rangle)$ is not empty. If $c\notin V_1(S,\langle v,v\rangle)$, then choose any $d\in V_2(S,\langle w*,w*\rangle)$, in which case $d\notin V_2(S,\langle 2\mathbb{N},2\mathbb{N}\rangle)$, and then we have $\langle v,w\rangle\vec{c},c\cong\langle 2\mathbb{N},2\mathbb{N}\rangle\vec{d},d$. The argument for R is very similar. Condition 8 is analogous. $\Box$

Theorem 3.A.1 *There is no sentence φ of $\mathcal{L}$ such that for every constant domain 2D-centered model $\mathcal{M} = \langle \mathrm{W}, w*, \mathcal{R}_{\Box}, \mathcal{R}_{\mathcal{D}}, \mathscr{D}, V\rangle$, $\mathcal{M} \vDash \varphi$ if and only if there is a pair of possible worlds $\langle w, w\rangle \in W$, such that for two predicate symbols R and S, $V(R, \langle w*, w*\rangle) \cap V(S, \langle w, w\rangle) = \varnothing$.*

Proof. By construction of the models, in $\mathcal{M}_2$ there is a pair $\langle 2\mathbb{N}, 2\mathbb{N}\rangle \in W_2$ such that $V_2(R, \langle w*, w*\rangle) \cap V_2(S, \langle 2\mathbb{N}, 2\mathbb{N}\rangle) = \varnothing$, but in $\mathcal{M}_1$ there is no pair of worlds $\langle w, w\rangle \in W_1$ such that $V(R, \langle w*, w*\rangle) \cap V(S, \langle w, w\rangle) = \varnothing$, since $V_1(R, \langle w*, w*\rangle)$ contains only odd numbers, and for any $\langle w, w\rangle \in W_1$, $V_1(S, \langle w, w\rangle)$ contains at least one odd number for each coordinate of the pair $\langle w, w\rangle$. □

Chapter 4

The Logic of Sequence Frames

Introduction

Two-dimensional modal logics have been the object of increasing philosophical interest given intuitive interpretations assigned to the modal operators. In particular, that such logics provide us with a logical analysis of metaphysical necessity, actuality, and a priori knowledge is now an important facet of prominent philosophical views in philosophy of language, epistemology, and metaphysics. Logics developed with the purpose of shedding light on a priori reasoning and its relation with the modal notions of necessity and actuality arguably originated with Davies and Humberstone (1980), and have recently been investigated by Restall (2012), Fritz (2013, 2014), Fusco (forthcoming), and others. Furthermore, the semantic treatment of epistemic and indexical terms in the works of Evans (1979), Kaplan (1989), and Chalmers (2004, 2014) make essential use of elements present in two-dimensional modal logics, while Weatherson (2001) and Wehmeier (2013) adopt similar two-dimensional semantics to provide formal treatments of both subjunctive and indicative conditionals.

The goal of this chapter is to generalize two-dimensional modal logics with actuality operators to *any finite dimension*. To be more precise, the logics investigated here are generalizations of the logic for epistemic two-dimensional semantics studied by Fritz (2014) under the name **2Dg**, which in a certain sense is the same logic as those investigated proof-theoretically in Restall (2012) and chapter 2—although the logics in chapter 2 contain first-order quantifiers. The formal language defined here contains several modal operators, one for each dimension, including indexed boxes, $\Box_i$, as well as actuality operators, $@_i$. As will be seen below, the resulting logics can be characterized as logics of generalized diagonal sequences, and therefore as natural generalizations of two-dimensional modal logics in which formulas in the scope of certain operators are evaluated at the diagonal points of *square models*, that is, models based on frames containing ordered pairs of worlds. Notwithstanding the formal character and scope of this chapter, whose principal concern is with model- and proof-theoretic investigations of certain classes of frames and models for modal logics, the development of n-dimensional modal logics is in fact philosophically motivated and arises from considerations regarding the expressive power of modal languages for necessity, actuality, and apriority. In what follows we briefly describe some of these motivations as well as the plan for the rest of the chapter.

Background

Crossley and Humberstone (1977) introduced the actuality operator, @, to remedy an expressive deficit in the first-order modal language,[1] i.e. the language of first-order logic augmented by the modal operators $\Box$ and $\Diamond$. What they observed was that there is no formula in that language that expresses the following truth condition:

(1) $\exists w \forall \mathsf{x}(R\mathsf{x}\text{-at-}z \rightarrow S\mathsf{x}\text{-at-}w)$,

[1]See also Hazen (1976).

where z is the "actual world" of the underlying frame.[2] According to (1), there is a world w such that every object that is R in the actual world z is S in w. As Crossley and Humberstone (1977: 12) point out, there are readings of English sentences with truth conditions corresponding to (1), such as "It is possible for every red thing to be shiny". But the most obvious attempts to represent (1) in the object language all fail: $\Diamond\forall\mathsf{x}(R\mathsf{x} \to S\mathsf{x})$ expresses the truth condition $\exists w\forall\mathsf{x}(R\mathsf{x}\text{-at-}w \to S\mathsf{x}\text{-at-}w)$, $\forall\mathsf{x}\Diamond(R\mathsf{x} \to S\mathsf{x})$ the truth condition $\forall\mathsf{x}\exists w(R\mathsf{x}\text{-at-}w \to S\mathsf{x}\text{-at-}w)$ and, if we assume that truth in a model is defined as *real-world truth*,[3] i.e. truth at the actual world of the model—in which case formulas are initially evaluated at z—$\forall\mathsf{x}(R\mathsf{x} \to \Diamond S\mathsf{x})$ expresses the truth condition $\forall\mathsf{x}(R\mathsf{x}\text{-at-}z \to \exists w(S\mathsf{x}\text{-at-}w))$. None of these truth conditions are equivalent to (1). Yet, in the presence of the actuality operator (1) can be expressed by the formula $\Diamond\forall\mathsf{x}(@R\mathsf{x} \to S\mathsf{x})$. This is so because the semantic evaluation clause for the actuality operator invariably takes back the evaluation of a formula in its scope to the actual world of the frame. Let $\mathfrak{M} = (W, z, \mathcal{R}, \mathcal{D}, V)$ be a model for the first-order modal language containing @, where W is a set of possible worlds, $z \in W$, $\mathcal{R} \subseteq W \times W$, $\mathcal{D}$ is a domain of objects, and V is a valuation function defined as usual.[4] For any formula φ,

$$\mathfrak{M}, w \vDash @\varphi \iff \mathfrak{M}, z \vDash \varphi.$$ [5]

[2]It is common to notate the actual world as w^* (see, for instance, Crossley and Humberstone (1977) and Davies and Humberstone (1980)), or even as @, if another symbol is used for the actuality operator (see, for instance, Wehmeier (2014)).

[3]This terminology comes from Crossley and Humberstone (1977: 15), where *real-world validity* is defined as truth at the actual world of every model, and *general validity* as truth at every world of every model. The distinction is not without a difference: the formula $@\varphi \to \varphi$ for example, being real-world but not generally valid. Since, however, there can be worlds at which $@\varphi \to \varphi$ is false, real-world validity gives rise to contingent logical truths, a point explored by Zalta (1988) in his defense of real-world validity over general validity as the correct generalization of the Tarskian notion of logical truth for modal languages. A reply to Zalta can be found in Hanson (2006), but we also direct the reader to Nelson and Zalta (2012), French (2012), Hanson (2014), and Wehmeier (2014).

[4]We shall not be concerned with first-order models in this chapter apart from this section. For simplicity, though, one can just assume that the constant symbols in the first-order modal languages mentioned here are interpreted rigidly, even though this undermines somewhat the purpose of extending two-dimensional modal languages with the first-order quantifiers, where it is possible to distinguish, say, the metaphysical rigidity of proper names, subsumed under the constant symbols of the language, from their epistemic non-rigidity.

[5]For simplicity, we leave the assignment function implicit here.

So even if the formula $@R\mathsf{x}$ occurs within the scope of $\Diamond$, the semantic entry for @ makes us consider the objects in the domain of the world introduced by $\Diamond$ that are R in the actual world z, thereby, as it were, temporarily suspending the scope of the $\Diamond$ operator.

There is, however, a subtle difference in the semantic clause for @ when we move to a full-blooded two-dimensional modal logic.[6] In this case, formulas are evaluated against pairs of possible worlds, (w, v), where w is taken intuitively as a counterfactual world and v as a world *considered as actual* (or an *epistemic scenario*, such as in Fritz (2014)), and so the set of evaluation points W in the frames can now be defined as the Cartesian product of some non-empty underlying set of worlds, say, S. This means that rather than having a single fixed actual world as a part of the frame, any world can be actual as long as it occupies the second coordinate of the pair of worlds relative to which a formula is evaluated. That is, to use a metaphor from Humberstone (2004: 26), we first *dethrone* the single actual world from the identity of the frames or models, and formulate the semantic clause for @ as

$$\mathfrak{M}, (w, v) \vDash @\varphi \Longleftrightarrow \mathfrak{M}, (v, v) \vDash \varphi,^{7}$$

where v can be any element of S. Then the truth conditions for $\Box_1\varphi$, where $\Box_1$ is now used for the metaphysical necessity operator, can be given with the aid of its corresponding accessibility relation, $\mathcal{R}_{\Box_1}$, which relates a pair (w, v) to any pair (x, y) such that $y = v$. That is, the actual world is held fixed while the counterfactual world varies. Additionally, two-dimensional modal languages are equipped with a diagonal necessity operator, interpreted intuitively as an apriority operator, which is sometimes taken as a primitive operator in the language (Restall (2012), Fritz (2014), also see chapter 2), or defined in terms of other operators (Davies and Humberstone (1980)). Here we notate it as $\Box_2$, and its corresponding

[6]We distinguish the number of dimensions based on the evaluation points in the frames: if formulas are evaluated against a possible world w, the semantics is one-dimensional. By contrast, if formulas are evaluated against a pair (w, v) of possible worlds, we say that the semantics is two-dimensional; similarly for triples (w, v, u), and so on.

[7]This can be compared with Davies and Humberstone (1980: 4), although they use a different notation.

accessibility relation will be $\mathcal{R}_{\Box_2}$, which in turn relates a pair (w, v) to any pair (x, y) such that $x = y$. Intuitively, the semantic entry for the apriority operator then says that a formula is true a priori if and only if, no matter which world turns out to be actual, that formula remains true in that world. The reason why we call this a diagonal operator is because when models are defined with a domain of pairs of possible worlds the operator $\Box_2$ projects precisely along the pairs consisting of identical coordinates. This can be illustrated by distributing the pairs constructed out of possible worlds w, v, u in a 2-D matrix, in which the X axis contains counterfactual worlds and the Y axis worlds taken as actual. $\Box_2\varphi$ is then true if φ holds along the diagonal points of the matrix, as seen in Figure 1 below.

$$\begin{pmatrix} & w & v & u \\ w & \varphi & - & - \\ v & - & \varphi & - \\ u & - & - & \varphi \end{pmatrix}$$

Figure 4.0.1: 2-D matrix

It is possible to subsume the semantic clauses for $\Box_1$ and $\Box_2$ under a single schema as follows. Let σ and τ be sequences (of length n) of possible worlds. Then:

$$\text{(GB) } \mathfrak{M}, \sigma \vDash \Box_i\varphi \Longleftrightarrow \text{for every } \tau \in W, \text{ if } \sigma\mathcal{R}_{\Box_i}\tau, \text{ then } \mathfrak{M}, \tau \vDash \varphi,$$

where $\sigma\mathcal{R}_{\Box_i}\tau$ if and only if the first i coordinates in τ are all identical and σ and τ are identical beyond i, i.e. for all $j > i$, the jth coordinate of σ is the same as the jth coordinate of τ.[8] It is simple to check that (GB) delivers the correct semantic entries for both $\Box_1$- and $\Box_2$-formulas. And since there is no single distinguished point in the underlying frames anymore, we can add to them a set $D = \{(z, z) \mid z \in S\}$ of diagonal points, and define truth in a model relative to the elements of D, in which case D is the set of distinguished points of the underlying frame. This is by and large the approach used by Fritz (2014) to

[8]Such sequences are defined more carefully in Definition 4.1.2.

define a class of what he calls *matrix frames with distinguished elements* for the propositional two-dimensional language.

Of a first-order two-dimensional framework we may ask whether the following relative of the truth condition in (1) is expressible in it:

(1*) $\exists(w, z)\forall \mathsf{x}(R\mathsf{x}\text{-at-}(z, z) \rightarrow S\mathsf{x}\text{-at-}(w, z))$.

The answer is positive if @ is present in the language, for (1*) is expressed by the formula $\Diamond_1 \forall \mathsf{x}(@R\mathsf{x} \rightarrow S\mathsf{x})$. In fact, it would seem that standard proofs for the analogous inexpressibility result in the one-dimensional setting would transfer, more or less directly, to the two-dimensional setting if the latter did not contain @ in the language. This can be done with Wehmeier's (2003) proof, for example, under the obvious generalization of the models for the two-dimensional case. Now, assuming the availability of @, and given that $\Diamond_2$ quantifies over diagonal pairs of worlds in W, it seems reasonable to expect that the truth condition in (2) be expressible, too:

(2) $\exists(w, w)\forall \mathsf{x}(R\mathsf{x}\text{-at-}(z, z) \rightarrow S\mathsf{x}\text{-at-}(w, w))$.

But it is not obvious whether (2) can be expressed by a formula in this language; the most obvious candidate, $\Diamond_2 \forall \mathsf{x}(@R\mathsf{x} \rightarrow S\mathsf{x})$, expresses the truth condition $\exists(w, w)\forall \mathsf{x}(R\mathsf{x}\text{-at-}(w, w) \rightarrow S\mathsf{x}\text{-at-}(w, w))$. This is so because within the scope of $\Diamond_2$, @ is idle. Thus, the expressive deficit of the first-order (one-dimensional) modal language that motivated the introduction of @ in the first-place seems to reoccur in the two-dimensional case: while @ is able to, as it were, temporarily suspend the scope of $\Diamond_1$, there is apparently no analogous device in the two-dimensional language that has the same effect on the scope of $\Diamond_2$. Furthermore, there are readings of English sentences such as "It is not a priori that not every red thing is shiny" whose truth conditions correspond to (2). That is, besides the apparent inability to express truth conditions that, on the face of it, should be expressible with the two-dimensional operators, it does not seem to be possible to formalize relevant portions of the kind of natural-language discourse targeted by two-dimensional semantics.

One way to fix this is to add a single distinguished point (pair of worlds) back in the frames and a new operator, say, A, that forces formulas in its scope to be evaluated at that distinguished point. In other words, A does just what @ did in the one-dimensional case (before it was dethroned). If we let this point be (z, z), the semantic clause for A will be:

$$\mathfrak{M}, (w, v) \models \mathsf{A}\varphi \Longleftrightarrow \mathfrak{M}, (z, z) \models \varphi.$$

Now the truth condition (2) can be expressed by the formula $\Diamond_2 \forall \mathsf{x}(\mathsf{A}R\mathsf{x} \rightarrow S\mathsf{x})$. Alternatively, we can take a point z from the underlying set S as distinguished, and instead of adding A to the language, add an operator D with the following semantic clause:

$$\mathfrak{M}, (w, v) \models \mathsf{D}\varphi \Longleftrightarrow \mathfrak{M}, (w, z) \models \varphi.$$

The truth condition (2) can then be expressed by the formula $\Diamond_2 \forall \mathsf{x}(\mathsf{D}@R\mathsf{x} \rightarrow S\mathsf{x})$, and A can be simulated in this language by the compound operator $\mathsf{D}@$.

Yet another option in a similar spirit is to dethrone the single distinguished element of S and thereby move to three-dimensional frames, just as the original actual world was dethroned in the introduction of two-dimensional frames. In this setting, formulas are evaluated relative to triples of worlds, that is, $W = S^3$, and a set of distinguished elements is defined in the frames as $D = \{(z, z, z) \mid z \in S\}$. In this new language we can have three $\Box_i$ operators whose semantic entries can be fully specified by the (GB) clause above, this time for $i \in \{1, 2, 3\}$. Accordingly, $\Box_1\varphi$ is true at a triple (w, v, u) if and only if φ is true at any $\mathcal{R}_{\Box_1}$-related triple (x, y, z), where $y = v$ and $z = u$; $\Box_2\varphi$ is true at a triple (w, v, u) if and only if φ is true at any $\mathcal{R}_{\Box_2}$-related triple (x, y, z), where $x = y$ and $z = u$ (i.e. at any triple (x, x, u)); and, finally, $\Box_3\varphi$ is true at a triple (w, v, u) if and only if φ is true at any $\mathcal{R}_{\Box_3}$-related triple (x, y, z), where $x = y = z$ (i.e. at any triple (x, x, x)). Now, just as Fritz (2014) adds an accessibility relation for the actuality operator in the two-dimensional frames,[9] we can relabel the actuality operator @ as $@_2$, add a new actuality operator, $@_3$, to the language, and accessibility relations to describe their semantic clauses schematically as follows:

$$\text{(GA) } \mathfrak{M}, \sigma \vDash @_i\varphi \Longleftrightarrow \text{for every } \tau \in W, \text{ if } \sigma\mathcal{R}_{@_i}\tau, \text{ then } \mathfrak{M}, \tau \vDash \varphi,$$

where $\sigma\mathcal{R}_{@_i}\tau$ if and only if the first i coordinates in τ are all identical and σ and τ are identical beyond $i - 1$. Thus, according to (GA), $@_2\varphi$ is true at a triple (w, v, u) if and only if φ is true at any $\mathcal{R}_{@_2}$-related triple (x, y, z), where $x = y = v$ and $z = u$ (i.e. at the triple (v, v, u)); and $@_3\varphi$ is true at a triple (w, v, u) if and only if φ is true at any $\mathcal{R}_{@_3}$-related triple (x, y, z), where $x = y = z = u$ (i.e. at the triple (u, u, u)). Additionally, (GA) gives us an $@_1$ operator which turns out to be redundant, for $@_1\varphi$ is true at a triple (w, v, u) if and only if φ is true at any $\mathcal{R}_{@_1}$-related triple (x, y, z), where $x = w$, $y = v$, and $z = u$ (i.e. at the triple (w, v, u) itself). If we now adapt the truth condition (2) to the three-dimensional environment we obtain

$$\text{(2*) } \exists(w, w, z)\forall \mathsf{x}(R\mathsf{x}\text{-at-}(z, z, z) \rightarrow S\mathsf{x}\text{-at-}(w, w, z)),$$

which is expressible by the formula $\Diamond_2\forall\mathsf{x}(@_3R\mathsf{x} \rightarrow S\mathsf{x})$.

What motivated the introduction of the original actuality operator, and the subsequent move to two dimensions, is really the same thing that motivates the introduction of the new actuality operator into the two-dimensional language, and the subsequent move to three dimensions. And even though it is not obvious what natural language correspondents for

[9]There are many advantages in doing this, also illustrated by the fact that the axioms involving actuality operators can then be defined as Sahlqvist formulas, and so the Sahlqvist completeness theorem can be applied (see §4.1.3).

operators such as $\Box_3$ and $@_3$ might be, it is clear that the original expressive deficit in the first-order modal language, which reoccurred in the two-dimensional case, seems to appear once again in the three-dimensional setting since $@_3$ does not seem able to reset the point of evaluation within the scope of $\Diamond_3$. That is to say, the following truth condition is apparently not expressible by any formula in the three-dimensional language under consideration:

(3) $\exists(w, w, w)\forall \mathsf{x}(R\mathsf{x}\text{-at-}(z, z, z) \rightarrow S\mathsf{x}\text{-at-}(w, w, w))$.

The most obvious candidate, $\Diamond_3\forall\mathsf{x}(@_3R\mathsf{x} \rightarrow S\mathsf{x})$, is unsuccessful for the same reason $\Diamond_2\forall\mathsf{x}(@R\mathsf{x} \rightarrow S\mathsf{x})$ fails to express the truth condition in (2).

It is now evident that analogous cases of expressive deficit will seem to occur for any finite dimension n. To be more specific, where $n \geq 2$, a first-order n-dimensional language contains operators $\Box_i$ and $@_i$ for each $i \in \{1, ..., n\}$, and a frame for this language is defined by a set $W = S^n$, where S is a non-empty set of worlds, relations $\mathcal{R}_{\Box_i}$ and $\mathcal{R}_{@_i}$ for each $i \in \{1, ..., n\}$, a set $D = \{\overline{s} \mid s \in S\}$ of distinguished points, where $\overline{s}$ is that sequence $\sigma \in W$ for which $\sigma_i = s \in S$ for all $1 \leq i \leq n$, besides a domain of objects $\mathcal{D}$. The truth conditions for $\Box_i$- and $@_i$-formulas are set by (GB) and (GA). Then, where $\sigma = \overline{s}$ and $\tau = \overline{t}$, the following truth condition would not seem to be expressible by any formula in the n-dimensional language:

(n) $\exists\sigma\forall\mathsf{x}(R\mathsf{x}\text{-at-}\tau \rightarrow S\mathsf{x}\text{-at-}\sigma)$.

In chapter 2, some of the philosophical ramifications of this general case for two-dimensional semantics were explored. In particular, the move from two to three-dimensions was motivated by the fact that some of the other solutions to the expressive deficit of the first-order two-dimensional language—namely, the ones involving the addition of the operators A or D—would result in the frames validating $\mathsf{A}\varphi \rightarrow \Box_2\mathsf{A}\varphi$ or $\mathsf{D}@\varphi \rightarrow \Box_2\mathsf{D}@\varphi$, respectively. But according to the intuitive interpretation we assign to these operators, such formulas should not be valid: if φ is true in the *real* actual world, it does not follow intuitively that it is a priori that it is true in that world. However, a similar issue ultimately arises in the three-dimensional language, as $@_3\varphi \to \Box_2 @_3\varphi$, too, would be valid in its respective class of frames.

In this chapter, however, we will not be concerned with philosophical implications of employing multidimensional frameworks to model a priori knowledge of finite or ideal agents. Rather, we shall focus on the logical generalization to higher-dimensions of logics for necessity, actuality, and a priori knowledge, which do raise a variety of questions from a purely logical point of view. The n-dimensional frames with distinguished elements defined earlier are natural generalizations of the two-dimensional matrix frames with distinguished elements defined in Fritz (2014). Fritz proves that such frames, which he defines for a propositional two-dimensional language, admit of a finite axiomatization through a more general class of *matrix frames*, which do not contain distinguished elements. Fritz calls the logic of such frames **2Dg**. It is natural to ask, in light of the constructions above, whether **2Dg** is a special case of a more general framework, and whether similar results transfer to n-dimensional frames defined for n-dimensional propositional languages.[10] Are these frames finitely axiomatizable? Are the logics decidable? We settle these and more questions in the course of this chapter. In §4.1 we work towards proving a completeness theorem for the logic of n-dimensions relative to n-dimensional frames without distinguished elements, and then we use this result to show that n-dimensional frames with distinguished elements are also complete. In §4.2 we prove that the resulting logics are decidable by means of n-dimensional tableaux and a systematic procedure guaranteeing termination for every tableau so constructed. At the end of the chapter we suggest some additional questions to be investigated in future work.

[10] It is reasonable to investigate the propositional languages first, especially because the whole technology consisting of finitely many $\Box_i$ and $@_i$ operators is already present at the propositional level.

4.1 The Logic of Sequence Frames

4.1.1 Syntax and Semantics

Definition 4.1.1 (Multidimensional language) Let PROP be a denumerable set of propositional variables $p, q, \ldots$. The language $\mathcal{L}_n^@$ is recursively generated by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_i\varphi \mid @_i\varphi$$

for all $1 \leq i \leq n$, where $n \geq 2$. The other Boolean connectives are defined as usual, and $\Diamond_i\varphi := \neg\Box_i\neg\varphi$. Finally, let $\mathcal{O} \in \{\Box, @\}$.

Definition 4.1.2 (Sequences) Let S be a non-empty set and $W = S^n$, i.e. W is the n-fold Cartesian product of S, so that W contains sequences $\sigma = (s_1, \ldots, s_n)$ of elements $s_1, \ldots, s_n \in S$. When $\sigma = (s_1, \ldots, s_n)$, we write σ_i for s_i. For $s \in S$, $\overline{s}$ is that sequence σ for which $\sigma_i = s$ for all $1 \leq i \leq n$. Moreover, we say that a tuple σ is *i-diagonal* just in case $\sigma_1 = \sigma_2 = \ldots = \sigma_i$, and for any sequence $\sigma \in W$, let σ_z^i be that sequence τ for which $\tau_1 = \tau_2 = \ldots = \tau_i = z$ and for $j > i$, $\tau_j = \sigma_j$, so that σ_z^i is always i-diagonal. We say that σ and τ are *identical beyond* i if for all $j > i$, $\sigma_j = \tau_j$.[11]

Now we define *sequence frames* for n-dimensional modal logics. The number of dimensions dictates the length of the sequences in the frame, which can be thought of as sequences of possible worlds. Strictly speaking, all that is needed to fully specify the desired frames are pairs (S, n), where S is a non-empty set. Yet, it will be helpful to add the relations $\mathcal{R}_{\Box_i}$ and $\mathcal{R}_{@_i}$ to the official definition of the frames, as these play an essential role in the Sahlqvist completeness theorem proved in §4.1.3, besides making several definitions easier

[11] To illustrate some of the items defined above, in the case of basic modal logic σ is always a one-place sequence, while in a two-dimensional modal logic σ is an ordered pair. Also, if, for example, $\sigma = (x, u, z, w)$, then $\sigma_u^3 = (u, u, u, w)$.

to articulate. Thus the official definition of n-dimensional sequence frames is the following:

Definition 4.1.3 (n-dimensional sequence frame) Let $n \geq 2$. An *n-dimensional sequence frame* for $\mathcal{L}_n^{@}$ is a triple, $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$, such that $W = S^n$ for some non-empty set S, $\mathcal{R}_{\Box_i} \subseteq W \times W$ and $\mathcal{R}_{@_i} \subseteq W \times W$ for all $1 \leq i \leq n$, such that for any sequences $\sigma, \tau \in W$,

- $\sigma\mathcal{R}_{\Box_i}\tau$ iff (i) τ is i-diagonal, and (ii) σ and τ are identical beyond i.
- $\sigma\mathcal{R}_{@_i}\tau$ iff (i) τ is i-diagonal, and (ii) σ and τ are identical beyond $i-1$.

Additionally, let F be the class of n-dimensional sequence frames.

To make the behaviour of each accessibility relation more explicit, we register the main characteristic properties that hold in every n-dimensional sequence frame. We begin with the properties involving only the $\mathcal{R}_{\Box_i}$ relations, moving in turn to the properties involving only the $\mathcal{R}_{@_i}$ relations, and then to the properties involving both families of relations. The proofs of these facts are straightforward, and so we omit them.

Proposition 4.1.1 *Let $1 \leq j \leq i \leq n$ and $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ be an* n-dimensional sequence frame*. Then:*

(1) $\mathcal{R}_{\Box_1}$ is an equivalence relation.

(2) $\mathcal{R}_{\Box_i}$ is serial, i.e. for all sequences σ there is a τ such that $\sigma\mathcal{R}_{\Box_i}\tau$.

(3) (Upward transitivity) If $\sigma\mathcal{R}_{\Box_j}\tau$ and $\tau\mathcal{R}_{\Box_i}\upsilon$, then $\sigma\mathcal{R}_{\Box_i}\upsilon$.

(4) (Upward Euclideanity) If $\sigma\mathcal{R}_{\Box_j}\tau$ and $\sigma\mathcal{R}_{\Box_i}\upsilon$, then $\tau\mathcal{R}_{\Box_i}\upsilon$.

(5) (Downward weak density) If $\sigma\mathcal{R}_{\Box_i}\tau$, there is a υ such that $\sigma\mathcal{R}_{\Box_j}\upsilon$ and $\upsilon\mathcal{R}_{\Box_i}\tau$.

(6) (Downward shift reflexivity) If $\sigma\mathcal{R}_{\Box_i}\tau$, then $\tau\mathcal{R}_{\Box_j}\tau$.

(7) (Strictly decreasing weak density) When $1 < i \leq n$, if $\sigma\mathcal{R}_{\Box_{i-1}}\tau$, there is a υ such that $\sigma\mathcal{R}_{\Box_i}\upsilon$ and $\upsilon\mathcal{R}_{\Box_{i-1}}\tau$.

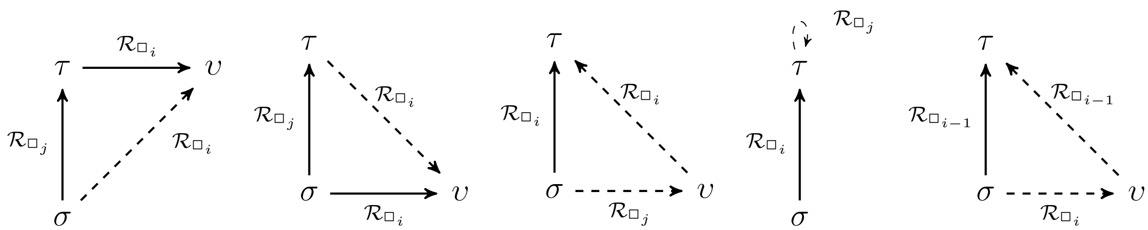

Figure 4.1.1: Upward transitivity, upward Euclideanity, downward weak density, downward shift reflexivity, and strictly decreasing weak density properties.

Proposition 4.1.2 *Let $1 \leq i \leq n$ and $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ be an n-*dimensional sequence frame. *Then:*

(1) $\mathcal{R}_{@_1}$ is reflexive.

(2) $\mathcal{R}_{@_i}$ is serial.

(3) $\mathcal{R}_{@_i}$ is functional, i.e. if $\sigma\mathcal{R}_{@_i}\tau$ and $\sigma\mathcal{R}_{@_i}\upsilon$, then $\tau = \upsilon$.

(4) (Upward-downward transitivity) When $1 < i \leq n$, if $\sigma\mathcal{R}_{@_i}\tau$ and $\tau\mathcal{R}_{@_{i-1}}\upsilon$, then $\sigma\mathcal{R}_{@_i}\upsilon$.

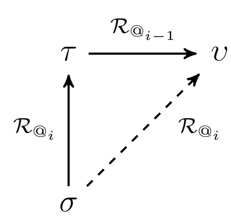

Figure 4.1.2: Upward-downward transitivity.

Remark 4.1.1 *By Proposition 4.1.2(3), it follows that each $\mathcal{R}_{@_i}$ is a function. Additionally, because $\mathcal{R}_{@_1}$ is reflexive, $\mathcal{R}_{@_1}$ is the identity function on the frame.*

Proposition 4.1.3 *Let $1 \leq j < i \leq n$ and $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ be an n-*dimensional sequence frame. *Then:*

(1) (Strictly decreasing act-box) When $1 < i \leq n$, if $\sigma\mathcal{R}_{@_i}\tau$, then $\sigma\mathcal{R}_{\Box_{i-1}}\tau$.

(2) (Act-box) If $\sigma\mathcal{R}_{@_i}\tau$, then $\sigma\mathcal{R}_{\Box_i}\tau$.

(3) (Mixed upward transitivity) If $\sigma\mathcal{R}_{\Box_j}\tau$ and $\tau\mathcal{R}_{@_i}\upsilon$, then $\sigma\mathcal{R}_{@_i}\upsilon$.

(4) (Mixed shift reflexivity) If $\sigma\mathcal{R}_{\Box_i}\tau$, then $\tau\mathcal{R}_{@_i}\tau$.

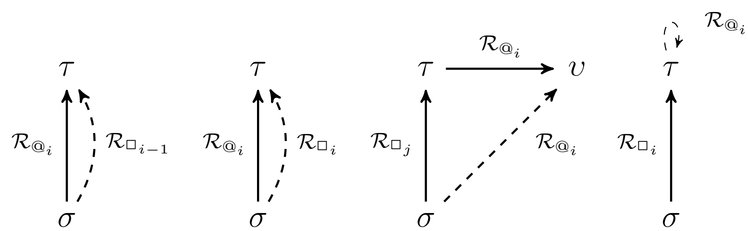

Figure 4.1.3: Strictly decreasing act-box, act-box, mixed upward transitivity, and mixed shift reflexivity properties.

Note that downward shift reflexivity now follows immediately from mixed shift reflexivity together with act-box and strictly decreasing act-box. Similarly, the properties (5) and (7) in Proposition 4.1.1 are provable from the other properties involving the $\mathcal{R}_{@_i}$ relations. With respect to (3) and (4), that is, upward transitivity and upward Euclideanity, they can also be derived from the other properties alongside (simple) transitivity and the Euclidean property. Thus, in order to characterize sequence frames syntactically we only need to add the modal axioms for transitivity and the Euclidean property alongside the other axioms listed below. Moreover, from the properties listed in Proposition 4.1.1 one can show that sequence frames have the property of *left commutativity* as well as the *Church-Rosser* property, that is:

Corollary 4.1.1 *Let $1 \leq j \leq i \leq n$, and $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ be an n-*dimensional sequence frame*. Then:*

(1) (Left commutativity) If $\sigma\mathcal{R}_{\Box_j}\tau$ and $\tau\mathcal{R}_{\Box_i}\upsilon$, then there is a σ' such that $\sigma\mathcal{R}_{\Box_i}\sigma'$ and $\sigma'\mathcal{R}_{\Box_j}\upsilon$.

(2) (Church-Rosser) If $\sigma\mathcal{R}_{\Box_j}\tau$ and $\sigma\mathcal{R}_{\Box_i}\upsilon$, then there is a σ' such that $\tau\mathcal{R}_{\Box_i}\sigma'$ and $\upsilon\mathcal{R}_{\Box_j}\sigma'$.

Proof. (1). If $\sigma\mathcal{R}_{\Box_j}\tau$ and $\tau\mathcal{R}_{\Box_i}\upsilon$, it follows that $\sigma\mathcal{R}_{\Box_i}\upsilon$, by upward transitivity. But then $\upsilon\mathcal{R}_{\Box_j}\upsilon$, by downward shift reflexivity. Therefore, there is a sequence σ', namely, υ itself, such that $\sigma\mathcal{R}_{\Box_i}\sigma'$ and $\sigma'\mathcal{R}_{\Box_j}\upsilon$.

(2). If $\sigma\mathcal{R}_{\Box_j}\tau$ and $\sigma\mathcal{R}_{\Box_i}\upsilon$, then $\tau\mathcal{R}_{\Box_i}\upsilon$, by the upward Euclidean property. But then $\upsilon\mathcal{R}_{\Box_j}\upsilon$, by downward shift reflexivity. Therefore, there is a sequence σ', namely, υ itself, such that $\tau\mathcal{R}_{\Box_i}\sigma'$ and $\upsilon\mathcal{R}_{\Box_j}\sigma'$. $\Box$

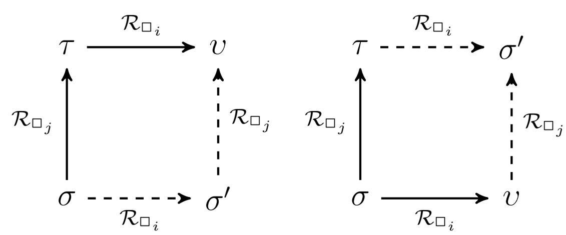

Figure 4.1.4: Left commutativity and Church-Rosser properties.

These properties are known for playing an important role in characterizing product frames for product logics (see Gabbay et al (2003: 222)). Note, however, that the property of *right commutativity*, which holds in product frames, does not hold in general for n-dimensional sequence frames.[12] Consider, for instance, the two-dimensional sequence frame $\mathfrak{F} = (W, \mathcal{R}_{\Box_1}, \mathcal{R}_{\Box_2}, \mathcal{R}_{@_1}, \mathcal{R}_{@_2})$ displayed in Figure 6, where $W = S^2$ and $S = \{w, v\}$. Then each of the properties listed in Proposition 4.1.1, 4.1.2, and 4.1.3 hold in that frame, but while $(v, w)\mathcal{R}_{\Box_2}(v, v)$ and $(v, v)\mathcal{R}_{\Box_1}(w, v)$, there is no pair σ' such that $(v, w)\mathcal{R}_{\Box_1}\sigma'$ and $\sigma'\mathcal{R}_{\Box_2}(w, v)$.

Even though no accessibility relation except for $\mathcal{R}_{\Box_1}$ is in general reflexive or symmetric, there is at least a weak sense in which n-dimensional sequence frames can be seen as generalizations to n dimensions of **S5**-frames, a sense that emerges when we restrict attention to the relations between sequences that are i-diagonal for some i. Consider the following definition:

[12] Right commutativity in our framework corresponds to the property that for all σ, τ, υ, if $\sigma\mathcal{R}_{\Box_i}\tau$ and $\tau\mathcal{R}_{\Box_j}\upsilon$, then there is a σ' such that $\sigma\mathcal{R}_{\Box_j}\sigma'$ and $\sigma'\mathcal{R}_{\Box_i}\upsilon$, for $1 \leq j \leq i \leq n$.

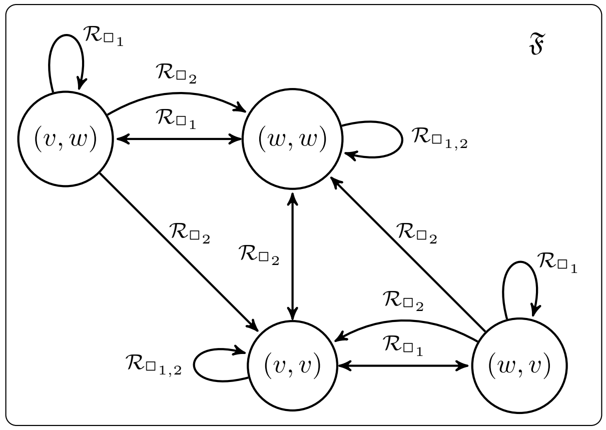

Figure 4.1.5: Two-dimensional sequence frame displaying only the $\mathcal{R}_{\Box_i}$ relations falsifying right commutativity.

Definition 4.1.4 Let S be non-empty and $W = S^n$ as before. $\mathcal{R}_{\Box_i}$ is *pseudo-reflexive* if it is reflexive on W^i; it is *pseudo-symmetric* if for all $\sigma, \tau \in W^i$, $\sigma\mathcal{R}_{\Box_i}\tau$ implies $\tau\mathcal{R}_{\Box_i}\sigma$; it is *pseudo-transitive* if for all $\sigma, \tau, \upsilon \in W^i$, $\sigma\mathcal{R}_{\Box_i}\tau$ and $\tau\mathcal{R}_{\Box_i}\upsilon$ implies $\sigma\mathcal{R}_{\Box_i}\upsilon$; and *pseudo-Euclidean* if for all $\sigma, \tau, \upsilon \in W^i$, $\sigma\mathcal{R}_{\Box_i}\tau$ and $\sigma\mathcal{R}_{\Box_i}\upsilon$ implies $\tau\mathcal{R}_{\Box_i}\upsilon$. Finally, $\mathcal{R}_{\Box_i}$ is a *pseudo-equivalence relation* if it is pseudo-reflexive and pseudo-Euclidean.

Note that if $n = 1$ (and the language is one-dimensional), the pseudo notions coincide with the original notions, so they are legitimate generalizations of the latter. Now it can be shown that every n-dimensional sequence frame contains a subframe on which every relation $\mathcal{R}_{\Box_i}$ is an equivalence relation:

Proposition 4.1.4 *Let* $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ *be an n-dimensional sequence frame. For each* $1 \leq i \leq n$, $\mathcal{R}_{\Box_i}$ *is an equivalence relation on* W^i. *In other words, for each* $1 \leq i \leq n$, $\mathcal{R}_{\Box_i}$ *is a pseudo-equivalence relation on* W.

Proof. Obviously, $\mathcal{R}_{\Box_i}$ is pseudo-reflexive, since any member of W^i is i-diagonal and identical with itself beyond i. Moreover, it is easy to see that $\mathcal{R}_{\Box_i}$ is pseudo-Euclidean, for if $\sigma\mathcal{R}_{\Box_i}\tau$ and $\sigma\mathcal{R}_{\Box_i}\upsilon$, then σ, τ, and υ are all identical beyond i, so that $\tau\mathcal{R}_{\Box_i}\upsilon$ as long as $\upsilon \in W^i$. □

In addition, observe that any n-dimensional sequence frame has a certain kind of universality property governing its last accessibility relation, $\mathcal{R}_{\Box_n}$, in the sense that every point in W is $\mathcal{R}_{\Box_n}$-related to every n-diagonal point. This property, which we may call *shift universality*, plays an important role in the completeness theorem below, for we use the fact that if a point is n-diagonal, then it is $\mathcal{R}_{\Box_n}$-accessible to any point in the frame. Yet, shift universality is not modally definable. To verify this, let $\mathfrak{F}_1 = (W_1, \mathcal{R}_1)$ be a usual Kripke frame with points w, u, v such that $\mathcal{R}_1 = \{(w, v), (v, v), (u, v)\}$, and $\mathfrak{F}_2 = (W_2, \mathcal{R}_2)$ a frame with points a, b, c such that $\mathcal{R}_2 = \{(a, b), (b, b), (c, b)\}$. Then the first-order formula expressing shift universality, that is, $\forall x \exists y (y\mathcal{R}x \rightarrow \forall z(z\mathcal{R}x))$, is valid in both $\mathfrak{F}_1$ and $\mathfrak{F}_2$, but not in their disjoint union, as illustrated in Figure 7. Therefore, by the Goldblatt-Thomason theorem—which states that a first-order definable class of frames is modally definable if and only if it is closed under taking disjoint unions, generated subframes, bounded morphic images, and reflects ultrafilter extensions—shift universality is not modally definable.[13]

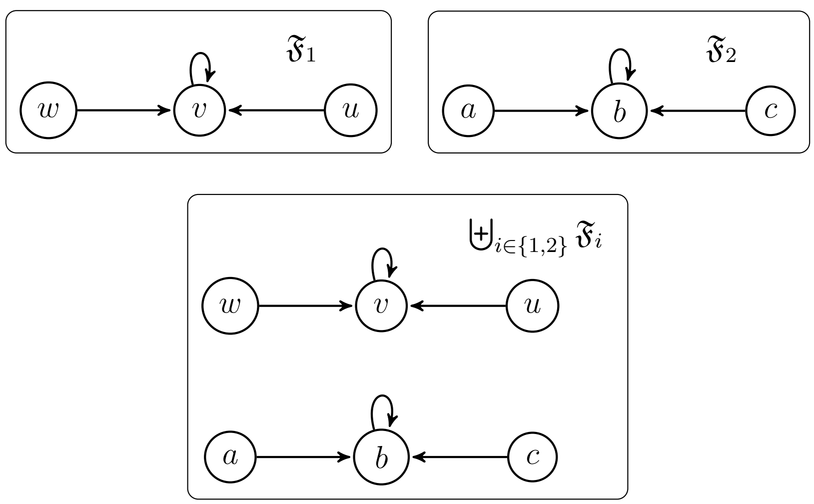

Figure 4.1.6: Non-definability of shift universality.

Next we define concepts such as models based on n-dimensional sequence frames, truth of a sentence at a sequence in a model, as well as the core logical notions of validity and consequence.

[13]The Goldblatt-Thomason theorem is proved in Blackburn et al. (2001: 180-183), §3.8, Theorem 3.19. The reader can also find definitions for the notions of disjoint unions, generated subframes, bounded morphic images, and ultrafilter extensions in Blackburn et al. (2001), §3.3. The notions of generated subframes and bounded morphic images will be used later in the proof of Lemma 4.1.5.

Definition 4.1.5 (n-dimensional sequence models) An *n-dimensional sequence model* is a quadruple, $\mathfrak{M} = (W, (\mathcal{R}_{\Box_i})_{1 \leq i \leq n}, (\mathcal{R}_{@_i})_{1 \leq i \leq n}, V)$, where V is a function assigning to each $p \in \mathsf{PROP}$ a subset $V(p) \subseteq W$. We say that $\mathfrak{M}$ is *based on* an n-dimensional sequence frame, $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1 \leq i \leq n}, (\mathcal{R}_{@_i})_{1 \leq i \leq n})$.

Definition 4.1.6 (Truth) We define 'φ is true at σ in a model $\mathfrak{M}$', written $\mathfrak{M}, \sigma \vDash \varphi$, by recursion on φ. For a sequence $\sigma \in W$, and a valuation V in $\mathfrak{M}$,

$$
\begin{array}{lcl}
\mathfrak{M}, \sigma \vDash p & \iff & \sigma \in V(p) \\
\mathfrak{M}, \sigma \vDash \neg\varphi & \iff & \mathfrak{M}, \sigma \nvDash \varphi \\
\mathfrak{M}, \sigma \vDash (\varphi \wedge \psi) & \iff & \mathfrak{M}, \sigma \vDash \varphi \text{ and } \mathfrak{M}, \sigma \vDash \psi \\
\mathfrak{M}, \sigma \vDash \Box_i\varphi & \iff & \text{for every } \tau \in W, \text{ if } \sigma\mathcal{R}_{\Box_i}\tau, \text{ then } \mathfrak{M}, \tau \vDash \varphi \\
\mathfrak{M}, \sigma \vDash @_i\varphi & \iff & \text{for every } \tau \in W, \text{ if } \sigma\mathcal{R}_{@_i}\tau, \text{ then } \mathfrak{M}, \tau \vDash \varphi
\end{array}
$$

Definition 4.1.7 (Logical notions) A sentence φ is *satisfiable* in a model $\mathfrak{M}$ iff there is a sequence σ in $\mathfrak{M}$ such that $\mathfrak{M}, \sigma \vDash \varphi$, and *satisfiable* in a frame $\mathfrak{F}$ iff it is satisfiable in a model based on $\mathfrak{F}$. A sentence φ is *valid in an n-dimensional sequence model* $\mathfrak{M}$, written $\mathfrak{M} \vDash \varphi$, iff $\mathfrak{M}, \sigma \vDash \varphi$ for every $\sigma \in W$. A sentence φ is *valid at a point σ in a frame $\mathfrak{F}$*, written $\mathfrak{F}, \sigma \vDash \varphi$, iff φ is true at σ in every model $\mathfrak{M}$ based on $\mathfrak{F}$, and φ is *valid in a frame* $\mathfrak{F}$, written $\mathfrak{F} \vDash \varphi$, iff it is valid at every point in $\mathfrak{F}$. A sentence φ is *valid on a class of frames* C, written $\mathsf{C} \vDash \varphi$, iff φ is valid in every $\mathfrak{F} \in \mathsf{C}$. A sentence φ is a *logical consequence* of a set of sentences Γ over a class of frames C if and only if for every $\mathfrak{M}$ in C and sequence σ in $\mathfrak{M}$, if $\mathfrak{M}, \sigma \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathfrak{M}, \sigma \vDash \varphi$.

4.1.2 Axiomatization

In this section we axiomatize n-dimensional sequence frames. The logic of n-dimensional sequence frames is called $\mathbf{S@}_n$:

Definition 4.1.8 (The logic $\mathbf{S@}_n$) Let $1 \leq i \leq n$, where $n \geq 2$. $\mathbf{S@}_n$ is the classical normal modal logic defined by the following axioms:

($\mathbf{T}_{\Box_1}$) $\Box_1 p \rightarrow p$.

($\mathbf{4}_{\Box_i}$) $\Box_i p \rightarrow \Box_i \Box_i p$.

($\mathbf{5}_{\Box_i}$) $\Diamond_i p \rightarrow \Box_i \Diamond_i p$.

($\mathbf{T}_{@_1}$) $@_1 p \rightarrow p$.

($\mathbf{D}_{@_i}$) $@_i p \rightarrow \neg @_i \neg p$.

($\mathbf{F}_{@_i}$) $\neg @_i \neg p \rightarrow @_i p$.

($\mathbf{A}_i$) $\Box_{i-1} p \rightarrow @_i p$, for $1 < i \leq n$.

($\mathbf{AT}_i$) $\Box_i p \rightarrow @_i p$.

($\mathbf{A4}_i$) $@_i p \rightarrow \Box_j @_i p$, for $1 \leq j < i \leq n$.

($\mathbf{AR}_i$) $\Box_i (@_i p \rightarrow p)$.

Where $\vdash_n$ denotes the provability relation in $\mathbf{S@}_n$, $\Gamma \vdash_n \varphi$ if and only if there are formulas $\psi_1, ..., \psi_n$ in Γ such that $\vdash_n \psi_1 \wedge ... \wedge \psi_n \rightarrow \varphi$.

As derived rules we can also add the *rule of regularity* and the *rule of congruentiality*, similarly to the basic modal case:

(**RR**) If $\vdash_n p \rightarrow q$, then $\vdash_n \Box_i p \rightarrow \Box_i q$.

(**RC**) If $\vdash_n p \leftrightarrow q$, then $\vdash_n \Box_i p \leftrightarrow \Box_i q$.

Moreover, left commutativity $\Box_i \Box_j p \rightarrow \Box_j \Box_i p$ and the Church-Rosser axiom $\Diamond_j \Box_i p \rightarrow \Box_i \Diamond_j p$ are both derivable from the axioms above.

With the exception of $\mathbf{T}_{@_1}$, and by setting $n = 2$, the axioms above are just the ones found in Fritz (2014: 391), for the logic **2Dg**. Note that by setting $n = 2$ there are two box-like operators in the language, namely, $\Box_1$, corresponding to the (metaphysical) necessity operator $\Box$ in **2Dg**, and the final box-like operator $\Box_2$, corresponding to the a priori operator A in **2Dg**. The reason why $\mathbf{T}_{@_1}$ is not found in **2Dg** is because its language has a single actuality operator which, provided $n = 2$, corresponds to $@_2$ in $\mathcal{L}_2^@$. Having a single actuality operator in the language is, after all, expected since **2Dg** is designed to be a logic of necessity, actuality, and the a priori, and hence $@_2$ is just the traditional actuality operator found in other two-dimensional modal logics, such as Davies and Humberstone's (1980) logic for deep necessity, for example. As mentioned before, we have added $@_1$ to the language for the sake of generality, so that there are n boxes and actuality operators. Nevertheless, it is now easy to see that $p \to @_1 p$ is derivable from the axioms above, and hence $p \leftrightarrow @_1 p$ is also derivable, by $\mathbf{T}_{@_1}$, which in turn means that $@_1$ is directly eliminable from the language $\mathcal{L}_2^@$ (in fact, $\mathcal{L}_n^@$) without loss of expressive power. Therefore, by discounting the $@_1$ operator, $\mathbf{S@}_n$ is a generalization of **2Dg** to n dimensions with many actuality operators.

4.1.3 Completeness

The axioms listed above are all Sahlqvist formulas, and hence we can compute their locally corresponding frame conditions by the general Sahlqvist algorithm for converting modal formulas into first-order formulas (see Blackburn et al. (2001, chapter 3.6) for a general description), from which it follows that $\mathbf{S@}_n$ is strongly complete with respect to the class of frames it defines.

Informally, we say that a class of frames C is *defined* by a set of formulas Γ (respectively, formula φ) if for every frame $\mathfrak{F}$, $\mathfrak{F} \in \mathsf{C}$ just in case $\mathfrak{F} \vDash \Gamma$ (respectively, $\mathfrak{F} \vDash \varphi$). Additionally, let $\psi(x)$ be a formula of first-order logic where x is the only free variable in ψ. Then $\psi(x)$ *locally corresponds* to a modal formula φ if $\psi(x)$ expresses a condition satisfiable on a point w in a first-order structure $\mathfrak{F}$ just in case φ is valid at w in $\mathfrak{F}$, where $\mathfrak{F}$ is taken as a Kripke frame. That is, for any frame $\mathfrak{F}$ and point w in $\mathfrak{F}$, $\mathfrak{F}, w \vDash \varphi$ just in case $\mathfrak{F}, \alpha[x/w] \vDash \psi$, where $\alpha[x/w]$ is the assignment function that sends x to w. Then the following first-order formulas locally correspond to the axioms of $\mathbf{S@}_n$:

$\mathbf{T}_{\Box_1}$	$\Box_1 p \to p$	$w\mathcal{R}_{\Box_1}w$
$\mathbf{4}_{\Box_i}$	$\Box_i p \to \Box_i\Box_i p$	$\forall vz((w\mathcal{R}_{\Box_i}v \wedge v\mathcal{R}_{\Box_i}z) \to w\mathcal{R}_{\Box_i}z)$
$\mathbf{5}_{\Box_i}$	$\Diamond_i p \to \Box_i\Diamond_i p$	$\forall vz((w\mathcal{R}_{\Box_i}v \wedge w\mathcal{R}_{\Box_i}z) \to v\mathcal{R}_{\Box_i}z)$
$\mathbf{T}_{@_1}$	$@_1 p \to p$	$w\mathcal{R}_{@_1}w$
$\mathbf{D}_{@_i}$	$@_i p \to \neg @_i \neg p$	$\exists v(w\mathcal{R}_{@_i}v)$
$\mathbf{F}_{@_i}$	$\neg @_i \neg p \to @_i p$	$\forall vz((w\mathcal{R}_{@_i}v \wedge w\mathcal{R}_{@_i}z) \to v = z)$
$\mathbf{A}_i$	$\Box_{i-1} p \to @_i p$	$\forall v(w\mathcal{R}_{@_i}v \to w\mathcal{R}_{\Box_{i-1}}v)$, for $1 < i \leq n$
$\mathbf{AT}_i$	$\Box_i p \to @_i p$	$\forall v(w\mathcal{R}_{@_i}v \to w\mathcal{R}_{\Box_i}v)$
$\mathbf{A4}_i$	$@_i p \to \Box_j @_i p$	$\forall vz((w\mathcal{R}_{\Box_j}v \wedge v\mathcal{R}_{@_i}z) \to w\mathcal{R}_{@_i}z)$, for $1 \leq j < i \leq n$
$\mathbf{AR}_i$	$\Box_i(@_i p \to p)$	$\forall v(w\mathcal{R}_{\Box_i}v \to v\mathcal{R}_{@_i}v)$

Theorem 4.1.1 (Sahlqvist completeness) $\mathbf{S@}_n$ *is sound and strongly complete with respect to the class of frames* $\mathsf{F}_{\mathbf{S@}_n}$ *(that is, the class of first-order frames defined by* $\mathbf{S@}_n$*).*

Proof. By Theorem 4.42 in Blackburn et al. (2001), the proof of which can be found in chapter 5. □

Let an $\mathbf{S@}_n$-frame be any frame in $\mathsf{F}_{\mathbf{S@}_n}$. It is clear that the axioms above are sound, too, with respect to the class of all n-dimensional sequence frames, given their correspondence to the first-order properties of these frames. An important question is whether the axioms above are also sufficient, that is, whether they settle the completeness problem for $\mathbf{S@}_n$ with respect to n-dimensional sequence frames. This question is settled here by generalizing the approach in Fritz (2014), in which a completeness theorem for **2Dg** is derived by first passing through an intermediate class of frames—in that case, the frames defined in Restall (2012). But rather than using an intermediate class of frames such as Restall's, we take the point-generated subframe of a frame in $\mathsf{F}_{\mathbf{S@}_n}$ and construct a bounded morphism from a sequence frame onto this point-generated subframe. Still, the proof strategy extends the main ideas employed in Fritz (2014), Lemmas 2.5-8.

First we prove a series of lemmas in order to establish the main completeness result for $\mathbf{S@}_n$. In particular, Lemmas 4.1.1 to 4.1.4 are instrumental in the proof of Lemma 4.1.5, which is in turn the principal lemma for the completeness theorem, namely, Theorem 4.1.3.

Lemma 4.1.1 *The following Sahlqvist formulas are derivable in* $\mathbf{S@}_n$. *Let* $1 \leq j < i \leq n$:

(1) (Seriality) $\vdash_n \Box_i p \to \Diamond_i p$.

(2) (Upward transitivity) $\vdash_n \Box_i p \to \Box_j \Box_i p$. *This formula corresponds to the frame condition* $\forall vz((w\mathcal{R}_{\Box_j} v \wedge v\mathcal{R}_{\Box_i} z) \to w\mathcal{R}_{\Box_i} z)$.

(3) (Upward Euclideanity) $\vdash_n \Diamond_i p \to \Box_j \Diamond_i p$. *This formula corresponds to the frame condition* $\forall vz((w\mathcal{R}_{\Box_j} v \wedge w\mathcal{R}_{\Box_i} z) \to v\mathcal{R}_{\Box_i} z)$.

Proof. (1). This is derivable from $\mathbf{AT}_i$ and its dual, $@_i p \to \Diamond_i p$.

(2). Consider the following derivation:

1. $\vdash_n \Diamond_i p \to \Box_i \Diamond_i p$ $\quad$ $\mathbf{5}_{\Box_i}$

2. $\vdash_n \Box_i \Diamond_i p \to @_i \Diamond_i p$ $\quad$ $\mathbf{AT}_i$

3. $\vdash_n \Diamond_i p \to @_i \Diamond_i p$ $\quad$ $1, 2$

4. $\vdash_n \neg @_i \neg \Box_i \neg p \to \Box_i \neg p$ $\quad$ 3

5. $\vdash_n @_i \Box_i p \to \Box_i p$ $\quad$ $\mathbf{D}_{@_i}, 4$

6. $\vdash_n \Box_j @_i \Box_i p \to \Box_j \Box_i p$ $\quad$ $\mathbf{Nec_j}, \mathbf{K}_{\Box_j}, 5$

7. $\vdash_n @_i \Box_i p \to \Box_j @_i \Box_i p$ $\quad$ $\mathbf{A4}_i$

8. $\vdash_n @_i \Box_i p \to \Box_j \Box_i p$ $\quad$ $6, 7$

9. $\vdash_n \Box_i p \to \Box_i \Box_i p$ $\quad$ $\mathbf{4}_{\Box_i}$

10. $\vdash_n \Box_i \Box_i p \to @_i \Box_i p$ $\quad$ $\mathbf{AT}_i$

11. $\vdash_n \Box_i p \to @_i \Box_i p$ $\quad$ $9, 10$

12. $\vdash_n \Box_i p \to \Box_j \Box_i p$ $\quad$ $8, 11$

(3). Consider the following derivation:

1. $\vdash_n \Box_i \Diamond_i p \to \Diamond_i \Diamond_i p$ $\quad$ (1)

2. $\vdash_n \Diamond_i \Diamond_i p \to \Diamond_i p$ $\quad$ Dual of $\mathbf{4}_{\Box_i}$

3. $\vdash_n \Box_i \Diamond_i p \to \Diamond_i p$ $\quad$ $1, 2$

4. $\vdash_n \Box_j \Box_i \Diamond_i p \to \Box_j \Diamond_i p$ $\quad$ $\mathbf{Nec}_j, 3, \mathbf{K}_{\Box_j}$

5. $\vdash_n \Box_i \Diamond_i p \to \Box_j \Box_i \Diamond_i p$ $\quad$ (2)

6. $\vdash_n \Diamond_i p \to \Box_i \Diamond_i p$ $\quad$ $\mathbf{5}_{\Box_i}$

7. $\vdash_n \Diamond_i p \to \Box_j \Diamond_i p$ $\quad$ $4, 5, 6$

$\square$

For the next lemmas, $\mathcal{R}_{\mathcal{O}_i}[X]$ is the image of the set X under the relation $\mathcal{R}_{\mathcal{O}_i}$, and for each function $\mathcal{R}_{@_i}$, $\mathcal{R}_{@_i}(w)$ is the unique v such that $w\mathcal{R}_{@_i}v$, and $im(\mathcal{R}_{@_i})$ is the image of the function $\mathcal{R}_{@_i}$.

Lemma 4.1.2 *Let* $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ *be an* $\mathbf{S}@_n$*-frame,* $w \in W$*, and* $\mathfrak{F}^w = (W^w, (\mathcal{R}^w_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}^w_{@_i})_{1\leq i\leq n})$ *the subframe of* $\mathfrak{F}$ *generated by* w*. Then:*

(1) $W^w = \mathcal{R}^w_{\Box_1}[\mathcal{R}^w_{\Box_2}[...[\mathcal{R}^w_{\Box_i}[...[\mathcal{R}^w_{\Box_n}[\{w\}]]]]]]$.

(2) $v\mathcal{R}^w_{\Box_n}u$ *if and only if* $u \in im(\mathcal{R}^w_{@_n})$.

Proof. (1). One inclusion is clear. For the other direction, let

$$Z = \mathcal{R}^w_{\Box_1}[\mathcal{R}^w_{\Box_2}[...[\mathcal{R}^w_{\Box_i}[...[\mathcal{R}^w_{\Box_n}[\{w\}]]]]]].$$

We first prove that (1a) $w \in Z$, and then that (1b) Z is closed under every relation $\mathcal{R}^w_{\mathcal{O}_i}$ in the frame $\mathfrak{F}^w$.

(1a). Let z be the element in W^w such that $\mathcal{R}^w_{@_n}(w) = z$, and for each i such that $1 \leq i < n$, let $\mathcal{R}^w_{@_i}(w) = z_i$ and $\mathcal{R}^w_{@_i}(z_i) = v_i$. Then $w\mathcal{R}^w_{\Box_n}z$, by $\mathbf{AT}_n$, and both $w\mathcal{R}^w_{\Box_{n-1}}z_{n-1}$ and $z_{n-1}\mathcal{R}^w_{\Box_{n-1}}v_{n-1}$, by $\mathbf{AT}_{n-1}$. So $w\mathcal{R}^w_{\Box_{n-1}}v_{n-1}$, by transitivity of $\mathcal{R}^w_{\Box_{n-1}}$. But $w\mathcal{R}^w_{\Box_{n-1}}z$ follows by $\mathbf{A}_n$, and so $z\mathcal{R}^w_{\Box_{n-1}}v_{n-1}$, since $\mathcal{R}^w_{\Box_{n-1}}$ is Euclidean. Then

$$w\mathcal{R}^w_{\Box_n}z\mathcal{R}^w_{\Box_{n-1}}v_{n-1}.$$

Moreover, both $w\mathcal{R}^w_{\Box_{n-2}}z_{n-2}$ and $z_{n-2}\mathcal{R}^w_{\Box_{n-2}}v_{n-2}$ follow by $\mathbf{AT}_{n-2}$. So $w\mathcal{R}^w_{\Box_{n-2}}v_{n-2}$, by transitivity of $\mathcal{R}^w_{\Box_{n-2}}$. Also, both $w\mathcal{R}^w_{\Box_{n-2}}z_{n-1}$ and $z_{n-1}\mathcal{R}^w_{\Box_{n-2}}v_{n-1}$, by $\mathbf{A}_{n-1}$. So $w\mathcal{R}^w_{\Box_{n-2}}v_{n-1}$, again by transitivity of $\mathcal{R}^w_{\Box_{n-2}}$. Hence $v_{n-1}\mathcal{R}^w_{\Box_{n-2}}v_{n-2}$, as $\mathcal{R}^w_{\Box_{n-2}}$ is Euclidean. Then

$$w\mathcal{R}^w_{\Box_n} z\mathcal{R}^w_{\Box_{n-1}} v_{n-1}\mathcal{R}^w_{\Box_{n-2}} v_{n-2}.$$

By repeating this argument multiple times, it follows that there is a chain

$$w\mathcal{R}^w_{\Box_n} z\mathcal{R}^w_{\Box_{n-1}} v_{n-1}\mathcal{R}^w_{\Box_{n-2}} v_{n-2}...\mathcal{R}^w_{\Box_3} v_3\mathcal{R}^w_{\Box_2} v_2.$$

Now, by $\mathbf{A}_2$, both $w\mathcal{R}^w_{\Box_1} z_2$ and $z_2\mathcal{R}^w_{\Box_1} v_2$, whence $w\mathcal{R}^w_{\Box_1} v_2$ and so $v_2\mathcal{R}^w_{\Box_1} w$, by transitivity and symmetry of $\mathcal{R}^w_{\Box_1}$, respectively, thereby extending the chain above to

$$(*)\ w\mathcal{R}^w_{\Box_n} z\mathcal{R}^w_{\Box_{n-1}} v_{n-1}\mathcal{R}^w_{\Box_{n-2}} v_{n-2}...\mathcal{R}^w_{\Box_3} v_3\mathcal{R}^w_{\Box_2} v_2\mathcal{R}^w_{\Box_1} w.$$

Therefore, $w \in Z$.

(1b). For $\mathcal{R}^w_{\Box_1}$, suppose that $v \in \mathcal{R}^w_{\Box_1}[Z]$. Then there is a $u \in Z$ such that $u\mathcal{R}^w_{\Box_1} v$. But since $u \in Z$, it follows by an argument similar to that which established $(*)$ in (1a) that there are $z_n, ..., z_2 \in W^w$ such that

$$w\mathcal{R}^w_{\Box_n} z_n\mathcal{R}^w_{\Box_{n-1}} z_{n-1}\mathcal{R}^w_{\Box_{n-2}} z_{n-2}...\mathcal{R}^w_{\Box_3} z_3\mathcal{R}^w_{\Box_2} z_2\mathcal{R}^w_{\Box_1} u.$$

By transitivity of $\mathcal{R}^w_{\Box_1}$, it then follows that $z_2\mathcal{R}^w_{\Box_1} v$, and so $v \in Z$.

For $\mathcal{R}^w_{\Box_i}$, where $1 < i \leq n$, suppose that $v \in \mathcal{R}^w_{\Box_i}[Z]$. Then there is a $u \in Z$ such that $u\mathcal{R}^w_{\Box_i} v$. But since $u \in Z$, it follows by an argument similar to the one establishing $(*)$ in (1a), that there are $z_n, ..., z_2 \in W^w$ such that

$$w\mathcal{R}^w_{\Box_n} z_n\mathcal{R}^w_{\Box_{n-1}} z_{n-1}\mathcal{R}^w_{\Box_{n-2}} z_{n-2}...\mathcal{R}^w_{\Box_3} z_3\mathcal{R}^w_{\Box_2} z_2\mathcal{R}^w_{\Box_1} u.$$

By upward transitivity, it follows that $z_2\mathcal{R}^w_{\Box_i} v$. Now, if $i = 2$, then $z_3\mathcal{R}^w_{\Box_2} v$, by transitivity of $\mathcal{R}^w_{\Box_2}$, from which it follows that $v \in Z$, as $\mathcal{R}^w_{\Box_1}$ is reflexive. If, on the other hand, $2 < i < n$, since $z_3\mathcal{R}^w_{\Box_2} z_2$, it follows that $z_3\mathcal{R}^w_{\Box_i} v$, by upward transitivity; since $z_4\mathcal{R}^w_{\Box_3} z_3$ and $z_3\mathcal{R}^w_{\Box_i} v$, it follows that $z_4\mathcal{R}^w_{\Box_i}v$ by upward transitivity; after multiple repetitions of this argument, since $z_i\mathcal{R}^w_{\Box_{i-1}}z_{i-1}$ and $z_{i-1}\mathcal{R}^w_{\Box_i}v$, it follows that $z_i\mathcal{R}^w_{\Box_i}v$ by upward transitivity. Then $v\mathcal{R}^w_{@_i}v$, by $\mathbf{AR}_i$, whence $v\mathcal{R}^w_{\Box_{i-1}}v$, by $\mathbf{A}_i$, and so $v\mathcal{R}^w_{@_{i-1}}v$, by $\mathbf{AR}_{i-1}$, whence $v\mathcal{R}^w_{\Box_{i-2}}v$, by $\mathbf{A}_{i-1}$, and so on. After multiple repetitions of this argument, $v\mathcal{R}^w_{@_3}v$, by $\mathbf{AR}_3$, whence $v\mathcal{R}^w_{\Box_2}v$, by $\mathbf{A}_3$. Also $v\mathcal{R}^w_{\Box_1}v$, since $\mathcal{R}^w_{\Box_1}$ is reflexive, and so $v \in Z$. In case $i = n$, since $w\mathcal{R}^w_{\Box_n}z_n$, and $z_n\mathcal{R}^w_{\Box_n}v$ follows as in the argument above, it then follows that $w\mathcal{R}^w_{\Box_n}v$ by transitivity of $\mathcal{R}^w_{\Box_n}$. The rest of the argument for the descending chain is exactly as the case $2 < i < n$ above.

For $\mathcal{R}^w_{@_1}$, suppose that $v \in \mathcal{R}^w_{@_1}[Z] = Z$, since $\mathcal{R}^w_{@_1}$ is the identity function. So $v \in Z$.

For $\mathcal{R}^w_{@_i}$, where $1 < i \leq n$, suppose that $v \in \mathcal{R}^w_{@_i}[Z]$. Then there is a $u \in Z$ such that $u\mathcal{R}^w_{@_i}v$. So $u\mathcal{R}^w_{\Box_i}v$, by $\mathbf{AT}_i$. That $v \in Z$ now follows from the argument for $\mathcal{R}^w_{\Box_i}$.

(2). For the left-to-right direction, suppose that $v\mathcal{R}^w_{\Box_n}u$. By $\mathbf{AR}_n$, $u\mathcal{R}^w_{@_n}u$, so $u \in im(\mathcal{R}^w_{@_n})$. For the converse, assume that $u \in im(\mathcal{R}^w_{@_n})$, and let z be the element in W^w such that $\mathcal{R}^w_{@_n}(w) = z$. Now, consider any point $v \in W^w$. We show that (2a) $v\mathcal{R}^w_{\Box_n}z$, and then that (2b) $z\mathcal{R}^w_{\Box_n}u$, from which $v\mathcal{R}^w_{\Box_n}u$ follows by transitivity of $\mathcal{R}^w_{\Box_n}$, thereby proving the lemma.

(2a) Since $v \in W^w$, it follows by an argument similar to the one establishing $(*)$ in (1a) that there are $z_n, ..., z_2 \in W^w$ such that

$$w\mathcal{R}^w_{\Box_n}z_n\mathcal{R}^w_{\Box_{n-1}}z_{n-1}\mathcal{R}^w_{\Box_{n-2}}z_{n-2}...\mathcal{R}^w_{\Box_3}z_3\mathcal{R}^w_{\Box_2}z_2\mathcal{R}^w_{\Box_1}v.$$

By symmetry of $\mathcal{R}^w_{\Box_1}$, it follows that $v\mathcal{R}^w_{\Box_1}z_2$. Now consider $z_3' \in \mathcal{R}^w_{@_2}(z_3)$. By $\mathbf{AT}_2$, it follows that $z_3\mathcal{R}^w_{\Box_2}z_3'$. So $z_3\mathcal{R}^w_{\Box_2}z_2$ and $z_3\mathcal{R}^w_{\Box_2}z_3'$, from which $z_2\mathcal{R}^w_{\Box_2}z_3'$ follows from the Euclidean property of $\mathcal{R}^w_{\Box_2}$. So $v\mathcal{R}^w_{\Box_1}z_2$ and $z_2\mathcal{R}^w_{\Box_2}z_3'$, hence $v\mathcal{R}^w_{\Box_2}z_3'$, by upward transitivity. Now consider $z_4' = \mathcal{R}^w_{@_3}(z_4)$. By $\mathbf{AT}_3$, it follows that $z_4\mathcal{R}^w_{\Box_3}z_4'$. So $z_4\mathcal{R}^w_{\Box_3}z_3$ and $z_4\mathcal{R}^w_{\Box_3}z_4'$, from which $z_3\mathcal{R}^w_{\Box_3}z_4'$ follows from the Euclidean property of $\mathcal{R}^w_{\Box_3}$. Since $z_3\mathcal{R}^w_{\Box_2}z_3'$, it follows that $z_3'\mathcal{R}^w_{\Box_3}z_4'$, by upward Euclideanity. And since $v\mathcal{R}^w_{\Box_2}z_3'$, it follows that $v\mathcal{R}^w_{\Box_3}z_4'$, by upward transitivity. By repeating this argument multiple times, it follows that $v\mathcal{R}^w_{\Box_{n-1}}z_n'$. Now consider $z = \mathcal{R}^w_{@_n}(w)$. By $\mathbf{AT}_n$, it follows that $w\mathcal{R}^w_{\Box_n}z$. So $w\mathcal{R}^w_{\Box_n}z_n$ and $w\mathcal{R}^w_{\Box_n}z$, whence $z_n\mathcal{R}^w_{\Box_n}z$, by the Euclidean property of $\mathcal{R}^w_{\Box_n}$. Then $z_n\mathcal{R}^w_{\Box_{n-1}}z'_n$ (by a previous iteration of the argument) and $z_n\mathcal{R}^w_{\Box_n}z$, so $z'_n\mathcal{R}^w_{\Box_n}z$ follows by upward Euclideanity. Therefore, $v\mathcal{R}^w_{\Box_{n-1}}z'_n$ and $z'_n\mathcal{R}^w_{\Box_n}z$, from which $v\mathcal{R}^w_{\Box_n}z$ follows by upward transitivity.

(2b). Since $u \in W^w$, by hypothesis, it follows by an argument similar to the one establishing $(*)$ in (1a) that there are $z_n, ..., z_2 \in W^w$ such that

$$w\mathcal{R}^w_{\Box_n}z_n\mathcal{R}^w_{\Box_{n-1}}z_{n-1}\mathcal{R}^w_{\Box_{n-2}}z_{n-2}...\mathcal{R}^w_{\Box_3}z_3\mathcal{R}^w_{\Box_2}z_2\mathcal{R}^w_{\Box_1}u.$$

Consider $z = \mathcal{R}^w_{@_n}(w)$. Then $w\mathcal{R}^w_{\Box_n}z$, by $\mathbf{AT}_n$. Since $\mathcal{R}^w_{\Box_n}$ is Euclidean, it follows that $z\mathcal{R}^w_{\Box_n}z_n$. Since $u \in im(\mathcal{R}^w_{@_n})$, there is a $u_n \in W^w$ such that $u_n\mathcal{R}^w_{@_n}u$. By $\mathbf{AT}_n$, $u_n\mathcal{R}^w_{\Box_n}u$, whence $u\mathcal{R}^w_{@_n}u$, by $\mathbf{AR}_n$, and so $u\mathcal{R}^w_{\Box_n}u$, by $\mathbf{AT}_n$. By $\mathbf{A}_n$, in turn, it also follows that $u_n\mathcal{R}^w_{\Box_{n-1}}u$, so $u\mathcal{R}^w_{@_{n-1}}u$ by $\mathbf{AR}_{n-1}$, whence $u\mathcal{R}^w_{\Box_{n-2}}u$ by $\mathbf{A}_{n-1}$. Then $u\mathcal{R}^w_{@_{n-2}}u$ by $\mathbf{AR}_{n-1}$, whence $u\mathcal{R}^w_{\Box_{n-3}}u$ by $\mathbf{A}_{n-2}$. By repeating this argument multiple times, it follows that $u\mathcal{R}^w_{\Box_3}u$ by $\mathbf{A}_4$ and, similarly, that $u\mathcal{R}^w_{\Box_2}u$ by $\mathbf{A}_3$. Now, since $z_2\mathcal{R}^w_{\Box_1}u$ and $u\mathcal{R}^w_{\Box_2}u$, $z_2\mathcal{R}^w_{\Box_2}u$ follows by upward transitivity. And because $z_3\mathcal{R}^w_{\Box_2}z_2$ and $z_2\mathcal{R}^w_{\Box_2}u$, $z_3\mathcal{R}^w_{\Box_2}u$ follows by transitivity of $\mathcal{R}^w_{\Box_2}$. But since $z_3\mathcal{R}^w_{\Box_2}u$ and $u\mathcal{R}^w_{\Box_3}u$, $z_3\mathcal{R}^w_{\Box_3}u$ follows by upward transitivity, and so $z_4\mathcal{R}^w_{\Box_3}u$ follows by transitivity of $\mathcal{R}^w_{\Box_3}$. By repeating this argument multiple times it follows that $z_n\mathcal{R}^w_{\Box_{n-1}}u$. But since $u\mathcal{R}^w_{\Box_n}u$, it follows that $z_n\mathcal{R}^w_{\Box_n}u$ by upward transitivity. And since $z\mathcal{R}^w_{\Box_n}z_n$, it follows that $z\mathcal{R}^w_{\Box_n}u$ by transitivity of $\mathcal{R}^w_{\Box_n}$, as desired. □

Lemma 4.1.3 *Let* $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ *be an* $\mathbf{S}@_n$*-frame, and* $v \in W$*. Then* $\mathcal{R}_{@_i}(\mathcal{R}_{@_{i+1}}(v)) = \mathcal{R}_{@_{i+1}}(v)$*, for* $1 \leq i < n$*.*

Proof. Let $\mathcal{R}_{@_{i+1}}(v) = z$. Then $v\mathcal{R}_{\Box_{i+1}}z$, by $\mathbf{AT}_{i+1}$, and so $z\mathcal{R}_{@_{i+1}}z$ follows by $\mathbf{AR}_{i+1}$. Now $z\mathcal{R}_{\Box_i}z$ follows by $\mathbf{A}_{i+1}$, and $z\mathcal{R}_{@_i}z$ follows by $\mathbf{AR}_i$. So $v\mathcal{R}_{@_{i+1}}z\mathcal{R}_{@_i}z$, and hence

$$\mathcal{R}_{@_i}(\mathcal{R}_{@_{i+1}}(v)) = \mathcal{R}_{@_i}(z) = z = \mathcal{R}_{@_{i+1}}(v),$$

as desired. $\square$

In order to establish the main completeness result of this section we now want to show the existence and relevant properties of certain families of surjective functions, which are essential to construct the relevant bounded morphism from a sequence frame onto an $\mathbf{S}@_n$-frame (or, more precisely, a point-generated subframe of an $\mathbf{S}@_n$-frame). In Fritz's (2014), Lemma 2.8, completeness proof for the two-dimensional case, this is done as follows. Let $\mathfrak{F} = (W, \mathcal{R}_{\Box_1}, \mathcal{R}_{\Box_2}, \mathcal{R}_{@_1}, \mathcal{R}_{@_2})$ be an $\mathbf{S}@_2$-frame, and $\mathfrak{F}^w = (W^w, \mathcal{R}^w_{\Box_1}, \mathcal{R}^w_{\Box_2}, \mathcal{R}^w_{@_1}, \mathcal{R}^w_{@_2})$ the subframe of $\mathfrak{F}$ generated by w. Validity is preserved by taking generated subframes,[14] and hence the $\mathbf{S}@_2$ axioms hold in $\mathfrak{F}^w$. Now, for every element $w \in W^w$ there is a surjective function $g^1_w : W^w \to \mathcal{R}^w_{\Box_1}[g^0_\emptyset(w)]$ such that $g^1_w(w) = \mathcal{R}^w_{@_2}(g^0_\emptyset(w))$, where $g^0_\emptyset = Id_{W^w}$. The family g^1 of surjections exists since for every $w \in W^w$, $\mathcal{R}^w_{\Box_1}[g^0_\emptyset(w)] \subseteq W^w$, and also because $\mathcal{R}^w_{@_2}$ is a function such that $\mathcal{R}^w_{@_2} \subseteq \mathcal{R}^w_{\Box_1}$, by $\mathbf{A}_2$. We then construct a two-dimensional sequence frame, $\mathfrak{F}' = (W', \mathcal{R}'_{\Box_1}, \mathcal{R}'_{\Box_2}, \mathcal{R}'_{@_1}, \mathcal{R}'_{@_2})$, where $W' = W^w \times W^w$, and define a map f from W' onto W^w by $f((w, v)) = g^1_v(w)$, which is a bounded morphism from $\mathfrak{F}'$ onto $\mathfrak{F}^w$—this can be verified by a standard back-and-forth argument. Since we know by Sahlqvist completeness (Theorem 4.1.1) that $\mathbf{S}@_2$ is strongly complete with respect to $\mathbf{S}@_2$-frames, completeness of $\mathbf{S}@_2$ relative to the class of two-dimensional sequence frames now follows since modal satisfaction is invariant under taking bounded morphisms between models, and the fact that every formula in the language is satisfiable in an $\mathbf{S}@_2$-model if and only if it is satisfiable in a point-generated submodel of it.[15] This is by and large Fritz's argument except for minor notational differences, the absence of an $\mathcal{R}_{@_i}$ relation, and the fact that he uses an intermediate class of frames from Restall (2012) indirectly instead of the point-generated subframes $\mathfrak{F}^w$.

To see how this method can be generalized for n dimensions it is instructive first to briefly

[14] This is proved in Blackburn et al. (2001: 140), Theorem 3.14(ii).

[15] See Blackburn et al. (2001), §4.2.1, for the definitions of bounded morphisms between models and generated submodels, as well as the relevant invariance results.

mention the adaptation of the argument above for three dimensions, as the definition of the bounded morphism in this case will need multiple families of surjections which are constructed in stages, just as in the n case. So, let $\mathfrak{F}$ be an $\mathbf{S}@_3$-frame, and $\mathfrak{F}^w$ the subframe of $\mathfrak{F}$ generated by w. In order to prove that $\mathbf{S}@_3$ is complete relative to three-dimensional sequence frames, we construct a three-dimensional sequence frame, which is defined as $\mathfrak{F}' = (W', (\mathcal{R}'_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}'_{@_i})_{1\leq i\leq n})$, where $W' = W^w \times W^w \times W^w$, and a bounded morphism from $\mathfrak{F}'$ onto $\mathfrak{F}^w$. This is done as follows. Let $g^0_\emptyset = Id_{W^w}$. Then for every $w \in W^w$ there is a surjective function $g^1_w : W^w \to \mathcal{R}^w_{\Box_2}[g^0_\emptyset(w)]$ such that

(i) $g^1_w(w) = \mathcal{R}^w_{@_3}(g^0_\emptyset(w))$.

The family g^1 of surjections exists since for any $w \in W^w$, $\mathcal{R}^w_{\Box_2}[g^0_\emptyset(w)] \subseteq W^w$, and also because $\mathcal{R}^w_{@_3}$ is a function such that $\mathcal{R}^w_{@_3} \subseteq \mathcal{R}^w_{\Box_2}$, by $\mathbf{A}_3$. Additionally, for every $w, v \in W^w$ there is a surjective function $g^2_{w,v} : W^w \to \mathcal{R}^w_{\Box_1}[g^1_v(w)]$ such that

(ii) $g^2_{w,v}(w) = \mathcal{R}^w_{@_2}(g^1_v(w))$.

The family g^2 of surjections exists since for any $w, v \in W^w$, $\mathcal{R}^w_{\Box_1}[g^1_v(w)] \subseteq W^w$, and also because $\mathcal{R}^w_{@_2}$ is a function such that $\mathcal{R}^w_{@_2} \subseteq \mathcal{R}^w_{\Box_1}$, by $\mathbf{A}_2$. Then for every $w \in W^w$, since $g^2_{w,w}(w) = \mathcal{R}^w_{@_2}(g^1_w(w))$, by (ii), $g^1_w(w) = \mathcal{R}^w_{@_3}(g^0_\emptyset(w))$, by (i), and $\mathcal{R}^w_{@_2}(\mathcal{R}^w_{@_3}(w)) = \mathcal{R}^w_{@_3}(w)$, by Lemma 4.1.3, it follows that

(iii) $g^2_{w,w}(w) = \mathcal{R}^w_{@_3}(w)$.

The map f from W' onto W^w defined by $f((w, v, z)) = g^2_{v,z}(w)$ is then a bounded morphism from $\mathfrak{F}'$ onto $\mathfrak{F}^w$, which can again be verified by a standard back-and-forth argument.

The aim of the next two lemmas is to register the existence of similar surjections in the general n-dimensional case as well as to construct the desired bounded morphism for the completeness proof.

Lemma 4.1.4 *Let $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ be an $\mathbf{S}@_n$-frame. Then:*

(1) There is a sequence $g^0, ..., g^{n-1}$ of families of surjective functions defined on W, where the family g^i is indexed by W^i, $g^0_\emptyset = Id_W$, and for each $1 \leq i \leq (n-1)$ and every $w_1, ..., w_i \in W$,

$$g^i_{w_1,...,w_i} : W \to \mathcal{R}_{\Box_{n-i}}[g^{i-1}_{w_2,...,w_i}(w_1)]$$

such that

$$g^i_{w_1,...,w_i}(w_1) = \mathcal{R}_{@_{n-(i-1)}}(g^{i-1}_{w_2,...,w_i}(w_1)).$$

(2) If an initial segment of $w_1, ..., w_{n-1}$ is such that $w_1 = ... = w_j = z$, for $1 \leq j \leq (n-1)$, and $z \in W$, then

$$g^{n-1}_{w_1,...,w_{n-1}}(z) = \mathcal{R}_{@_{j+1}}(g^{n-(j+1)}_{w_{j+1},...,w_{n-1}}(w_j)).$$

Proof. (1). The proof is by induction on $1 \leq i \leq (n-1)$. For $i = 1$, since for any $w_1 \in W$, $\mathcal{R}_{\Box_{n-1}}[g^0_\emptyset(w_1)] \subseteq W$, and $\mathcal{R}_{@_n}$ is a function such that $\mathcal{R}_{@_n} \subseteq \mathcal{R}_{\Box_{n-1}}$, by $\mathbf{A}_n$, there is a family of surjective functions

$$g^1_{w_1} : W \to \mathcal{R}_{\Box_{n-1}}[g^0_\emptyset(w_1)]$$

such that

$$g^1_{w_1}(w_1) = \mathcal{R}_{@_n}(g^0_\emptyset(w_1)).$$

Assume the induction hypothesis, for $i = k$, where $1 \leq k < (n-1)$, that there is family of surjective functions

$$g^k_{w_1,...,w_k} : W \to \mathcal{R}_{\Box_{n-k}}[g^{k-1}_{w_2,...,w_k}(w_1)]$$

such that

$$g^k_{w_1,\dots,w_k}(w_1) = \mathcal{R}_{@_{n-(k-1)}}(g^{k-1}_{w_2,\dots,w_k}(w_1)).$$

We show that the lemma holds for $i = k+1$. Since, by the induction hypothesis, there are functions $g^k_{w_1,\dots,w_k}$, and both $\mathcal{R}_{\Box_{n-(k+1)}}[g^k_{w_2,\dots,w_{k+1}}(w_1)] \subseteq W$ and $\mathcal{R}_{@_{n-k}} \subseteq \mathcal{R}_{\Box_{n-(k+1)}}$, by $\mathbf{A}_{n-k}$, it follows that there is a family of surjective functions

$$g^{k+1}_{w_1,\dots,w_{k+1}} : W \to \mathcal{R}_{\Box_{n-(k+1)}}[g^k_{w_2,\dots,w_{k+1}}(w_1)]$$

such that

$$g^{k+1}_{w_1,\dots,w_{k+1}}(w_1) = \mathcal{R}_{@_{n-k}}(g^k_{w_2,\dots,w_{k+1}}(w_1)),$$

as desired.

(2). The proof is by induction on j. For $j = 1$, it follows from (1) that

$$g^{n-1}_{w_1,\dots,w_{n-1}}(z) = \mathcal{R}_{@_2}(g^{n-2}_{w_2,\dots,w_{n-1}}(w_1)).$$

Now let $j = k$, where $1 \leq k < (n-1)$, and assume that

$$g^{n-1}_{w_1,\dots,w_{n-1}}(z) = \mathcal{R}_{@_{k+1}}(g^{n-(k+1)}_{w_{k+1},\dots,w_{n-1}}(w_k)).$$

Suppose that $w_1 = \dots = w_{k+1} = z$. Then the family of surjections

$$g^{n-(k+1)}_{w_{k+1},\dots,w_{n-1}} : W \to \mathcal{R}_{\Box_{n-(n-(k+1))}}[g^{(n-(k+1))-1}_{w_{k+2},\dots,w_{n-1}}(w_{k+1})]$$

is such that

$$g^{n-(k+1)}_{w_{k+1},\dots,w_{n-1}}(w_k) = \mathcal{R}_{@_{n-((n-(k+1))-1)}}(g^{(n-(k+1))-1}_{w_{k+2},\dots,w_{n-1}}(w_{k+1})),$$

by (1), and hence

$$g^{n-1}_{w_1,\ldots,w_{n-1}}(z) = \mathcal{R}_{@_{k+1}}(\mathcal{R}_{@_{n-((n-(k+1))-1)}}(g^{(n-(k+1))-1}_{w_{k+2},\ldots,w_{n-1}}(w_{k+1}))).$$

Note, moreover, that $n-((n-(k+1))-1) = k+2$, and hence

$$g^{n-1}_{w_1,\ldots,w_{n-1}}(z) = \mathcal{R}_{@_{n-((n-(k+1))-1)}}(g^{(n-(k+1))-1}_{w_{k+2},\ldots,w_{n-1}}(w_{k+1})),$$

by Lemma 4.1.3, as desired. □

Lemma 4.1.5 *Let* $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ *be an* $\mathbf{S}@_n$*-frame,* $w \in W$*, and* $\mathfrak{F}^w = (W^w, (\mathcal{R}^w_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}^w_{@_i})_{1\leq i\leq n})$ *the subframe of* $\mathfrak{F}$ *generated by* w*. Then* $\mathfrak{F}^w$ *is a bounded morphic image of a sequence frame.*

Proof. Let $\mathfrak{F}$, $w \in W$, and $\mathfrak{F}^w$ be as in the hypothesis. Because generated subframes preserve validity between frames, all of the $\mathbf{S}@_n$ axioms hold in $\mathfrak{F}^w$. So in what follows we construct a sequence frame, $\mathfrak{F}' = (W', (\mathcal{R}'_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}'_{@_i})_{1\leq i\leq n})$, such that $W' = (W^w)^n$, where $(W^w)^n$ is the n-fold Cartesian product of W^w, and a surjective bounded morphism $f : \mathfrak{F}' \twoheadrightarrow \mathfrak{F}^w$. Then, where $\sigma \in (W^w)^n$, let $f : (W^w)^n \to W^w$ be defined as

$$f(\sigma) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1),$$

where the sequence $g^0, \ldots, g^{n-1}$ of families of surjective functions is defined on W^w. We prove that f so defined is a bounded morphism from $\mathfrak{F}'$ onto $\mathfrak{F}^w$ by checking the back and forth conditions for each accessibility relation as follows:

[$\mathcal{R}_{\Box_1}$] Suppose that $\sigma\mathcal{R}'_{\Box_1}\sigma'$. Then σ and σ' are identical beyond 1. So $f(\sigma) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1)$ $\in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$, and $f(\sigma') = g^{n-1}_{\sigma'_2,\ldots,\sigma'_n}(\sigma'_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$, from which it follows that $f(\sigma)\mathcal{R}^w_{\Box_1}f(\sigma')$. Conversely, suppose that $f(\sigma)\mathcal{R}^w_{\Box_1}z$. Since

$$f(\sigma) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)],$$

it follows that $z \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$. And since $g^{n-1}_{\sigma_2,\ldots,\sigma_n}$ is a surjection, there is a $v \in W^w$ such that $g^{n-1}_{\sigma_2,\ldots,\sigma_n}(v) = z$, whence $f(\sigma^1_v) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(v) = z$. Furthermore, $\sigma\mathcal{R}'_{\Box_1}\sigma^1_v$.

[$\mathcal{R}_{\Box_i}, 1 < i < n$] Suppose that $\sigma\mathcal{R}'_{\Box_i}\sigma'$. Then σ' is i-diagonal and identical with σ beyond i. So $f(\sigma') = g^{n-1}_{\sigma'_2,\ldots,\sigma'_n}(\sigma'_1) = \mathcal{R}^w_{@_i}(g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i))$, by Lemma 4.1.4(2), and $f(\sigma) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1)$ $\in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$. Since $g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)\mathcal{R}^w_{@_i}f(\sigma')$, it follows that $g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)\mathcal{R}^w_{\Box_i}f(\sigma')$, by $\mathbf{AT}_i$. Also, $g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i) \in \mathcal{R}^w_{\Box_i}[g^{(n-i)-1}_{\sigma_{i+2},\ldots,\sigma_n}(\sigma_{i+1})]$, since the sequences σ and σ' are identical beyond i, whence $g^{(n-i)-1}_{\sigma_{i+2},\ldots,\sigma_n}(\sigma_{i+1})\mathcal{R}^w_{\Box_i}f(\sigma')$, by transitivity of $\mathcal{R}^w_{\Box_i}$. But

$$g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2) \in \mathcal{R}^w_{\Box_2}[g^{n-3}_{\sigma_4,\ldots,\sigma_n}(\sigma_3)],$$
$$g^{n-3}_{\sigma_4,\ldots,\sigma_n}(\sigma_3) \in \mathcal{R}^w_{\Box_3}[g^{n-4}_{\sigma_5,\ldots,\sigma_n}(\sigma_4)],$$
$$\ldots,$$
$$g^{n-i}_{\sigma_{i+1},\ldots,\sigma_n}(\sigma_i) \in \mathcal{R}^w_{\Box_i}[g^{(n-i)-1}_{\sigma_{i+2},\ldots,\sigma_n}(\sigma_{i+1})].$$

So $g^{n-i}_{\sigma_{i+1},\ldots,\sigma_n}(\sigma_i)\mathcal{R}^w_{\Box_i}f(\sigma')$, by the Euclidean property of $\mathcal{R}^w_{\Box_i}$. Now, it follows that $g^{n-(i-1)}_{\sigma_i,\ldots,\sigma_n}(\sigma_{i-1}) \in \mathcal{R}^w_{\Box_{i-1}}[g^{n-i}_{\sigma_{i+1},\ldots,\sigma_n}(\sigma_i)]$. So $g^{n-(i-1)}_{\sigma_i,\ldots,\sigma_n}(\sigma_{i-1})\mathcal{R}^w_{\Box_i}f(\sigma')$, by upward Euclideanity. By an analogous argument, since $g^{n-(i-2)}_{\sigma_{i-1},\ldots,\sigma_n}(\sigma_{i-2}) \in \mathcal{R}^w_{\Box_{i-2}}[g^{n-(i-1)}_{\sigma_i,\ldots,\sigma_n}(\sigma_{i-1})]$, it follows that $g^{n-(i-2)}_{\sigma_{i-1},\ldots,\sigma_n}(\sigma_{i-2})\mathcal{R}^w_{\Box_i}f(\sigma')$, by upward Euclideanity. By multiple repetitions of this argument, $g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)\mathcal{R}^w_{\Box_i}f(\sigma')$. Since $g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$, by upward Euclideanity, again, $g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1)\mathcal{R}^w_{\Box_i}f(\sigma')$, and therefore $f(\sigma)\mathcal{R}^w_{\Box_i}f(\sigma')$. Conversely, suppose that $f(\sigma)\mathcal{R}^w_{\Box_i}z$. Then $f(\sigma) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$. So

$$g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)\mathcal{R}^w_{\Box_1}f(\sigma)\mathcal{R}^w_{\Box_i}z,$$

from which $g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)\mathcal{R}^w_{\Box_i}z$ follows by upward transitivity. Now, $g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2) \in \mathcal{R}^w_{\Box_2}[g^{n-3}_{\sigma_4,\ldots,\sigma_n}(\sigma_3)]$, whence

$$g^{n-3}_{\sigma_4,\ldots,\sigma_n}(\sigma_3)\mathcal{R}^w_{\Box_2}g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)\mathcal{R}^w_{\Box_i}z,$$

from which $g^{n-3}_{\sigma_4,\ldots,\sigma_n}(\sigma_3)\mathcal{R}^w_{\Box_i}z$ follows by upward transitivity. Similarly, it follows that $g^{n-3}_{\sigma_4,\ldots,\sigma_n}(\sigma_3) \in \mathcal{R}^w_{\Box_3}[g^{n-4}_{\sigma_5,\ldots,\sigma_n}(\sigma_4)]$, whence

$$g^{n-4}_{\sigma_5,\ldots,\sigma_n}(\sigma_4)\mathcal{R}^w_{\Box_3}g^{n-3}_{\sigma_4,\ldots,\sigma_n}(\sigma_3)\mathcal{R}^w_{\Box_i}z,$$

from which $g^{n-4}_{\sigma_5,\ldots,\sigma_n}(\sigma_4)\mathcal{R}^w_{\Box_i}z$ follows by upward transitivity. By repeating this argument, $g^{n-i}_{\sigma_{i+1},\ldots,\sigma_n}(\sigma_i)\mathcal{R}^w_{\Box_i}z$, but since $g^{n-i}_{\sigma_{i+1},\ldots,\sigma_n}(\sigma_i) \in \mathcal{R}^w_{\Box_i}[g^{(n-i)-1}_{\sigma_{i+2},\ldots,\sigma_n}(\sigma_{i+1})]$, it follows that

$$g^{(n-i)-1}_{\sigma_{i+2},\ldots,\sigma_n}(\sigma_{i+1})\mathcal{R}^w_{\Box_i}g^{n-i}_{\sigma_{i+1},\ldots,\sigma_n}(\sigma_i)\mathcal{R}^w_{\Box_i}z,$$

and so $g^{(n-i)-1}_{\sigma_{i+2}\ldots\sigma_n}(\sigma_{i+1})\mathcal{R}^w_{\Box_i}z$ follows by transitivity of $\mathcal{R}^w_{\Box_i}$. Then $z \in \mathcal{R}^w_{\Box_i}[g^{(n-i)-1}_{\sigma_{i+2},\ldots,\sigma_n}(\sigma_{i+1})]$. Now, since $g^{n-i}_{\sigma_{i+1},\ldots,\sigma_n}$ is a surjective function, it follows that there is a $v \in W^w$ such that

$$(*)\ g^{n-i}_{\sigma_{i+1}\ldots\sigma_n}(v) = z.$$

So consider the i-diagonal sequence $(v, ..., v, \sigma_{i+1}, ..., \sigma_n) \in (W^w)^n$, which is identical with σ beyond i. Then $\sigma\mathcal{R}'_{\Box_i}(v, ..., v, \sigma_{i+1}, ..., \sigma_n)$, and so:

$$\begin{aligned} f((v, ..., v, \sigma_{i+1}, ..., \sigma_n)) &= g^{n-1}_{v,\ldots,v,\sigma_{i+1},\ldots,\sigma_n}(v) && \text{[by def. of } f\text{]} \\ &= \mathcal{R}^w_{@_i}(g^{n-i}_{\sigma_{i+1},\ldots,\sigma_n}(v)) && \text{[by Lemma 4.1.4(2)]} \\ &= \mathcal{R}^w_{@_i}(z). && \text{[by } (*)\text{]} \end{aligned}$$

So $z\mathcal{R}^w_{@_i}f((v, ..., v, \sigma_{i+1}, ..., \sigma_n))$, whence $z\mathcal{R}^w_{\Box_i}f((v, ..., v, \sigma_{i+1}, ..., \sigma_n))$, by $\mathbf{AT}_i$. Then, by transitivity of $\mathcal{R}^w_{\Box_i}$, $f(\sigma)\mathcal{R}^w_{\Box_i}f((v, ..., v, \sigma_{i+1}, ..., \sigma_n))$ follows. Furthermore, $z\mathcal{R}^w_{@_i}z$ follows from the original hypothesis and $\mathbf{AR}_i$, in which case both $z\mathcal{R}^w_{@_i}f((v, ..., v, \sigma_{i+1}, ..., \sigma_n))$ and $z\mathcal{R}^w_{@_i}z$, from which $f((v, ..., v, \sigma_{i+1}, ..., \sigma_n)) = z$ follows by $\mathbf{F}_{@_i}$, as desired.

$[\mathcal{R}_{\Box_n}]$ Suppose that $\sigma\mathcal{R}'_{\Box_n}\sigma'$. So σ' is n-diagonal. Then $f(\sigma') = g^{n-1}_{\sigma'_2,\ldots,\sigma'_n}(\sigma'_1) = \mathcal{R}^w_{@_n}(\sigma'_n)$, by Lemma 4.1.4(2), whence $f(\sigma') \in im(\mathcal{R}^w_{@_n})$. It follows that $f(\sigma)\mathcal{R}^w_{\Box_n}f(\sigma')$, by Lemma 4.1.2(2). Conversely, suppose that $f(\sigma)\mathcal{R}^w_{\Box_n}z$. So $z\mathcal{R}^w_{@_n}z$, by $\mathbf{AR}_n$, in which case there is a $v \in W^w$, namely, u, such that

$$(\star)\ \mathcal{R}^w_{@_n}(v) = z,$$

as $\mathcal{R}^w_{@_n}$ is a function. Now consider the sequence $\sigma^n_v \in (W^w)^n$. Then:

$$\begin{aligned} f(\sigma^n_v) &= g^{n-1}_{v,\ldots,v}(v) && \text{[by def. of } f\text{]} \\ &= \mathcal{R}^w_{@_n}(v) && \text{[by Lemma 4.1.4(2)]} \\ &= z. && \text{[by } (\star)\text{]} \end{aligned}$$

Additionally, $\sigma\mathcal{R}'_{\Box_n}\sigma^n_v$.

$[\mathcal{R}_{@_1}]$ Suppose that $\sigma\mathcal{R}'_{@_1}\sigma'$. Then $\sigma = \sigma'$. Since $\mathcal{R}^w_{@_1}$ is reflexive, it follows that $f(\sigma)\mathcal{R}^w_{@_1}f(\sigma')$. Now suppose that $f(\sigma)\mathcal{R}^w_{@_1}z$. Because $\mathcal{R}^w_{@_1}$ is reflexive, $f(\sigma)\mathcal{R}^w_{@_1}f(\sigma)$, hence $f(\sigma) = z$, as $\mathcal{R}^w_{@_1}$ is a function. Also, $\sigma\mathcal{R}'_{@_1}\sigma$.

$[\mathcal{R}_{@_i}, 1 < i < n]$ Suppose that $\sigma\mathcal{R}'_{@_i}\sigma'$. Then σ' is i-diagonal and identical with σ beyond $i-1$. So

$$f(\sigma') = g^{n-1}_{\sigma'_2,\ldots,\sigma'_n}(\sigma'_1) = \mathcal{R}^w_{@_i}(g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)),$$

by Lemma 4.1.4(2), and $f(\sigma) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$. Then $f(\sigma)\mathcal{R}^w_{\Box_1}g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)$, by symmetry of $\mathcal{R}^w_{\Box_1}$. Also,

$$g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2) \in \mathcal{R}^w_{\Box_2}[g^{n-3}_{\sigma_4,\ldots,\sigma_n}(\sigma_3)],$$

$$g^{n-3}_{\sigma_4,\ldots,\sigma_n}(\sigma_3) \in \mathcal{R}^w_{\Box_3}[g^{n-4}_{\sigma_5,\ldots,\sigma_n}(\sigma_4)],$$

$$\ldots,$$

$$g^{n-(i-1)}_{\sigma_i,\ldots,\sigma_n}(\sigma_{i-1}) \in \mathcal{R}^w_{\Box_{i-1}}[g^{n-i}_{\sigma_{i+1},\ldots,\sigma_n}(\sigma_i)].$$

Since σ' is identical with σ beyond $i-1$, it follows that $g^{n-(i-1)}_{\sigma_i,\ldots,\sigma_n}(\sigma_{i-1}) \in [g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)]_{\mathcal{R}^w_{\Box_{i-1}}}$. And since $g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)\mathcal{R}^w_{@_i}f(\sigma')$, it follows that $g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)\mathcal{R}^w_{\Box_{i-1}}f(\sigma')$, by $\mathbf{A}_i$, and so

$$g^{n-(i-1)}_{\sigma_i,\ldots,\sigma_n}(\sigma_{i-1})\mathcal{R}^w_{\Box_{i-1}}f(\sigma'),$$

as $\mathcal{R}^w_{\Box_{i-1}}$ is Euclidean. But since $g^{n-(i-2)}_{\sigma_{i-1},\ldots,\sigma_n}(\sigma_{i-2}) \in \mathcal{R}^w_{\Box_{i-2}}[g^{n-(i-1)}_{\sigma_i,\ldots,\sigma_n}(\sigma_{i-1})]$, it follows that $g^{n-(i-2)}_{\sigma_{i-1},\ldots,\sigma_n}(\sigma_{i-2})\mathcal{R}^w_{\Box_{i-1}}f(\sigma')$, by upward Euclideanity. And since we have $g^{n-(i-3)}_{\sigma_{i-2},\ldots,\sigma_n}(\sigma_{i-3}) \in \mathcal{R}^w_{\Box_{i-3}}[g^{n-(i-2)}_{\sigma_{i-1}\ldots\sigma_n}(\sigma_{i-2})]$, it follows that

$$g^{n-(i-3)}_{\sigma_{i-2},\ldots,\sigma_n}(\sigma_{i-3})\mathcal{R}^w_{\Box_{i-1}}f(\sigma'),$$

again by upward Euclideanity. By repeating this argument, $g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)\mathcal{R}^w_{\Box_{i-1}}f(\sigma')$, follows by upward Euclideanity. Then $f(\sigma)\mathcal{R}^w_{\Box_{i-1}}f(\sigma')$, follows from upward Euclideanity, too. But since $g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)\mathcal{R}^w_{@_i}f(\sigma')$, it follows that $f(\sigma')\mathcal{R}^w_{@_i}f(\sigma')$, by $\mathbf{AT}_i$ and $\mathbf{AR}_i$, respectively. So $f(\sigma)\mathcal{R}^w_{@_i}f(\sigma')$, by $\mathbf{A4}_i$. Conversely, suppose that $f(\sigma)\mathcal{R}^w_{@_i}z$. Consider the sequence $\sigma' \in (W^w)^n$ such that σ' is i-diagonal and identical with σ beyond $i-1$. Then $\sigma\mathcal{R}'_{@_i}\sigma'$. So

$$f(\sigma') = g^{n-1}_{\sigma'_2,\ldots,\sigma'_n}(\sigma'_1) = \mathcal{R}^w_{@_i}(g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)),$$

by Lemma 4.1.4(2), and $f(\sigma) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$. By the same argument as in the forth direction, it follows that $f(\sigma)\mathcal{R}^w_{@_i}f(\sigma')$. But since $f(\sigma)\mathcal{R}^w_{@_i}z$, by assumption, it follows that $f(\sigma') = z$, by $\mathbf{F}_{@_i}$.

$[\mathcal{R}_{@_n}]$ Suppose that $\sigma\mathcal{R}'_{@_n}\sigma'$. Then σ' is n-diagonal and identical with σ beyond $(n-1)$. So

$$f(\sigma') = g^{n-1}_{\sigma'_2,\ldots,\sigma'_n}(\sigma'_1) = \mathcal{R}^w_{@_n}(\sigma'_n),$$

by Lemma 4.1.4(2), and $f(\sigma) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$. Since $\sigma_n = \sigma'_n$, it follows that $\sigma_n\mathcal{R}^w_{@_n}f(\sigma')$, and so both $\sigma_n\mathcal{R}^w_{\Box_{n-1}}f(\sigma')$, by $\mathbf{A}_n$, and $f(\sigma')\mathcal{R}^w_{@_n}f(\sigma')$, by $\mathbf{AR}_n$. Now, $g^1_{\sigma_n}(\sigma_{n-1}) \in \mathcal{R}^w_{\Box_{n-1}}[\{\sigma_n\}]$, so it follows that $g^1_{\sigma_n}(\sigma_{n-1})\mathcal{R}^w_{\Box_{n-1}}f(\sigma')$, as $\mathcal{R}^w_{\Box_{n-1}}$ is Euclidean. Also, $g^2_{\sigma_{n-1},\sigma_n}(\sigma_{n-2}) \in \mathcal{R}^w_{\Box_{n-2}}[g^1_{\sigma_n}(\sigma_{n-1})]$. So $g^2_{\sigma_{n-1},\sigma_n}(\sigma_{n-2})\mathcal{R}^w_{\Box_{n-1}}f(\sigma')$ by upward Euclideanity. Analogously,

$$g^3_{\sigma_{n-2},\sigma_{n-1},\sigma_n}(\sigma_{n-3}) \in \mathcal{R}^w_{\Box_{n-3}}[g^2_{\sigma_{n-1},\sigma_n}(\sigma_{n-2})].$$

So $g^3_{\sigma_{n-2},\sigma_{n-1},\sigma_n}(\sigma_{n-3})\mathcal{R}^w_{\Box_{n-1}}f(\sigma')$, again by upward Euclideanity. By repetitions of this argument, it follows that $g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)\mathcal{R}^w_{\Box_{n-1}}f(\sigma')$, by upward Euclideanity. Since we have $f(\sigma)\mathcal{R}^w_{\Box_1}g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)$, by symmetry of $\mathcal{R}^w_{\Box_1}$, $f(\sigma)\mathcal{R}^w_{\Box_{n-1}}f(\sigma')$ follows by upward transitivity. Therefore, $f(\sigma)\mathcal{R}^w_{@_n}f(\sigma')$, by $\mathbf{A4}_n$. Conversely, suppose that $f(\sigma)\mathcal{R}^w_{@_n}z$. Now consider the sequence $\sigma' \in (W^w)^n$ such that σ' is n-diagonal and identical with σ beyond $n-1$. Also $\sigma\mathcal{R}'_{@_n}\sigma'$. So

$$f(\sigma') = g^{n-1}_{\sigma'_2,\ldots,\sigma'_n}(\sigma'_1) = \mathcal{R}^w_{@_n}(\sigma'_n),$$

by Lemma 4.1.4(2), and $f(\sigma) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$. By the same argument as in the forth direction, it follows that $f(\sigma)\mathcal{R}^w_{@_n}f(\sigma')$. But since $f(\sigma)\mathcal{R}^w_{@_n}z$, it follows that $f(\sigma') = z$, by $\mathbf{F}_{@_n}$.

This concludes the back and forth cases. It remains to show that f is surjective. Consider any $u \in W^w$. By Lemma 4.1.2(1),

$$u \in \mathcal{R}^w_{\Box_1}[\mathcal{R}^w_{\Box_2}[\ldots[\mathcal{R}^w_{\Box_i}[\ldots[\mathcal{R}^w_{\Box_n}[\{w\}]]]]]],$$

so there are elements $v_n, ..., v_2 \in W^w$, and a chain

$$w\mathcal{R}^w_{\Box_n} v_n \mathcal{R}^w_{\Box_{n-1}} v_{n-1} \mathcal{R}^w_{\Box_{n-2}} v_{n-2} ... \mathcal{R}^w_{\Box_3} v_3 \mathcal{R}^w_{\Box_2} v_2 \mathcal{R}^w_{\Box_1} u.$$

Consider the mapping $g^1_{v_n} : W^w \to \mathcal{R}^w_{\Box_{n-1}}[\{v_n\}]$. As this is surjective, and $v_{n-1} \in \mathcal{R}^w_{\Box_{n-1}}[\{v_n\}]$, there is a $z_{n-1} \in W^w$ such that $g^1_{v_n}(z_{n-1}) = v_{n-1}$. Now consider the mapping $g^2_{z_{n-1},v_n} : W^w \to \mathcal{R}^w_{\Box_{n-2}}[g^1_{v_n}(z_{n-1})]$. As this is surjective, and $v_{n-2} \in \mathcal{R}^w_{\Box_{n-2}}[g^1_{v_n}(z_{n-1})]$, there is a $z_{n-2} \in W^w$ such that $g^2_{z_{n-1},v_n}(z_{n-2}) = v_{n-2}$. After multiple repetitions of this argument, consider the mapping $g^{n-1}_{z_2,...,z_{n-1},v_n} : W^w \to \mathcal{R}^w_{\Box_1}[g^{n-2}_{z_3,...,z_{n-1},w}(z_2)]$. By a previous iteration of the argument, $g^{n-2}_{z_3,...,z_{n-1},v_n}(z_2) = v_2$, and so $u \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{z_3,...,z_{n-1},v_n}(z_2)]$. As $g^{n-1}_{z_2,...,z_{n-1},v_n}$ is also surjective, there is a $z_1 \in W^w$ such that $g^{n-1}_{z_2,...,z_{n-1},v_n}(z_1) = u$. Therefore, there is a sequence $(z_1, z_2, ..., z_{n-1}, v_n) \in (W^w)^n$ such that

$$f((z_1, z_2, ..., z_{n-1}, v_n)) = g^{n-1}_{z_2,...,v_n}(z_1) = u,$$

as desired. □

This is sufficient to prove that $\mathbf{S@}_n$ is the multidimensional logic of sequence frames:

Theorem 4.1.2 (Soundness and completeness) $\mathbf{S@}_n$ *is sound and strongly complete with respect to the class* F *of sequence frames.*

Proof. Soundness is clear given the correspondence between the axioms of $\mathbf{S@}_n$ and the properties of n-dimensional sequence frames. For strong completeness, by Theorem 4.1.1, $\mathbf{S@}_n$ is strongly complete with respect to the class of $\mathbf{S@}_n$-frames, that is, $\mathsf{F}_{\mathbf{S@}_n}$. So it suffices to prove that a set of formulas is satisfiable in a sequence frame if and only if it is satisfiable in an $\mathbf{S@}_n$-frame. Suppose that a set of formulas Γ is satisfiable in a sequence frame $\mathfrak{F} \in \mathsf{F}$. Because every sequence frame is an $\mathbf{S@}_n$-frame, it follows that Γ is also satisfiable in an $\mathbf{S@}_n$-frame. Conversely, suppose that Γ is satisfiable in an $\mathbf{S@}_n$-frame, say, $\mathfrak{F}$. Then Γ is satisfiable in a model $\mathfrak{M} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, V)$ based on $\mathfrak{F}$. So there is a point w in $\mathfrak{M}$ such that $\mathfrak{M}, w \vDash \Gamma$. Let $\mathfrak{M}^w = (W^w, (\mathcal{R}^w_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}^w_{@_i})_{1\leq i\leq n}, V^w)$ be the submodel of $\mathfrak{M}$ generated by w. Since modal satisfaction is invariant under taking generated submodels, and $w \in W^w$, it follows that $\mathfrak{M}^w, w \vDash \Gamma$, and so Γ is satisfiable in the point-generated subframe $\mathfrak{F}^w$ of $\mathfrak{F}$. But, by Lemma 4.1.5, $\mathfrak{F}^w$ is a bounded morphic image of a sequence frame, say, $\mathfrak{F}' = (W', (\mathcal{R}'_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}'_{@_i})_{1\leq i\leq n})$. Let $f : \mathfrak{F}' \twoheadrightarrow \mathfrak{F}^w$ be a surjective bounded morphism from $\mathfrak{F}'$ onto $\mathfrak{F}^w$, and $\mathfrak{M}' = (W', (\mathcal{R}'_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}'_{@_i})_{1\leq i\leq n}, V')$ a model based on $\mathfrak{F}'$ such that for each $p \in \mathsf{PROP}$, $V'(p) = \{\sigma \in W' \mid f(\sigma) \in V^w(p)\}$. Because f is surjective, and $w \in W^w$, let σ' be a sequence in W' such that $f(\sigma') = w$. Then $\mathfrak{M}^w, f(\sigma') \vDash \Gamma$. But as $\mathfrak{M}^w$ is a bounded morphic image of $\mathfrak{M}'$, this implies that $\mathfrak{M}', \sigma' \vDash \Gamma$, since modal satisfaction is invariant under bounded morphisms between models. Therefore, Γ is satisfiable in the sequence frame $\mathfrak{F}'$. □

Recall that in Fritz (2014: 386) a class of frames with distinguished elements is derived from matrix frames—or, according to our nomenclature, two-dimensional sequence frames (again, for a language without $@_1$). The logic of these frames is then derived syntactically from **2Dg**, and a completeness proof is presented in Fritz (2014: 394). Given the completeness result established just above, we show that Fritz's argument can once more be generalized for the n-dimensional case. We say that an *n-dimensional sequence frame with distinguished elements* is a quadruple,

$$\mathfrak{F}^D = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, D)$$

where $(W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ is an n-dimensional sequence frame such that $W = S^n$, and $D = \{\overline{s} \mid s \in S\}$. Let F^D be the class of n-dimensional sequence frames with distinguished elements.

Models based on such frames with distinguished elements are defined in the obvious way, and the logical notions of validity and consequence are now relativized to generalized diagonal points $\overline{s}$, as expected. Even though, as pointed out by Fritz (2014: 286), such logics characterizing frames with distinguished elements are not, in general, normal, for they do not have to be closed under the rule of generalization (or necessitation) for each $\mathcal{O}_i$, they can be defined syntactically from their normal counterparts, as it were, as follows:

Definition 4.1.9 $\vdash_n^D \varphi$ iff $\vdash_n @_n\varphi$.

Let $\mathbf{S}@_n^D$ be the logic characterizing n-dimensional sequence frames with distinguished elements. Given the strong completeness of $\mathbf{S}@_n$ relative to the class of n-dimensional sequence frames, the argument for the strong completeness of $\mathbf{S}@_n^D$ relative to F^D is then a simple adaptation of the argument in Fritz (2014: 394), Theorem 2.11.

Theorem 4.1.3 (Soundness and completeness) $\mathbf{S}@_n^D$ *is sound and strongly complete with respect to* F^D.

4.2 n-dimensional Tableaux

It is very natural, as will be seen below, to define indexed tableau systems for $\mathbf{S}@_n$ following the style of Melvin Fitting's prefixed tableaux for modal logics.[16] In fact, generalizations of indexed tableau systems for a variety of two-dimensional modal logics appear in chapter 2 as well as in Gilbert (2016).

In order to define indices and index-sequences for the tableaux, we apply similar conventions as found in Definition 4.1.2 for sequences of possible worlds. A nice feature of these tableaux is their simplicity, for we only need, in effect, a single rule for each modal operator in the language rather than multiple rules corresponding to multiple properties of the accessibility relations, as is usually the case. Next we define the notions of *index-sequences, indexed*

[16] These can be found in several papers as well as in Fitting and Mendelsohn (1998).

formulas, and *root* of an n-dimensional tableau. These will consist in slight modifications of the notions appearing in Definition 4.1.2 for sequences of possible worlds.

Definition 4.2.1 (Index-sequence, formula, root) An *index* is a natural number greater than 0. Let $\mathtt{s} = (x_1, ..., x_n)$ be an *index-sequence*, where each x_i is an index. We will often write $\mathtt{s}_i$ for x_i. An *indexed formula* is an expression, $[\varphi]\mathtt{s}$, where $\mathtt{s}$ is an index-sequence and φ is a formula of $\mathcal{L}_n^@$. All indexed formulas are enclosed within brackets. The *root* of an n-dimensional tableau always contains the negation of the formula we are attempting to prove indexed by $(1, 2, ..., n)$, that is, an index-sequence with the natural numbers ordered from 1 to n.

Definition 4.2.2 (Notation) Let $\mathtt{s}_z^i$ be that index-sequence $\mathtt{t}$ for which $\mathtt{t}_1 = \mathtt{t}_2 = ... = \mathtt{t}_i = z$ and for $j > i$, $\mathtt{t}_j = \mathtt{s}_j$. Additionally, for an index x, $\overline{x}$ is that sequence $\mathtt{s}$ for which $\mathtt{s}_i = x$ for all $i \in \{1, ..., n\}$.

As per usual, a branch of a tableau is *closed* if for some formula, φ, and index-sequence $\mathtt{s}$, both $[\varphi]\mathtt{s}$ and $[\neg\varphi]\mathtt{s}$ occur on the branch. A tableau is *closed* just in case all of its branches are closed. Otherwise, the tableau is *open.* Finally, a *tableau proof* of a sentence, φ, is a closed tableau for $[\neg\varphi]\mathtt{s}$, where $\mathtt{s}$ is the index-sequence $(1, 2, ..., n)$. The tableau rules for $\mathbf{S}@_n$ are displayed in Figure 8 with some provisos for the modal rules explained below.

The rules $(\neg\neg)$, $(\wedge)$, and $(\vee)$ comprise the portion from the classical propositional calculus. The necessity rules, (ν_i), are applied to any index z already occurring on the branch, and the possibility rules, (π_i), may be used provided the index z is new to the branch. These are just the usual restrictions on indexed-tableau rules but generalized for n-dimensional modal logic. In the case of $(@_i)$, the rule says to take the ith element of the sequence in question and copy it $(i-1)$ times towards the very first element in the sequence. For instance, the following is a proof of $\Diamond_3 p \rightarrow \Box_2\Box_1@_3\Diamond_3 p$ in a four-dimensional tableau:

$$(\neg\neg) : \frac{[\neg\neg\varphi]\mathsf{s}}{[\varphi]\mathsf{s}}$$

$$(\wedge) : \frac{[\varphi \wedge \psi]\mathsf{s}}{\begin{array}{c}[\varphi]\mathsf{s}\\ [\psi]\mathsf{s}\end{array}} \quad \frac{[\neg(\varphi \vee \psi)]\mathsf{s}}{\begin{array}{c}[\neg\varphi]\mathsf{s}\\ [\neg\psi]\mathsf{s}\end{array}} \quad \frac{[\neg(\varphi \to \psi)]\mathsf{s}}{\begin{array}{c}[\varphi]\mathsf{s}\\ [\neg\psi]\mathsf{s}\end{array}} \quad \frac{[\varphi \leftrightarrow \psi]\mathsf{s}}{\begin{array}{c}[\varphi \to \psi]\mathsf{s}\\ [\psi \to \varphi]\mathsf{s}\end{array}}$$

$$(\vee) : \frac{[\varphi \vee \psi]\mathsf{s}}{[\varphi]\mathsf{s} \mid [\psi]\mathsf{s}} \quad \frac{[\neg(\varphi \wedge \psi)]\mathsf{s}}{[\neg\varphi]\mathsf{s} \mid [\neg\psi]\mathsf{s}} \quad \frac{[\varphi \to \psi]\mathsf{s}}{[\neg\varphi]\mathsf{s} \mid [\psi]\mathsf{s}}$$

$$\frac{[\neg(\varphi \leftrightarrow \psi)]\mathsf{s}}{[\neg(\varphi \to \psi)]\mathsf{s} \mid [\neg(\psi \to \varphi)]\mathsf{s}}$$

$$(\nu_i) : \frac{[\Box_i\varphi]\mathsf{s}}{[\varphi]\mathsf{s}^i_z} \quad \frac{[\neg\Diamond_i\varphi]\mathsf{s}}{[\neg\varphi]\mathsf{s}^i_z} \qquad (\pi_i) : \frac{[\Diamond_i\varphi]\mathsf{s}}{[\varphi]\mathsf{s}^i_z} \quad \frac{[\neg\Box_i\varphi]\mathsf{s}}{[\neg\varphi]\mathsf{s}^i_z}$$

$$(@_i) : \frac{[@_i\varphi]\mathsf{s}}{[\varphi]\mathsf{s}^i_{\mathsf{s}_i}} \quad \frac{[\neg @_i\varphi]\mathsf{s}}{[\neg\varphi]\mathsf{s}^i_{\mathsf{s}_i}}$$

Figure 4.2.1: n-Dimensional Tableau Rules.

1. $[\neg(\Diamond_3 p \to \Box_2\Box_1 @_3\Diamond_3 p)](1,2,3,4)$
2. $[\Diamond_3 p](1,2,3,4)$
3. $[\neg\Box_2\Box_1 @_3\Diamond_3 p](1,2,3,4)$
4. $[p](5,5,5,4)$
5. $[\neg\Box_1 @_3\Diamond_3 p](6,6,3,4)$
6. $[\neg @_3\Diamond_3 p](7,6,3,4)$
7. $[\neg\Diamond_3 p](3,3,3,4)$
8. $[\neg p](5,5,5,4)$

$\times$

Items 2 and 3 result from the rule $(\wedge)$ applied to the negated conditional. Item 4 results from applying the (π_3) rule to item 2, and so the index-sequence $(5,5,5,4)$ appears on the tableau because the index 5 is new, that is, it had no previous occurrences. Items 5 and 6 result from applications of (π_2) and (π_1) to items 3 and 5, respectively. The index-sequence $(6,6,3,4)$ appears in 5 because we needed a new index for $\neg\Box_2$ from item 3, and the index-sequence

$(7,6,3,4)$ occurs because we needed a new index for $\neg\Box_1$ from item 5. Finally, 7 results from 6 by an application of $(@_3)$, and so the index 3 is copied down in the index-sequence twice, and 8 results by applying (ν_3), and so any index could have been chosen to compose its index-sequence. The index-sequence $(5,5,5,4)$, therefore, is chosen so that the tableau closes.

4.2.1 Soundness

Definition 4.2.3 (Satisfiability) Let F be a set of indexed formulas. We say F is *satisfiable* in a model $\mathfrak{M} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, V)$, where $W = S^n$, if there is a function f assigning to each single index $\mathtt{s}_i$ occurring in a sequence $\mathtt{s}$ in F a possible world $f(\mathtt{s}_i) \in S$, and, where g is a function such that $g(\mathtt{s}) = (f(\mathtt{s}_1), ..., f(\mathtt{s}_n))$,

- If $[\varphi]\mathtt{s} \in F$, then φ is true at $g(\mathtt{s})$, i.e. $\mathfrak{M}, g(\mathtt{s}) \vDash \varphi$.
- If the index-sequences $\mathtt{s}$ and $\mathtt{s}^i_z$ are in F, then $g(\mathtt{s})\mathcal{R}_{\Box_i} g(\mathtt{s})^i_{f(z)}$. If, moreover, $z = \mathtt{s}_i$, then also $g(\mathtt{s})\mathcal{R}_{@_i} g(\mathtt{s})^i_{f(z)}$.

Definition 4.2.4 A tableau branch $\mathfrak{b}$ is *satisfiable* if the set of indexed formulas on it is satisfiable in some model, and a tableau is *satisfiable* if some branch of it is satisfiable.

It follows immediately from the definitions that a closed tableau is not satisfiable. Then the following lemma can be established by induction on formulas:

Lemma 4.2.1 *If one of the rules is applied to a tableau that is satisfiable in an n-dimensional sequence model $\mathfrak{M} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, V)$, it results in another tableau satisfiable in $\mathfrak{M}$.*

Theorem 4.2.1 (Soundness) *If φ has an n-dimensional tableau proof, then φ is valid on the class of n-dimensional sequence frames* F.

Proof. Suppose φ has a tableau proof, in which case there is a closed tableau, $\mathcal{T}$, beginning with $[\neg\varphi]\mathtt{s}$, where $\mathtt{s}$ is the sequence $(1, 2, ..., n)$. For a contradiction, assume that φ is not valid. Thus, there is an n-dimensional sequence model $\mathfrak{M} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, V)$, where $W = S^n$, for some non-empty set S, such that $\mathfrak{M}, \sigma \not\vDash \varphi$ for some sequence $\sigma \in W$. Define a function f such that for each $\mathtt{s}_i \in \mathtt{s}$ and $\sigma_i \in \sigma$, $f(\mathtt{s}_i) = \sigma_i$, such that $g(\mathtt{s}) = (f(\mathtt{s}_1), ..., f(\mathtt{s}_n)) = \sigma$. Then $\{[\neg\varphi]\mathtt{s}\}$ is satisfiable in $\mathfrak{M}$. Moreover, since the one point tableau $[\neg\varphi]\mathtt{s}$ is satisfiable in $\mathfrak{M}$, $\mathcal{T}$ is also satisfiable in $\mathfrak{M}$, by Lemma 4.2.1. Therefore, $\mathcal{T}$ is both closed and satisfiable, which is impossible, whence $\mathfrak{M}, \sigma \vDash \varphi$, as desired. □

4.2.2 Completeness

We establish completeness by constructing a systematic tableau proof procedure in the style of Fitting (1983, ch. 8) producing a tableau proof in case there is one, and directing us to a countermodel otherwise. As it is shown below, this will also give us a decision procedure for the validities.

Systematic proof procedure

For the purposes of a systematic proof procedure, we do not want to apply the rules for a single occurrence of an indexed formula more than one time, so we need to make sure that each occurrence of an indexed formula is used only once. This is not difficult to keep track of, as we can just introduce a device to declare formulas *used*—a check mark, for instance, as in Smullyan (1968). The procedure is defined by stages, and for stage $n = 1$, introduce $[\neg\varphi]\mathtt{s}$ at the tableau's root, where $\mathtt{s}$ is the sequence $(1, 2, ..., n)$. Next, suppose n stages of the procedure have been completed. If the tableau is closed, or every occurrence of an indexed formula is used, then stop. If, on the other hand, the tableau remains open, then we proceed to stage $n + 1$ as follows: take the highest occurrence of an indexed formula in the tree, say, $[\psi]\mathtt{s}$, that is not yet used.[17] Now for each open branch $\mathfrak{b}$ through the occurrence of $[\psi]\mathtt{s}$, do the following:

I If $[\psi]\mathtt{s}$ is atomic or a negation thereof, then declare it used. This ends stage $n+1$.

II If $[\psi]\mathtt{s}$ is $[\neg\neg\chi]\mathtt{t}$, add $[\chi]\mathtt{t}$ to the end of $\mathfrak{b}$.

III If $[\psi]\mathtt{s}$ is $[(\zeta \wedge \chi)]\mathtt{t}$, add both $[\zeta]\mathtt{t}$ and $[\chi]\mathtt{t}$ to the end of $\mathfrak{b}$ (analogously for the other conjunctive cases).

IV If $[\psi]\mathtt{s}$ is $[(\zeta\vee\chi)]\mathtt{t}$, split the end of $\mathfrak{b}$ into $\mathfrak{b}_1$ and $\mathfrak{b}_2$, adding $[\zeta]\mathtt{t}$ and $[\chi]\mathtt{t}$ to the end of $\mathfrak{b}_1$ and $\mathfrak{b}_2$, respectively (analogously for the other disjunctive cases).

V If $[\psi]\mathtt{s}$ is $[\Diamond_i\chi]\mathtt{t}$, take the smallest integer, z, which is new to $\mathfrak{b}$, and add $[\chi]\mathtt{t}^i_z$ to the end of $\mathfrak{b}$ (analogously for $\neg\Box_i$).

VI If $[\psi]\mathtt{s}$ is $[\Box_i\chi]\mathtt{t}$, for every index z occurring on the branch, add $[\chi]\mathtt{t}^i_z$ to the end of $\mathfrak{b}$ (analogously for $\neg\Diamond_i$), and then add a fresh occurrence of $[\Box_i\chi]\mathtt{t}$ to the end of $\mathfrak{b}$. If, however, all possible $[\chi]\mathtt{t}^i_z$ already occur on the branch, do nothing (not even checking off the original boxed formula).

VII If $[\psi]\mathtt{s}$ is $[@_i\chi]\mathtt{t}$, add $[\chi]\mathtt{t}^i_{\mathtt{t}_i}$ to the end of $\mathfrak{b}$ (analogously for $\neg@_i$).

Once this procedure is completed for each open branch $\mathfrak{b}$ through $[\psi]\mathtt{s}$, mark that formula as used. This completes stage $n+1$. Now, either the systematic procedure resulted in a closed tableau, producing a proof; the procedure terminated, producing an open branch; or it does not terminate, producing a possibly infinite open branch.

Let $\mathfrak{b}$ be a *complete* open branch of a tableau $\mathcal{T}$ if any application of a rule to an indexed formula occurring on the branch would only introduce indexed formulas already occurring

[17] If there are multiple occurrences of unused formulas at the same level, chose the unused formula occurring on the leftmost branch.

on it. In order to prove completeness we show that if $\mathfrak{b}$ is any complete open branch of a tableau, then $\mathfrak{b}$ is satisfiable in an n-dimensional sequence model constructed out of the formulas and index-sequences occurring on $\mathfrak{b}$. Once completeness is established we come back to the case where the systematic procedure does not terminate.

Definition 4.2.5 We define a frame $\mathfrak{F}_{\mathfrak{b}} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ induced by $\mathfrak{b}$ as follows. Let the set of indices in $\mathfrak{b}$ be $S = \{x \mid \text{ for some } \mathsf{s} \in \mathfrak{b} \text{ and some } i, x = \mathsf{s}_i\}$, and let $W = S^n$. For every $\mathsf{s}, \mathsf{s}^i_z \in W$, set $\mathsf{s}\mathcal{R}_{\Box_i}\mathsf{s}^i_z$, and when $z = \mathsf{s}_i$, set $\mathsf{s}\mathcal{R}_{@_i}\mathsf{s}^i_z$, too. A model $\mathfrak{M}_{\mathfrak{b}} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, V)$, based on $\mathfrak{F}_{\mathfrak{b}}$, is defined as follows: for each $p \in \mathsf{PROP}$, let $\mathsf{s} \in V(p)$ provided that $[p]\mathsf{s}$ occurs on $\mathfrak{b}$; otherwise set $\mathsf{s} \notin V(p)$.

It is simple to verify that the frames defined above are in fact n-dimensional sequence frames. The following lemma is then established by induction on formulas:

Lemma 4.2.2 (Truth lemma) *Let $\mathfrak{b}$ be a complete open branch of a tableau. Then, let $\mathfrak{F}_{\mathfrak{b}} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ be a frame induced by $\mathfrak{b}$, and $\mathfrak{M}_{\mathfrak{b}} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, V)$ a model based on $\mathfrak{F}_{\mathfrak{b}}$. For every indexed formula, $[\varphi]\mathsf{s}$,*

$$[\varphi]\mathsf{s} \text{ occurs on } \mathfrak{b} \Leftrightarrow \mathfrak{M}_{\mathfrak{b}}, \mathsf{s} \models \varphi.$$

Theorem 4.2.2 (Completeness) *If φ is valid on the class of n-dimensional sequence frames F, then φ has an n-dimensional tableau proof.*

Proof. We prove the contrapositive. Suppose φ does not have a tableau proof, in which case any tableau $\mathcal{T}$ beginning with $[\neg\varphi]\mathsf{s}$ remains open. Let $\mathfrak{b}$ be a complete open branch of $\mathcal{T}$ such that $[\neg\varphi]\mathsf{s}$ occurs on $\mathfrak{b}$. By Lemma 4.2.2, there is an n-dimensional sequence frame $\mathfrak{F}_{\mathfrak{b}} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$, induced by $\mathfrak{b}$, and index-sequence $\mathsf{s} \in W$, such that for an n-dimensional sequence model, $\mathfrak{M}_{\mathfrak{b}} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, V)$, based on $\mathfrak{F}_{\mathfrak{b}}$, we have $\mathfrak{M}_{\mathfrak{b}}, \mathsf{s} \not\models \varphi$. Consequently, φ is not valid on the class of n-dimensional sequence frames F. □

n-dimensional tableaux can be easily adapted for n-dimensional sequence frames with distinguished elements. For an index x, let $\overline{x}$ be that index-sequence $\mathtt{s} = (x_1, ..., x_n)$ for which $x_i = x$ for all $1 \leq i \leq n$. Then a tableau proof in this case for a sentence φ is a closed tableau for $[\neg\varphi]\overline{x}$. The soundness and completeness results above (as well as the decidability argument below) are also straightforward to adapt.

4.2.3 Decidability

In this section we prove decidability for $\mathbf{S}@_n$ by showing that the systematic tableau procedure used to prove completeness is in fact a decision procedure. For this purpose we can adapt the argument in Fitting (1983: 410-413). Say we are attempting a proof by way of the systematic procedure described above. However, we now modify the procedure so that no branch contains multiple occurrences of the same indexed formulas: if the procedure says we should add an indexed formula to the end of a branch that already contains an occurrence of it, refrain from adding the repeated occurrence of that indexed formula. This involves a simple modification to our step VI above—which is completely analogous to Fitting's (1983: 411). Now, we prove that the (modified) systematic procedure we have described guarantees termination for n-dimensional tableaux, whereby they either deliver a proof or a countermodel.

Proposition 4.2.1 (Termination) *Let φ be an $\mathcal{L}_n^@$-formula. A systematic attempt to prove φ terminates.*

Proof. For a contradiction, suppose a systematic attempt at proving φ never terminates. Then there is a sequence $\mathcal{T}_0, \mathcal{T}_1, ...$ of tableaux which successively properly extend each other. Let $\mathcal{T}$ be their limit, i.e.

$$\mathcal{T} = \bigcup_{n\geq 0} \mathcal{T}_n.$$

Then $\mathcal{T}$ is a finitely branching infinite labeled tree. Hence, it follows by König's lemma that $\mathcal{T}$ has an infinite branch, say, $\mathfrak{b}$. Consider the set $\Sigma = \{\mathtt{s} \mid [\psi]\mathtt{s} \in \mathfrak{b}$ for some formula $\psi\}$. Because $\mathfrak{b}$ is infinite, the set Σ must be infinite, too. Otherwise, the set of indexed formulas, say, Φ, occurring on $\mathfrak{b}$ would be finite, since $\Phi \subseteq sb(\varphi) \times \Sigma$, where $sb(\varphi)$ is the set of subformulas of φ, which is itself finite. Moreover, because no multiple occurrences of the same indexed formulas are allowed, it is not possible for some index-sequence in Σ to occur infinitely many times postfixing a formula on $\mathfrak{b}$.

Let the *height of an index-sequence* $\mathtt{s}$, written $height(\mathtt{s})$, be the greatest natural number occurring in $\mathtt{s}$, and for every $k \in \mathbb{N}$, define $\Sigma_k = \{\mathtt{s} \mid \mathtt{s} \in \Sigma$ and $height(\mathtt{s}) = k\}$. Then we let the set Σ of all index-sequences on $\mathfrak{b}$ be the union of all sets Σ_k. There are two cases that could allow Σ to be infinite:

(1) For some height j, $card(\Sigma_j) = \aleph_0$. But this is impossible. Consider the set $S_j = \{\mathtt{s} \mid height(\mathtt{s}) = j\}$ of all possible sequences (with length n) of height j. Note that $card(S_j) = j^n - (j-1)^n$. Since $card(S_j)$ is finite, and $\Sigma_j \subseteq S_j$, there is no height j such that $card(\Sigma_j) = \aleph_0$.[18]

(2) For each height j, $card(\Sigma_j) < \aleph_0$. Because Σ is infinite, and Σ is the union of all Σ_k, each of which is of finite cardinality, there must be infinitely many heights j such that $\Sigma_j \neq \emptyset$. Let the *modal degree* of a formula occurring on $\mathfrak{b}$ be the number of modal operators occurring in that formula. Because the modal degree of formulas gets smaller as the index-sequences increase in height, there is a height k such that any formula on the branch $\mathfrak{b}$ postfixed by an index-sequence of that height has modal degree 0. And since the modal rules do not apply in this case, there cannot be further index-sequences on $\mathfrak{b}$ with height $> k$, in which case there are not infinitely many heights j such that $\Sigma_j \neq \emptyset$, a contradiction. □

Corollary 4.2.1 (Decidability) $\mathbf{S@}_n$ *is decidable.*

[18] The original argument for (1) was much more complicated and, in fact, unnecessary, in light of $card(S_j)$ being finite, as a referee very helpfully pointed out.

Proof. Immediately from the fact that systematic tableau constructions always terminate in a finite number of steps. □

Conclusion

We have developed and proved several results about n-dimensional sequence modal logics with actuality operators. These are natural generalizations of two-dimensional modal logics known in the philosophical literature as logics for a priori knowledge, necessity, and actuality. In particular, the structures investigated here were shown to be extensions to any finite dimension of the structures studied in Fritz (2014) for the system **2Dg**. The completeness argument from Fritz was also generalized to any finite dimension, and it can now be seen as a special case of the completeness proof presented in §4.1.3 by just setting $n = 2$ (given that $@_1$ is directly eliminable from the language). Additionally, we have developed sound and complete tableau calculi for the logics herein considered, and used these to show decidability by means of a systematic tableau construction. There are, of course, many questions left open by the present chapter which can be settled by future research, namely, questions in proof-theory, model-theory, and even in complexity theory. Some examples of these include the following:

- Fritz's axiom system **2Dg** was naturally generalized for any arbitrary dimension, and the tableau calculi presented here also involved a natural generalization of tableaux for basic modal logic. Is there a natural generalization for n dimensions of the hypersequents developed in Restall (2012) for two-dimensional modal logic? What about different proof systems for modal logic such as natural deduction systems?

- What are the n-dimensional modal logics generated with multiple actuality operators such as A and D? There are many possibilities here, including classes of frames defined with a single distinguished point $z \in S$, and so formulas in the scope of $\mathsf{D}@_n$ would be evaluated relative to the generalized diagonal point $(z, ..., z)$; or even frames with a distinguished point $(z, ..., z)$ for a language containing A, and so any formula in the scope of A would be evaluated relative to that point. It is expected that the main theorems proved here, such as completeness and decidability, would transfer to these cases, although it would be interesting to compare the expressive power of such languages relative to distinct notions of validity. One relevant question would be under which circumstances these actuality operators are eliminable from their respective languages, that is, whether the eliminability of the actuality operator, for instance, from the basic (one-dimensional) modal language when real-world validity is assumed (see, for example, Hazen et al. (2013)) carries over to n-dimensions.

- Antonelli and Thomason (2002) proved that adding propositional quantifiers to a modal logic with two **S5** modalities results in a system that is (recursively) intertranslatable with full second-order logic. What about in this framework? Even though $\Box_1$ is an equivalence relation, any $\Box_i$, for $i > 2$, is not. So, in case there is a translation, it has to be modified accordingly, for the procedure described by Antonelli and Thomason does not seem to generalize for $\mathbf{S}@_2$ (or $\mathbf{S}@_n$) with propositional quantification. Still, is there any recursive translation with full second-order logic, such as in Antonelli and Thomason's case, that shows second-order $\mathbf{S}@_2$ and hence $\mathbf{S}@_n$ $(1 < n)$ to be undecidable?

- Even though the validity problem for n-dimensional sequence modal logics was shown to be decidable, what is its complexity?

- Is the sequence modal logic of ω-dimensions complete or decidable?

- Standefer (forthcoming) investigates relevant logics with the actuality operator. It would be interesting to see the relevant counterparts of multidimensional modal logics with, possibly, many actuality operators.

Bibliography

Antonelli, G. A., and Thomason, R. H. (2002). Representability in Second-Order Propositional Poly-Modal Logic. *The Journal of Symbolic Logic*, 67: 1039-1054.

Blackburn, P., de Rijke, M., and Venema, Y. (2001). *Modal Logic*. Cambridge: Cambridge University Press.

Boolos, G. (1984). To Be is to be a Value of a Variable (or to be Some Values of Some Variables). *Journal of Philosophy*, 81: 430-49.

Bricker, P. (1989). Quantified Modal Logic and the Plural *De Re*. *Midwest studies in philosophy*, XIV: 372-394.

Chalmers, D. (1996). *The Conscious Mind: In Search of a Fundamental Theory*. Oxford: Oxford University Press.

Chalmers, D. (2004). Epistemic Two-Dimensional Semantics. *Philosophical Studies*, 118: 153-226.

Chalmers, D. (2011a). Propositions and Attitude Ascriptions: A Fregean Account. *Nous*, 45(4): 595-639.

Chalmers, D. (2011b). Actuality and Knowability. *Analysis*, 71(3): 411-419.

Chalmers, D. (2012). *Constructing the World*. Oxford: Oxford University Press.

Chalmers, D. and Jackson, F. (2001). Conceptual Analysis and Reductive Explanation. *Philosophical Review*, 110: 315-61.

Chalmers, D., and Rabern, B. (2014). Two-Dimensional Semantics and the Nesting Problem. *Analysis*, 74(2): 210-224.

Cresswell, M. (1990). *Entities and Indices*. Dordrecht: Kluwer Academic Publishers.

Crossley, J. N. and Humberstone, L. (1977). The Logic of "Actually". *Reports on Mathematical Logic*, 8: 11-29.

Davies, M. (2004). Reference, Contingency, and the Two-Dimensional Framework. *Philosophical Studies*, 118: 83-131.

Davies, M., and Humberstone, L. (1980). Two Notions of Necessity. *Philosophical Studies*, 38(1): 1-30.

Dorr, C. (2011). *De Re* A Priori Knowledge. *Mind*, 120: 939-91.

Dummett, M. (1993). Could There be Unicorns? In M. Dummett, *The Seas of Language*, Oxford: Clarendon Press, pp. 328–348.

Evans, G. (1979). Reference and Contingency. *The Monist*, 62: 161-189.

Forbes, G. (1985). *The Metaphysics of Modality*. Oxford: Clarendon Press.

Forbes, G. (1989). *Languages of Possibility*. Oxford: Basil Blackwell.

Fitting, M. (1983). *Proof Methods for Modal and Intuitionistic Logics*. Dordrecht: D. Reidel Publishing Co.

Fitting, M. and Mendelsohn, R.L. (1998). *First-Order Modal Logic*. Kluwer, Dordrecht.

French, R. (2012). An Argument Against General Validity? *Thought*, 1(1): 4-9.

Fritz, P. (2013). A Logic for Epistemic Two-Dimensional Semantics. *Synthese*, 190: 1753-1770.

Fritz, P. (2014). What is the Correct Logic of Necessity, Actuality, and Apriority? *The Review of Symbolic Logic*, 7(3): 385-414.

Fusco, M. (forthcoming). A Two-Dimensional Logic for Diagonalization and the A Priori. *Synthese*.

Gabbay, D. M., Kurucz, A., Wolter, F., & Zakharyaschev, M. (2003). *Many-Dimensional Modal Logics: Theory and Applications*. Studies in Logic and the Foundations of Mathematics; Vol. 148, Elsevier.

Gilbert, D. (2016). Two-Dimensional Tableaux. *Australasian Journal of Logic*, 13(7):

143-170.

Hanson, W. (2006). Actuality, Necessity, and Logical Truth. *Philosophical Studies*, 130: 437-459.

Hanson, W. (2014). Logical Truth in Modal Languages: a Reply to Nelson and Zalta. *Philosophical Studies*, 167(2): 327-339.

Hazen, A. P. (1976). Expressive Completeness in Modal Language. *Journal of Philosophical Logic*, 5: 25-46.

Hazen, A. P. (1978). The Eliminability of the Actuality Operator in Propositional Modal Logic. *Notre Dame Journal of Formal Logic*, 4: 617–622.

Hazen, A. P., Rin, B., and Wehmeier, K. (2013). Actuality in Propositional Modal Logic. *Studia Logica*, 101: 487-503.

Hodes, H. (198a). Some Theorems on the Expressive Limitations of Modal Languages. *Journal of Philosophical Logic*, 13: 13-26.

Hodes, H. T. (1984b). Axioms for Actuality. *Journal of Philosophical Logic*, 13: 27-34.

Holliday, W. and Perry, J. (2014). Roles, rigidity, and quantification in epistemic logic. In *Trends in Logic, Outstanding Contributions: Johan van Benthem on Logic and Information Dynamics*, eds. A. Baltag and S. Smets, Springer, pp. 591-629.

Humberstone, L. (1982). Scope and subjunctivity. *Philosophia* 12(1-2): 99-126.

Humberstone, L. (2004). Two-Dimensional Adventures. *Philosophical Studies*, 118: 17-65.

Humberstone, L. (2008). Can Every Modifier be Treated as a Sentence Modifier? *Philosophical Perspectives*, 22: 241-275.

Jackson F. (1998). *From Metaphysics to Ethics: A Defense of Conceptual Analysis*. Oxford University Press.

Kamp, J.A.W. (1971). Formal Properties of "Now". *Theoria*, 37: 227-273.

Kaplan, D. (1978). Dthat. In P. Cole (Ed.), *Syntax and Semantics*. New York: Academic Press.

Kaplan, D. (1989). Demonstratives. In J. Almog, J. Perry and H. Wettstein (eds.), *Themes from Kaplan*, Oxford: Oxford University Press, 481-563.

Kocurek, A. (2018). On the Expressive Power of First-Order Modal Logic with Two-Dimensional Operators. *Synthese*, 195(10): 4373-4417.

Kripke, S. (1958). A Completeness Theorem in Modal Logic. *Journal of Symbolic Logic*, 24(1): 1-14.

Kripke, S. (1963). Semantical Considerations on Modal Logic. *Acta Philosophica Fennica*, 16: 83-94.

Kripke, S. (1980). *Naming and Necessity*. Cambridge, MA: Harvard University Press.

Lampert, F. (2018a). Actuality, Tableaux, and Two-Dimensional Modal Logics. *Erkenntnis*, 83: 403-443.

Lampert, F. (2018a). Actuality and the A Priori. *Philosophical Studies*, 175: 809-830.

Lampert, F. (2020). The Logic of Sequence Frames. *The Review of Symbolic Logic*, doi:10.1017/S175502032000032.

Lewis, D. (1970). How to Define Theoretical Terms. *Journal of Philosophy*, 67: 427-46.

Linnebo, Ø. (2016). Plurals and Modals. *Canadian Journal of Philosophy*, 46(4-5): 654-676.

Meyer, U. (2013). Counterpart Theory and the Actuality Operator. *Mind*, 122(485): 27-42.

Montague, R. (1968). Pragmatics. In R. Klibansky (ed.), *Contemporary Philosophy: A Survey*, Florence: La Nuova Italia Editrice, pp. 102-122. Reprinted in R. H. Thomason (ed.), *Formal Philosophy: Selected Papers of Richard Montague*, New Haven, CT: Yale University Press, 1974, pp. 95-118.

Nelson, M. and Zalta, E. (2012). A Defense of Contingent Logical Truths. *Philosophical Studies* 157(1): 153-162.

Poggiolesi, F., and Restall, G. (2012). Interpreting and Applying Proof Theories for Modal Logic. In G. Restall and G. Russell (Eds.), *New Waves in Philosophical Logic*, Basingstoke, Houndmills: Palgrave Macmillan, pp. 39-62.

Priest, G. (2008). *An Introduction to Non-Classical Logics. From If to Is.* Cambridge: Cambridge University Press, 2nd edition.

Prior, A. (1968). Now. *Nous*, 2: 101-119.

Ray, G. and Lampert, F. (manuscript). *Natural* Natural Deduction for Modal Logics.

Read, S. (2015). Semantic Pollution and Syntactic Purity. *The Review of Symbolic Logic*, 8(4): 649-661.

Restall, G. (2012). A Cut-Free Sequent System for Two-Dimensional Modal Logic, and Why it Matters. *Annals of Pure and Applied Logic*, 163: 1611-1623.

Salmon, N. (1989). The Logic of What Might Have Been. *Philosophical Review*, 98: 3-34.

Segerberg, K. (1973). Two-Dimensional Modal Logic. *Journal of Philosophical Logic*, 2: 77-96.

Sider, T. (2010). *Logic for Philosophers.* New York: Oxford University Press.

Smullyan, R. (1968). *First-Order Logic.* New York, Springer-Verlag.

Soames, S. (2005). *Reference and Description: The Case Against Two-Dimensionalism.* Princeton University Press.

Stalnaker, R. C. (1978). Assertion. In P. Cole (Ed.), *Syntax and Semantics: Pragmatics*, Vol. 9, New York: Academic Press, pp. 315-339.

Standefer, S. (forthcoming). Actual Issues for Relevant Logics. *Ergo*.

Stephanou, Y. (2005). First-Order Modal Logic with an Actually Operator. *Notre Dame Journal of Formal Logic*, 46(4): 381-405.

Urquhart, A. (2015). First Degree Formulas in Quantified S5. *Australasian Journal of Logic*, 12(5): 204-210.

Uzquiano, G. (2011). Plural Quantification and Modality. *Proceedings of the Aristotelian Society*, 111(2): 219-50.

Van Benthem, J. (2010). Frame Correspondences in Modal Predicate Logic. In *Proofs, Categories and Computations: Essays in Honor of Grigori Mints.* London: College Publications.

Vlach, F. (1973). 'Now' and 'Then': A Formal Study in the Logic of Tense Anaphora. PhD Thesis, Los Angeles: UCLA.

Weatherson, B. (2001). Indicative and Subjunctives. *Philosophical Quarterly*, 51: 200-216.

Wehmeier, K. (2003). World Traveling and Mood Swings. In Benedikt Löwe, Thoralf Räsch, and Wolfgang Malzkorn (eds.), *Foundations of the Formal Sciences II*, Kluwer (Trends in Logic), pp. 257-260.

Wehmeier, K. (2004). In the mood. *Journal of Philosophical Logic*, 33: 607-630.

Wehmeier, K. (2005). Modality, mood, and descriptions. In Reinhard Kahle (ed.), *Intensionality: An Interdisciplinary Discussion*, Lecture Notes in Logic, Wellesley Mass.: AK Peters, pp. 187-216.

Wehmeier, K. (2013). Subjunctivity and Conditionals. *The Journal of Philosophy*, 110: 117-142.

Wehmeier, K. (2014). Nothing but d-Truth. *Analytic Philosophy*, 55(1): 114-117.

Williamson, T. (2003). Everything. *Philosophical Perspectives*, 17: 415-65.

Williamson, T. (2010). Necessitism, Contingentism, and Plural Quantification. *Mind*, 119(475): 657-748.

Williamson, T. (2013a). *Modal Logic as Metaphysics*. Oxford: Oxford University Press.

Williamson, T. (2013b). Gettier Cases in Epistemic Logic. *Inquiry*, 56(1): 1-14.

Zalta, E. (1988). Logical and Analytic Truths that are Not Necessary. *Journal of*

Philosophy 85: 57-74.