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On Cohomology of The Space of Linear Generic Points in Three Dimensional Projective Space

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On Cohomology of The Space of Linear Generic Points in Three Dimensional Projective Space

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy

in

Mathematics

by

Yuzhe Bai

Committee in Charge:

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Professor David Meyer
Professor Dragos Oprea

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The Dissertation of Yuzhe Bai is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

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Dedication

I would like to dedicate this thesis to my loving parents.

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Abstract of The Dissertation

On Cohomology of The Space of Linear Generic Points in Three Dimensional Projective Space

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Professor Justin Roberts, Chair

In this paper, we first introduce a type of generalized configuration spaces C_n^m and $\mathbb{P}C_n^m$, namely, it consists the n-tuples in affine or projective m spaces where any $m+1$ points are not contained in a hyperplane. The spaces can be considered as complements of affine and projective algebraic sets respectively and hence are quasi-

projective varieties. There is a free action of $AGL_{m+1}(\mathbb{C})$ and $PGL_{m+1}(\mathbb{C})$ on C_n^m and $\mathbb{P}C_n^m$ respectively, which gives the spaces a structure as trivial bundles. Also, similar to the classic configuration spaces, there is a natural S_n action on these spaces and that puts an induced S_n-represnetation structure on the cohomology.

Based on these structures, there are a few result in fundamental group and cohomology group of these spaces studied by other mathematicians. We will introduce some of the results and focus on the cohomology of $\mathbb{P}C_n^m$. To go one more step further, let X_n^m be the quotient of $\mathbb{P}C_n^m$ by $PGL_{m+1}(\mathbb{C})$, we will study the cohomology of X_n^3 as a S_n representation for $n = 6$ and 7. In the case that $n = 7$, our result is based on the assumption that the cohomology groups $H^k(X_n^m)$ are pure of Hodge-Tate type (k,k). The calculation is done by the technique of twisted point counting introduced by [3]. To utilize the point counting technique, we will discuss the background that makes it possible and introduce the Grothendiecck-Lefchetz formula which is the main foundation of the calculation.

Chapter 1

Introduction of the generic configuration spaces and result about them

The configuration space $\mathrm{Conf}_n(X)$ of a given topological space X is defined as the subspace of X^n which consists of the ordered n-tuples of distinct points in X. When X is affine or projective, we can generalize the conditions to some other intuitively generic cases. In this thesis, we will study the cases where X is the complex affine space $\mathbb{C}^n$ or complex projective space $\mathbb{CP}^n$. We denote C_n^m, where $n \geq m+1$, the subspace of $(\mathbb{C}^m)^n$ consisting of the n-tuples of points in $\mathbb{C}^m$ such that each $m+1$ points don't lie on the same affine hyperplane in $\mathbb{C}^m$. We can define $\mathbb{P}C_n^m$ corresponding to the projective case in the same manner.

These collection of generalized configuration spaces are studied by many mathematicians, but not much is learned of the topological properties of the topological properties of them. Moulton has studied the fundamental groups of C_n^2 and found a series of groups project into the fundamental groups. In recent years, there are several studies about the cohomology of the projective case $\mathbb{P}C_n^m$ for $m = 2$. Namely, Ashraf and Bercenau calculated the cohomology of $\mathbb{P}C_3^2$, Das and O'connor[4] computed the cohomology for $\mathbb{P}C_6^2$ as well as Begvall[2] studied the cohomology of $\mathbb{P}C_7^2$.

In this thesis, we will talk about some basic structure of C_n^m and $\mathbb{P}C_n^m$ as well as some results about the topology by other mathematicians. At the end, we will mainly focus on the cohomology of the $\mathbb{P}C_n^m$ where $m = 3$ and $n \leq 7$.

1.1 Basic structures of C_n^m and $\mathbb{P}C_n^m$

As we defined earlier, we can see that when $m = 1$, C_n^m is the classic configuration space $\mathrm{Conf}_n(\mathbb{C})$ and for general C_n^m, it is also a subspace of the configuration space $\mathrm{Conf}_n(\mathbb{C}^m)$. Therefore, the natural S_n action on the configuration space will also work on C_n^m, that is, for a permutation σ, σ sends an n-tuple $(x_1,...,x_n)$ to $(x_{\sigma(1)},...,x_{\sigma(n)})$. Moreover, C_n^m can be viewed as a complement of affine algebraic set. Namely, define $\Delta = (\mathbb{C}^m)^n - C_n^m$. We have

$$\Delta = \{(x_1,...,x_n) \in \mathbb{C}^m : \exists \{i_1,...i_{m+1}\} \in [n] \text{ s.t. } \mathrm{span}\{x_{i_1},...x_{i_{m+1}}\} \neq \mathbb{C}^m\}.$$

Then for any of the x_i, we can write $x_i = (z_{i,1},...z_{i,m})$ with respect to the standard basis. Consider the $(m+1) \times n$ matrix

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ z_{1,1} & \cdots & z_{n,1} \\ \vdots & \ddots & \vdots \\ z_{1,m} & \cdots & z_{n,m} \end{pmatrix}.$$

The product of all possible $(m+1) \times (m+1)$ minors of A will be the defining polynomial for the algebraic set Δ.

The exact same argument can be used for the projective case. That is, $\Delta' = (\mathbb{CP}^m)^n - \mathbb{P}C_n^m$. For any point $x_i \in \mathbb{CP}^m$ the i^{th}, we have $x_i = [z_{i,1} : ... : z_{i,m+1}]$. Define the matrix

$$B = \begin{pmatrix} z_{1,1} & \cdots & z_{n,1} \\ \vdots & \ddots & \vdots \\ z_{1,m+1} & \cdots & z_{n,m+1} \end{pmatrix}.$$

Then Δ' is a projective algebraic set that is defined by the product of all the $(m+1)\times(m+1)$ minors of B.

Remark 1.1.1. *These observations make C_n^m and $\mathbb{P}C_n^m$ quasiprojective varities, and in fact smooth schemes over $\mathbb{Z}$.*

Moreover, based on the definition of C_n^m, the affine general linear group $AGL_{m+1}(\mathbb{C})$ can act on C_n^m diagonally. The action is free for all $n \geq m+1$ and is transitive when $n = m+1$. Let $\overline{C_n^m} = C_n^m / AGL_{m+1}(\mathbb{C})$. We have

$$\overline{C_n^m} \simeq \{(x_1, ...x_n) \in C_n^m : x_1 = \overrightarrow{0}, x_i = e_{i-1} \text{ if } 2 \leq i \leq m+1\}$$

where e_i's are the standard basis vector of $\mathbb{C}^m$.

By the action, C_n^m is a principal $AGL_{m+1}(\mathbb{C})$ bundle over $\overline{C_n^m}$ with a section. Hence, we have the following lemma,

Lemma 1.1.2. $C_n^m \simeq \overline{C_n^m} \times AGL_{m+1}(\mathbb{C})$.

Corollary 1.1.2.1. $\pi_1(C_{m+1}^m) \simeq \mathbb{Z}$

Proof. When $n = m+1$, since the action of $AGL_n(\mathbb{C})$ on C_n^m is free and transitive, the orbit map is a diffeomorphism. Hence, C_n^m is diffeomorphic to $AGL_{m+1}(\mathbb{C})$. Also, since $\mathbb{C}^m \to AGL_{m+1}(\mathbb{C}) \to GL_m(\mathbb{C})$ is a fibration, the fundamental group $\pi_1(C_{m+1}^m)$ of C_{m+1}^m is isomorphic to the fundamental group of $GL_m(\mathbb{C})$, which is $\mathbb{Z}$. $\square$

Remark 1.1.3. *Notice that the action of $AGL_{m+1}(\mathbb{C})$ commutes with the action of S_n, hence the action of $AGL_{m+1}(\mathbb{C})$ will descend to the quotient space C_n^m/S_n and the covering map will be $AGL_{m+1}(\mathbb{C})$-equivariant. Similarly, the action of S_n will descend to $\overline{C_n^m}$ and the covering map is S_n-equivariant.*

Similar to the affine case, the projective general linear group $PGL_{m+1}(\mathbb{C})$ acts on $\mathbb{P}C_n^m$ diagonally. The action is free for all $n \geq m+2$ and is transitive when $n = m+2$. Let $\overline{\mathbb{P}C_n^m} \simeq \mathbb{P}C_n^m/PGL_{m+1}(\mathbb{C})$. Let $w = [1:1:...:1] \in \mathbb{CP}^m$. Then we have

$$\overline{\mathbb{P}C_n^m} = \{(x_1,...x_n) \in \mathbb{P}C_n^m : x_i = e_i' \text{ if } 1 \leq i \leq m+1, x_{m+2} = w\}$$

where e_i' is the i^{th} basis vector of the standard basis.

By similar argument, we have the following lemma

Lemma 1.1.4. $\mathbb{P}C_n^m \simeq \overline{\mathbb{P}C_n^m} \times PGL_{m+1}(\mathbb{C})$.

Corollary 1.1.4.1. $\pi(\mathbb{P}C_{m+1}^m) \simeq \mathbb{Z}/(m+1)\mathbb{Z}$

Proof. Again, since when $n = m+2$ the action is free and transitive, $\mathbb{P}C_n^m$ is diffeomorphic to $PGL_{m+1}(\mathbb{C})$. Now, we have $\pi(\mathbb{P}C_{m+1}^m) \simeq \pi_1(PGL_{m+1}(\mathbb{C})) \simeq \mathbb{Z}/(m+1)\mathbb{Z}$. □

The exact same argument can be made as the last remark with respect to $PGL_{m+1}(\mathbb{C})$ and S_n in the projective case.

1.2 Some results of C_n^m

Because of the similarity of the structure of C_n^m to the structure of the configuration space, we want to calculate the fundamental group of C_n^m in the way we calculate the fundamental group of $\mathrm{Conf}_n(\mathbb{C})$. Namely, the forgetting map $\pi : \mathrm{Conf}_n(\mathbb{C}) \to \mathrm{Conf}_{n-1}(\mathbb{C})$ by forgetting the last entry for each $n-tuple$ is a fibration where the fiber is $\mathbb{C}$ minus $n-1$ points. The fibration will give us a long exact sequence which result in the following sequence

$$1 \to F_n \to P_n \to P_{n-1} \to 1,$$

where F_n is the free group of rank n and P_n is the classical pure braid group.

However, although we can still define the forgetting map on C_n^m and $\mathbb{P}C_n^m$ by omitting the last entry, the map is not generally a fibration. For example, when $m = 2$, the forgetting map will no longer be a fibration when $n = 5$ for the affine case. We can see an example in figure 1.1. When $n = 5$, the fiber can be considered as the complement of a hyperplane arrangement generated by four points that no three of them are not collinear. Six lines will be generated by these four points and when we move the points in the base, it is possible that some of the lines are parallel to each other, which change the number of connected component in the fiber and make the map a fibration no more.

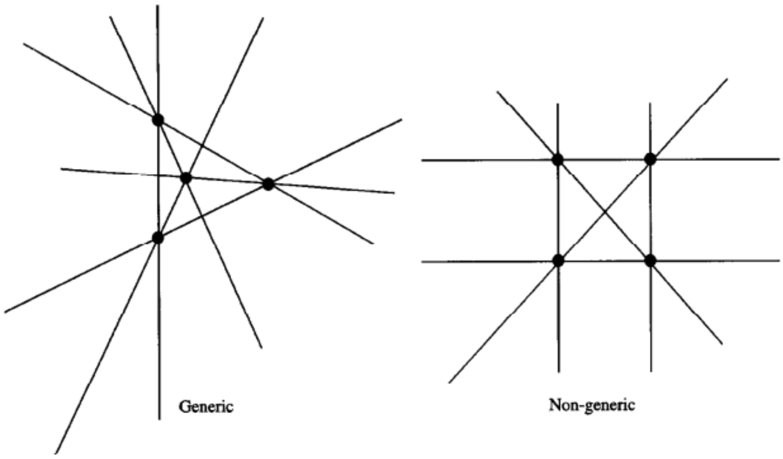

Figure 1.1 Parallel Problem When $n = 5$

Things become a little bit better in the projective case since we don't need to worry about parallel lines. However, when n gets to 7, the fiber can be considered as the complement of the hyperplane arrangement created by six points in general position. There is a neighborhood that moving some points may cause three lines intersecting at a single point that is not from those six points like what happened in figure 1.2. Therefore, the fibration argument won't help us a lot.

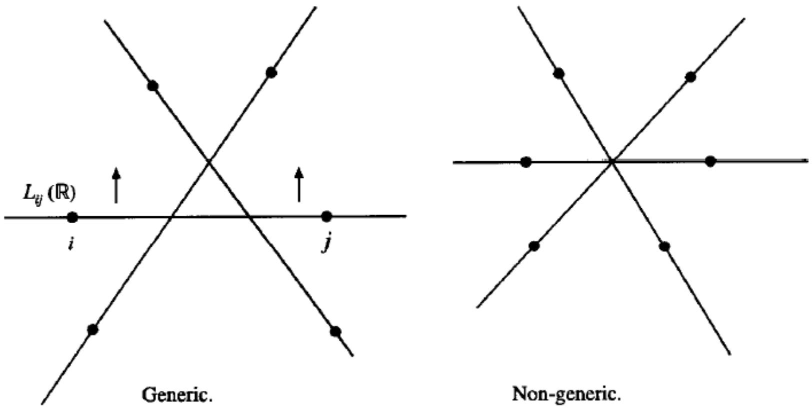

Figure 1.2 Intersection Problem When $n = 6$

Even though we cannot use the exact same argument as for pure braid group, there are some work around that help us gain some knowledge of the fundamental groups. For the affine case, Moulton[10] provided a finitely presented group that surjects onto $\pi_1(C_n^2)$ and conjectured that they are actually isomorphic. Moreover, for some special cases, we can use some other way to derive the fundamental groups. As an example, we will show that for all $m \geq 2$, $\pi_1(C_{m+2}^m) \simeq \mathbb{Z}^{m+3}$.

Notice that $C_{m+2}^m \simeq AGL_{m+1}(\mathbb{C}) \times \overline{C}_{m+2}^m$. Since

$$\overline{C_n^m} \simeq \{(x_1, ...x_n) \in C_n^m : x_1 = \overrightarrow{0}, x_i = e_{i-1} \text{ if } 2 \leq i \leq m+1\},$$

$\overline{C}_{m+2}^m$ can be viewed as a complex hyperplane complement where the hyperplane arrangement M is given by the set of points $S = \{0, e_1, ..., e_m\}$. In fact, by the points in the set S, the hyperplanes in M are defined by real equations. Hence, we can instead consider the complex hyperplane complement M as a complexified real hyperplane arrangement M'. Then by [11], we can compute the fundamental group of the complement of M by the structure of the complement of M'.

Generally, let $\mathscr{M}$ be a complex hyperplane arrangement in $\mathbb{C}^n$ where each of the hyperplane is defined by a real polynomial and $\mathscr{M}'$ be the real hyperplane arrangement defined by the same polynomials in $\mathbb{R}^n$. Let X be the complement of $\mathscr{M}'$ in $\mathbb{R}^n$. Then X is the disjoint union of codimension 0 connected components in $\mathbb{R}^n$ that were separated by hyperplanes in $\mathscr{M}'$. Then we can define a groupoid base on $\mathscr{M}'$ and X as following.

- For each connected component in X, we call it a chamber and pick a point in it and we ambiguously denote both of them by c
- For each hyperplane, we call it a cut denoted by l
- For any two adjacent chambers c_1, c_2, which means we can go from c_1 to c_2 by across only one cut, we define e_{12} to be the edge from c_1 to c_2 pass through the cut l_{12} that separates them
- A path p from one chamber c_i to another chamber c_j is a composition of connected edges $e_1^{n_1}, ..., e_k^{n_k}$ where e_m are edges between adjacent chambers and $n_m = \pm 1$. e_i^{-1} means the inverse edge of e_i.

Then we have a groupoid where vertices are the points we picked from each chamber, and elements are the paths and $e_{i,j}$ where c_i, c_j are adjacent form a generating set. We call a path p positive path if $p = e_1, ..., e_k$ and call it negative if $p = e_1^{-1}...e_k^{-1}$. A path from c_1 to c_2 is shortest if the path only passes through the cuts that separating c_1 and c_2 and passes through them exact once. Let $\sim$ be the relation such that for any two positive paths p_1 and p_2 who have the same starting and ending points, $p_1 \sim p_2$.

Let the groupoid G be the one we constructed above and Y be the complement of corresponding complex hyperplane arrangement. Then by Salvetti[11], we have the following theorem,

Theorem 1.2.1. *$\pi_1(Y)$ is isomorphic to any vertex group of $G/\sim$.*

Back to our case, M' can be considered as the boundary of a m-simplex whose faces are extended to a hyperplane. Thus, each pair of hyperplanes intersect at a codimensional two space and no three hyperplanes will share a codimensional two intersection. The vertex group with respect to M' will have $m+2$ generators and they commute with each other. Therefore, $\pi_1(\overline{C}^m_{m+2}) \simeq \mathbb{Z}^{m+2}$. As a result, we have the following lemma,

Lemma 1.2.2. *For all $m \geq 2$, $\pi_1(C^m_{m+2}) \simeq \mathbb{Z}^{m+3}$.*

Using this method, we can also try to calculate the fundamental groups of the generic fibers where there are no singularity. However, the obstacles will still hold us back from finding out the actual fundamental groups of C^m_n as a whole and that remain an open problem.

There are also results about the cohomology of C^m_n and $\mathbb{P}C^m_n$. Ashraf and Bercenau [1] calculated the cohomology of $\mathbb{P}C^2_3$ and B^2_3 where the result is given as following.

Theorem 1.2.3. *The Poincaré series of B^2_3 is given by*

$$P_{B^2_3}(t) = \frac{(1+t^5)(1-t^4)}{1-t^2} = 1+t^2+t^5+t^7.$$

Theorem 1.2.4. *The Poincaré series of $\mathbb{P}C^2_3$ is given by*

$$P_{\mathbb{P}C^2_3}(t) = \frac{(1-t^2)(1-t^4)(1-t^6)}{(1-t^2)^3} = 1+2t^2+2t^4+t^6.$$

1.3 Cohomology of $\mathbb{P}C^2_n$ and $\mathbb{P}C^3_n$

From now on, we will focus on the projective case $\mathbb{P}C^m_n$, especially when $m = 2$ and 3. For brevity, we will use B^m_n to denote $\mathbb{P}C^m_n/S_n$ and X^m_n to denote $\overline{\mathbb{P}C}^m_n$.

By the transitive and free action of $PGL_{m+1}(\mathbb{C})$ on $\mathbb{P}C^m_{m+2}$, we have the following proposition.

Proposition 1.3.1. *For a chosen basepoint $x \in \mathbb{P}C^m_{m+2}$, the orbit map $PGL_{m+1} \to \mathbb{P}C^m_{m+2}$ which is given by $g \to g \cdot x$ is a homeomorphism.*

Proposition 1.3.2. *The S_{m+2}-action on $\mathbb{P}C^m_{m+2}$ is homotopically trivial. In particular,*

$$H^*(\mathbb{P}C^m_{m+2}(\mathbb{C});\mathbb{Q}) \cong H^*(PGL_{m+1}(\mathbb{C});\mathbb{Q})$$

is trivial as an S_{m+2}-representation.

Proof. Since the action of $PGL_{m+1}(\mathbb{C})$ is free and transitive, for any point $x \in \mathbb{P}C^m_{m+2}$ and $\sigma \in S_{m+2}$, there is a corresponding element $g_\sigma \in PGL_{m+1}(\mathbb{C})$ which acts on x same as σ. By fixing x, we can define a homomorphism from S_{m+2} to $PGL_{m+1}(\mathbb{C})$ and the action of S_{m+2} goes through this homomorphism. The conclusion comes from the fact that $PGL_{m+1}(\mathbb{C})$ is a path-connected group. □

We will talk about the case that $m = 2$ which is done by Das and O'Connor, and show the result on the case $m = 3$. To compute the cohomology of F^m_n in these cases, the following remark will come in handy.

Remark 1.3.3.

$$H^*(PGL_3(\mathbb{C});\mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } * = 0,3,5,8 \\ 0 \text{ otherwise.} \end{cases}$$

Where the generators are in degree 3 and 5 have Hodge weight 2 and 3 respectively. Also, we have

$$H^*(PGL_4(\mathbb{C});\mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } * = 0,3,5,7,8,10,12,15 \\ 0 \text{ otherwise.} \end{cases}$$

Where the generators are in degree 3, 5 and 7 have Hodge weight 2, 3, and 4 respectively.

For $n > m+2$, the action of $PGL_{m+1}(\mathbb{C})$ on $\mathbb{P}C_n^m$ is no longer transitive, but it is still free. Which makes the the bundle $F_n^m \to X_n^m$ principal. Furthermore, we will show that there always exists a section. Therefore, we have the proposition as following.

Proposition 1.3.4. *For $n > m+2$, $\mathbb{P}C_n^m$ is a trivial bundle over X_n^m.*

Proof. Let $x \in \mathbb{P}C_{m+2}^m$ be the point formed by the standard bases. For any point $y = (y_1,...,y_n) \in \mathbb{P}C_n^m$, there is an unique element $g(y) \in PGL_{m+1}(\mathbb{C})$ that maps the first $m+2$ coordinates to x. Then $y \to g(y)y$ give us a section. Hence the $\mathbb{P}C_n^m$ is a principle bundle with a section, which implies that it is a trivial $PGL_{m+1}(\mathbb{C})$ bundle over X_n^m. □

By the argument above, we can identify the quotient X_n^m with the fiber of the projection $\mathbb{P}C_n^m \to \mathbb{P}C_{m+2}^m$ to the first $m+2$ coordinates. Obviously, and inclusion map $[m+2] \to [n]$ will give us a projection on the the corresponding coordinates. In fact, for a choice of basepoint in $\mathbb{P}C_{m+2}^m$ and an inclusion $[m+2] \to [n]$, there is an corresponding injection

$$H^*(PGL_{m+1}(\mathbb{C});\mathbb{Q}) \cong H^*(\mathbb{P}C_{m+2}^m;\mathbb{C}) \to H^*(\mathbb{P}C_n^m;\mathbb{Q}).$$

Because $\mathbb{P}C_{m+2}^m$ is connected, the injection is independent from the choice of the basepoint.

Also we have the following regarding the cohomology of the direct product based on Lemma 2.5.3.

Proposition 1.3.5. *With the trivial action on $H^*(PGL_{m+1};\mathbb{Q})$, the isomorphism*

$$H^*(\mathbb{P}C_n^m;\mathbb{Q}) \cong H^*(PGL_{m+1}(\mathbb{C});\mathbb{Q}) \otimes H^*(X_n^m;\mathbb{Q})$$

is S_n-equivariant.

In this thesis, we will focus on find the cohomology of B_n^m, $\mathbb{P}C_n^m$, and X_n^m for $m = 3$ and $n = 6,7$. We will use the technique called "twisted point count". The techniqual background and the method will be showed in next Chapter and the calculation will be showed in the last chapter. The case of $m = 2$ and $n = 5$ was calculated by Kisin and Lehrer[7] and the $m = 2$, $n = 6$ case is calculated by Das and O'Connor[4] by the same method. Let U be the trivial representation and other irreducible representation will be subscripted by the corresponding partitions.

Theorem 1.3.6. *With terminology above, as S_5-representations, we have*

$$H^*(X_5^2;\mathbb{Q}) \cong \begin{cases} U & \textit{if } * = 0, \\ S_{3^1 2^1} & \textit{if } * = 1, \\ S_{3^1} & \textit{if } * = 2, \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 1.3.7. *Similarly, as S_6-representations, we have*

$$H^*(X_6^2;\mathbb{Q}) \cong \begin{cases} U & \textit{if } * = 0, \\ S_{3^2} \oplus S_{4^1 2^1} & \textit{if } * = 1, \\ S_{5^1} \oplus S_{4^1}^{\oplus 2} \oplus S_{3^1} \oplus S_{3,3} \oplus S_{3,2,1}^{\oplus 2} & \textit{if } * = 2, \\ S_{5^1} \oplus S_{4^1}^{\oplus 3} \oplus S_{3^1}^{\oplus 3} \oplus S_{3^2} \oplus S_{2^3} \oplus S_{4^1,2^1}^{\oplus 2} & \\ \qquad \oplus S_{2^2}^{\oplus 2} \oplus S_{3^1 2^1}^{\oplus 3} & \textit{if } * = 3 \\ U \oplus S_{1^6} \oplus S_{5^1} \oplus S_{2^1} \oplus S_{4^1} \oplus S_{3^1}^{\oplus 2} \oplus S_{3^2}^{\oplus 2} & \\ \qquad \oplus S_{2^3}^{\oplus 3} \oplus S_{4^1 2^1}^{\oplus 2} \oplus S_{2^2} \oplus S_{3^1 2^1}^{\oplus 3} & \textit{if } * = 4 \\ 0 & \textit{otherwise.} \end{cases}$$

Then since we have

$$B_n^2 = \mathbb{P}C_n^2/S_n = (PGL_3(\mathbb{C}) \times X_n^2)/S_n,$$

we can use transfer to derive the cohomologies for B_n^2 for $n = 5$ and $n = 6$.

Corollary 1.3.7.1. *With terminology as above, we have*

$$H^*(B_5^2;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } * = 0,3,5,8 \\ 0 & \text{otherwise,} \end{cases}$$

$$H^*(B_6^2;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } * = 0,3,4,5,7,8,9,12 \\ 0 & \text{otherwise,} \end{cases}$$

The first isomorphism is induced by the orbit map and hence is an isomorphism of mixed Hodge structures. Similarly, the inclusion of $H^*(PGL_3(\mathbb{C}))$ into $H^*(B_6)$ preserves the mixed Hodge structures, and the extra generator in $H^4(B_6)$ has weight 4.

The following theorem is the main result of this paper.

Theorem 1.3.8.

$$H^*(X_6^3;\mathbb{Q}) \cong \begin{cases} U & \text{if } * = 0, \\ S_{4^1 2^1} & \text{if } * = 1, \\ S_{4^1} \oplus S_{3^1 2^1} & \text{if } * = 2, \\ S_{4^1} \oplus S_{3^2} \oplus S_{2^2} & \text{if } * = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Which as a direct consequence, we have

Corollary 1.3.8.1.

$$H^*(B_6^3;\mathbb{Q}) \cong H^*(PGL_4(\mathbb{C})) \cong \begin{cases} \mathbb{Q} & \text{if } * = 0,3,5,7,8,10,12,15 \\ 0 & \text{otherwise,} \end{cases}$$

Also, Bergvall [2] calculated the cohomology for X_7^2 and we will calculate the cohomology for X_7^3 in the third chapter of this thesis. These two spaces will have the same cohomology. The result is given by table 1.1 where the irreducible representations of S_7 are demonstrated by the corresponding partitions.

Theorem 1.3.9. *The cohomology of X_7^3 as S_7 representation is given by the table* 1.1.

Table 1.1 The cohomology of X_7^2 and X_7^3 as S_7-representation

	U	6^1	$5^1 2^1$	5^1	$4^1 3^1$	$4^1 2^1$	4^1	3^2	$3^1 2^2$	$3^1 2^1$	3^1	2^3	2^2	2^1	1^7
H^0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0
H^2	0	1	1	3	1	3	1	3	1	1	0	0	0	0	0
H^3	0	3	6	9	7	15	10	9	6	12	3	5	3	0	0
H^4	3	9	21	19	20	47	27	25	29	42	20	17	13	6	1
H^5	3	14	34	31	31	78	42	44	48	75	34	30	29	13	1
H^6	2	9	18	25	23	50	31	34	28	52	19	23	22	9	4

It is curious that these two spaces have the same cohomology even if they share the same dimension. Study can be made about the potential relation between them. Moreover, we may also study if similar situation will happen for other pairs of m's and n's.

Chapter 2

Background on Grothendieck-Lefschetz and Twisted Point Counts

The main technique we are going to use is introduced by Church, Ellenberg and Farb [3] and also used by Das and O'Connors. In the following sections, we will make an exposition of the context and summarise the important definitions and results that we need to do our calculations.

2.1 Fixed points of Frobenius morphism and Grothendieck-Lefschetz formula

For a finite field $\mathbb{F}_q$ where $q = p^r$ for some prime number p, let X be a variety defined over $\mathbb{F}_q$. We will have the geometric Frobenius morphism $\mathrm{Frob}_q : X \to X$ acts on X via coordinates by sending $x \in X$ to x^q. Moreover, for the algebraic closure $\overline{\mathbb{F}}_q$ for $\mathbb{F}_q$, the Frobenius map will also acts on, $x \in X(\overline{\mathbb{F}}_q)$, the set of $\overline{\mathbb{F}}_q$-points of X the same way. If a point $x \in X(\overline{\mathbb{F}}_q)$ is fixed by Frob_q, then we can see that for any coordinate x_i of x, we have $x_i^q = x_i$, which implies that $x_i \in \mathbb{F}_q$ for all x_i. Then we can see that the points that lie in $X(\mathbb{F}_q)$ are precisely the points that are fixed by the Frobenius morphism. Namely,

$$\left|X(\mathbb{F}_q)\right| = \mathrm{Fix}(\mathrm{Frob}_q : X(\overline{\mathbb{F}}_q) \to X(\overline{\mathbb{F}}_q)).$$

For an endomorphism $f : Y \to Y$ of a compact topological space Y, the classical Lefschetz fixed point formula will relates the number of fixed points of f and the trace of the induced action of f^* on the rational cohomology $H^*(Y;\mathbb{Q})$ of Y. Namely,

$$\#\mathrm{Fix}(f : Y \to Y) = \sum_{i\geq 0}(-1)^i \mathrm{Tr}(f : H^i(Y;\mathbb{Q})). \tag{2.1}$$

An analogue made by Grothendieck[5] was that the fixed points of Frob_q could be analyzed in the same fashion, but instead based on the étale cohomology of the base change $X_{/\overline{F}_q}$ of X to $\overline{\mathbb{F}}_q$, which we denote as $H^i_{\text{ét}}(X_{/\overline{F}_q};\mathbb{Q}_l)$ of X, where l is coprime to q. After this point, we will follow the notation from [3] and use $H^i_{\text{ét}}(X;\mathbb{Q}_l)$ as an abbreviation for $H^i_{\text{ét}}(X_{/\overline{F}_q};\mathbb{Q}_l)$. In this paper, we will treat the étale cohomology as a black box and omit the definition of it as we will not use the actual étale cohomology directly but use it as a tool to build up the bridge between point counting and the singular cohomology.

Grothendieck-Lefschetz fixed point theorem is the main result of the analogue introduced by Grothendieck, which relates the number of fixed points of a morphism $f : X \to X$ with the trace of its action on the étale cohomology $H^i_{\text{ét}}(X;\mathbb{Q}_l)$ of X, exactly the same as the classic Lefschetz fixed point formula. With f being the Frobenius map Frob_q, we will have the following fundamental formula which holds for any smooth projective variety X over $\mathbb{F}_q$:

$$\left|X(\mathbb{F}_q)\right| = \#\mathrm{Fix}(\mathrm{Frob}_q) = \sum_{i\geq 0}(-1)^i \mathrm{Tr}(\mathrm{Frob}_q : H^i_{\text{ét}}(X;\mathbb{Q}_l)). \tag{2.2}$$

In the case that X is not projective, we will use the compactly-supported étale cohomology instead and if X is smooth, we can use Poincaré duality to make a switch back.

2.2 Formula for twisted coefficients

Much more subtle counts of $\mathbb{F}_q$-points can be obtained by using a version of Grothendieck-Lefschetz with twisted coefficients. For any appropriate coefficients $\mathscr{F}$ on a smooth projective variety X defined over $\mathbb{F}_q$ (namely a so-called l-adic sheaf), we have a version of (2.2) with coefficients in $\mathscr{F}$:

$$\sum_{x\in X(\mathbb{F}_q)} \mathrm{Tr}(\mathrm{Frob}_q|\mathscr{F}_x) = \sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}_q : H^i_{\text{ét}}(X;\mathscr{F})). \tag{2.3}$$

If Frob_q fixes a point $x \in X(\mathbb{F}_q)$, it acts on the stalk $\mathscr{F}_x$ of $\mathscr{F}$ at x, and the local contribution on the left side of equation (2.3) are the trace of Frob_q on each of these stalks. On the right side, we have the étale cohomology of $X_{/\overline{\mathbb{F}}_q}$ with coefficients in $\mathscr{F}$. For non-projective X, again, we need to correct the formula by using the compactly-supported version of $H^i_{\text{ét}}(X;\mathscr{F})$.

2.3 Comparison theorem with singular cohomology

Let X be a smooth scheme such that we have a S_n action on it and $Y = X/S_n$. Then given a finite-dimensional S_n representation V over $\mathbb{Q}_l$, let $\mathscr{V}$ be the associated locally constant sheaf on $Y(\mathbb{F}_q)$. Then by Theorem 21.3 and 21.5 from [9], there is a corresponding locally constant sheaf $\mathscr{V}_{an}$ on $Y(\mathbb{C})$ whose pullback to $X(\mathbb{C})$ is trivial and

$$H^*_{\text{ét}}(Y;\mathscr{V}) \cong H^*(Y(\mathbb{C});\mathscr{V}_{an}) \cong H^*_{sing}(X(\mathbb{C});V)^{S_n} \tag{2.4}$$

where the local coefficients V are given by the action $\pi_1(Y(\mathbb{C})) \to S_n$ acting on V. Then by transfer, we get the following isomorphism

$$H^*_{sing}(X(\mathbb{C});V)^{S_n} \cong H^*_{sing}(X(\mathbb{C});\mathbb{Q}_l) \otimes_{S_n} V.$$

Moreover, V can be considered as defined over $\mathbb{Q}$ in the sense that $V = V_{\mathbb{Q}} \otimes \mathbb{Q}_l$ for some rational S_n-representation $V_{\mathbb{Q}}$. Then we also have the following isomorphism,

$$H^*_{sing}(X(\mathbb{C});\mathbb{Q}_l) \otimes_{S_n} V = (H^*_{sing}(X(\mathbb{C});\mathbb{Q}) \otimes_{S_n} V) \otimes \mathbb{Q}_l.$$

As a result, we can see that

$$H^*_{\text{ét}}(Y;\mathscr{V}) \cong (H^*_{sing}(X(\mathbb{C});\mathbb{Q}) \otimes_{S_n} V) \otimes \mathbb{Q}_l.$$

By similar reasons, we can see that on the left hand side of equation (2.4), we have

$$H^*_{\text{ét}}(Y;\mathscr{V}) \cong H^*_{\text{ét}}(X;V)^{S_n} \cong H^*_{\text{ét}}(X;\mathbb{Q}_l) \otimes_{S_n} V \cong (H^*_{\text{ét}}(X;\mathbb{Q}_l) \otimes_{S_n} V) \otimes \mathbb{Q}_l.$$

Hence, we can see that in terms of S_n-representation, the character χ^i_w, which we will talk about in the next section, of the étale cohomology is precisely the characters of the degree i, weight w part of the singular cohomology $H^*_{sing}(X(\mathbb{C}))$. This helps us relates the S_n-representation structure of the étale cohomology of $\mathbb{P}C^m_n$ and the S_n-representation of its singular cohomology.

2.4 Applying Grothendieck-Lefschetz trace formula

In our case, we need to use the formula for twisted coefficient to do point counts for the varieties $B^m_n(\mathbb{F}_q)$ to calculate the cohomology $H^*(\mathbb{P}C^m_n;\mathbb{Q})$ as S_n-representation for the case. Das and O'Connor[4] computed the case where $m = 2$, $n = 5$ and 6. We will do the computation for $m = 3$, $n = 6$ and 7.

Since $\mathbb{P}C^m_n$ and B^m_n are varieties of complex points defined over $\mathbb{Z}$, we can define them over $\mathbb{F}_q$ and we will keep the same notation to make our life easier. Because $B^m_n = \mathbb{P}C^m_n/S_n$, the S_n action on $\mathbb{P}C^m_n$ define an S_n-Galois covering $\mathbb{P}C^m_n \to B^m_n$. By Milne [9], we will obtain a natural correspondence between the set of S_n-representaions with finite dimensions up to conjugacy and the set of finite dimensional local systems on B_n^m up to isomorphism whose pullbacks to $\mathbb{P}C_n^m$ are trivial. Then for an irreducible finite dimensional S_n-representation V, let $\mathscr{V}$ be the corresponding local system, by equation (2.3), we have

$$\sum_{p\in X(\mathbb{F}_q)} \mathrm{Tr}(\mathrm{Frob}_q|\mathscr{V}_p) = \sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}_q : H^{2n-i}_{\text{ét},c}(X;\mathscr{V})), \tag{2.5}$$

where $H^*_{\text{ét},c}$ denotes the compactly supported étale cohomology.

The action of Frob_q on the stalk $\mathscr{V}_p \simeq V$ is given as follows. For a point $p \in B_n^m(\mathbb{F}_q)$, we can view p as a set $\{p_1,...,p_n\} \subseteq \mathbb{P}^2(\overline{\mathbb{F}}_q)$ (i.e. no $m+1$ points are in the same hyperplane) that is fixed by the Frobenius morphism as a set. Then Frob_q acts on p by permutation, so it will determine a unique permutation $\sigma_p \in S_n$ up to conjugacy. Frob_q acts on the S_n-representation $V \simeq \mathscr{V}_p$ as σ_p. If χ_V is the character for the corresponding representation, then we have $\mathrm{Tr}(\mathrm{Frob}_q|\mathscr{V}_p) = \chi_V(\sigma_p)$, and we can transfer the left-hand side of equation (2.5) into

$$\sum_{p\in B_n^m(\mathbb{F}_q)} \chi_V(\sigma_p).$$

For a conjugacy class $C \in \mathrm{Class}(S_n)$, let 1_C be the indicator function for C and denote $p^m_{n,C}(q) = \left|p \in B_n^m(\mathbb{F}_q)|\sigma_p \in C\right|$. We can see that $\chi_V(\sigma_p) = \sum_C \chi_V(C)1_C(\sigma_p)$ and

$$\begin{aligned} \sum_{p\in B_n^m(\mathbb{F}_q)} \chi_V(\sigma_p) =& \sum_p \sum_C \chi_V(C)1_C(\sigma_p) \\ =& \sum_C \chi_V(C) \sum_p 1_C(\sigma_p) \\ =& \sum_C \chi_V(C) p^m_{n,C}(q) \end{aligned} \tag{2.6}$$

Now, for the right-hand side of (2.5), let $\tilde{\mathscr{V}}$ be the pullback of $\mathscr{V}$ to $\mathbb{P}C_n^m$. Then by transfer, we have

$$H_{\text{ét},c}^{2mn-i}(B_n^m;\mathscr{V}) \cong H_{\text{ét},c}^{2mn-i}(\mathbb{P}C_n^m;\tilde{\mathscr{V}})^{S_n}$$

Moreover, by the argument above, we know that $\tilde{\mathscr{V}}$ is a constant sheaf. Then

$$H_{\text{ét},c}^{2mn-i}(\mathbb{P}C_n^m;\tilde{\mathscr{V}}) \cong (H_{\text{ét},c}^{2mn-i}(\mathbb{P}C_n^m;\mathbb{Q}_l)\otimes V)$$

as S_n-representation. By combining these, we have

$$H_{\text{ét},c}^{2mn-i}(B_n^m;\mathscr{V}) \cong (H_{\text{ét},c}^{2mn-i}(\mathbb{P}C_n^m;\mathbb{Q}_l)\otimes V)^{S_n} \cong H_{\text{ét},c}^{2mn-i}(\mathbb{P}C_n^m;\mathbb{Q}_l)\otimes_{\mathbb{Q}_l[S_n]} V.$$

Now, because V is an irreducible S_n-representation, it is self-dual, so we have the following identification

$$H_{\text{ét},c}^{2mn-i}(\mathbb{P}C_n^m;\mathbb{Q}_l)\otimes_{\mathbb{Q}_l[S_n]} V \cong \text{Hom}_{\mathbb{Q}_l[S_n]}(V, H_{\text{ét},c}^{2mn-i}(\mathbb{P}C_n^m;\mathbb{Q}_l)).$$

At last, since $\mathbb{P}C_n^m$ is smooth, we are able to apply Poincaré duality to get the following isomorphism

$$\text{Hom}_{\mathbb{Q}_l[S_n]}(V, H_{\text{ét},c}^{2mn-i}(\mathbb{P}C_n^m;\mathbb{Q}_l)) \cong \text{Hom}_{\mathbb{Q}_l[S_n]}(V, \text{Hom}(H_{\text{ét}}^{i}(\mathbb{P}C_n^m;\mathbb{Q}_l);\mathbb{Q}_l(-mn))).$$

where $\mathbb{Q}_l(-mn)$ is the mn^{th} cyclotomic character, i.e. the vector space $\mathbb{Q}_l$ that acted on by Frob_q as multiplying q^{mn}.

Denote $H_w^i(\mathbb{P}C_n^m)$ the subspace of $H_{\text{ét}}^i(\mathbb{P}C_n^m;\mathbb{Q}_l)$ on which Frob_q acts by q^w and letting $\chi_w^i(\mathbb{P}C_n^m)$ be the character of the corresponding representation, the equations above let us compute the trace of Frob_q as:

$$
\begin{aligned}
&\mathrm{Tr}(\mathrm{Frob}_q : \mathrm{Hom}_{\mathbb{Q}_l[S_n]}(V, \mathrm{Hom}_{\mathbb{Q}_l[S_n]}(H^i_{\text{ét}}(\mathbb{P}C^m_n; \mathbb{Q}_l); \mathbb{Q}_l(-mn)))\\
=&\mathrm{Tr}(\mathrm{Frob}_q : \mathrm{Hom}_{\mathbb{Q}_l[S_n]}(V, \oplus_w \mathrm{Hom}_{\mathbb{Q}_l[S_n]}(H^i_w(\mathbb{P}C^m_n; \mathbb{Q}_l); \mathbb{Q}_l(-mn)))
\end{aligned}
$$

Since Frob_q acts on $\mathbb{Q}_l(-mn)$ as multiplication by q^{mn} and on $H^i_w(\mathbb{P}C^m_n; \mathbb{Q}_l)$ by q^w, then it acts on $\mathrm{Hom}_{\mathbb{Q}_l[S_n]}(H^i_w(\mathbb{P}C^m_n; \mathbb{Q}_l); \mathbb{Q}_l(-mn))$ by q^{mn-w}. Hence, we have

$$
\begin{aligned}
&\mathrm{Tr}(\mathrm{Frob}_q : \mathrm{Hom}_{\mathbb{Q}_l[S_n]}(V, \oplus_w \mathrm{Hom}_{\mathbb{Q}_l[S_n]}(H^i_w(\mathbb{P}C^m_n; \mathbb{Q}_l); \mathbb{Q}_l(-mn)))\\
=&\sum_w q^{mn-w} \dim(\mathrm{Hom}_{\mathbb{Q}_l[S_n]}(V, \mathrm{Hom}_{\mathbb{Q}_l[S_n]}(H^i_w(\mathbb{P}C^m_n; \mathbb{Q}_l); \mathbb{Q}_l(-mn)))\\
=&\sum_w q^{mn-w} \langle \chi_V, \overline{\chi}^i_w(\mathbb{P}C^m_n) \chi_{\mathbb{Q}_l(-mn)} \rangle_{S_n}\\
=&\sum_w q^{mn-w} \langle \chi_V, \chi^i_w(\mathbb{P}C^m_n) \rangle_{S_n}
\end{aligned}
$$

The right-hand side of equation (2.5) then becomes

$$
\sum_{i,w} q^{mn-w} (-1)^i \langle \chi_V, \chi^i_w(\mathbb{P}C^m_n) \rangle \tag{2.7}
$$

Combining equations (2.6) and (2.7) gives

$$
\sum_C \chi_V(C) p_{n,C}(q) = \sum_{i,w} q^{mn-w} (-1)^i \langle \chi_V, \chi^i_w(\mathbb{P}C^m_n) \rangle. \tag{2.8}
$$

Since both sides of this equation are linear over the space of class functions on S_n, and since the irreducible characters from a basis for this space, the equation (2.8) holds for a general class function χ:

$$\sum_C \chi(C) p_{n,C}(q) = \sum_{i,w} q^{mn-w}(-1)^i \langle \chi, \chi_w^i(\mathbb{P}C_n^m)\rangle. \tag{2.9}$$

2.5 Representation polynomials

To help us calculate the cohomologies, we need to define a class of polynomials that is analogous to the Poincaré polynomials. Let X be a variety defined over $\mathbb{F}_q$ that has an action on it by a group G. Define $H_w^i(X)$ like what we did above, that is, to be the q^w-eigenspace with respect to the Frobenius morphism. Let χ_w^i to be the corresponding character of $H_w^i(X)$ as a G-representation. Let

$$P_X(x,t) = \sum_{i,w} \chi_w^i(X) x^i t^w.$$

$P_X(x,t)$ is a polynomial with two variables with coefficients in the ring of class functions on G. Moreover, for a direct product $X \times Y$, whose cohomology satisfies the Kunneth formula in terms of G-representation, i.e., $H^*_{\text{ét}}(X \times Y) \simeq H^*_{\text{ét}}(X) \otimes H^*_{\text{ét}}(Y)$, we can derive that

$$P_{X\times Y} = P_X \cdot P_Y.$$

Then by letting $x = -1$ and $t = q^{-1}$, we can write equation 2.9 as

$$\sum_C \chi(C) \frac{p^m_{n,C}(q)}{q^{mn}} = \langle \chi, P_{\mathbb{P}C_n^m}(-1, q^{-1})\rangle.$$

In our case, by the decomposition formula and Lemma 2.5.3, we have

$$\begin{aligned} P_{\mathbb{P}C_n^3}(x,t) =& P_{PGL_4}(x,t) \cdot P_{X_n^3}(x,t) \\ =& (1 + x^3t^2 + x^5t^3 + x^7t^4 + x^8t^5 + x^{10}t^6 + x^{12}t^7 + x^{15}t^9) \cdot P_{X_n^3}(x,t). \end{aligned} \tag{2.10}$$

To get $P_{X_n^3}(x,t)$, especially in our case $n=6$, we need the following facts.

Let k be a field, and let $L:\mathbb{A}^n \to \mathbb{A}^1$ be a nontrivial k-linear form. If $H \subseteq \mathbb{A}^n$ is the hyperplane defined by $L=0$, this form restricts to a map $L:\mathbb{A}^n - H \to \mathbb{A}^1 - \{0\}$. The fibers of this map are $\mathbb{A}^{n-1}$, so on cohomology, L induces an isomorphism

$$L^* : H^1_{\text{ét}}(\mathbb{A}^1 - \{0\};\mathbb{Q}_l) \cong H^1_{\text{ét}}(\mathbb{A}^n - H;\mathbb{Q}_l).$$

Moreover, by Kim[6] and Lehrer[8], we have

Proposition 2.5.1. *Let k be a field, and fix a prime l different from the characteristic of k. Given a finite set off hyperplanes $H_1,...,H_m$ in $\mathbb{A}^n$ defined over k, let $\mathscr{A}$ be the complement $\mathscr{A} := \mathbb{A}^n - \bigcup H_j$. Then:*

(i). $H^1_{ét}(\mathscr{A};\mathbb{Q}_l)$ is spanned by the images of the m maps

$$H^1_{ét}(\mathbb{A}^n - H_j;\mathbb{Q}_l) \to H^1_{ét}(\mathscr{A};\mathbb{Q}_l)$$

induced by the inclusion of $\mathscr{A}$ into $\mathbb{A}^n - H_j$ for $j = 0,...,m$.

(ii). $H^i_{ét}(\mathscr{A};\mathbb{Q}_l)$ is generated by $H^1_{ét}(\mathscr{A};\mathbb{Q}_l)$ under cup product.

Let $k = \overline{F}_q$ and suppose that L is defined over $\mathbb{F}_q$. Consider the Frobenius map $\text{Frob}_q : \mathscr{A} \to \mathscr{A}$. We will derive the following corollary from Proposition 2.5.1.

Corollary 2.5.1.1. *The induced action of $Frob_q$ on $H^i_{ét}(\mathscr{A};\mathbb{Q}_l)$ is scalar multiplication by q^i.*

Therefore, we can derive the following lemma.

Lemma 2.5.2. $H^i_{ét}(X_6^3;\mathbb{Q}_l) = H^i_i(X_6^3)$.

Proof. Let $x = (x_1,...,x_5) \in \mathbb{P}C_5^3$ and let $\mathscr{M}_x$ be the arrangement of planes consist planes that formed by points in x, namely $M_x = \{P_{i,j,k} | 1 \le i < j < k \le 5\}$ where $P_{i,j,k}$ is the plane that contains x_i, x_j and x_k. Then we can see that the fiber of the map $\mathbb{P}C_6^3 \to \mathbb{P}C_5^3$ over x is

$$\mathbb{P}^3 \backslash \bigcup_{P \in \mathscr{M}_x} P \simeq \mathbb{A}^3 \backslash \bigcup_{p \in \mathscr{M}'_x} p$$

where $\mathscr{M}'_x$ is the arrangement of planes by letting one of the plane $p \in \mathscr{M}_x$ to be the plane at infinity defining $\mathbb{A}^3 \simeq \mathbb{P}^3 \backslash p$.

Then X_6^3 can be viewed as the hyperplane complement of a hyperplane arrangement in the affine space $\mathbb{A}^3$. Then by 2.5.1, the geometric Frobenius map Frob_q acts as multiplication by q^i on $H^*_{\text{ét}}(X_6^3; \mathbb{Q}_l)$. Therefore we established our claim. $\square$

In the case when $n = 7$, we will need to conjecture that $H^i_{\text{ét}}(X_7^3; \mathbb{Q}_l) = H^i_i(X_7^3)$ at the moment.

Therefore, the polynomial for X_n^3 for $n = 6$ and $n = 7$ can be written as

$$P_{X_n^3}(x,t) = \sum_k \chi^3_{n,k} x^k t^k \tag{2.11}$$

where $\chi^3_{n,k}$ is the character of $H^k_{\text{ét}}(X_n^3; \mathbb{Q}_l)$ as S_n-representation. Then by combining the equations above, we have

$$\begin{aligned} \sum_C \chi_V(C) p^3_{n,c}(q) =& q^{3n} \sum_k q^{-k} (-1)^k (\langle \chi_V, \chi^3_{n,k} \rangle - \langle \chi_V, \chi^3_{n,k-2} \rangle + \langle \chi_V, \chi^3_{n,k-3} \rangle \\ & - \langle \chi_V, \chi^3_{n,k-4} \rangle - \langle \chi_V, \chi^3_{n,k-5} \rangle + \langle \chi_V, \chi^3_{n,k-6} \rangle - \langle \chi_V, \chi^3_{n,k-7} \rangle \\ & + \langle \chi_V, \chi^3_{n,k-9} \rangle). \end{aligned} \tag{2.12}$$

Because $\chi^3_{n,k} = 0$ for $n < 0$, we can find the characters inductively by using the character table for irreducible S_n-representations after gain complete knowledge about $p^3_{n,c}(q)$.

Now, we want to establish that $H^*_{\text{ét}}(\mathbb{P}C_n^m) \simeq H^*_{\text{ét}}(X_n^m) \otimes H^*_{\text{ét}}(PGL_{m+1}(\mathbb{C}))$ in terms of S_n-representation for $m = 3$.

Lemma 2.5.3. *For any pairs of $I,J : [5] \hookrightarrow [n]$, which induce the map $f_I, f_J : \mathbb{P}C_n^3 \to \mathbb{P}C_5^3$, the induced map on cohomology always have the same image. Namely, $f_I^*\left(H_{ét}^*(\mathbb{P}C_5^3;\mathbb{Q}_l)\right) = f_J^*\left(H_{ét}^*(\mathbb{P}C_5^3;\mathbb{Q}_l)\right)$.*

Proof. Since any injection $[5] \to [n]$ will factor through an injection $[5] \to [6]$, we only need to consider the case that $n = 6$. Since the Frobenius morhpism acts on $H^i_{\text{ét}}(X_6^3)$ by q^i and the Kunneth formula

$$H^*_{\text{ét}}(\mathbb{P}C_6^3) \simeq H^*_{\text{ét}}(X_6^3) \otimes H^*_{\text{ét}}(PGL_4(\mathbb{C}))$$

provides that the two sides have the same weight structure. Then we can see that for any generator in the sense of cup product of $H^*_{\text{ét}}(\mathbb{P}C_5^3) = H^*_{\text{ét}}(PGL_4(\mathbb{C}))$ with weight w and degree i, the corresponding part of cohomology of $H^*_{\text{ét}}(\mathbb{P}C_5^3)$ and the weight w, degree i part of $H^*_{\text{ét}}(\mathbb{P}C_6^3)$ always have the same dimension. Since the induced maps f_I^* and f_J^* are injections maintaining the weight structure, they will always have the same image. $\square$

Therefore, different injections $[5] \to [n]$ yields the same image for the corresponding map $H^*_{\text{ét}}(PGL_4(\mathbb{C})) = H^*_{\text{ét}}(\mathbb{P}C_5^3) \to H^*_{\text{ét}}(\mathbb{P}C_n^3)$. Moreover, the image is trivial as S_n-representation and that implies that $H^*_{\text{ét}}(\mathbb{P}C_6^3) \simeq H^*_{\text{ét}}(X_6^3) \otimes H^*_{\text{ét}}(PGL_4(\mathbb{C}))$ in terms of S_n-representation. In fact, although we are dealing with étale cohomology here, the statement also holds in singular cohomology by comparison. Therefore, we get Proposition 1.3.5.

Back to $p^3_{n,C}(q)$, since the space of class functions is spanned by the characters χ_V of irreducible representations, we can decompose $1_C = \sum_j \alpha_j \chi_{V_i}$ where $\alpha_j \in \mathbb{Q}$ for all j. Then

Table 2.1 Point counts for $B_5^2(\mathbb{F}_q)$ twisted by conjugacy classes of S_5

Conjugacy Class (C)	$p^2_{5,C}(q)$
e	$\frac{1}{120}(q-3)(q-2)(q-1)^2q^3(q+1)(q^2+q+1)$
(12)	$\frac{1}{12}(q-1)^3q^4(q+1)(q^2+q+1)$
$(12)(34)$	$\frac{1}{8}(q-2)(q-1)^2q^3(q+1)^2(q^2+q+1)$
(123)	$\frac{1}{6}(q-1)^2q^4(q+1)^2(q^2+q+1)$
$(123)(45)$	$\frac{1}{6}(q-1)^3q^4(q+1)(q^2+q+1)$
(1234)	$\frac{1}{4}(q-1)^2q^4(q+1)^2(q^2+q+1)$
(12345)	$\frac{1}{5}(q-1)^2q^3(q+1)(q^2+1)(q^2+q+1)$

we can will have

$$\begin{aligned}
p^3_{n,c}(q) &= \sum_{C'} 1_C(C')p^3_{n,C'}(q) \\
&= \sum_{i,w} q^{3n-w}(-1)^i\langle 1_C, \chi^i_w\rangle \\
&= \sum_{i,w} q^{3n-w}(-1)^i\langle \sum_j \alpha_j\chi_{V_j}, \chi^i_w\rangle \\
&= \sum_j \alpha_j \sum_{i,w} q^{3n-w}(-1)^i\langle \chi_{V_j}, \chi^i_w\rangle
\end{aligned}$$

which implies that $p^3_{n,c}(q)$ is a polynomial with rational coefficients.

The results for $p^2_{n,c}$, which derived by Das and O'connor, and $p^3_{n,c}$, which derived by me, are given by the tables 2.1, 2.2, 2.3 and 2.4.

Table 2.2 Point counts for $B_6^2(\mathbb{F}_q)$ twisted by conjugacy classes of S_6

Conjugacy Class (C)	$p^2_{6,C}(q)$
e	$\frac{1}{720}(q-3)(q-2)(q-1)^2q^3(q+1)(q^2+q+1)(q^2-9q+21)$
(12)	$\frac{1}{48}(q-1)^3q^4(q+1)(q^2+q+1)(q^2-3q+3)$
$(12)(34)$	$\frac{1}{16}(q-2)(q-1)^2q^3(q+1)^2(q^2+q+1)(q^2-q-3)$
$(12)(34)(56)$	$\frac{1}{48}(q-1)^2q^3(q+1)(q^2+q+1)(q^4-6q^2+q+8)$
(123)	$\frac{1}{18}(q-1)^2q^6(q+1)^2(q^2+q+1)$
$(123)(45)$	$\frac{1}{6}(q-1)^3q^6(q+1)(q^2+q+1)$
$(123)(456)$	$\frac{1}{18}(q-1)^2q^3(q+1)(q^2+q+1)(q^4-2q^3-3q+9)$
(1234)	$\frac{1}{8}(q-1)^2q^4(q+1)^2(q^2+q+1)(q^2+q-1)$
$(1234)(56)$	$\frac{1}{8}(q-1)^2q^3(q+1)(q^2+q+1)(q^4-2q^2-q-2)$
(12345)	$\frac{1}{5}(q-1)^2q^3(q+1)(q^2+1)(q^2+q+1)^2$
(123456)	$\frac{1}{6}(q-1)^2q^3(q+1)(q^2+q+1)(q^4+q-1)$

Table 2.3 Point counts for $B_6^3(\mathbb{F}_q)$ twisted by conjugacy classes of S_6

Conjugacy Class (C)	$p^3_{6,C}(q)$
e	$\frac{1}{720}(q-4)(q-3)(q-2)(q-1)^3q^6(q+1)^2(q^2+1)(q^2+q+1)$
(12)	$\frac{1}{48}(q-2)(q-1)^4q^7(q+1)^2(q^2+1)(q^2+q+1))$
$(12)(34)$	$\frac{1}{16}(q-2)(q-1)^3q^7(q+1)^3(q^2+1)(q^2+q+1)$
$(12)(34)(56)$	$\frac{1}{48}(q-2)(q-1)^3q^6(q+1)^2(q^2+1)(q^2+q+1)(q^2-q-4)$
(123)	$\frac{1}{18}(q-1)^4q^7(q+1)^3(q^2+1)(q^2+q+1)$
$(123)(45)$	$\frac{1}{6}(q-1)^4q^7(q+1)^3(q^2+1)(q^2+q+1)$
$(123)(456)$	$\frac{1}{18}(q-1)^3q^6(q+1)^2(q^2+1)(q^2+q+1)(q^3-q-3)$
(1234)	$\frac{1}{8}(q-1)^3q^8(q+1)^2(q^2+q+1)(q^3+q^2+q+1)$
$(1234)(56)$	$\frac{1}{8}(q-1)^4q^8(q+1)^2(q^2+1)(q^2+q+1)$
(12345)	$\frac{1}{5}(q-1)^3q^6(q+1)^2(q^2+1)(q^2+q+1)(q^3+q^2+q+1)$
(123456)	$\frac{1}{6}(q-1)^3q^6(q+1)^2(q^2+1)(q^2+q+1)(q^3+q-1)$

Table 2.4 Point counts for $B_7^3(\mathbb{F}_q)$ twisted by conjugacy classes of S_7

Conjugacy Class (C)	$p_{7,C}^3(q)$
c	$\frac{1}{5040}(q-5)(q-3)(q-1)^3q^6(q+1)^2(q^2+q+1)(q^2+1)$ $\left(q^4-20q^3+148q^2-468q+498\right)$
(12)	$\frac{1}{240}(q-3)(q-2)(q-1)^4q^7(q+1)^2(q^2+1)(q^2+q+1)$ (q^2-4q+6)
$(12)(34)$	$\frac{1}{48}(q-1)^4q^6(q+1)^3(q^2+1)(q^2+q+1)(q^4-4q^3+12q-6)$
$(12)(34)(56)$	$\frac{1}{48}(q-2)(q-1)^3q^6(q+1)^3(q^2+1)(q^2+q+1)(q^2-q-4)$ (q^2-6)
(123)	$\frac{1}{72}(q-1)^5q^9(q+1)^3(q^2+1)(q^2+q+1)$
$(123)(45)$	$\frac{1}{12}(q-1)^5q^9(q+1)^3(q^2+1)(q^2+q+1)$
$(123)(45)(67)$	$\frac{1}{24}(q-1)^5q^9(q+1)^3(q^2+1)(q^2+q+1)$
$(123)(456)$	$\frac{1}{18}(q-1)^3q^6(q+1)^2(q^2+1)(q^2+q+1)$ $(q^6-q^5-q^4-8q^3+9q^2+6q+18)$
(1234)	$\frac{1}{24}(q-1)^4q^9(q+1)^3(q+2)(q^2+1)(q^2+q+1)$
$(1234)(56)$	$\frac{1}{8}(q-1)^4q^6(q+1)^3(q^2+1)(q^2+q+1)(q^4-2q^2-2)$
$(1234)(567)$	$\frac{1}{12}(q-1)^5q^9(q+1)^3(q^2+1)(q^2+q+1)$
(12345)	$\frac{1}{10}(q-1)^3q^7(q+1)^3(q^2+1)^2(q^2+q+1)^2$
$(12345)(67)$	$\frac{1}{10}(q-1)^4q^7(q+1)^2(q^2+1)^2(q^2+q+1)^2$
(123456)	$\frac{1}{6}(q-1)^3q^8(q+1)^3(q^2+1)(q^2+q+1)(q^3+q-1)$
(1234567)	$\frac{1}{7}(q-1)^3q^6(q+1)^2(q^2+1)(q^2+q+1)(q^2-q+1)$ $(q^4+q^3+q^2+q+1)$

Chapter 3

Preparation for Twisted Point-Counting

3.1 Basic structures for $B_n^3(\mathbb{F}_q)$

Let q be a power of a odd prime number, we know that the Frobenius morhpism Frob_q will acts on a point $p \in B_n^3(\mathbb{F}_q)$ as a permutation. Each point will define a unique representation up to conjugacy. Namely, it will be unique up to cycle type and and let's denote the corresponding cycle type by σ_p. We want to count the number of points

$$p_{n,c}^3(q) = \left|\{p \in B_n^3(\mathbb{F}_q) | \sigma_p \in C\}\right|$$

for each conjugacy class C of S_n.

To make our life easier, we will abbreviate the Frobenius automorphism Frob_q into f or f_q when we need to emphasize the power from this point. For a point p in $\mathbb{P}^N(\mathbb{F}_q)$, we denote $\{f(p)\}$ the orbit of p under the Frobenius morphism. The orbit is finite for all p and we call $|\{f(p)\}|$ the order of p and denote it by $\mathrm{ord}p$. We call a point p with order n a q^n-point and sometime write $p^{(n)}$ when we want to emphasize the order.

To count the number of points corresponding to a certain cycle type, we need to know how to count the number of q^n-points in $\mathbb{P}^N(\mathbb{F}_q)$ or some subspaces of it. Notice that for any

q^n-point p, we have $p \in (\mathbb{P}^N(\overline{\mathbb{F}}_q))^{f^n}$ and $(\mathbb{P}^N(\overline{\mathbb{F}}_q))^{f^n} \cong \mathbb{P}^N(\mathbb{F}_{q^n})$. Let's denote $\mathbb{P}^N(\overline{\mathbb{F}}_q^{(n)})$ the set of q^n-points in $\mathbb{P}^N(\mathbb{F}_q)$. We can see that

$$\mathbb{P}^N(\mathbb{F}_{q^n}) = \bigsqcup_{k|n} \mathbb{P}^N(\overline{\mathbb{F}}_q^{(k)}).$$

Then we have

$$\mathbb{P}^N(\overline{\mathbb{F}}_q^{(n)}) = \mathbb{P}^N(\mathbb{F}_{q^n}) \backslash \bigcup_{k|n, k\neq n} \mathbb{P}^N(\overline{\mathbb{F}}_q^{(k)})$$

Then we can count the number of q^n-points recursively by the following equation

$$\left|\mathbb{P}^N(\overline{\mathbb{F}}_q^{(n)})\right| = \left|\mathbb{P}^N(\mathbb{F}_{q^n})\right| - \sum_{k|n, k\neq n} \left|\mathbb{P}^N(\overline{\mathbb{F}}_q^{(k)})\right|. \tag{3.1}$$

For a linear subspace of $P \subseteq \mathbb{P}^N(\overline{\mathbb{F}}_q)$, the order P is also denoted by $\mathrm{ord}P$ which is the smallest integer n such that $f^n(P) = P$. Similar to the points, we will call such a space q^n-space, which are planes and lines in this paper as we are working in three dimensional space, if $\mathrm{ord}P = n$. By the projective duality, there is a f-equivariant correspondence between the set of q^n-hyperplanes and $\mathbb{P}^N(\overline{\mathbb{F}}_q^{(n)})$. Namely, one direct result is the number of q^n-hyperplanes is equal to the number of q^n-points. To be more general, the number of k-dimensional q^n-subspace is equal to the number of $N-k$-dimensional q^n-subspace in $\mathbb{P}^N(\overline{\mathbb{F}}_q)$. Besides, the duality also gives us the following results. First, for any q^k-subspace P, there is an f^k-equivariant correspondence between P and $\mathbb{P}^{N-m}(\overline{\mathbb{F}}_{q^k})$ where m is the codimension of P. Secondly, for a q-point p, the number of k-dimensional q-subspace containing p is equal to the number of q-point on a k-dimensional q-subspace.

To count the number of points, for example, for $p^3_{7,C}(q)$ where $C = (1234)(56)$, we need to find the number of ways to choose a q^4-point a, a q^2-point b and a q-point c. This will give us a set $p = \{f(a)\} \bigcup \{f(b)\} \bigcup \{f(c)\}$ that corresponding to cycle type C. However, different choices of points may result in a same set, namely, different point from the same orbit will not change the resulting set. As a result, we need to correct for the overcounting. Besides overcounting, we also need to make sure the points we choose satisfy our requirement, that is, not four points can be on a same plane.

To achieve that in this example, we first of all need to choose a q^4-point a whose orbit are not contained in a same line. Then for the q^2-point b, we first of all need to make sure that it is not contained in the existing plane, and also make sure the line $\langle b, f(b)\rangle$ doesn't intersect with the any of the six lines decided by the orbit of a. At last, we need our q-point c not on any of the ten existing planes and then divded by $3 \cdot 2$ to correct for the overcounting.

To count for $p^m_{n,C}$ generally, we need to know how the order of points, lines and planes interact with each other. For points p_1, p_2, and p_3 in $\mathbb{P}^3(\overline{\mathbb{F}}_q)$, we use $\langle p_1, p_2, p_3\rangle$ to denote the space spanned by these three points. Usually $\langle p_1, p_2, p_3\rangle$ is a plane and $\langle p_1, p_2\rangle$ is a line. Dually, for three planes P_1, P_2, and P_3 in $\mathbb{P}^3(\mathbb{F}_p)$, $\langle P_1, P_2, P_3\rangle$ is usually a point and $\langle P_1, P_2\rangle$ is a line. We will discuss how the order of p_1, p_2, and p_3 affect the order of $\langle p_1, p_2, p_3\rangle$ and $\langle p_1, p_2\rangle$.

By the properties of projective spaces, we will have the following observation.

Remark 3.1.1. *Two different lines l_1 and l_2 intersect if and only if they are in a same plane P.*

Lemma 3.1.2. *For two different points p_1, p_2 in $\mathbb{P}^3(\mathbb{F}_q)$ where $ord(p_1) = n_1$ and $ord(p_2) = n_2$. Let $n = ord(l)$, where $l = \langle p_1, p_2\rangle$. Then $n | lcm(n_1, n_2)$ and for each i, we either have $n_i | n$ or $n | n_i$. Similar conclusion can be make for two different lines l_1, l_2 in a same plane P. Notice that $\langle l_1, l_2\rangle$ can be considered as a point that is their intersection or a plane that contains both of them.*

Proof. We will only show the case for the points, the case for lines can be proved with the exact same procedure. Let $k = \mathrm{lcm}(n_1, n_2)$, then

$$f^k(l) = \langle f^k(p_1), f^k(p_2)\rangle = \langle p_1, p_2\rangle = l,$$

$\square$

so we have $n|k$.

For the second part, if $f^n(p_i) = p_i$, then $n_i|n$. If not, then $f^n(p_i) \in f^n(l) = l$. Then l also equal to $\langle p_i, f^n(p_i)\rangle$ and we have

$$f^{n_i}(l) = \langle f^{n_i}(p_i), f^{n_i}(f^n(p_i))\rangle = \langle p_i, f^n(p_i)\rangle = l.$$

Therefore, $n|n_i$.

Corollary 3.1.2.1. *For three noncollinear points p_1, p_2, and p_3 in $\mathbb{P}^1(\mathbb{F}_q)$ where $ord(p_i) = n_i$. Let $n = ord(P)$ where $P = \langle p_1, p_2, p_3\rangle$. We have $n|lcm(p_1, p_2, p_3)$*

The next lemma can also be useful in our calculation.

Lemma 3.1.3. *Let p be a q^k-point that is in a q^m-plane P where $m < k$, then let $d = \gcd(k,m)$, there exists a plane P' such that $f^d(P') = P'$ and $p \in P'$.*

Proof. If $m = d$, then we are done. Otherwise, we can see that for each $i \in \mathbb{N}$ such that $i < m/d$, p is in $f^{id}(P)$. Then p will be in their intersection which has order that divides d. Since $d < m < k$, the intersection must be a line, so we can always construct a plane using the intersection to create a plane P' that have the same order. $\square$

3.2 Generic points

When choosing a q^n-point p, we would like to make sure that the points $\{f^i(p)\}_{0\leq i<n}$ are in general position, that is, no four points are in a same projective plane, no three points are collinear. To make sure selected points satisfy our requirement, we will need to understand what does the term "in general position" really mean for each n and determine the number of points that satisfy the condition. In [4], Das and $O'Connor$ set up a collection of sets that

can be described in a consistent way. An analog will work for $n < 6$ in our condition but things will get much more complicated when $n = 6$. Nevertheless, we will still use the same terminology.

Definition 3.2.1. A q^n-point p in $\mathbb{P}^k(\overline{\mathbb{F}}_q)$ is called a generic point if all point in its orbit $\{f(p)\}$ are in general position, namely, for any $m \leq \min(n, k+1)$ points in the orbit, these points will form a codimensional $k+1-m$ subspace in $\mathbb{P}^n(\overline{\mathbb{F}}_q)$. We will denote the set of generic q^n-points by $\mathbb{P}^k(\overline{\mathbb{F}}_q^{(n,\mathrm{gen})})$.

In our case, where $n = 3$, a point being generic means any four points from the orbit cannnot be in a same plane, and no three points are collinear, which is precisely what. In this paper, when we say generic points, we usually mean the generic points in $\mathbb{P}^3(\overline{\mathbb{F}}_q)$. There are several cases that we will consider the generic points in $\mathbb{P}^2(\overline{\mathbb{F}}_q)$. Those cases will either be obvious under context or noted specially.

Now, we want to know what does a q^n-point being generic actually mean geometrically and how to calculate the cardinality of $\mathbb{P}^3(\overline{\mathbb{F}}_q^{(n,\mathrm{gen})})$ for different n. The following lemma describes the generic points in a way that we can use to compute $\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(n,\mathrm{gen})})\right|$.

Lemma 3.2.1. *For $3 \leq n \leq 6$, we can describe $\mathbb{P}^3(\overline{\mathbb{F}}_q^{(n,gen)})$ as followings:*

- *when $n = 3$, $p \in \mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,gen)})$ if and only if p is not on any q-line;*
- *when $n = 4, 5$, $p \in \mathbb{P}^3(\overline{\mathbb{F}}_q^{(n,gen)})$ if and only if p is not on any q-plane;*
- *when $n = 6$, $p \in \mathbb{P}^3(\overline{\mathbb{F}}_q^{(6,gen)})$ if and only if p doesn't fell into any one of the following conditions:*
    - *p is in a q-plane;*
    - *p is in a q^2-line*
    - *p is in a non-generic q^3-line, which is a q^3-line containing a q-point.*

Proof. It is obvious that when p does not satisfy the requirement from the lemma, the points from $\{f(p)\}$ cannot be in general position. Then we only need to show that when the points from $\{f(p)\}$ are not in the general position, then the point p cannot fulfill the requirement to be a generic point as we described above.

Due to the cyclic structure of the orbit under the Frobenius action, we can treat the case that four points in the orbit of p on a same plane as a cyclically ordered partition of n into four parts. For example, the case that $p, f(p), f^3(p), f^4(p)$ are on the same plane where $n = 6$ can be demonstrated by the partition $(1,2,1,2)$ where the i^{th} part of the partition demonstrates the cyclic distance of the i^{th} point to the next one. Moreover, since the parts are ordered cyclically in the partitions, we only need to consider a few cases for each n.

When $n = 3$, a q^3-point is in a q-line if and only if every point from its orbit are collinear. When $n = 4$ and 5, we only have one partition for each, and it is trivial to see that in each of the cases, all points in the orbits will be on a same plane. Then p must be on a q-plane.

When $n = 6$, we have three different partitions, which are $(1,1,1,3)$, $(1,1,2,2)$, $(1,2,1,2)$. In the case of $(1,1,1,3)$, without loss of generality, we suppose $\{f^i(p)\}_{0 \le i \le 3}$ are on the same plane P, then by applying the map f, we can see that $\{f^i(p)\}_{1 \le i \le 4}$ are on the same plane P' whose intersection with P contains $f(p)$, $f^2(p)$ and $f^3(p)$. If these three points are on a same line, then this line will be a q-line and every point in the orbit of p will be on it, then they will also be on a q-plane. If these three points are not collinear, then we can see that $P' = P$. Using the same argument, we can see that ever point in the orbit of p will be contained in P, which implies that P is a q-plane and p is not q-generic.

In the case $(1,2,1,2)$, let P be the plane that contains $\{p, f(p), f^3(p), f^4(p)\}$, we can see that $f(P)$ contains $\{f(p), f^2(p), f^4(p), f^5(p)\}$ and $f(P)$ contains $\{p, f^2(p), f^3(p), f^5(p)\}$. Then let $l = \langle p, f^3(p)\rangle$, we can see that l is a q-lines or a q^3-line. If l is a q-line, then p must be in a q-plane. If l is a q^3-line, then the three lines from $\{f(l)\}$ are the intersections of the three pairs of planes from $\{f(P)\}$ respectively, which means the three lines in $\{f(l)\}$ must intersect at a q-point. Therefore, p is contained in a non-generic q^3-lines.

In the case $(1,1,2,2)$, let's suppose $\{p, f(p), f^2(p), f^4(p)\}$ are in the same plane P, then $f^2(P)$ will contain $\{p, f^2(p), f^3(p), f^4(p)\}$. Their intersection contains $\{p, f^2(p), f^4(p)\}$. If the three points are not collinear, then we can see that all points from $\{f(p)\}$ will be in the plane $\langle p, f^2(p), f^4(p)\rangle$, hence p is in a q-plane. Otherwise, these three points will form a line which has order 1 or 2. No matter what, p will violate the condition for being generic.

When $n = 7$, we can see that if three points in $\{f(p)\}$ is collinear then all seven points must be collinear, let's start with the assumption that no three points from $\{f(p)\}$ are collinear. There are five different partitions $(1,1,1,4)$, $(1,1,2,3)$, $(1,1,3,2)$, $(1,2,1,3)$ and $(1,2,2,2)$ with $n = 7$. The first partition will result in all points in a same plane obviously. As for $(1,2,1,3)$, let $P = \langle p, f(p), f^3(p), f^4(p)\rangle$. We can see that there are three points from $\{f(p)\}$ are in both P and $f^3(P)$. Since no three points are collinear, we must have $P = f^3(P)$ and it will contain five points from $\{f(p)\}$. Which implies that P contains all points from $\{f(p)\}$ and p is in a q-plane. Similarly, for $(1,2,2,2)$, let $P = \langle p, f(p), f^3(p), f^5(p)\rangle$. We will have $P = f^2(P)$ and it will contain five points from the orbit. Hence p is in a q-plane in this case.

To deal with the partition $(1,1,2,3)$, let

$$P = \langle p, f(p), f^2(p), f^4(p)\rangle \text{ and } P' = \langle f^3(p), f^5(p), f^6(p)\rangle.$$

Then we can see that $f^3(P) = \langle p, f^3(p), f^4(p), f^5(p)\rangle$, which implies that $\langle p, f^4(p)\rangle$ intersect $\langle f^3(p), f^5(p)\rangle$ at a point a. By the same logic, since $f^5(P) = \langle p, f^2(p), f^5(p), f^6(p)\rangle$, and $f^6(P) = \langle p, f(p), f^3(p), f^6(p)\rangle$, $\langle p, f^2(p)\rangle$ intersects $\langle f^5(p), f^6(p)\rangle$ at a point b and $\langle p, f(p)\rangle$ intersects $\langle f^3(p), f^6(p)\rangle$ at a point c. Moreover, we also see that a is also the intersection of $\langle p, f^4(p)\rangle$ and $\langle f^1(p), f^2(p)\rangle$, b is also the intersection of $\langle p, f^2(p)\rangle$ and $\langle f(p), f^4(p)\rangle$, and c is also the intersection of $\langle p, f(p)\rangle$ and $\langle f^2(p), f^4(p)\rangle$. Hence, a, b, and

c are not collinear. Since they are contained in both P and P', we have to have $P = P'$. As a result, P is a q-plane.

For $(1,1,3,2)$, let $P = \langle p, f(p), f^2(p), f^5(p)\rangle$ and let $P' = \langle f^3(p), f^4(p), f^6(p)\rangle$. Follow the same method as $(1,1,2,3)$, P and P' must share three noncollinear points, which implies $P = P'$ and P is a q-plane. As a result, if we have four points from $\{f(p)\}$ that are in a same plane, then p must be in a q-plane when p is a q^7-point.

□

Remark 3.2.2. *Dual to generic points, we can also define generic lines and planes in a similar manner. We will list the conditions for a few kinds of lines and planes to be generic that we will use when we count the cardinality of* $\mathbb{P}^3(\overline{\mathbb{F}}_q^{(n,gen)})$ *and the numbers of twisted points:*

- *a q^n-line is generic if it doesn't contain a q-point when* $2 \le n \le 5$, *and doesn't contain any q-point as well as q^2-point when* $n = 6$. *Notice that although we don't have generic q^2-line in* $\mathbb{P}^2(\overline{\mathbb{F}}_q)$, *we do have such lines in* $\mathbb{P}^3(\overline{\mathbb{F}}_q)$.
- *a q^n-plane is generic if it doesn't contain any q-point when* $n = 4,5$.

3.3 Counting generic points

By Definition 3.2.1, it is crucial to find the number of generic q^n-points to calculate $p^3_{n,c}$, and by Lemma 3.2.1, we can find those numbers geometrically. Since we will also need the number of points in $\mathbb{P}^2(\overline{\mathbb{F}}_q^{(n,\mathrm{gen})})$, we will refer to the the following proposition by Das and O'Connor[4].

Proposition 3.3.1. *When* $m = 2$, *for each* $n \ge 3$, *let* $\mathbb{P}^2(\overline{\mathbb{F}}_q^{(n,gen)})$ *denote the set of generic q^n-points in plane (no three points in an orbit are collinear). Then for* $n < 6$,

$$\left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(n,gen)})\right| = \left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(n)})\right| - (q^2+q+1)\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(n)})\right|.$$

For $n = 6$,

$$\left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(6,gen)})\right| = \left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(6)})\right| - (q^2+q+1)\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(6)})\right| - (q^4-q)\left|\mathbb{P}^1(\overline{\mathbb{F}}_{q^2}^{(3)})\right|.$$

We will use these result when we do twist points counting. The next remark is also crucial for us to compute the number of generic points for $m = 3$ as well as some of the point-counting.

Remark 3.3.2. *For a finite field $\mathbb{F}_k$, the number of lines in $\mathbb{P}^n(\mathbb{F}_k)$ is $\frac{(k^{n+1}-1)(k^n-1)}{(k-1)^2(k+1)}$. We can derive the result by choosing two different points in the space and then divide the number of choices by the number of overcounting.*

Proposition 3.3.3. *For $\mathbb{P}^3(\overline{F}_q^{(n,gen)})$, the set of generic q^n-points in $\mathbb{P}^3(\overline{\mathbb{F}}_q)$, we can calculate the cardinality of these sets as follow.*

When $n = 3$, we have

$$\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,gen)})\right| = \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3)})\right| - (q^2+1)(q^2+q+1)\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(3)})\right|.$$

When $n = 4,5,7$, we have

$$\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(n,gen)})\right| = \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(n)})\right| - (q^3+q^2+q+1)\left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(n)})\right| + q(q^2+1)(q^2+q+1)\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(n)})\right|.$$

When $n = 6$, we have

$$\begin{aligned}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(6,gen)})\right| = &\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(6)})\right| - (q^3+q^2+q+1)\left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(6)})\right| + q(q^2+1)(q^2+q+1)\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(6)})\right| \\ &- q^4(q-1)^2(q^2+q+1)\left|\mathbb{P}^1(\overline{\mathbb{F}}_{q^2}^{(3)})\right| \\ &- q^3(q+1)^2(q-1)^2(q^2+1)\left|\mathbb{P}^1(\overline{\mathbb{F}}_{q^3}^{(2)})\right|.\end{aligned}$$

Proof. In the case that $n=3$, we want to exclude the q^3-points that are on a q-line, that means, the q^3-points p such that all the points in the orbit $\{f(p)\}$ are on a same q-line. To exclude all such points, we just need to find the number of q-lines in the space and multiply the number of q^3-points on a q-line. Moreover, there is no double counting since q-lines will only intersect at q-points. Thus, we have the above result.

When $n=4,5$, we first of all need to get rid of all the q^n-points that are on the q-planes. However, since each pair of q-planes will intersect at a q-line, we multiple counted all of the q-lines. To adjust for that, notice that each q-lines is contained in $q+1$ many q-planes. Therefore, for each q-line, we add back q times the number of q^n-points in the line. Again, since q-lines will only intersect each other at q-points, we don't need to do more adjustment. As a result, we will derive the desirable result.

When $n=6$, we need additionally throw away the q^6-points that are in q^2-lines which are not in q-planes and the q^6-points such that the three q^3-lines formed by the orbit intersect at a q-point.

For the first case, we need to find out the number of q^2-lines which are not in a q-plane. By remark 3.3.2, there are $(q^4+1)(q^4+q^2+1)$ lines in $\mathbb{P}^3(\mathbb{F}_{q^2})$ including q^2-lines and q-lines. By the same remark, there are $(q^2+1)(q^2+q+1)$ many q-lines in the space. Moreover, in a q-plane, there are q^4+q^2+1 many q^2-lines and q-lines in total and q^2+q+1 many q-lines. Since intersections of q^2-lines are either q^2-point or q-point, we don't need to worry about overcounting. For each of the q^2-lines $l^{(2)}$, the Frob_{q^2}-equivariant isomorphism from $l^{(2)}$ to $\mathbb{P}^1(\overline{\mathbb{F}}_{q^2})$ will identifies the q^6 points in l with the $(q^2)^3$-points in $\mathbb{P}^1(\overline{\mathbb{F}}_{q^2})$. Since there are q^3+q^2+q+1 many q-planes, and each pair of q-planes only intersect at a q-plane, we need to rule out

$$\begin{aligned}
&\left((q^4+1)(q^4+q^2+1)-(q^2+1)(q^2+q+1)-(q^3+q^2+q+1)(q^4-q)\right)\left|\mathbb{P}^1(\overline{\mathbb{F}}_{q^2}^{(3)})\right| \\
&=q^4(q-1)^2(q^2+q+1)\left|\mathbb{P}^1(\overline{\mathbb{F}}_{q^2}^{(3)})\right|
\end{aligned}$$

q^6-points in this case.

For the second case, we first need to find the number of non-generic q^3-lines that are not in a q-plane. Since a non-generic q^3-lines can always be constructed by connecting a q-point and a q^3-point, we start the search by find all such pairs and there are

$$\left|\mathbb{P}^3(\mathbb{F}_q)\right| \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3)})\right|$$

of them. However, these pairs contains a lot of q-lines and also have a great amount of overcounting. Since for each q-line, there are $\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(3)})\right|$ many q^3-points and $\left|\mathbb{P}^1(\mathbb{F}_q)\right|$ many q-points on it and there are total $(q^2+1)(q^2+q+1)$ many q-lines, by the way we find all those pairs, there are

$$(q^2+1)(q^2+q+1)\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(3)})\right| \left|\mathbb{P}^1(\mathbb{F}_q)\right|$$

counts that are q-lines for the pair. Because any non-generic q^3-line contain only one q-point, there are $\left|\mathbb{P}^1(\mathbb{F}_{q^3})\right| - 1$ many q^3-points on it. Therefore, there are

$$\frac{\left|\mathbb{P}^3(\mathbb{F}_q)\right| \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3)})\right| - (q^2+1)(q^2+q+1)\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(3)})\right| \left|\mathbb{P}^1(\mathbb{F}_q)\right|}{\left|\mathbb{P}^1(\mathbb{F}_{q^3})\right| - 1}$$

non-generic q^3-lines in total. Now, we need to rule out all non-generic q^3-lines that are contained in a q-plane. Since in a plane, every two lines must intersect at a point, for any generic q^3-point p, we can get a generic q^3-line l by letting $l = \langle p, f^2(a)\rangle$ and for any generic q^3-line l, we can get a generic q^3-point by letting $p = \langle l, f(l)\rangle$. These will create a one-to-one correspondence between the generic q^3-point in the plane and the generic q^3-lines in the plane. Hence, there are totally q^6+q^3+1 lines altogether in a q-plane with order 1 or 3, q^2+q+1 lines that are q-lines and $\left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right|$ are generic q^3-lines, where there are

q^3+q^2+q+1 many q-planes in total. Therefore, we have

$$\frac{\left|\mathbb{P}^3(\mathbb{F}_q)\right|\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3)})\right|-(q^2+1)(q^2+q+1)\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(3)})\right|\left|\mathbb{P}^1(\mathbb{F}_q)\right|}{\left|\mathbb{P}^1(\mathbb{F}_{q^3})\right|-1}$$
$$-(q^3+q^2+q+1)\left((q^6+q^3+1)-(q^2+q+1)-\left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right|\right)$$
$$=q^3(q+1)^2(q-1)^2(q^2+1)$$

non-generic q^3-lines. Similar to the argument we used in the last case, the Frob_{q^3}-equivariant isomorphism from the q^3-lines to $\mathbb{P}^1(\overline{\mathbb{F}}_{q^3})$ will identifies the q^6 points in the q^3-lines with the $(q^3)^2$-points in $\mathbb{P}^1(\overline{\mathbb{F}}_{q^3})$. Therefore, we will exclude

$$q^3(q+1)^2(q-1)^2(q^2+1)\left|\mathbb{P}^1(\overline{\mathbb{F}}_{q^3}^{(2)})\right|$$

q^6 points in this case. Moreover, notice that if a non-generic q^3-line intersects a q^2-line at a q^6-point that fell into both of the cases, then the q^6-point must be in a q-plane, so these two cases won't affect each other (see figure 3.1, where p' is the q-point contained in all three q^3-lines). Therefore, we got the result. □

Lemma 3.3.4. *If p is a q-generic q^n-point where $n>3$, the plane $P=\langle p,f(p),f^2(P)\rangle$ is a q-generic plane.*

Proof. By contradiction, suppose P contains a q-point p', then p' either on the line $\langle p,f(p)\rangle$ or not. If p' is on $\langle p,f(p)\rangle$, then we can see that $p,f(p),f^2(p)$ are collinear, hence not q-generic. Otherwise, $\langle p,f(p),p'\rangle$ defines the same plane. we can see that

$$P=\langle p,f(p),p'\rangle=\langle f(p),f^2(p),p'\rangle=f\langle p,f(p),p'\rangle=f(P).$$

Then p cannot be q-generic. □

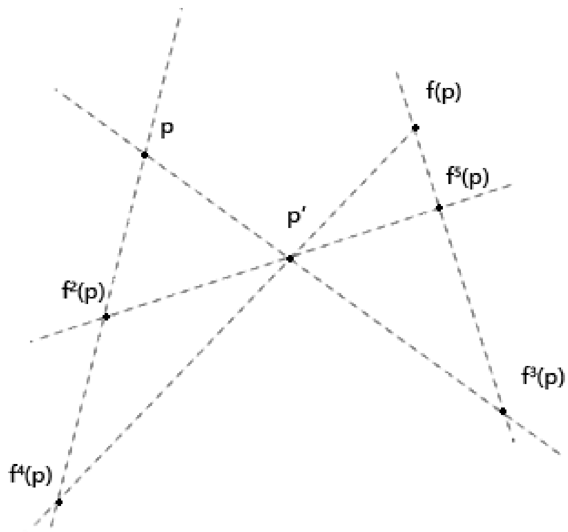

Figure 3.1 When a q^2-line intersect a q^3-line at a q^6-point

Remark 3.3.5. *In fact, using the same way as we decide the conditions for generic position, we can see that for q^n-points p where $4 \leq n \leq 6$, the q^n-planes formed by $\{f(p)\}$ will always be $q-$generic.*

Chapter 4

Twisted Point-Counting

4.1 Computation for twisted point counting for $B_6^3(\mathbb{F}_q)$

Now, we will start the calculation for the twisted point counting. To improve the readability, the number of generic q^n-points will be given as below.

$$
\begin{aligned}
\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right| &= (q-1)^2q^3(q+1)^2(q^2+1) \\
&= q^9 - q^7 - q^5 + q^3 \\
\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(4,\mathrm{gen})})\right| &= (q-1)^3q^6(q+1)(q^2+q+1) \\
&= q^{12} - q^{11} - q^{10} + q^8 + q^7 - q^6 \\
\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(5,\mathrm{gen})})\right| &= (q-1)^3q^2(q+1)^2(q^2+1)(q^2+q+1) \\
&= q^{11} - q^9 - q^8 - q^7 + q^6 + q^5 + q^4 - q^2 \\
\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(6,\mathrm{gen})})\right| &= (q-1)^3q^6(q+1)^2(q^2+1)(q^2+q+1)(q^3+q-1) \\
&= q^{18} - 2q^{15} - 2q^{14} + q^{13} + q^{12} + 3q^{11} - q^9 - q^8 - q^7 + q^6 \\
\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(7,\mathrm{gen})})\right| &= (q-1)^3q^6(q+1)^2(q^2+1)(q^2+q+1)(q^2-q+1)(q^4+q^3+q^2+q+1) \\
&= q^{21} - q^{17} - q^{16} - q^{15} + q^{12} + q^{11} + q^{10} - q^6
\end{aligned}
$$

4.1.1 Cycle type e

For $p \in B_6(\mathbb{F}_q)$ with cycle type $\sigma_p = e$, each $p_i \in p$ is a $q-$point. An ordering of p gives an element $\tilde{p} \in F_6(\mathbb{F}_q) \simeq \mathrm{PGL}_4(\mathbb{F}_q) \times X_6(\mathbb{F}_q)$. Therefore, we only need to calculate $\left|X_6(\mathbb{F}_q)\right|$. That is, we want to count the number of ways to pick a q-point that is not on the existing ten q-planes formed by a subset $S = \{p_1, p_2, p_3, p_4, p_5\} \in \mathbb{P}^3(\mathbb{F}_q)$. We can see that there are ten q-lines, which are lines that contains two points in S and each of them is contained in three q-planes, let's call them kind one lines. Moreover, for a pair of planes that only share one common points p_i in S, they will intersect at a line that going through p_i. For each point in S, there are three pairs of such planes. Hence, we have fifteen such lines that only goes through one point in S. We call this kind of lines as kind 2 lines.

Now, we need to analyze how those lines intersect with each other. First of all, two kind 1 lines intersect each other if and only if they share a point in S since otherwise we will have four points in the same plane. Then, we want to show that two kind 2 lines going through the same point in S cannot be the same, that is, there are indeed fifteen kind 2 lines. Without loss of generality, let's say both of them goes through p_2 and l_1 is in the plane $\langle p_1, p_2, p_3\rangle$ and l_2 is in the plane $\langle p_2, p_3, p_4\rangle$. Notice that this two planes must have two common points in S otherwise they will just be the pair of planes that decide a kind 2 lines. If l_1 and l_2 are the same, then since it in both of the planes and both of the planes contain p_3, then either these two planes are the same or the line contain p_3, both of this cases will fell into the result that we have four points in a plane. Moreover, we want to show that two kind 2 lines that build with different planes and goes through two different points in S will not intersect. Namely, for example, let l_1 be the intersection of $\langle p_1, p_2, p_3\rangle$ and $\langle p_2, p_4, p_5\rangle$, and let l_2 be the intersection of $\langle p_2, p_3, p_4$ and $\langle p_1, p_3, p_5$. Then we can see that l_1 and l_2 don't share a common building plane. By contradiction, suppose that l_1 and l_2 intersect, then their intersection will be in both $\langle p_1, p_2, p_3\rangle$ and $\langle p_2, p_3, p_4\rangle$. Then either $l_1 = l_2 = \langle p_2, p_3\rangle$ or these two planes are the same, and no matter what we will have four points in a same plane. Lastly, we need to show that a kind 2 line l will not intersect a kind 1 line that is not inside any of the two planes we used to build l and not going through the same point in S as l. Without loss of generality, let's say l is the lines passing through p_2 in $\langle p_1, p_2, p_3\rangle$ the possible kind 1 lines are $\langle p_1, p_4\rangle$, $\langle p_1, p_5\rangle$, $\langle p_3, p_4\rangle$ and $\langle p_3, p_5\rangle$. Again, without loss of generality and by contradiction, suppose l intersects $\langle p_1, p_4\rangle$. Then either p_1 is on l or p_4 is in $\langle p_1, p_2, p_3\rangle$. We will reach a contradiction whatsoever.

At this point, we are safe to consider the intersections of kind 2 lines and kind 1 lines. Without loss of generality, let l_1, l_2, l_3 be the kind 2 lines in $\langle p_1, p_2, p_3\rangle$ such that l_i goes through p_i. Then we can see that l_1 is in $\langle p_1, p_4, p_5\rangle$, l_2 is in $\langle p_2, p_4, p_5\rangle$, and l_3 is in $\langle p_3, p_4, p_5\rangle$. It is obvious that l_1, l_2, and l_3 all intersect with $\langle 4,5\rangle$. Therefore, since $\langle 4,5\rangle$ only intersect $\langle p_1, p_2, p_3\rangle$ at one point, the four lines l_1, l_2, l_3 and $\langle 4,5\rangle$ all intersect at a single point. We can see that for all kind 1 lines, there is one such point on the line that is contained in one kind 1 lines and three kind 2 lines.

Therefore, there are $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 10\left|\mathbb{P}^2(\mathbb{F}_q)\right| + 10\cdot(3-1)\left|\mathbb{P}^1(\mathbb{F}_q)\right| + 15\cdot(2-1)\left|\mathbb{P}^1(\mathbb{F}_q)\right| - 5\cdot(-6+2\cdot 4+3+1) - 10\cdot(-4+2\cdot 1+3+1)$ choices for the last point. Hence,

$$\begin{aligned}\left|X_6^3(\mathbb{F}_q)\right| =& \left|\mathbb{P}^3(\mathbb{F}_q)\right| - 10\left|\mathbb{P}^2(\mathbb{F}_q)\right| + 35\left|\mathbb{P}^1(\mathbb{F}_q)\right| - 50\\ =& q^3 - 9q^2 + 26q - 24\end{aligned}$$

Since we have an order for the q-points, we should divide the result by 6! to correct for overcounting. Then we have the point count

$$\begin{aligned}p_{6,e}^3 =& \frac{1}{6!}\left(\left|PGL_4(\mathbb{F}_q)\right|\left|X_6^3(\mathbb{F}_q)\right|\right)\\ =& \frac{1}{720}(q-4)(q-3)(q-2)(q-1)^3q^6(q+1)(q^2+q+1)(q^3+q^2+q+1).\end{aligned}$$

Remark 4.1.1. *By using the trace formula on X_6^3 with trivial $\mathbb{Q}_l$-coefficients, we have*

$$\left|X_6^3(\mathbb{F}_q)\right| = q^3\left(\dim H^0_{\acute{e}t}(X_6^3) - \frac{1}{q}\dim H^1_{\acute{e}t}(X_6^3) + \frac{1}{q^2}\dim H^2_{\acute{e}t}(X_6^3) - \frac{1}{q^3}\dim H^3_{\acute{e}t}(X_6^3)\right).$$

Hence the Poincaré polynomial of X_6^3 is $1+9x+26x^2+24x^3$.

4.1.2 Cycle type (12)

After choosing the q^2-point a, the first q-point b can be choose arbitrarily away from the q-line $\langle a, f(a)\rangle$, so we have $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^1(\mathbb{F}_q)\right|$ choices. Once we picked b, we can form a q-plane and the second q-point c can be choose anywhere away from this q-plane, so we have $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right|$ choices. Now, we have two q-planes intersect at line $\langle a, f(a)\rangle$ and a pair of q^2-planes intersect at $\langle b, c\rangle$. Since the pair of q^2-planes already contain a q-lines, containing a q-point outside the q-line $\langle b, c\rangle$ will result in the four points in a same plane. Hence, the two q^2-planes contain no q-point away from $\langle b, c\rangle$, so we can choose the third q-point d anywhere away from the two q-planes and the q-line $\langle b, c\rangle$. Notice that the two q-planes intersect at the q-line $\langle a, f(a)\rangle$ and $\langle b, c\rangle$ intersect each of the q-planes at a single points. Thus, we can choose d in $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 2\left|\mathbb{P}^2(\mathbb{F}_q)\right| + \left|\mathbb{P}^1(\mathbb{F}_q)\right| - \left|\mathbb{P}^1(\mathbb{F}_q)\right| + 2$ ways.

Now, with one q^2-point and three q-points, we have three pairs of q^2-planes where each pair of the planes each contains two q-points and one q^2-point from $\{f(a)\}$. By the same argument before, these planes contain no q-point away from their intersection. Besides the q^2-planes, we also have four q-planes. One of them contains all the three q-points, let's call it $P = \langle b, c, d\rangle$. For the other three q-planes, let's call them P_b, P_c, and P_d intersect at the line $\langle a, f(a)\rangle$. Then for each P_α, $\alpha \in \{b, c, d\}$, P_α intersect P at a q-line l_α going through α. Since for each of P_α, we can consider it as $P_\alpha = \langle a, f(a), \alpha\rangle$. Since l_α is the intersection of P_α and P, it is in both of them hence it must intersect $\langle a, f(a)\rangle$. Since we choose α arbitrarily and $\langle a, f(a)\rangle$ intersect P at a single point, l_b, l_c, l_d and $\langle a, f(a)\rangle$ intersect at a single point. Then we can choose any q-point outside these four q-planes and the q-line $\langle a, f(a)\rangle$.

As a result, by inclusion-exclusion formula, there are $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 4\left|\mathbb{P}^2(\mathbb{F}_q)\right| + 5\left|\mathbb{P}^1(\mathbb{F}_q)\right| - 2$ ways to choose the last q-point. Notice the the order of selection for the q-point and the choice from the orbit of a won't affect the result, so we need to divide the result by $2 \cdot 4!$ to correct the overcounting. Therefore,

$$\begin{aligned}
p^3_{6,(12)} =& \frac{1}{2\cdot 4!}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| \left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^1(\mathbb{F}_q)\right|\right)\left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right|\right) \\
& \left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 2\left|\mathbb{P}^2(\mathbb{F}_q)\right| + \left|\mathbb{P}^1(\mathbb{F}_q)\right| - \left|\mathbb{P}^1(\mathbb{F}_q)\right| + 2\right) \\
& \left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 4\left|\mathbb{P}^2(\mathbb{F}_q)\right| + 3\left|\mathbb{P}^1(\mathbb{F}_q)\right| - \left|\mathbb{P}^1(\mathbb{F}_q)\right| + 1\right) \\
=& \frac{1}{48} q(q^3-1)(q^2+1)(q^3+q^2)q^3(q^3-q^2-q+1)(q^3-3q^2+2q) \\
=& \frac{1}{48}(q-2)(q-1)^4 q^7 (q+1)^2(q^2+1)(q^2+q+1)
\end{aligned}$$

4.1.3 Cycle type (12)(34)

For the first q^2-point a, we can choose arbitrarily. To choose the second q^2-point b, we want to make sure that the line $l = \langle b, f(b)\rangle$ will not intersect the line $l' = \langle a, f(a)\rangle$. Since both l and l' are q-lines, they intersect if and only if they are in the same q-plane. To form a q-plane with l' in it, we can choose any q-point outside q' and divides the total number of choices by the number of q-point on a q-plane outside a q-line. Hence, there are

$$\frac{\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^1(\mathbb{F}_q)\right|}{\left|\mathbb{P}^2(\mathbb{F}_q)\right| - \left|\mathbb{P}^1(\mathbb{F}_q)\right|} = \frac{(q^3+q^2+q+1)-(q+1)}{(q^2+q+1)-(q+1)} = q+1$$

q-planes that contain l' and by the duality, each plane contains q^2+q+1 many q-lines hence q^2+q many q-lines other than l'. Therefore, we need to rule out in total $(q+1)(q^2+q)+1$ many q-lines including l'. Since the intersections of q-lines are always q-points, we don't need to care about how they intersect each other to pick b. Hence, there are

$$\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| - \left[(q+1)(q^2+q)+1\right]\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|$$

different choices for b.

With the chosen pairs of q^2-points, we will have two pairs q^2-planes one intersect at the q-line $\langle a, f(a)\rangle$ and the other intersect at the q-line $\langle b, f(b)\rangle$. These q^2-planes will contain no q-point outside the two q-lines since otherwise the orbit of a and orbit of b will all in a same plane. Then for the first q-point c, it can picked anywhere outside the two q-lines which don't intersect. Thus, we have $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 2\left|\mathbb{P}^1(\mathbb{F}_q)\right|$ options.

Now, with the existing points, we will the same two pairs q^2-planes we considered earlier that only contain points for $\{f(a)\}$ and $\{f(b)\}$, two pairs of q^2-planes that contains one point from each of $\{f(a)\}$ and $\{f(b)\}$ as well as c, and two q-planes $\langle a, f(a), c\rangle$ and $\langle b, f(b), c\rangle$. The first kind of q^2-planes still contain no q-points outside $l' = \langle a, f(a)\rangle$ and $l = \langle b, f(b)\rangle$. Each pair of the second of q^2-planes will have an intersection that is a q-line. Let's call them l_1 and l_2. The two q-planes will also intersect at a line that going through c, let's call it l''. To utilize inclusion-exclusion formula, we need to know how these q-lines intersect with each other. In fact, since both l_1 and l_2 are intersections of two planes that contain c, both of the lines will go through c, so we only need to know if any of these three lines are actually the same. Let's say l_1 is the intersection of $\langle a, b, c\rangle$ and $\langle f(a), f(b), c\rangle$, then l_2 is the intersection of $\langle a, f(b), c\rangle$ and $\langle b, f(a), c\rangle$. Since l_1 intersect $\langle a, b\rangle$, $l_1 = l_2$ will make $a, f(a), b, f(b)$ in the same plane. Hence, $l_1 \neq l_2$. Since l'' is the intersection of $\langle a, f(a), c\rangle$ and $\langle b, f(b), c\rangle$, both $l'' = l_1$ and $l'' = l_2$ while result in the same condition.

Thus, we can pick the last q-point anywhere other than the two q-planes and the two q-lines where the two q-lines intersect at the intersection of the two q-planes at a single point.

Hence, we have

$$\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 2\left|\mathbb{P}^2(\mathbb{F}_q)\right| + \left|\mathbb{P}^1(\mathbb{F}_q)\right| - 2\left|\mathbb{P}^1(\mathbb{F}_q)\right| + 2$$

ways to choose the second q-point. Since the different choice from orbits and the order of choosing will create overcounting, we need to correct it doing division by $2^2 \cdot 2! \cdot 2!$ As a result,

$$\begin{aligned}
p^3_{6,(12)(34)} =& \frac{1}{2^2 \cdot 2! \cdot 2!} \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| \left(\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| - \left[(q+1)(q^2+q)+1\right] \left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right| \right) \\
& \left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 2\left|\mathbb{P}^1(\mathbb{F}_q)\right|\right) \left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 2\left|\mathbb{P}^2(\mathbb{F}_q)\right| + \left|\mathbb{P}^1(\mathbb{F}_q)\right| - 2\left|\mathbb{P}^1(\mathbb{F}_q)\right| + 2\right) \\
=& \frac{1}{16} q(q^3-1)(q^2+1)\left(q(q^3-1)(q^2+1) - q(q+1)^2(q^2-q) - (q^2-q)\right) \\
& (q^3+q^2-q-1)\left(q^3-q^2-2q\right) \\
=& \frac{1}{16}(q-2)(q-1)^3 q^7 (q+1)^3 (q^2+1)(q^2+q+1)
\end{aligned}$$

4.1.4 Cycle type (12)(34)(56)

For the first two q^2-points a and b, we con just follow the same procedure as we did in the previous part. Hence we have $\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| - \left[(q+1)(q^2+q)+1\right] \left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|$ different choices for b. For the last q^2-point c, we first of all don't want it to be on the four q^2-planes and the line $l = \langle c, f(c)\rangle$ doesn't intersect any of the six lines formed by $\{f(a)\}$ and $\{f(b)\}$. For the two q-lines $l' = \langle a, f(a)\rangle$ and $l'' = \langle b, f(b)\rangle$, we can do the same thing as we did to find b. However, it is possible for a line that passes through both l' and l'', so we need to find out how many lines are counted twice. Since the intersection between q-lines must be q-points, then there is a one-to-one correspondence between the q-lines intersect both l' and l'' and the pair of q-points one from each of l' and l''. Therefore, there are $\left|\mathbb{P}^1(\mathbb{F}_q)\right|^2 = (q+1)^2$ lines counted twice. Hence, there are total

$$2(q+1)(q^2+q)-2(q+1)^2=(2q-2)(q+1)^2$$

such q-lines that intersect one of l' or l and not including l' and l''. Notice that when l intersect l' or l'', then it will intersect one pair of the q^2-planes at their intersection at a q-point and the other pair of q^2-planes at a q^2-point on each plane of the pair, and if l intersect both, then it will not intersect the plane anywhere else, so we exclude

$$(2q-2)(q+1)^2\left(\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|-2\right)+(q+1)^2\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|$$

q^2-points outside the q^2-planes in total.

Moreover, We have two pairs of q^2-lines which contain one point from each of $\{f(a)\}$ and $\{f(b)\}$. For one such q^2-lines L, l intersects L if and only if l intersects $f(L)$ since l is a q-line. Let's say $L=\langle a,b\rangle$ and $L'=\langle a,f(b)\rangle$, we want to show that if l intersect L, then it cannot intersect L'. If l intersect both L and L', then either a is in l or $f(b)$ is in L. The latter case will result in a contradiction as we have three points collinear. The former one will make $l=L$ since l is a q-line. This will belong to the part that c is in the q^2-planes.

Now, if l intersect L, the intersection must be a q^2-point because otherwise L and $f(L)$ will intersect and that will make the four points from the two orbits in the same plane. Let p be the intersection of l and L, then $f(p)$ will be the intersection of l and $f(L)$. By the same reason, both L and L' cannot intersect, so every point on them is q^2-point. We can see that there is a one-to-one correspondence between q-lines passing through L and q^2-points on L. Therefore, we can find $2\left(\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right|-2\right)$ such q-lines for each orbit in total excluding l and l''. Moreover, l will only intersect with the planes at L and $f(L)$ or L' and $f(L')$, where the intersections will be q^2-points. Hence, we can exclude

$$2\left(\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right|-2\right)\left(\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|-2\right)$$

options for c.

At this point, we can turn our view back to the q^2-planes and try to find out how many choices we should rule out for c. Each of the q^2-plane will contain one of the q-lines l' and l'', so they will contain no q-point outsides these two lines. Besides, it will contain L and L' or $f(L)$ and $f(L')$ and these three lines it contains will intersect pairwisely at three different points from $\{f(a)\}$ and $\{f(b)\}$. Thus, for each plane, there are $\left|\mathbb{P}^2(\mathbb{F}_{q^2})\right|-3\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right|+3$ points outside the three lines it contains. So we can rule out

$$4\left(\left|\mathbb{P}^2(\mathbb{F}_{q^2})\right|-3\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right|+3\right)$$

more points at this stage. As for the lines, each q-lines contain $\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|$ many q^2-points and each q^2-lines contain $\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right|$ many q^2-points. These lines intersect at points from $\{f(a)\}$ and $\{f(b)\}$ where each of the points is in three lines. Hence, there are

$$2\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|+4\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right|-8$$

q^2-points on the lines.

As a result, to can choose c in

$$\begin{aligned}
&\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right|-(2q-2)(q+1)^2\left(\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|-2\right)-(q+1)^2\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|\\
&-2\left(\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right|-2\right)\left(\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|-2\right)\\
&-4\left(\left|\mathbb{P}^2(\mathbb{F}_{q^2})\right|-3\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right|+3\right)-\left(2\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|+4\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right|-8\right)
\end{aligned}$$

different ways. Since our way of selection distinguish the points from an orbit and also gives an order for the three q^2-points, we need to divide the result by $2^3 \cdot 3!$ to correcct the overcounting. Thus, we have

$$
\begin{aligned}
p^3_{6,(12)(34)(56)} =& \frac{1}{2^3 \cdot 3!} \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| \left(\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| - (q+1)(q^2+q)\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right| - \left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|\right) \\
& \Bigg(\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| - (2q-2)(q+1)^2\left(\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right| - 2\right) - (q+1)^2\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right| - \\
& 2\left(\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right| - 2\right)\left(\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right| - 2\right) - \\
& 4\left(\left|\mathbb{P}^2(\mathbb{F}_{q^2})\right| - 3\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right| + 3\right) - \\
& \left(2\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right| + 4\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right| - 8\right)\Bigg) \\
=& \frac{1}{48} q(q^3-1)(q^2+1)\left(q(q^3-1)(q^2+1) - q(q+1)^2(q^2-q) - (q^2-q)\right) \\
& (q+1)^2(q-1)(q-2)\left(q^2-q-4\right) \\
=& \frac{1}{48} q(q^3-1)(q^2+1)q^5(q-1)(q+1)^2(q-1)(q-2)\left(q^2-q-4\right) \\
=& \frac{1}{48}(q-2)(q-1)^3 q^6 (q+1)^2 (q^2+1)(q^2+q+1)(q^2-q-4)
\end{aligned}
$$

4.1.5 Cycle type (123)

We first of all choose a generic q^3-point a, then the orbit of a form a q-plane P. The first q-point b can be chosen anywhere away from P, so there are $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right|$ choices. These four existing points will form a q-plane and three q^3-planes that contains two points from $\{f(a)\}$ and b. The three q^3-planes contain no q-points other than p else b will be in P. Therefore, we can choose c anywhere outside P and not b, so we have $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right| - 1$ different choices c. Now, we have one q-plane P, two groups of q^3-planes that contain two points in $\{f(a)\}$ and one of b or c, and one group of q^3-planes that contain a point from $\{f(a)\}$ and both b and c. The first kind of q^3-plane will still have no q-point other than b or c, and the second kind of q^3-plane will hold no q-point outside the line $\langle b, c\rangle$ as otherwise all five points must be in a same plane. Thus, we can choose the last q-point d anywhere away from the plane P and the line $\langle b,c\rangle$. Since the line $\langle b,c\rangle$ intersect P at a single q-point, by inclusion-exclusion formula, there are $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right| - \left|\mathbb{P}^1(\mathbb{F}_q)\right| + 1$ choices for the d. Because we choose the points in a way that distinguish the points in $\{f(a)\}$ and give an order for b, c and d, we need to do a division by $3\cdot 3!$ to correct the overcounting. Hence, there are in total

$$\begin{aligned}
p^3_{6,(123)} =& \frac{1}{3\cdot 3!}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right|(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right|)(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right| - 1) \\
& (\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right| - \left|\mathbb{P}^1(\mathbb{F}_q)\right| + 1) \\
=& \frac{1}{18}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| q^3(q^3-1)(q^3-q) \\
=& \frac{1}{18}(q-1)^4 q^7 (q+1)^3 (q^2+1)(q^2+q+1)
\end{aligned}$$

4.1.6 Cycle type (123)(45)

We first of all choose a generic q^3-point a. Then to pick a q^2-point b, we want to make sure it is not on the plane P formed by the $\{f(a)\}$ and the line $l = \langle b, f(b)\rangle$ doesn't intersect any of the existing three q^3-lines formed by $\{f(a)\}$. Let's rule out the second case first. Without loss of generality, suppose l intersects the line $l' = \langle a, f(a)\rangle$, then their intersection p is either a q-point or a q^3-point since l is a q-line. If the intersection is a q-point, then we will have

$$l' = \langle a, p\rangle = \langle f(a), p\rangle = f(l').$$

Which makes l' a q-line and a non-generic, hence not possible. If the intersection is a q^3-point, then l must also intersect $f(l')$ and $f^2(l')$ since l is a q-line, which implies that it must be inside the q-plane P. Thus, we only need to rule out the q^2-points that are in P, there are $\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| - \left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(2)})\right|$ choices for b.

At this stage, we have one q-plane P, a group of q^3-planes that contain the q-line l and one point from $\{f(a)\}$, and one group of q^6-planes that contain two points from $\{f(a)\}$ and one point from $\{f(b)\}$. The q^3-planes cannot contain any q-point outside l or a will not be generic. Now, we want to show that the q^6-planes cannot contain any q-points. Without loss of generality, let's pick the q^6-plane $P' = \langle a, f(a), b\rangle$ and suppose that it contains a q-point p. Since a is a generic q^3-point, p cannot be on the line $\langle a\rangle$. If p is on the line $\langle a,b\rangle$ or $\langle f(a),b\rangle$, the line will contain a q-point, a q^2-point and a q^3-point, which makes the line a q-line and a non-generic. Hence, p must be outside the three lines in P'. Then

$$P' = \langle a, f(a), p, b\rangle = f^3(P') = \langle a, f(a), p, f^3(b)\rangle = \langle a, f(a), p, f(b)\rangle$$

which give us the condition that four points in a same plane. As a result, the q^6-planes cannot contain any q-point, so we can choose any q-point away from P and l. Since l intersect P at a single q-point, by inclusion-exclusion formula, there are $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right| - \left|\mathbb{P}^1(\mathbb{F}_q)\right| + 1$ choices for the q-point. Because we choose the points in the manner that will distinguish the points in $\{f(a)\}$ and $\{f(b)\}$, we need to divide everything by $3\cdot 2$ to correct the overcounting. As a result, we have

$$\begin{aligned}
p^3_{6,(123)(45)} =& \frac{1}{3\cdot 2}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right|\left(\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| - \left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(2)})\right|\right)\\
&\left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right| - \left|\mathbb{P}^1(\mathbb{F}_q)\right| + 1\right)\\
=& \frac{1}{6}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right|(q^6-q^3)(q^3-q)\\
=& \frac{1}{6}(q-1)^4 q^7 (q+1)^3 (q^2+1)(q^2+q+1)
\end{aligned}$$

choices after correction for overcounting.

4.1.7 Cycle type (123)(456)

Let P be the plane formed by the $\{f(a)\}$ where a is the first generic q^3-point we chose. Then to choose the second generic q^3-point b, we want to make sure that b does not fell into one of the following three conditions, which are

(1). b is inside the q-plane P;

(2). a line formed by $\{f(b)\}$ intersects a line formed by $\{f(a)\}$;

(3). a point in $\{f(a)\}$ is contained in the q-plane formed by $\{f(b)\}$.

For the first condition, we only need to rule out the generic q^3-points in P. For the second case, without loss of generality and by contradiction, let's suppose the q^3-line $l = \langle b, f(b)\rangle$ intersects the q^3-line $l' = \langle a, f(a)\rangle$, where b is not in P. Let P' be the plane that contains both l and l'. P' must be a q^3-plane since otherwise $P' = P$. Then the three planes from $\{f(P')\}$ will intersect at a q-point p and points from $\{f(b)\}$ will be on the pair intersection of the planes in $\{f(P')\}$. In fact, there is an one-to-one correspondence between the q-point outside P and such orbits of planes. A q^3 point fell into the second condition if and only if it is on a q^3-line that contains a q-point and one point from $\{f(a)\}$. There are $3\left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right|\right)$ such lines on each of such lines, there are $\mathbb{P}^1(\mathbb{F}_{q^3}) - 2$ many q^3-points outside P. Hence we can rule out

$$3(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right|)(\mathbb{P}^1(\mathbb{F}_{q^3}) - 2)$$

points in this case.

As for the last condition, if a point in $\{f(a)\}$ is contained in the q-plane formed by $\{f(b)\}$, then it must be on the intersection of two q-planes, which will be a q-line. Then this condition will make a non-generic, so it cannot happen. Therefore, there are

$$\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right| - \left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right| - 3(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right|)(\mathbb{P}^1(\mathbb{F}_{q^3}) - 2)$$

choices for b. Since the points in $\{f(a)\}$ and $\{f(b)\}$ are distinguished by our selection procedure and an order of a and b is given, we need to divide the result by $3^2 \cdot 2!$ to correct the overcounting. Therefore,

$$\begin{aligned}
p^3_{6,(123)(456)} =& \frac{1}{3^2 \cdot 2!} \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| \\
& \left(\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| - \left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| - 3(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right|)(\mathbb{P}^1(\mathbb{F}_{q^3}) - 2)\right) \\
=& \frac{1}{18} \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| (q-1)q^3(q^2+q+1)(q^3-q-3) \\
=& \frac{1}{18}(q-1)^3 q^6 (q+1)^2 (q^2+1)(q^2+q+1)(q^3-q-3)
\end{aligned}$$

4.1.8 Cycle type (1234)

Once we chose the generic q^4-point a, all the planes formed will be generic q^4-planes. Hence, we can choose the first q-point b arbitrarily. Now, with the q-point b, we will get four more q^4-planes, which are $\langle a, f(a), b\rangle$, $\langle f(a), f^2(a), b\rangle$, $\langle f^2(a), f^3(a), b\rangle$ and $\langle f^3(a), a, b\rangle$, and two q^2-planes, $\langle a, f^2(a), b\rangle$ and $\langle f(a), f^3(a), b\rangle$. These q^4-planes can only contain one q-point b since otherwise a will not be generic. and for the two q^2-planes, they will intersect at a q-line containing b. Therefore, we have $\left|\mathbb{P}^3(\mathbb{F}_q) - \mathbb{P}^1(\mathbb{F}_q)\right|$ choices for the second q-point c. Then we have

$$\begin{aligned}
p^3_{6,(1234)} =& \frac{1}{4 \cdot 2!} \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(4,\text{gen})})\right| \left|\mathbb{P}^3(\mathbb{F}_q)\right| \left(\left|\mathbb{P}^3(\mathbb{F}_q) - \mathbb{P}^1(\mathbb{F}_q)\right|\right) \\
=& \frac{1}{8}(q-1)^3 q^8 (q+1)^2 (q^2+q+1)(q^3+q^2+q+1)
\end{aligned}$$

4.1.9 Cycle type (1234)(56)

After choosing the generic q^4-point a, we will have a group of q^4-planes, and let's first show that there are no q^2-points on these planes outside the pair of q^2-lines $\langle a, f^2(a)\rangle$ and

$\langle f(a), f^3(a)\rangle$. Without loss of generality, let's suppose the plane $P = \langle a, f(a), f^2(a)\rangle$ contain a q^2-point p that is not on the q^2-line $\rangle a, f^2(a)\rangle$. First of all, p cannot be on the line $\langle a, f(a)\rangle$ or $\langle f(a), f^2(a)\rangle$ or a will not be generic. Then p is not on any of the three lines in P and we will have

$$P = \langle a, f(a), f^2(a), p\rangle = \langle f^2(a), f^3(a), a, p\rangle = f^2(P)$$

which makes a non-generic. Hence, to choose the q^2-points b, we only need to rule out the cases that the line $l = \langle b, f(b)\rangle$ intersects some lines formed by $\{f(b)\}$. There are two cases, l intersects a q^4-line or a q^2-line. If l intersect at a q^4-line, without loss of generality, let's suppose it intersects $\langle a, f(a)\rangle$. Then since l is a q-line, it will intersect all the q^4-lines, then l must be one of the q^2-lines $\langle a, f^2(a)\rangle$ and $\langle f(a), f^3(a)\rangle$.

Now, for the other case, we can see that $L = \langle a, f^2(a)\rangle$ and $f(L) = \langle f(a), f^3(a)\rangle$ doesn't intersect since a is generic. If l intersect one of L and $f(L)$ then it will intersect both of them since l is a q-line, and the intersections will be q^2-points from a same orbit. Hence, there is an one-to-one correspondence between the q-lines that intersect L and the q^2-points on L. Since there are $\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right|$ many q^2-points on L and $\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right| - 2$ many q^2-points on each of the q-lines outside L and $f(L)$. Then we can rule out

$$\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right| \left(\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right| - 2\right)$$

choices for b that are not on the q^2-lines. As a result, we have

$$\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| - 2\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right| - \left|\mathbb{P}^1(\mathbb{F}_{q^2})\right| \left(\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right| - 2\right)$$

different choices for b. Because the choice distinguish the points in $\{f(a)\}$ and $\{f(b)\}$, we need to get the result divided by $4\cdot 2$ for correction. Hence, we have

$$\begin{aligned}
p^3_{6,(1234)(56)} =& \frac{1}{4\cdot 2}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(4,\mathrm{gen})})\right|\left(\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| - 2\left|\mathbb{P}^1(\mathbb{F}_{q^2})\right| - \left|\mathbb{P}^1(\mathbb{F}_{q^2})\right|\left(\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right| - 2\right)\right) \\
=& \frac{1}{8}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(4,\mathrm{gen})})\right| q^2(q^2+1)(q+1)(q-1) \\
=& \frac{1}{8}(q-1)^4 q^8 (q+1)^2 (q^2+1)(q^2+q+1)
\end{aligned}$$

4.1.10 Cycle type (12345)

We first choose any generic q^5-point a. Since all the planes formed by $\{f(a)\}$ will be generic, we can choose the q-point arbitrarily. We have totally

$$\begin{aligned}
p^3_{6,(12345)} =& \frac{1}{5}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(5,\mathrm{gen})})\right|\left|\mathbb{P}^3(\mathbb{F}_q)\right| \\
=& \frac{1}{5}(q-1)^3 q^6 (q+1)^2 (q^2+1)(q^2+q+1)(q^3+q^2+q+1)
\end{aligned}$$

4.1.11 Cycle type (123456)

We can choose any generic q^6-point.

$$\begin{aligned}
p^3_{6,(123456)} =& \frac{1}{6}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(6,\mathrm{gen})})\right| \\
=& \frac{1}{6}(q-1)^3 q^6 (q+1)^2 (q^2+1)(q^2+q+1)(q^3+q-1)
\end{aligned}$$

4.2 Proof for Theorem 1.3.8

For irreducible S_6-representation S, denote $P_S(q) = \sum_C \chi_S(C)\cdot p^3_{6,C}$, where U is the trivial representation and other irreducible representation will be subscripted by its corresponding partition. Then we have the following,

$$
\begin{aligned}
&P_U(q) = q^{18} - q^{16} - q^{15} - q^{14} + q^{13} + q^{12} + q^{11} - q^9, \\
&P_{S_{41^21}}(q) = -q^{17} + q^{15} + q^{14} + q^{13} - q^{12} - q^{11} - q^{10} + q^8, \\
&P_{S_{41}}(q) = q^{16} - q^{15} - q^{14} + 2q^{11} - q^8 - q^7 + q^6, \\
&P_{S_{32}}(q) = -q^{15} + q^{13} + q^{12} + q^{11} - q^{10} - q^9 - q^8 + q^6, \\
&P_{S_{31^21}}(q) = q^{16} - q^{14} - q^{13} - q^{12} + q^{11} + q^{10} + q^9 - q^7, \\
&P_{S_{21}}(q) = -q^{15} + q^{13} + q^{12} + q^{11} - q^{10} - q^9 - q^8 + q^6,
\end{aligned}
$$

where the polynomials for the other irreducible representations are 0. Then combine with equation (2.12), we can derive the result for Theorem 1.3.8.

4.3 Computation for twisted point counting for $B_7^3(\mathbb{F}_q)$

Since we have the result for $B_6^3(\mathbb{F}_q)$, for some cases that contains a q-point for $B_7^3(\mathbb{F}_q)$, we only need to find the number of ways to choose the additional q-point. Since for each of the cases we already have 20 planes, we just need to rule out the q-points on those planes. However, as we can see from the calculation below, the projection map $X_7^3 \to X_6^3$ by forgetting the last point is not a fibration anymore, so we need to use some other ways to deal with some of the cases.

4.3.1 Cycle type e

Instead of based on the case that we already have six points in general position, let's start from the case that we have the first five q-points $\{a,b,c,d,e\}$ in general position. Then we know have ten q-planes and ten q-lines. There are

$$(q-1)^3q^6(q+1)(q^2+q+1)(q^3+q^2+q+1)(q^3+q^2+q-4)(q^3+q^2+q-5) =$$
$$= |PGL_4(\mathbb{C})|\,(q^3+q^2+q-4)(q^3+q^2+q-5)$$

7-tuples total to start with. Let H be the unions of these planes, then there are

$$q^3+q^2+q+1-(q-4)(q-3)(q-2) = 10q^2-25q+25$$

q-points in H. For the last two points g and h, there are three cases to fail the general position condition:

- at least one of $\{g,h\}$ is in H;
- the q-line $l = \langle g,h\rangle$ passing through one point from $\{a,b,c,d,e\}$, and both g and h are not in H;
- l intersects at least one of the ten q-lines away from $\{a,b,c,d,e\}$, but not containing any point from $\{a,b,c,d,e\}$, and both g and h are not in H.

We can see that these three cases are mutually exclusive, so we can calculate them separately.

For the first one, there are

$$(10q^2-25q+20)(q^3+q^2+q-5)$$

ways to have g inside H and

$$(10q^2-25q+20)(10q^2-25q+19)$$

ways to have both g and h inside H. Hence, there are

$$(10q^2-25q+20)(2q^3-8q^2+27q-29)$$

pairs falling into the first condition.

For the second case, notice that if l intersect a point p from $\{a,b,c,d,e\}$, then it cannot intersect any of the q-lines otherwise g and h will be in H. Since all the union of the intersections of the four planes that don't contain p is union of the q-lines, l will intersect all those four planes each at a distinct point. There are q^2+q+1 lines passing through p and $6q-5$ lines that are contained in H. Hence there are

$$5(q^2+q+1-6q+5)(q-4)(q-5)$$

pairs are in this case.

For the third case, if l intersect a lines l' formed by $\{a,b,c,d,e\}$, then it will be in a same plane as l'. Now, there are $q-2$ planes that contain l' and not in H. For such a plane P, P will intersect any planes that don't contain l' at a line and every intersection of two such q-planes at distinct points. There are

$$q^2+q+1-8(q+1)+2\cdot(3)+6+1+2\cdot 3=q^2-7q+12$$

q-points in P that not on H (See figure 4.1).

Therefore, we have

$$(q-2)(q^2-7q+12)(q^2-7q+11)$$

pairs of q-points forming such a line l. Notice that we didn't exclude the lines that going through the two points in $\{a,b,c,d,e\}$ on l', so we need to add them back later.

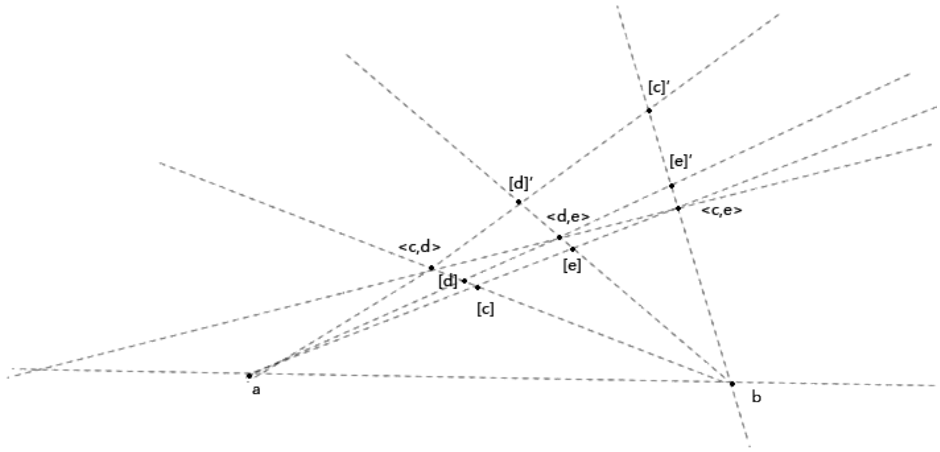

Figure 4.1 l intersect l'

Moreover, it is possible that l intersect two lines, for example $\langle a,b\rangle$ and $\langle c,d\rangle$. Namely, l can intersect a lines l' and another line l'' such that l' and l'' don't contain a common point in $\{a,b,c,d,e\}$. Without loss of generality, let's consider the case that l intersect $l' = \langle a,b\rangle$ and $l'' = \langle c,d\rangle$. Any pair of point one from each lines will give us a q-line that goes through both of them. We don't want the point we choose to be in $\{a,b,c,d\}$ and for l, there is a point that is contained in $\langle l'e\rangle$ and vice versa, so we also need to exclude this point. Thus, there are $(q-2)^2$ line intersect both of them.

l then will intersect four other planes that don't contain either l' or l'', which are $\langle a,c,e\rangle$, $\langle a,d,e\rangle$, $\langle b,c,e\rangle$ and $\langle b,d,e\rangle$. Notice that there are two q-lines L and L' which are the intersections of $\langle a,c,e\rangle$ with $\langle b,d,e\rangle$ and $\langle b,c,e\rangle$ and $\langle a,d,e\rangle$ respectively. Both of L and L' contain e and the line l may go through one or two of them. In fact, consider the plane $\langle L,L'\rangle$, this plane will intersect both l' and l'' at a single point and the line connecting these two points will not go through e. Hence, we have one line that intersect both L and L'. Then pick a point on l' that is not in H, this point and l'' will form a plane that intersect both L and L'' at a single point. Then for each point, except one, we picked on l', there is one line intersect l', l'' and L (L'). The one exceptional point together with l'' will form the plane

$\langle c,d,e\rangle$. Then we have $2(q-2)-2$ lines passing through one of L and L' and one line that intersect both of them. Thus, there are

$$(q-2)^2-2(q-3)-1=q^2-6q+9$$

lines that intersect both l' and l'' but intersect the four planes each at a separate point. Hence, there are

$$(q^2-6q+9)(q-5)(q-6)+(2q-6)(q-4)(q-5)+(q-3)(q-4)$$

pairs of points that can form a line l intersect both l' and l''.

Since we have ten lines and fifteen pairs of lines where each pair is counted twice, we will exclude

$$\begin{aligned}&10(q-2)(q^2-7q+12)(q^2-7q+11)-15(q^2-6q+9)(q-5)(q-6)\\&-15(2q-6)(q-4)(q-5)-15(q-3)(q-4)\\=&10q^5-175q^4+1225q^3-4280q^2+7420q-5070\end{aligned}$$

pair of points in this case.

As a result, there are

$$\begin{aligned}&(q^3+q^2+q-4)(q^3+q^2+q-5)-(10q^2-25q+20)(2q^3-8q^2+27q-29)\\&+15(q^2+q+1-6q+5)(q-4)(q-5)\\&-(10q^5-175q^4+1225q^3-4280q^2+7420q-5070)\\=&q^6-28q^5+323q^4-1952q^3+6462q^2-11004q+7470\\=&(q-3)(q-5)\left(q^4-20q^3+148q^2-468q+498\right)\end{aligned}$$

pairs of points that fulfill the general position condition Hence,

$$
\begin{aligned}
p_{7,e}^{3} =& \frac{1}{7!}\left|PGL_4(\mathbb{C})\right|(q^6-28q^5+303q^4-1674q^3+5004q^2-7582q+4528)\\
=& \frac{1}{5040}(q-5)(q-3)(q-1)^3q^6(q+1)^2(q^2+q+1)(q^2+1)\\
& \left(q^4-20q^3+148q^2-468q+498\right)
\end{aligned}
$$

Remark 4.3.1. *Notice that $\left|X_7^3(\mathbb{F}_q)\right|$ is not divisible by $\left|X_6^3\right|(\mathbb{F}_q)$ for general q. We can verify that $X_7^3 \to X_6^3$ is not a fibration.*

4.3.2 Cycle type (12)

Similar to the last one, we start from having five q-points $\{a,b,c,d,e\}$ in general position, there are total

$$|PGL_4(\mathbb{C})|(q^6+q^4-q^3-q)$$

tuples to start with. Let H be the union of all the planes formed by $\{a,b,c,d,e\}$, there are

$$10(q^4-q)-20(q^2-q)-15(q^2-q)=10q^4-35q^2+25q$$

q^2-points in H. Again, for the q^2-point g, we have three cases to make the general position condition fail:

- g is in H;
- the q-line $l=\langle g,f(g)\rangle$ passing through one point from $\{a,b,c,d,e\}$, and g is not in H;
- l intersects at least one of the ten q-lines away from $\{a,b,c,d,e\}$, but not containing any point from $\{a,b,c,d,e\}$, and g is not in H.

Then for the first case, we have

$$10q^4 - 35q^2 + 25q$$

q^2-points.

For the second case, for each point p in $\{a,b,c,d,e\}$, there are $(q^2+q+1-6q+7)$ many q-lines going through p and not in H. Hence, we have

$$5(q^2+q+1-6q+5)(q^2-q)$$

q^2-points in this case.

For the last case, for any line $l' = \langle \alpha, \beta \rangle$, where $\alpha, \beta \in \{a,b,c,d,e\}$ we have

$$(q-2)\left((q^4-q)-8(q^2-q)\right)$$

q^2-points that form lines that intersect l' including at α and β. Also, there are $15(q-2)^2$ lines intersect a pair of lines that don't intersect each other. Therefore we have

$$10(q-2)\left((q^4-q)-8(q^2-q)\right) - 15(q-2)^2(q^2-q) - 20(q^2-5q+6)(q^2-q)$$

q^2-points for this one.

Therefore, we have

$$\begin{aligned}
&(q^6+q^4-q^3-q)-(10q^4-35q^2+25q)+15(q^2+q+1-6q+5)(q^2-q)\\
&-10(q-2)\left((q^4-q)-8(q^2-q)\right)+15(q-2)^2(q^2-q)\\
=&q^6-10q^5+41q^4-86q^3+90q^2-36q\\
=&q\,(q-1)\,(q-2)\,(q-3)\left(q^2-4q+6\right)
\end{aligned}$$

choices for g. Hence, we have

$$\begin{aligned} p_{7,(12)}^{3} =& \frac{1}{5!\cdot 2}\left|PGL_4(\mathbb{C})\right| q\,(q-1)\,(q-2)\,(q-3)\left(q^2-4q+6\right) \\ =& \frac{1}{240}(q-3)(q-2)(q-1)^4q^7(q+1)^2(q^2+1)(q^2+q+1)(q^2-4q+6). \end{aligned}$$

4.3.3 Cycle type (12)(34)

For this one, let's start with two q^2-points a, b, and one q-point c in the general position. Then we have two q-planes, two pairs of q^2-planes each contains an orbit of a or b and one of the other q^2-point, and two pairs of q^2-planes that contain one point from each orbits. Moreover, there are two q-lines, two pairs of q^2-lines that contain one point from each orbit of q^2-points, and two pairs of q^2-lines that contain one point from a q^2-orbit and c. Let H be the union of these planes, then H contain

$$q^3+q^2+q+1-q^3+q^2+2q = 2q^2+3q+1$$

q-points. For the last two q-point d and e, we have $(q^3+q^2+q)(q^3+q^2+q-1)$ choices in total. Let $l = \langle d,e\rangle$, we will still consider the three cases as previous ones.

For the first case, there are

$$(2q^2+3q)(q^3+q^2+q-1)$$

pairs of points to have d inside H, and

$$(2q^2+3q)(2q^2+3q-1)$$

pairs of points to have both d and e in H. Therefore, we rule out

$$(2q^2+3q)(2q^3-q-1)$$

pairs of points in the first case.

For the second case, if l contains a q^2-point p, then it must also contain $f(p)$, then both d and e will be in H, so we only consider the case that l going through c. There are q^2+q+1 lines going through c and $2q+3$ of them are in H. Since the only two q-planes both contain c, l doesn't intersect H at another q-point other than c. Then we rule out

$$(q^2-q-2)q(q-1)$$

pairs of points for this one.

For the last case, notice that if l intersects with a q^2-line l', then it must also intersect $f(l')$. Then we have four conditions,

- l intersects at least one q-line;
- l intersects a pair of q^2-lines which contains one point each of the orbits of a and b

Let's start with the first one, without loss of generality, let's suppose l intersect $l' = \langle a, f(a) \rangle$. Then let P be a plane that containing l' and not in H. P will intersect each plane at a separated line and each of intersections of planes at a distinct point. Then there are two q-lines and two q-points off the line in P that are in H. Hence, we have $q^2+q+1-2(q+1)-1$ points in P that are not in H. Since we have q such plane P, there are

$$q(q^2-q-2)(q^2-q-3)$$

pairs of points that form a line intersects l'. The only possible line that can be paired with l' is $l'' = \langle b, f(b) \rangle$. There are q^2 lines that intersect both of them and one such a line will go through c. l will intersect the planes $\langle a,b,c \rangle$, $\langle a, f(b), c \rangle$, $\langle f(a), b, c \rangle$ and $\langle f(a), f(b), c \rangle$.

Let L be the intersection of $\langle a,b,c \rangle$ and $\langle f(a), f(b), c \rangle$, and let L' be the intersection of $\langle a, f(b), c \rangle$ and $\langle f(a), b, c \rangle$. L and L' will be two q-lines that containing c. Moreover, these four q^2-planes contain no q-points outside L and L'. Consider plane $P' = \langle L, L' \rangle$. P' will intersect L and L' each at a point and the line connecting these two points will not go through c. Then there is exactly one line that intersect l', l'', L, and L' altogether. Moreover, for each point p on l' except one, the plane $\langle p, l'' \rangle$ will intersect L and L' each at a point. So there are $2q-2$ lines that intersect one of L or L'. Then there are q^2-2q lines that don't intersect either of L and L''. Hence, we have

$$
\begin{aligned}
&(q^2-2q+1)(q-1)(q-2)+(2q-2)(q-2)(q-3)+(q-3)(q-4)\\
=&q^4-3q^3-2q^2+8q+2
\end{aligned}
$$

pairs of points that form a line intersect l' and l''.

For the first kind of q^2-lines, without loss of generality, let's suppose l intersect $l' = \langle a, b \rangle$. Then since l is a q-line, l must also intersect $f(l')$. Then there are q^2-2 such lines in total. l will intersect two q-planes $\langle a, f(a), c \rangle$ and $\langle b, f(b), c \rangle$ and a pair of q^2-planes $\langle a, f(b), c \rangle$ and $\langle f(a), b, c \rangle$. Let L be the intersection of the two q-planes and L' be the intersection of the pair of q^2-planes. Consider a plane P' containing L. L will intersect l' and $f(l')$ at a pair of q^2-points. The q-line connecting these two points will not going through c except for one plane. Similar argument apply for planes containing L'. Moreover, there is one plane that containing both L and L' so there is one line that intersect both L and L'. There are $q-3$ planes containing L but not L' and not contained in H. For L', there are $q-1$ such planes. Therefore, there are $2q-4$ lines intersect with one of L or L' and one line that intersect both. We have $q^2-2-(2q-4)-1 = q^2-2q+1$ lines that doesn't intersect either L or L'. As a result, there are

$$
\begin{aligned}
&(q^2-2q+1)(q-1)(q-2)+(q-3)q(q-1)+(q-1)(q-2)(q-3)+(q-1)(q-2)\\
=&(q-1)\left(q^3-2q^2-2q+2\right)
\end{aligned}
$$

pairs for points that generates a line that intersect l' and l''.

Therefore, we have

$$
\begin{aligned}
&(q^3+q^2+q)(q^3+q^2+q-1)-(2q^2+3q)(2q^3-q-1)-(q^2-q-2)q(q-1)\\
&-2q(q^2-q-2)(q^2-q-3)+\left(q^4-3q^3-2q^2+8q+2\right)\\
&-2\left(q-1\right)\left(q^3-2q^2-2q+2\right)\\
=&\left(q-1\right)\left(q+1\right)\left(q^4-4q^3+12q-6\right)
\end{aligned}
$$

pairs of points to choose. As a result,

$$
\begin{aligned}
p_{7,(12)(34)^3}=&\frac{1}{2^2\cdot 2!\cdot 3!}\left|PGL_4(\mathbb{C})\right|\left(q-1\right)\left(q+1\right)\left(q^4-4q^3+12q-6\right)\\
=&\frac{1}{48}(q-1)^4q^6(q+1)^3(q^2+1)(q^2+q+1)(q^4-4q^3+12q-6)
\end{aligned}
$$

4.3.4 Cycle type (12)(34)(56)

Since we already know $p^3_{6,(12)(34)(56)}$, we only need to find the number of possible ways to find the q-point d given three q^2-points a, b, and c. With these points, we will have six pairs of q^2-planes which contains a q-line that formed by an orbit and q^2-point from another orbit and four pairs of q^2-planes which contains a q^2-point from each of the three orbits. The first kind of q^2-planes contain no q-point outside the q-lines. For each pair of the second kind of q^2-planes, they will intersect at a q-lines that is not intersecting any q-line formed by an orbit. Moreover, the four intersections will not intersect each other as well. Therefore, we need to rule out all the q-points on the seven q-lines. There are $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 7\left|\mathbb{P}^1(\mathbb{F}_q)\right|$. To correct for the overcounting, we need to divides the result by $2^3 \cdot 3!$ Hence, we have

$$\begin{aligned}
p^3_{7,(12)(34)(56)} =& \frac{1}{2^3 \cdot 3!}(48 \cdot p^3_{6,(12)(34)(56)}) \left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 7\left|\mathbb{P}^1(\mathbb{F}_q)\right|\right) \\
=& \frac{1}{48}(q-2)(q-1)^3 q^6 (q+1)^2 (q^2+1)(q^2+q+1)(q^2-q-4) \\
& (q^3+q^2-6q-6) \\
=& \frac{1}{48}(q-2)(q-1)^3 q^6 (q+1)^3 (q^2+1)(q^2+q+1)(q^2-q-4)(q^2-6)
\end{aligned}$$

4.3.5 Cycle type (123)

We can solve this base on $p^3_{6,(123)}$. Suppose we already have one generic q^3-point a and three q-points b, c, and d. Then we will have two q-planes, three groups of q^3-planes which contains one point from $\{f(a)\}$ and two points from $\{b,c,d\}$, and three groups of q^3-planes which contains two points from $\{f(a)\}$ and one point from $\{b,c,d\}$. The first kind of q^3-planes cannot contain any q-point outside the q-line formed by two points from $\{b,c,d\}$ and the second kind of q^3-planes cannot contain any q-points other than the intersection of the planes from its orbit, and that will be a point from $\{b,c,d\}$. Therefore, we only need to rule out the q-points that are on the two q-planes. Since the two q-planes will intersect at a q-line, there are $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 2\left|\mathbb{P}^2(\mathbb{F}_1)\right| + \left|\mathbb{P}^1(\mathbb{F}_1)\right|$ choices for the last q-point. The process of selection will distinguish the points in $\{f(a)\}$ and choose an order for the q-points, so to correct the overcounting, the result need to be divided by $3 \cdot 4!$ .Therefore,

$$\begin{aligned}
p^3_{7,(123)} =& \frac{1}{3 \cdot 4!}(18 \cdot p^3_{6,(123)}) \left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 2\left|\mathbb{P}^2(\mathbb{F}_1)\right| + \left|\mathbb{P}^1(\mathbb{F}_1)\right|\right) \\
=& \frac{1}{72}(q-1)^4 q^7 (q+1)^3 (q^2+1)(q^2+q+1)(q^3-q^2) \\
=& \frac{1}{72}(q-1)^5 q^9 (q+1)^3 (q^2+1)(q^2+q+1)
\end{aligned}$$

4.3.6 Cycle type (123)(45)

Based on $p^3_{6,(123)(45)}$, we only need to find the number of choices for the second q-point d when we already have a q^3-point a, a q^2-point b and a q-point c. Now, we have two q-planes, one group of q^3-planes that contains the line $\langle b, f(b)\rangle$, one group of q^3-planes that contain c, one group of q^6-planes that contain two points in $\{f(a)\}$ and one point in $\{f(b)\}$, and one group of q^6-planes that contain one point from $\{f(a)\}$, one point from $\{f(b)\}$, and c. For the first kind of q^3-planes, they contain no q-point outside $\langle b, f(b)\rangle$, and the second kind of q^3-planes contain not q-point other than c. The first kind of q^6-planes contain no q-point and the second kind of q^6-planes contain no q-point other than c. Therefore, we have $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 2\left|\mathbb{P}^2(\mathbb{F}_q)\right| + \left|\mathbb{P}^1(\mathbb{F}_q)\right|$. Since the choice from an orbit and the order of choice won't change the result, to correct the overcounting, we need to divides the total number of choices by $3\cdot 2\cdot 2!$. Therefore,

$$
\begin{aligned}
p^3_{7,(123)(45)} =& \frac{1}{3\cdot 2\cdot 2!}(6\cdot p^3_{6,(123)(45)})\left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - 2\left|\mathbb{P}^2(\mathbb{F}_q)\right| + \left|\mathbb{P}^1(\mathbb{F}_q)\right|\right)\\
=& \frac{1}{12}(q-1)^4 q^7 (q+1)^3 (q^2+1)(q^2+q+1)(q^3-q^2)\\
=& \frac{1}{12}(q-1)^5 q^9 (q+1)^3 (q^2+1)(q^2+q+1)
\end{aligned}
$$

4.3.7 Cycle type (123)(45)(67)

First we can choose a generic q^3-point a freely, and that will give us a q-plane. For the q^2-point b, we only need to make sure that it is not on the existing q-plane P. Hence, there are $\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| - \left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(2)})\right|$ choices for b. Now, we will have one group of q^6-planes, one q-plane, and three q^3-planes. The three q^3-planes cannot contain any q^2-point outside the q-line $\langle b, f(b)\rangle$ and the q^6-planes cannot contain any q^2-point other than b and $f(b)$. Now, we only need to make sure the second q^2-point c not on the q-plane P and the line

$l = \langle c, f(c) \rangle$ doesn't intersect the line $l' = \langle b, f(b) \rangle$. Thus, we need to rule out the q-lines that intersects l' and not in P.

For a q-line that intersects l' and not in P, there are two possibilities. The first is that it intersect l' and P at two different q-points other than the intersection of l and P. There are $\left(\left|\mathbb{P}^2(\mathbb{F}_q)\right| - 1\right)\left(\left|\mathbb{P}^1(\mathbb{F}_q)\right| - 1\right)$ such lines. Also, l can go through the intersection of l' and P. There are $q^2 - 1$ such lines not contained in the planes we already have. Then there are $\left(\left|\mathbb{P}^2(\mathbb{F}_q)\right| - 1\right)\left(\left|\mathbb{P}^1(\mathbb{F}_q)\right| - 1\right) + (q^2 - 1)$ choices for the q-lines l. Therefore, we can choose c in

$$\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| - \left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(2)})\right| - \left(\left(\left|\mathbb{P}^2(\mathbb{F}_q)\right| - 1\right)\left(\left|\mathbb{P}^1(\mathbb{F}_q)\right| - 1\right) + q^2\right)\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|$$

ways. Since the choice from an orbit and the order of choice won't change the result, to correct the overcounting, we need to divides the total number of choices by $3 \cdot 2 \cdot 2 \cdot 2!$. Hence,

$$\begin{aligned}
p^3_{7,(123)(45)(67)} =& \frac{1}{3 \cdot 2 \cdot 2 \cdot 2!}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right|\left(\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| - \left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(2)})\right|\right) \\
&\left(\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)})\right| - \left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(2)})\right| - \left(\left(\left|\mathbb{P}^2(\mathbb{F}_q)\right| - 1\right)\left(\left|\mathbb{P}^1(\mathbb{F}_q)\right| - 1\right) + q^2\right)\left|\mathbb{P}^1(\overline{\mathbb{F}}_q^{(2)})\right|\right) \\
=& \frac{1}{24}q^3(q+1)^2(q-1)^2(q^2+1)\left(q^6 - q^3\right) \\
&\left(q^6 - q^3 - \left((q^2+q)q + q^2\right)(q^2 - q)\right) \\
=& \frac{1}{24}(q-1)^5 q^9 (q+1)^3 (q^2+1)(q^2+q+1)
\end{aligned}$$

4.3.8 Cycle type (123)(456)

For this case, let's start from the beginning, i.e. for two generic q^3-points a, b and one q-point c, how many ways do we have to put them in non-general position. There

$$\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right|\left(\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right| - 3\right)\left|\mathbb{P}^3(\mathbb{F}_q)\right|$$

tuples in total. To violate the general position condition, we have six cases, and we will use Δ_i to demonstrate the set of tuples in the i^{th} case. Let $A = \langle a, f(a), f^2(a)\rangle$ and $B = \langle b, f(b), f^2(b)\rangle$ as well as $l_a = \langle a, f(a)\rangle$ and $l_b = \langle b, f(b)\rangle$. Then we have the cases as following:

- Δ_1 consists all the tuples such that c is in A;
- Δ_2 consists all the tuples such that c is in B;
- Δ_3 consists all the tuples such that $A = B$;
- Δ_4 consists all the tuples such that a line in $\{f(l_a)\}$ intersect a line in $\{f(l_b)\}$ where $A \neq B$;
- Δ_5 consists all the tuples such that one point in $\{f(a)\}$, two points in $\{f(b)\}$ and c are in a same plane where the points in $\{f(a)\}$ and c are not in B;
- Δ_6 consists all the tuples such that one point in $\{f(b)\}$, two points in $\{f(a)\}$ and c are in a same plane where the points in $\{f(b)\}$ and c are not in A.

For the first two cases, we start by choosing any two generic q^3-points and put c in the corresponding plane, then we have

$$|\Delta_1| = |\Delta_2| = \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| (\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| - 3) \left|\mathbb{P}^2(\mathbb{F}_q)\right|.$$

The third case can be derived by first choosing a, then choose b from A and choose c arbitrarily. Hence,

$$|\Delta_3| = \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| (\left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| - 3) \left|\mathbb{P}^3(\mathbb{F}_q)\right|.$$

To get case four, we can first choose a arbitrarily, then pick any q-point p that is not on A and choose any q^3-points on the three q^3-lines that connecting the p and a point from $\{f(a)\}$

that is not in A. Lastly, choose an arbitrary c. Hence,

$$|\Delta_4| = 3\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right|\left(|\mathbb{P}^3(\mathbb{F}_q)| - |\mathbb{P}^2(\mathbb{F}_q)|\right)\left(\left|\mathbb{P}^1(\mathbb{F}_{q^3})\right| - 2\right)|\mathbb{P}^3(\mathbb{F}_q)|.$$

For the last two cases, we first choose one generic q^3-point and choose any c outside the plane. Then choose the other generic q^3-point on the three q^3-planes outside the q-plane. Hence,

$$|\Delta_5| = |\Delta_6| = 3\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right|\left(|\mathbb{P}^3(\mathbb{F}_q)| - |\mathbb{P}^2(\mathbb{F}_q)|\right)(q^6 - q^4 - 2q^3 + q + 1).$$

To apply inclusion-exclusion formula, we also need to know about their intersections. For case one and two, the intersection will be c in both A and B. Then either $A = B$ and they contain c or A and B are different plane and c is in their intersection. Hence,

$$\begin{aligned}\left|\Delta_1 \bigcap \Delta_2\right| =& \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right|\left(\left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right| - 3\right)|\mathbb{P}^2(\mathbb{F}_q)| \\ &+ \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right|\left(\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right| - \left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right|\right)|\mathbb{P}^1(\mathbb{F}_q)|.\end{aligned}$$

Both case one and case two intersect case three will result all the points in the same plane. Therefore,

$$\begin{aligned}&\left|\Delta_1 \bigcap \Delta_3\right| = \left|\Delta_2 \bigcap \Delta_3\right| = \left|\Delta_1 \bigcap \Delta_2 \bigcap \Delta_3\right| \\ &= \left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right|\left(\left|\mathbb{P}^2(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right| - 3\right)|\mathbb{P}^2(\mathbb{F}_q)|.\end{aligned}$$

Case four and one of the first two cases will happen when c is in the corresponding plane, hence,

$$\left|\Delta_1 \bigcap \Delta_4\right| = \left|\Delta_2 \bigcap \Delta_4\right| = 3\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\mathrm{gen})})\right|\left(|\mathbb{P}^3(\mathbb{F}_q)| - |\mathbb{P}^2(\mathbb{F}_q)|\right)\left(\left|\mathbb{P}^1(\mathbb{F}_{q^3})\right| - 2\right)|\mathbb{P}^2(\mathbb{F}_q)|$$

To get the intersection of Δ_1 and Δ_5 as well as Δ_2 and Δ_6, without loss of generality, let's consider the first pair. To have c in A, we want to find the number of generic q^3-points in a q^3-plane, which contains two points in $\{f(b)\}$ and c, and also in a q-plane that contains c. Without loss of generality, let $P = \langle b, f(b), c\rangle$. There are q^2+q+1 many q-planes contain c and they each intersect P at a distinct q^3-line. These q^3-line all contain c, so they each contain q^3-q^2 generic q^3-points. Therefore,

$$\left|\Delta_1 \bigcap \Delta_6\right| = \left|\Delta_2 \bigcap \Delta_5\right| = 3\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| \left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right|\right)(q^2+q+1)(q^3-q^2).$$

For case four and case five or case six happen together, we either have two points from $\{f(a)\}$, two points from $\{f(b)\}$ and c in a same plane, or l_a intersect $l \in \{f(l_b)\}$ and $\langle b', c\rangle$ intersects l_a where $b' \in \{f(b)\}$ and $b' \notin l$. Since l_a is a q^3-line that formed by a generic q^3-point, the q^3-plane $\langle l_a, b'\rangle$ contains only one q-point, so we only have one choice for c. Hence,

$$\left|\Delta_4 \bigcap \Delta_5\right| = \left|\Delta_4 \bigcap \Delta_6\right| = 6\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| \left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right|\right)\left(\left|\mathbb{P}^1(\mathbb{F}_{q^3})\right| - 2\right).$$

Case five and six happen at the same time will make two points from $\{f(a)\}$, two points from $\{f(b)\}$ and c in a same plane, so

$$\left|\Delta_5 \bigcap \Delta_6\right| = 3\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| \left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right|\right)\left(\left|\mathbb{P}^1(\mathbb{F}_{q^3})\right| - 2\right)$$

By the definition, we can see that Δ_3 will not intersect Δ_4, Δ_5 and Δ_6. We also have three triple intersections where $|\Delta_1 \bigcap \Delta_2 \bigcap \Delta_3|$ been calculated. Case one, two, and four happen the same time when c is in the intersection of A and B. We have

$$\left|\Delta_1 \bigcap \Delta_2 \bigcap \Delta_4\right| = 3\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| \left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right|\right)\left(\left|\mathbb{P}^1(\mathbb{F}_{q^3})\right| - 2\right)\left|\mathbb{P}^1(\mathbb{F}_q)\right|.$$

Case four, five, and six happen the same time when we have two points in $\{f(a)\}$, two points in $\{f(b)\}$ and c are in a same plane, hence,

$$\left|\Delta_4 \bigcap \Delta_5 \bigcap \Delta_6\right| = 3\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})})\right| \left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^2(\mathbb{F}_q)\right|\right) \left(\left|\mathbb{P}^1(\mathbb{F}_{q^3})\right| - 2\right).$$

Then by applying inclusion-exclusion formula, we can get the final result

$$\begin{aligned} p^3_{7,(123)(456)} =& \frac{1}{18}(q-1)^3 q^6 (q+1)^2 (q^2+1)(q^2+q+1) \\ & (q^6 - q^5 - q^4 - 8q^3 + 9q^2 + 6q + 18). \end{aligned}$$

4.3.9 Cycle type (1234)

Let's start with have one generic q^4-point a and one q-point b in generic position. There are two orbits of q^4-planes of $P_1 = \langle a, f(a), b\rangle$ and $P_2 = \langle a, f(a), f^2(a)\rangle$, and a pair of q^2-planes $\{f(P_3 = \langle a, f^2(a), c\rangle)\}$. Planes from $\{f(P_1)\}$ contain no q-points and planes from $\{f(P_2)\}$ contains no q-point other than b. The pair of q^2-planes will intersect at a q-line that containing b and contain no other q-point outside the intersection. Thus, there are $q+1$ many q-points on the union of planes H. For the last two points c and d, let $l = \langle c, d\rangle$. There are three cases to fail the general position condition:

- at least one of c or d is in H;
- l going through one existing point and c, d are not in H;
- l intersects at least one existing line but not go through any existing point, and c, d are not in H.

In the first case, we have $q(q^3+q^2+q-1)$ pairs for c to be in H, and $q(q-1)$ pairs for c and d both in H. Hence we rule out

$$q(2q^3+2q^2+q-1)$$

pairs in this case.

In the second case, since a is generic, l cannot contain a, so the only possible case is l contains b. There are q^2+q lines going through b and not in H. These lines intersect with H at no q-point other than b, hence we rule out

$$(q^2+q)q(q-1)$$

pairs in this case.

For the third one, notice that if l intersect a q^4-line l', then it must intersect with all lines in $\{f(l)\}'$. Then l intersect a q^4-line if and only if l goes through b, which is already considered, so we only need to consider the case that l intersect $l' = \langle a, f^2(a)\rangle$. By the same argument, l intersect l' if and only if it intersect $f(l')$. Then for any q^2-point on l', we can get a such q-line and there is exactly one such q-lines that is going through b and inside H. Therefore, since l' contain no q-point, we will get rid of

$$(q^2+1-1)(q+1)q$$

pairs to avoid this case.

As a result, there are

$$
\begin{aligned}
&(q^3+q^2+q)(q^3+q^2+q-1)-q(2q^3+2q^2+q-1)-(q^2+q)q(q-1)\\
&-(q^2+1-1)(q+1)q\\
=&(q-1)q^3(q+1)(q+2)
\end{aligned}
$$

pairs of q-points satisfying the general position condition. Hence,

$$
\begin{aligned}
p_{7,(1234)(56)} =&\frac{1}{4\cdot 3!}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(4,\text{gen})})\right|\left|\mathbb{P}^3(\mathbb{F}_q)\right|(q-1)q^3(q+1)(q+2)\\
=&\frac{1}{24}(q-1)^4q^9(q+1)^3(q+2)(q^2+1)(q^2+q+1)
\end{aligned}
$$

4.3.10 Cycle type (1234)(56)

We start with the same setup as the last one, let's call the q^2-point we want to find out c and $l=\langle c, f(c)\rangle$. q^4-planes in $\{f(P_1)\}$ have no q^2-point outside the pair of q^2-lines $l'=\langle a, f^2(a)\rangle$ and $f(l')=\langle f(a), f^3(a)\rangle$. $\langle P_2, f^2(P)\rangle$ defines another q^2-line l'' where $f(l'')=\langle f(P_2), f^3(P_2)\rangle$. l'' and $f(l'')$ intersect at b and will not intersect any of $\{f(l')\}$. Hence, we have

$$2q^4+(q^2-q)+2q^2$$

q^2-points in H.

For the first case, it is obvious that we will rule out

$$2q^4+(q^2-q)+2q^2$$

q^2-points.

For the second case, there is again only one q-lines through b inside H and since l will intersect the planes from $\{f(P_1)\}$ outside $\{f(l')\}$, the intersections are always q^4-points.

Hence, we will rule out

$$(q^2+q)(q^2-q)$$

q^2-points in this case.

For the last one, by the same reasoning, l cannot intersect any of the q^4-lines. Thus, we only need to consider the case that l intersect with l'. However, in this case, it is possible that l intersect both l' and l''. Consider the q-plane $P' = \langle l'', f(l'')\rangle$. If l intersect l'', then l must be in P'. Since l'' doesn't intersect l' or $f(l')$, P' will intersect l' and $f(l')$ each at a q^2-point and it cannot going through b. Hence, there is exactly one line passing both l' and l''. As a result, we can rule out

$$(q^2+1-2)(q^2-q-2)+(q^2-q-4)$$

q^2-points in this case.

Therefore, we have

$$\begin{aligned}
&q^6+q^4-q^3-q-(2q^4+(q^2-q)+2q^2)-(q^2+q)(q^2-q)\\
&-(q^2+1-2)(q^2-q-2)-(q^2-q-4)\\
=&(q-1)(q+1)(q^4-2q^2-2)
\end{aligned}$$

q^2-point to choose. Thus,

$$\begin{aligned}
p_{7,(1234)(56)} =&\frac{1}{4\cdot 2}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(4,\text{gen})})\right|\left|\mathbb{P}^3(\mathbb{F}_q)\right|(q-1)(q+1)(q^4-2q^2-2)\\
=&\frac{1}{8}(q-1)^4q^6(q+1)^3(q^2+1)(q^2+q+1)(q^4-2q^2-2)
\end{aligned}$$

4.3.11 Cycle type (1234)(567)

Since choosing a q^4-points won't put any constraint on choosing a q^3-points, so we can choose both of the points arbitrarily. To correct the overcounting, we just need to divide the result by the number of elements in each of the orbits, so we divide the result by $4 \cdot 3$. Thus,

$$\begin{aligned} p^3_{7,(1234)(567)} =& \frac{1}{4 \cdot 3} \left| \mathbb{P}^3(\overline{\mathbb{F}}_q^{(4,\text{gen})}) \right| \left| \mathbb{P}^3(\overline{\mathbb{F}}_q^{(3,\text{gen})}) \right| \\ =& \frac{1}{12}(q-1)^5 q^9 (q+1)^3 (q^2+1)(q^2+q+1) \end{aligned}$$

4.3.12 Cycle type (12345)

The generic q^5-point a can be chosen arbitrarily. Then the q^5-planes are all generic, so there is no restriction on the choice of first q-point b. With b, we will get ten more q^5-planes where each of them contains only one q-point which is b. Hence, we can choose any q-point other than b. To correct for the overcounting, we need to ignore the order of choices and the different choices of the elements from the orbit, so we divide the result by $5 \cdot 2!$. As a result,

$$\begin{aligned} p^3_{7,(12345)} =& \frac{1}{5 \cdot 2!} \left| \mathbb{P}^3(\overline{\mathbb{F}}_q^{(5,\text{gen})}) \right| \left| \mathbb{P}^3(\mathbb{F}_q) \right| \left(\left| \mathbb{P}^3(\mathbb{F}_q) \right| - 1 \right) \\ =& \frac{1}{10}(q-1)^3 q^7 (q+1)^3 (q^2+1)^2 (q^2+q+1)^2 \end{aligned}$$

4.3.13 Cycle type (12345)(67)

We can choose the generic q^5-point arbitrarily and it will put no constraint on the choice of the q^2-point. To correct the overcounting, we just need to divide the result by the number of elements in each of the orbits, so we divide the result by $5 \cdot 2$. Hence,

$$\begin{aligned} p^3_{7,(12345)(67)} =& \frac{1}{5 \cdot 2} \left| \mathbb{P}^3(\overline{\mathbb{F}}_q^{(5,\text{gen})}) \right| \left| \mathbb{P}^3(\overline{\mathbb{F}}_q^{(2)}) \right| \\ =& \frac{1}{10}(q-1)^4 q^7 (q+1)^2 (q^2+1)^2 (q^2+q+1)^2 \end{aligned}$$

4.3.14 Cycle type (123456)

After choosing a generic q^6-point, we can check that all the eighteen q^6-planes are generic. Hence, there are no q-points on those planes. Moreover, we have a pair of q^2-planes that intersect each other on a q-line. Thus, there are $\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^1(\mathbb{F}_q)\right|$ many choices for the q-point. Since we only need to correct the overcounting caused by distinguishing points in the orbit of q^6-point, we divide the result by 6 to get the correct result. Therefore, we have

$$\begin{aligned}
p^3_{7,(123456)} =& \frac{1}{6}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(6,\mathrm{gen})})\right| \left(\left|\mathbb{P}^3(\mathbb{F}_q)\right| - \left|\mathbb{P}^1(\mathbb{F}_q)\right|\right) \\
=& \frac{1}{6}(q-1)^3 q^8 (q+1)^3 (q^2+1)(q^2+q+1)(q^3+q-1)
\end{aligned}$$

4.3.15 Cycle type (1234567)

Since we can choose the generic q^7-point arbitrarily, the overcounting is only caused by different pick from the orbit, we have

$$\begin{aligned}
p^3_{7,(1234567)} =& \frac{1}{7}\left|\mathbb{P}^3(\overline{\mathbb{F}}_q^{(7,\mathrm{gen})})\right| \\
=& \frac{1}{7}(q-1)^3 q^6 (q+1)^2 (q^2+1)(q^2+q+1)(q^2-q+1)(q^4+q^3+q^2+q+1)
\end{aligned}$$

4.4 Proof for Theorem 1.3.9

Similar to the proof for 1.3.8, with the character table of S_7, we have

$$\begin{aligned}
P_U(q) &= q^{21} - q^{19} - q^{18} + 2q^{17} - 2q^{16} + q^{14} - 2q^{13} + 3q^{12} - 2q^{11} + 2q^{10} - q^9 \\
&\quad - q^8 + 3q^7 - 2q^6 \\
P_{S_{6^1}} &= q^{19} - 3q^{18} + 8q^{17} - 12q^{16} + 2q^{15} + 9q^{14} - 6q^{13} + 12q^{12} - 17q^{11} + 3q^{10} \\
&\quad - 2q^9 + 14q^7 - 9q^6 \\
P_{S_{5^1 2^1}} &= -q^{20} + q^{19} - 5q^{18} + 21q^{17} - 28q^{16} + q^{15} + 19q^{14} - 11q^{13} + 32q^{12} - 36q^{11} \\
&\quad + 4q^{10} - 10q^9 - 3q^8 + 34q^7 - 18q^6 \\
P_{S_{5^1}} &= 3q^{19} - 9q^{18} + 16q^{17} - 25q^{16} + 12q^{15} + 24q^{14} - 19q^{13} + 19q^{12} - 46q^{11} + 10q^{10} \\
&\quad + 3q^9 + 6q^8 + 31q^7 - 25q^6 \\
P_{S_{4^1 3^1}} &= -q^{20} + q^{19} - 6q^{18} + 20q^{17} - 24q^{16} + 8q^{15} + 18q^{14} - 19q^{13} + 22q^{12} - 40q^{11} \\
&\quad + 11q^{10} - q^9 + 3q^8 + 31q^7 - 23q^6 \\
P_{S_{4^1 2^1}} &= 3q^{19} - 15q^{18} + 44q^{17} - 66q^{16} + 15q^{15} + 49q^{14} - 31q^{13} + 63q^{12} - 96q^{11} + 16q^{10} \\
&\quad - 13q^9 + 3q^8 + 78q^7 - 50q^6 \\
P_{S_{4^1}} &= q^{19} - 10q^{18} + 26q^{17} - 33q^{16} + 13q^{15} + 26q^{14} - 25q^{13} + 29q^{12} - 56q^{11} + 15q^{10} \\
&\quad - q^9 + 4q^8 + 42q^7 - 31q^6 \\
P_{S_{3^2}} &= 3q^{19} - 9q^{18} + 22q^{17} - 38q^{16} + 15q^{15} + 31q^{14} - 21q^{13} + 29q^{12} - 62q^{11} + 12q^{10} \\
&\quad - q^9 + 9q^8 + 44q^7 - 34q^6 \\
P_{S_{3^1 2^2}} &= q^{19} - 6q^{18} + 28q^{17} - 43q^{16} + 4q^{15} + 26q^{14} - 14q^{13} + 44q^{12} - 53q^{11} + 8q^{10} \\
&\quad - 14q^9 - q^8 + 48q^7 - 28q^6
\end{aligned}$$

$$
\begin{aligned}
P_{S_{3^1 2^1}} &= q^{19} - 12q^{18} + 41q^{17} - 64q^{16} + 21q^{15} + 46q^{14} - 30q^{13} + 54q^{12} - 97q^{11} + 18q^{10} \\
&\quad - 11q^{9} + 10q^{8} + 75q^{7} - 52q^{6} \\
P_{S_{3^1}} &= -3q^{18} + 20q^{17} - 31q^{16} + 2q^{15} + 17q^{14} - 8q^{13} + 32q^{12} - 36q^{11} + 5q^{10} - 12q^{9} \\
&\quad - q^{8} + 34q^{7} - 19q^{6} \\
P_{S_{2^3}} &= -5q^{18} + 17q^{17} - 25q^{16} + 11q^{15} + 18q^{14} - 15q^{13} + 19q^{12} - 41q^{11} + 10q^{10} - 2q^{9} \\
&\quad + 6q^{8} + 30q^{7} - 23q^{6} \\
P_{S_{2^2}} &= -3q^{18} + 13q^{17} - 26q^{16} + 12q^{15} + 19q^{14} - 9q^{13} + 17q^{12} - 41q^{11} + 6q^{10} - 4q^{9} \\
&\quad + 9q^{8} + 29q^{7} - 22q^{6} \\
P_{S_{2^1}} &= 6q^{17} - 13q^{16} + 3q^{15} + 7q^{14} - 2q^{13} + 10q^{12} - 16q^{11} + 2q^{10} - 4q^{9} + 3q^{8} \\
&\quad + 13q^{7} - 9q^{6} \\
P_{S_{1^7}} &= q^{17} - q^{16} + 3q^{15} - 4q^{13} - 2q^{12} - 4q^{11} + 4q^{10} + 3q^{9} + 3q^{8} + q^{7} - 4q^{6}.
\end{aligned}
$$

Then combine with equation (2.12), we will derive the result for table (1.1).

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