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Stabilization of Nonlinear Systems With Limited Information Feedback

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***Abstract*—This note is concerned with the problem of stabilizing a nonlinear continuous-time system by using sampled encoded measurements of the state. We demonstrate that global asymptotic stabilization is possible if a suitable relationship holds between the number of values taken by the encoder, the sampling period, and a system parameter, provided that a feedback law achieving input-to-state stability with respect to measurement errors can be found. The issue of relaxing the latter condition is also discussed.**

***Index Terms*—Asymptotic stability, encoding, input-to-state stability, limited information, measurement errors, nonlinear system.**

I. INTRODUCTION

In recent years, extensive research activity has been devoted to the question of how much information a feedback controller really needs in order to stabilize a given system. Questions of this kind are motivated by applications where communication capacity is limited (e.g., a large number of systems sharing the same network cable or wireless medium, microsystems with a large number of sensors and actuators on a single chip) as well as situations where security considerations compel one to transmit as little information as possible. Among the many references on this subject, the ones particularly close in spirit to the present work are [3], [4], [10], [16], [17], [21], [24], and [27].

All results developed in the aforementioned papers are limited to linear systems. The work reported here is one of the first steps toward understanding information-based control aspects for nonlinear systems. (The recent independent work [18] also treats the nonlinear problem, using a different approach.) Specifically, in this note we extend the result and the control scheme described in [16] to nonlinear systems, characterizing the amount of information sufficient for global asymptotic stabilization. As we will see, the main issues that arise in this process are the following:

1) Due to limited information available to the controller, the control law is based on estimated values of the state. The propagation of the error between the actual state and the estimated state is not as straightforward to characterize as in the linear case, because it is not decoupled from the state equations. Our analysis of the error dynamics relies on the Lipschitz property of the right-hand side, and in this sense our solution bears a conceptual resemblance to the construction given in [25][1] (see also [26]).
2) Unlike in the linear case, a feedback law that stabilizes the system in the case of perfect information is not necessarily robust with respect to the estimation error. Our main result is obtained under the assumption that the feedback law provides *input-to-state stability* (ISS) in the sense of [23] with respect to measurement errors. This requirement is quite restrictive in general, although some results on designing such control laws are available [5]–[7], [11]. (The same assumption was also used in the context of nonlinear control with limited information in [15].) An alternative result relying on a more easily verifiable assumption is presented in Section IV.

The setup considered in this note is as follows. The system to be stabilized is

$$\dot{x} = f(x, u) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control variable, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz function satisfying $f(0,0) = 0$. Control inputs considered in this note are piecewise Lipschitz continuous. The term "limited information feedback" refers to the following scenario.

- SAMPLING: Measurements are to be received by the controller at discrete times $0, \tau, 2\tau, \ldots$, where $\tau > 0$ is a fixed *sampling period*.
- ENCODING: At each of these sampling times, the measurement received by the controller must be a number in the set $\{0, 1, \ldots, N\}$, where N is a fixed positive integer.

Thus, the data available to the controller consists of a stream of integers

$$q_0(x(0)), q_1(x(\tau)), q_2(x(2\tau)), \ldots$$

where $q_k(\cdot) : \mathbb{R}^n \to \{0, 1, \ldots, N\}$ is, for each k, some *encoding function*. For different values of k, we can use different encoding functions. As we will see, it is natural to use the previous values $q_i(x(i\tau))$, $i = 0, \ldots, k-1$ to define the function $q_k(\cdot)$. We assume that the controller knows the initial encoding function $q_0(\cdot)$ as well as the rule that defines $q_k(\cdot)$ on the basis of the previously received encoded measurements, so that for each k the function q_k is known to the controller. In other words, there is a communication protocol satisfying the constraints just described upon which the process and the controller agree in advance.

In what follows, we find it convenient to use the infinity norm $\|x\|_\infty := \max\{|x_i| : 1 \le i \le n\}$ on $\mathbb{R}^n$. We let $B^n_\infty(x_0, r)$ denote a ball with respect to this norm with radius r and center x_0, i.e., the hypercubic box centered at x_0 with edges $2r$

$$B^n_\infty(x_0, r) := \{x \in \mathbb{R}^n : \|x - x_0\|_\infty \le r\}.$$

II. CONTROL STRATEGY AND ASSUMPTIONS

In this section, we describe the proposed control strategy based on limited information feedback, stating and briefly discussing the corresponding assumptions along the way. Our first goal is to obtain an upper bound on the size of the state. We do this by "zooming out," i.e., expanding the support of the encoding function, fast enough to dominate the growth of the state for the uncontrolled system (no feedback is applied at this stage). The following assumption is needed to execute this task.

Assumption 1: The unforced system

$$\dot{x} = f(x, 0) \tag{2}$$

is forward complete. This means that for every initial state $x(0)$ the solution of (2), which we denote by $\xi(x(0), \cdot)$, is defined for all $t \ge 0$.

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[1]We thank an anonymous referee for this remark.

Set the control u equal to 0. Let $\mu_0 := 1$. Pick a sequence $\mu_1, \mu_2, \ldots$ that increases fast enough to dominate the rate of growth of $\|x(t)\|_\infty$ at the times $\tau, 2\tau, \ldots$; for example, we can define $\mu_1 := 2\max_{\|x(0)\|_\infty \le \tau, t\in[0,\tau]} \|\xi(x(0),t)\|_\infty$, $\mu_2 := 2\max_{\|x(0)\|_\infty \le 2\tau, t\in[0,2\tau]} \|\xi(x(0),t)\|_\infty$, and so on. This construction guarantees the existence of an integer $k_0 \ge 0$ such that $\|x(k_0\tau)\|_\infty \le \mu_{k_0}$. For $k = 0, 1, \ldots$, define the encoding function q_k by the formula

$$q_k(x) := \begin{cases} 1, & \text{if } x \in B_\infty^n(0, \mu_k) \\ 0, & \text{otherwise.} \end{cases}$$

Then, we can take k_0 to be the smallest k for which $q_k(x(k\tau)) = 1$. We have thus obtained the bound

$$\|x(k_0\tau)\|_\infty \le E_0 := \mu_{k_0} \tag{3}$$

using the encoded state measurements with $N = 1$.

Remark 1: Constructing the sequence $\mu_1, \mu_2, \ldots$ requires an upper bound on the size of the reachable set of (2) from a given compact initial set in a given time. In practice, some known structure of the system can be utilized to obtain such an over-approximation to the reachable set. We also note that computation and analysis of reachable sets and their approximations is an active area of research (see, e.g., [14] for ellipsoidal approximation techniques and [1] for related results involving Lyapunov functions). □

The inequality (3) means that the state of the system at time $t = k_0\tau$ lies in $B_\infty^n(0, E_0)$. In other words, $\hat{x}(k_0\tau) := 0$ can be viewed as an estimate of $x(k_0\tau)$ with estimation error of infinity norm at most E_0. Our goal now is to generate state estimates with estimation errors approaching 0 as $t \to \infty$, while at the same time using these estimates to compute the feedback law. The next assumption was already discussed in the Introduction.

Assumption 2: System (1) admits a locally Lipschitz feedback law $u = k(x)$ which satisfies $k(0) = 0$ and renders the closed-loop system input-to-state stable (ISS) with respect to measurement errors. Written in terms of the infinity norm and for piecewise continuous inputs (which is sufficient for our purposes), this condition means that there exist functions[2] $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial condition $x(t_0)$ and every piecewise continuous signal e the corresponding solution of the system

$$\dot{x} = f(x, k(x+e)) \tag{4}$$

satisfies

$$\|x(t)\|_\infty \le \beta(\|x(t_0)\|_\infty, t - t_0) + \gamma\left(\sup_{s\in[t_0,t]} \|e(s)\|_\infty\right) \qquad \forall\, t \ge t_0. \tag{5}$$

Take κ to be some class $\mathcal{K}_\infty$ function with the property that $\kappa(r) \ge \max_{\|x\|_\infty \le r} \|k(x)\|_\infty$ for all $r \ge 0$. Then, we have

$$\|k(x)\|_\infty \le \kappa(\|x\|_\infty) \qquad \forall\, x. \tag{6}$$

Let L be the Lipschitz constant for the function f on the region

$$\{(x,u) : \|x\|_\infty \le D, \|u\|_\infty \le \kappa(D)\} \tag{7}$$

[2]Recall that a function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then it is said to be of class $\mathcal{K}_\infty$. A function $\beta : [0,\infty)\times[0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \ge 0$ and $\beta(r,t)$ decreases to 0 as $t \to \infty$ for each fixed $r \ge 0$. We will write $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$, etc.

where

$$D := \beta(E_0, 0) + \gamma(N^{1/n}E_0) + N^{1/n}E_0. \tag{8}$$

Define

$$\Lambda := e^{L\tau} \ge 1. \tag{9}$$

For $t \in [k_0\tau, k_0\tau + \tau)$, let $u(t) = 0$. At time $t = k_0\tau + \tau$, consider the box $B_\infty^n(0, \Lambda E_0)$.

Assumption 3: The number $N^{1/n}$ is an odd integer. This assumption is made mostly for notational convenience. If $N^{1/n}$ is not an integer, we can work with some $N' \le N$ such that $(N')^{1/n}$ is an integer. The reason for taking this integer to be odd is to ensure that the control strategy described later preserves the equilibrium at the origin. By making slight modifications, we can also achieve the desired properties when this integer is even.

Assumption 3 allows us to define the encoding function q_{k_0+1} as follows: Divide $B_\infty^n(0, \Lambda E_0)$ into N equal hypercubic boxes, numbered from 1 to N in some specific way, and let $q_{k_0+1}(x)$ be the number of the box that contains x, if $x \in B_\infty^n(0, \Lambda E_0)$, and 0, otherwise. In case x lies on the boundary of several boxes, the value $q_{k_0+1}(x)$ can be chosen arbitrarily among the candidates. If $q_{k_0+1}(x(k_0\tau + \tau)) > 0$, then the encoded measurement specifies a box with edges at most $2\Lambda E_0/N^{1/n}$ which contains $x(k_0\tau + \tau)$. Letting $\hat{x}(k_0\tau + \tau)$ be the center of this box, we obtain

$$\|\hat{x}(k_0\tau + \tau) - x(k_0\tau + \tau)\|_\infty \le \frac{\Lambda E_0}{N^{1/n}}.$$

If $q_{k_0+1}(x(k_0\tau + \tau)) = 0$, we interpret this as an error and return to the "zooming-out" stage described earlier.

For $t \in [k_0\tau + \tau, k_0\tau + 2\tau)$, we apply the control law

$$u(t) = k(\hat{x}(t)) \tag{10}$$

where $\hat{x}(\cdot)$ is the solution of the "copy" of the system (1), given by

$$\dot{\hat{x}} = f(\hat{x}, u) \tag{11}$$

with the initial condition $\hat{x}(k_0\tau + \tau)$ specified earlier. At time $t = k_0\tau + 2\tau$, we consider the box $B_\infty^n(\hat{x}(k_0\tau + 2\tau^-), \Lambda^2 E_0/N^{1/n})$. To define the encoding function q_{k_0+2}, divide this box into N equal hypercubic boxes and let $q_{k_0+2}(x)$ be the number of the box that contains x, or let $q_{k_0+2}(x) = 0$ if $x \notin B_\infty^n(\hat{x}(k_0\tau + 2\tau^-), \Lambda^2 E_0/N^{1/n})$. If $q_{k_0+2}(x(k_0\tau + 2\tau)) > 0$, then the encoded measurement singles out a box with edges at most $2\Lambda^2 E_0/(N^{1/n})^2$ which contains $x(k_0\tau + 2\tau)$. Let $\hat{x}(k_0\tau + 2\tau)$ be the center of this box to obtain

$$\|\hat{x}(k_0\tau + 2\tau) - x(k_0\tau + 2\tau)\|_\infty \le \frac{\Lambda^2 E_0}{(N^{1/n})^2}$$

and continue. If $q_{k_0+2}(x(k_0\tau + 2\tau)) = 0$, go back to the "zooming-out" stage.

Repeating this procedure, we see that as long as the encoded measurements received by the controller are positive, the upper bounds on the norm of the estimation error $\|\hat{x} - x\|_\infty$ at the sampling times $k_0\tau, k_0\tau + \tau, k_0\tau + 2\tau, \ldots$ form a geometric progression with ratio $\Lambda/(N^{1/n})$. The goal of forcing the estimation error to approach 0 motivates our final assumption.

Assumption 4: We have

$$\Lambda < N^{1/n}.$$

In view of the definition of Λ via the formula (9), this inequality characterizes the tradeoff between the amount of information provided by the encoder at each sampling time and the required sampling frequency (see also Remark 2 in Section III). This relationship depends explicitly on the Lipschitz constant L which, as we will see, can be interpreted as a measure of expansiveness of (1).

III. Main Result

Theorem 1: Under Assumptions 1–4, the control law described in Section II globally asymptotically stabilizes the system (1).

Proof: We first show that $\|x(t)\|_\infty < D$ and $\|\hat{x}(t)\|_\infty < D$ for all $t \geq k_0\tau$, where D is defined by (8) and E_0 is defined by (3). Suppose that this is not true. Then, since x is continuous with $\|x(k_0\tau)\|_\infty \leq E_0 < D$ and $\hat{x}$ is continuous from the right with $\hat{x}(k_0\tau) = 0$, the time

$$\bar{t} := \min\{t > k_0\tau : \max\{\|x(\bar{t})\|_\infty, \|\hat{x}(\bar{t})\|_\infty\} \geq D\} \tag{12}$$

is well defined. We have $\|x(t)\|_\infty < D$ and $\|\hat{x}(t)\|_\infty < D$ for all $t \in [k_0\tau, \bar{t})$. Formulas (10) and (6) imply that (x, u) and $(\hat{x}, u)$ stay inside the region (7) on the interval $[k_0\tau, \bar{t})$. Let us label the estimation error as

$$e := \hat{x} - x. \tag{13}$$

We know that $\|e(k_0\tau)\|_\infty \leq E_0$. Combining the equation (valid between the sampling times)

$$\dot{e} = f(\hat{x}, u) - f(x, u)$$

with the formula

$$\|f(\hat{x}, u) - f(x, u)\|_\infty \leq L\|e\|_\infty$$

and applying the Bellman–Gronwall lemma, we conclude that for every interval $(t_1, t_2] \subset [k_0\tau, \bar{t})$ not containing any sampling times we have

$$\|e(t_2)\|_\infty \leq e^{L(t_2-t_1)}\|e(t_1)\|_\infty \leq \Lambda\|e(t_1)\|_\infty$$

where the last inequality follows from (9). This in turn guarantees that at each sampling time $k\tau \in [k_0\tau, \bar{t})$, we have $q_k(x(k\tau)) > 0$ and the upper bound on $\|e\|_\infty$ is divided by $N^{1/n}$. Invoking Assumption 4, we arrive at the bound $\|e(t)\|_\infty \leq \Lambda E_0$ for all $t \in [k_0\tau, \bar{t})$. If $\bar{t}$ is not a sampling time, then e is continuous at $\bar{t}$; if $\bar{t}$ is a sampling time, then e can only decrease at $\bar{t}$. In either case, we actually have

$$\|e(t)\|_\infty \leq \Lambda E_0 \qquad \forall\, t \in [k_0\tau, \bar{t}]. \tag{14}$$

Now, Assumption 2 expressed by (5) with $t_0 = k_0\tau$ implies that $\|x(t)\|_\infty \leq \beta(E_0, 0) + \gamma(\Lambda E_0) < D$ for all $t \in [k_0\tau, \bar{t}]$, where the last inequality follows from Assumption 4. Using (13) and (14), we also obtain $\|\hat{x}(t)\|_\infty \leq \beta(E_0, 0) + \gamma(\Lambda E_0) + \Lambda E_0 < D$ for all $t \in [k_0\tau, \bar{t}]$. This yields a contradiction with the definition (12) of $\bar{t}$.

We have thus established that all of the previous estimates are valid with $\bar{t} = \infty$. In particular, the upper bound on $\|e\|_\infty$ is divided by $N^{1/n}$ at the sampling times $k_0\tau + \tau, k_0\tau + 2\tau, \ldots$ and grows by the factor of Λ on every interval between these times. By Assumption 4, we have $e(t) \to 0$ as $t \to \infty$. The evolution of x is governed by (4), and in view of the ISS property of this system with respect to e we conclude that x converges to 0 as well.

It remains to prove that the origin is a stable equilibrium of the closed-loop system in the sense of Lyapunov. The fact that it is an equilibrium is clear from the conditions $f(0,0) = 0$ and $k(0) = 0$ and from Assumption 3 (the latter ensures that if $x \equiv 0$ then $\hat{x} \equiv 0$ because one of the boxes is always centered at the origin). Take an arbitrary $\varepsilon > 0$. We need to find a $\delta > 0$ such that the solutions of (4) with initial conditions in $B^n_\infty(0, \delta)$ remain in $B^n_\infty(0, \varepsilon)$ for all time. With reference to (5), choose two positive numbers ρ and η such that

$$\beta(\rho, 0) + \gamma(\eta) \leq \varepsilon. \tag{15}$$

Note that $\beta(r, 0) \geq r$ for all $r \geq 0$ (just apply (5) with $e \equiv 0$ and $t = t_0$), hence $\rho < \varepsilon$. Choose a sufficiently large integer $\bar{k} \geq 0$ such that

$$\frac{\Lambda^{\bar{k}+1}E_0}{(N^{1/n})^{\bar{k}}} \leq \eta. \tag{16}$$

In view of Assumption 1 and the fact that the origin is an equilibrium of the unforced system (2), there exists a class $\mathcal{K}_\infty$ function ν such that all solutions of (2) satisfy $\|\xi(x(0), t)\|_\infty \leq \nu(\|x(0)\|_\infty)$ for all $t \in [0, \bar{k}\tau]$. (Just take some function $\nu \in \mathcal{K}_\infty$ satisfying $\nu(r) \geq \max_{\|x(0)\|_\infty \leq r, t \in [0, \bar{k}\tau]} \|\xi(x(0), t)\|_\infty$ for all $r \geq 0$.) Choose a sufficiently small $\delta > 0$ such that

$$\nu(\delta) < \min\left\{\frac{\Lambda^{\bar{k}-1}}{(N^{1/n})^{\bar{k}-1}}, \rho\right\}.$$

This inequality and Assumption 4 ensure that if $\|x(0)\|_\infty \leq \delta$, then $k_0 = 0$, $\hat{x}(t) = 0$ for all $t \in [0, \bar{k}\tau)$, and $\|x(t)\|_\infty < \rho$ for all $t \in [0, \bar{k}\tau]$. Inequality (16) and our previous analysis imply that $\|e(t)\|_\infty \leq \eta$ for all $t \geq \bar{k}\tau$. By time-invariance of the system (4), inequality (5) remains valid with t_0 replaced by $\bar{k}\tau$. In view of (15), we conclude that $\|x(t)\|_\infty < \varepsilon$ for $t \geq \bar{k}\tau$, and the proof is complete. □

Remark 2: Instead of binary encoding functions, we could let q_k, $k \leq k_0$ be encoding functions taking $N + 1$ values, similar to the functions q_k, $k \geq k_0 + 1$ but defined for the boxes $B^n_\infty(0, \mu_k)$. While this would make the "zooming-out" stage slightly more complicated, it has the advantage that the upper bound on $\|e(k_0\tau)\|_\infty$ would then be divided by $N^{1/n}$. This would allow us to eliminate $N^{1/n}$ from the formula for D and obtain the bound $\|e(t)\|_\infty < E_0$ for all $t \geq k_0\tau$, thus rendering Λ independent of N and making the relationship expressed by Assumption 4 more transparent (although Λ would still implicitly depend on the initial state through E_0). Another way to achieve the same goal is to leave the functions q_k, $k \leq k_0$ as they are but observe that by virtue of Assumption 1, $x(k_0\tau + \tau)$ belongs to a bounded region whose size is determined by E_0 and which could be used to define D and to bound $\|e(t)\|_\infty$ for $t \geq k_0\tau + \tau$. In both cases, the functions q_k, $k \geq k_0 + 1$ would need to be redefined appropriately. Thus, for a fixed τ, our method can be applied for sufficiently large N, while it is clear from (9) that for a fixed N, Assumption 4 is satisfied for sufficiently small τ. □

Remark 3: In Theorem 1, we were only concerned with the behavior of the state x. Since the control law applied to the system for $t \geq k_0\tau$ is a dynamic one with the (discontinuous) state $\hat{x}$, it is more accurate to refer to stability properties of the resulting composite system. Global asymptotic stability of this overall system (appropriately defined to take into account the constraint $\hat{x}(k_0\tau) = 0$ imposed by the construction) can be easily deduced from the proof of Theorem 1. □

IV. Alternative ISS Assumption

The construction given earlier relies on ISS of the closed-loop system (4) with respect to the estimation error e (Assumption 2). In this section, we propose a different approach, which centers around the behavior of the estimated state $\hat{x}$ (and is inspired by the analysis of supervisory control algorithms for uncertain systems described in

[8]). In view of (10), (11), (13), and continuity of x, the evolution of $\hat{x}$ for $t \geq k_0\tau$ is described by the impulsive system

$$\begin{cases} \dot{\hat{x}} = f(\hat{x}, k(\hat{x})), & t \neq k_0\tau + k\tau, k = 1, 2, \dots \\ \hat{x}(t) = \hat{x}(t^-) + \Delta e(t), & t = k_0\tau + k\tau, k = 1, 2, \dots \end{cases} \tag{17}$$

where

$$\Delta e(t) := \begin{cases} e(t) - e(t^-), & t = k_0\tau + k\tau, k = 1, 2, \dots \\ 0, & \text{elsewhere.} \end{cases} \tag{18}$$

The initial condition for (17) is $\hat{x}(k_0\tau) = 0$. Assuming as before that the feedback law $k(\cdot)$ is locally Lipschitz with $k(0) = 0$, let us replace the ISS condition appearing in Assumption 2 by the following one.

Assumption 2′: There exist functions $\hat{\beta} \in \mathcal{KL}$ and $\hat{\gamma} \in \mathcal{K}_\infty$ such that for every $t_0 \geq k_0\tau$ and every input Δe the corresponding solution of (17) satisfies

$$\|\hat{x}(t)\|_\infty \leq \hat{\beta}(\|\hat{x}(t_0)\|_\infty, t - t_0) + \hat{\gamma}\left(\sup_{s\in[t_0,t]} \|\Delta e(s)\|_\infty\right) \qquad \forall\, t \geq t_0. \tag{19}$$

Using the bound $\sup_{s\in[t_0,t]} \|\Delta e(s)\|_\infty \leq 2\sup_{s\in[t_0,t]} \|e(s)\|_\infty$ and the relation $x = \hat{x} - e$, it is not difficult to verify that (19) implies (5) with

$$\beta(r, t) := \hat{\beta}(2r, t) \quad \gamma(r) := \hat{\gamma}(2r) + r + \hat{\beta}(2r, 0). \tag{20}$$

This leads to an alternative version of Theorem 1. Note that the stated implication relies on the fact that e in (5) and Δe in (19) are related via (18). In general, Assumption 2′ does not imply Assumption 2 for arbitrary e, as will become clear later.

We can obtain a similar result by proceeding from Assumption 2′ directly, as follows. Let L' be the Lipschitz constant for the function f on the region

$$\{(x, u) : \|x\|_\infty \leq D', \|u\|_\infty \leq \kappa(D')\}$$

where κ is given by (6) and

$$D' := \hat{\gamma}(2N^{1/n}E_0) + N^{1/n}E_0. \tag{21}$$

Now, redefine Λ to be

$$\Lambda := e^{L'\tau} \geq 1. \tag{22}$$

Note that this number is in general smaller than the one given by (9) using L obtained from (7) and (8) and β and γ defined by (20). By the same reasoning as in the proof of Theorem 1 (using Assumption 2′ instead of Assumption 2 and interchanging the roles of x and $\hat{x}$), we arrive at the following result.

Theorem 2: Under Assumptions 1, 2′, 3, and 4 with the new definition of Λ given by (22), the control law described in Section II globally asymptotically stabilizes the system (1).

Assumption 2′ expresses an ISS property for the impulsive system with inputs (17) which apparently has not been studied in the literature. However, employing available results for the corresponding discrete-time system, we can obtain the following sufficient condition for this property.

Lemma 1: Suppose that there exists a positive–definite radially unbounded C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\frac{\partial V}{\partial \hat{x}} f(\hat{x}, k(\hat{x})) \leq -V(\hat{x}) \qquad \forall\, \hat{x} \tag{23}$$

and for every $\varepsilon > 0$ there exists a continuous function $W_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(\hat{x} + d) - (\varepsilon + 1)V(\hat{x}) \leq W_\varepsilon(d) \qquad \forall\, \hat{x}, d. \tag{24}$$

Then, Assumption 2′ holds.

Proof (Sketch): We associate to the impulsive system (17) the discrete-time system

$$x_{k+1} = \varphi(x_k) + d_{k+1} \tag{25}$$

where $\varphi(\cdot)$ is the flow of the system

$$\dot{\hat{x}} = f(\hat{x}, k(\hat{x})) \tag{26}$$

for τ units of time and $d_i := \Delta e(k_0\tau + i\tau),\ i \geq 1$. Clearly, (23) guarantees that (26) is globally asymptotically stable. In view of this, we can show that the impulsive system (17) satisfies (19) if and only if the discrete-time system (25) is ISS in the sense of [12], which in the present context means

$$\|x_k\|_\infty \leq \bar{\beta}(\|x_0\|_\infty, k) + \bar{\gamma}\left(\sup_{1\leq i\leq k} \|d_i\|_\infty\right)$$

for some $\bar{\beta} \in \mathcal{KL}$ and $\bar{\gamma} \in \mathcal{K}_\infty$. From (25) and (24), we have

$$\begin{aligned} V(x_{k+1}) - V(x_k) &= V(\varphi(x_k) + d_{k+1}) - V(x_k) \\ &\leq (\varepsilon + 1)V(\varphi(x_k)) \\ &\quad - V(x_k) + W_\varepsilon(d_{k+1}). \end{aligned}$$

Combining this with (23) and the definition of φ, we obtain

$$V(x_{k+1}) - V(x_k) \leq ((\varepsilon + 1)e^{-\tau} - 1)V(x_k) + W_\varepsilon(d_{k+1}).$$

Since $(\varepsilon + 1)e^{-\tau} - 1 < 0$ for ε small enough, we can find functions $\alpha, \sigma \in \mathcal{K}_\infty$ such that

$$V(x_{k+1}) - V(x_k) \leq -\alpha(\|x_k\|_\infty) + \sigma(\|d_{k+1}\|_\infty).$$

According to the main result of [12], this implies that (25) is ISS. Alternatively, using the exponential decay of V along solutions of (26), it is not difficult to establish (19) directly by computing the desired functions $\hat{\beta}$ and $\hat{\gamma}$. □

It is known that a Lyapunov function V satisfying (23) can always be found provided that the system (26) is globally asymptotically stable (see, e.g., [22]). Thus, Lemma 1 says that Assumption 2′ holds if we can also satisfy the condition expressed by (24). It is straightforward to check that this additional condition holds if V is quadratic or, more generally, takes the form

$$V(\hat{x}) = \sum_{i=1}^{m} (\hat{x}^T P_i \hat{x})^i, \qquad P_i \geq 0, \quad i = 1, \dots, m$$

where m is a positive integer. It is not hard to show that both conditions of Lemma 1 are satisfied if the system (26) is globally exponentially stable and its right-hand side is globally Lipschitz. Indeed, [13, Th. 4.14] guarantees the existence of a function V satisfying $c_1|\hat{x}|^2 \leq V(\hat{x}) \leq c_2|\hat{x}|^2$, $\nabla V(\hat{x}) f(\hat{x}, k(\hat{x})) \leq -c_3|\hat{x}|^2$, and $|\nabla V(\hat{x})| \leq c_4|\hat{x}|$, where $c_i > 0, i = 1, 2, 3, 4$. The mean value theorem gives

$$\begin{aligned} V(\hat{x} + d) - (\varepsilon + 1)V(\hat{x}) &= V(\hat{x} + d) - V(\hat{x}) - \varepsilon V(\hat{x}) \\ &= \nabla V(z)d - \varepsilon V(\hat{x}) \end{aligned}$$

where z is a point on the line segment between $\hat{x}$ and $\hat{x} + d$. Therefore

$$|V(\hat{x} + d) - (\varepsilon + 1)V(\hat{x})| \leq c_4|\hat{x}||d| + c_4|d|^2 - \varepsilon c_1|\hat{x}|^2$$

which is bounded for each d uniformly over $\hat{x}$, and so

$$W_\varepsilon(r) := \sup_{\hat{x},d:|d|\leq r} \{V(\hat{x}+d) - (\varepsilon+1)V(\hat{x})\}$$

is well defined. In particular, we conclude that Assumption $2'$ holds if (26) is a linear asymptotically stable system (in this case we can also explicitly solve (17) and confirm this fact directly). There exist other conditions guaranteeing that the hypotheses of Lemma 1 are satisfied, which are reported in [9].

On the other hand, these conditions are not sufficient for Assumption 2 to hold. For example, let the right-hand side of (1) be $f(x,u) := -x - x^2 + xu$ and consider the feedback law $k(x) := x$. Then the system (26) is $\dot{\hat{x}} = -\hat{x}$ and so Assumption $2'$ holds in view of the previous remarks. On the other hand, the system (4) is $\dot{x} = -x + xe$ which is not ISS (just note that $e \equiv 2$ produces unbounded solutions), so Assumption 2 is not satisfied. In fact, constructive sufficient conditions for ISS with respect to measurement errors for general nonlinear systems are lacking. This makes Assumption $2'$ more attractive because it is easier to test. We note, however, that Assumption 2 does not seem to imply Assumption $2'$.

V. Concluding Remarks

We studied the problem of stabilizing a nonlinear system using sampled encoded measurements of its state. The result and the method of proof presented here are extensions of those described in [16] for the case of linear systems. Our sufficient condition for global asymptotic stabilizability involves a relationship between the number of values taken by the encoder and the sampling frequency, and relies on input-to-state stabilizability with respect to measurement errors. The stabilizing control law that we constructed takes the form of a "certainty equivalence" discontinuous dynamic feedback. Similar techniques can be used for the case when *control signals* rather than state measurements are encoded (so that measurement errors are replaced by *actuator* errors), and also in situations where the sampling interval and the number of values taken by the encoder may vary with time (as long as a suitable relationship between them is satisfied "on the average").

The limited information feedback control strategy proposed in this note is not intended for practical implementation. Our main goal was to identify the difficulties associated with extending relevant existing results from linear to nonlinear systems, as well as possible tools that can be used to overcome them. Compared with the analysis given in [16], the key complications in the nonlinear context are that the propagation of the estimation error is not as straightforward to characterize and that a suitable form of robustness with respect to this error is needed from the controller. Our control scheme is not claimed to be optimal in any sense. Known results for linear systems suggest that to obtain *necessary and sufficient* conditions for nonlinear stabilizability, additional structure must be imposed on the system. For some classes of systems (e.g., piecewise linear systems) it may be possible to characterize the propagation of the estimation error more succinctly, thus reducing the conservatism of the condition expressed by Assumption 4.

An alternative analysis method presented in Section IV provides additional insight into the issue of robustness required from the controller. It involves an input-to-state stability property for impulsive systems, which has independent interest and is under current investigation [9]. We also mention two related results recently obtained by C. De Persis on the basis of the method presented in this note. First, exploiting the exponential decay of $e(t)$ and the results of [2], one can replace ISS in Assumption 2 by the weaker integral-ISS property plus an additional technical condition on the corresponding nonlinear gain [19]. Second, if one assumes only that the control law globally asymptotically stabilizes the system in the absence of measurement errors, then asymptotic stabilization with encoded state feedback can still be achieved at the expense of increasing the sampling rate according to the size of the initial condition [20].

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Adaptive Control of a Class of Slowly Time Varying Systems With Modeling Uncertainties

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***Abstract*—In a recent work, a new linear adaptive controller based on certainty-equivalence and backstepping design, which promises a level of transient and asymptotic performance comparable to that of the tuning functions adaptive backstepping controller without using high order nonlinearities, was proposed for linear time invariant systems. The proposal was supplemented with robustness and performance analysis in the presence of modeling uncertainties. In this note, the same idea is used to develop a new linear adaptive controller for slowly time varying systems with modeling uncertainties. The new adaptive control scheme guarantees robustness with respect to modeling errors via normalizing damping, parameter projection, and static normalization. Use of normalizing damping is essential in protecting the "linearity" of the system, which plays a key role in reaching the stability and robustness results.**

***Index Terms*—Backstepping, robust adaptive control, time-varying systems.**

I. Introduction

In a recent work, a new "linear" adaptive controller which combines beneficial features of the two dominant approaches of adaptive control, the certainty equivalence approach [1], [2] and the backstepping approach [3] was proposed [4], [5] for linear-time invariant (LTI) plants. The new design promises a high level of transient and asymptotic performance comparable to that of the tuning functions adaptive backstepping controller [3] with a simple "linear" certainty equivalence adaptive structure that avoids high order nonlinearities that may lead to poor robustness and localization of stability in the presence of high frequency unmodeled dynamics [6], [7]. The proposed design is supplemented with robustness and performance analysis in the presence of modeling uncertainties. In this note, the same idea is used to develop a new linear adaptive controller for linear time varying (LTV) systems. As in the LTI case, the "linearity" of the controller, parameter projection, and static normalization are the main features that are used to provide stability and robustness beside computable performance bounds provided by the backstepping algorithm.

Considering the works on adaptive control of slowly time varying plants in the literature, the main contribution of our work is providing computable bounds for systematic improvement of the transient performance with or without adaptation of the parameter estimates while keeping the system stable and robust against modeling errors. Most of the previous works on adaptive control of LTV plants are based on the certainty equivalence approach that combines a controller structure with a robust adaptive law [8]–[10]. These controllers can not guarantee good transient behavior in general [11]. Our controller, on the other hand, provides improved transient performance beside guaranteeing stability and robustness. Another major difference of our work from most of the previous ones is direct involvement of unmodeled dynamics and external disturbances in the problem definition. Hence, our controller is designed directly for a class of LTV plants with modeling errors.

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II. Notation

The following notation is used throughout the note, unless otherwise stated. We denote $|\cdot|$ as the Euclidean norm for vectors, and $\|\cdot\|$ as matrix Frobenius norm. Unless specifically declared otherwise, $\hat{x}$ denotes the estimate of x, and $\tilde{x} = x - \hat{x}$ denotes the estimation error. We denote I_n as an $n \times n$ identity matrix, $0_{m\times n}$ as an $m \times n$ zero matrix, 0_n as the $n \times 1$ zero matrix. For a square matrix A, the minimum and maximum eigenvalues of A are denoted by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively.

$\bar{S}(\mu)$ denotes the set $\{x\colon [0,\infty) \rightarrow \Re^n \,|\, x \in \mathcal{L}_{2e}, \int_t^{t+T} x^T(\tau)x(\tau)d\tau \leq \mu T + c_1, \forall t, T \geq 0\}$ for a given constant $\mu \geq 0$, where $c_1 \geq 0$ is a finite constant. The set $\mathcal{S}(\mu)$ for a given constant $\mu \geq 0$, the norms/measures $\|x\|_\infty, \|x\|_p, p \geq 1, \|(x)_t\|_{2\delta}$, $\mathrm{MSE}(x)$ for a function $x : [0,\infty) \rightarrow \Re$ are defined as in [1]. For an exponentially stable linear time varying differential operator $H(s,t)$ [8], if $H(s-(\delta/2),t)$ is exponentially stable as well, i.e., if the corresponding impulse response(transition matrix) $h(t,\tau)$ satisfies $\|h(t,\tau)\| \leq \kappa_h e^{-\gamma_h(t-\tau)}, \forall t,\tau$ for some positive constants κ_h, γ_h, we define the equation shown at the bottom of the next page. For $\delta = 0$, we simply use $\|(H)_t\|_2$ and $\|(H)_t\|_\infty$ instead of $\|(H)_t\|_{2\delta}$ and $\|(H)_t\|_{\infty\delta}$.

III. Plant Model and Parameterization

The plant under consideration is the following single-input–single-output (SISO) LTV system:

$$\dot{x}(t) = \begin{bmatrix} a(t) & \begin{matrix} I_{n-1} \\ 0_{n-1}^T \end{matrix} \end{bmatrix} x(t) + b_{p\mu}(t)c_{p\mu}^T(t)x_\mu(t) + b(t)u(t) \tag{1}$$

$$\dot{x}_\mu(t) = A_\mu(t)x_\mu(t) + b_\mu(t)u(t) \tag{2}$$

$$y(t) = x_1(t) + c_\mu^T(t)x_\mu(t) + d(t) \tag{3}$$

where $y \in \Re$ is the plant output; $u \in \Re$ is the control input; $x = [x_1,\ldots,x_n]^T, x_\mu = [x_{\mu 1},\ldots,x_{\mu n_\mu}]^T$ are the states of the modeled part of the plant and unmodeled plant dynamics, respectively; $a(t) = [a_{n-1}(t),\ldots,a_0(t)]^T \in \Re^n, b(t) = [0_{\rho-1}^T, b_m(t),\ldots,b_0(t)]^T \in \Re^n$ are unknown plant parameter vectors; $\rho = n-m > 0$ is the relative degree of the nominal plant; $A_\mu(t) \in \Re^{n_\mu\times n_\mu}, b_\mu(t) \in \Re^{n_\mu}, c_\mu(t) \in \Re^{n_\mu}, b_{p\mu}(t) \in \Re^n, c_{p\mu}(t) \in \Re^{n_\mu}$ are unknown parameter matrices for the unmodeled dynamics; $d \in \Re$ is the output disturbance.