ACCURATE ITERATIVE ANALYSIS OF THE K-V EQUATIONS

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Those working with alternating-gradient (A-G) systems look for simple, accurate ways to analyze A-G performance for matched beams. The useful K-V equations [1] are easily solved in the smooth approximation [2], [3], [4]. This approximate solution becomes quite inaccurate for applications with large focusing fields and phase advances. Results of efforts to improve the accuracy [5], [6] have tended to be indirect or complex. Our generalizations presented previously [7] gave better accuracy in a simple explicit format. However, the method used to derive our results (expansion in powers of a small parameter) was complex and hard to follow; also, reference [7] only gave low-order correction formulas.

The present paper uses a straightforward iteration method and obtains equations of higher order than shown in our previous paper.

The K-V equations for the envelopes a(z) and b(z) are

$$a(z)'' = -K(z)\,a + \frac{\epsilon^2}{a^3} + \frac{2Q}{a+b} \qquad (1)$$

$$b(z)'' = +K(z)\,b + \frac{\epsilon^2}{b^3} + \frac{2Q}{a+b} \qquad (2)$$

with input parameters: normalized beam current Q; emittance ϵ; and A-G focus function K(z). The z origin is located at the midpoint of a quadrupole and K(z) is assumed here to be symmetric about z=0, periodic over a cell length 2L, and antisymmetric about L/2. Thus

$$K(z-2L)=K(z),\quad K(-z)=K(z),\quad K(z-L)=-K(z). \qquad (3)$$

We solve for the x and y beam envelopes a(z) and b(z), assumed to be matched to the lattice, i.e., periodic over 2L. To aid the solution of Eqs. (1) and (2), we define in Eqs. (4)-(19) the operators on even periodic functions $\langle\ldots\rangle$, $\{\ldots\}$, $\int$ and $\iint$; the even periodic functions h(z), g(z), δ(z) and ρ(z); and the constants k, α, β, q, A, K_{eff}, Φ, and ρ_m. In Eq. (19), h_1 is the first Fourier coefficient of h(z).

The operator $\langle\ldots\rangle$ performs an average over a cell length 2L while the operator $\{\ldots\}$ removes the average part of a periodic function: e.g., $2\{\cos^2 x\}=\{1+\cos 2x\}=\cos 2x$. The operator $\iint$ operates on periodic functions that have no average. It gives the repeated indefinite integral and removes the average part, if any, of the result.

Table 1: Definitions to be used in this paper

$\langle f\rangle \equiv (1/2L)\int_0^{2L} f(z)dz,$	(4)	$\delta(z) \equiv \iint\{hg\},$	(11)
$\{f\} \equiv f-\langle f\rangle .$	(5)	$A \equiv \langle a(z)\rangle,$	(12)
		$\rho(z) \equiv (a(z)-A)/A,$	(13)
For even $\psi(z) \ni \langle\psi\rangle = 0$:		$\rho_b(z) \equiv (b(z)-A)/A,$	(14)
$\int\psi \equiv \int_0^z \psi(z')dz'$ and	(6)	$\alpha \equiv \frac{3\epsilon^2}{A^4},\quad \beta \equiv \alpha\frac{L^2}{\pi^2},$	(15)
$\iint\psi \equiv \left\{\int_0^z dz'\int_0^{z'}\psi(z'')dz''\right\}.$	(7)	$q \equiv Q/A^2,$	(16)
$k \equiv K^{max},$	(8)	$K^{eff} \equiv k^2\langle[\int h]^2\rangle,$	(17)
$h(z) \equiv K(z)/k,$	(9)	$\Phi \equiv 3k^2\langle g^2\rangle,$	(18)
$g \equiv \iint h,$	(10)	$\rho_m \equiv h_1 k L^2/\pi^2.$	(19)

DECOUPLING AND DECOMPOSING

With the quadrupole symmetries of Eq. (3), our matched beam assumption implies b(z)=a(z+L), so that Eqs. (1) and (2) are decoupled. We have $\langle a\rangle = \langle b\rangle \equiv A$, and

$$a = A(1+\rho),\quad b = A(1+\rho_b). \qquad (20)$$

The Q terms in Eqs. (1) and (2) can be expanded as

$$\frac{2Q}{a+b} = \frac{Q}{A}\left(1-(\rho+\rho_b)/2+\ldots\right) = \frac{Q}{A}\left(1-k^2\delta(z)+\ldots\right), \qquad (21)$$

since [8]

$$(\rho+\rho_b)/2 = k^2\delta(z)+\ldots \qquad (22)$$

with δ(z) [Eq. (11)] derived from the lattice waveform h(z).

This decouples Eqs. (1) and (2). After the decoupled version of Eq. (1) is solved for a(z), then b(z) is found by symmetry. Equation (2) is no longer needed.

Substituting a = A(1+ρ) in the first three terms of Eq. (1), expanding $1/a^3$, dividing by A, and using (21) and (15), the first K-V equation is equivalent to

$$\rho(z)'' = -kh(z) - kh(z)\rho + \frac{\alpha}{3}\left(1-3\rho+6\rho^2-10\rho^3+15\rho^4\ldots\right) + q\left(1-k^2\delta(z)\ldots\right). \qquad (23)$$

To solve for the ripple ρ(z) and the mean radius A (which appears in the definitions of α and q), we decompose Eq. (23) into a pair of equations. Averaging Eq. (23),

$$0 = -k\langle h\rho\rangle + \frac{\alpha}{3} + 2\alpha\langle\rho^2\rangle - \frac{10}{3}\alpha\langle\rho^3\rangle + 5\alpha\langle\rho^4\rangle\ldots + q. \qquad (24)$$

Subtracting Eq. (24) from (23),

$$\rho'' = -kh(z) - k\{h\rho\} - \alpha\rho + 2\alpha\{\rho^2\} - \frac{10}{3}\alpha\{\rho^3\} + 5\alpha\{\rho^4\}\ldots - qk^2\delta(z)\ldots, \qquad (25)$$

with {..} from Eq. (5). There are now two equations, each containing A and ρ(z). Because of our periodicity constraint these have the essence of the K-V equations (1) and (2).

ITERATIVE SOLUTION OF K-V EQUATIONS

On the right of Eq. (25), the kh(z) term dominates the terms involving the unknown function ρ(z). They are omitted for the initial integrations, which give $\rho_{(0)}$. Then we insert $\rho_{(0)}$ into (25) and integrate again to get $\rho_{(1)}$. The process is repeated for $\rho_{(2)}$. The resulting terms of greatest significance are:

$$\rho_{(0)} = -kg, \qquad (26a)$$

$$\rho_{(1)} = \rho_{(0)} + \alpha k\iint g + k^2\delta + \frac{10}{3}\alpha k^3\iint g^3, \qquad (26b)$$

$$\rho_{(2)} = \rho_{(1)} - \alpha^2 k\iiiint g - k^3\iint h\delta - 2\alpha k^3\iint g\delta. \qquad (26c)$$

To complete the approximate solution of the K-V equations, ρ(z) from Eq. (26) is put in the matching equation (24). From

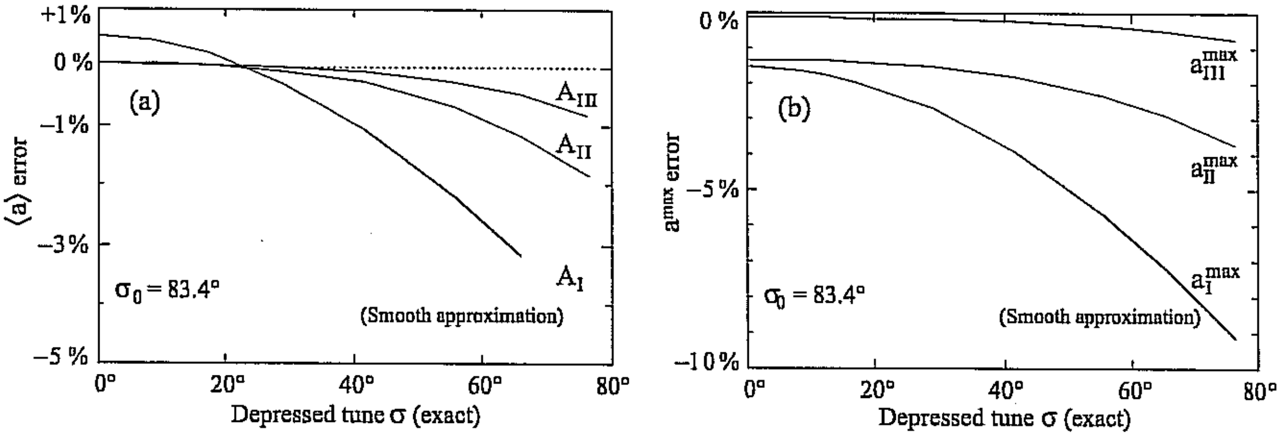

Fig. 1. Accuracy of: (a) mean radius A from Eqs. (30), (32), or (33) and (b) maximum radius a^{max} from Eqs. (36), (37), or (38). Input quantities are Q, ϵ, and quadrupole voltage V_Q (proportional to K). Quadrupole dimensions, etc., are in Ref. [8]. V_Q, fixed at 20 kV, gives phase advance $\sigma_0 = 83.37°$; ϵ, Q are varied so that depressed tune σ ranges between 0° and 76.5°; exact σ_0 and σ are obtained numerically.

Eq. (26) we discarded items, such as $2\alpha k^2 \iint g^2$, that would give terms in (24) higher than third power in the parameters k^2, α, and q. A miniscule term, $qk^2\iint\delta(z)$, in $\rho_{(0)}$ is also omitted.

The order of a term in the matching equation is reckoned by counting the number of factors k^2, α, and q. (These would become small parameters in a non-dimensional formalism [8]. Here, we prefer to retain physical units for length z, etc.)

Third Order: Inserting Eq. (26) into Eq. (24) yields seven terms [8] through third order. Some terms combine, with result

$$K_\dagger^{eff} - \frac{\epsilon_{III}^2}{A_{III}^4} - \frac{Q}{A_{III}^2} = 0, \quad (27)$$

where

$$K_\dagger^{eff} \equiv \langle [\int K(z)]^2 \rangle \left[1 + \frac{1}{24}\Phi\left(1 + \frac{20}{27}c_3\right)\right]; \quad (28)$$

$$\epsilon_{III}^2 \equiv \epsilon^2\left[1 + \Phi\left(1 + \tfrac{1}{2}\Phi + 3\beta_I\right)\right]. \quad (29)$$

Here c_3 is of order unity [8]. Roman-numeral subscripts on A and ϵ signify the order of approximation—third order in this case. The subscript on $\beta \sim A^{-4}$ indicates that A_I [Eq. (33)] is used to approximate A. The matching equation (27) is in the standard form of the smooth approximation, Eq. (33), and can be solved to find the third-order A:

$$A_{III}^2 = (Q/2K_\dagger^{eff}) + \left[(Q/2K_\dagger^{eff})^2 + \epsilon_{III}^2/K_\dagger^{eff}\right]^{1/2}. \quad (30)$$

If the input quantity is the mean radius A_{inp}, then Eq. (27) gives the allowable Q to third order,

$$Q_{III} = A_{inp}^2 K_\dagger^{eff} - \epsilon_{III}^2/A_{inp}^2.$$

Second Order: There are two second-order terms. One yields the correction to K^{eff} seen in Eq. (28). The other term is $\alpha k^2\langle g^2\rangle$, or, using definition (18), $\frac{\alpha}{3}\Phi$. We define

$$\epsilon_{II}^2 \equiv \epsilon^2(1+\Phi), \quad (31)$$

and get

$$K_\dagger^{eff} - \frac{\epsilon_{II}^2}{A_{II}^4} - \frac{Q}{A_{II}^2} = 0. \quad (32)$$

Eq. (32) can be solved for A_{II} or Q_{II} in the same way as for the third order, giving useful approximations when K(z) and ϵ produce σ_0 and σ less than about 80°.

First Order: The three terms of lowest order produce what is called the first-order matching equation in this paper (Ref. [7] used another terminology). This is the classic smooth approximation. These terms give $k^2\langle[\int h]^2\rangle = \alpha/3 + q$, or, using the definitions (15), (16), and (17)

$$K^{eff} - \frac{\epsilon^2}{A_I^4} - \frac{Q}{A_I^2} = 0. \quad (33)$$

First, second, and third-order results for A, from (33), (32) and (30), are plotted in Fig. 1a. The smooth approximation is relatively inaccurate except near the point where its error curve crosses the 0 % line.

MAXIMUM RADIUS

Knowing the matched mean radius A, one can complete the solution for the envelope $a(z) = A[1+\rho(z)]$ using ρ(z) from Eq. (26); b(z) can be found by changing the sign of the terms that contain odd powers of k.

Some terms of Eq. (26) can be written in exact form [8] for models such as FODO, but Fourier expansion is more useful in general:

$$h(z) = h_1\left[\cos\frac{\pi z}{L} + \tfrac{1}{3}c_3\cos 3\frac{\pi z}{L} + \tfrac{1}{5}c_5\cos 5\frac{\pi z}{L} \ldots\right]. \quad (34)$$

Values (usually of order unity) of h_1 and c_n for both FODO and smooth profiles are given in Ref. [8]. With the definition

$$\beta_I \equiv 3\frac{L^2}{\pi^2}\frac{\epsilon^2}{A_I^4} \quad (35)$$

we have

$$a_{III}^{max} = A_{III}\left[1 + \rho_m\left(1 + \tfrac{1}{27}c_3 + \tfrac{1}{125}c_5\right) + \tfrac{1}{8}\rho_m^2\left(1 + \tfrac{25}{54}c_3\right) + \beta_I\rho_m\left(1 + \tfrac{5}{2}\rho_m^2 + \beta_I\right)\right] \quad (36)$$

using results from Ref [8]. The accuracy of Eq. (36) is shown in Fig. 1b, along with that of the truncations

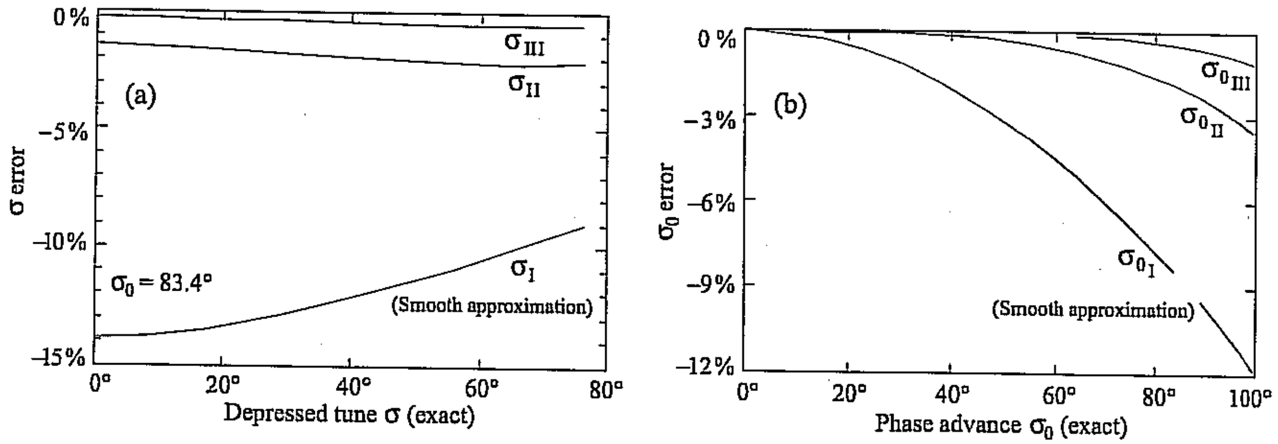

Fig. 2. (a) Accuracy of depressed tune σ from Eqs. (40), (41), and (42). V_Q is fixed at 20 kV as in Fig. 1.
(b) Accuracy of phase advance σ_0 from Eqs. (43), (44), and (45). V_Q ranges from 5 kV to about 22 kV.

$$a_{II}^{max} = A_{II}\left[1+\rho_m\left(1+\tfrac{1}{27}c_3+\tfrac{1}{125}c_5\right)+\beta_I\rho_m\right] \qquad (37)$$

and (the smooth approximation)

$$a_I^{max} = A_I\left[1+\rho_m\right]. \qquad (38)$$

PHASE ADVANCES

From the well-known phase-amplitude result [9], the phase advance per quadrupole cell of length 2L is

$$\sigma = \in\int_0^{2L}\frac{dz}{a^2} = 2L\in\langle a^{-2}\rangle.$$

We approximate a(z) by $A_{III}[1+\rho(z)]$ with A_{III} from Eq. (30) and $\rho(z)$ to third order from Eq. (26). Subscripts are omitted for brevity. Expanding a^{-2} and taking the average gives

$$\sigma = 2L\frac{\in}{A_{III}^2}\left[1+3\langle\rho^2\rangle-4\langle\rho^3\rangle+5\langle\rho^4\rangle-\cdots\right]. \qquad (39)$$

(The 2ρ term has zero average by definition.) Ref. [8] shows that to third-order accuracy

$$\sigma_{III} = 2L\frac{\in}{A_{III}^2}\left[1+\Phi\left(1+\tfrac{3}{4}\Phi+2\beta_I\right)\right]. \qquad (40)$$

Errors with respect to exact values from simulations are shown in Fig. 2a. Useful accuracy is retained after dropping two terms and using A_{II} from Eq. (32):

$$\sigma_{II} = 2L\frac{\in}{A_{II}^2}(1+\Phi). \qquad (41)$$

Figure 2a shows large errors for the first-order result (smooth approximation):

$$\sigma_I = 2L\frac{\in}{A_I^2}. \qquad (42)$$

The undepressed σ_0 is found by setting Q = 0 in Eq. (27), then eliminating $\in$ from Eq. (40). Details are in Ref. [8]. The result is

$$\sigma_{0_{III}} = 2L\left(K_\dagger^{eff}\right)^{1/2}\left[1+\tfrac{1}{2}\Phi+\tfrac{7}{8}\Phi^2\right]. \qquad (43)$$

This equation is used to calculate σ_0 as a function of the strength of the quadrupole field gradient. Figure 2b shows its accuracy and also illustrates the second-order case

$$\sigma_{0_{II}} = 2L\left(K_\dagger^{eff}\right)^{1/2}\left[1+\tfrac{1}{2}\Phi\right] \qquad (44)$$

and the smooth approximation,

$$\sigma_{0_I} = 2L\left(K^{eff}\right)^{1/2}. \qquad (45)$$

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