## Title

Horospherical flows in infinite volume rank one homogeneous spaces: effective equidistribution and applications



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# Horospherical flows in infinite volume rank one homogeneous spaces: effective equidistribution and applications

A dissertation submitted in partial satisfaction of the
requirements for the degree
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in

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by

Jacqueline Warren

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The dissertation of Jacqueline Warren is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

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2021

DEDICATION

To my parents.

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ABSTRACT OF THE DISSERTATION

**Horospherical flows in infinite volume rank one homogeneous spaces: effective equidistribution and applications**

by

Jacqueline Warren

Doctor of Philosophy in Mathematics

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Professor Amir Mohammadi, Chair

We prove effective equidistribution of horospherical flows in $\mathrm{SO}(n,1)^{\circ}/\Gamma$ when $\Gamma$ is geometrically finite and the frame flow is exponentially mixing for the Bowen-Margulis-Sullivan measure. We also discuss settings in which such an exponential mixing result is known to hold.

As a significant part of the proof, we establish quantitative nondivergence of horospherical orbits, and show that the Patterson-Sullivan measure satisfies certain friendly-

like properties when $\Gamma$ is geometrically finite. We also prove that a much stronger result, called global friendliness, if all cusps are assumed to be of maximal rank. The proof strategy of the equidistribution theorems combines these with the "banana trick" of Margulis.

As an application, we study the distribution of non-discrete orbits of geometrically finite groups in $\mathrm{SO}(n,1)$ acting on the quotient of $\mathrm{SO}(n,1)$ by a horospherical subgroup. In particular, this can be identified with $\Gamma$ acting on the "light cone" in $\mathbb{R}^{n+1}$, or on certain wedge products. We obtain asymptotics for the distribution of orbits of geometrically finite groups when all cusps have maximal rank. When we also have that the Bowen-Margulis-Sullivan measure is exponentially mixing, we obtain a quantitative ratio theorem, using global friendliness of the PS measure.

# Chapter 1

# Introduction

The group $G = \mathrm{SO}(n,1)^\circ$ with $n \geq 2$, can be considered as the group of orientation preserving isometries of the hyperbolic space $\mathbb{H}^n$. Let $\Gamma \subseteq G$ be a geometrically finite and Zariski dense subgroup of $G$ with infinite covolume, which may be thought of as having a finite sided fundamental domain. In particular, when $\Gamma$ is geometrically finite, $\mathbb{H}^n/\Gamma$ has only finitely many cusps.

In this thesis, we establish an effective rate of equidistribution of orbits under the action of a horospherical subgroup $U \subseteq G$ under a certain exponential mixing assumption (Assumption 1.1.2). As an application, we will study the distribution of the orbits of $\Gamma$ acting on $U\backslash G$, which will be identified with the "light cone" in $\mathbb{R}^{n+1}$.

## 1.1 Effective Equidistribution

An early result on the equidistribution of horocyclic flows in $G/\Gamma$ for $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma$ a lattice was obtained by Dani and Smillie in [DS84]. They proved that if $U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$ and if $x$ does not have a closed $U$-orbit in $G/\Gamma$, then for every $f \in C_c(X)$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(u_t x) dt = m(f), \tag{1.1}$$

where $m$ denotes the normalized Haar probability measure on $X$. The lattice case is well-understood in general, thanks to Ratner's celebrated theorems on unipotent flows, [Rat91].

Results such as these are not considered to be *effective*, because they do not address the rate of convergence, and this is important in many applications. Burger proved effective equidistribution of horocyclic flows for $\mathrm{SL}_2(\mathbb{R})/\Gamma$ when $\Gamma$ is a uniform lattice or convex cocompact with critical exponent at least $1/2$ in [Bur90]. Sarnak proved effective equidistribution of translates of closed horocycles when $\Gamma$ is a non-uniform lattice in [Sar81]. More general results were obtained for non-uniform lattices using representation theoretic methods by Flaminio and Forni in [FF03], and also by Strömbergsson in [Str13]. The case where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ was also obtained independently by Sarnak and Ubis in [SU15]. The higher dimensional setting has recently been considered by Katz [Kat19] and McAdam [McA19]. McAdam proved equidistribution of abelian horospherical flows in $\mathrm{SL}_n(\mathbb{R})/\Gamma$ for $n \geq 3$ when $\Gamma$ is a cocompact lattice or $\mathrm{SL}_n(\mathbb{Z})$, and Katz proved equidistribution in greater generality when $\Gamma$ is a lattice in a semi-simple linear group without compact factors.

In infinite volume, we cannot hope for a result such as equation (1.1) for the Haar measure: by the Hopf ratio ergodic theorem, for almost every point,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(u_t x)dt = 0.$$

This tells us that this is not the correct measure to consider. A key characteristic of the Haar probability measure in the lattice case is that it is the unique $U$-invariant ergodic Radon measure that is not supported on a closed $U$ orbit, [DS84, Fur73]. By [Bur90, Rob03, Win15], the measure with this property in the infinite volume setting is the *Burger-Roblin* (BR) measure, which is defined fully in Chapter 3. The correct normalization will be given by the *Patterson-Sullivan* (PS) measure, which is a geometrically defined measure on $U$ orbits. This is also defined in Chapter 3.

Maucourant and Schapira proved equidistribution of horocycle flows on geometrically finite quotients of $\mathrm{SL}_2(\mathbb{R})$ in [MS14], and in [MO16], Mohammadi and Oh generalize these results to geometrically finite quotients of $\mathrm{SO}(n,1)^\circ$ for $n \geq 2$. These results are not effective, but will be useful for our applications to understanding the distribution of orbits of a geometrically finite group. Oh and Shah also proved equidistribution on the unit tangent bundle of geometrically finite hyperbolic manifolds in [OS13]. In [Edw19], Edwards proves effective results for geometrically finite quotients of $\mathrm{SL}_2(\mathbb{R})$.

We will use the result of Mohammadi and Oh, but we require some notation in order to state this result.

Let $U = \{u_{\mathbf{t}} : \mathbf{t} \in \mathbb{R}^{n-1}\}$ denote the expanding horospherical flow, which is parametrized in Chapter 3. Let $B_U(r)$ denote the ball in $U$ of radius $r$ with the max norm on $\mathbb{R}^{n-1}$. We denote by $\Lambda(\Gamma)$ the set of limit points of $\Gamma$, and $\Lambda_r(\Gamma)$ denotes the set of *radial* limit points, defined fully in Chapter 3. The notation $x^{\pm}$ is also defined in that chapter. Here, $m^{\mathrm{BR}}$ denotes the BR measure and $\mu^{\mathrm{PS}}$ denotes the PS measure. These measures are also defined in Chapter 3.

Specifically, Mohammadi and Oh proved the following:

**Theorem 1.1.1.** *[MO16, Theorem 4.6] Let $\Gamma$ be geometrically finite. Fix $x \in G/\Gamma$ such that $x^- \in \Lambda_r(\Gamma)$. Then for any $\psi \in C_c(G/\Gamma)$, we have that*

$$\lim_{T\to\infty} \frac{1}{\mu_x^{\mathrm{PS}}(B_U(T))} \int_{B_U(T)} \psi(ux)du = m^{\mathrm{BR}}(\psi).$$

We will extend these results to geometrically finite quotients of $\mathrm{SO}(n,1)^{\circ}$, under the assumption of exponential mixing of the frame flow for the *Bowen-Margulis-Sullivan* (BMS) measure, which is defined in Chapter 3. More explicitly, in Chapter 6 (but not 4 or 5), we will assume the following holds, where $\{a_s : s \in \mathbb{R}\}$ denotes the frame flow on $G/\Gamma$:

**Assumption 1.1.2** (Exponential Mixing). *There exist $c, \kappa > 0$ and $\ell \in \mathbb{N}$ which depend only on $\Gamma$, such that for $\psi, \varphi \in C_c^{\infty}(G/\Gamma)$ and $s > 0$,*

$$\left| \int_X \psi\left(a_s x\right) \varphi\left(x\right) dm^{\mathrm{BMS}}\left(x\right) - m^{\mathrm{BMS}}\left(\psi\right) m^{\mathrm{BMS}}\left(\varphi\right) \right| < cS_\ell(\psi)S_\ell(\varphi)e^{-\kappa s}.$$

Assumption 1.1.2 is known to hold when $\Gamma$ is convex cocompact by [SW20], which covers the case when $G/\Gamma$ has no cusps. In [MO15], Mohammadi and Oh prove such a result for geometrically finite $\Gamma$ under a spectral gap assumption (see Definition 2.0.1), using decay of matrix coefficients. Edwards and Oh recently proved effective mixing for the geodesic flow on the unit tangent bundle of a geometrically finite hyperbolic manifold when the critical exponent is larger than $(n-1)/2$ in [EO19]. Further details about this assumption are discussed in Chapter 2.

We will need to restrict consideration to points satisfying the following geometric property, which means that the point does not travel into a cusp "too fast". Here, $d$ is a left-invariant Riemannian metric on $G/\Gamma$ that projects to the hyperbolic distance on $\mathbb{H}^n$.

**Definition 1.1.3.** For $0 < \varepsilon < 1$ and $s_0 \geq 1$, we say that $x \in G/\Gamma$ with $x^- \in \Lambda(\Gamma)$ is $(\varepsilon, s_0)$**-Diophantine** if for all $s \geq s_0$,

$$d(\mathcal{C}_0, a_{-s}x) < (1-\varepsilon)s,$$

where $\mathcal{C}_0$ is a compact set arising from the thick-thin decomposition, and is fully defined in §3.2. We say that $x \in G/\Gamma$ with $x^- \in \Lambda(\Gamma)$ is $\varepsilon$-**Diophantine** if $x$ is $(\varepsilon, s_0)$-Diophantine for some $s_0$, and simply **Diophantine** if it is $(\varepsilon, s_0)$-Diophantine for some $\varepsilon$ and $s_0$. ◀

*Remark.* A point $x \in G/\Gamma$ is Diophantine if and only if $x^- \in \Lambda_r(\Gamma)$, because Definition 1.1.3 precisely says that $x^- \notin \Lambda_{bp}(\Gamma)$, by the construction of the thick-thin decomposition. Here, $\Lambda_{bp}(\Gamma)$ denotes the *bounded parabolic* limit points. These are defined fully in Chapter 3. ◀

In the case that $\Gamma$ is a lattice, the condition $x^- \in \Lambda(\Gamma)$ is always satisfied. Also, if $\Gamma$ is convex cocompact, every point $x \in G/\Gamma$ with $x^- \in \Lambda(\Gamma)$ will be Diophantine, because all limit points are radial in this case (see Chapter 3).

Note that $x$ is $(\varepsilon, s_0)$-Diophantine if $(1-\varepsilon)s$ is a bound on the asymptotic excursion rate of the geodesic $\{a_{-s}x\}$, i.e.

$$\limsup_{s\to\infty} \frac{d(\mathcal{C}_0, a_{-s}x)}{s} \leq 1-\varepsilon. \tag{1.2}$$

Sullivan's logarithm law for geodesics when $\Gamma$ is geometrically finite with $\delta_\Gamma > (n-1)/2$ was shown in [KO21, SV95] (and is a strengthening of Sullivan's logarithm law for non-compact lattices ([Sul82, §9])), and implies that for almost all $x \in G/\Gamma$,

$$\limsup_{s\to\infty} \frac{d(\mathcal{C}_0, a_{-s}x)}{\log s} = \frac{1}{2\delta_\Gamma - k}, \tag{1.3}$$

where $k$ is the maximal cusp rank. In [KO21], Kelmer and Oh showed a strengthening of the above, considering excursion to individual cusps and obtaining a limit for the shrinking target problem of the geodesic flow. Note also that the result stated in [KO21] is for $x \in \mathrm{T}^1(G/\Gamma)$, but since the distance function there is assumed to be $K$-invariant, where $\mathbb{H}^n = K\backslash G$, and the set $\mathcal{C}_0$ is $K$-invariant as well (see §3.2), we can deduce the form above.

It follows from (1.3) that the limit on the left hand side of (1.2) is zero for almost every point $x \in G/\Gamma$ (with respect to the invariant volume measure) in this case. Moreover, for any $\varepsilon$, the Hausdorff dimension of the set of directions in $\mathrm{T}^1(\mathbb{H}^n/\Gamma)$ around a fixed point in $\mathbb{H}^n/\Gamma$ that do not satisfy (1.2) is computed in [MP93, Theorem 1]. For geometrically finite $\Gamma$, the Hausdorff dimension of the set of directions around a fixed point that do not satisfy (1.2) can be found in [HV62, SV95].

## 1.2 Statements of Main Equidistribution Theorems

We will establish the following two theorems. Here, $m^{\mathrm{BR}}$ denotes the BR measure, $m^{\mathrm{BMS}}$ denotes the Bowen-Margulis-Sullivan (BMS) measure, and $\mu^{\mathrm{PS}}$ denotes the PS measure. These measures are defined in Chapter 3. Throughout the paper, the notation

$$x \ll y$$

means there exists a constant $c$ such that

$$x \leq cy.$$

If a subscript is denoted, e.g. $\ll_{\Gamma}$, this explicitly indicates that this constant depends on $\Gamma$.

**Theorem 1.2.1.** *Assume that $\Gamma$ satisfies Assumption 1.1.2. For any $0 < \varepsilon < 1$ and $s_0 \geq 1$, there exist constants $\ell = \ell(\Gamma) \in \mathbb{N}$ and $\kappa = \kappa(\Gamma, \varepsilon) > 0$ satisfying: for every $\psi \in C_c^{\infty}(G/\Gamma)$, there exists $c = c(\Gamma, \operatorname{supp}\psi)$ such that every $x \in G/\Gamma$ that is $(\varepsilon, s_0)$-Diophantine, and for every $r \gg_{\Gamma,\varepsilon} s_0$,*

$$\left| \frac{1}{\mu_x^{\mathrm{PS}}(B_U(r))} \int_{B_U(r)} \psi(u_{\boldsymbol{t}} x) d\mu_x^{\mathrm{PS}}(\boldsymbol{t}) - m^{\mathrm{BMS}}(\psi) \right| \leq c S_\ell(\psi) r^{-\kappa},$$

*where $S_\ell(\psi)$ is the $\ell$-Sobolev norm.*

For the Haar measure, we will prove the following equidistribution result:

**Theorem 1.2.2.** *Assume that $\Gamma$ satisfies Assumption 1.1.2. For any $0 < \varepsilon < 1$ and $s_0 \geq 1$, there exist $\ell = \ell(\Gamma) \in \mathbb{N}$ and $\kappa = \kappa(\Gamma, \varepsilon) > 0$ satisfying: for every $\psi \in C_c^{\infty}(G/\Gamma)$,* *there exists $c = c(\Gamma, \operatorname{supp}\psi)$ such that for every $x \in G/\Gamma$ that is $(\varepsilon, s_0)$-Diophantine, and for all $r \gg_{\Gamma,\operatorname{supp}\psi,\varepsilon} s_0$,*

$$\left|\frac{1}{\mu_x^{\mathrm{PS}}(B_U(r))}\int_{B_U(r)} \psi(u_{\boldsymbol{t}}x)d\boldsymbol{t} - m^{\mathrm{BR}}(\psi)\right| \leq cS_\ell(\psi)r^{-\kappa},$$

*where $S_\ell(\psi)$ is the $\ell$-Sobolev norm.*

Note that the assumption that $x$ is Diophantine is required to obtain quantitative nondivergence of $U$ orbits in Chapter 4, which is key in proving the above theorems. The dependence on a Diophantine condition is necessary, and is analogous to known effective equidistribution results for when $\Gamma$ is a non-cocompact lattice (see [McA19, Str13]).

A key step towards proving Theorem 1.2.1 is the following.

**Theorem 1.2.3.** *Assume that $\Gamma$ satisfies Assumption 1.1.2. There exist $\kappa = \kappa(\Gamma)$ and $\ell = \ell(\Gamma)$ which satisfy the following: for any $\psi \in C_c^\infty(X)$, there exists $c = c(\Gamma, \operatorname{supp}\psi) > 0$ such that for any $f \in C_c^\infty(B_U(r))$, $0 < r < 1$, $x \in \operatorname{supp} m^{\mathrm{BMS}}$, and $s \gg_\Gamma d(\mathcal{C}_0, x)$, we have*

$$\left|\int_U \psi(a_s u_{\boldsymbol{t}} x) f(\boldsymbol{t}) d\mu_x^{\mathrm{PS}}(\boldsymbol{t}) - \mu_x^{\mathrm{PS}}(f) m^{\mathrm{BMS}}(\psi)\right| < cS_\ell(\psi)S_\ell(f)e^{-\kappa s}.$$

In Chapter 6, we also prove an analogous statement for the Haar measure. Such a result is proven in [MO15] under a spectral gap assumption on $\Gamma$, but we show in this paper how to prove it whenever the frame flow is exponentially mixing.

The proof will use similar techniques as in [MO16, OS13]; in particular, we will rely on Margulis' "thickening trick" from his thesis, [Mar04].

In the proofs of our main theorems (Theorems 1.2.1 and 1.2.2), we use partition of unity arguments. In particular, the bounds we get are on slightly bigger sets. As a result, we need an effective bound on the PS measure of a small neighborhood of a boundary of a ball relative to the PS measure of that ball. The following theorem achieves this. It is shown using [DFSU20, Lemma 3.8] and [SV95, Theorem 2], together with quantitative nondivergence established in Chapter 4:

**Theorem 1.2.4.** *There exists a constant $\alpha = \alpha(\Gamma) > 0$, such that for every $x \in G/\Gamma$ that is $(\varepsilon, s_0)$-Diophantine, for every $0 < s \le T^{\frac{\varepsilon}{1-\varepsilon}}$, every $0 < \xi \ll_\Gamma 1$, and every $T \gg_{\Gamma,\varepsilon} s_0$,*

$$\frac{\mu^{\mathrm{PS}}_{a_{-s}x}(B_U(\xi + T))}{\mu^{\mathrm{PS}}_{a_{-s}x}(B_U(T))} - 1 \ll_\Gamma \xi^\alpha.$$

## 1.3 Distribution of Orbits

We often seek to understand a group through the distribution of its orbits on a given space. As an application of the equidistribution theorems stated above, we will study the distribution of the orbits of a geometrically finite group $\Gamma$ acting on $\mathbb{R}^{n+1}$ and other spaces.

When $\Gamma$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$ acting on $\mathbb{R}^2$, this question was considered by Ledrappier [Led99], who proved that

$$\lim_{T \to \infty} \frac{1}{T} \sum_{\gamma \in \Gamma, \|\gamma\| \le T} f(X\gamma) = c(\Gamma) \int_{\mathbb{R}^2} \frac{f(Y)}{|X||Y|} dY$$

for compactly supported functions $f$ and $X \in \mathbb{R}^2$, where $c(\Gamma)$ is some constant depending on the covolume of the lattice $\Gamma$, and $\|\gamma\|$ denotes the $\ell_2$ norm on $\Gamma$. Nogueira [Nog02] independently obtained this result for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ using different methods. More recently, Macourant and Weiss obtained a quantitative version of this theorem for cocompact lattices in $\mathrm{SL}_2(\mathbb{R})$, and also for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ in [MW12]. The case of lattices in $\mathrm{SL}_n(\mathbb{R})$ acting on different spaces $V$ has also been considered, see for instance [Gor04, GM05].

In [Pol10], Pollicott proved a similar quantitative theorem for the action of a lattice in $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$. In the $p$-adic case, Ledrappier and Pollicott [LP05] considered lattices in $\mathrm{SL}_2(\mathbb{Q}_p)$ acting on $\mathbb{Q}_p^2$.

Similar questions have been studied extensively for lattices in a wide variety of groups $G$. For instance, Gorodnik and Weiss consider in [GW07] second countable, locally compact groups $G$ with a general axiomatic approach, with several examples. More recently, Gorodnik and Nevo comprehensively studied the action of a lattice in a connected algebraic Lie group acting on infinite volume homogeneous varieties in [GN14], including obtaining quantitative results under appropriate assumptions.

The case when $\Gamma$ has infinite covolume was recently studied by Maucourant and Schapira in [MS14], where they obtained an asymptotic version of Ledrappier's result for convex cocompact subgroups of $\mathrm{SL}_2(\mathbb{R})$, with a scaling factor permitted. Moreover, they prove that an ergodic theorem like Ledrappier's in the lattice case cannot be obtained in the infinite volume setting, because there is not even a ratio ergodic theorem. More specifically, [MS14, Prop. 1.5] shows that if $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ is geometrically finite with $-I$ the unique torsion element, then there exist small bump functions $f$ and $g$ such that for

$\overline{\nu}$-almost every $v$ (where $\overline{\nu}$ is defined in Chapter 9),

$$\frac{\sum_{\gamma\in\Gamma_T} f(v\gamma)}{\sum_{\gamma\in\Gamma_T} g(v\gamma)}$$

does not have a limit. Thus, it is impossible to obtain an ergodic theorem in this setting with a normalization factor that does not depend on the functions. The key obstruction is the fluctuating behaviour of the PS measure. However, they show that with an additional averaging to address these fluctuations, there is a Log-Cesaro convergence, see [MS14, Theorem 1.6].

As a consequence of a more general ratio theorem that we will discuss later in this section, we will obtain the following asymptotic behaviour for $\Gamma$ orbits acting on

$$V = \mathbf{e}_{n+1}G \setminus \{0\},$$

which is similar to a result of Maucourant and Schapira for $n = 2$. Note that $V$ consists of null vectors of a certain quadratic form and corresponds to the upper half of the "light cone" in the usual representation of $\mathrm{SO}(n,1)$; see §9.1 for more details.

**Proposition 1.3.1.** *Let $\Gamma$ be convex cocompact. For any $\overline{\varphi} \in C_c(V)$ and every $v \in V$ with $v^- \in \Lambda(\Gamma)$, as $T \to \infty$, we have that*

$$\frac{1}{T^{\delta_\Gamma/2}} \sum_{\gamma\in\Gamma_T} \overline{\varphi}(v\gamma) \asymp \int_V \overline{\varphi}(u) \frac{d\overline{\nu}(u)}{(\|v\|_2 \, \|u\|_2)^{\delta_\Gamma/2}},$$

*where the implied constant depends on $v$ and $\Gamma$. Here, $\delta_\Gamma$ denotes the critical exponent of $\Gamma$, $\|u\|_2$ denotes the Euclidean norm of $u \in \mathbb{R}^{n+1}$, and $\Gamma_T = \{\gamma \in \Gamma : \|\gamma\| \leq T\}$, where $\|\gamma\|$ denotes the max norm of $\gamma$ as a matrix in $\mathrm{SL}_{n+1}(\mathbb{R})$. The notation $v^- \in \Lambda_r(\Gamma)$ is discussed in Chapter 9.*

Here, the notation $a \asymp b$ means that there exists a constant $\lambda > 1$ such that

$$\lambda^{-1} \leq \frac{a}{b} \leq \lambda.$$

The precise definition of the measure $\overline{\nu}$ is discussed in Chapter 9. It is the push-forward of the measure $\nu$ defined in §3.3, which is part of the product structure of the BR measure, defined fully in that section.

Recall that $U = \{u_{\mathbf{t}} : \mathbf{t} \in \mathbb{R}^{n-1}\}$ is the expanding horospherical subgroup for the frame flow $A$. Let $P \subset G$ be the parabolic subgroup which contains the contracting horospherical subgroup. Parametrizations of these groups are given in Chapter 3.

Proposition 1.3.1 is obtained by counting orbit points in $U\backslash G$. We will also establish a stronger version, specifically showing that a more precise ratio tends to 1. When Assumption 1.1.2 holds, we obtain a quantitative version of this statement. We need to define additional notation in order to state this result.

Let $UAK$ be the Iwasawa decomposition of $\mathrm{SL}_{n+1}(\mathbb{R})$, and let $\Psi : U\backslash G \to G$ be the map

$$\Psi(Ug) = ak,$$

where $g = uak$ in the Iwasawa decomposition.

We view $G$ as embedded in $\mathrm{SL}_{n+1}(\mathbb{R})$. For $g \in G$, let $\|g\|$ denote the max norm as a matrix in $\mathrm{SL}_{n+1}(\mathbb{R})$. The following "product" is useful for our statements (a similar definition exists in the $\mathrm{SL}_2(\mathbb{R})$ case). For $x, y \in U\backslash G$, let

$$x \star y := \sqrt{\frac{1}{2}\,\|\Psi(x)^{-1}E_{1,n+1}\Psi(y)\|}, \tag{1.4}$$

where $E_{1,n+1}$ is the $(n+1)\times(n+1)$ matrix with one in the $(1,n+1)$-entry and zeros everywhere else. For $x \in U\backslash G$ and $g \in G$, $x \star xg$ measures the difference between the $U$ components of the Iwasawa decomposition of $x$ and $xg$. More specifically, it measures the $(1,n+1)$ component of $g$.

For $L \subseteq G$, define

$$L_T := \{g \in L : \|g\| \leq T\}$$

and recall that

$$B_U(T) := \{u_{\mathbf{t}} \in U : \|\mathbf{t}\| \leq T\},$$

where $\|\mathbf{t}\|$ denotes the max norm of $\mathbf{t} \in \mathbb{R}^{n-1}$. Let $\pi_U : G \to U\backslash G$ denote the natural projection map.

We will be interested in the following quantity:

$$I(\varphi, T, x) := \int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x \star \pi_U(p)}\right)\right)\varphi(\pi_U(p))d\nu(p). \tag{1.5}$$

Here, $\varphi$ is a function on $U\backslash G$, $x \in U\backslash G$, $T > 0$, $\mu^{\mathrm{PS}}$ denotes the PS measure, fully defined in §3.1, and $\nu$ is defined in §3.3.

For two functions of $T$, $a(T), b(T)$, we write

$$a(T) \sim b(T) \iff \lim_{T\to\infty}\frac{a(T)}{b(T)} = 1.$$

We can now state a qualitative version of our ratio theorem. We emphasize to the reader that Assumption 1.1.2 is not required for this result.

**Theorem 1.3.2.** *Let $\Gamma$ be geometrically finite. For any $\varphi \in C_c(U\backslash G)$ and every $x \in U\backslash G$ such that $\Psi(x)^- \in \Lambda_r(\Gamma)$,*

$$\sum_{\gamma \in \Gamma_T} \varphi(x\gamma) \sim I(\varphi, T, x).$$

*The notation $g^-$ for $g \in G$ is defined in Chapter 3.*

By the shadow lemma, Proposition 3.2.1, we obtain the following corollary of Theorem 1.3.2 , which will in turn imply Proposition 1.3.1:

**Corollary 1.3.3.** *Assume that $\Gamma$ is convex cocompact. For any $\varphi \in C_c(U\backslash G)$ and every $x \in U\backslash G$ such that $\Psi(x)^- \in \Lambda(\Gamma)$, as $T \to \infty$,*

$$\frac{1}{T^{\delta_\Gamma/2}} \sum_{\gamma \in \Gamma_T} \varphi(x\gamma) \asymp \int_P \frac{\varphi(\pi_U(p))}{(x \star \pi_U(p))^{\delta_\Gamma}} d\nu(p),$$

*where the implied constant depends on $x$ and $\Gamma$.*

*Remark.* The proof also works for $\Gamma$ geometrically finite when the geodesic of $\Psi(x)\Gamma$ is bounded. We must then assume that $\Psi(x)^- \in \Lambda_r(\Gamma)$. ◀

**Theorem 1.3.4.** *Assume that Assumption 1.1.2 holds. For any $0 < \varepsilon < 1$, there exist $\ell = \ell(\Gamma) \in \mathbb{N}$ and $\kappa = \kappa(\Gamma, \varepsilon)$ satisfying: for every $\varphi \in C_c^\infty(U\backslash G)$ and for every $x \in U\backslash G$ such that $\Psi(x)\Gamma$ is $\varepsilon$-Diophantine, and for all $T \gg_{\Gamma, \operatorname{supp}\varphi, x} 1$,*

$$\left| \frac{\sum_{\gamma \in \Gamma_T} \varphi(x\gamma)}{\int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x \star \pi_U(p)}\right)\right) \varphi(\pi_U(p)) d\nu(p)} - 1 \right|$$
$$\ll_{\Gamma, \operatorname{supp}\varphi, x} T^{-\kappa}\left(1 + S_\ell(\varphi)\nu(\varphi \circ \pi_U)^{-1}\right).$$

The dependencies in this statement are quite explicit. The dependence of $T$ on $x$ in Theorem 1.3.4 arises from the constant in Lemma 7.0.2, which is explicitly defined in that proof, and the precise Diophantine nature of $x$, through Theorem 1.2.2 (i.e. the $\varepsilon$ and $s_0$ that appear in Definition 1.1.3). The implied dependence on $x$ in the conclusion is discussed at the end of section Chapter 8.

If the support of the function is small enough, then we can get a more explicit estimate, showing that this is a sort of ergodic theorem:

For $x \in U\backslash G$ and a compact set $H \subset U\backslash G$, let $\mathcal{R}(H, x) := \max\limits_{y,z\in H} \frac{x\star y}{x\star z}$.

**Theorem 1.3.5.** *Assume that Assumption 1.1.2 holds. For any $0 < \varepsilon < 1$, there exist $\ell = \ell(\Gamma) \in \mathbb{N}$ and $\kappa = \kappa(\Gamma, \varepsilon)$ satisfying: for every $x \in U\backslash G$ such that $\Psi(x)\Gamma$ is $\varepsilon$-Diophantine and every compact $\Omega \subset G$, there exists $T_0 = T_0(x, \Omega)$ so that for every $T \geq T_0$, there exists $\eta = \eta(T, \ell, \kappa, n, \Omega) > 0$ such that if $\varphi \in C_c^\infty(U\backslash G)$ with $\Psi(\operatorname{supp}\varphi) \subseteq \Omega$ and satisfies $\mathcal{R}(\operatorname{supp}\varphi, x) - 1 < \eta$, then for every $y \in \operatorname{supp}\varphi$,*

$$\left| \frac{1}{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star y}\right)\right)} \sum_{\gamma\in\Gamma_T} \varphi(x\gamma) - \int_P \varphi(\pi_U(p))d\nu(p) \right| \ll_{\Gamma,\Omega,x} S_\ell(\varphi)T^{-\kappa}.$$

## 1.4 Organization of the Dissertation

This dissertation is organized as follows. In Chapter 2, we discuss under what conditions Assumption 1.1.2 is known to hold. In Chapter 3, we set out notation used in the article, and define the measures we will be using, along with proving some important facts about them. In Chapter 4, we prove quantitative nondivergence of horospherical orbits of Diophantine points, which is needed in the following chapters. In Chapter 5, we control the PS measure of the boundary of a set by proving Theorem 1.2.4. In Chapter 6, we use Margulis' "thickening trick" to prove Theorem 1.2.3 and an analogous result for the Haar measure, which are key in the proofs of Theorems 1.2.1 and 1.2.2, which are also contained in this chapter. Theorems 1.2.1 and 1.2.2 also rely on quantitative nondivergence.

In Chapter 7, we begin the study of the distribution of the orbits of $\Gamma$ acting on $U\backslash G$. Specifically, in this chapter, we explore the duality between $\Gamma$ orbits on $U\backslash G$ and of $U$ orbits on $G/\Gamma$, and prove key lemmas that are common to the proofs of both Theorems 1.3.2 and 1.3.4. We also use a thickening argument, inspired by Ledrappier, to reduce the problem to that of equidistribution of $U$ orbits on $G/\Gamma$. In Chapter 8, we prove Theorem 1.3.2, using the equidistribution theorem of Mohammadi and Oh, Theorem 1.1.1. In this chapter, we also prove Theorem 1.3.4 using Theorem 1.2.2. In Chapter 9, we consider two specific examples, and prove Proposition 1.3.1. Finally, several technical details of the proof of Theorem 1.2.4 are in the appendix, Chapter 10.

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This chapter contains material from the following, which has been submitted for publication: N. Tamam, J. M. Warren, *"Effective equidistribution of horospherical flows in infinite volume rank one homogeneous spaces"*, arXiv:2007.03135. The dissertation author was one of the primary investigators and authors of this paper, and was supported in part by the National Science and Engineering Research Council of Canada (NSERC)

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# Chapter 2

# Known Exponential Mixing Results

We will assume the existence of an exponential mixing result in the proofs of our main theorems (see Assumption 1.1.2). In this section we elaborate on the conditions under which such a result is known. Here we assume that $\Gamma$ is a Zariski dense discrete subgroup of $G$.

There is a natural action of $G$ on $\mathbb{H}^n$ and $\partial\mathbb{H}^n$, the hyperbolic $n$-space and its boundary, respectively. Let $\Lambda(\Gamma) \subseteq \partial(\mathbb{H}^n)$ denote the limit set of $X$, i.e., the set of all accumulation points of $\Gamma z$ for some $z \in \mathbb{H}^n \cup \partial(\mathbb{H}^n)$. The *convex core* of $X$ is the image in $X$ of the minimal convex subset of $\mathbb{H}^n$ which contains all geodesics connecting any two points in $\Lambda(\Gamma)$. We say that $\Gamma$ is *convex cocompact* if the convex core of $\mathbb{H}^n/\Gamma$ is compact, and *geometrically finite* if a unit neighborhood of the convex core of $\Gamma$ has finite volume.

For $\Gamma$ convex cocompact, Assumption 1.1.2 was proved by Sarkar and Winter in [SW20, Theorem 1.1].

Fix a point $w_o \in \mathrm{T}^1(\mathbb{H}^n)$ and denote $M = \mathrm{Stab}_G(w_o)$. Denote by $\hat{G}$ and $\hat{M}$ the unitary dual of $G$ and $M$, respectively. A representation $(\pi, \mathcal{H}) \in \hat{G}$ is called *tempered* if for any $K$-finite $v \in \mathcal{H}$, the associated matrix coefficient function $g \mapsto \langle \pi(g)v, v \rangle$ belongs to $L^{2+\varepsilon}(G)$ for any $\varepsilon > 0$, and *non-tempered* otherwise. The non-tempered part of $\hat{G}$ consists of the trivial representation, and complementary series representations $\mathcal{U}(v, s-n+1)$ parameterized by $v \in \hat{M}$ and $s \in I_v$, where $I_v \subseteq (\frac{n-1}{2}, n-1)$ is an interval depending on $v$ (see Hirai [Hir62]).

**Definition 2.0.1.** The space $L^2(X)$ has a **spectral gap** if there exist $\frac{n-1}{2} < s_0 = s_0(\Gamma) < \delta$ and $n_0 = n_0(\Gamma) \in \mathbb{N}$ such that

1. the multiplicity of $\mathcal{U}(v, \delta_\Gamma - n + 1)$ contained in $L^2(X)$ is at most $dim(v)^{n_0}$ for any $v \in \hat{M}$;
2. $L^2(X)$ does not weakly contain any $\mathcal{U}(v, s-n+1)$ with $s \in (s_0, \delta)$ and $v \in \hat{M}$.

◀

According to [MO15, Theorem 3.27], if $\delta_\Gamma > \frac{n-1}{2}$ for $n = 2, 3$, or if $\delta_\Gamma > n-2$ for $n \geq 4$, then $L^2(X)$ has a spectral gap. If $\delta_\Gamma \leq \frac{n-1}{2}$, then there is no spectral gap, but it was conjectured that whenever $\delta_\Gamma > \frac{n-1}{2}$, $L^2(X)$ has a spectral gap (see [MO15]). Note that if there are cusps of maximal rank $n-1$, it follows that $\delta_\Gamma > \frac{n-1}{2}$.

For $\Gamma$ geometrically finite such that $L^2(X)$ has a spectral gap and $\delta_\Gamma > \frac{n-1}{2}$, Mohammadi and Oh stated in [MO15, Theorem 1.6] an exponential mixing result similar to Assumption 1.1.2. In their statement the constant $c$ depends on $\Gamma$ and the support of the functions. The dependence on the support of the functions arises in the last part of the proof (see [MO15, §6.3]), and can be omitted by using the following lemma (the BR-measure is defined in §3.3). Thus, we see that a result of the form needed in Assumption 1.1.2 holds in this case.

**Lemma 2.0.2.** *If $\delta > (n-1)/2$, then there exists $c = c(\Gamma) > 0$ such that any $B \subset X$ of diameter smaller than 1 satisfies*

$$m^{\mathrm{BR}}(B) \leq c.$$

*Proof.* For any $g \in G$ denote

$$\Phi_0(g) = |\nu_{g(o)}|,$$

where $o$ is the projection of $w_o$ onto $\mathbb{H}^n$ and for any $x \in \mathbb{H}^n$, $\nu_x$ is the Patterson-Sullivan density defined in §3.1. Since $\Phi_0$ is $\Gamma$-invariant, it can be considered as a smooth function on $X$. Moreover, by assuming $B$ contains $K = \mathrm{Stab}_G(o)$ and using the Cauchy Schwartz inequality, we get

$$\begin{aligned} m^{\mathrm{BR}}(B) &= \int_B \Phi_0(g) dm^{\mathrm{Haar}}(g) \\ &\leq \sqrt{dm^{\mathrm{Haar}}(B)}\, \|\Phi_0\|_2 \\ &\ll \|\Phi_0\|_2 . \end{aligned}$$

According to [Sul79, §7] and by the assumption $\delta > (n-1)/2$, we have that $\phi_0 \in L^2(X)$. $\square$

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# Chapter 3

# Notation and Preliminaries

Recall that $G = \mathrm{SO}(n,1)^\circ$ and $\Gamma \subseteq G$ is a geometrically finite Kleinian subgroup of $G$. Denote

$$X := G/\Gamma.$$

$G$ acts transitively on $\mathbb{H}^n$, the hyperbolic $n$-space. Fix a reference point $o \in \mathbb{H}^n$ and let $K = \mathrm{Stab}_G(o)$, then $K\backslash G = \mathbb{H}^n$. Let $\pi : G \to \mathbb{H}^n$ be the projection

$$\pi(g) = g(o). \tag{3.1}$$

We will abuse notation and also write $\pi$ for the induced map from $G/\Gamma$ to $\mathbb{H}^n/\Gamma$. For convenience, we will assume throughout the paper that we have chosen $o$ so that $o\Gamma \in \pi(\mathcal{C}_0)$, where $\mathcal{C}_0$ is defined in §3.2. This says that $o\Gamma$ is in the convex core of $\mathbb{H}^n/\Gamma$.

Let $d$ denote the left $G$-invariant metric on $G$ which induces the hyperbolic metric on $K\backslash G = \mathbb{H}^n$.

Recall that $\Lambda(\Gamma) \subseteq \partial(\mathbb{H}^n)$ denotes the limit set of $\Gamma$, which is the set of accumulation points of the $\Gamma$ orbit of $o$. We denote the Hausdorff dimension of $\Lambda(\Gamma)$ by $\delta_\Gamma$. It is equal to the critical exponent of $\Gamma$ (see [Pat88]).

We say that a limit point $\xi \in \Lambda(\Gamma)$ is *radial* if there exists a compact subset of $X$ so that some (and hence every) geodesic ray toward $\xi$ has accumulation points in that set. An element $g \in G$ is called *parabolic* if the set of fixed points of $g$ in $\partial(\mathbb{H}^n)$ is a singleton. We say that a limit point is *parabolic* if it is fixed by a parabolic element of $\Gamma$. A parabolic limit point $\xi \in \Lambda(\Gamma)$ is called *bounded* if the stabilizer $\Gamma_\xi$ acts cocompactly on $\Lambda(\Gamma) - \{\xi\}$.

We denote by $\Lambda_r(\Gamma)$ and $\Lambda_{bp}(\Gamma)$ the set of all radial limit points and the set of all bounded parabolic limit points, respectively. By [Bow93], since $\Gamma$ is geometrically finite, we have that

$$\Lambda(\Gamma) = \Lambda_r(\Gamma) \cup \Lambda_{bp}(\Gamma).$$

Fix $w_o \in \mathrm{T}^1(\mathbb{H}^n)$ and let $M = \mathrm{Stab}_G(w_o)$ so that $\mathrm{T}^1(\mathbb{H}^n)$ may be identified with $M\backslash G$. For $w \in \mathrm{T}^1(\mathbb{H}^n)$,

$$w^\pm \in \partial\mathbb{H}^n$$

denotes the forward and backward endpoints of the geodesic $w$ determines. For $g \in G$, we define

$$g^\pm := w_o^\pm g.$$

Without loss of generality, we may assume that $w_o^\pm \in \Lambda(\Gamma)$, and hence every $\gamma \in \Gamma$ will satisfy $\gamma^\pm \in \Lambda(\Gamma)$.

Let $A = \{a_s \,:\, s \in \mathbb{R}\}$ be a one parameter diagonalizable subgroup such that $M$ and $A$ commute, and such that the right at action on $M\backslash G = \mathrm{T}^1(\mathbb{H}^n)$ corresponds to unit speed geodesic flow. We parametrize $A$ by $A = \{a_s : s \in \mathbb{R}\}$, where

$$a_s = \begin{pmatrix} e^s & & \\ & I & \\ & & e^{-s} \end{pmatrix} \tag{3.2}$$

and $I$ denotes the $(n-1) \times (n-1)$ identity matrix, and parametrize $M$ by

$$M = \left\{ \begin{pmatrix} 1 & & \\ & m & \\ & & 1 \end{pmatrix} : m \in \mathrm{SO}(n-1) \right\}.$$

Let $U$ denote the expanding horospherical subgroup

$$U = \{g \in G \,:\, a_{-s} g a_s \to e \text{ as } s \to +\infty\},$$

let $\tilde{U}$ be the contracting horospherical subgroup

$$\tilde{U} = \{g \in G \,:\, a_s g a_{-s} \to e \text{ as } s \to +\infty\},$$

and let $P = MA\tilde{U}$ be the parabolic subgroup.

The group $U$ is a connected abelian group, isomorphic to $\mathbb{R}^{n-1}$. We may use the parametrization $\mathbf{t} \mapsto u_{\mathbf{t}}$ so that for any $s \in \mathbb{R}$,

$$a_s u_{\mathbf{t}} a_{-s} = u_{e^s \mathbf{t}}. \tag{3.3}$$

Similarly, we parametrize $\tilde{U}$ by $\mathbf{t} \mapsto v_{\mathbf{t}} \in \tilde{U}$ so that for $s \in \mathbb{R}$,

$$a_s v_{\mathbf{t}} a_{-s} = v_{e^{-s} \mathbf{t}}. \tag{3.4}$$

More explicitly, if $\mathbf{t} \in \mathbb{R}^{n-1}$ is viewed as a row vector,

$$u_{\mathbf{t}} = \begin{pmatrix} 1 & \mathbf{t} & \frac{1}{2}\|\mathbf{t}\|^2 \\ & I & \mathbf{t}^T \\ & & 1 \end{pmatrix} \tag{3.5}$$

and

$$v_{\mathbf{t}} = \begin{pmatrix} 1 & & \\ \mathbf{t}^T & I & \\ \frac{1}{2}|\mathbf{t}|^2 & \mathbf{t} & 1 \end{pmatrix}.$$

For a subset $H$ of $G$ and $\eta > 0$, $H_\eta$ denotes the closed $\eta$-neighborhood of $e$ in $H$, i.e.

$$H_\eta = \{h \in H \,:\, d(h, e) \leq \eta\}.$$

For any $r > 0$ let

$$B_U(r) = \{u_{\mathbf{t}} \,:\, \|\mathbf{t}\| \leq r\} \quad \text{and} \quad B_{\tilde{U}}(r) = \{v_{\mathbf{t}} \,:\, \|\mathbf{t}\| \leq r\},$$

where $\|\mathbf{t}\|$ is the sup-norm of $\mathbf{t} \in \mathbb{R}^{n-1}$.

**Lemma 3.0.1.** *For $0 < \eta < 1/4$ and $p \in P_\eta$, there exists $\rho_p : B_U(1) \to B_U(1 + O(\eta))$ that is a diffeomorphism onto its image and a constant $D = D(\eta) < 3\eta$ such that*

$$u_{\boldsymbol{t}} p^{-1} \in P_D u_{\rho_p(\boldsymbol{t})}.$$

*Explicitly, if $p = a_s v_{\boldsymbol{r}}$, then* $\rho_p(\boldsymbol{t}) = \dfrac{\boldsymbol{t} + \frac{1}{2}\|\boldsymbol{t}\|^2 \boldsymbol{r}}{e^s(1 - (\boldsymbol{t} \cdot \boldsymbol{r}) + \frac{1}{4}\|\boldsymbol{r}\|^2\|\boldsymbol{t}\|^2)}$.

*Proof.* For $s \in \mathbb{R}$ and $\mathbf{r} \in \mathbb{R}^{n-1}$, let $p = a_s v_{\mathbf{r}}$. Then $p^{-1} = \begin{pmatrix} e^{-s} & & \\ -e^{-s}\mathbf{r}^T & I & \\ \frac{1}{2}e^{-s}\|\mathbf{r}\|^2 & -\mathbf{r} & e^s \end{pmatrix}$, so

$$u_{\mathbf{t}}p^{-1} = \begin{pmatrix} e^{-s}(1 - (\mathbf{t}\cdot\mathbf{r}) + \frac{1}{4}\|\mathbf{r}\|^2\|\mathbf{t}\|^2) & \mathbf{t} - \frac{1}{2}\|\mathbf{t}\|^2\mathbf{r} & \frac{1}{2}e^s\|\mathbf{t}\|^2 \\ -e^{-s}\mathbf{r}^T + \frac{1}{2}e^{-s}\|\mathbf{r}\|^2\mathbf{t}^T & I - \mathbf{t}^T\mathbf{r} & e^s\mathbf{t}^T \\ \frac{1}{2}e^{-s}\|\mathbf{r}\|^2 & -\mathbf{r} & e^s \end{pmatrix}.$$

Now, if $p' = a_{s'}v_{\mathbf{r}'}$, we obtain that

$$p'u_{\mathbf{t}'} = \begin{pmatrix} e^{s'} & e^{s'}\mathbf{t}' & \frac{1}{2}e^{s'}\|\mathbf{t}'\|^2 \\ \mathbf{r}'^T & \mathbf{r}'^T\mathbf{t}' + I & \frac{1}{2}\|\mathbf{t}'\|^2\mathbf{r}'^T + \mathbf{t}'^T \\ \frac{1}{2}e^{-s'}\|\mathbf{r}'\|^2 & \frac{1}{2}e^{-s'}\|\mathbf{r}'\|^2\mathbf{t}' + e^{-s'}\mathbf{r}' & e^{-s'}(\frac{1}{4}\|\mathbf{r}'\|^2\|\mathbf{t}'\|^2 + (\mathbf{r}'\cdot\mathbf{t}') + 1) \end{pmatrix}.$$

We wish to solve for $\mathbf{t}'$.

Setting entries equal yields

$$\mathbf{t} + \frac{1}{2}\|\mathbf{t}\|^2\mathbf{r} = e^{s'}\mathbf{t}'$$

and

$$e^{s'} = e^{-s}\left(1 - (\mathbf{t}\cdot\mathbf{r}) + \frac{1}{4}\|\mathbf{r}\|^2\|\mathbf{t}\|^2\right). \tag{3.6}$$

Combining these implies that

$$\mathbf{t}' = \frac{\mathbf{t} + \frac{1}{2}\|\mathbf{t}\|^2\mathbf{r}}{e^s(1 - (\mathbf{t}\cdot\mathbf{r}) + \frac{1}{4}\|\mathbf{r}\|^2\,\|\mathbf{t}\|^2)}.$$

We define $\rho_p(\mathbf{t})$ to be this quantity. One can directly check that it satisfies the claim.

□

## 3.1 Patterson-Sullivan and Lebesgue Measures

A family of finite measures $\{\mu_x : x \in \mathbb{H}^n\}$ on $\partial(\mathbb{H}^n)$ is called a $\Gamma$-*invariant conformal density of dimension* $\delta_\mu > 0$ if for every $x, y \in \mathbb{H}^n$, $\xi \in \partial(\mathbb{H}^n)$ and $\gamma \in \Gamma$,

$$\gamma_* \mu_x = \mu_{x\gamma} \text{ and } \frac{d\mu_y}{d\mu_x}(\xi) = e^{-\delta_\mu \beta_\xi(y,x)}, \tag{3.7}$$

where $\gamma_*\mu_x(F) = \mu_x(F\gamma)$ for any Borel subset $F$ of $\partial(\mathbb{H}^n)$.

We let $\{\nu_x\}_{x\in\mathbb{H}^n}$ denote the Patterson-Sullivan density on $\partial\mathbb{H}^n$, that is, the unique (up to scalar multiplication) conformal density of dimension $\delta_\Gamma$.

For each $x \in \mathbb{H}^n$, we denote by $m_x$ the unique probability measure on $\partial(\mathbb{H}^n)$ which is invariant under the compact subgroup $\mathrm{Stab}_G(x)$. Then $\{m_x : x \in \mathbb{H}^n\}$ forms a $G$-invariant conformal density of dimension $n-1$, called the Lebesgue density. Fix $o \in \mathbb{H}^n$.

For $x, y \in \mathbb{H}^n$ and $\xi \in \partial(\mathbb{H}^n)$, the *Busemann function* is given by

$$\beta_\xi(x,y) := \lim_{t\to\infty} d(x,\xi_t) - d(y,\xi_t)$$

where $\xi_t$ is a geodesic ray towards $\xi$.

For $g \in G$, we can define measures on $Ug$ using the conformal densities defined previously. The Patterson-Sullivan measure (abbreviated as the PS-measure):

$$d\mu^{\mathrm{PS}}_{Ug}(u_{\mathbf{t}}g) := e^{\delta_\Gamma \beta_{(u_{\mathbf{t}}g)^+}(o, u_{\mathbf{t}}g(o))} d\nu_o((u_{\mathbf{t}}g)^+), \tag{3.8}$$

and the Lebesgue measure

$$\mu^{\mathrm{Leb}}_{Ug}(u_{\mathbf{t}}g) := e^{(n-1)\beta_{(u_{\mathbf{t}}g)^+}(o, u_{\mathbf{t}}g(o))} dm_o((u_{\mathbf{t}}g)^+).$$

We similarly define the opposite PS measure on $\tilde{U}g$:

$$d\mu_{\tilde{U}g}^{\mathrm{PS}\,-}(v_{\mathbf{t}}g) := e^{\delta_\Gamma \beta_{(v_{\mathbf{t}}g)^-}(o, v_{\mathbf{t}}g(o))} d\nu_o((v_{\mathbf{t}}g)^-). \tag{3.9}$$

The conformal properties of $m_x$ and $\nu_x$ imply that these definitions are independent of the choice of $o \in \mathbb{H}^n$.

We often view $\mu_{Ug}^{\mathrm{PS}}$ as a measure on $U$ via

$$d\mu_g^{\mathrm{PS}}(\mathbf{t}) := d\mu_{Ug}^{\mathrm{PS}}(u_{\mathbf{t}}g),$$

and similarly for $\mu_{\tilde{U}g}^{\mathrm{PS}\,-}$ on $\tilde{U}$. For $g \in G, s \in \mathbb{R}$, and $E \subseteq U$ a Borel subset (or $E \subseteq \tilde{U}$ for $\mu^{\mathrm{PS}\,-}$), these measures satisfy:

$$\mu_g^{\mathrm{Leb}}(E) = e^{(n-1)s}\mu_{a_{-s}g}^{\mathrm{Leb}}(a_{-s}Ea_s), \tag{3.10}$$

$$\mu_g^{\mathrm{PS}}(E) = e^{\delta_\Gamma s}\mu_{a_{-s}g}^{\mathrm{PS}}(a_{-s}Ea_s), \tag{3.11}$$

$$\mu_g^{\mathrm{PS}\,-}(E) = e^{\delta_\Gamma s}\mu_{a_sg}^{\mathrm{PS}\,-}(a_sEa_{-s}). \tag{3.12}$$

In particular,

$$\mu_g^{\mathrm{PS}}(B_U(e^s)) = e^{\delta_\Gamma s}\mu_{a_{-s}g}^{\mathrm{PS}}(B_U(1)) \text{ and } \mu_g^{\mathrm{PS}\,-}(B_{U^-}(e^s)) = e^{\delta_\Gamma s}\mu_{a_sg}^{\mathrm{PS}\,-}(B_U(1)).$$

The measure

$$d\mu_{Ug}^{\mathrm{Leb}}(u_{\mathbf{t}}g) = d\mu_U^{\mathrm{Leb}}(u_{\mathbf{t}}) = d\mathbf{t}$$

is independent of the orbit $Ug$ and is simply the Lebesgue measure on $U \equiv \mathbb{R}^{n-1}$ up to a scalar multiple.

We will need the following fundamental results, which are stated for $\mu^{\mathrm{PS}}$ and $U$, but also hold if we replace them with $\mu^{\mathrm{PS}\,-}$ and $\tilde{U}$.

**Lemma 3.1.1.** *The map $g \mapsto \mu_g^{\mathrm{PS}}$ is continuous, where the topology on the space of regular Borel measures on $U$ is given by $\mu_n \to \mu \iff \mu_n(f) \to \mu(f)$ for all $f \in C_c(U)$.*

*Proof.* This is clear from the definition of the PS measure, since it is defined using the Busemann function and stereographic projection. □

**Corollary 3.1.2.** *For any compact set $\Omega \subseteq G$ and any $r > 0$,*

$$0 < \inf_{g\in\Omega, g^+\in\Lambda(\Gamma)} \mu_g^{\mathrm{PS}}(B_U(r)g) \leq \sup_{g\in\Omega, g^+\in\Lambda(\Gamma)} \mu_g^{\mathrm{PS}}(B_U(r)g) < \infty.$$

To define the PS measure on $Ux$ for $x \in X$, note that

$$\text{if } x^- \in \Lambda_r(\Gamma), \text{ then } u \mapsto ux \text{ is injective,} \tag{3.13}$$

and we can define the PS measure on $Ux \subseteq X$, denoted $\mu_x^{\mathrm{PS}}$, simply by pushforward of $\mu_g^{\mathrm{PS}}$, where $x = g\Gamma$. In general, defining $\mu_x^{\mathrm{PS}}$ requires more care, see e.g. [MO16, §2.3] for more details. As before, we can view $\mu_x^{\mathrm{PS}}$ as a measure on $U$ via

$$d\mu_x^{\mathrm{PS}}(\mathbf{t}) = d\mu_x^{\mathrm{PS}}(u_{\mathbf{t}}x).$$

## 3.2 Thick-thin Decomposition and the Height Function

There exists a finite set of $\Gamma$-representatives $\xi_1, \ldots, \xi_q \in \Lambda_{bp}(\Gamma)$. For $i = 1, \ldots, q$, fix $g_i \in G$ such that $g_i^- = \xi_i$, and for any $R > 0$, set

$$\mathcal{H}_i(R) := \bigcup_{s>R} K a_{-s} U g_i, \quad \text{and} \quad \mathcal{X}_i(R) := \mathcal{H}_i(R)\Gamma \tag{3.14}$$

(recall, $K = \mathrm{Stab}_G(o)$). Each $\mathcal{H}_i(R)$ is a horoball of depth $R$.

The *rank* of $\mathcal{H}_i(R)$ is the rank of the finitely generated abelian subgroup $\Gamma_{\xi_i} = \mathrm{Stab}_\Gamma(\xi_i)$. We say that the cusp has maximal rank if $\operatorname{rank}\Gamma_\xi = n-1$. It is known that each rank is strictly smaller than $2\delta_\Gamma$.

We denote

$$\operatorname{supp} m^{\mathrm{BMS}} := \left\{ g\Gamma \in X \;:\; g^{\pm} \in \Lambda(\Gamma) \right\}.$$

(For now, this is simply notation. The measure $m^{\mathrm{BMS}}$ will be defined in the next section, and this set is its support. It projects onto the convex core of $\mathbb{H}^n/\Gamma$.) Note that the condition $g^{\pm} \in \Lambda(\Gamma)$ is independent of the choice of representative of $x = g\Gamma$ in the above definition, because $\Lambda(\Gamma)$ is $\Gamma$-invariant. Thus, the notation $x^{\pm} \in \Lambda(\Gamma)$ is well-defined, even though $x^{\pm}$ itself is not.

According to [Bow93], there exists $R_0 \geq 1$ such that $\mathcal{X}_1(R_0), \ldots, \mathcal{X}_q(R_0)$ are disjoint, and for some compact set $\mathcal{C}_0 \subset G/\Gamma$,

$$\operatorname{supp} m^{\mathrm{BMS}} \subseteq \mathcal{C}_0 \sqcup \mathcal{X}_1(R_0) \sqcup \cdots \sqcup \mathcal{X}_q(R_0).$$

For $1 \leq i \leq q$ and $R \geq R_0$, denote

$$\mathcal{X}(R) := \mathcal{X}_1(R) \sqcup \cdots \sqcup \mathcal{X}_q(R), \quad \mathcal{C}(R) := \operatorname{supp} m^{\mathrm{BMS}} - \mathcal{X}(R).$$

We will need a version of Sullivan's shadow lemma, obtained by Maucourant and Schapira (see Proposition 5.1 and Remark 5.2 in [MS14]).

**Proposition 3.2.1.** *There exists a constant $\lambda = \lambda(\Gamma) \geq 1$ such that for all $x \in \operatorname{supp} m^{\mathrm{BMS}}$ and all $T > 0$, we have*

$$\lambda^{-1} T^{\delta_\Gamma} e^{(k_1(x,T)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{-\log T} x))} \leq \mu_x^{\mathrm{PS}}(B_U(T)) \tag{3.15}$$

$$\leq \lambda T^{\delta_\Gamma} e^{(k_1(x,T)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{-\log T} x))} \tag{3.16}$$

*and*

$$\lambda^{-1} T^{\delta_\Gamma} e^{(k_2(x,T)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log T} x))} \leq \mu_x^{\mathrm{PS}\,-}(B_{\tilde{U}}(T)) \tag{3.17}$$

$$\leq \lambda T^{\delta_\Gamma} e^{(k_2(x,T)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log T} x))},$$

*where $k_1(x,T)$ is the rank of $\mathcal{X}_i(R_0)$ if $a_{-\log T} x \in \mathcal{X}_i(R_0)$ for some $1 \leq i \leq \ell$ and equals 0 if $a_{-\log T} x \in \mathcal{C}_0$, and $k_2(x,T)$ is defined analogously for $a_{\log T} x$. Recall the definition of $\pi$ from* (3.1) *as the projection from $G$ to $\mathbb{H}^n$.*

*Remark.* When $\Gamma$ is convex cocompact, $\mathcal{C}_0 = \operatorname{supp} m^{\mathrm{BMS}}$, and the shadow lemma simplifies to

$$\lambda^{-1} T^{\delta_\Gamma} \leq \mu_x^{\mathrm{PS}}(B_U(T)) \leq \lambda T^{\delta_\Gamma}.$$

◀

**Definition 3.2.2.** For $x \in G/\Gamma$, we define the *height* of $x$ by

$$\operatorname{height}(x) = d(\pi(\mathcal{C}_0), \pi(x)), \tag{3.18}$$

where $\pi : G/\Gamma \to \mathbb{H}^n/\Gamma$ is the projection map as in (3.1), recalling that $\mathbb{H}^n/\Gamma \cong K\backslash G/\Gamma$.

◀

**Lemma 3.2.3.** *For any $x \in \operatorname{supp} m^{\mathrm{BMS}}$ and $R \geq R_0$, we have that*

$$x \in \mathcal{C}(R) \iff \operatorname{height}(x) \leq R - R_0.$$

*Proof.* The claim follows from the disjointness of $\mathcal{X}_i(R_0)$, $1 \leq i \leq q$ from $\mathcal{C}_0$, and the fact that $\mathcal{X}_i(R) \subseteq \mathcal{X}_i(R_0)$:

If $x \in \mathcal{C}(R)$, then either $x \in \mathcal{C}_0$, in which case $\operatorname{height}(x) = 0$ and we are done, or $x \in \mathcal{X}_i(R_0)$. Assume the latter, then the Busemann function between $x$ and the boundary of $\mathcal{X}_i(R_0)$ (which intersects $\mathcal{C}_0$) is at most $R - R_0$. Thus, we may deduce the claim in this case.

Next, assume $x \in \mathcal{X}_i(R)$ for some $i$. The Busemann function between two points in different horoballs is at least $R - R_0$. Since a point from $\mathcal{X}_i(R)$ cannot go into $\mathcal{C}_0$ without passing through $\mathcal{X}_i(R_0)$, this is a lower bound for the distance between the base points, i.e. the height. □

**Corollary 3.2.4.** *Let $x \in G/\Gamma$ be $(\varepsilon, s_0)$-Diophantine. Then*

$$\operatorname{height}(x) < (2 - \varepsilon)s_0.$$

*Proof.* By Definition 1.1.3,

$$d(\mathcal{C}_0, a_{-s_0}x) < (1 - \varepsilon)s_0.$$

Hence, we have that

$$\begin{aligned}
\operatorname{height}(x) &\leq d(\mathcal{C}_0, x) \\
&< d(\mathcal{C}_0, a_{-s_0}x) + d(a_{-s_0}x, x) \\
&< (1-\varepsilon)s_0 + s_0.
\end{aligned}$$

$\square$

The *injectivity radius* at $x \in X$ is defined to be the supremum over all $\varepsilon > 0$ such that the map

$$h \mapsto hx \text{ is injective on } G_\varepsilon.$$

We denote the injectivity radius at $x$ by

$$\operatorname{inj}(x).$$

The injectivity radius of a set $\Omega$ is defined to be

$$\inf_{x\in\Omega} \operatorname{inj}(x).$$

By the proof of [MO20, Proposition 6.7], there exists a constant $\sigma = \sigma(\Gamma) > 0$ such that for all $x \in \operatorname{supp} m^{\mathrm{BMS}}$,

$$\sigma^{-1} \operatorname{inj}(x) \leq e^{-\operatorname{height}(x)} \leq \sigma \operatorname{inj}(x). \tag{3.19}$$

The following fact is well-known, but we include a proof for completion.

**Lemma 3.2.5.** *There exists $T_0 = T_0(\Gamma) > 0$ which satisfies the following. Let $x \in G/\Gamma$ with $x^- \in \Lambda(\Gamma)$, and let $R > 0$ be such that $d(\mathcal{C}_0, x) < R$. Then there exists $\boldsymbol{t} \in B_U(2(R + T_0))$ such that*

$$(u_{\boldsymbol{t}}x)^{\pm} \in \Lambda(\Gamma).$$

*In particular, for every $0 < \varepsilon < 1$, $s_0 \geq 1$, and $(\varepsilon, s_0)$-Diophantine point $x$, there exists $|\boldsymbol{t}| \ll_\Gamma s_0$ such that*

$$(u_{\boldsymbol{t}}x)^{\pm} \in \Lambda(\Gamma).$$

*Proof.* Let $g, h' \in G$ be such that $x = g\Gamma$, $h'^- = g^-$, $h'\Gamma \in K\mathcal{C}_0$, and

$$d(g, h') \leq \text{height}(x) < R.$$

Since $K\mathcal{C}_0$ is a compact set, by [MO16, Lemma 3.3], there exists a constant $T_0$, which only depends on $\mathcal{C}_0$ (i.e., on $\Gamma$) such that for some $\mathbf{t} \in B_U(T_0)$,

$$(u_{\mathbf{t}}h')^{\pm} \in \Lambda(\Gamma).$$

Fix $h := u_{\mathbf{t}}h'$ and observe that

$$d(g, h) < R + T_0. \tag{3.20}$$

We must flow $h\Gamma$ with an element of $A$ so that it lies on $Ux$.

Because $h^- = g^-$, if $s = \beta_{g^-}(h, g)$, then

$$a_s h \in Ug.$$

Since $\beta_{g^-}(h, g) \leq d(h, g)$, we arrive at

$$\begin{aligned} d(g, a_s h) &\leq d(g, h) + d(h, a_s h) \\ &\leq 2d(g, h) \\ &\leq 2\left(R + T_0\right). \end{aligned}$$

For $(\varepsilon, s_0)$-Diophantine $x$, observe that

$$\begin{aligned} d(\mathcal{C}_0, x) &\leq d(\mathcal{C}_0, a_{-s_0} x) + d(a_{-s_0}, x) \\ &< (1 - \varepsilon) s_0 + s_0 \\ &< 2s_0, \end{aligned}$$

so we see that $R = 2s_0$ works for all such points. $\square$

## 3.3 Bowen-Margulis-Sullivan and Burger-Roblin Measures

Recall $\pi : G \to \mathbb{H}^n$ from (3.1). In this section, we will abuse notation and write $\pi$ for the restriction of $\pi$ to $\mathrm{T}^1(\mathbb{H}^n) \cong M\backslash G$. Recalling the fixed reference point $o \in \mathbb{H}^n$ as before, the map

$$w \mapsto (w^+, w^-, s := \beta_{w^-}(o, \pi(w)))$$

is a homeomorphism between $\mathrm{T}^1(\mathbb{H}^n)$ and

$$(\partial(\mathbb{H}^n) \times \partial(\mathbb{H}^n) - \{(\xi, \xi) : \xi \in \partial(\mathbb{H}^n)\}) \times \mathbb{R}.$$

This homeomorphism allows us to define the Bowen-Margulis-Sullivan (BMS) and Burger-Roblin (BR) measures on $\mathrm{T}^1(\mathbb{H}^n)$, denoted by $\tilde{m}^{\mathrm{BMS}}$ and $\tilde{m}^{\mathrm{BR}}$ respectively:

$$d\tilde{m}^{\mathrm{BMS}}(w) := e^{\delta_\Gamma \beta_{w^+}(o,\pi(w))} e^{\delta_\Gamma \beta_{w^-}(o,\pi(w))} d\nu_o(w^+) d\nu_o(w^-) ds,$$

$$d\tilde{m}^{\mathrm{BR}}(w) := e^{(n-1)\beta_{w^+}(o,\pi(w))} e^{\delta_\Gamma \beta_{w^-}(o,\pi(w))} dm_o(w^+) d\nu_o(w^-) ds.$$

The conformal properties of $\{\nu_x\}$ and $\{m_x\}$ imply that these definitions are independent of the choice of $o \in \mathbb{H}^n$. Using the identification of $\mathrm{T}^1(\mathbb{H}^n)$ with $M\backslash G$, we lift the above measures to $G$ so that they are all invariant under $M$ from the left. By abuse of notation, we use the same notation ($\tilde{m}^{\mathrm{BMS}}$ and $\tilde{m}^{\mathrm{BR}}$). These measures are left $\Gamma$-invariant, and hence induce locally finite Borel measures on $X$, which are the Bowen-Margulis-Sullivan measure $m^{\mathrm{BMS}}$ and the Burger-Roblin measure $m^{\mathrm{BR}}$, respectively.

Note that

$$\operatorname{supp} m^{\mathrm{BMS}} := \left\{ x \in X \ : \ x^{\pm} \in \Lambda(\Gamma) \right\}$$

and

$$\operatorname{supp} m^{\mathrm{BR}} = \left\{ x \in X \ : \ x^{-} \in \Lambda(\Gamma) \right\}.$$

Recall $P = MA\tilde{U}$, which is exactly the stabilizer of $w_o^+$ in $G$. We can define another measure $\nu$ on $Pg$ for $g \in G$, which will give us a product structure for $\tilde{m}^{\mathrm{BMS}}$ and $\tilde{m}^{\mathrm{BR}}$ that will be useful in our approach. For any $g \in G$ define

$$d\nu(pg) := e^{\delta_\Gamma \beta_{(pg)^-}(o,pg(o))} d\nu_o(w_o^- pg) dm ds, \tag{3.21}$$

on $Pg$, where $s = \beta_{(pg)^-}(o, pg(o))$, $p = mav \in MA\tilde{U}$ and $dm$ is the probability Haar measure on $M$.

Then for any $\psi \in C_c(G)$ and $g \in G$, we have

$$\tilde{m}^{\mathrm{BMS}}(\psi) = \int_{Pg} \int_U \psi(u_{\mathbf{t}} pg) d\mu_{pg}^{\mathrm{PS}}(\mathbf{t}) d\nu(pg), \tag{3.22}$$

and

$$\tilde{m}^{\mathrm{BR}}(\psi) = \int_{Pg} \int_U \psi(u_{\mathbf{t}} pg) d\mathbf{t} d\nu(pg). \tag{3.23}$$

**Lemma 3.3.1.** *There exists a constant $\lambda = \lambda(\Gamma) > 1$ such that for all $g \in \operatorname{supp} \tilde{m}^{\mathrm{BMS}}$ and all $0 < \varepsilon < \operatorname{inj}(g)$, we have*

$$\begin{aligned} &\lambda^{-1} \varepsilon^{\delta_\Gamma + \frac{1}{2}(n-1)(n-2)+1} e^{(k_2(x,\varepsilon)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log \varepsilon} x))} \\ &\leq \nu(P_\varepsilon g) \\ &\leq \lambda \varepsilon^{\delta_\Gamma + \frac{1}{2}(n-1)(n-2)+1} e^{(k_2(x,\varepsilon)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log \varepsilon} x))}, \end{aligned}$$

*where $x = g\Gamma$, and $k_2(x, \varepsilon)$ is as defined in Proposition 3.2.1.*

*Proof.* Let $x = g\Gamma$. By Proposition 3.2.1, there exists $\tilde{\lambda} > 1$ such that for all such $\varepsilon$,

$$\tilde{\lambda}^{-1} \varepsilon^{\delta_\Gamma} e^{(k_2(x,\varepsilon)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log \varepsilon} x))} \leq \mu_g^{\mathrm{PS}\,-}(B_{\tilde{U}}(\varepsilon)) \leq \tilde{\lambda} \varepsilon^{\delta_\Gamma} e^{(k_2(x,\varepsilon)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log \varepsilon} x))} \tag{3.24}$$

From (3.21), if $m$ denotes the probability Haar measure on $M$ we then have

$$\begin{aligned} \nu(P_\varepsilon g) &\leq \int_{A_\varepsilon} \int_{M_\varepsilon} \mu_g^{\mathrm{PS}\,-}(B_{\tilde{U}}(\varepsilon)) dm ds \\ &\leq C \tilde{\lambda} \varepsilon^{\delta_\Gamma + \frac{1}{2}(n-1)(n-2)+1} e^{(k_2(x,\varepsilon)-\delta_\Gamma) d(\pi(\mathcal{C}_0), \pi(a_{\log \varepsilon} x))}, \end{aligned}$$

where $C$ is determined by the scaling of the probability Haar measures on $A$ and $M$. The lower bound follows similarly. Then, $\lambda = \max\{C\tilde{\lambda}, \tilde{\lambda}\}$ satisfies the conclusion of the lemma. $\square$

## 3.4 Admissible Boxes and Smooth Partitions of Unity

Recall that for $\eta > 0$ we denoted by $G_\eta$ the closed $\eta$-neighborhood of $e$ in $G$. For $x \in X$ and $\eta_1 > 0$, $\eta_2 \geq 0$ less than $\operatorname{inj}(x)$, we call

$$B = B_U(\eta_1)P_{\eta_2}x$$

an *admissible box* (with respect to the PS measure) if $B$ is the injective image of $B_U(\eta_1)P_{\eta_2}$ in $X$ under the map $h \mapsto hx$ and

$$\mu^{\mathrm{PS}}_{px}(B_U(\eta_1)px) \neq 0$$

for all $p \in P_{\eta_2}$. For $g \in G$, we say that $B = B_U(\eta_1)P_{\eta_2}g$ is an admissible box if $B = B_U(\eta_1)P_{\eta_2}g\Gamma$ is one.

Note that if $B_U(\eta_1)P_{\eta_2}g$ is an admissible box, then there exists $\varepsilon > 0$ such that $B_U(\eta_1+\varepsilon)P_{\eta_2+\varepsilon}g$ is also an admissible box. Moreover, every point has an admissible box around it by [OS13, Lemma 2.17].

The error terms in our main theorems are in terms of Sobolev norms, which we define here. For $\ell \in \mathbb{N}$, $1 \leq p \leq \infty$, and $\psi \in C^\infty(X) \cap L^p(X)$ we consider the following Sobolev norm

$$S_{p,\ell}(\psi) = \sum \|U\psi\|_p$$

where the sum is taken over all monomials $U$ in a fixed basis of $\mathfrak{g} = \operatorname{Lie}(G)$ of order at most $\ell$, and $\|\cdot\|_p$ denotes the $L^p(X)$-norm. Since we will be using $S_{2,\ell}$ most often, we set

$$S_\ell = S_{2,\ell}.$$

Our proofs will require constructing smooth indicator functions and partitions of unity with controlled Sobolev norms. We prove such lemmas below.

**Lemma 3.4.1** ([KM96, Lemma 2.4.7])**.**

1. *Let $X, Y$ be Riemannian manifolds, and let $\varphi \in C_c^\infty(X)$, $\psi \in C_c^\infty(Y)$. Consider $\varphi \cdot \psi$ as a function on $X \times Y$. Then*

$$S_\ell(\varphi \cdot \psi) \leq c(X, Y) S_\ell(\varphi) S_\ell(\psi),$$

*where $c(X, Y)$ is a constant depending only on $X$ and $Y$ (independent of $\varphi, \psi$).*

2. *Let $X$ be a Riemannian manifold of dimension $N$ and let $x \in X$. Then for any $0 < r < 1$, there exists a non-negative function $f \in C_c^\infty(X)$ such that $\operatorname{supp}(f)$ is contained in the ball of radius $r$ centered at $x$, $\int_X f = 1$, and*

$$S_\ell(f) \leq c(X, x) r^{-\ell + N/2},$$

*where $c(X, x)$ is a constant depending only on $X$ and $x$, not $r$.*

**Lemma 3.4.2.** *Let $H$ be a horospherical subgroup of $G$ (that is, $U$ or $\tilde{U}$). For every $\xi_1, \xi_2 > 0$ and $g \in G$, there exists a non-negative smooth function $\chi_{\xi_1,\xi_2}$ defined on $H_{\xi_1+\xi_2} g$ such that $0 \leq \chi_{\xi_1,\xi_2} \leq 1$, $S_\ell(\chi_{\xi_1,\xi_2}) \ll_{n,\Gamma} \xi_1^{n-1} \xi_2^{-\ell-(n-1)/2}$, and*

$$\chi_{\xi_1,\xi_2}(h) = \begin{cases} 0 & \text{if } h \notin H_{\xi_1+\xi_2} g \\ \\ 1 & \text{if } h \in H_{\xi_1-\xi_2} g \end{cases}.$$

*Proof.* According to Lemma 3.4.1(2), there exists $c_1 = c_1(n) > 0$ such that for every $\xi > 0$, there exists a non-negative smooth function $\sigma_\xi$ defined on $H_\xi$ such that

$$\int_H \sigma_\xi(h) dm^{\mathrm{Haar}}(h) = 1, \quad S_\ell(\sigma_\xi) < c_1 \xi^{-\ell-(n-1)/2}. \tag{3.25}$$

For $g \in \Omega$, let $\chi_{\xi_1,\xi_2} = \mathbf{1}_{H_{\xi_1} g} * \sigma_{\xi_2}$. Then for any $h \in H$, we have $0 \leq \chi_{\xi_1,\xi_2}(h) \leq 1$ and

$$\chi_{\xi_1,\xi_2}(h) = \begin{cases} 0 & \text{if } h \notin H_{\xi_1+\xi_2} g \\ \\ 1 & \text{if } h \in H_{\xi_1-\xi_2} g \end{cases}$$

Since for some $c_2 = c_2(\Gamma) > 0$

$$S_{1,0}(\mathbf{1}_{H_{\xi_1} g_0}) = m^{\mathrm{Haar}}(H_{\xi_1}) < c_2 {\xi_1}^{n-1},$$

by the properties of the Sobolev norm and (3.25) we arrive at

$$S_\ell(\chi_{\xi_1,\xi_2}) \leq S_{1,0}(\mathbf{1}_{H_{\xi_1} g_0}) S_\ell(\sigma_{\xi_2}) < c_1 c_2 {\xi_1}^{n-1} {\xi_2}^{-\ell-(n-1)/2}.$$

□

**Lemma 3.4.3.** *Let $H$ be a horospherical subgroup of $G$, $r > 0$, $\ell \in \mathbb{N}$, and let $E \subset H$ be bounded. Then, there exists a partition of unity $\sigma_1, \ldots, \sigma_k$ of $E$ in $H_r E$, i.e.*

$$\sum_{i=1}^{k} \sigma_i(x) = \begin{cases} 0 & \textit{if } x \notin H_r E \\ \\ 1 & \textit{if } x \in E, \end{cases}$$

*such that for some $u_1, \ldots, u_k \in E$ and all $1 \leq i \leq k$*

$$\sigma_i \in C_c^\infty(H_r u_i), \quad S_\ell(\sigma_i) \ll_n r^{-\ell+n-1}.$$

*Moreover, if there exists $R > r$ such that $E = H_R$, then $k \ll_n \left(\frac{R}{r}\right)^{n-1}$.*

*Proof.* Let $\{u_1, \ldots, u_k\}$ be a maximal $\frac{r}{4}$-separated set in $E$. Then

$$E \subseteq \bigcup_{i=1}^{k} H_{r/2} u_i. \tag{3.26}$$

Let $1 \leq i \leq k$. According to [HÖ3, Theorem 1.4.2] there exists $\chi_i \in C_c^{\infty}(H_r u_i)$ such that $0 \leq \chi_i \leq 1$, $\chi_i(u) = 1$ for any $u \in H_{r/2} u_i$, and for $1 \leq m \leq \ell$

$$|\chi_i^{(m)}| \ll r^{-m} \tag{3.27}$$

(where the implied constant depends only on $n$). Let $\sigma_i$ be defined by

$$\sigma_i = \chi_i(1 - \chi_{i-1}) \cdots (1 - \chi_1).$$

Then, each $\sigma_i \in C_c^{\infty}(H_r u_i)$ and

$$1 - \sum_{i=1}^{k} \sigma_i = \prod_{i=1}^{k} (1 - \chi_i) = 0 \text{ on } \bigcup_{i=1}^{k} H_r u_i$$

implies that $\sum_{i=1}^{k} \sigma_i = 1$ on $\bigcup_{i=1}^{k} H_{r/2} u_i$.

By the rules for differentiating a product and (3.27) for $1 \leq m \leq \ell$ we have

$$|\sigma_i^{(m)}| \leq C r^{-m},$$

where $C$ is the multiplicity of the cover in (3.26). By Besicovitch covering theorem, $C$ is bounded by a constant which depends only on $n$. Using the definition of the Sobolev norm we arrive at

$$S_\ell(\sigma_i) \ll_n r^{-\ell+n-1}.$$

Now, assume there exists $R > r$ such that $E = H_R$. Since the geometry of $H$ is of an Euclidean space of dimension $\dim H$, we then have

$$k \ll_n \left(\frac{R}{r}\right)^{n-1}.$$

$\square$

**Lemma 3.4.4.** *Let $H$ be either $U$ or $G$. There exists $\ell' = \ell'(H) > 0$ such that for any integer $\ell > \ell'$, $\eta > 0$, $H \in \{U, G\}$, and $f \in C_c^\infty(H)$, there exist functions $f_{\eta,\pm} \in C_c^\infty(H)$ which are supported on an $2\eta$ neighborhood of* $\operatorname{supp} f$*, and for any $h \in H$ satisfy*

1. $f_{\eta,-}(h) \leq \min_{w\in H_\eta} f(wh) \leq \max_{w\in H_\eta} f(wh) \leq f_{\eta,+}(h)$
2. $|f_{\eta,\pm}(h) - f(h)| \ll_{\operatorname{supp} f} \eta S_\ell(f)$
3. $S_\ell(f_{\eta,\pm}) \ll_{H,\operatorname{supp} f} \eta^{-2\ell} S_\ell(f)$.

*Proof.* First, according to [Aub82], there exists $\ell' \in \mathbb{N}$ such that any $\ell > \ell'$ satisfies $S_{\infty,1}(\psi) \ll_{\operatorname{supp}\psi} S_\ell(\psi)$ for any $\psi \in C_c^\infty(H)$.

Let $f'_{\eta,\pm}$ be defined by

$$f'_{\eta,+}(h) := \sup_{w\in H_\eta} f(wh) \text{ and } f'_{\eta,-}(h) := \inf_{w\in H_\eta} f(wh)$$

for any $h \in H$.

As before, we use Lemma 3.4.1(2) to deduce that there exist $c_1 = c_1(H) > 0$, $n_1 = n_1(H)$ and a non-negative smooth function $\sigma_\eta$ supported on $H_\eta$ such that

$$\int_H \sigma_\eta(h) dm^{\text{Haar}}(h) = 1, \quad S_\ell(\sigma_\eta) < c_1 \eta^{-\ell-n_1}.$$

Define $f_{\eta,\pm}$ by

$$f_{\eta,\pm} := f'_{2\eta,\pm} * \sigma_\eta.$$

Then, $f_{\eta,\pm}$ are smooth functions which are supported on an $2\eta$ neighborhood of $\operatorname{supp} f$.

Moreover, for any for any $h \in H$

$$\begin{aligned}
f'_{\eta,+}(h) &= \int_{H_\eta} f'_{\eta,+}(h)\sigma_\eta(u^{-1})dm^{\mathrm{Haar}}(u) \\
&\leq \int_{H_\eta} f'_{2\eta,+}(uh)\sigma_\eta(u^{-1})dm^{\mathrm{Haar}}(u) && \text{by definition of } f'_{2\eta,+} \qquad (3.28) \\
&= f_{\eta,+}(h) \\
&\leq \int_{H_\eta} f'_{3\eta,+}(h)\sigma_\eta(u^{-1})dm^{\mathrm{Haar}}(u) && \text{by (3.28) and definition of } f'_{3\eta,+} \\
&= f'_{3\eta,+}(h).
\end{aligned}$$

In a similar way, one can show

$$f'_{3\eta,-} \leq f_{\eta,-} \leq f'_{\eta,-},$$

proving the first inequality.

By the mean value theorem, for any $h \in H$, $w \in H_{3\eta}$

$$|f(wh) - f(h)| \ll \eta S_{\infty,1}(f) \ll_{\operatorname{supp} f} S_\ell(f).$$

Since $f'_{3\eta,-} \leq f_{\eta,-} \leq f_{\eta,+} \leq f'_{3\eta,+}$, there exist some $w_+, w_- \in H_{3\eta}$ such that

$$|f_{\eta,\pm}(h) - f(h)| \leq |f(w_\pm h) - f(h)|,$$

and we have the second inequality.

Now, we have

$$S_\ell(f_{\eta,\pm}) \leq S_{\infty,1}(f'_{2\eta,\pm})S_\ell(\sigma_\eta) \ll_{H,\operatorname{supp} f} S_\ell(f)\eta^{-\ell-n_1+1}.$$

By choosing $\ell' > n_1$, we may deduce the last inequality. $\square$

The results below will be necessary for our application to understanding the orbits of a geometrically finite subgroup $\Gamma \subseteq \mathrm{SO}(n,1)^\circ$ acting on $U\backslash G$. The following lemma is an immediate consequence of the product rule.

**Lemma 3.4.5.** *Let $X$ be a Riemannian manifold and let $\varphi, \psi \in C_c^\infty(X)$. For any $\ell \in \mathbb{N}$,*

$$S_\ell(\varphi \cdot \psi) \ll_\ell S_\ell(\varphi) S_\ell(\psi).$$

**Lemma 3.4.6.** *For any $\ell'$ there exists $\ell > \ell'$ which satisfies the following. Let $X, Y$ be Riemannian manifolds, $\varphi \in C_c^\infty(X)$, and $\psi : Y \to X$ be a smooth function. Then*

$$S_{\ell'}(\varphi \circ \psi) \ll_{\ell', \psi} S_\ell(\varphi).$$

*Proof.* By the chain rule, such that for any $1 \leq k \leq \ell'$,

$$\begin{aligned}
\left\|(\varphi \circ \psi)^{(k)}\right\|_2 &\ll_{\psi,k} \sum_{i=0}^{k} \left\|\varphi^{(k)} \circ \psi\right\|_2 \\
&\ll_{\psi,k} \sum_{i=0}^{k} \left\|\varphi^{(k)}\right\|_\infty m^{\mathrm{Haar}}(\operatorname{supp}\varphi) \\
&\ll_{\psi,k} S_{\infty,\ell'}(\varphi) m^{\mathrm{Haar}}(\operatorname{supp}\varphi) \\
&\ll_{\psi,k} S_\ell(\varphi),
\end{aligned}$$

where in the last line, we have used [Aub82] to choose $\ell > \ell'$ satisfying

$$S_{\infty,\ell'}(f) m^{\mathrm{Haar}}(\operatorname{supp} f) \ll S_\ell(f)$$

for any $f$, where the implied constant is global. $\square$

**Lemma 3.4.7.** *Let $H$ be a Riemannian manifold of dimension $N$, $0 < r < 1$, $\ell \in \mathbb{N}$, and $E$ a bounded subset of $H$. Then, there exists a partition of unity $\sigma_1, \ldots, \sigma_k$ of $E$ in $H_r(E) = \{g \in G : d_H(g, E) \leq r\}$ where $d_H$ denotes the Riemannian metric on $H$, i.e.*

$$\sum_{i=1}^{k} \sigma_i(x) = \begin{cases} 0 & \text{if } x \notin H_r(E) \\ \\ 1 & \text{if } x \in E, \end{cases}$$

*such that for some $u_1, \ldots, u_k \in E$ and all $1 \leq i \leq k$*

$$\sigma_i \in C_c^\infty(H_r(u_i)), \quad S_\ell(\sigma_i) \ll_N r^{-\ell+N/2}.$$

*Moreover,*

$$\sum_{i=1}^{k} S_\ell(\sigma_i) \ll_{N,E} r^{-\ell+N/2}.$$

*Proof.* According to Lemma 3.4.1(2) there exists a non-negative smooth function $\sigma$ supported on $H_{r/2}$ such that

$$\int_H \sigma(h) dm^{\text{Haar}}(h) = 1, \quad S_\ell(\sigma) \ll_N r^{-\ell+N/2}.$$

Since $H$ is a Riemannian manifold and $E$ is bounded, there exists a smooth partition of unity, $f_i : H \to \mathbb{R}$, $i = 1, \ldots, k$, such that each $f_i$ is supported on a ball of radius $r/2$ with a center $u_i \in E$ and

$$\sum_{i=1}^{k} f_i(x) = \begin{cases} 0 & \text{if } x \notin H_r(E) \\ \\ 1 & \text{if } x \in H_{r/2}(E). \end{cases}$$

For $i = 1, \ldots, k$ define $\sigma_i$ by

$$\sigma_i := f_i * \sigma.$$

We will show that $\sigma_1, \ldots, \sigma_k$ satisfy the claim.

By definition, for $i = 1, \ldots, k$, $\sigma_i$ is supported on a ball of radius $r$ and centered at a point in $E$. By Young's convolution inequality, we have

$$S_\ell(\sigma_i) \le S_{1,0}(f_i) S_\ell(\sigma) \ll_N r^{-\ell+N/2}. \tag{3.29}$$

For any $h \in E$, $h^{-1}E$ contains the identity, and so we have $h^{-1}H_{r/2}(E) \supseteq H_{r/2}$. Thus,

$$\begin{aligned}
\sum_{i=1}^{k} \sigma_i(h) &= \sum_{i=1}^{k} \int_H f_i\,(x)\,\sigma(hx^{-1}) dm^{\text{Haar}}(x) \\
&= \int_H \sum_{i=1}^{k} f_i\,(x)\,\sigma\left(hx^{-1}\right) dm^{\text{Haar}}(x) \\
&= \int_{H_r(E)} \sigma\left(hx^{-1}\right) dm^{\text{Haar}}(x) \\
&= 1.
\end{aligned}$$

If $h \notin H_r(E)$, then we have $h^{-1}H_{r/2}(E) \cap H_{r/2} = \emptyset$. Hence, the above computation yields

$$\sum_{i=1}^{k} \sigma_i(h) = 0.$$

Note that by (3.29), and since $f_i$ is a partition of unity, we may also deduce

$$\begin{aligned}
\sum_{i=1}^{k} S_\ell(\sigma_i) &\le S_\ell(\sigma) \sum_{i=1}^{k} S_{1,0}(f_i) \\
&= S_\ell(\sigma) \int_H \sum_{i=1}^{k} f_i(x) dm^{\text{Haar}}(x) \\
&\le S_\ell(\sigma) m^{\text{Haar}}(H_r(E)) \\
&\ll_{N,E} r^{-\ell+N/2}.
\end{aligned}$$

$\square$

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This chapter contains material from the following, which has been submitted for publication: N. Tamam, J. M. Warren, *"Effective equidistribution of horospherical flows in infinite volume rank one homogeneous spaces"*, arXiv:2007.03135. The dissertation author was one of the primary investigators and authors of this paper, and was supported in part by the National Science and Engineering Research Council of Canada (NSERC) PGSD3-502346-2017 during this work.

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# Chapter 4

# Quantitative Nondivergence

In this chapter, we prove a quantitative nondivergence result that is crucial in our proofs. We use the notation established in §3.2. The results in this chapter hold for any $\Gamma$ that is geometrically finite, without need for Assumption 1.1.2.

Recall from Chapter 1 that for $0 < \varepsilon < 1$ and $s_0 \geq 1$, we say that $x \in X = G/\Gamma$ is $(\varepsilon, s_0)$-*Diophantine* if for all $\tau > s_0$,

$$d(\mathcal{C}_0, a_{-\tau}x) < (1 - \varepsilon)\tau, \tag{4.1}$$

where $\mathcal{C}_0$ is the compact set defined in §3.2. Let $R_0$ and $q$ also be as defined in §3.2.

This chapter is dedicated to the proof of the following theorem, which says (in a quantitative way) that most of the $U$ orbit of a Diophantine point is not in the cusp:

**Theorem 4.0.1.** *There exists* $\beta > 0$ *satisfying the following: for every* $0 < \varepsilon < 1$ *and* $s_0 \geq 1$, *and for every* $(\varepsilon, s_0)$*-Diophantine element* $x \in X$, *every* $R \geq R_0$, *every $T \gg_{\Gamma,\varepsilon} s_0$, and every $0 < s \leq T^{\varepsilon}$, we have*

$$\mu^{\mathrm{PS}}_{a_{-\log s}x}\left(B_U(T/s)a_{-\log s}x \cap \mathcal{X}(R)\right) \ll_{n,\Gamma} \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T/s)a_{-\log s}x)e^{-\beta R}.$$

We now follow the notation of Mohammadi and Oh in [MO20, §6]. Equip $\mathbb{R}^{n+1}$ with the Euclidean norm. Recall from §3.2 that for $1 \leq i \leq q$, $g_i^- = \xi_i$. Without loss of generality, we may further assume that $g_i$ satisfies $\left\|g_i^{-1}e_1\right\| =\ 1$. Let

$$v_i = g_i^{-1}e_1.$$

**Lemma 4.0.2.** *For any $i = 1, \ldots, q$, $\Gamma v_i$ is a discrete subset of $\mathbb{R}^{n+1}$.*

*Proof.* Since $\xi_i$ is assumed to be a bounded parabolic limit point, by definition we have that $(\Lambda(\Gamma) \setminus \{\xi_i\})/\Gamma_{\xi_i} = (\Lambda(\Gamma) \setminus \{\xi_i\})/\Gamma_{v_i}$ is compact, where

$$G_{v_i} = g_i^{-1}MUg_i \quad \text{ and } \quad \Gamma_{v_i} = \Gamma \cap G_{v_i}.$$

If $\gamma \in \Gamma_{v_i}$, then $\mathcal{H}_i(R_0)\gamma = \mathcal{H}_i(R_0)$. Therefore, the visual map induces a homeomorphism between $\mathcal{H}_i(R_0)/\Gamma_{v_i}$ and $(\partial\mathbb{H}^n \setminus \{\xi_i\})/\Gamma_{v_i}$. It follows that the quotient of $\{g^+ \in \Lambda : g \in \mathcal{H}_i(R_0)\}$ by the action of $\Gamma_{v_i}$ is compact. Using the Iwasawa decomposition, it follows that there exists a compact set $U_0 \subset U$ such that for any $g = kaug_i \in \mathcal{H}_i(R_0)$ such that $g^+ \in \Lambda(\Gamma)$, $k \in K$, $a \in A$, and $u \in U$, there exist $\gamma \in \Gamma_{v_i}$, $k' \in K$, $u' \in U_0$ so that $g\gamma = k'au'g_i$.

Since $\xi_i$ is assumed to be a parabolic limit point, there exists a parabolic element $\gamma_0 \in \Gamma_{\xi_i}$, i.e. $\gamma_0 = g_i^{-1}mug_i$.

Assume by contradiction that there exists an infinite sequence $\{\gamma_j\} \in \Gamma$ such that $\{\gamma_j v_i\}$ converges to 0. Using the Iwasawa decomposition we get that for all $j$ there exist $a_{t_j} \in A$, $k_j \in K$, and uniformly bounded $u_j \in U$ such that $\gamma_j = k_j a_{t_j} u_j g_i$. Since

$$\|\gamma_j v_i\| = \left\|k_j a_{t_j} u_j e_1\right\| = e^{t_j},$$

we may deduce that $t_j \to -\infty$. In particular, $\gamma_j \in H_i(R_0)$ for all large enough $j$.

We have

$$\begin{aligned}\gamma_j \gamma_0 \gamma_j^{-1} &= (k_j a_{t_j} u_j g_i)(g_i^{-1} m u g_i)(g_i^{-1} u_j^{-1} a_{t_j}^{-1} k_j^{-1}) \\ &= k_j a_{t_j} u_j m u u_j^{-1} a_{t_j}^{-1} k_j^{-1}.\end{aligned}$$

Since $u_j m u_j^{-1} = m_j u_j' \in MU$, with $u_j'$ uniformly bounded, and since $M$ centralizes $A$, we have

$$\gamma_j \gamma_0 \gamma_j^{-1} = k_j m_j a_{t_j} u_j' u a_{t_j}^{-1} k_j^{-1}.$$

Since $u_j' u$ is in a bounded subset of $U$, we get that $a_{t_j} u_j' u a_{t_j}^{-1} \to e$ as $t_j \to -\infty$. Since $K$ and $M$ are compact, it then follows that the sequence $\gamma_j \gamma_0 \gamma_j^{-1}$ has a convergent subsequence. This contradicts the discreteness of $\Gamma$, since the $\gamma_j$'s were assumed to be distinct.

$\square$

For any $g \in G$, we have that $g\Gamma \in \mathcal{X}_i(R)$, if and only if there exists $\gamma \in \Gamma$ such that

$$\|g\gamma v_i\| \leq e^{-R}. \tag{4.2}$$

Indeed, by the Iwasawa decomposition and (3.14), if $g\Gamma \in \mathcal{X}_i(R)$, then there exist $\gamma \in \Gamma$, $k \in K$, $s > R$, and $u \in U$, such that

$$\|g\gamma v_i\| = \|ka_{-s}ug_iv_i\| = \|a_{-s}e_i\| = e^{-s}.$$

Moreover, it follows from [MO20, Lemma 6.4, Lemma 6.5] that the $\gamma$ in (4.2) is unique. Note that both lemmas are proved under the additional assumption that $n = 3$, but this assumption is not needed in their proof.

On the other hand, by [MO20, Lemma 6.5] and Lemma 4.0.2, there exists a constant $\eta_0 = \eta_0(\Gamma) > 0$ such that if $g\Gamma \notin \mathcal{X}_i(R_0)$, then for any $\gamma \in \Gamma$,

$$\|g\gamma v_i\| > \eta_0. \tag{4.3}$$

**Lemma 4.0.3.** *There exists $c = c(\Gamma) > 0$ which satisfies the following. Let $\varepsilon, s_0 > 0$ and let $g \in G$. If $x = g\Gamma$ is $(\varepsilon, s_0)$-Diophantine, then for any $T \gg_{\Gamma,\varepsilon} s_0$,*

$$\sup_{\|\boldsymbol{t}\| \leq T} \inf_{\gamma \in \Gamma} \inf_{i=1,\ldots,q} \|u_{\boldsymbol{t}} g\gamma v_i\| > cT^{\varepsilon}. \tag{4.4}$$

*Proof.* Fix $T > T_0 = \max\left\{s_0, \eta_0^{\frac{1}{\varepsilon - 1}}\right\}$. We will first show that

$$\inf_{\gamma \in \Gamma} \inf_{i=1,\ldots,q} \|a_{-\log T} g\gamma v_i\| > cT^{\varepsilon-1}, \tag{4.5}$$

for some constant $1 > c = c(\Gamma) > 0$.

There are two cases to consider. If $a_{-\log T}x \notin \mathcal{X}_i(R_0)$, then (4.5) follows from (4.3) and the choice of $T$.

Otherwise, $a_{-\log T}x \in \mathcal{X}_i(R)$ for some maximal $R > R_0$. According to Lemma 3.2.3, we have

$$d(x, \mathcal{C}_0) \geq R - R_0.$$

Then, because $x$ is $(\varepsilon, s_0)$ Diophantine and $T > s_0$, by (4.1), we may deduce that

$$R - R_0 < (1 - \varepsilon) \log T.$$

Hence, $a_{-\log T}x \notin \mathcal{X}_i((1 - \varepsilon) \log T + R_0)$, so (4.2) implies (4.5).

Now, fix $\gamma \in \Gamma$ and $1 \leq i \leq q$, and let

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} = a_{-\log T} g\gamma v_i.$$

According to (4.5), there exists $1 \leq k \leq n$ such that $|x_k| > cT^{\varepsilon-1}$. If $|x_1| > cT^{\varepsilon-1}$, then it follows from the action of $a_{-\log T}$ on $\mathbb{R}^{n+1}$ that

$$\|g\gamma v_i\| \geq |cTx_1| > cT^{\varepsilon}.$$

Otherwise, there exists $2 \leq k \leq n$ such that $|x_k| > cT^{\varepsilon-1}$. Then, for any $\mathbf{t} \in \mathbb{R}^{n-1}$, the first coordinate of $u_{\mathbf{t}} a_{-\log T} g\gamma v_i$ is

$$x_1 + \mathbf{t} \cdot \mathbf{x}' + \frac{1}{2} \|\mathbf{t}\|^2 x_{n+1}, \quad \text{where} \quad \mathbf{x}' = \begin{pmatrix} x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

In particular, by taking $t_k = \pm T$ (the $k$-th entry in $\mathbf{t}$) one can ensure that

$$\|a_{\log T} u_{\mathbf{t}} a_{-\log T} g\gamma v_i\| > cT^{\varepsilon}.$$

$\square$

A measure $\mu$ is called *D-Federer* if for all $v \in \operatorname{supp}(\mu)$ and $0 < \eta \leq 1$,

$$\mu(B(v, 3\eta)) \leq D\mu(B(v, \eta)).$$

It is proved in the appendix, Chapter 10, (specifically Corollary 10.1.4) that there exists $D = D(\Gamma) > 0$ such that:

$$\text{if } x \in X \text{ satisfies } x^- \in \Lambda(\Gamma)\text{, then } \mu_x^{\mathrm{PS}} \text{ is } D\text{-Federer.} \tag{4.6}$$

For $f : \mathbb{R}^d \to \mathbb{R}$ and $B \subset \mathbb{R}$, let

$$\|f\|_B := \sup_{x \in B} |f(x)|.$$

Recall that $U \cong \mathbb{R}^{n-1}$.

**Lemma 4.0.4.** *Let $y \in \operatorname{supp} m^{\mathrm{BMS}}$ and let $f : B_U(\eta) \to \mathbb{R}^{n-1}$ be such that there exists $b \neq 0$ so that for every coordinate function $f_i : B_U(\eta) \to \mathbb{R}$, there exist $a_i \in \mathbb{R}$, such that*

$$f_i(\boldsymbol{t}) = a_i + b\boldsymbol{t}_i.$$

*Then for $0 < \eta \leq 1$ and $0 < \varepsilon < 1$, we have*

$$\mu_y^{\mathrm{PS}}\left(\{\boldsymbol{t} \in B_U(\eta) : \|f(\boldsymbol{t})\| < \varepsilon\}\right) \ll_\Gamma \left(\frac{\varepsilon}{\|f\|_{B_U(\eta)}}\right)^\sigma \mu_y^{\mathrm{PS}}(B_U(\eta)), \tag{4.7}$$

*where $\|f(x)\|$ denotes the max norm.*

*Proof.* First, note that if $\|f\|_{B_U(\eta)} < 2\varepsilon$, then the result holds by assuming that the implied coefficient in (4.7) is bigger than $2^\sigma$: in this case, the right hand side is grater or equal to

$$2^\sigma \left(\frac{\varepsilon}{\|f\|_{B_U(\eta)}}\right)^\sigma \mu_y^{\mathrm{PS}}(B_U(\eta)) \geq 2^\sigma \left(\frac{\varepsilon}{2\varepsilon}\right)^\sigma \mu_y^{\mathrm{PS}}(B_U(\eta))$$
$$\geq \mu_y^{\mathrm{PS}}\left(\{\mathbf{t} \in B_U(\eta) : \|f(\mathbf{t})\| < \varepsilon\}\right),$$

as desired. Thus, we now assume that

$$\|f\|_{B_U(\eta)} \geq 2\varepsilon. \tag{4.8}$$

If $\|f(\mathbf{t})\| \geq \varepsilon$ for all $\mathbf{t} \in B_U(\eta)$ such that $(u_{\mathbf{t}}y)^+ \notin \Lambda(\Gamma)$, then there is nothing to prove. So assume that $\|f(\mathbf{t})\| < \varepsilon$ and $(u_{\mathbf{t}}y)^+ \in \Lambda(\Gamma)$. Since each $f_i$ is linear, for all $\mathbf{t}' \in B_U(\eta)$ with $\|f(\mathbf{t}')\| < \varepsilon$ we get that for all $1 \leq i \leq n-1$,

$$|f_i(\mathbf{t}')| = |a_i + b\mathbf{t}'| < \varepsilon$$

$$|b(t_i' - t_i)| = |f_i(\mathbf{t}') - f_i(\mathbf{t})| < 2\varepsilon.$$

Therefore,

$$\|f(x)\| < \varepsilon \implies x \in B_U(2\varepsilon/b)z.$$

Thus, by (4.6), we have that there exists $\sigma = \sigma(\Gamma) > 0$ so that

$$\mu_y^{\mathrm{PS}}(\{x \in B_U(\eta)y : \|f(x)\| < \varepsilon\}) \leq \mu_z^{\mathrm{PS}}(B_U(2\varepsilon/b))$$
$$\ll_\Gamma \left(\frac{2\varepsilon}{b\eta}\right)^\sigma \mu_z^{\mathrm{PS}}(B_U(\eta))$$
$$\ll_\Gamma \left(\frac{2\varepsilon}{b\eta}\right)^\sigma \mu_y^{\mathrm{PS}}(B_U(3\eta))$$
$$\ll_\Gamma \left(\frac{6\varepsilon}{b\eta}\right)^\sigma \mu_y^{\mathrm{PS}}(B_U(\eta))$$

Assuming $\|f(\mathbf{t})\| < \varepsilon$ for some $\mathbf{t} \in B_U(\eta)$ (otherwise, as before, there is nothing to prove), for any $\mathbf{t}'' \in B_U(\eta)$ and $1 \le i \le n-1$ we have

$$|f_i(\mathbf{t}'')| \le |f_i(\mathbf{t}'') - f_i(\mathbf{t})| + |f_i(\mathbf{t})| < 2b\eta + \varepsilon.$$

Thus, $\|f\|_{B_U(\eta)y} - \varepsilon \le 2b\eta$, so by (4.8),

$$\frac{1}{2}\|f\|_{B_U(\eta)} \le 2b\eta,$$

which completes the proof. □

A function $f$ which satisfies (4.7) with the implied constant $C$ for any $\varepsilon > 0$ and any ball $B \subset U \subset \mathbb{R}^m$, is called $(C,\sigma)$-*good on* $U$ *with respect to* $\mu$. Observe that

$$\text{if } g \text{ is } (C,\sigma)\text{-good and if } |g(x)| \le |f(x)| \text{ for } \mu\text{-a.e. } x, \text{ then } f \text{ is } (C,\sigma)\text{-good.} \tag{4.9}$$

In the proof of the following theorem we use similar ideas to the ones which appear in the proof of [KLW04, Lemma 5.2]. Note that the proof in this case is simplified by the third assumption, reflecting our rank one setting.

**Proposition 4.0.5.** *Given positive constants $C, \beta, D$, and $0 < \eta < 1$, there exists $C' = C'(C,\beta,D) > 0$ with the following property. Suppose $\mu$ is a $D$-Federer measure on $\mathbb{R}^m$, $f : \mathbb{R}^m \to \mathrm{SL}_k(\mathbb{R})$ is a continuous map, $0 \le \varrho \le \eta$, $z \in \operatorname{supp}\mu$, $\Lambda \subset \mathbb{R}^k$, $B = B(z, r_0) \subset \mathbb{R}^m$, and $\tilde{B} = B(z, 3r_0)$ satisfy:*

1. *For any $v \in \Lambda$, the function $\boldsymbol{t} \mapsto \|f(\boldsymbol{t})v\|$ is $(C,\beta)$-good on $\tilde{B}$ with respect to $\mu$.*

2. *For any $v \in \Lambda$, there exists $\boldsymbol{t} \in B$ such that $\|f(\boldsymbol{t})v\| \ge \varrho$.*

*3. For any* $\boldsymbol{t} \in B$*, there is at most one* $v \in \Lambda$ *which satisfies* $\|f(\boldsymbol{t})v\| < \eta$*.*

*Then, for any* $0 < \varepsilon < \varrho$*,*

$$\mu\left(\{\boldsymbol{t} \in B \,:\, \exists v \in \Lambda \text{ such that } \|f(\boldsymbol{t})v\| < \varepsilon\}\right) \leq C' \left(\frac{\varepsilon}{\varrho}\right)^{\beta} \mu(B).$$

*Proof.* For any $\mathbf{t} \in B$, denote

$$f_\Lambda(\mathbf{t}) = \min\left\{\|f(\mathbf{t})v\| \,:\, v \in \Lambda\right\}.$$

Let

$$E = \{\mathbf{t} \in B \,:\, f_\Lambda(\mathbf{t}) < \varrho\} \cap \operatorname{supp} \mu,$$

and for each $v \in \Lambda$, define

$$E_v = \{\mathbf{t} \in B \,: \|f(\mathbf{t})v\| < \varrho\} \cap \operatorname{supp} \mu.$$

Observe that by assumption (3), the $E_v$'s are a disjoint cover of $E$. For each $t \in E_v$, define

$$r_{\mathbf{t},v} = \sup\{r \,:\, \|f(\mathbf{s})v\| < \varrho \text{ for all } \mathbf{s} \in B(\mathbf{t}, r_{\mathbf{t},v})\}.$$

By assumption (2), we know that for every $\mathbf{t} \in E$, the set $B(\mathbf{t}, r_{\mathbf{t},v})$ does not contain $B$. Thus, since $\mathbf{t} \in B$, we deduce that $r_{\mathbf{t},v} < 2r_0$. For any fixed $r_{\mathbf{t},v} < r'_{\mathbf{t},v} < 2r_0$, we have that

$$B(\mathbf{t}, r'_{\mathbf{t},v}) \subset B(z, 3r_0) = \tilde{B}, \tag{4.10}$$

and by the definition of $r_{\mathbf{t},v}$, there exists $\mathbf{s} \in B(\mathbf{t}, r'_{\mathbf{t},v})$ such that

$$\|f(\mathbf{s})v\| \geq \varrho.$$

Note that $\{B(\mathbf{t}, r_{\mathbf{t},v}) : \mathbf{t} \in E, v \in \Lambda\}$ is a cover of $E$. By the Besicovitch covering theorem, there exists a countable subset $I \subset E \times \Lambda$ such that $\{B(\mathbf{t}, r_{\mathbf{t},v}) : (\mathbf{t}, v) \in I\}$ is a cover of $E$ with a covering number bounded by a constant which only depends on $m$. Thus,

$$\sum_{(\mathbf{t},v)\in I} \mu\left(B(\mathbf{t}, r_{\mathbf{t},v})\right) \ll_m \mu\left(\bigcup_{(\mathbf{t},v)\in I} B(\mathbf{t}, r_{\mathbf{t},v})\right). \tag{4.11}$$

By assumption (3) and the continuity of $f$, for any $(\mathbf{t}, v) \in I$ and $\mathbf{s} \in E \cap B(\mathbf{t}, r_{\mathbf{t},v})$,

$$f_\Lambda(\mathbf{s}) = \|f(\mathbf{s})v\|.$$

Thus,

$$\begin{aligned}\mu\left(\{\mathbf{s} \in B(\mathbf{t}, r_{\mathbf{t},v}) \,:\, f_\Lambda(\mathbf{s}) < \varepsilon\}\right) &= \mu\left(\{\mathbf{s} \in B(\mathbf{t}, r_{\mathbf{t},v}) \,:\, \|f(\mathbf{s})v\| < \varepsilon\}\right) \\ &\le \mu\left(\left\{\mathbf{s} \in B(\mathbf{t}, r'_{\mathbf{t},v}) \,:\, \|f(\mathbf{s})v\| < \varepsilon\right\}\right).\end{aligned}$$

Thus, assumption (1) and the assumption that $\mu$ is $D$-Federer together imply that

$$\begin{aligned}\mu\left(\{\mathbf{s} \in B(\mathbf{t}, r_{\mathbf{t},v}) \,:\, f_\Lambda(\mathbf{s}) < \varepsilon\}\right) &\le \mu\left(\left\{\mathbf{s} \in B(\mathbf{t}, r'_{\mathbf{t},v}) \,:\, \|f(\mathbf{s})v\| < \varepsilon\right\}\right) \\ &\le C\left(\frac{\varepsilon}{\varrho}\right)^\beta \mu(B(\mathbf{t}, r'_{\mathbf{t},v})) \\ &\le CD\left(\frac{\varepsilon}{\varrho}\right)^\beta \mu(B(\mathbf{t}, r_{\mathbf{t},v})). \end{aligned} \tag{4.12}$$

Since $E$ covers the set of points for which $f_\Lambda$ is less than $\varepsilon$, we may now conclude

$$\begin{aligned}
&\mu\left(\{\mathbf{t} \in B \ : \ f_\Lambda(\mathbf{t}) < \varepsilon\}\right) \\
&\leq \sum_{(\mathbf{t},v)\in I} \mu\left(\{\mathbf{s} \in B(\mathbf{t}, r_{\mathbf{t},v}) \ : \ f_\Lambda(\mathbf{t}) < \varepsilon\}\right) \\
&\leq CD \sum_{(\mathbf{t},v)\in I} \left(\frac{\varepsilon}{\varrho}\right)^\beta \mu(B(\mathbf{t}, r_{\mathbf{t},v})) && \text{by (4.12)} \\
&\ll_m CD \left(\frac{\varepsilon}{\varrho}\right)^\beta \mu\left(\bigcup_{(\mathbf{t},v)\in I} B(\mathbf{t}, r_{\mathbf{t},v})\right) && \text{by (4.11)} \\
&\ll_m CD \left(\frac{\varepsilon}{\varrho}\right)^\beta \mu\left(\tilde{B}\right) && \text{by (4.10)} \\
&\ll_m CD^2 \left(\frac{\varepsilon}{\varrho}\right)^\beta \mu\left(B\right) && \mu \text{ is } D\text{-Federer.}
\end{aligned}$$

□

*Remark.* Fix $x \in X$ such that $x^- \in \Lambda(\Gamma)$. Since the PS-measure $\mu_x^{\mathrm{PS}}$ is supported on $Ux \cap \operatorname{supp} m^{\mathrm{BMS}}$, it follows from Lemma 4.0.4 and (4.6) that Proposition 4.0.5 holds for $\mu_x^{\mathrm{PS}}$ and function $f$ which satisfies the assumption of Lemma 4.0.4. ◀

We are now ready to prove Theorem 4.0.1.

*Proof of Theorem 4.0.1.* Let $x_0 = a_{-\log s}x$, and fix $g \in G$ such that $x = g\Gamma$. By Lemma 4.0.3, for all $T \gg_{\Gamma,\varepsilon} s_0$, we have (4.4), that is, that

$$\sup_{\|\mathbf{t}\|\leq T} \inf_{\gamma\in\Gamma} \inf_{i=1,\ldots,q} \|u_{\mathbf{t}} g\gamma v_i\| > T^\varepsilon.$$

Let $f : \mathbb{R}^{n-1} \to \mathrm{SL}_{n+1}(\mathbb{R})$ be defined by

$$f(\mathbf{t}) = u_{\mathbf{t}} a_{-\log s} g.$$

We first show that parts (1), (2), and (3) of Proposition 4.0.5 for $\mu = \mu_{x_0}^{\mathrm{PS}}$, $f$, $\varrho = 1$, $z = x_0$, $r = T/s$, $\eta = e^{-R_0}$, and

$$\Lambda = \Gamma \{v_1, \dots, v_q\}.$$

Note that $0 \notin \Lambda$.

It follows from the action of $u_{\mathbf{t}}$ on $\mathbb{R}^{n+1}$ that for any $v \in \mathbb{R}^{n+1}$ there exists $v' = (v'_1, \dots, v'_{n+1})^T \in \mathbb{R}^{n+1}$ such that

$$f(\mathbf{t})v = \left(v'_1 + \mathbf{t} \cdot v'' + \frac{1}{2} \|\mathbf{t}\|^2 v'_{n+1}, v'_2 - t_1 v'_{n+1}, \dots, v'_n - t_{n-1} v'_{n+1}, v'_{n+1}\right)^T, \tag{4.13}$$

where $v'' = (v'_2, \dots, v'_n)^T$. Thus, if $v'_{n+1} \neq 0$, then $\mathbf{t} \mapsto f(\mathbf{t})v$ is bounded from below by a function which satisfies the assumption of Lemma 4.0.4. Therefore, by (4.9), for any $v \in \Lambda$ the function $\mathbf{t} \mapsto \|f(\mathbf{t})v\|$ is $(C, \beta)$-good on $\tilde{B}$ with respect to $\mu_{x_0}^{\mathrm{PS}}$, for some $C = C(\Gamma) \geq 1$, $\beta = \beta(\Gamma) > 0$, which proves (1) of Proposition 4.0.5. Note that these constants are uniform across all $v$ so that $v'_{n+1} \neq 0$.

On the other hand, if $v \neq 0$ and

$$v'_{n+1} = 0,$$

then $\mathbf{t} \mapsto f(\mathbf{t})v$ is bounded below by some positive constant, and since positive constant functions are $(C, \beta)$-good for any $C \geq 1, \beta > 0$, we conclude that so is this function by (4.9).

By (3.3), we have

$$u_{\mathbf{t}} a_{-\log s} = a_{-\log s} u_{s\mathbf{t}}.$$

Since multiplication by $a_{-\log s}$ only changes the matrix entries by scaling, using (4.4), for $i = 1, \ldots, q$ we get

$$\sup_{\|\mathbf{t}\| \leq T/s} \|a_{-\log s} u_{s\mathbf{t}} g\gamma v_i\| > s^{-1} \sup_{\|\mathbf{t}\| \leq T} \|u_{\mathbf{t}} g\gamma v_i\| > s^{-1} T^{\varepsilon}.$$

Thus, for any $s \leq T^{\varepsilon}$, $\mathbf{t} < T/s$, and $v \in \Lambda$,

$$\|f(\mathbf{t})v\| \geq 1,$$

which establishes (2) of Proposition 4.0.5.

Since $\eta = e^{-R_0}$ and $\mathcal{H}_i(R_0)$'s are pairwise disjoint, part (3) of Proposition 4.0.5 follows from the uniqueness of $\gamma$ in (4.2) and (4.3).

According to 4.6, the measure $\mu_{x_0}^{\mathrm{PS}}$ is $D$-Federer for any $D > 0$. Thus, we may now use (4.2) and Proposition 4.0.5 to deduce

$$\begin{aligned}
&\mu_{x_0}^{\mathrm{PS}}\left(B_U(T/s) \cap \mathcal{H}_i(R)\right) \\
&= \mu_{x_0}^{\mathrm{PS}}\left(\left\{\mathbf{t} \in B_U(T/s) \ : \ \exists \gamma \in \Gamma, 1 \leq i \leq q \text{ such that } \|f(\mathbf{t})\gamma v_i\| < e^{-R}\right\}\right) \\
&\ll e^{-R\beta} \mu_{x_0}^{\mathrm{PS}}(B_U(T/s)x_0),
\end{aligned}$$

where the implied constant depends on $n$ and $\Gamma$. $\square$

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This chapter contains material from the following, which has been submitted for publication: N. Tamam, J. M. Warren, *"Effective equidistribution of horospherical flows in infinite volume rank one homogeneous spaces"*, arXiv:2007.03135. The dissertation author was one of the primary investigators and authors of this paper, and was supported in part by the National Science and Engineering Research Council of Canada (NSERC) PGSD3-502346-2017 during this work.

# Chapter 5

# Friendliness Properties of the PS Measure

In this chapter we prove several key properties of the PS-measure, including that slightly enlarging a ball does not increase the measure too much and that scaling the size of the ball has a bounded multiplicative increase on the measure. Note that the results in this chapter hold for any $\Gamma$ that is geometrically finite; we do not require Assumption 1.1.2. In the setting that all cusps have maximal rank, or balls are centered at BMS points, stronger statements hold. See the appendix, specifically Chapter 10, for more details.

The main results in this chapter are the following, which both establish control over the measure of a slightly enlarged ball. Many technical details of the proofs are hidden in Proposition 5.0.4, which is proved in the appendix, Chapter 10.

**Theorem 5.0.1.** *There exists a constant $\alpha' = \alpha'(\Gamma) > 0$, such that for every $x \in G/\Gamma$ that is $(\varepsilon, s_0)$-Diophantine, for every $0 < s \leq T^{\frac{\varepsilon}{1-\varepsilon}}$, every $0 < \xi \ll_\Gamma 1$, and every $T \gg_{\Gamma,\varepsilon} s_0$,*

$$\frac{\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(\xi + T))}{\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T))} - 1 \ll_\Gamma \xi^{\alpha'}.$$

**Theorem 5.0.2.** *There exist $\alpha' = \alpha'(\Gamma) > 0$, $\theta' = \theta'(\Gamma) \geq \alpha'$, $\omega' = \omega'(\Gamma) \geq 2\delta_\Gamma$, such that for any $g \in G$ with $g^- \in \Lambda(\Gamma)$ and $0 < \xi < \eta \ll_\Gamma e^{-\operatorname{height}(g\Gamma)}$, we have that*

$$\frac{\nu(P_{\xi+\eta}g)}{\nu(P_\eta g)} - 1 \ll_\Gamma e^{\omega' \operatorname{height}(g\Gamma)} \frac{\xi^{\alpha'}}{\eta^{\theta'}}.$$

Theorem 5.0.2 will be obtained as a corollary of the following:

**Proposition 5.0.3.** *There exist constants $\alpha = \alpha(\Gamma) > 0$, $\theta = \theta(\Gamma) \geq \alpha$, and $\omega = \omega(\Gamma) \geq 2\delta_\Gamma$ such for $x \in G/\Gamma$ which satisfies $x^+ \in \Lambda(\Gamma)$, and $0 < \xi < \eta \ll_\Gamma e^{-\operatorname{height}(x)}$ we have*

$$\frac{\mu^{\mathrm{PS}}_x(B_U(\xi + \eta))}{\mu^{\mathrm{PS}}_x(B_U(\eta))} - 1 \ll_\Gamma e^{\omega \operatorname{height}(x)} \frac{\xi^{\alpha}}{\eta^{\theta}}.$$

We first show how to obtain Theorem 5.0.2 from Proposition 5.0.3.

*Proof of Theorem 5.0.2 assuming Proposition 5.0.3.* Using the product structure of $\nu$, we can write

$$\nu(P_\eta g) = \int_{A_\eta} \int_{M_\eta} \mu^{\mathrm{PS}\,-}_g(B_{\tilde{U}}(\eta))dmds.$$

Then, by an analogous statement to Proposition 5.0.3 for $\mu^{\mathrm{PS}\,-}$, there exists a constant

$c_0 = c_0(\Gamma) > 0$ such that

$$\begin{aligned}
\nu(P_{\eta+\xi}g) &= \int_{A_{\xi+\eta}} \int_{M_{\xi+\eta}} \mu_g^{\mathrm{PS}-}(B_{\tilde{U}}(\xi+\eta))dmds \\
&\leq \int_{A_{\xi+\eta}} \int_{M_{\xi+\eta}} \mu_g^{\mathrm{PS}-}(B_{\tilde{U}}(\eta)) \left[1 + c_0 \frac{\xi^\alpha}{\eta^\theta} e^{\omega' \operatorname{height}(g\Gamma)}\right] dmds \\
&= \left[1 + c_0 \frac{\xi^\alpha}{\eta^\theta} e^{\omega' \operatorname{height}(g\Gamma)}\right] \left[\frac{(\xi+\eta)^{\frac{1}{2}(n-1)(n-2)+1}}{\eta^{\frac{1}{2}(n-1)(n-2)+1}} \nu(P_\eta g)\right]. \\
&\leq \left[1 + c_0 \frac{\xi^\alpha}{\eta^\theta} e^{\omega' \operatorname{height}(g\Gamma)}\right] \left[1 + c_1 \frac{\xi}{\eta}\right] \nu(P_\eta g),
\end{aligned}$$

where $c_1 > 0$ is an absolute constant (which depends only on $n$) arising from the binomial theorem. Therefore,

$$\frac{\nu(P_{\xi+\eta}g)}{\nu(P_\eta g)} - 1 \ll_\Gamma e^{\omega' \operatorname{height}(g\Gamma)} \frac{\xi^\alpha}{\eta^\theta} \cdot \frac{\xi}{\eta} + \frac{\xi}{\eta} + e^{\omega' \operatorname{height}(g\Gamma)} \frac{\xi^\alpha}{\eta^\theta}. \tag{5.1}$$

Since $\xi < \eta$, the first term on the left hand side of (5.1) is dominated by the last term, and so

$$\frac{\nu(P_{\xi+\eta}g)}{\nu(P_\eta g)} - 1 \ll_\Gamma \frac{\xi}{\eta} + e^{\omega' \operatorname{height}(g\Gamma)} \frac{\xi}{\eta^\theta}.$$

Since $e^{\omega \operatorname{height}(g\Gamma)} \geq 1$, if we define

$$\alpha' = \min\{1, \alpha\}, \quad \theta' = \max\{1, \theta\},$$

then both terms are dominated by

$$e^{\omega \operatorname{height}(g\Gamma)} \frac{\xi^{\alpha'}}{\eta^{\theta'}},$$

which completes the proof. $\square$

The following result, showing that the PS measure is not concentrated near hyperplanes, is proved in the appendix to improve the readability of this chapter. See Proposition 10.2.2 for the proof. This result builds upon the work of Das, Fishman, Simmons, and Urbański in [DFSU20], where it is shown that the PS density $\nu_o$ is *friendly* when $\Gamma$ is geometrically finite.

For a hyperplane $L \subset U \cong \mathbb{R}^{n-1}$ and $\xi > 0$, define

$$\mathcal{N}_U(L, \xi) := \{u_{\mathbf{t}} y : y \in L, \mathbf{t} \in B_U(\xi)\}.$$

**Proposition 5.0.4.** *Let $\Gamma$ be geometrically finite and Zariski dense. There exist constants $\alpha = \alpha(\Gamma) > 0, \omega = \omega(\Gamma) \geq 0,$ and $\theta = \theta(\Gamma) > \alpha$ satisfying the following: for any $x \in G/\Gamma$ with $x^+ \in \Lambda(\Gamma)$, and for every $\xi > 0$ and $0 < \eta \ll_\Gamma e^{-\operatorname{height}(x)}$, we have that for every hyperplane $L$,*

$$\mu_x^{\mathrm{PS}}(\mathcal{N}_U(L, \xi) \cap B_U(\eta)) \ll_\Gamma e^{\omega \operatorname{height}(x)} \frac{\xi^\alpha}{\eta^\theta} \mu_x^{\mathrm{PS}}(B_U(\eta)).$$

We are now ready to prove Proposition 5.0.3.

*Proof of Proposition 5.0.3.* It follows from the geometry of $B_U(\xi + \eta)x - B_U(\eta)x$ that there exist hyperplanes $L_1, \ldots, L_m$, where $m$ only depends on $n$, such that

$$B_U(\xi + \eta)x - B_U(\eta)x \subseteq \bigcup_{i=1}^m \mathcal{N}_U(L_i, 2\xi).$$

For any $0 < \xi < \eta \ll_\Gamma e^{-\operatorname{height}(x)}$, we have that

$$\begin{aligned}\frac{\mu_x^{\mathrm{PS}}(B_U(\xi+\eta))}{\mu_x^{\mathrm{PS}}(B_U(\eta))} - 1 &= \frac{\mu_x^{\mathrm{PS}}(B_U(\xi+\eta) - B_U(\eta))}{\mu_x^{\mathrm{PS}}(B_U(\eta))} \\ &\leq \sum_{i=1}^{m} \frac{\mu_x^{\mathrm{PS}}(\mathcal{N}(L_i,\xi) \cap B(x,\xi+\eta))}{\mu_x^{\mathrm{PS}}(B_U(\eta))} \qquad \text{by Proposition 5.0.4} \\ &\ll_\Gamma m e^{\omega \operatorname{height}(x)} \frac{\xi^\alpha}{\eta^\theta} \cdot \frac{\mu_x^{\mathrm{PS}}(B_U(2\eta))}{\mu_x^{\mathrm{PS}}(B_U(\eta))}\end{aligned}$$

By (4.6), $\mu_x^{\mathrm{PS}}$ is $D$-Federer (see Corollary 10.1.3 for more detail), in particular

$$\mu_x^{\mathrm{PS}}(B_U(2\eta)) \ll_\Gamma \mu_x^{\mathrm{PS}}(B_U(\eta)).$$

Thus, we obtain

$$\frac{\mu_x^{\mathrm{PS}}(B_U(\xi+\eta))}{\mu_x^{\mathrm{PS}}(B_U(\eta))} - 1 \ll_\Gamma e^{\omega \operatorname{height}(x)} \frac{\xi^\alpha}{\eta^\theta},$$

and relabeling the constants completes the proof. □

In (4.6), we saw that $\mu_x^{\mathrm{PS}}$ is Federer when $x \in \operatorname{supp} m^{\mathrm{BMS}}$. Below, we show that $\mu_x^{\mathrm{PS}}$ satisfies a similar condition for sufficiently large balls when $x$ is Diophantine, but not necessarily a BMS point.

**Corollary 5.0.5.** *There exists a constant $\sigma = \sigma(\Gamma) \geq \delta_\Gamma$ such that for every $c \geq 1$ and every $x \in G/\Gamma$ that is $(\varepsilon, s_0)$-Diophantine, if $T \gg_{\Gamma,\varepsilon} s_0$, then*

$$\mu_x^{\mathrm{PS}}(B_U(cT)) \ll_\Gamma c^\sigma \mu_x^{\mathrm{PS}}(B_U(T)).$$

*Proof.* By Lemma 3.2.5, for some $T_0 \gg_{\Gamma,\varepsilon} s_0$ there exists

$$y \in B_U(T_0)x \cap \operatorname{supp} m^{\mathrm{BMS}}.$$

Then for $T \geq T_0$, we have

$$B_U(T - T_0)y \subseteq B_U(T)x \subseteq B_U(T + T_0)y.$$

Since $c \geq 1$, we therefore have that for $T \geq 2T_0$,

$$\begin{aligned}
\mu_x^{\mathrm{PS}}(B_U(cT)) &\leq \mu_y^{\mathrm{PS}}(B_U(cT + T_0)) \\
&\leq \mu_y^{\mathrm{PS}}(B_U((c+1)T)) \\
&\ll_\Gamma (c+1)^\sigma \mu_y^{\mathrm{PS}}(B_U(T/2)) && \text{by (4.6)} \\
&\ll_\Gamma (c+1)^\sigma \mu_y^{\mathrm{PS}}(B_U(T - T_0)) \\
&\ll_\Gamma (c+1)^\sigma \mu_x^{\mathrm{PS}}(B_U(T)) \\
&\ll_\Gamma (2c)^\sigma \mu_x^{\mathrm{PS}}(B_U(T)) && \text{since } c \geq 1 \\
&\ll_\Gamma c^\sigma \mu_x^{\mathrm{PS}}(B_U(T)).
\end{aligned}$$

□

*Remark.* Observe that if $x$ is $(\varepsilon, s_0)$-Diophantine and $T \gg_{\Gamma,\varepsilon} s_0$, then $T$ is sufficiently large to use Corollary 5.0.5 on $a_{-s}x$ for $s > 0$. To see this, observe that, in the notations of the proof of Corollary 5.0.5, $T_0$ is such that for any $(\varepsilon, s_0)$-Diophantine point $x$, there exists $y \in \operatorname{supp} m^{\mathrm{BMS}}$ and $\mathbf{t} \leq T_0$ so that $x = u_{\mathbf{t}}y$. Then

$$a_{-s}x = a_{-s}u_{\mathbf{t}}y = u_{e^{-s}\mathbf{t}}a_{-s}y.$$

Thus, the distance to the nearest BMS point in the $U$ orbit shrinks, and so $T$ is still sufficiently large. ◀

**Proposition 5.0.6.** *Let $H_R = \{y \in G/\Gamma : \mathrm{height}(y) \leq R\}$. There exist constants $\alpha = \alpha(\Gamma) > 0$, and $\omega = \omega(\Gamma) \geq 0$ such that for every $x \in G/\Gamma$ that is $(\varepsilon, s_0)$-Diophantine and for every $0 < \xi < 1/2$, and $T \gg_{\Gamma,\varepsilon} s_0$,*

$$\frac{\mu_x^{\mathrm{PS}}((B_U(\xi+T) \cap H_R) - (B_U(T) \cap H_R))}{\mu_x^{\mathrm{PS}}(B_U(T))} \ll_\Gamma e^{\omega R}\xi^\alpha.$$

*Proof.* Let $T_0 \gg_{\Gamma,\varepsilon} s_0$ satisfy the conclusion of Lemma 3.2.5. For $T \geq T_0$, let

$$E_T := \mathcal{N}(L,\xi) \cap B_U(T) \cap H_R \cap \operatorname{supp} m^{\mathrm{BMS}},$$

and observe that $\mu_x^{\mathrm{PS}}(E_T) = \mu_x^{\mathrm{PS}}(\mathcal{N}(L,\xi) \cap B_U(T) \cap H_R)$.

Let $c_1 = c_1(\Gamma) > 0$ be the implied constant in Proposition 5.0.4. Fix $r = c_1 e^{-R}$ and let $\{u_1, \ldots, u_k\}$ be a maximal $\frac{r}{2}$-separated set in $E_{T-\frac{r}{4}}$. Then,

$$E_T \subseteq \bigcup_{i=1}^{k} B_U(r)u_i.$$

Note also that by (4.6), we have that there exists a constant $c_2 = c_2(\Gamma) > 0$ such that for all $u_i$,

$$\mu_{u_i}^{\mathrm{PS}}(B_U(r)) = \mu_{u_i}^{\mathrm{PS}}(B_U(8(r/8)) \leq c_2\mu_{u_i}^{\mathrm{PS}}(B_U(r/8)). \tag{5.2}$$

Therefore,

$$\begin{aligned}
&\mu_x^{\mathrm{PS}}(\mathcal{N}(L,\xi) \cap B_U(T) \cap H_R) \\
&\leq \sum_{i=1}^{k} \mu_{u_i}^{\mathrm{PS}}(\mathcal{N}(L,\xi) \cap B_U(r)) \\
&\ll_\Gamma e^{\omega R} \frac{\xi^\alpha}{r^\theta} \sum_{i=1}^{k} \mu_{u_i}^{\mathrm{PS}}(B_U(r)) && \text{by Proposition 5.0.4} \\
&\ll_\Gamma e^{(\omega+\theta)R} \xi^\alpha \sum_{i=1}^{k} \mu_{u_i}^{\mathrm{PS}}(B_U(r/8)) && \text{by (5.2)} \\
&\ll_\Gamma e^{(\omega+\theta)R} \xi^\alpha \mu_x^{\mathrm{PS}}(B_U(T+1)) && \text{as the 1/8 balls are disjoint.}
\end{aligned}$$

By Corollary 5.0.5, there exists $\sigma = \sigma(\Gamma) \geq \delta_\Gamma$ so that

$$\mu_x^{\mathrm{PS}}(B_U(T+1)) \subseteq \mu_x^{\mathrm{PS}}(B_U(2T)) \ll_\Gamma 2^\sigma \mu_x^{\mathrm{PS}}(B_U(T)).$$

Let

$$\omega' = \omega + \theta.$$

It follows from the geometry of $B_U(\xi+T)x - B_U(T)x$ that there exist $L_1, \ldots, L_m$, where $m$ only depends on $n$, such that

$$B_U(\xi+T)x - B_U(T)x \subseteq \bigcup_{i=1}^{m} \mathcal{N}_U(L_i, 2\xi).$$

Thus, we also have

$$(B_U(\xi+T)x - B_U(T)x) \cap H_R \subseteq \bigcup_{i=1}^{m} \mathcal{N}_U(L_i, 2\xi).$$

We arrive at

$$\frac{\mu_x^{\mathrm{PS}}((B_U(\xi+T)\cap H_R)-(B_U(T)\cap H_R))}{\mu_x^{\mathrm{PS}}(B_U(T))} \leq \sum_{i=1}^{m} \frac{\mu_x^{\mathrm{PS}}(\mathcal{N}(L_i,2\xi)\cap B_U(\xi+T))}{\mu_x^{\mathrm{PS}}(B_U(T))}$$
$$\ll_\Gamma m e^{\omega' R}\xi^\alpha \frac{\mu_x^{\mathrm{PS}}(B_U(\xi+T))}{\mu_x^{\mathrm{PS}}(B_U(T))}$$

By Corollary 5.0.5 again, we conclude that

$$\frac{\mu_x^{\mathrm{PS}}((B_U(\xi+T)\cap H_R)-(B_U(T)\cap H_R))}{\mu_x^{\mathrm{PS}}(B_U(T))} \ll_\Gamma e^{\omega' R}\xi^\alpha,$$

which completes the proof.

□

We are now ready to prove Theorem 5.0.1. This will follow by combining Proposition 5.0.6 with quantitative nondivergence, Theorem 4.0.1.

*Proof of Theorem 5.0.1.* Observe that by Lemma 3.2.3,

$$\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T+\xi)) = \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T+\xi)\cap H_{R-R_0}) + \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T+\xi)\cap \mathcal{X}(R)) \quad (5.3)$$

By Theorem 4.0.1, for $T \gg_{\Gamma,\varepsilon} s_0$, $0 < s \leq T^{\frac{\varepsilon}{1-\varepsilon}}$, and any $R \geq R_0$,

$$\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T+\xi)\cap \mathcal{X}(R)) = \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U((s(T+\xi)/s)\cap \mathcal{X}(R))$$
$$\ll_\Gamma \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T+\xi))e^{-\beta R}$$
$$\ll_\Gamma \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T))e^{-\beta R} \qquad \text{by Corollary 5.0.5}$$

Observe that use of Corollary 5.0.5 is justified if $T \gg_{\Gamma,\varepsilon} s_0$ by the remark after that statement. Similarly, by Proposition 5.0.6 and the same reasoning as in the remark, for

$T \gg_{\Gamma,\varepsilon} s_0$, we have

$$\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T+\xi) \cap H_{R-R_0}) \ll_\Gamma e^{\omega R}\xi^\alpha \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T)) + \mu^{\mathrm{PS}}_{a_{-s}x}(B_U(T) \cap H_{R-R_0}).$$

Putting this together with (5.3), we conclude

$$\begin{aligned}
&\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T+\xi)) \\
&= \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T+\xi) \cap H_{R-R_0}) + \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T+\xi) \cap \mathcal{X}(R)) \\
&\ll_\Gamma \left[e^{\omega R}\xi^\alpha \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T)) + \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T) \cap H_{R-R_0})\right] + e^{-\beta R}\mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T)) \\
&\ll_\Gamma \left(e^{\omega R}\xi^\alpha + e^{-\beta R} + 1\right) \mu^{\mathrm{PS}}_{a_{-\log s}x}(B_U(T))
\end{aligned}$$

Taking $R = -\frac{\alpha}{\omega+\beta}\log\xi$ implies the result, provided that $\xi$ is sufficiently small so that this is larger than $R_0$. Note that since $\alpha, \omega, \beta, R_0$ are all constants depending only on $\Gamma$, this is equivalent to requiring $\xi \ll_\Gamma 1$. □

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# Chapter 6

# Proofs of the Equidistribution Theorems

In this chapter, we prove our effective equidistribution theorems. Recall the notation of §3.2. In particular, $d$ denotes the hyperbolic distance, height$(x)$ is the height of a point $x$ into the cusps, and $\mathcal{C}_0$ is the fixed compact set in $G/\Gamma$ which is defined in §3.2. In this chapter, Assumption 1.1.2 is critical.

## 6.1 Proof of Theorem 1.2.3

We will first prove the following proposition, which is a form of Theorem 1.2.3 for $G$. Theorem 1.2.3 will follow by a partition of unity argument.

**Proposition 6.1.1.** *There exist $\kappa = \kappa(\Gamma)$ and $\ell = \ell(\Gamma)$ which satisfy the following: let*

$0 < r < 1$, $\psi \in C_c^\infty(G)$ *supported on an admissible box, and* $f \in C_c^\infty(B_U(r))$. *Then, there exists* $c = c(\Gamma, \operatorname{supp}\psi) > 0$ *such that for any* $g \in \operatorname{supp}\tilde{m}^{\mathrm{BMS}}$*, and* $s \gg_\Gamma \operatorname{height}(g\Gamma)$*, we have*

$$\begin{aligned}&\left|\sum_{\gamma\in\Gamma}\int_U \psi(a_s u_{\boldsymbol{t}} g\gamma) f(\boldsymbol{t}) d\mu_g^{\mathrm{PS}}(\boldsymbol{t}) - \mu_g^{\mathrm{PS}}(f)\tilde{m}^{\mathrm{BMS}}(\psi)\right| \\ &< c S_\ell(\psi) S_\ell(f) e^{-\kappa s}\mu_g^{\mathrm{PS}}(B_U(1)).\end{aligned}$$

*Proof.* Without loss of generality assume that $f$ and $\psi$ are non-negative functions.

**Step 1: Setup and approximations.**

Let $\kappa', \ell'$ satisfy the conclusion of Assumption 1.1.2, and let $\ell > \ell'$ satisfy the conclusion of Lemma 3.4.4. Observe that $\ell$ can be increased if necessary while maintaining this property.

Because $\psi$ is supported on an admissible box, there exists $0 < \eta_0 < 1/2$ (depending on $\operatorname{supp}\psi$) such that $G_{3\eta_0}\operatorname{supp}\psi$ is still an admissible box. For $0 < \eta < \eta_0$, let $\psi_{\eta,\pm}$ satisfy the conclusion of Lemma 3.4.4 for $G, 3\eta$, and $\psi$. In particular, for all small $\eta > 0$

$$S_{\ell'}(\psi_{\eta,\pm}) \ll_{\operatorname{supp}\psi} \eta^{-2\ell} S_\ell(\psi). \tag{6.1}$$

Since $\psi$ is uniformly continuous and the BMS-measure is finite, we may deduce from Lemma 3.4.4(2) that

$$\left|\tilde{m}^{\mathrm{BMS}}(\psi_{\eta,\pm}) - \tilde{m}^{\mathrm{BMS}}(\psi)\right| \ll_{\operatorname{supp}\psi,\Gamma} \eta S_\ell(\psi). \tag{6.2}$$

According to Lemma 3.0.1, for any $p \in P_\eta$, there exists $\rho_p : B_U(1) \to B_U(1+O(\eta))$

that is a diffeomorphism onto its image and a constant $D = D(\eta) < 3\eta$ such that

$$u_t p^{-1} \in P_D u_{\rho_p(t)}. \tag{6.3}$$

**Step 2: Assuming that $f$ is supported on a small ball.**

We start by proving that there exists $\kappa > 0$ such that if $f \in C_c^\infty(B_U(r_1))$, where $r_1 \leq \mathrm{inj}(g)$, then for $s > 0$,

$$\begin{aligned} &\sum_{\gamma\in\Gamma} \int_U \psi(a_s u_{\mathbf{t}} g\gamma) f(\mathbf{t}) d\mu_g^{\mathrm{PS}}(\mathbf{t}) - \tilde{m}^{\mathrm{BMS}}(\psi)\mu_g^{\mathrm{PS}}(f) \\ &\ll_{\Gamma,\operatorname{supp}\psi} S_\ell(\psi)\, S_\ell(f)\, e^{-2\kappa s}\mu_g^{\mathrm{PS}}(B_U(1)). \end{aligned} \tag{6.4}$$

For any $s > 0$ and $\gamma \in \Gamma$, from (6.3) we have that

$$\begin{aligned} &\int_{B_U(r_1)} \psi(a_s u_{\mathbf{t}} g\gamma) f(\mathbf{t}) d\mu_g^{\mathrm{PS}}(\mathbf{t}) \\ &= \frac{1}{\nu(P_\eta g)} \int_{P_\eta g} \int_{B_U(r_1)} \psi(a_s u_{\mathbf{t}} p^{-1} pg\gamma) f(\mathbf{t}) d\mu_g^{\mathrm{PS}}(\mathbf{t}) d\nu(pg) \\ &\leq \frac{1}{\nu(P_\eta g)} \int_{P_\eta g} \int_{B_U(r_1)} \psi_{\eta,+}(a_s u_{\rho_p(\mathbf{t})} pg\gamma) f(\mathbf{t}) d\mu_g^{\mathrm{PS}}(\mathbf{t}) d\nu(pg), \end{aligned}$$

where the last inequality follows since $a_s P_{3\eta} a_{-s} \subset P_{3\eta}$ for any positive $s$.

**Step 2.1: Use the product structure of the BMS measure.**

For any $p \in P_\eta$, $(u_{\mathbf{t}} g)^+ = (u_{\rho_p(\mathbf{t})} pg)^+$, the measures $d\mu_g^{\mathrm{PS}}(\mathbf{t})$ and $d(\rho_{p*}\mu_{pg}^{\mathrm{PS}}(\mathbf{t})) = d\mu_{pg}^{\mathrm{PS}}(\rho_p(\mathbf{t}))$ are absolutely continuous with each other, and the Radon-Nikodym derivative at $\mathbf{t}$ is given by

$$\frac{d\mu_g^{\mathrm{PS}}(\mathbf{t})}{d\mu_{pg}^{\mathrm{PS}}(\rho_p(\mathbf{t}))} = e^{\delta_\Gamma \beta_{(u_{\mathbf{t}} g)^+}(u_{\mathbf{t}} g(o), u_{\rho_p(\mathbf{t})} pg(o))}. \tag{6.5}$$

Let $0 < \xi < \eta$. Let $\chi_{\eta,\xi}$ satisfy the conclusion of Lemma 3.4.2 for $H = P$, $\xi_1 = \eta - \xi$, $\xi_2 = \xi$, and $g$. Let $\varphi_{\eta,g}$ be the function defined on $B_U(1)P_\eta g$ given by

$$\varphi_{\eta,g}(u_{\rho_p(\mathbf{t})}pg) := \frac{f(\mathbf{t})\chi_{\eta,\xi}(pg)}{\nu(P_\eta g)e^{\delta_\Gamma \beta_{(u_{\mathbf{t}}g)^+}(u_{\mathbf{t}}g(o),u_{\rho_p(\mathbf{t})}pg(o))}}.$$

We will need a bound on $S_\ell(\varphi_{\eta,g})$. To that end, note that

$$\begin{aligned}\left|\beta_{(u_{\mathbf{t}}g)^+}(u_{\mathbf{t}}g(o), u_{\rho_p(\mathbf{t})}pg(o))\right| &\leq d(u_{\mathbf{t}}g(o), u_{\rho_p(\mathbf{t})}pg(o)) \\ &= d(g(o), u_{-\mathbf{t}}u_{\rho_p(\mathbf{t})}pg(o)).\end{aligned}$$

Since $u_{-\mathbf{t}}u_{\rho_p(\mathbf{t})}p \in G_{5\eta}$, the above is bounded by some absolute constant (depending only on $\Gamma$) for all $\eta < \frac{1}{2}$.

Thus, because the Busemann function is Lipschitz, we have that for all $p \in P_\eta$,

$$S_\ell(\beta_{(u_{\mathbf{t}}g_0)^+}(u_{\mathbf{t}}g(o), u_{\rho_p(\mathbf{t})}pg(o))) \ll_\Gamma 1. \tag{6.6}$$

By [KM96, Lemma 2.4.7(a)], Lemma 3.4.2, (6.6), and Lemma 3.3.1, we have

$$\begin{aligned}S_\ell(\varphi_{\eta,g}) &\ll_{\Gamma,\ell} \nu(P_\eta g)^{-1}S_\ell(f)S_\ell(\chi_{\eta,\xi}) \\ &\ll_{\Gamma,\ell} \eta^{-(\delta_\Gamma+\frac{1}{2}(n-1)(n-2)+1)}e^{(\delta_\Gamma-k_2(x,\eta))d(\pi(\mathcal{C}_0),\pi(a_{\log\eta}g))}S_\ell(\chi_{\eta,\xi})S_\ell(f) \\ &\ll_{\Gamma,\ell} e^{\delta_\Gamma(|\log\eta|+\mathrm{height}(g\Gamma))}\eta^{-(\delta_\Gamma+\frac{1}{2}(n-1)(n-2)+1)}\eta^{n-1}\xi^{-\ell-(n-1)/2}S_\ell(f) \\ &\ll_{\Gamma,\ell} e^{\delta_\Gamma\,\mathrm{height}(g\Gamma)}\eta^{-(2\delta_\Gamma+\frac{1}{2}(n-1)(n-2)+1)}\eta^{n-1}\xi^{-\ell-(n-1)/2}S_\ell(f) \\ &\ll_{\Gamma,\ell} e^{\delta_\Gamma\,\mathrm{height}(g\Gamma)}\eta^{4n-\frac{1}{2}n^2-3-2\delta_\Gamma}\xi^{-\ell-(n-1)/2}S_\ell(f)\end{aligned} \tag{6.7}$$

Note that the dependence on $\ell$ arises from the exponential of the Busemann function in the denominator.

Also, using the product structure of $\tilde{m}^{\mathrm{BMS}}$ in (3.22), we get

$$\begin{aligned}
&\frac{1}{\nu(P_\eta g)}\int_{P_\eta g}\int_{B_U(r_1)}\psi_{\eta,+}(a_s u_{\rho_p(\mathbf{t})}pg\gamma)f(\mathbf{t})d\mu_g^{\mathrm{PS}}(\mathbf{t})d\nu(pg)\\
&=\frac{1}{\nu(P_\eta g)}\int_{P_\eta g}\int_{B_U(r_1)}\psi_{\eta,+}(a_s u_{\rho_p(\mathbf{t})}pg\gamma)f(\mathbf{t})\frac{d\mu_g^{\mathrm{PS}}(\mathbf{t})}{d\mu_{pg}^{\mathrm{PS}}(\rho_p(\mathbf{t}))}d\mu_{pg}^{\mathrm{PS}}(\rho_p(\mathbf{t}))d\nu(pg)\\
&\leq\int_G\psi_{\eta,+}(a_s h\gamma)\varphi_{\eta,g}(h)d\tilde{m}^{\mathrm{BMS}}(h).
\end{aligned}$$

**Step 2.2: Use the exponential mixing assumption.**

By defining $\Psi_{\eta,+}(h\Gamma)=\sum_{\gamma\in\Gamma}\psi_{\eta,+}(h\gamma)$ and $\Phi_{\eta,g}(h\Gamma):=\sum_{\gamma\in\Gamma}\varphi_{\eta,g}(h\gamma)$, we obtain

$$\sum_{\gamma\in\Gamma}\int_G\psi_{\eta,+}(a_s h\gamma)\varphi_{\eta,g}(h)d\tilde{m}^{\mathrm{BMS}}(h)\leq\int_X\Psi_{\eta,+}(a_s x)\Phi_{\eta,g}(x)dm^{\mathrm{BMS}}(x)$$

for any positive $s$. Note that

$$S_{\ell'}(\Psi_{\eta,+})=S_{\ell'}(\psi_{\eta,+})\text{ and }S_{\ell'}(\Phi_{\eta,g})=S_{\ell'}(\varphi_{\eta,g}).\tag{6.8}$$

In particular, (6.1) and (6.7) imply

$$\begin{aligned}
&S_{\ell'}(\Psi_{\eta,+})\ll_{\operatorname{supp}\psi}\eta^{-2\ell}S_\ell(\psi)\text{ and}\\
&S_{\ell'}(\Phi_{\eta,g})\ll_\Gamma e^{\delta_\Gamma\operatorname{height}(g\Gamma)}\eta^{4n-\frac{1}{2}n^2-3-2\delta_\Gamma}\xi^{-\ell-(n-1)/2}S_\ell(f).
\end{aligned}\tag{6.9}$$

By Assumption 1.1.2,

$$\begin{aligned}
&\int\Psi_{\eta,+}\left(a_s x\right)\Phi_{\eta,g}\left(x\right)dm^{\mathrm{BMS}}\left(x\right)-m^{\mathrm{BMS}}\left(\Psi_{\eta,+}\right)m^{\mathrm{BMS}}\left(\Phi_{\eta,g}\right)\\
&\ll_\Gamma S_{\ell'}\left(\Psi_{\eta,+}\right)S_{\ell'}\left(\Phi_{\eta,g}\right)e^{-\kappa' s}.
\end{aligned}$$

Then, by (6.9), there exists $c_1 = c_1(\Gamma, \operatorname{supp}\psi)$ such that

$$\sum_{\gamma\in\Gamma} \int_{B_U(r_1)} \psi(a_s u_{\mathbf{t}} g\gamma) f(\mathbf{t}) d\mu_g^{\mathrm{PS}}(\mathbf{t})$$
$$< m^{\mathrm{BMS}}\left(\Psi_{\eta,+}\right) m^{\mathrm{BMS}}\left(\Phi_{\eta,g}\right) + c_1 e^{\delta_\Gamma \operatorname{height}(g\Gamma)} \eta^{4n-\frac{1}{2}n^2-3-2\delta_\Gamma} \xi^{-\ell-(n-1)/2} S_\ell\left(\psi\right) S_\ell\left(f\right) e^{-\kappa' s}.$$

**Step 2.3: Rewrite in terms of $\psi$ and $f$.**

Using Lemma 3.4.2 and (6.5), one can calculate

$$\begin{aligned} m^{\mathrm{BMS}}(\Phi_{\eta,g}) &= \int_G \varphi_{\eta,g}(h) d\tilde{m}^{\mathrm{BMS}}(h) \\ &= \frac{1}{\nu(P_\eta g)} \int_{Pg} \int_U \frac{f(\mathbf{t})\chi_{\eta,\xi}(p)}{e^{\delta_\Gamma \beta_{(u_{\mathbf{t}}g)^+}(u_{\mathbf{t}}g(o), u_{\rho_p(\mathbf{t})}pg(o))}} d\mu_{pg}^{\mathrm{PS}}(\rho_p(\mathbf{t})) d\nu(pg) \\ &= \frac{1}{\nu(P_\eta g)} \int_{Pg} \int_U f(\mathbf{t})\chi_{\eta,\xi}(p) d\mu_g^{\mathrm{PS}}(\mathbf{t}) d\nu(pg) \\ &\leq \frac{\nu(P_{\eta+\xi}g)}{\nu(P_\eta g)} \int_{B_U(r_1)} f(\mathbf{t}) d\mu_g^{\mathrm{PS}}(\mathbf{t}). \end{aligned}$$

Thus, by Proposition 5.0.2, there exist $\alpha, \theta, \omega, c_0 > 0$ depending only on $\Gamma$ such that

$$\begin{aligned} m^{\mathrm{BMS}}(\Phi_{\eta,g}) &\leq \left(1 + c_2 e^{\omega \operatorname{height}(g\Gamma)} \frac{\xi^\alpha}{\eta^\theta}\right) \int_{B_U(r_1)} f(\mathbf{t}) d\mu_g^{\mathrm{PS}}(\mathbf{t}) \\ &= \left(1 + c_2 e^{\omega \operatorname{height}(g\Gamma)} \frac{\xi^\alpha}{\eta^\theta}\right) \int_{B_U(r_1)} f(\mathbf{t}) d\mu_g^{\mathrm{PS}}(\mathbf{t}) \\ &= \left(1 + c_2 e^{\omega \operatorname{height}(g\Gamma)} \frac{\xi^\alpha}{\eta^\theta}\right) \mu_g^{\mathrm{PS}}(f). \end{aligned}$$

Using (6.2), we get that there exists $c_3 = c_3(\Gamma, \operatorname{supp}\psi)$ such that

$$\begin{aligned} m^{\mathrm{BMS}}(\Psi_{\eta,+}) &\leq \int_G \psi_{\eta,+}(g) d\tilde{m}^{\mathrm{BMS}}(g) \\ &< \tilde{m}^{\mathrm{BMS}}(\psi) + c_3 \eta S_\ell(\psi). \end{aligned}$$

To summarize, we have

$$\begin{aligned} &\sum_{\gamma\in\Gamma} \int_{B_U(r_1)} \psi(a_s u_{\mathbf{t}} g\gamma) f(\mathbf{t}) d\mu_g^{\mathrm{PS}}(\mathbf{t}) \\ &\leq \frac{1}{\nu(P_\eta g)} \sum_{\gamma\in\Gamma} \int_{P_\eta g} \int_{B_U(r_1)} \psi_{\eta,+}(a_s u_{\rho_p(\mathbf{t})} pg\gamma) f(\mathbf{t}) d\mu_g^{\mathrm{PS}}(\mathbf{t}) d\nu(pg) \\ &\leq \sum_{\gamma\in\Gamma} \int_G \psi_{\eta,+}(a_s h\gamma) \varphi_{\eta,g}(h) d\tilde{m}^{\mathrm{BMS}}(h) \\ &\leq \int_X \Psi_{\eta,+}(a_s x) \Phi_{\eta,g}(x) dm^{\mathrm{BMS}}(x) \\ &< m^{\mathrm{BMS}}\left(\Psi_{\eta,+}\right) m^{\mathrm{BMS}}\left(\Phi_{\eta,g}\right) + c_1 \eta^{4n-\frac{1}{2}n^2-3-\delta_\Gamma-2\ell} \xi^{-\ell-(n-1)/2} S_\ell\left(\psi\right) S_\ell\left(f\right) e^{-\kappa' s} \\ &< \left(\tilde{m}^{\mathrm{BMS}}(\psi) + c_3\eta S_\ell(\psi)\right) \left(\left(1 + c_2 e^{\omega \operatorname{height}(g\Gamma)} \frac{\xi^\alpha}{\eta^\theta}\right) \mu_g^{\mathrm{PS}}(f)\right) \\ &\quad + c_1 e^{\delta_\Gamma \operatorname{height}(g\Gamma)} \eta^{4n-\frac{1}{2}n^2-3-2\delta_\Gamma} \xi^{-\ell-(n-1)/2} S_\ell\left(\psi\right) S_\ell\left(f\right) e^{-\kappa' s}. \end{aligned}$$

It follows from the proof of Lemma 3.4.4 that $\tilde{m}^{\mathrm{BMS}}(\psi) \ll_{\operatorname{supp}\psi} S_\ell(\psi)$ and $\mu_g^{\mathrm{PS}}(f) \ll S_\ell(f)\mu_g^{\mathrm{PS}}(B_U(1))$. Then, using Proposition 3.2.1 we arrive at

$$\begin{aligned} &\sum_{\gamma\in\Gamma} \int_{B_U(r_1)} \psi(a_s u_{\mathbf{t}} g\gamma) f(\mathbf{t}) d\mu_g^{\mathrm{PS}}(\mathbf{t}) - \mu_g^{\mathrm{PS}}(f) \tilde{m}^{\mathrm{BMS}}(\psi) \\ &\ll_\Gamma \left( e^{\omega \operatorname{height}(g\Gamma)} \frac{\xi^\alpha}{\eta^\theta} + e^{\delta_\Gamma \operatorname{height}(g\Gamma)} \eta^{4n-\frac{1}{2}n^2-3-2\delta_\Gamma} \xi^{-\ell-(n-1)/2} e^{-\kappa' s} \right) \\ &\qquad \cdot S_\ell\left(\psi\right) S_\ell\left(f\right) \mu_g^{\mathrm{PS}}(B_U(1)) \end{aligned}$$

Define

$$\kappa = \frac{3\alpha\theta\kappa'}{2\theta(2\ell + n - 1) + 9\alpha(2\delta_\Gamma + 3 + n^2/2 - 4n)},$$

and note that by making $\ell$ larger if necessary, we guarantee $\kappa > 0$. Recall from (3.19) that

$$e^{-\operatorname{height}(g\Gamma)} \ll_\Gamma \operatorname{inj}(g).$$

For $s \geq \max\{\theta, \omega\} \operatorname{height}(g\Gamma)/\kappa$, choose

$$\eta = e^{-\kappa s/\theta},\ \xi = e^{-4\kappa s/\alpha}. \tag{6.10}$$

Note that $\eta < \operatorname{inj}(g\Gamma)$ by choice of $s$, $\omega \operatorname{height}(g\Gamma) \leq \kappa s$, and $\xi < \eta$ since by Proposition 5.0.2, $\alpha < \theta$. By Proposition 5.0.2 we have $\omega > \delta_\Gamma$, therefore $\delta_\Gamma \operatorname{height}(g\Gamma) \leq \kappa s$. Note also that $\max\{\theta, \omega\} \operatorname{height}(g\Gamma)/\kappa \ll_\Gamma \operatorname{height}(g\Gamma)$.

With these choices, we obtain

$$e^{\omega \operatorname{height}(g\Gamma)} \left(\frac{\xi}{\eta^{\theta'}}\right)^{\alpha'} + e^{\delta_\Gamma \operatorname{height}(g\Gamma)} \eta^{4n - \frac{1}{2}n^2 - 3 - 2\delta_\Gamma} \xi^{-\ell-(n-1)/2} e^{-\kappa' s} \leq 2e^{-2\kappa s}. \tag{6.11}$$

In a similar way, using $\psi_{\eta,-}$, one can show a lower bound, proving (6.4).

**Step 3: Covering argument for general $f$.**

We now deduce the claim by decomposing $f$ into a sum of functions, each defined on a ball of radius $r_1$ in $U$.

Let $u_1, \ldots, u_k$ and $\sigma_1, \ldots, \sigma_k \in C_c^\infty(B_U(r))$ satisfy the conclusion of Lemma 3.4.7 for $E = B_U(r)$ and $r_1$. For $1 \leq i \leq k$, let

$$f_i := f\sigma_i.$$

Then, $f \leq \sum_{i-1}^{k} f_i$, and by Lemma 3.4.7 and [KM96, Lemma 2.4.7(a)]

$$S_\ell(f_i) \ll_\Gamma S_\ell(f) S_\ell(\sigma_i) \ll_\Gamma r_1^{-\ell+n-1} S_\ell(f). \tag{6.12}$$

Since each $f_i$ is supported on $B_U(r_1)u_i$ for some $u_i \in B_U(1)$, by (6.4) we have

$$\sum_{\gamma\in\Gamma} \int_{B_U(r_1)} \psi(a_s u_{\mathbf{t}} g\gamma) f_i(\mathbf{t}) d\mu_g^{\mathrm{PS}}(\mathbf{t}) - \tilde{m}^{\mathrm{BMS}}(\psi)\mu_g^{\mathrm{PS}}(f_i)$$

$$\ll_{\Gamma, \operatorname{supp}\psi} \mu_g^{\mathrm{PS}}(B_U(1)) S_\ell(\psi) S_\ell(f_i) e^{-2\kappa s}.$$

Summing the above expressions for $i = 1, \ldots, k$, we get

$$\sum_{\gamma\in\Gamma} \int_{B_U(r)} \psi(a_s u_{\mathbf{t}} g\gamma) f(\mathbf{t}) d\mu_g^{\mathrm{PS}}(\mathbf{t}) - \tilde{m}^{\mathrm{BMS}}(\psi)\mu_g^{\mathrm{PS}}(f)$$

$$\ll k r_1^{-\ell+n-1} S_\ell(\psi) S_\ell(f) e^{-2\kappa s} \mu_g^{\mathrm{PS}}(B_U(1))$$

$$\ll \left(\frac{r}{r_1}\right)^{n-1} r_1^{-\ell+n-1} S_\ell(\psi) S_\ell(f) e^{-2\kappa s} \mu_g^{\mathrm{PS}}(B_U(1))$$

$$\ll r_1^{-\ell} S_\ell(\psi) S_\ell(f) e^{-2\kappa s} \mu_g^{\mathrm{PS}}(B_U(1))$$

$$\ll S_\ell(\psi) S_\ell(f) e^{-\kappa s} \mu_g^{\mathrm{PS}}(B_U(1)),$$

where the first inequality is by Lemma 3.4.7, the second inequality follows from $r_1 = \operatorname{inj}(g) > e^{-\kappa s/\ell}$, the third is by (6.10) and because $r < 1$, and the implied constants depend on $\Gamma$ and $\operatorname{supp}\psi$.

As before, using similar arguments, one can show a lower bound, proving the claim.

$\square$

We will now use a partition of unity argument to prove Theorem 1.2.3. For the reader's convenience, we restate it below.

**Theorem 6.1.2.** *There exist $\kappa = \kappa(\Gamma)$ and $\ell = \ell(\Gamma)$ which satisfy the following: for any $\psi \in C_c^\infty(X)$, there exists $c = c(\Gamma, \operatorname{supp}\psi) > 0$ such that for any $f \in C_c^\infty(B_U(r))$, $0 < r < 1$, $x \in \operatorname{supp} m^{\mathrm{BMS}}$, and $s \gg_\Gamma \mathrm{height}(x)$, we have*

$$\left| \int_U \psi(a_s u_{\boldsymbol{t}} x) f(\boldsymbol{t}) d\mu_x^{\mathrm{PS}}(\boldsymbol{t}) - \mu_x^{\mathrm{PS}}(f) m^{\mathrm{BMS}}(\psi) \right| < c S_\ell(\psi) S_\ell(f) e^{-\kappa s}.$$

*Proof.* According to [OS13, Lemma 2.17], there exists an admissible box $B_y$ around $y$, for any $y \in X$. Then, $\{B_y : y \in \operatorname{supp}\psi\}$ is an open cover of the compact set $\operatorname{supp}\psi$. Hence, there exists a minimal sub-cover $B_{y_1}, \ldots, B_{y_k}$. Using a similar construction to one in Lemma 3.4.7, there exist $\sigma_1, \ldots, \sigma_k$, a partition of unity for $\operatorname{supp}\psi$, such that for $i = 1, \ldots, k$ we have $\sigma_i \in C_c^\infty(B_{y_i})$ and for $i = 1, \ldots, k$ and $m = 1, \ldots, \ell$

$$|\sigma_i^{(m)}| \ll_{\operatorname{supp}\psi, \Gamma} 1 \tag{6.13}$$

(the implied constant depends on the chosen sub-cover).

Define $\psi_i = \psi \sigma_i$. Then

$$\psi = \sum_{i=1}^{k} \psi_i, \tag{6.14}$$

and by (6.13) and the product rule, we have

$$S_\ell(\psi_i) \ll_{\operatorname{supp}\psi, \Gamma} S_\ell(\psi). \tag{6.15}$$

According to Proposition 6.1.1 and Proposition 3.2.1, there exist $c = c(\Gamma, \operatorname{supp}\psi) > 0$,

$\lambda = \lambda(\Gamma) > 1$ such that for $s \gg_\Gamma \operatorname{height}(x)$,

$$\begin{aligned}
&\int_{B_U(r)} \psi(a_s u_{\mathbf{t}} x) f(\mathbf{t}) d\mathbf{t} \\
&= \sum_{i=1}^{k} \int_{B_U(r)} \psi_i(a_s u_{\mathbf{t}} x) f(\mathbf{t}) d\mathbf{t} \\
&\leq \sum_{i=1}^{k} m^{\mathrm{BMS}}(\psi_i)\mu_x^{\mathrm{PS}}(f) + cS_\ell(\psi_i)S_\ell(f)e^{-\kappa s}\mu_x^{\mathrm{PS}}(B_U(1)) \\
&\leq \sum_{i=1}^{k} m^{\mathrm{BMS}}(\psi_i)\mu_g^{\mathrm{PS}}(f) + c\lambda S_\ell(\psi_i)S_\ell(f)e^{-\kappa s+(n-1-\delta_\Gamma)d(\pi(\mathcal{C}_0),\pi(x))} \\
&\ll_{\Gamma,\operatorname{supp}\psi} m^{\mathrm{BMS}}(\psi)\mu_g^{\mathrm{PS}}(f) + c\lambda S_\ell(\psi)S_\ell(f)e^{-\kappa s+(n-1-\delta_\Gamma)\operatorname{height}(x)}.
\end{aligned}$$

where the last line follows by the definition of $\operatorname{height}(x)$ and equations (6.14) and (6.15). Moreover, we may assume that $s \geq \frac{2(n-1-\delta_\Gamma)}{\kappa}\operatorname{height}(x)$ without changing the assumption $s \gg_\Gamma \operatorname{height}(x)$. Then

$$e^{-\kappa s+(n-1-\delta_\Gamma)\operatorname{height}(x)} \ll_\Gamma e^{-\kappa s/2},$$

as desired. □

We will now use Theorem 1.2.3 to prove a similar result for the Haar measure. This will be necessary for the proof of Theorem 1.2.2. Note that such a result is proven in [MO15] under a spectral gap assumption on $\Gamma$, but we show here how to prove it whenever the frame flow is exponentially mixing.

**Theorem 6.1.3.** *There exists $\kappa = \kappa(\Gamma) < 1$ and $\ell = \ell(\Gamma)$ that satisfy the following: let $0 < r < 1$, let $f \in C_c^\infty(B_U(r))$, and let $\psi \in C_c^\infty(X)$ be supported on an admissible box. Then there exists $c = c(\Gamma, \operatorname{supp}\psi) > 0$ such that for every $x \in \operatorname{supp} m^{\mathrm{BMS}}$ and*

$s \gg_{\Gamma,\operatorname{supp}\psi} \operatorname{height}(x)$,

$$\left| e^{(n-1-\delta_\Gamma)s} \int_{B_U(r)} \psi(a_s u_{\boldsymbol{t}} x) f(\boldsymbol{t}) d\boldsymbol{t} - \mu_x^{\mathrm{PS}}(f) m^{\mathrm{BR}}(\psi) \right| < c S_\ell(\psi) S_\ell(f) e^{-\kappa s}.$$

*Proof.* **Step 1: Setup and approximations.**

Assume $s \gg_\Gamma \operatorname{height}(x)$, and let $\kappa, \ell'$ satisfy the conclusion of Theorem 1.2.3, and $\ell > \ell'$ satisfy the conclusion of Lemma 3.4.4.

Since $\psi$ is assumed to be supported on an admissible box, there exist $r_0, \eta, \varepsilon_0, \varepsilon_1 > 0$ (depending only on $\operatorname{supp}\psi$) and $z \in X$ such that

$$\operatorname{supp}\psi = B_U(r_0) P_\eta z,$$

and

$$G_{\varepsilon_0} \operatorname{supp}\psi \subset B_U(r_0 + \varepsilon_1) P_{\eta+\varepsilon_1} z,$$

where $B_U(r+\varepsilon_1)P_{\eta+\varepsilon_1} z$ is also an admissible box. Denote $\eta' = \eta + \varepsilon_1$ and $r_0' = r_0 + \varepsilon_1$.

Without loss of generality, assume that $f$ is a non-negative function. Continuously extend $\psi$ to $P_{\eta'}$ by defining $\psi = 0$ on $P_{\eta'} \setminus P_\eta$.

For $0 < \varepsilon < \varepsilon_0$ let $\psi_{\varepsilon,\pm}$ and $f_{\varepsilon,\pm}$ for Lemma 3.4.4 for $G, \varepsilon, \psi$ and $U, \varepsilon, f$, respectively. By Lemma 3.4.4,

$$S_{\ell'}(\psi_{\varepsilon,\pm}) \ll_{\Gamma,\operatorname{supp}(\psi)} \varepsilon^{-2\ell} S_\ell(\psi) \quad \text{and} \quad S_{\ell'}(f_{\varepsilon,\pm}) \ll_\Gamma \varepsilon^{-2\ell} S_\ell(f). \tag{6.16}$$

Moreover, by Lemma 3.4.4(2),

$$\|f_{\varepsilon,\pm} - f\|_\infty \le \varepsilon S_\ell(f). \tag{6.17}$$

For $p \in P_{\eta'}$, define

$$\varphi(p) := \mu_{pz}^{\mathrm{PS}}(B_U(r_0')pz). \tag{6.18}$$

**Step 1.1: Construct a smooth approximation to $1/\varphi$.**

Since the Busemann function is smooth and $\varphi$ is bounded below by a positive quantity on $P_{\eta'}$ by Corollary 3.1.2, the mean value theorem implies that for any $0 < \varepsilon < \varepsilon_0$ and all $p, p' \in P_\varepsilon$, there exists a constant $d = d(\Gamma, \operatorname{supp}\psi)$ such that

$$\left|\frac{1}{\varphi(p)} - \frac{1}{\varphi(p')}\right| \le \frac{d\varepsilon}{\varphi(p)}. \tag{6.19}$$

By Lemma 3.4.2, for any $\xi > 0$, there exists a non-negative smooth function $\chi_\xi$ with

$$1_{P_{\varepsilon-\xi}} \le \chi_\xi \le \mathbf{1}_{P_\varepsilon} \tag{6.20}$$

and $S_{\ell'}(\chi_\xi) \ll_{\Gamma,n} (\varepsilon - \xi/2)^{n-1}(\xi/2)^{-\ell'-(n-1)/2}$. Define

$$\sigma(p) := \frac{1}{\varphi} * \frac{\chi_\xi}{m(P_{\varepsilon-\xi})} \tag{6.21}$$

where $m$ denotes the probability Haar measure on $P$. Then, assuming $\varepsilon_0 < 1/2$ and $\xi \le \varepsilon^2$, by (6.19), (6.20), and (6.21), we have that

$$\begin{aligned}
\frac{1-d\varepsilon}{\varphi(p)} &\le \frac{1}{m(P_{\varepsilon-\xi})}\int_{pP_{\varepsilon-\xi}} \frac{1}{\varphi(p')}dp' && (6.22)\\
&\le \sigma(p)\\
&\le \frac{1}{m(P_{\varepsilon-\xi})}\int_{pP_\varepsilon} \frac{1+d\varepsilon}{\varphi(p')}dp'\\
&\le \left(\frac{\varepsilon}{\varepsilon-\xi}\right)^n \frac{1+d\varepsilon}{\varphi(p)}\\
&\le \frac{1+d'\varepsilon}{\varphi(p)}, && (6.23)
\end{aligned}$$

for some absolute constant $d' > 0$.

For $upz \in B_U(r_0')P_{\eta'}z$ and $0 < \varepsilon < \varepsilon_0$, let

$$\Psi_{\varepsilon,\pm}(upz) = \sigma(p) \int_{Upz} \psi_{c_1\varepsilon,\pm}(u_{\mathbf{t}}pz)d\mathbf{t}.$$

Then, by (6.19),

$$\begin{aligned} \sup_{w \in G_\varepsilon} \Psi_{\varepsilon,\pm}(wupz) &= \sup_{w \in P_\varepsilon} \sigma(wp) \int_{Uwpz} \psi_{c_1\varepsilon,+}(u_{\mathbf{t}}wpz)d\mathbf{t} \\ &\le (1 + d'\varepsilon)\Psi_{2\varepsilon,\pm}. \end{aligned} \tag{6.24}$$

**Step 2: Bounding with PS measure.**

Let

$$P(f, \psi, x; s) = \{p \in P_\eta : a_s \operatorname{supp}(f)x \cap B_U(r_0)pz \neq \emptyset\}.$$

By [MO15, Lemma 6.2], there exists an absolute constant $c_1 > 0$ such that

$$\begin{aligned} &e^{(n-1)s} \int_{B_U(r)} \psi(a_s u_{\mathbf{t}} x) f(\mathbf{t}) d\mathbf{t} \\ &\le (1 + c_1\varepsilon) \sum_{p \in P(f,\psi,x;s)} f_{c_1 e^{-s}\eta}(a_{-s}pz) \int_{Upz} \psi_{c_1\varepsilon,+}(u_{\mathbf{t}}pz)d\mathbf{t}. \end{aligned} \tag{6.25}$$

It now follows from [MO15, Lemma 6.5], (6.22), and (6.24) that there exists an absolute constant $c_2 > 0$ such that

$$\begin{aligned} &e^{-\delta_\Gamma s} \sum_{p \in P(f,\psi,x;s)} f_{c_1 e^{-s}\eta}(a_{-s}pz) \int_{Upz} \psi_{c_1\varepsilon,+}(u_{\mathbf{t}}pz)d\mathbf{t} \\ &\le \frac{(1 + c_2\varepsilon)(1 + d'\varepsilon)}{1 - d\varepsilon} \int_U \Psi_{2c_2\varepsilon,+}(a_s u_{\mathbf{t}} x) f_{(c_1+c_2)e^{-s}\varepsilon_0,+}(\mathbf{t}) d\mu_x^{\mathrm{PS}}(\mathbf{t}). \end{aligned}$$

Note that (6.22) is needed because our definition of $\Psi_{\varepsilon,+}$ is not identical to $\Psi$ as defined in [MO15, Lemma 6.5]. The latter is bounded above by $\frac{1}{1-d\varepsilon}\Psi_{\varepsilon,+}$ by (6.22).

Combining the above with (6.25), we get that there exist constants $c_3 > 0$, $c_4 = c_4(\Gamma, \operatorname{supp}\psi) > 0$ such that

$$\begin{aligned} &e^{(n-1-\delta_\Gamma)s}\int_{B_U(r)}\psi(a_s u_{\mathbf{t}} x) f(\mathbf{t}) d\mathbf{t} \\ &\leq (1+c_4\varepsilon)\int_U \Psi_{c_3\varepsilon,+}(a_s u_{\mathbf{t}} x) f_{c_3 e^{-s}\varepsilon_0,+}(\mathbf{t}) d\mu_x^{\mathrm{PS}}(\mathbf{t}). \end{aligned}$$

It follows from Theorem 1.2.3 that for some constant $c_5 = c_5(\Gamma, \operatorname{supp}\psi) > 0$

$$\begin{aligned} &e^{(n-1-\delta_\Gamma)s}\int_{B_U(r)}\psi(a_s u_{\mathbf{t}} x) f(\mathbf{t}) d\mathbf{t} \\ &\leq (1+c_4\varepsilon)\left(\mu_x^{\mathrm{PS}}(f_{c_3 e^{-s}\varepsilon_0,+}) m^{\mathrm{BMS}}(\Psi_{c_3\varepsilon,+}) + c_5 S_{\ell'}(\Psi_{c_3\varepsilon,+}) S_{\ell'}(f_{c_3 e^{-s}\varepsilon_0,+}) e^{-\kappa s}\right). \end{aligned} \tag{6.26}$$

**Step 3: Bounding the error terms.**

We now show how to bound the various error terms to obtain the desired conclusion.

To compute $m^{\mathrm{BMS}}(\Psi)$, we use (3.22), (6.17), and (6.23) to deduce that for some $c_6 = c_6(\Gamma, \operatorname{supp}\psi)$, if $\xi = \varepsilon^2$,

$$\begin{aligned} &m^{\mathrm{BMS}}(\Psi_{c_3\varepsilon,+}) \\ &\leq (1+d'\varepsilon)\int_{P_{\eta'}z}\int_{B_U(r_0')}\frac{1}{\mu_{pz}^{\mathrm{PS}}(B_U(r_0')pz)}\int_{B_U(r_0')pz}\psi_{c_1\varepsilon,\pm}(u_{\mathbf{t}} pz) d\mathbf{t} d\mu_{pz}^{\mathrm{PS}}(\mathbf{t}) d\nu(pz) \\ &\leq (1+d'\varepsilon)\int_{P_{\eta'}z}\int_{B_U(r_0')pz}\psi_{c_1\varepsilon,\pm}(u_{\mathbf{t}} pz) d\mathbf{t} d\nu(pz) \\ &\leq (1+d'\varepsilon)\left(m^{\mathrm{BR}}(\psi) + c_6\varepsilon S_\ell(\psi)\right). \end{aligned} \tag{6.27}$$

By Proposition 3.2.1, if $s$ is sufficiently large so that $r + c_3 e^{-s}\varepsilon_0 \leq 1$ (note that this requirement on $s$ depends only on $\Gamma$ and $\operatorname{supp}\psi$), we have that

$$\mu_x^{\mathrm{PS}}(B_U(r+c_3e^{-s}\varepsilon_0)) \leq \mu_x^{\mathrm{PS}}(B_U(1)) \ll_\Gamma e^{(n-1-\delta_\Gamma)d(\pi(\mathcal{C}_0),\pi(x))}. \tag{6.28}$$

Hence, by (6.17) and (6.28), we have

$$\begin{aligned}\mu_x^{\mathrm{PS}}(f_{c_3e^{-s}\varepsilon_0,+}) - \mu_x^{\mathrm{PS}}(f) &\ll_\Gamma e^{-s}\varepsilon_0 S_\ell(f)\mu_x^{\mathrm{PS}}(B_U(r+c_3e^{-s}\varepsilon_0)) \\ &\ll_\Gamma e^{-s}\varepsilon_0 S_\ell(f) e^{(n-1-\delta_\Gamma)d(\pi(\mathcal{C}_0),\pi(x))}. \end{aligned} \tag{6.29}$$

According to [KM96, Lemma 2.4.7(a)] and (6.16), if $\xi = \varepsilon^2$ and

$$\varepsilon = e^{-\frac{\kappa s}{2(n+4\ell)}}, \tag{6.30}$$

then

$$\begin{aligned} S_{\ell'}(\Psi_{c_3\varepsilon,+}) &\ll_\Gamma S_{\ell'}(\psi_{c_3\varepsilon,+})S_{\ell'}(\sigma) \\ &\ll_\Gamma (m(P_{\varepsilon-\xi}))^{-1}(\varepsilon-\xi/2)^{n-1}\xi^{-\ell'-(n-1)/2}\varepsilon^{-2\ell}S_\ell(\psi) \\ &\ll \varepsilon^{-1-2\ell}\xi^{-\ell'-(n-1)/2}S_\ell(\psi) \\ &\leq e^{\kappa s/2}S_\ell(\psi). \end{aligned} \tag{6.31}$$

Using (6.26), (6.27), (6.29), and (6.31), we obtain

$$\begin{aligned} & e^{(n-1-\delta_\Gamma)s}\int_{B_U(r)} \psi(a_s u_{\mathbf{t}} x) f(\mathbf{t}) d\mathbf{t} - \mu_x^{\mathrm{PS}}(f) m^{\mathrm{BR}}(\psi) \\ & \leq (1+c_4\varepsilon)\Bigg[ d'\varepsilon\mu_x^{\mathrm{PS}}(f)m^{\mathrm{BR}}(\psi) + (1+d'\varepsilon)\Bigg\{ c_6\varepsilon\mu_x^{\mathrm{PS}}(f)S_\ell(\psi) \\ & \qquad + (e^{-s}\varepsilon_0 m^{\mathrm{BR}}(\psi)S_\ell(f) + c_6e^{-s}\varepsilon_0\varepsilon S_\ell(f)S_\ell(\psi))e^{(n-1-\delta_\Gamma)d(\pi(\mathcal{C}_0),\pi(x))}\Bigg\} \\ & \qquad + c_8 S_\ell(\psi)S_\ell(f)e^{-\kappa s/2}\Bigg] \end{aligned} \tag{6.32}$$

These remaining error terms can be controlled as follows. Using (6.28), we can deduce

$$\mu_x^{\mathrm{PS}}(f) \le \|f\|_\infty \, \mu_x^{\mathrm{PS}}(B_U(r)) \ll_\Gamma S_\ell(f) e^{(n-1-\delta_\Gamma)(\pi(\mathcal{C}_0),\pi(x))}. \tag{6.33}$$

We also have that

$$m^{\mathrm{BR}}(\psi) \ll_{\Gamma,\operatorname{supp}\psi} S_\ell(\psi). \tag{6.34}$$

Combining (6.32), (6.33), and (6.34) implies

$$\begin{aligned} &e^{(n-1-\delta_\Gamma)s} \int_{B_U(r)} \psi(a_s u_{\mathbf{t}} x) f(\mathbf{t}) d\mathbf{t} - \mu_x^{\mathrm{PS}}(f) m^{\mathrm{BR}}(\psi) \\ &\ll_{\Gamma,\operatorname{supp}\psi} S_\ell(\psi) S_\ell(f) \left[ d'\varepsilon + (1+d'\varepsilon)\left(c_6\varepsilon(1+e^{-s}\varepsilon_0) + e^{-s}\varepsilon_0\right) e^{(n-1-\delta_\Gamma)d(\pi(\mathcal{C}_0),\pi(x))} + c_8 e^{-\kappa s/2} \right] \end{aligned} \tag{6.35}$$

Finally, by the choice of $\varepsilon$ in (6.30) and because we may assume without loss of generality that $\kappa < 1$, we obtain from (6.35) that there exists $\kappa' < 1$ such that

$$\begin{aligned} &e^{(n-1-\delta_\Gamma)s} \int_{B_U(r)} \psi(a_s u_{\mathbf{t}} x) f(\mathbf{t}) d\mathbf{t} - \mu_x^{\mathrm{PS}}(f) m^{\mathrm{BR}}(\psi) \\ &\ll_{\Gamma,\operatorname{supp}\psi} S_\ell(\psi) S_\ell(f) e^{-\kappa' s + (n-1-\delta_\Gamma)d(\pi(\mathcal{C}_0),\pi(x))}. \end{aligned}$$

Recall that $d(\pi(\mathcal{C}_0),\pi(x)) = \operatorname{height}(x)$. Thus, if we assume that $s \ge \frac{2(n-1-\delta_\Gamma)}{\kappa'} \operatorname{height}(x)$ (which means $s \gg_\Gamma \operatorname{height}(x)$), then

$$e^{-\kappa' s + (n-1-\delta_\Gamma)\operatorname{height}(x)} \ll_\Gamma e^{-\kappa'/2s},$$

which completes the proof. $\square$

## 6.2 Proof of Theorem 1.2.1

In this section, we prove Theorem 1.2.1, which is restated below for the reader's convenience. The proof relies on the quantitative nondivergence result in Theorem 4.0.1 and Theorem 1.2.3. As a reminder, throughout the section, we must assume that Assumption 1.1.2 holds.

**Theorem 6.2.1.** *For any $0 < \varepsilon < 1$ and $s_0 \geq 1$, there exist constants $\ell = \ell(\Gamma) \in \mathbb{N}$ and $\kappa = \kappa(\Gamma, \varepsilon) > 0$ satisfying: for every $\psi \in C_c^\infty(G/\Gamma)$, there exists $c = c(\Gamma, \operatorname{supp}\psi)$ such that every $x \in G/\Gamma$ that is $(\varepsilon, s_0)$-Diophantine, and for every $T$ with $T^{1-\varepsilon/2} \gg_\Gamma s_0$,*

$$\left| \frac{1}{\mu_x^{\mathrm{PS}}(B_U(T))} \int_{B_U(T)} \psi(u_{\boldsymbol{t}}x) d\mu_x^{\mathrm{PS}}(\boldsymbol{t}) - m^{\mathrm{BMS}}(\psi) \right| \leq c S_\ell(\psi) r^{-\kappa},$$

*where $S_\ell(\psi)$ is the $\ell$-Sobolev norm.*

*Proof.* Let $\beta > 0$ satisfy the conclusion of Theorem 4.0.1 for $\varepsilon$ and $s_0$. Let $\kappa' > 0$, $\ell \in \mathbb{N}$ satisfy the conclusion of Theorem 1.2.3.

Since $x$ is $(\varepsilon, s_0)$-Diophantine, by Theorem 4.0.1, for $T_0 \gg_\Gamma s_0$ and $R \geq R_0$,

$$\mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0 \cap \mathcal{X}(R)) \ll \mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0) e^{-\beta R}, \tag{6.36}$$

where

$$s_\varepsilon := \frac{\varepsilon}{2} \log T, \quad T_0 := T e^{-s_\varepsilon} = T^{1-\varepsilon/2}, \quad x_0 := a_{-s_\varepsilon} x. \tag{6.37}$$

By (3.3) and (3.11), we have

$$\frac{1}{\mu_x^{\mathrm{PS}}(B_U(T))} \int_{B_U(T)} \psi(u_{\mathbf{t}}x) d\mu_x^{\mathrm{PS}}(\mathbf{t}) = \frac{1}{\mu_{x_0}^{\mathrm{PS}}(B_U(T_0))} \int_{B_U(T_0)} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0) d\mu_{x_0}^{\mathrm{PS}}(\mathbf{t}).$$

Fix $R > R_0$, and define

$$Q_0 = B_U(T_0)x_0 \cap \mathcal{C}(R).$$

By the definition of $\mathcal{C}(R)$,

$$Q_0 \subseteq \operatorname{supp} m^{\mathrm{BMS}}.$$

Let $\rho > 0$ be smaller than half of the injectivity radius of $Q_0$.

First, by Lemma 3.4.7, there exist $\{y : y \in I_0\} \subseteq Q_0$ and $f_y \in C_c^\infty(B_U(2\rho)y)$ satisfying

$$S_\ell(f_y) \ll \rho^{-\ell+n-1} \tag{6.38}$$

and

$$\sum_y f_y = 1 \text{ on } E_1 := \bigcup_{y \in I_0} B_U(\rho)y \supseteq Q_0$$

and 0 outside of

$$E_2 = \bigcup_{y \in I_0} B_U(2\rho)y.$$

Observe that

$$Q_0 \subseteq E_1 \subseteq E_2 \subseteq B_U(T_0 + 2\rho)x_0. \tag{6.39}$$

Thus,

$$\int_{u_{\mathbf{t}}x_0 \in E_1} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0) d\mu_{x_0}^{\mathrm{PS}}(\mathbf{t}) \leq \sum_{y \in I_0} \int_{u_{\mathbf{t}}x_0 \in B_U(2\rho)y} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0) f_y(u_{\mathbf{t}} x_0) d\mu_{x_0}^{\mathrm{PS}}(\mathbf{t})$$

Because $Q_0 \subseteq \operatorname{supp} m^{\mathrm{BMS}}$, we may use Proposition 3.2.1 to deduce that there exists

$\lambda = \lambda(\Gamma) \geq 1$ such that for any $y \in I_0$, we have

$$\begin{aligned}
\mu_y^{\mathrm{PS}}(B_U(\rho)) &\geq \lambda^{-1}\rho^{\delta_\Gamma} e^{(k(y,\rho)-\delta_\Gamma)d(\pi(\mathcal{C}_0),\pi(a_{-\log\rho}y))} \\
&\geq \lambda^{-1}\rho^{\delta_\Gamma} e^{-\delta_\Gamma d(\pi(\mathcal{C}_0),\pi(a_{-\log\rho}y))} \\
&\geq \lambda^{-1}\rho^{\delta_\Gamma} e^{-\delta_\Gamma(-\log\rho)} e^{-\delta_\Gamma \operatorname{height}(y)} && \text{since } \rho < 1 \\
&\gg_\Gamma \lambda^{-1}\rho^{2\delta_\Gamma} e^{-\delta_\Gamma \operatorname{height}(y)} \\
&\geq \lambda^{-1}\rho^{2\delta_\Gamma} e^{-\delta_\Gamma R},
\end{aligned}$$

where the last line follows by Lemma 3.2.3.

Since $e^{s_\varepsilon} = T^{\varepsilon/2}$, it follows from (6.38) and the above, that if we choose $\rho$ and $R$ such that

$$e^{\delta_\Gamma R}\rho^{n-1-\ell-2\delta_\Gamma} \ll_\Gamma T^{\varepsilon\kappa'/4}, \tag{6.40}$$

then, by the choice of $f_y$, we have

$$S_\ell(f_y) \ll \mu_y^{\mathrm{PS}}(B_U(\rho))e^{\kappa' s_\varepsilon/2} \ll \mu_y^{\mathrm{PS}}(f_y)e^{\kappa' s_\varepsilon/2} \tag{6.41}$$

where the implied constant is absolute.

If we further assume that

$$T \gg_\Gamma e^{2R/\varepsilon} \tag{6.42}$$

(with the implied constant coming from Theorem 1.2.3), then $s_\varepsilon \gg_\Gamma R$, and by (6.41), Theorem 1.2.3, and Lemma 3.2.3, there exist $c_1, c_2 > 0$ which depend only on $\Gamma$ and supp $\psi$ such that

$$\begin{aligned}
&\sum_{y\in I_0}\int_{u_{\mathbf{t}}x_0\in B_U(2\rho)y}\psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)f_y(u_{\mathbf{t}}x_0)d\mu^{\mathrm{PS}}_{x_0}(\mathbf{t})\\
&\le\sum_{y\in I_0}\left(m^{\mathrm{BMS}}(\psi)\mu^{\mathrm{PS}}_y(f_y)+c_1S_\ell(\psi)S_\ell(f_y)e^{-\kappa' s_\varepsilon}\right)\\
&\le\sum_{y\in I_0}\mu^{\mathrm{PS}}_y(f_y)\left(m^{\mathrm{BMS}}(\psi)+c_2S_\ell(\psi)e^{-\kappa' s_\varepsilon/2}\right)
\end{aligned}$$

By Lemma 3.2.3 and Theorem 5.0.1, there exists $c_3=c_3(\Gamma)>0$ such that if $T_0\gg s_0$, then there exist $\alpha=\alpha(\Gamma)>0$, $c_3=c_3(\Gamma)>0$ such that

$$\begin{aligned}
\sum_{y\in I_0}\mu^{\mathrm{PS}}_y(f_y)&\le\mu^{\mathrm{PS}}_{x_0}(B_U(T_0+2\rho)\cap\mathcal{C}(R+1))\\
&\ll_\Gamma(1+c_3(2\rho)^\alpha)\mu^{\mathrm{PS}}_{x_0}B_U(T_0)
\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}
&\sum_{y\in I_0}\int_{u_{\mathbf{t}}x_0\in B_U(2\rho)y}\psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)f_y(u_{\mathbf{t}}x_0)d\mu^{\mathrm{PS}}_{x_0}(\mathbf{t}) \qquad (6.43)\\
&\le\mu^{\mathrm{PS}}_{x_0}(B_U(T_0))\,(1+c_3\,(2\rho)^\alpha)\left(m^{\mathrm{BMS}}(\psi)+c_2S_\ell(\psi)e^{-\kappa' s_\varepsilon/2}\right)
\end{aligned}$$

Fix

$$\kappa:=\kappa'\varepsilon/4,\quad R>\frac{\kappa}{\beta}\log T,\quad \rho<T^{-\frac{\kappa}{\alpha}} \qquad (6.44)$$

such that $\rho$ also satisfies the assumption of Theorem 5.0.1, and $R$ which satisfies (6.40) and (6.42). Thus, (6.43) and (6.44) imply

$$\begin{aligned}
&\frac{1}{\mu^{\mathrm{PS}}_{x_0}(B_U(T_0))}\int_{u_{\mathbf{t}}x_0\in E_1}\psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)d\mu^{\mathrm{PS}}_{x_0}(\mathbf{t})-m^{\mathrm{BMS}}(\psi)\\
&\ll_{\Gamma,\operatorname{supp}\psi}S_\ell(\psi)T^{-\kappa}, \qquad (6.45)
\end{aligned}$$

where we have used that by [Aub82], $\|\psi\|_\infty \ll_{\operatorname{supp}\psi} S_\ell(\psi)$, so $m^{\mathrm{BMS}}(\psi) \ll_{\operatorname{supp}\psi} S_\ell(\psi)$.

By (6.36),

$$\begin{aligned}\int_{B_U(T_0)x_0\setminus E_1} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0)\, d\mu^{\mathrm{PS}}_{x_0}(\mathbf{t}) &\le \|\psi\|_\infty \mu^{\mathrm{PS}}_{x_0}(B_U(T_0)\setminus E_1)\\ &\ll_{\operatorname{supp}\psi} S_\ell(\psi)\mu^{\mathrm{PS}}_{x_0}(B_U(T_0)x_0)e^{-\beta R}\\ &\ll_{\operatorname{supp}\psi} S_\ell(\psi)\mu^{\mathrm{PS}}_{x_0}(B_U(T_0)x_0)T^{-\kappa},\end{aligned}$$

where we have again used that by [Aub82], $\|\psi\|_\infty \ll_{\operatorname{supp}\psi} S_\ell(\psi)$. Combining the above with (6.45) implies that

$$\begin{aligned}&\frac{1}{\mu^{\mathrm{PS}}_{x_0}(B_U(T_0))}\int_{B_U(T_0)} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0)\, d\mu^{\mathrm{PS}}_{x_0}(\mathbf{t}) - m^{\mathrm{BMS}}(\psi)\\ &\ll_{\Gamma,\operatorname{supp}\psi} S_\ell(\psi)T^{-\kappa}\end{aligned}$$

The lower bound is obtained similarly, as is shown in the proof of Theorem 1.2.2 in the next section. □

## 6.3 Proof of Theorem 1.2.2

In this section, we will prove Theorem 1.2.2 using Theorem 6.1.3. We will use a partition of unity argument for a cover of the intersection of $B_U(r)x$ with a fixed compact set by small balls centered at PS-points. Assumption 1.1.2 is required.

We will need the following lemma.

**Lemma 6.3.1.** *There exists an absolute constant $c > 0$ satisfying the following: for $x \in X$, $y \in Ux$, $\psi \in C_c^\infty(X)$ supported on an admissible box of diameter smaller than*

*$1$, $0 < \rho < \operatorname{inj}(y)$, $f \in C_c^\infty(B_U(\rho)y)$ such that $0 \le f \le 1$, and $s > 0$ which satisfies $ce^{-s}\varepsilon < \rho$, we have*

$$e^{(n-1-\delta_\Gamma)s} \int_{Ux} \psi(a_s u_{\boldsymbol{t}} y) f(u_{\boldsymbol{t}} y) d\boldsymbol{t} \ll_{\Gamma, \operatorname{supp}\psi} S_\ell(\psi) \mu_y^{\mathrm{PS}}(B_U(2\rho)y),$$

*where $\ell \in \mathbb{N}$ satisfies the conclusion of Lemma 3.4.4.*

*Proof.* Assume that for $0 < \varepsilon_0, \varepsilon_1 < 1$, $\psi$ is supported on the admissible box $B_U(\varepsilon_0)P_{\varepsilon_1}z$ for $z \in X$. Without loss of generality, we may assume that $\psi$ is non-negative. Fix $y \in Ux$.

For small $\eta > 0$, $h \in G_\eta \operatorname{supp}(\psi)$, and $p \in P$, let

$$\psi_{\eta,+}(h) := \sup_{w \in G_\eta} \psi(wh), \quad \Psi_{\eta,+}(ph) := \int_{Uph} \psi_{\eta,+}(u_{\mathbf{t}} ph) d\mathbf{t}$$

and for $upz \in B_U(\varepsilon_0)P_{\varepsilon_1}z$ let

$$\tilde{\Psi}_{\eta,+}(upz) := \frac{1}{\mu_{pz}^{\mathrm{PS}}(B_U(\varepsilon_0)pz)} \Psi_{\eta,+}(pz).$$

By choice of $\ell$, for any $\eta > 0$ and $h \in G_\eta \operatorname{supp}(\psi)$,

$$|\psi_{\eta,+}(h) - \psi(h)| \ll \eta S_\ell(\psi),$$

and

$$|\psi(z)| \le S_{\infty,0}(\psi) \ll S_\ell(\psi),$$

where the implied constants depend on $\operatorname{supp}\psi$. Since the diameter of $\operatorname{supp}\psi$ is smaller than 1, we may assume that the implied constant in the above is absolute. Then, for any

$u \in U$ such that $a_s uy = u'pz \in B_U(\varepsilon_0)P_{\varepsilon_1}z$ and $0 < \eta < 1$, we have

$$\begin{aligned}\left|\tilde{\Psi}_{\eta,+}(a_s uy)\right| &= \left|\frac{1}{\mu_{pz}^{\mathrm{PS}}(B_U(\varepsilon_0)pz)}\int_{Upz}\psi_{\eta,+}(u_{\mathbf{t}}pz)d\mathbf{t}\right| \\ &= \frac{\mu_{pz}^{\mathrm{Leb}}(B_U(\varepsilon_0)pz)}{\mu_{pz}^{\mathrm{PS}}(B_U(\varepsilon_0)pz)}S_\ell(\psi) \\ &\ll S_\ell(\psi), \end{aligned} \tag{6.46}$$

where the implied constant depends only on $\operatorname{supp}\psi$.

For small $\eta > 0$ and $uy \in B_U(\eta + \varepsilon_0)y$, let

$$f_{\eta,+}(uy) := \sup_{w \in B_U(\eta)} f(wuy)$$

Using Lemma 6.2 and Lemma 6.5 from [MO15], we get that for some absolute constant $c > 0$,

$$\begin{aligned} e^{(n-1-\delta_\Gamma)s}\int_{B_U(\rho)y}\psi(a_s u_{\mathbf{t}}y)f(u_{\mathbf{t}}y)d\mathbf{t} &\ll \int_U \tilde{\Psi}_{c\varepsilon,+}(a_s u_{\mathbf{t}}y)f_{ce^{-s}\varepsilon,+}(u_{\mathbf{t}}y)d\mu_y^{\mathrm{PS}}(\mathbf{t}) \\ &\le \int_{B_U(\rho+ce^{-s}\varepsilon)y}\tilde{\Psi}_{c\varepsilon,+}(a_s u_{\mathbf{t}}y)d\mu_y^{\mathrm{PS}}(\mathbf{t}), \end{aligned}$$

where the implied constant is absolute. Then, by (6.46) we get

$$\begin{aligned} e^{(n-1-\delta_\Gamma)s}\int_{B_U(\rho)y}\psi(a_s u_{\mathbf{t}}y)f(u_{\mathbf{t}}y)d\mathbf{t} &\ll_{\operatorname{supp}\psi} \mu_y^{\mathrm{PS}}(B_U(\rho+ce^{-s}\varepsilon))S_\ell(\psi) \\ &\le \mu_y^{\mathrm{PS}}(B_U(2\rho))S_\ell(\psi). \end{aligned}$$

□

We are now ready to prove Theorem 1.2.2. For the reader's convenience, we restate that theorem below:

**Theorem 6.3.2.** *For any $0 < \varepsilon < 1$ and $s_0 \geq 1$, there exist $\ell = \ell(\Gamma) \in \mathbb{N}$ and $\kappa = \kappa(\Gamma, \varepsilon) > 0$ satisfying: for every $\psi \in C_c^\infty(G/\Gamma)$, there exists $c = c(\Gamma, \operatorname{supp}\psi)$ such that for every $x \in G/\Gamma$ that is $(\varepsilon, s_0)$-Diophantine, and for all $T$ such that $T^{1-\varepsilon/2} \gg_{\Gamma, \operatorname{supp}\psi} s_0$,*

$$\left| \frac{1}{\mu_x^{\mathrm{PS}}(B_U(T))} \int_{B_U(T)} \psi(u_{\boldsymbol{t}} x) d\boldsymbol{t} - m^{\mathrm{BR}}(\psi) \right| \leq c S_\ell(\psi) r^{-\kappa},$$

*where $S_\ell(\psi)$ is the $\ell$-Sobolev norm .*

*Proof.* We keep the notation of Chapter 4. By an argument similar to the proof of Theorem 1.2.3, we may assume that $\psi$ is supported on an admissible box. Because $\psi$ is compactly supported, we may also assume $\psi \geq 0$.

Let $\beta > 0$ satisfy the conclusion of Theorem 4.0.1 for $\varepsilon$ and $s_0$. Let $\kappa' > 0$, $\ell \in \mathbb{N}$ satisfy the conclusion of Theorem 6.1.3.

Since $x$ is $(\varepsilon, s_0)$-Diophantine, by Theorem 4.0.1, for $T_0 \gg_\Gamma s_0$ and $R \geq R_0$, we have

$$\mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0 \cap \mathcal{X}(R)) \ll \mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0) e^{-\beta R}, \tag{6.47}$$

where

$$s_\varepsilon := \frac{\varepsilon}{2} \log T, \quad x_0 := a_{-s_\varepsilon} x, \quad \text{and} \quad T_0 = T^{1-\frac{\varepsilon}{2}}. \tag{6.48}$$

Observe that by (3.3), (3.10), and (3.11),

$$\frac{1}{\mu_x^{\mathrm{PS}}(B_U(T))} \int_{B_U(T)} \psi(u_{\mathbf{t}} x) d\mathbf{t} = \frac{e^{(n-1-\delta)s_\varepsilon}}{\mu_{x_0}^{\mathrm{PS}}(B_U(T_0))} \int_{B_U(T_0)} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0) d\mathbf{t}.$$

Fix $R > R_0$ and define

$$Q_0 := B_U(T_0)x_0 \cap \mathcal{C}(R).$$

Since for any $R \geq R_0$ the set $\mathcal{C}(R)$ is in the convex core of $\mathbb{H}^n/\Gamma$,

$$Q_0 \subseteq \operatorname{supp} m^{\text{BMS}}. \tag{6.49}$$

Let $\rho > 0$ be smaller than half of the injectivity radius of $Q_0$.

First, by Lemma 3.4.7, there exist $\{y : y \in I_0\} \subseteq Q_0$ and $f_y \in C_c^\infty(B_U(2\rho)y)$ satisfying

$$S_\ell(f_y) \ll \rho^{-\ell+n-1} \tag{6.50}$$

and

$$\sum_y f_y = 1 \text{ on } E_1 := \bigcup_{y \in I_0} B_U(\rho)y \supseteq Q_0$$

and 0 outside of

$$E_2 = \bigcup_{y \in I_0} B_U(2\rho)y.$$

By replacing references to Theorem 1.2.3 with references to Theorem 6.1.3, the exact same argument as in the proof of Theorem 1.2.1 will establish that for $T \gg_\Gamma e^{2R/\varepsilon}$ and

$$\kappa = \frac{\beta\varepsilon}{2}, \quad R = \frac{\kappa \log T}{\beta}, \quad \rho \leq T^{-\kappa/\alpha} \tag{6.51}$$

we get that if we assume without loss of generality that $\kappa' < 2\beta$ and also that $T \gg_\Gamma 1$,

$$\frac{e^{(n-1-\delta_\Gamma)s_\varepsilon}}{\mu_{x_0}^{\text{PS}}(B_U(T_0)x_0)} \int_{u_{\mathbf{t}}x_0 \in E_1} \psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)d\mathbf{t} - m^{\text{BR}}(\psi) \ll_{\Gamma, \operatorname{supp}\psi} S_\ell(\psi)T^{-\kappa}. \tag{6.52}$$

We now want to bound the integral over $B_U(T_0)x_0 \setminus E_1$. Using Lemma 3.4.7 again, we may deduce that there exist $\{y : y \in I_1\} \subseteq B_U(T_0)x_0 \setminus E_1$ and $f_y \in C_c^\infty(B_U(\rho/4)y)$

satisfying $\sum_{y \in I_1} f_y = 1$ on $\bigcup_{y \in I_1} B_U(\rho/8)y$ and 0 outside of

$$\bigcup_{y \in I_1} B_U(\rho/4)y.$$

In particular, by the definition of $E_1$, we have

$$E_3 := \bigcup_{y \in I_1} B_U(\rho/2)y \subseteq (B_U(T_0)x_0 \setminus Q_0) \cup (B_U(T_0 + \rho/2)x_0 \setminus B_U(T_0))x_0.$$

Using Lemma 6.3.1, we arrive at

$$\begin{aligned}
&e^{(n-1-\delta_\Gamma)s_\varepsilon} \int_{B_U(T_0)\setminus E_1} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0) d\mathbf{t} \\
&\leq e^{(n-1-\delta_\Gamma)s_\varepsilon} \sum_{y \in I_1} \int_{B_U(\rho/2)y} \psi(a_{s_\varepsilon} u_{\mathbf{t}} y) f_y(u_{\mathbf{t}} y) d\mathbf{t} \\
&\ll \sum_{y \in I_1} S_\ell(\psi) \mu_y^{\mathrm{PS}}(B_U(\rho/2)y) \\
&\leq S_\ell(\psi)(\mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0 \setminus Q_0) + \mu_{x_0}^{\mathrm{PS}}((B_U(T_0 + \rho/2) \setminus B_U(T_0))x_0)).
\end{aligned}$$

Thus, by Theorem 5.0.1 there exists $\alpha = \alpha(\Gamma) > 0$ such that using equations (6.47), (6.51), we arrive at

$$\begin{aligned}
&e^{(n-1-\delta_\Gamma)s_\varepsilon} \int_{B_U(T_0)\setminus E_1} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0) d\mathbf{t} \\
&\ll_{\Gamma, \operatorname{supp}\psi} S_\ell(\psi) \mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0) \left(e^{-\beta R} + \rho^\alpha\right) \\
&\ll_{\Gamma, \operatorname{supp}\psi} S_\ell(\psi) \mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0) T^{-\kappa}.
\end{aligned}$$

Using (6.52), we may now deduce

$$\frac{e^{(n-1-\delta_\Gamma)s_\varepsilon}}{\mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0)} \int_{B_U(T_0)} \psi(a_{s_\varepsilon} u_{\mathbf{t}} x_0) d\mathbf{t} - m^{\mathrm{BR}}(\psi) \ll_{\Gamma, \operatorname{supp}\psi} S_\ell(\psi) T^{-\kappa}.$$

On the other hand, define

$$Q_1 := B_U(T_0 - 2\rho)x_0 \cap \mathcal{C}(R).$$

As before, according to Lemma 3.4.7, there exist $\{y : y \in I_1\} \subseteq Q_1$ and $f_y \in C_c^\infty(B_U(2\rho)y)$ satisfying

$$S_\ell(f_y) \ll \rho^{-\ell+n-1}$$

and

$$\sum_{y\in I_1} f_y = 1 \text{ on } E_4 := \bigcup_{y\in I_1} B_U(\rho)y$$

and 0 outside of

$$\bigcup_{y\in I_1} B_U(2\rho)y \subseteq B_U(T_0)x_0. \tag{6.53}$$

Hence,

$$\int_{u_{\mathbf{t}}x_0\in B_U(T_0)x_0} \psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)d\mathbf{t} \geq \sum_{y\in I_1}\int_{u_{\mathbf{t}}x_0\in B_U(2\rho)y} \psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)f_y(u_{\mathbf{t}}x_0)d\mathbf{t}.$$

By Theorem 6.1.3 we have

$$\int_{u_{\mathbf{t}}x_0\in B_U(T_0)x_0} \psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)d\mathbf{t} \geq \mu_{x_0}^{\mathrm{PS}}(B_U(T_0-2\rho))(m^{\mathrm{BR}}(\psi) - c_2S_\ell(\psi)e^{-\kappa' s_\varepsilon/2}),$$

and by Theorem 5.0.1 we arrive at

$$\geq \mu_{x_0}^{\mathrm{PS}}(B_U(T_0))\left(1 - c_3(2\rho)^\alpha\right)\left(m^{\mathrm{BR}}(\psi) - c_2S_\ell(\psi)e^{-\kappa' s_\varepsilon/2}\right).$$

Hence, (6.51) implies

$$\frac{e^{(n-1-\delta_\Gamma)s_\varepsilon}}{\mu_{x_0}^{\mathrm{PS}}(B_U(T_0)x_0)}\int_{u_{\mathbf{t}}x_0\in E_1} \psi(a_{s_\varepsilon}u_{\mathbf{t}}x_0)d\mathbf{t} - m^{\mathrm{BR}}(\psi) \gg_{\Gamma,\operatorname{supp}\psi} S_\ell(\psi)T^{-\kappa}.$$

$\square$

*Remark.* The dependence of $T$ on $\operatorname{supp}\psi$ in the previous proof arises from Theorem 6.1.3, through the quantity $s_\varepsilon$. Upon closer inspection, one can verify that this means $T$ depends on $\operatorname{supp}\psi$ through the maximum height of elements in $\operatorname{supp}\psi$. In particular, we may choose a larger compact set containing $\operatorname{supp}\psi$ and have $T$ depend on that compact set, rather than $\operatorname{supp}\psi$ specifically. ◀

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# Chapter 7

# Duality between $G/\Gamma$ and $U\backslash G$

In this chapter, we begin the setup necessary to use the equidistribution results obtained in previous chapters to understand the distribution of the orbits of a geometrically finite subgroup $\Gamma \subseteq \mathrm{SO}(n,1)^\circ$ acting on $U\backslash G$. Throughout the remainder of the dissertation, we will assume that Assumption 1.1.2 holds.

The goal of this chapter is to prove the following proposition, which shows that one can use equidistribution results of $U$ orbits in $G/\Gamma$ in order to study the distribution of the points in $x\Gamma_T$ for $x \in U\backslash G$.

Recall that for $x, y \in U\backslash G$, we defined

$$x \star y := \sqrt{\frac{1}{2}\,\|\Psi(x)^{-1}E_{1,n+1}\Psi(y)\|}, \tag{7.1}$$

where $E_{1,n+1}$ is the $(n+1)\times(n+1)$ matrix with one in the $(1, n+1)$-entry and zeros everywhere else.

Recall the Iwasawa decomposition $G = \mathrm{SO}(n,1)^\circ = U \times A \times K$. Define a continuous section by $\Psi : U\backslash G \to AK$ by

$$\Psi(Ug) = ak,$$

where $g = uak$ is the Iwasawa decomposition of $g$.

For $\varphi \in C_c(U\backslash G)$, define

$$R_\varphi := \max_{y\in\operatorname{supp}\varphi}(x \star y), \quad r_\varphi := \min_{y\in\operatorname{supp}\varphi}(x \star y). \tag{7.2}$$

**Proposition 7.0.1.** *Let $\eta > 0$, $\Omega \subset U\backslash G$ be a compact set, $\varphi \in C(\Omega)$, and $\psi \in C(B_U(\eta))$ be a non-negative function such that $\int_U \psi = 1$. Fix $x \in U\backslash G$. Define $F \in C_c(G/\Gamma)$ by*

$$F(g\Gamma) := \sum_{\gamma\in\Gamma} \psi(u(g\gamma))\varphi(\pi_U(g\gamma)).$$

*Then, for some $c = c(x,\Omega) > 0$,*

$$\int_{B_U\left(\frac{\sqrt{T}-c}{R_\varphi}-\eta\right)} F(u_{\boldsymbol{t}}\Psi(x)\Gamma)d\boldsymbol{t} \le \sum_{\gamma\in\Gamma_T} \varphi(x\gamma) \le \int_{B_U\left(\frac{\sqrt{T}+c}{r_\varphi}+\eta\right)} F(u_{\boldsymbol{t}}\Psi(x)\Gamma)d\boldsymbol{t}.$$

Observe that

$$g\Psi(Ug)^{-1} \in U.$$

Therefore, for any $g, h \in G$,

$$\Psi(Uh)g\Psi(Uhg)^{-1} = (h\Psi(Uh)^{-1})^{-1}(hg\Psi(Uhg)^{-1}) \in U.$$

Hence, for any $x \in U\backslash G$, we can define

$$u_x(g) := u_{c_x(g)} = \Psi(x)g\Psi(xg)^{-1} \in U. \tag{7.3}$$

It satisfies

$$c_e(u_{\mathbf{t}}g) = c_e(g) + \mathbf{t}, \quad c_e(a_s g) = e^s c_e(g) \tag{7.4}$$

and for any $x \in U\backslash G$,

$$c_x(g) = c_e(\Psi(x)g). \tag{7.5}$$

Observe that (7.5) implies that

$$c_e(hg) = c_e(h) + c_e(\Psi(Uh)g) = c_e(h) + c_{Uh}(g). \tag{7.6}$$

Note that for $g \in G$,

$$g = u_e(g)\Psi(Ug).$$

That is, $u_e(g)$ is the $U$ component of the Iwasawa decomposition of $G$, and $\Psi(Ug)$ is the $AK$ component.

**Lemma 7.0.2.** *For any compact $\Omega \subset U\backslash G$ and $x \in U\backslash G$ there exist $c = c(\Omega, x) > 0$ such that for any $xg \in \Omega$ and $T > c$, we have*

1. *If $\|g\| \leq T$, then $u_x(g) \in B_U\left(\frac{\sqrt{T}+c}{x \star xg}\right)$.*
2. *If $\|g\| \geq T$, then $u_x(g) \notin B_U\left(\frac{\sqrt{T}-c}{x \star xg}\right)$.*

*Proof.* We have $g = \Psi(x)^{-1}u_x(g)\Psi(xg)$. For $\mathbf{t} := c_x(g)$ we get

$$\begin{aligned} g &= \Psi(x)^{-1}u_x(g)\Psi(xg) \\ &= \Psi(x)^{-1}\left(I + \begin{pmatrix} 0 & \mathbf{t} & 0 \\ 0 & 0 & \mathbf{t}^T \\ 0 & 0 & 0 \end{pmatrix} + \|\mathbf{t}\|^2 E_{1,n+1}\right)\Psi(y) \end{aligned}$$

Denote

$$c_1 := \max_{y\in\Omega}\left\{\left\|\Psi(x)^{-1}\Psi(y)\right\|\right\},$$

$$c_2 := \max_{y\in\Omega, \|\mathbf{t}\|\le 1}\left\{\left\|\Psi(x)^{-1}\begin{pmatrix}0 & \mathbf{t} & 0\\ 0 & 0 & \mathbf{t}^T\\ 0 & 0 & 0\end{pmatrix}\Psi(y)\right\|\right\}.$$

Then, $c_1$ and $c_2$ are functions of $x$ and $\Omega$. By the triangle inequality,

$$\begin{aligned}\|g\| &\le \|\mathbf{t}\|^2(x\star xg)^2 + \|\Psi(x)^{-1}\Psi(xg)\| + \left\|\Psi(x)^{-1}\begin{pmatrix}0 & \mathbf{t} & 0\\ 0 & 0 & \mathbf{t}^T\\ 0 & 0 & 0\end{pmatrix}\Psi(xg)\right\|\\ &\le \|\mathbf{t}\|^2(x\star xg)^2 + c_1 + c_2\|\mathbf{t}\|.\end{aligned}$$

In a similar way

$$\begin{aligned}\|g\| &\ge \|\mathbf{t}\|^2(x\star xg)^2 - \|\Psi(x)^{-1}\Psi(xg)\| - \left\|\Psi(x)^{-1}\begin{pmatrix}0 & \mathbf{t} & 0\\ 0 & 0 & \mathbf{t}^T\\ 0 & 0 & 0\end{pmatrix}\Psi(xg)\right\|\\ &\ge \|\mathbf{t}\|^2(x\star xg)^2 - c_1 - c_2\|\mathbf{t}\|.\end{aligned}$$

We conclude that for any $g\in\Omega$,

$$\left|\|g\| - (x\star xg)^2\|\mathbf{t}\|^2\right| \le c_1 + c_2\|\mathbf{t}\|. \tag{7.7}$$

Assume $\|g\|\ge T\ge c_1$. Then, by (7.7)

$$0 \le (x\star xg)^2\|\mathbf{t}\|^2 + c_2\|\mathbf{t}\| + (c_1 - T).$$

Using the quadratic formula, we may deduce that the right hand side of the above equation is equal to zero when

$$\|t\| = \frac{-c_2 \pm \sqrt{c_2^2 + 4(T - c_1)(x \star xg)^2}}{2(x \star xg)^2}$$

Since $(x \star xg)^2$ and $\|t\|$ are non-negative, it follows that

$$\|t\| \geq \frac{-c_2 + \sqrt{c_2^2 + 4(T - c_1)(x \star xg)^2}}{2(x \star xg)^2}$$

Using the inequality $\sqrt{a \pm b} \geq \sqrt{a} - \sqrt{b}$, we arrive at

$$\|t\| \geq \frac{\sqrt{T}}{x \star xg} - \frac{c_2 + c_1 x \star xg}{(x \star xg)^2}$$

A similar computation shows that $\|g\| \leq T$ implies

$$\|t\| \leq \frac{\sqrt{T}}{x \star xg} + \frac{c_2 + c_1 x \star xg}{(x \star xg)^2}.$$

Letting $c$ be the maximum of $\frac{c_2 + c_1 x \star xg}{(x \star xg)^2}$ for $g \in \Omega$ completes the proof. $\square$

**Lemma 7.0.3.** *Let $\varphi \in C_c(U\backslash G)$ and suppose that $\psi \in C_c(U)$ satisfies*

$$\int_U \psi = 1.$$

*Define*

$$f(g) = \psi(u(g))\varphi(\pi_U(g)).$$

*Then for every $g \in G$,*

$$\varphi(\pi_U(g)) = \int_{\mathrm{supp}(\psi)u(g)^{-1}} f(u_{\boldsymbol{t}}g)d\boldsymbol{t}.$$

*Proof.* By the definition of $\psi$,

$$\begin{aligned}\varphi(\pi_U(g)) &= \varphi(\pi_U(g)) \int_{\mathrm{supp}(\psi)} \psi(u_{\mathbf{t}}) d\mathbf{t} \\ &= \varphi(\pi_U(g)) \int_{u(g)^{-1}\,\mathrm{supp}(\psi)} \psi(u(g)u_{\mathbf{t}}) d\mathbf{t}.\end{aligned}$$

Since $\pi_U(u_t g) = \pi_U(g)$, we have

$$\begin{aligned}\varphi(\pi_U(g)) &= \int_{u(g)^{-1}\,\mathrm{supp}(\psi)} \psi(u(u_{\mathbf{t}}g))\varphi(\pi_U(u_{\mathbf{t}}g)) d\mathbf{t} \\ &= \int_{u(g)^{-1}\,\mathrm{supp}(\psi)} f(u_{\mathbf{t}}g) d\mathbf{t}\end{aligned}$$

□

We are now ready to prove Proposition 7.0.1.

*Proof of Proposition 7.0.1.* Without loss of generality, we may assume that $\varphi \geq 0$. Define $f : G \to \mathbb{R}$ by

$$f(g) = \psi(u(g))\varphi(\pi_U(g)).$$

By Lemma 7.0.3, for every $g \in G$,

$$\varphi(\pi_U(g)) = \int_{u(g)^{-1}B_U(\eta)} f(u_{\mathbf{t}}g) d\mathbf{t}. \tag{7.8}$$

By Lemma 7.0.2, there exist $c > 0$ depending on $\Omega$ and $x$ such that for all $T \geq c$, if $\gamma \in \Gamma_T$ and $x\gamma \in \Omega$, then

$$u_x(\gamma)^{-1}B_U(\eta) \subseteq B_U\left(\frac{\sqrt{T}+c}{x \star x\gamma} + \eta\right). \tag{7.9}$$

Observe also that since $\mathrm{supp}(\psi) \subseteq B_U(\eta)$, if $u_{\mathbf{t}} \notin u_x(\gamma)^{-1}B_U(\eta)$, then

$$f(u_{\mathbf{t}}\Psi(x)\gamma) = \psi(u_{\mathbf{t}}u(\Psi(x)\gamma))\varphi(\pi_U(\Psi(x)\gamma)) = 0.$$

Thus, using (7.6) and Lemma 7.0.3, for $\gamma \in \Gamma_T$ with $x\gamma \in \Omega$, we have that

$$\begin{aligned} \varphi(x\gamma) &= \int_{u(\Psi(x)\gamma)^{-1}B_U(\eta)} f(u_{\mathbf{t}}\Psi(x)\gamma)d\mathbf{t} \\ &= \int_{u_x(\gamma)^{-1}B_U(\eta)} f(u_{\mathbf{t}}\Psi(x)\gamma)d\mathbf{t} \\ &= \int_{B_U\left(\frac{\sqrt{T}+c}{x\star x\gamma}+\eta\right)} f(u_{\mathbf{t}}\Psi(x)\gamma)d\mathbf{t}. \end{aligned} \tag{7.10}$$

Note that

$$F(g\Gamma) := \sum_{\gamma\in\Gamma} f(g\gamma)$$

Thus, from (7.10), for $r = r_\varphi := \min_{y\in\mathrm{supp}\,\varphi}(x \star y)$, we obtain

$$\begin{aligned} \sum_{\gamma\in\Gamma_T} \varphi(x\gamma) &\leq \sum_{\gamma\in\Gamma_T} \int_{B_U\left(\frac{\sqrt{T}+c}{r}+\eta\right)} f(u_{\mathbf{t}}\Psi(x)\gamma)d\mathbf{t} \\ &\leq \int_{B_U\left(\frac{\sqrt{T}+c}{r}+\eta\right)} F(u_{\mathbf{t}}\Psi(x)\Gamma)d\mathbf{t}. \end{aligned}$$

To obtain a lower bound, we must control the terms arising from $\gamma \in \Gamma \setminus \Gamma_T$ in the definition of $F$. Note that by Lemma 7.0.2, if $\gamma \in (\Gamma \setminus \Gamma_T)$ and $x\gamma \in \Omega$, then we see that

$$u_x(\gamma)^{-1}B_U(\eta) \cap B_U\left(\frac{\sqrt{T}-c}{x\star x\gamma} - \eta\right) = \emptyset. \tag{7.11}$$

Thus, similarly to the above, we obtain

$$\begin{aligned} \sum_{\gamma\in\Gamma_T} \varphi(x\gamma) &= \sum_{\gamma\in\Gamma_T} \int_{B_U\left(\frac{\sqrt{T}-c}{x\star xg}-\eta\right)} f(u_{\mathbf{t}}\Psi(x)\gamma)d\mathbf{t} \\ &\geq \sum_{\gamma\in\Gamma_T} \int_{B_U\left(\frac{\sqrt{T}-c}{R}-\eta\right)} f(u_{\mathbf{t}}\Psi(x)\gamma)d\mathbf{t}, \end{aligned}$$

where $R = R_\varphi := \max_{y \in \operatorname{supp} \varphi} (x \star y)$, completing the proof. $\square$

**Lemma 7.0.4.** *Let $\varphi \in C_c(U \backslash G)$ and $F$ be as defined in Proposition 7.0.1. Then,*

$$m^{\mathrm{BR}}(F) = \int_P \varphi(\pi_U(p)) d\nu(p) \tag{7.12}$$

*Proof.* By the definition of $F$ and the assumption that $\int_U \psi = 1$, by the product structure of the BR measure in (3.23), we obtain

$$\begin{aligned} m^{\mathrm{BR}}(F) &= \int_G \psi(u(g)) \varphi(\pi_U(g)) d\tilde{m}^{\mathrm{BR}}(g) \\ &= \int_P \int_U \psi(u_{\mathbf{t}} u(p)) \varphi(\pi_U(p)) d\mathbf{t} d\nu(p) \\ &= \int_P \varphi(\pi_U(p)) d\nu(p). \end{aligned}$$

$\square$

## 7.1 A "Nice" Partition of $\varphi$

In the later sections, we will require a partition of $\varphi$, say into $\varphi_1, \ldots, \varphi_k$, so that for each $i$, $R_{\varphi_i}$ and $r_{\varphi_i}$ are close. In this section, we construct such a partition.

For a set $H \subseteq G$, let

$$B(H, r) = \{g \in G : d(g, H) \leq r\},$$

where $d$ is the Riemannian metric on $G$. That is, $B(H, r)$ is the $r$-thickening of $H$ with respect to $d$. For $h \in G$, we denote $B(\{h\}, r)$ by $B(h, r)$ (in this case we get the Riemannian ball around the point $h$).

For $H \subseteq G$, denote by

$$\text{inj}(H)$$

the infimum over all $r > 0$ satisfying that for every $h \in H$,

$$\pi_\Gamma|_{B(h,r)} : B(h,r) \to G/\Gamma$$

is injective.

**Lemma 7.1.1.** *Fix $x \in U\backslash G$. For a compact set $H \subseteq G$, there exists $0 < \eta_0 = \eta(H) <$ $\text{inj}(H)$, $\beta = \beta(H) > 1$ so that for any $0 < \eta < \eta_0$ and $\varphi \in C_c(U\backslash G)$ with $\operatorname{supp}\varphi \subset$ $\pi_U(B(h,\eta))$ for some $h \in H$, we have that*

$$\frac{R_\varphi}{r_\varphi} - 1 \leq \|\Psi(x)^{-1}\|\beta\eta.$$

*Proof.* Since $B(H,1)$ is a compact set, by [EW11, Lemma 9.12], there exist constants $0 < \eta_0 = \eta(H) < \text{inj}(H)$, $\beta = \beta(H) > 1$, such that $\eta_0 < 1$ and for all $g, h \in B(H,1)$ with $d(g,h) \leq \eta_0$,

$$\beta^{-1}\|g - h\| \leq d(g,h) \leq \beta\|g - h\|. \tag{7.13}$$

Therefore, for any $h \in H$, we have

$$B(h,\eta) \subseteq \{g \in G : \|g - h\| \leq \beta\eta\}.$$

Note that for any $g \in G$,

$$E_{1,n+1}\Psi(\pi_U(g)) = E_{1,n+1}g.$$

Thus, if $\|g - h\| < \beta\eta$, then

$$\begin{aligned}\left\|\Psi(x)^{-1}E_{1,n+1}\Psi(\pi_U(g))\right\| &= \left\|\Psi(x)^{-1}E_{1,n+1}g\right\| \\ &\leq \left\|\Psi(x)^{-1}E_{1,n+1}h\right\| + \left\|\Psi(x)^{-1}E_{1,n+1}(g-h)\right\| \\ &\leq \left\|\Psi(x)^{-1}E_{1,n+1}\Psi(\pi_U(h))\right\| + \beta\eta\left\|\Psi(x)^{-1}\right\|,\end{aligned}$$

and similarly

$$\left\|\Psi(x)^{-1}E_{1,n+1}\Psi(\pi_U(g))\right\| \geq \left\|\Psi(x)^{-1}E_{1,n+1}\Psi(\pi_U(h))\right\| - \beta\eta\left\|\Psi(x)^{-1}\right\|.$$

Thus, it follows from (7.1) that for

$$R = \max_{y\in\pi_U(B(h,\eta))}(x \star y), \quad r = \min_{y\in\pi_U(B(h,\eta))}(x \star y),$$

we have

$$R - r \leq 2\beta\|\Psi(x)^{-1}\|\eta.$$

Since $r$ is bounded below by a constant depending on $H$, this implies that

$$\left(\frac{R}{r}\right) - 1 \ll_H \|\Psi(x)^{-1}\|\eta.$$

$\square$

**Corollary 7.1.2.** *Fix $x \in U\backslash G$ and $\varphi \in C_c(U\backslash G)$. Let $\eta_0 = \eta_0(\Psi(\operatorname{supp}\varphi))$ be as in Lemma 7.1.1. For any $0 < \eta < \eta_0$, there exist some $k$ and $\varphi_1, \ldots, \varphi_k \in C_c(U\backslash G)$ so that*

$$\sum_{i=1}^{k}\varphi_i = \varphi \quad \text{and } \frac{R_{\varphi_i}}{r_{\varphi_i}} - 1 \ll_{\Gamma,\operatorname{supp}\varphi} \eta.$$

*Moreover, if $\varphi \in C_c^\infty(U\backslash G)$, then we also have $\varphi_i \in C_c^\infty(U\backslash G)$, and*

$$\sum_{i=1}^{k} S_\ell(\varphi_i) \ll_{\ell,\operatorname{supp}\varphi} \eta^{-\ell+n(n+1)/4}S_\ell(\varphi). \tag{7.14}$$

*Proof.* For the first case (only assuming $\varphi \in C_c(U\backslash G)$, cover $\Psi(\operatorname{supp}\varphi)$ with balls of radius $\eta$, and let $\sigma_1, \ldots, \sigma_k$ be a partition of unity subordinate to this cover. Defining

$$\varphi_i = \varphi \cdot (\sigma_i \circ \Psi)$$

yields functions with the desired property, by Lemma 7.1.1.

Now, assume that $\varphi \in C_c^\infty(U\backslash G)$. We must be more careful in order to control Sobolev norms. By Lemma 3.4.7, for $0 < \eta \leq \eta_0$, there exist $h_1, \ldots, h_k \in \Psi(\operatorname{supp}\varphi)$ and $\sigma_1, \ldots, \sigma_k \in C_c^\infty(B(h_i, \eta))$ with

$$\sum_{i=1}^{k} \sigma_i = 1 \text{ on } \Psi(\operatorname{supp}\varphi) \text{ and } = 0 \text{ outside } B(\Psi(\operatorname{supp}\varphi), \eta) \tag{7.15}$$

and such that

$$\sum_{i=1}^{k} S_\ell(\sigma_i) \ll_{n, \operatorname{supp}\varphi} \eta^{-\ell + n(n+1)/4}. \tag{7.16}$$

Define

$$\varphi_i = \varphi \cdot (\sigma_i \circ \Psi).$$

Then, by Lemma 7.1.1,

$$\frac{R_{\varphi_i}}{r_{\varphi_i}} - 1 \ll_{\operatorname{supp}\varphi} \|\Psi(x)^{-1}\|\eta.$$

Since $\Psi$ is smooth, by Lemmas 3.4.5 and 3.4.6,

$$\begin{aligned} S_\ell(\varphi_i) &\ll_\ell S_\ell(\varphi) S_\ell(\sigma_i \circ \Psi) \\ &\ll_{\ell,\Psi} S_\ell(\varphi) S_\ell(\sigma_i). \end{aligned} \tag{7.17}$$

From (7.16) and (7.17), we conclude that

$$\sum_{i=1}^{k} S_\ell(\varphi_i) \ll_{\ell, n, \operatorname{supp}\varphi, \Psi} \eta^{-\ell + n(n+1)/4} S_\ell(\varphi).$$

$\square$

## 7.2 Comparing the PS Measure of the Balls Arising in Proposition 7.0.1

The purpose of Proposition 7.0.1 is to enable the use of the equidistribution theorems. However, in order to deduce useful bounds from this, we must show that

$$\mu_x^{\mathrm{PS}}\left(\frac{\sqrt{T}-c}{R_\varphi}-\eta\right) \text{ and } \mu_x^{\mathrm{PS}}\left(\frac{\sqrt{T}+c}{r_\varphi}+\eta\right)$$

are close to each other if $R_\varphi$ and $r_\varphi$ are close. The necessary machinery to do this is proven in the appendix, namely the following:

**Corollary 7.2.1.** *There exists a constant $\alpha = \alpha(\Gamma) > 0$ satisfying the following: let $0 < \varepsilon \leq 1$ and let $s_0 \geq 1$. There exists $T_0 = T_0(\Gamma, s_0) > 0$ so that for every $(\varepsilon, s_0)$-Diophantine point $x \in G/\Gamma$, all $T > 2T_0 + 1$, and all $\xi > 0$,*

$$\mu_x^{\mathrm{PS}}(B_U((1+2\xi)T)) - \mu_x^{\mathrm{PS}}(B_U(T)) \ll_\Gamma \left(\xi + \frac{T_0}{T-T_0}\right)^\alpha \mu_x^{\mathrm{PS}}(B_U(T)). \tag{7.18}$$

*In particular, if $x^- \in \Lambda_r(\Gamma)$, there exists $T_0 = T_0(x) > 0$ so that for all $T \geq 2T_0 + 1$ and all $\xi > 0$,* (10.19) *holds.*

We now interpret the above in a form that will be most convenient for our proofs. Note that the implied constant below depends on $x$ through the initial time in Corollary 7.2.1, and it can be made uniform over a compact set or over all points with the same Diophantine properties. However, this level of detail is not necessary for our results.

**Lemma 7.2.2.** *Let $\alpha = \alpha(\Gamma) > 0$ be as in Corollary 7.2.1. For every $x \in G/\Gamma$ with $x^- \in \Lambda_r(\Gamma)$, $c > 0$, $0 < \eta \leq 1$, and $0 < r_+ < \ell < r_-$ satisfying*

$$\frac{r_+}{r_-} < 1 + \eta,$$

*there exists $T_0 = T_0(x, r_+, r_-) > 0$ such that for any $T > T_0$,*

$$\begin{aligned}
&\left| \mu_x^{\mathrm{PS}} \left( B_U \left( \frac{\sqrt{T} \pm c}{r_\pm} \pm \eta \right) \right) - \mu_x^{\mathrm{PS}} \left( B_U \left( \frac{\sqrt{T}}{\ell} \right) \right) \right| \\
&\ll_{\Gamma, x} \left( \eta + \frac{c+1}{\sqrt{T}} \right)^\alpha \mu_x^{\mathrm{PS}} \left( B_U \left( \frac{\sqrt{T}}{\ell} \right) \right)
\end{aligned}$$

*Proof.* First, observe that by Corollary 7.2.1, there exists $T_1 = T_1(x)$ so that for all $T \geq 2T_1 + 1$ and all $\xi > 0$,

$$\frac{\mu_x^{\mathrm{PS}}(B_U(T + \xi)) - \mu_x^{\mathrm{PS}}(B_U(T))}{\mu_x^{\mathrm{PS}}(B_U(T))} \ll_\Gamma \left( \xi + \frac{T_1}{T - T_1} \right)^\alpha \ll_\Gamma \left( \xi + \frac{T_1}{T} \right)^\alpha. \tag{7.19}$$

This follows immediately from the fact that

$$\mu_x^{\mathrm{PS}}(B_U(T + \xi)) \leq \mu_x^{\mathrm{PS}}(B_U(1 + 2\xi)T).$$

Thus, if we assume that $T$ is sufficiently large so that $\sqrt{T}/\ell \geq 2T_1 + 1$ (and note that this condition can be taken to rely on $r_-$ rather than on $\ell$ specifically), and note that by the assumption,

$$1 \leq \frac{\ell}{r_+} \leq 1 + \eta,$$

we see from (7.19) that

$$\begin{aligned}
&\mu_x^{\mathrm{PS}}\left(B_U\left(\frac{\sqrt{T}+c}{r_+}+\eta\right)\right)-\mu_x^{\mathrm{PS}}\left(B_U\left(\frac{\sqrt{T}}{\ell}\right)\right)\\
&\ll_\Gamma \left(\frac{r_+^{-1}(\sqrt{T}+c)+\eta-\ell^{-1}\sqrt{T}}{\ell^{-1}\sqrt{T}}+\frac{T_1}{\ell^{-1}\sqrt{T}}\right)^\alpha \mu_x^{\mathrm{PS}}\left(B_U\left(\frac{\sqrt{T}}{\ell}\right)\right)\\
&\ll_\Gamma \left(\frac{\ell r_+^{-1}(\sqrt{T}+c)+\ell\eta-\sqrt{T}+T_1}{\sqrt{T}}\right)^\alpha \mu_x^{\mathrm{PS}}\left(B_U\left(\frac{\sqrt{T}}{\ell}\right)\right)\\
&\ll_\Gamma \left(\frac{(1+\eta)(\sqrt{T}+c)+\ell\eta-\sqrt{T}+T_1}{\sqrt{T}}\right)^\alpha \mu_x^{\mathrm{PS}}\left(B_U\left(\frac{\sqrt{T}}{\ell}\right)\right)\\
&\ll_\Gamma \left(\eta+\frac{c+\eta\ell+T_1}{\sqrt{T}}\right)^\alpha \mu_x^{\mathrm{PS}}\left(B_U\left(\frac{\sqrt{T}}{\ell}\right)\right)\\
&\ll_{\Gamma,x} \left(\eta+\frac{c+\eta\ell+1}{\sqrt{T}}\right)^\alpha \mu_x^{\mathrm{PS}}\left(B_U\left(\frac{\sqrt{T}}{\ell}\right)\right)
\end{aligned}$$

Note that the implied constant depends on $x$ because we have absorbed the constant $T_1$. Now, choose $T_0 \geq T_1$ so that $T \geq T_0$ implies $\frac{\ell}{\sqrt{T}} < 1$ (a condition which depends on $x$ and $r_-$ in this case), which implies the claim because we may then absorb this term into the $\eta$ term.

The second case can be shown in a similar way, with the choice of $T_0$ depending on $x$ and $r_+$ there. $\square$

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This chapter contains material from the following, which has been submitted for publication: N. Tamam, J. M. Warren, *"Distribution of orbits of geometrically finite groups acting on null vectors"*, arXiv:2009.11968. The dissertation author was one of the primary investigators and authors of this paper.

# Chapter 8

# Proofs of Orbit Distribution Theorems

In this chapter, we prove the main results about the distributions of orbits of $\Gamma$ acting on $U\backslash G$.

## 8.1 Proof of Theorem 1.3.2

This section is dedicated to the proof of Theorem 1.3.2, which is restated below for convenience. Note that we do *not* require Assumption 1.1.2 here (that is only necessary for quantitative bounds).

**Theorem 8.1.1.** *Let $\Gamma$ be geometrically finite. For any $\varphi \in C_c(U\backslash G)$ and every $x \in U\backslash G$*

*such that* $\Psi(x)^- \in \Lambda_r(\Gamma)$,

$$\sum_{\gamma\in\Gamma_T} \varphi(x\gamma) \sim I(\varphi, T, x).$$

We will need the following lemma. Theorem 1.3.2 will then follow by a partition of unity argument.

**Lemma 8.1.2.** *Let* $\varphi \in C_c(U\backslash G)$ *and let* $x \in U\backslash G$ *be such that* $\Psi(x)^- \in \Lambda_r(\Gamma)$. *Let* $R = R_\varphi$ *and* $r = r_\varphi$ *be as in (7.2). Let* $\eta > 0$, *and suppose that* $\frac{R}{r} < 1+\eta$ *and that* $B_U(\eta)\Psi(\operatorname{supp}\varphi)$ *injects into* $G/\Gamma$.

*Then for any* $\varepsilon > 0$, *there exists* $T_1 = T_1(x,\eta,\varphi) > 0$ *such that for all* $T \geq T_1$,

$$\begin{aligned}
&\left|\sum_{\gamma\in\Gamma_T} \varphi(x\gamma) - \int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star\pi_U(p)}\right)\right)\varphi(\pi_U(p))d\nu(p)\right| \\
&\ll_{\Gamma,x} \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right)\left[\left(\eta + \frac{c+1}{\sqrt{T}}\right)^{\alpha}\int_P \varphi(\pi_U(p))d\nu(p) + \varepsilon\right],
\end{aligned} \tag{8.1}$$

*where* $\alpha = \alpha(\Gamma)$ *is from Lemma 7.2.2, and* $c = c(x, \operatorname{supp}\varphi) > 0$ *is as in Proposition 7.0.1.*

*Remark.* Note that $T_1$ depends on $\eta$ through a non-canonical choice of bump function $\psi$, as seen in the proof. When we apply this lemma to a partition of unity, the same $\psi$ will be used for each part. ◀

*Proof.* Let $\psi \in C(B_U(\eta))$ be a non-negative function such that $\int_U \psi = 1$. Let $F$ and $c = c(x, \operatorname{supp}\varphi) > 0$ be as in the statement of Proposition 7.0.1 for this $\psi$, and let $\varepsilon > 0$.

By Theorem 1.1.1, there exists $T_1 = T_1(x, \psi, \varphi)$ such that for $T \geq T_1$,

$$\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}+c}{R} - \eta\right)\right)\left(m^{\mathrm{BR}}(F) - \varepsilon\right) \tag{8.2}$$

$$\leq \sum_{\gamma \in \Gamma_T} \varphi(x\gamma)$$

$$\leq \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}+c}{r} + \eta\right)\right)\left(m^{\mathrm{BR}}(F) + \varepsilon\right). \tag{8.3}$$

By combining the above with Lemma 7.2.2 (using $R, r$, and $\ell = x \star y$ for $y \in \operatorname{supp}\varphi$), we see that there exist constants $c_0 = c_0(\Gamma, x)$ and $T_2 = T_2(\Gamma, x, \operatorname{supp}\varphi) > 0$ such that for $T \geq T_2$ and any $y \in \operatorname{supp}\varphi$,

$$\left(1 - c_0\left(\eta + \frac{c+1}{\sqrt{T}}\right)^{\alpha}\right)\left(m^{\mathrm{BR}}(F) - \varepsilon\right)$$

$$\leq \frac{1}{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star y}\right)\right)} \sum_{\gamma \in \Gamma_T} \varphi(x\gamma) \tag{8.4}$$

$$\leq \left(1 + c_0\left(\eta + \frac{c+1}{\sqrt{T}}\right)^{\alpha}\right)\left(m^{\mathrm{BR}}(F) + \varepsilon\right).$$

By Lemma 7.0.4, $m^{\mathrm{BR}}(F) = \int_P \varphi(\pi_U(p))d\nu(p)$, and so by (8.4), for any $y \in \operatorname{supp}\varphi$, we obtain that

$$\left|\frac{1}{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star y}\right)\right)} \sum_{\gamma \in \Gamma_T} \varphi(x\gamma) - \int_P \varphi(\pi_U(p))d\nu(p)\right|$$

$$\ll_\Gamma \left(\eta + \frac{c+1}{\sqrt{T}}\right)^{\alpha} \int_P \varphi(\pi_U(p))d\nu(p) + \varepsilon.$$

Since the above holds for any $y \in \operatorname{supp}\varphi$, by bounding

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_T}\varphi(x\gamma)-\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right)\int_P\varphi(\pi_U(p))d\nu(p).\\
&\le\sum_{\gamma\in\Gamma_T}\varphi(x\gamma)-\int_P\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star\pi_U(p)}\right)\right)\varphi(\pi_U(p))d\nu(p)\\
&\le\sum_{\gamma\in\Gamma_T}\varphi(x\gamma)-\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{R}\right)\right)\int_P\varphi(\pi_U(p))d\nu(p),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\left|\sum_{\gamma\in\Gamma_T}\varphi(x\gamma)-\int_P\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star\pi_U(p)}\right)\right)\varphi(\pi_U(p))d\nu(p)\right|\\
&\ll_{\Gamma,x}\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right)\left[\left(\eta+\frac{c+1}{\sqrt{T}}\right)^{\alpha}\int_P\varphi(\pi_U(p))d\nu(p)+\varepsilon\right].
\end{aligned}$$

□

We are now ready to prove Theorem 1.3.2.

*Proof of Theorem 1.3.2.* By Corollary 7.1.2, there exists $\eta_0=\eta_0(\Psi(\operatorname{supp}\varphi))>0$ so that for every $0<\eta<\eta_0$, there exists $\{\varphi_i : 1\le i\le k\}$ that are a partition of $\varphi$, i.e.,

$$\varphi=\sum_{i=1}^{k}\varphi_i$$

so that all the $\varphi_i$ are supported on a small neighborhood of $\operatorname{supp}\varphi$, which we denote by $B$, and each $\varphi_i$ satisfies the assumptions of Lemma 8.1.2.

For any $1\le i\le k$ let,

$$R_i=R_{\varphi_i},\quad r_i=r_{\varphi_i}$$

as in (7.2).

Note that

$$R := \max_{y \in B}(x \star y), \quad r := \min_{y \in B}(x \star y)$$

satisfy $R \geq R_i \geq r_i \geq r$ for any $i$.

Fix $\varepsilon > 0$. By Lemma 8.1.2, there exists $T_1 > 0$ (depending on the $\varphi_i$'s, $x$, $\eta$, and $\varepsilon$) such that for all $T \geq T_1$ and for each $i$,

$$\begin{aligned} &\left| \sum_{\gamma \in \Gamma_T} \varphi_i(x\gamma) - \int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x \star y}\right)\right) \varphi_i(\pi_U(p)) d\nu(p) \right| \\ &\ll_{\Gamma,x} \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right) \left[\left(\eta + \frac{c+1}{\sqrt{T}}\right)^{\alpha} \int_P \varphi_i(\pi_U(p)) d\nu(p) + \frac{\varepsilon}{k}\right]. \end{aligned}$$

Summing over $i$, we obtain

$$\begin{aligned} &\left| \sum_{\gamma \in \Gamma_T} \varphi(x\gamma) - \int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x \star y}\right)\right) \varphi(\pi_U(p)) d\nu(p) \right| \\ &\ll_{\Gamma,x} \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right) \left[\left(\eta^{\alpha} + \frac{\sqrt{c}}{\sqrt{T}}\right) \int_P \varphi(\pi_U(p)) d\nu(p) + \varepsilon\right]. \end{aligned} \tag{8.5}$$

Recall that

$$I(\varphi, T, x) := \int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x \star y}\right)\right) \varphi(\pi_U(p)) d\nu(p).$$

By Corollary 10.1.4, there exists $\sigma = \sigma(\Gamma) > 0$ so that for any $y \in \operatorname{supp} \varphi$,

$$\frac{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right)}{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x \star y}\right)\right)} \ll_{\Gamma} \left(\frac{R}{r}\right)^{\sigma}.$$

Thus, from (8.5), we obtain

$$\begin{aligned} &\left| \frac{\sum_{\gamma \in \Gamma_T} \varphi(x\gamma)}{I(\varphi, T, x)} - 1 \right| \\ &\ll_{\Gamma} \left(\frac{R}{r}\right)^{\sigma} \nu(\varphi \circ \pi_U)^{-1} \left[\left(\eta + \frac{c+1}{\sqrt{T}}\right)^{\alpha} \int_P \varphi(\pi_U(p)) d\nu(p) + \varepsilon\right]. \end{aligned}$$

Since $\eta$ and $\varepsilon$ can be chosen arbitrarily small, the claim follows. $\square$

We will now deduce Corollary 1.3.3 using the shadow lemma, Proposition 3.2.1.

*Proof of Corollary 1.3.3.* Since $\Psi(x)^- \in \Lambda_r(\Gamma)$, there exists $r = r(x) \geq 0$ such that

$$B_U(r)\Psi(x)\Gamma \cap \operatorname{supp} m^{\mathrm{BMS}} \neq \emptyset.$$

Let $w \in B_U(r)\Psi(x)\Gamma \cap \operatorname{supp} m^{\mathrm{BMS}} \subseteq G/\Gamma$. Then for any $T \geq 0$,

$$\mu_w^{\mathrm{PS}}(B_U(T-r)) \leq \mu_{\Psi(x)\Gamma}^{\mathrm{PS}}(B_U(T)) \leq \mu_w^{\mathrm{PS}}(B_U(T+r)).$$

Thus, by Proposition 3.2.1, there exists $\lambda = \lambda(\Gamma) > 1$ such that for all $T \geq 0$,

$$\lambda^{-1}(T-r)^{\delta_\Gamma} \leq \mu_{\Psi(x)\Gamma}^{\mathrm{PS}}(B_U(T)) \leq \lambda(T+r)^{\delta_\Gamma}.$$

For every $y \in \operatorname{supp} \varphi$, we therefore have that for all $T \geq 2r$,

$$\frac{T^{\delta_\Gamma/2}}{(x \star y)^{\delta_\Gamma}} \ll_{\Gamma,x} \mu_{\Psi(x)\Gamma}^{\mathrm{PS}}\left(\frac{\sqrt{T}}{x \star y}\right) \ll_{\Gamma,x} \frac{T^{\delta_\Gamma/2}}{(x \star y)^{\delta_\Gamma}}. \tag{8.6}$$

By Theorem 1.3.2, there exists $T_0 = T_0(x, \varphi)$ such that for $T \geq T_0$,

$$\left|\frac{\sum_{\gamma \in \Gamma_T} \varphi(x\gamma)}{I(\varphi, T, x)} - 1\right| \leq 1/2.$$

Then

$$\frac{1}{\mu_{\Psi(x)\Gamma}^{\mathrm{PS}}\left(\frac{\sqrt{T}}{x \star y}\right)} \sum_{\gamma \in \Gamma_T} \varphi(x\gamma) \leq \frac{2}{\mu_{\Psi(x)\Gamma}^{\mathrm{PS}}\left(\frac{\sqrt{T}}{x \star y}\right)} I(\varphi, T, x)$$

so by (8.6), we obtain

$$\begin{aligned}\frac{1}{T^{\delta_\Gamma/2}}\sum_{\gamma\in\Gamma_T}\varphi(x\gamma) &\ll_{\Gamma,x} \frac{1}{T^{\delta_\Gamma/2}}\int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(\frac{\sqrt{T}}{x\star\pi_U(p)}\right)\varphi(\pi_U(p))d\nu(p)\\ &\ll_{\Gamma,x} \frac{1}{T^{\delta_\Gamma/2}}\int_P \frac{T^{\delta_\Gamma/2}}{(x\star\pi_U(p))^{\delta_\Gamma}}\varphi(\pi_U(p))d\nu(p)\\ &\ll_{\Gamma,x} \int_P \frac{\varphi(\pi_U(p))}{(x\star\pi_U(p))^{\delta_\Gamma}}d\nu(p). \end{aligned}\tag{8.7}$$

The lower bound is very similar. □

## 8.2 Proof of Theorem 1.3.5

In this section, we prove Theorem 1.3.5, restated below for convenience. Recall that for $x \in U\backslash G$ and a compact set $H \subset U\backslash G$, let

$$\mathcal{R}(H,x) := \max_{y,z\in H}\frac{x\star y}{x\star z}.$$

**Theorem 8.2.1.** *Assume that Assumption 1.1.2 holds. For any $0<\varepsilon<1$, there exist $\ell=\ell(\Gamma)\in\mathbb{N}$ and $\kappa=\kappa(\Gamma,\varepsilon)$ satisfying: for every $x\in U\backslash G$ such that $\Psi(x)\Gamma$ is $\varepsilon$-Diophantine and every compact $\Omega\subset G$, there exists $T_0=T_0(x,\Omega)$ so that for every $T\geq T_0$, there exists $\eta=\eta(T,\ell,\kappa,n,\Omega)>0$ such that if $\varphi\in C_c^\infty(U\backslash G)$ with $\Psi(\operatorname{supp}\varphi)\subseteq\Omega$ and satisfies $\mathcal{R}(\operatorname{supp}\varphi,x)-1<\eta$, then for every $y\in\operatorname{supp}\varphi$,*

$$\left|\frac{1}{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star y}\right)\right)}\sum_{\gamma\in\Gamma_T}\varphi(x\gamma)-\int_P\varphi(\pi_U(p))d\nu(p)\right| \ll_{\Gamma,\Omega,x} S_\ell(\varphi)T^{-\kappa}.$$

*Proof.* Fix $x\in U\backslash G$ such that $\Psi(x)\Gamma$ is $\varepsilon$-Diophantine. Let $0<\eta_1=\eta_1(\Omega)<1$ be such that for all $g\in\Omega$,

$$\pi_\Gamma|_{B(g,\eta_1)} : B(g,\eta_1)\to G/\Gamma$$

is injective, where $B(g,\eta_1) = \{h \in G : \|g-h\| \leq \eta_1\}$. Let $0 < \eta < \eta_1$. Then if $\Psi(\operatorname{supp}\varphi) \subset \Omega \subset G$, we have that

$$B := B_U(\eta)\Psi(\operatorname{supp}\varphi)$$

injects into $G/\Gamma$. Let $R = R_\varphi$, $r = r_\varphi$ as in (7.2). We are assuming that

$$\mathcal{R}(\operatorname{supp}\varphi, x) - 1 = \frac{R}{r} - 1 < \eta. \tag{8.8}$$

We will find $T_0 = T_0(x,\Omega)$ as in the statement of the theorem, and choose $\eta$ depending on $T \geq T_0$ later.

According to Lemma 3.4.1(2), there exists $\psi : U \to \mathbb{R}$ such that $\operatorname{supp}\psi = B_U(\eta)$ and

$$\int_U \psi = 1, \quad S_\ell(\psi) \ll \eta^{-\ell+n-1}. \tag{8.9}$$

We can now use Proposition 7.0.1 with the above $\psi$ and $\varphi$ to get an expression that we can estimate using the effective equidistribution theorem, Theorem 1.2.2.

Let $F$ and $c = c(\Omega, x)$ be as in Proposition 7.0.1 for $\psi, \varphi$. There exists $\ell, \kappa', c_2 = c_2(\Gamma, \operatorname{supp}\psi, x)$ as in the statement of Theorem 1.2.2 and $T_1 = T_1(x,\Omega) \geq c$ such that for all $T \geq T_0$,

$$\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}+c}{R} - \eta\right)\right)\left(m^{\mathrm{BR}}(F) - c_2 S_\ell(F) T^{-\kappa'}\right) \tag{8.10}$$

$$\leq \sum_{\gamma\in\Gamma_T} \varphi(x\gamma)$$

$$\leq \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}+c}{r} + \eta\right)\right)\left(m^{\mathrm{BR}}(F) + c_2 S_\ell(F) T^{-\kappa'}\right). \tag{8.11}$$

We now need to express $m^{\mathrm{BR}}(F)$ and $S_\ell(F)$ in terms of $\varphi$, and to compare the PS measures of the balls arising in (8.10) and (8.11).

Let $y \in \operatorname{supp}\varphi$. Note that, by assumption, $r \le x \star y \le R$. Hence, we may use Lemma 7.2.2 to deduce

$$\begin{aligned}&\left|\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}\pm c}{r_\pm}\pm\eta\right)\right)-\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star y}\right)\right)\right|\\&\ll_{\Gamma,x}\left(\eta+\frac{c+1}{\sqrt{T}}\right)^{\alpha}\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star y}\right)\right)\end{aligned}$$

According to Lemma 7.0.4, we have

$$m^{\mathrm{BR}}(F)=\int_P\varphi(\pi_U(p))d\nu(p).$$

Combining the above with (8.10) and (8.11) implies that, for some $c_0=c_0(\Gamma,x)$,

$$\begin{aligned}&\left(1-c_0\left(\eta+\frac{c+1}{\sqrt{T}}\right)^{\alpha}\right)\left(\int_P\varphi(\pi_U(p))d\nu(p)-c_2S_\ell(F)T^{-\kappa'}\right)\\&\le\frac{1}{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star y}\right)\right)}\sum_{\gamma\in\Gamma_T}\varphi(x\gamma)\\&\le\left(1+c_0\left(\eta+\frac{c+1}{\sqrt{T}}\right)^{\alpha}\right)\left(\int_P\varphi(\pi_U(p))d\nu(p)+c_2S_\ell(F)T^{-\kappa'}\right).\end{aligned}\tag{8.12}$$

We are left to find $S_\ell(F)$. Since $B\mapsto B\Gamma$ is injective and $f$ is supported on $B$, using Lemma 3.4.1(1), Lemma 3.4.6, and (8.9), we have

$$\begin{aligned}S_\ell(F)&=S_\ell(f)\\&\ll_n S_\ell(\psi)S_\ell(\varphi\circ\pi_U)\\&\ll_{n,\Gamma}\eta^{-\ell+n-1}S_\ell(\varphi).\end{aligned}\tag{8.13}$$

Finally, we need to put this all together. Combining (8.12) and (8.13), for any $y \in \operatorname{supp}\varphi$, we obtain that

$$\begin{aligned}
&\left| \frac{1}{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star y}\right)\right)} \sum_{\gamma\in\Gamma_T} \varphi(x\gamma) - \int_P \varphi(\pi_U(p))d\nu(p) \right| \\
&\ll_{\Gamma,x} \left(\eta + \frac{c+1}{\sqrt{T}}\right)^{\alpha} \int_P \varphi(\pi_U(p))d\nu(p) + \eta^{-\ell+n-1} S_\ell(\varphi) T^{-\kappa'} \\
&\ll_{\Gamma,\Omega,x} \left[\left(\eta + T^{-1/2}\right)^{\alpha} + \eta^{-\ell+n-1}T^{-\kappa'}\right] S_\ell(\varphi). \qquad (8.14)
\end{aligned}$$

Choose $\rho$ sufficiently small so that

$$(\ell - n + 1)\rho < \kappa'/2.$$

Let $\eta = T^{-\rho}$, for $T \geq T_0(x,\Omega) = \max\{T_1, T_2\}$. Let

$$\kappa = \min\{\rho\alpha, \alpha/2, \kappa'/2\}.$$

Then we conclude that

$$\begin{aligned}
&\left| \frac{1}{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star y}\right)\right)} \sum_{\gamma\in\Gamma_T} \varphi(x\gamma) - \int_P \varphi(\pi_U(p))d\nu(p) \right| \\
&\ll_{\Gamma,\Omega,x} T^{-\kappa} S_\ell(\varphi).
\end{aligned}$$

□

## 8.3 Proof of Theorem 1.3.4

In this section, we will use a partition of unity argument and the previous section to establish Theorem 1.3.4, which is restated below for convenience.

**Theorem 8.3.1.** *Assume that Assumption 1.1.2 holds. For any $0 < \varepsilon < 1$, there exist $\ell = \ell(\Gamma) \in \mathbb{N}$ and $\kappa = \kappa(\Gamma, \varepsilon)$ satisfying: for every $\varphi \in C_c^\infty(U\backslash G)$ and for every $x \in U\backslash G$ such that $\Psi(x)\Gamma$ is $\varepsilon$-Diophantine, and for all $T \gg_{\Gamma, \operatorname{supp}\varphi, x} 1$,*

$$\left| \frac{\sum_{\gamma \in \Gamma_T} \varphi(x\gamma)}{\int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x \star \pi_U(p)}\right)\right) \varphi(\pi_U(p)) d\nu(p)} - 1 \right|$$
$$\ll_{\Gamma, \operatorname{supp}\varphi, x} T^{-\kappa}\left(1 + S_\ell(\varphi)\nu(\varphi \circ \pi_U)^{-1}\right).$$

We begin by interpreting (8.14) in another form, as in the following lemma. This form will be easier to work with when using a partition of unity. Note that the main idea here is that for $\varphi$ of small support and for any $y \in \operatorname{supp}\varphi$, $x \star y$ is very close to both $R$ and $r$.

For $H \subseteq U\backslash G$ compact and $x \in U\backslash G$, define

$$R_H = \max_{y \in H} x \star y \quad \text{ and } r_H = \min_{y \in H} x \star y.$$

**Lemma 8.3.2.** *Let $\Omega \subseteq G$ be a compact set, let $x \in U\backslash G$ be such that $\Psi(x)\Gamma$ is $\varepsilon$-Diophantine, let $\varphi \in C_c^\infty(U\backslash G)$ with $\Psi(\operatorname{supp}\varphi) \subset \Omega$, and let $\eta > 0$ be smaller than the injectivity radius of $\Omega$. Let $R = R_{\pi_U(\Omega)}$ and $r = r_{\pi_U(\Omega)}$. Then for $T \gg_{\Gamma, \Omega, x} 1$,*

$$\left| \sum_{\gamma \in \Gamma_T} \varphi(x\gamma) - \int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x \star \pi_U(p)}\right)\right) \varphi(\pi_U(p)) d\nu(p) \right|$$
$$\ll_{\Gamma, \Omega, x} \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right)\left(\eta + T^{-1/2}\right)^\alpha \int_P \varphi(\pi_U(p)) d\nu(p)$$
$$+ \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right) \eta^{-\ell + (n-1)/2} S_\ell(\varphi) T^{-\kappa'}.$$

*Proof.* Following the arguments in the proof of Theorem 8.2.1 (more explicitly, the computations leading to (8.14)), one may deduce that for $T \gg_{\Gamma,\Omega,x} 1$,

$$\begin{aligned}
&\left| \frac{1}{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star y}\right)\right)} \sum_{\gamma\in\Gamma_T} \varphi(x\gamma) - \int_P \varphi(\pi_U(p))d\nu(p) \right| \\
&\ll_{\Gamma,\Omega,x} \left[ \left(\eta + T^{-1/2}\right)^{\alpha} + \eta^{-\ell+(n-1)/2} T^{-\kappa'} \right] S_\ell(\varphi).
\end{aligned}$$

Therefore, we may conclude

$$\begin{aligned}
&-\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{R}\right)\right)\left[\left(\eta + T^{-1/2}\right)^{\alpha} \int_P \varphi(\pi_U(p))d\nu(p) - \eta^{-\ell+(n-1)/2} S_\ell(\varphi) T^{-\kappa'}\right] \\
&\ll_{\Gamma,\operatorname{supp}\varphi,x} \sum_{\gamma\in\Gamma_T} \varphi(x\gamma) - \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{R}\right)\right) \int_P \varphi(\pi_U(p))d\nu(p) \\
&\leq \sum_{\gamma\in\Gamma_T} \varphi(x\gamma) - \int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star \pi_U(p)}\right)\right)\varphi(\pi_U(p))d\nu(p) \\
&\leq \sum_{\gamma\in\Gamma_T} \varphi(x\gamma) - \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right) \int_P \varphi(\pi_U(p))d\nu(p) \\
&\ll \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right)\left[\left(\eta + T^{-1/2}\right)^{\alpha} \int_P \varphi(\pi_U(p))d\nu(p) + \eta^{-\ell+(n-1)/2} S_\ell(\varphi) T^{-\kappa'}\right],
\end{aligned}$$

where the implied constant in the last line depends on $\Gamma, \operatorname{supp}\varphi$, and $x$. □

*Proof of Theorem 1.3.4.* **Step 1: Use an appropriate partition of $\varphi$.**

By Corollary 7.1.2, there exists a partition $\varphi_1, \dots, \varphi_k$ of $\varphi$ satisfying Lemma 8.3.2 with $\Omega = \Psi(\operatorname{supp}\varphi)$ and

$$\sum_{i=1}^{k} S_\ell(\varphi_i) \ll_{\ell,\operatorname{supp}\varphi} \eta^{-\ell+n(n+1)/4} S_\ell(\varphi). \tag{8.15}$$

Thus, by Lemma 8.3.2, we have that for each $\varphi_i$,

$$\begin{aligned}
&\left|\sum_{\gamma\in\Gamma_T}\varphi_i(x\gamma)-\int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star\pi_U(p)}\right)\right)\varphi_i(\pi_U(p))d\nu(p)\right|\\
&\ll_{\Gamma,\operatorname{supp}\varphi,x}\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r_i}\right)\right)\cdot \qquad (8.16)\\
&\left[\left(\eta+T^{-1/2}\right)^{\alpha}\int_P\varphi_i(\pi_U(p))d\nu(p)+\eta^{-\ell+(n-1)/2}S_\ell(\varphi_i)T^{-\kappa'}\right].
\end{aligned}$$

Let

$$r=\min\{r_1,\ldots,r_k\}.$$

Summing over $i$, using (8.15), and noting that $\eta<1$ yields

$$\begin{aligned}
&\left|\sum_{\gamma\in\Gamma_T}\varphi(x\gamma)-\int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star\pi_U(p)}\right)\right)\varphi(\pi_U(p))d\nu(p)\right|\\
&\ll_{\Gamma,\operatorname{supp}\varphi,x}\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right)\left(\eta+T^{-1/2}\right)^{\alpha}\int_P\varphi(\pi_U(p))d\nu(p)\\
&\qquad+\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right)\eta^{-2\ell+(n^2+3n-2)/4}S_\ell(\varphi)T^{-\kappa'}. \qquad (8.17)
\end{aligned}$$

**Step 2: Putting it together.**

Recall

$$I(\varphi,T,x):=\int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x\star\pi_U(p)}\right)\right)\varphi(\pi_U(p))d\nu(p).$$

Let

$$R=R_\varphi:=\max_{y\in\operatorname{supp}\varphi}x\star y.$$

By Corollary 10.1.4, we have that there exists $\sigma = \sigma(\Gamma) > 0$ so that

$$\begin{aligned}
\frac{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right)}{I(\varphi, T, x)} &\leq \frac{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right)}{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{R}\right)\right)\nu(\varphi\circ\pi_U)} \\
&\ll_\Gamma \left(\frac{R}{r}\right)^\sigma \nu(\varphi\circ\pi_U)^{-1} \\
&\ll_{\Gamma,\operatorname{supp}\varphi,x} \nu(\varphi\circ\pi_U)^{-1}, \qquad (8.18)
\end{aligned}$$

where the last line follows because $(R/r)^\sigma$ is simply a constant depending on $\operatorname{supp}\varphi, \Gamma$, and $x$.

From (8.17) and (8.18), we obtain that

$$\begin{aligned}
\left|\frac{\sum\limits_{\gamma\in\Gamma_T}\varphi(x\gamma)}{I(\varphi,T,x)} - 1\right| &\ll_{\Gamma,\operatorname{supp}\varphi,x} \frac{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right)}{I(\varphi, T, x)}\sqrt{T} + c\left(\eta + T^{-1/2}\right)^\alpha \int_P \varphi(\pi_U(p))d\nu(p) \\
&\quad + \frac{\mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{r}\right)\right)}{I(\varphi, T, x)}\eta^{-2\ell+(n^2+3n-2)/4}S_\ell(\varphi)T^{-\kappa'} \\
&\ll_{\Gamma,\operatorname{supp}\varphi,x} \sqrt{T} + c\left(\eta + T^{-1/2}\right)^\alpha + \nu(\varphi\circ\pi_U)^{-1}\eta^{-2\ell+(n^2+3n-2)/4}S_\ell(\varphi)T^{-\kappa'} \\
&\ll_{\Gamma,\operatorname{supp}\varphi,x} \sqrt{T} + c\left(\eta + T^{-1/2}\right)^\alpha + \frac{\eta^{-2\ell+(n^2+3n-2)/4}S_\ell(\varphi)T^{-\kappa'}}{\nu(\varphi\circ\pi_U)} \\
&\ll_{\Gamma,\operatorname{supp}\varphi,x} T^{-\kappa}\left(1 + S_\ell(\varphi)\nu(\varphi\circ\pi_U)^{-1}\right), \qquad (8.19)
\end{aligned}$$

where (8.19) follows by choosing $\eta = T^{-\rho}$, where $\rho = 1$ if $2\ell - \frac{n^2+3n-2}{4} < 0$, and

$$\rho = \frac{\kappa'}{4\ell - n + 1 - \frac{1}{2}n(n+1)}$$

otherwise, and letting

$$\kappa = \min\{\rho\alpha, \alpha/2, \kappa'/2\}.$$

□

*Remark.* Note that the implied dependence on $x$ is quite explicit. It arises from suppressing the factors $R_\varphi, r_\varphi, \|\Psi(x)^{-1}\|$, and $c$ throughout the argument. Specifically, $c$ is suppressed in the use of Lemma 8.3.2, and $r_\varphi, R_\varphi$ are suppressed in (8.19). Note that these constants depend on $x$ and $\operatorname{supp}\varphi$ through the $\star$ operation, as can be seen from the definitions and the proof of Lemma 7.0.2, and they can also be computed explicitly if desired. The factor of $\|\Psi(x)^{-1}\|$ is suppressed in the construction of the partition in Corollary 7.1.2. The implied constant from Theorem 1.2.2 also depends on $x$ through the explicit Diophantine behaviour of $x$, i.e. the $(\varepsilon, s_0)$. ◀

*Remark.* The suppressed constants $R_\varphi, r_\varphi, c$, and $\|\Psi(x)^{-1}\|$ mentioned in the previous remark are continuous functions of $x$ by definition of $\star$. This will be used in the next section. ◀

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This chapter contains material from the following, which has been submitted for publication: N. Tamam, J. M. Warren, *"Distribution of orbits of geometrically finite groups acting on null vectors"*, arXiv:2009.11968. The dissertation author was one of the primary investigators and authors of this paper.

# Chapter 9

# Applications

Let $V$ be a manifold on which $G$ acts smoothly and transitively from the right, so that $V$ may be identified with $H\backslash G$ for some closed subgroup $H$ of $G$ that is the stabilizer of a point $v_0 \in V$. Let $\sigma : H\backslash G \to V$ be the identification

$$\sigma(Hg) = v_0 \cdot g. \tag{9.1}$$

Note that $\sigma$ is smooth because $G$ acts smoothly.

Assume further that $U \subseteq H \subseteq UM$. In particular, $\pi_U(H)$ is compact in $U\backslash G$ (recall from Chapter 3 that $\pi_U : G \to U\backslash G$ is the quotient map). Define $\theta : U\backslash G \to H\backslash G$ by

$$\theta(Ug) = Hg. \tag{9.2}$$

We will now show that $\theta$ is smooth. Since $U$ is closed, $\pi_U : G \to U\backslash G$ is a smooth submersion. Thus, $\theta$ is smooth if and only if $\theta \circ \pi_U$ is smooth. Since $\theta \circ \pi_U = \pi_H$, the quotient map from $G \to H\backslash G$, it is smooth, which establishes the smoothness of $\theta$.

For $v, u \in V$, let $x, y \in U\backslash G$ be such that $u = \sigma(\theta(x)), v = \sigma(\theta(y))$. We may define

$$v \star u = x \star y.$$

This is well-defined because $UM$ stabilizes $E_{1,n+1}$, and $H \subseteq UM$ (see (1.4) for the definition of $\star$ on $U\backslash G$).

Recall the definition of $\Psi : U\backslash G \to G$ from Chapter 7:

$$\Psi(Ug) = ak,$$

where $g = uak$ is the Iwasawa decomposition of $g$.

**Definition 9.0.1.** A vector $v \in V$ is called **$\varepsilon$-Diophantine** if there exists $x \in U\backslash G$ such that $v = v_0 \cdot x$ and $\Psi(x)\Gamma$ is $\varepsilon$-Diophantine. Such $x$ is called an **$\varepsilon$-Diophantine representative** of $v$. ◀

*Remark.* Note that for any $g \in G$, $g^- \in \Lambda_r(\Gamma)$ if and only if $(umg)^- \in \Lambda_r(\Gamma)$ for all $um \in UM$, since $UM$ does not change $g^-$. Thus, for $v \in V$, we may define the notation

$$v^- \in \Lambda_r(\Gamma)$$

if for any representative $\Psi(x)$, $\Psi(x)^- \in \Lambda_r(\Gamma)$. Note also that since $\mathcal{C}_0$ is $M$ invariant and $A$ commutes with $M$, the definition of $v$ being $\varepsilon$-Diophantine is independent of the choice of a representative $x \in U\backslash G$. ◀

Observe that $\nu$ uniquely defines a measure on $U\backslash G$ by $\nu(\varphi \circ \pi_U)$ for any continuous function $\varphi$ defined on $U\backslash G$. One can use the push-forward of this measure to $H\backslash G$ and the identification of $V$ with $H\backslash G$ to uniquely define a measure on $V$. Denote this measure by $\bar{\nu}$.

**Corollary 9.0.2.** *For any $0 < \varepsilon < 1$, there exist $\ell = \ell(\Gamma) \in \mathbb{N}$ and $\kappa = \kappa(\Gamma, \varepsilon)$ satisfying: for every $\overline{\varphi} \in C_c^\infty(V)$ and $\varepsilon$-Diophantine $v \in V$ with Diophantine representative $x \in U\backslash G$ (i.e., $v_0 x = v$), and $T \gg_{\Gamma, \operatorname{supp}\overline{\varphi}, v} 1$,*

$$\left| \frac{\sum_{\gamma \in \Gamma_T} \overline{\varphi}(v\gamma)}{\int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{v \star u}\right)\right) \overline{\varphi}(u) d\overline{\nu}(u)} - 1 \right| \ll_{\Gamma, \operatorname{supp}\overline{\varphi}, x} T^{-\kappa}\left(1 + S_\ell(\overline{\varphi})\nu(\overline{\varphi})^{-1}\right).$$

*Proof.* Let $\ell'$ satisfy the conclusion of Theorem 1.3.4 and $\ell$ satisfy the conclusion of Lemma 3.4.6 for $\ell'$.

Recall the definitions of $\sigma : H\backslash G \to V$ in (9.1) and $\theta : U\backslash G \to H\backslash G$ in (9.2). Define $\varphi \in C_c^\infty(U\backslash G)$ by

$$\varphi = \overline{\varphi} \circ \sigma \circ \theta.$$

Let $x \in U\backslash G$ be an $\varepsilon$-Diophantine representative of $v$. In particular, note that $\sigma(\theta(x)) = \sigma(H\Psi(x)) = v$. Then, since

$$\varphi(x\gamma) = \overline{\varphi}(\sigma(\theta(x)) \cdot \gamma) = \overline{\varphi}(v \cdot \gamma),$$

by Theorem 1.3.4, for $T \gg_{\Gamma, \operatorname{supp}\overline{\varphi}, \varepsilon, x} 1$,

$$\begin{aligned} &T^{-\kappa}\left(1 + S_\ell(\varphi)\nu(\varphi \circ \pi_U)^{-1}\right) \\ &\gg_{\Gamma, \operatorname{supp}\varphi, x} T^{-\kappa} \left| \frac{\sum_{\gamma \in \Gamma_T} \varphi(x\gamma)}{\int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{x \star \pi_U(p)}\right)\right) \varphi(\pi_U(p)) d\nu(p)} - 1 \right| \\ &\gg_{\Gamma, \operatorname{supp}\overline{\varphi}, x} T^{-\kappa} \left| \frac{\sum_{\gamma \in \Gamma_T} \overline{\varphi}(v\gamma)}{\int_P \mu^{\mathrm{PS}}_{\Psi(x)\Gamma}\left(B_U\left(\frac{\sqrt{T}}{v \star u}\right)\right) \overline{\varphi}(u) d\overline{\nu}(u)} - 1 \right|. \end{aligned}$$

Note that the dependence of $T$ on $x$ is through $\varepsilon, s_0$ such that $x$ is $(\varepsilon, s_0)$-Diophantine, and by Remark , this is in fact independent of the choice of Diophantine representative $x$ of $v$. By Remark , the dependence on $x$ in the implied constant in the above inequality can be made uniform over all representatives of $v$, as they vary by elements in $M$, a compact set. Thus, both dependencies on $x$ can be replaced by dependence on $v$.

Observe that $\varphi$ can be viewed as a function on $U\backslash H \times H\backslash G \cong U\backslash G$ by

$$\varphi(y, x) = \mathrm{id}_{U\backslash H}(y) \cdot (\overline{\varphi} \circ \sigma)(x).$$

Therefore, Lemma 3.4.1 and Lemma 3.4.6 imply

$$S_{\ell'}(\varphi) \ll_H S_{\ell'}(\mathrm{id}_{U\backslash H}) S_{\ell'}(\overline{\varphi} \circ \sigma) \ll_{H,\sigma,\mathrm{supp}\,\varphi} S_\ell(\overline{\varphi}),$$

where the Sobolev norm of $\mathrm{id}_{U\backslash H}$ is finite since we are assuming $U\backslash H$ is compact. $\square$

In a similar way, one may deduce the following from Corollary 1.3.3 (see Remark for the notation $v^- \in \Lambda_r(\Gamma)$):

**Corollary 9.0.3.** *Assume that $\Gamma$ is convex cocompact. For any $\overline{\varphi} \in C_c(V)$ and every $v \in V$ with $v^- \in \Lambda(\Gamma)$, as $T \to \infty$,*

$$\frac{1}{T^{\delta_\Gamma/2}} \sum_{\gamma \in \Gamma_T} \overline{\varphi}(v\gamma) \asymp \int_P \frac{\overline{\varphi}(u)}{(v \star u)^{\delta_\Gamma}} d\overline{\nu}(u),$$

*where the implied constant depends on $v$ and $\Gamma$.*

## 9.1 Identification with Null Vectors

Let $G$ act on $\mathbb{R}^{n+1}$ by right matrix multiplication, and let

$$V = \mathbf{e}_{n+1}G \setminus \{0\}.$$

To better understand the set $V$, note that the representation of $\mathrm{SO}(n,1)$ we are using is

$$\mathrm{SO}(n,1) = \{A \in \mathrm{SL}_{n+1}(\mathbb{R}) : AJA^T = J\},$$

where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $P$ be such that

$$J' := \begin{pmatrix} -I_n & 0 \\ 0 & 1 \end{pmatrix} = PJP^T.$$

Then $VP$ is the upper half of the "light cone" in the standard representation of $\mathrm{SO}(n,1)$. In particular, this consists of null vectors of

$$Q'(x_1, \ldots, x_{n+1}) = x_{n+1}^2 - x_1^2 - \cdots - x_n^2$$

with $x_{n+1} > 0$. In our case, $V$ consists of null vectors of

$$Q(x_1, \ldots, x_{n+1}) = 2x_1x_{n+1} - x_2^2 - \cdots - x_n^2.$$

**Proposition 9.1.1.** *Let $\Gamma$ be convex cocompact. For any $\overline{\varphi} \in C_c(V)$ and every $v \in V$ with $v^- \in \Lambda(\Gamma)$, as $T \to \infty$, we have that*

$$\frac{1}{T^{\delta_\Gamma/2}} \sum_{\gamma \in \Gamma_T} \overline{\varphi}(v\gamma) \asymp \int_V \overline{\varphi}(u) \frac{d\overline{\nu}(u)}{(\|v\|_2 \, \|u\|_2)^{\delta_\Gamma/2}},$$

*where the implied constant depends on $v$ and $\Gamma$.*

The measure $\overline{\nu}$ is described more explicitly in (9.8), below.

Let

$$\mathbf{e}_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}.$$

Then

$$\mathrm{Stab}_G(\mathbf{e}_{n+1}) = UM, \tag{9.3}$$

and hence

$$A \times M\backslash K \cong UM\backslash G \cong V \tag{9.4}$$

via right matrix multiplication

$$UMg \mapsto \mathbf{e}_{n+1}g.$$

We will now interpret Corollary 9.0.2 in this setting. We start by understanding the measure $\overline{\nu}$.

We view $V$ as $(M\backslash K) \times \mathbb{R}^+$, via the "polar decomposition" of $v \in V$,

$$v = \|v\|_2 \mathbf{e}_{n+1} k = \mathbf{e}_{n+1} a_{-\log \|v\|_2} k, \tag{9.5}$$

where $\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$ and $\|\cdot\|_2$ denotes the Euclidean norm on $V$. We may also identify $M\backslash K$ with $\partial(\mathbb{H}^n)$ via

$$Mk \mapsto w_o^- k. \tag{9.6}$$

Thus, given $v \in V$, (9.5) and (9.6) uniquely determine a pair $(a_{-\log \|v\|_2}, Mk) \in A \times M\backslash K$, or equivalently, a pair $(a_{-\log \|v\|_2}, w_o^- k) \in A \times \partial(\mathbb{H}^n)$.

Viewing $\partial(\mathbb{H}^n)$ as $M\backslash K$ as in (9.6), we may in turn identify this with $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ via

$$w_o^- k \mapsto \mathbf{e}_{n+1} k.$$

Thus, $\nu_o$ uniquely determines a measure $\overline{\nu}_o$ on $\mathbb{S}^n \cap V$ via

$$d\overline{\nu}_o(\mathbf{e}_{n+1}k) = d\nu_o(w_o^- k). \tag{9.7}$$

Then, since $K$ stabilizes $o$ and $M$ stabilizes $w_o$, $\overline{\nu}$ can be described from (3.21): if $s = \beta_{(a_{-\log\|v\|_2}k)^-}(o, a_{-\log\|v\|_2}k(o)) = \log\|v\|_2$,

$$\begin{aligned} d\overline{\nu}(v) &:= d\nu(a_{-\log\|v\|_2}k) \\ &= e^{\delta_\Gamma \beta_{(a_{-\log\|v\|_2}k)^-}(o, a_{-\log\|v\|_2}k(o))} d\nu_o(w_o^- a_{-\log\|v\|_2}k) ds \\ &= e^{\delta_\Gamma s} d\nu_o(w_o^- k) ds \\ &= \|v\|_2^{\delta_\Gamma - 1} d\overline{\nu}_o(\mathbf{e}_{n+1}k) d\|v\|_2. \end{aligned}$$

For $v \in V$, define

$$v^- := \mathbf{e}_{n+1}k \in \mathbb{S}^n,$$

where $v$ corresponds to $(a_{-\log\|v\|_2}, Mk) \in A \times M\backslash K$. Then we have

$$d\overline{\nu}(v) = \|v\|_2^{\delta_\Gamma - 1} d\overline{\nu}_o(v^-) d\|v\|_2. \tag{9.8}$$

As discussed in the previous section, $v \star u$ may be computed by the formula in (1.4) for any choice of representatives of $v$ and $u$ in $U\backslash G$. In particular, if

$$v = \|v\|_2 \mathbf{e}_{n+1} k_v, \quad u = \|u\|_2 \mathbf{e}_{n+1} k_u,$$

then

$$v \star u = \sqrt{\frac{1}{2}\|v\|_2\|u\|_2 \max_{1\leq i,j\leq n+1} |(k_v^{-1})_{i,1}(k_u)_{n+1,j}|},$$

where $k_{i,j}$ denotes the $(i,j)$ entry of $k$. In particular

$$v \star u \asymp \sqrt{\|v\|_2 \, \|u\|_2}.$$

Putting this together with Corollary 9.0.3 yields the proposition.

## 9.2 Wedge Products

The previous example can be generalized to $\bigwedge^j \mathbb{R}^{n+1}$ for any $1 \leq j \leq n$. Fix $j$, and let

$$W = \bigwedge^j \mathbb{R}^{n+1}, \quad \text{and} \quad v_0 = v_0(j) = e_{n-j+1} \wedge \cdots \wedge e_{n+1},$$

with $G$ acting on $W$ by right multiplication. Then,

$$\mathrm{Stab}_{e_{n-j+1}\wedge\cdots\wedge e_{n+1}} = U \cdot M_j$$

for some $M_j \subseteq M$. Define

$$V = v_0 G \setminus \{0\}.$$

Fix a norm on $V$ which is invariant under $K$ such that $\|v_0\| = 1$.

Since any $v \in V$ can be written as

$$v = v_0 a_{-\log\|v\|} k,$$

where $k \in M_j \backslash K$, in a similar way to the construction in the previous section, one can show that if $a_{-\log\|v\|}k \in UP$ and can be written as $uamv \in UAM\tilde{U}$, then

$$d\overline{\nu}(v) = \|v\|^{\delta_\Gamma - 1} d\nu_o(v^-) d\|v\| \, dm,$$

where $v^- := w_o^- k$, and $dm$ is the push forward of the probability Haar measure on $M_j \backslash M$. $d\overline{\nu}(v)$ is zero if $a_{-\log\|v\|}k \notin UP$, because the original measure $\nu$ is supported on $P$.

Moreover, by earlier reasoning, $v \star u$ is well defined and, as in the previous section, we have that

$$v \star u \asymp \sqrt{\|v\| \, \|u\|}.$$

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# Chapter 10

# Appendix: Friendliness of the PS Measure

This chapter is dedicated to the technical details of the proof of Proposition 5.0.4. From the starting point of the fact that the PS density $\nu_o$ is *friendly* by [DFSU20], we will show that the PS measures $\mu_x^{\mathrm{PS}}$ satisfy a stronger condition than just that of friendliness, thanks to the shadow lemma.

For simplicity, in this chapter, we work in the Poincaré ball models of hyperbolic geometry $\mathbb{D}^n$, instead of $\mathbb{H}^n$. Recall that $\mathbb{D}^n$ and $\mathbb{H}^n$ are isometric via the Cayley transform.

Denote by $d_E$ the Euclidean metric on $\mathbb{R}^m$. For a subset $S \subseteq \mathbb{R}^m$ and $\xi > 0$, let

$$\mathcal{N}(S, \xi) = \{x \in \mathbb{R}^m \,:\, d_E(x, S) \leq \xi\}.$$

For $v \in \mathbb{R}^m$ and $r > 0$, let

$$B(v, r) = \{u \in \mathbb{R}^m \,:\, d_E(u, v) \leq r\}$$

be the Euclidean ball of radius $r$ around $v$.

**Definition 10.0.1.** Let $\mu$ be a measure defined on $\mathbb{R}^m$.

1. $\mu$ is called **Federer** (respectively, **doubling**) if for any $c > 1$, there exists $k_1 > 0$ such that for all $v \in \operatorname{supp}(\mu)$ and $0 < \eta \leq 1$ (respectively, $\eta > 0$),

$$\mu(B(v, c\eta)) \leq k_1 \mu(B(v, \eta)).$$

2. $\mu$ is called **decaying and nonplanar** if there exist $\alpha, c_2 > 0$ such that for all $v \in \operatorname{supp} \mu$, $\xi > 0$, $0 < \eta \leq 1$, and every affine hyperplane $L \subseteq \mathbb{R}^n$,

$$\mu(\mathcal{N}(L, \xi \left\| d_L \right\|_{\mu, B(v,\eta)}) \cap B(v, \eta)) \leq c_2 \xi^\alpha \mu(B(v, \eta)),$$

where

$$\left\| d_L \right\|_{\mu, B(v,\eta)} := \sup \left\{ d(\mathbf{y}, L) \,:\, \mathbf{y} \in B(v, \eta) \cap \operatorname{supp} \mu \right\}.$$

3. $\mu$ is called **friendly** if it is Federer, decaying, and nonplanar.

◀

**Theorem 10.0.2.** *[DFSU20, Theorem 1.9] Assume* $\Gamma$ *is geometrically finite and Zariski dense. Then the PS-densities* $\{\nu_x\}_{x \in \mathbb{D}^n}$ *are friendly. Moreover, in this case, the constants in Definition 10.0.1 only depend on* $\Gamma$*.*

Note that, as in [DFSU20, Definition 1.1(1.3)], using closed thickenings, one obtains Definition 10.0.1(2) by combining the separate definitions of decaying and of nonplanar from [DFSU20]. The above result for the case when $\Gamma$ is convex cocompact was proved in [SU40, Theorem 2].

In the case that all cusps have maximal rank (which vacuously includes the case of convex cocompact $\Gamma$), a stronger statement holds, see §10.3.

Because of the shadow lemma, Proposition 3.2.1, we will see that the leafwise PS measures $\{\mu_x^{\mathrm{PS}}\}$ satisfy a stronger condition than that of friendliness. In general, our proofs take the following form: we will begin by proving a statement for $\nu_o$, then for $\mu_x^{\mathrm{PS}}$ when $x^+ \in \Lambda(\Gamma)$, and then finally a nicer statement for $x \in \operatorname{supp} m^{\mathrm{BMS}}$ will be obtained by a flowing argument.

The next lemma and subsequent corollaries are necessary to move between these measures.

As in §3.1, we fix $o \in \mathbb{D}^n$.

For any $x \in \mathbb{D}^n$ define the **Gromov distance** at $x$ of $\xi, \eta \in \partial\mathbb{D}^n$ by

$$d_x(\xi,\eta) = \exp\left(-\frac{1}{2}\beta_\xi(x,y) - \frac{1}{2}\beta_\eta(x,y)\right),$$

where $y$ is on the ray joining $\xi$ and $\eta$. For any $x \in \mathbb{D}^n$, $\xi \in \partial\mathbb{D}^n$, and $r > 0$ let

$$B_x(\xi, r) := \{\eta \in \partial\mathbb{D}^n \ : \ d_x(\xi,\eta) \leq r\}.$$

For $v \in \mathrm{T}^1(\mathbb{D}^n)$, denote by $\mathrm{Pr}_{v^-} : Uv \to \partial\mathbb{D}^n \setminus \{v^-\}$ the projection $w \mapsto w^+$.

The next lemma follows from §1.6 in [Kai90], and [Sch04, Lemma 2.5, Theorem 3.4].

**Lemma 10.0.3.** *There exist constants $\alpha_0 > 0$, $c > 1$ such that for all $g \in G$ and $0 < \eta \leq \alpha_0$, we have*

$$B_{\pi(g)}\left(g^+, c^{-1}\eta\right) \subseteq \mathrm{Pr}_{g^-}\left(B_U\left(\eta\right)g\right) \subseteq B_{\pi(g)}\left(g^+, c\eta\right).$$

According to [DSU17, Lemma 3.5.1] for any $\xi, \eta \in \partial\mathbb{D}^n$

$$d_o(\xi,\eta) = \frac{1}{2} d_E(\xi,\eta). \tag{10.1}$$

Using the triangle inequality on the hyperbolic distance and the definition of the Busemann function, one can show that for any $x \in \mathbb{D}^n$ and $\xi, \eta \in \partial\mathbb{D}^n$

$$e^{-d(o,x)} \leq \frac{d_x(\xi,\eta)}{d_o(\xi,\eta)} \leq e^{d(o,x)}. \tag{10.2}$$

The following is a direct corollary of (10.1), (10.2), and Lemma 10.0.3.

**Corollary 10.0.4.** *There exist constants $\alpha_0 > 0$, $c > 1$ such that for all $g \in G$ and $0 < \eta \leq \alpha_0$, we have*

$$B\left(g^+, c^{-1}e^{-d(o,\pi(g))}\eta\right) \subseteq \mathrm{Pr}_{g^-}\left(B_U\left(\eta\right)g\right) \subseteq B\left(g^+, ce^{d(o,\pi(g))}\eta\right).$$

The next corollary will be necessary to obtain a nonplanarity result for $\mu_x^{\mathrm{PS}}$. It follows from Corollary 10.0.4 by covering the hyperplane with small balls using the fact that $\eta \leq 1$ to uniformly bound the $d(o,\pi(g'))$'s with $d(o,\pi(g))$, where $g'$ is the center of one of the balls in this cover.

**Corollary 10.0.5.** *Let $\alpha_0$ be as in Corollary 10.0.4. There exists a constant $c > 1$ so that for every $g \in G$, every $0 < \xi < \eta \leq \alpha_0$, and every hyperplane $L$ in $\mathbb{R}^{n-1}$, there exists a hyperplane $L'$ in $\partial(\mathbb{H}^n)$ so that*

$$\begin{aligned} &\mathcal{N}\left(L', c^{-1}e^{-d(o,\pi(g))}\xi\right) \cap B\left(g^+, c^{-1}e^{-d(o,\pi(g))}\eta\right) \\ &\subseteq \mathrm{Pr}_{g^-}\left(\mathcal{N}(L,\xi) \cap B_U(\eta)g\right) \\ &\subseteq \mathcal{N}\left(L', ce^{d(o,\pi(g))}\xi\right) \cap B\left(g^+, ce^{d(o,\pi(g))}\eta\right) \end{aligned}$$

## 10.1 The PS Measure is Federer

In this section, we prove more specific Federer statements for $\nu_o$ and $\mu_x^{\mathrm{PS}}$.

**Lemma 10.1.1.** *There exists a constant $\sigma \geq \delta_\Gamma$ depending only on $\Gamma$ such that for any $\lambda \in \Lambda(\Gamma), \eta > 0$ and $c \geq 1$, we have that*

$$\nu_o(B(\lambda, c\eta)) \ll_\Gamma c^\sigma \nu_o(B(\lambda,\eta)).$$

*Proof.* We will prove this for the balls $B_o(\lambda, c\eta)$, and $B_o(\lambda,\eta)$ using the Gromov distance. It then immediately follows for the Euclidean balls $B(\lambda, c\eta)$ and $B(\lambda,\eta)$ by the Federer condition and (10.1).

Let $\{\lambda_t\}_{t\geq 0}$ be a geodesic ray joining $o$ to $\lambda$. By the shadow lemma for $\nu_o$, [SV95, Theorem 2] (see also [Sch04, Theorem 3.2]), we have that for any $\eta > 0$,

$$\begin{aligned} \eta^{\delta_\Gamma} e^{(k(\lambda_{-\log\eta})-\delta_\Gamma)d(\pi(\mathcal{C}_0),\lambda_{-\log\eta})} &\ll_\Gamma \nu_o\left(B_o\left(\lambda,\eta\right)\right) \\ &\ll_\Gamma \eta^{\delta_\Gamma} e^{(k(\lambda_{-\log\eta})-\delta_\Gamma)d(\pi(\mathcal{C}_0),\lambda_{-\log\eta})}. \end{aligned} \tag{10.3}$$

Here, $k(\lambda_{-\log\eta})$ denotes the rank of the cusp that $\lambda_{-\log\eta}$ lies in; if it is in $\pi(\mathcal{C}_0)$, it is defined to be zero. (Recall the definition of $\mathcal{C}_0$ from §3.2.) Note also that we have absorbed a constant depending on $\operatorname{diam}\pi(\mathcal{C}_0)$ (hence only on $\Gamma$) in order to write the distance from $\pi(\mathcal{C}_0)$ rather than from the fixed reference point $o$.

It follows from (10.3) that it is enough to show that for some $\sigma \geq \delta_\Gamma$,

$$
\begin{aligned}
\nu_o\left(B_o\left(\lambda, c\eta\right)\right) &\ll_\Gamma (c\eta)^{\delta_\Gamma} e^{(k(\lambda_{-\log c\eta})-\delta_\Gamma)d(\pi(\mathcal{C}_0),\lambda_{-\log c\eta})} \\
&\ll_\Gamma c^\sigma \eta^{\delta_\Gamma} e^{(k(\lambda_{-\log\eta})-\delta_\Gamma)d(\pi(\mathcal{C}_0),\lambda_{-\log\eta})} \\
&\ll_\Gamma c^\sigma \nu_o\left(B_o\left(\lambda, \eta\right)\right).
\end{aligned}
$$

Equivalently, it is enough to show that

$$
\begin{aligned}
&(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log\eta}) \qquad (10.4)\\
&\ll_\Gamma (\sigma - \delta_\Gamma)\log c.
\end{aligned}
$$

**Case 1:** Assume $k(\lambda_{-\log c\eta}) \leq k(\lambda_{-\log\eta})$.

Then

$$
\begin{aligned}
&(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log\eta}) \\
&\leq (k(\lambda_{-\log\eta}) - \delta_\Gamma)(d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - d(\pi(\mathcal{C}_0), \lambda_{-\log\eta})) \\
&\leq (k(\lambda_{-\log\eta}) - \delta_\Gamma)\log c \\
&\leq (n - 1 - \delta_\Gamma)\log c.
\end{aligned}
$$

**Case 2:** $k(\lambda_{-\log c\eta}) > k(\lambda_{-\log\eta})$ and $k(\lambda_{-\log\eta}) = 0$.

Then, $d(\pi(\mathcal{C}_0), \lambda_{-\log\eta}) = 0$ and

$$0 < d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) \leq d(\pi(\mathcal{C}_0), \lambda_{-\log\eta}) + d(\lambda_{-\log\eta}, \lambda_{-\log c\eta}) \leq \log c.$$

Therefore,

$$\begin{aligned}
&(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log\eta}) \\
&\leq (k(\lambda_{-\log c\eta}) - \delta_\Gamma)d((\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) \\
&\leq (k(\lambda_{-\log c\eta}) - \delta_\Gamma)\log c \\
&\leq (n - 1 - \delta_\Gamma)\log c.
\end{aligned}$$

**Case 3:** Assume $k(\lambda_{-\log c\eta}) > k(\lambda_{-\log\eta})$ and $k(\lambda_{-\log\eta}) > 0$. In particular, $\lambda_{-\log\eta}$ and $\lambda_{-\log c\eta}$ are in two different cusps, and hence there exists $1 < r < c$ such that $\lambda_{-\log r\eta} \in \pi(\mathcal{C}_0)$. Then,

$$d(\pi(\mathcal{C}_0), \lambda_{-\log\eta}) \leq d(\lambda_{-\log r\eta}, \lambda_{-\log\eta}) \leq \log r \leq \log c$$

$$d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) \leq d(\lambda_{-\log r\eta}, \lambda_{-\log c\eta}) \leq \log(c/r) \leq \log c$$

Note that since $k(\lambda_{-\log c\eta}) \geq 2$, we have $\delta_\Gamma > 1$, because $\delta_\Gamma > k/2$, where $k$ is the maximal cusp rank. We arrive at

$$\begin{aligned}
(k(\lambda_{-\log\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log\eta}) &\geq (1 - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log\eta}) \\
&\geq (1 - \delta_\Gamma)\log c \\
(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0, \lambda_{-\log c\eta}) &\leq (n - 1 - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) \\
&\leq (n - 1 - \delta_\Gamma)\log c.
\end{aligned}$$

It follows that

$$
\begin{aligned}
&(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log \eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \\
&\leq (n-1-\delta_\Gamma)\log c - (1-\delta_\Gamma)\log c \\
&\leq (n-2)\log c.
\end{aligned}
$$

Thus, choosing

$$\sigma = \max\{n-1-\delta_\Gamma, n-2\} + \delta_\Gamma$$

completes the proof. □

When $c < 1$, we obtain a similar result, with a slightly more involved argument.

**Lemma 10.1.2.** *There exists a constant $\sigma > 0$ depending only on $\Gamma$ such that for any $\lambda \in \Lambda(\Gamma), \eta > 0$ and $0 < c < 1$, we have that*

$$\nu_o(B(\lambda, c\eta)) \ll_\Gamma c^\sigma \nu_o(B(\lambda, \eta)).$$

*Proof.* The proof is extremely similar to that of Lemma 10.1.1.

By the shadow lemma, as in the proof of Lemma 10.1.1, it is enough to show that for some $\sigma > 0$,

$$
\begin{aligned}
&(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log \eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \\
&\ll_\Gamma (\delta_\Gamma - \sigma)|\log c|.
\end{aligned} \tag{10.5}
$$

**Case 1:** Assume $k(\lambda_{-\log c\eta}) \leq k(\lambda_{-\log \eta})$.

Then

$$(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log \eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta})$$

$$\leq (k(\lambda_{-\log c\eta}) - \delta_\Gamma)(d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) - d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}))$$

$$\leq |k(\lambda_{-\log \eta}) - \delta_\Gamma||\log c|$$

Let $k$ be the maximal cusp rank. Since $|k - \delta_\Gamma| < \delta_\Gamma$, we get that

$$\sigma := \delta_\Gamma - |k - \delta_\Gamma| > 0$$

satisfies the claim.

**Case 2:** $k(\lambda_{-\log c\eta}) > k(\lambda_{-\log \eta})$ and $k(\lambda_{-\log \eta}) = 0$.

Then, $d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) = 0$ and

$$0 < d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) \leq d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) + d(\lambda_{-\log \eta}, \lambda_{-\log c\eta}) \leq |\log c|.$$

Therefore,

$$(k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log \eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log \eta})$$

$$\leq (k(\lambda_{-\log c\eta}) - \delta_\Gamma)d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta})$$

$$\leq |k(\lambda_{-\log c\eta}) - \delta_\Gamma||\log c|,$$

and the claim follows as in Case 1.

**Case 3:** Assume $k(\lambda_{-\log c\eta}) > k(\lambda_{-\log \eta})$ and $k(\lambda_{-\log \eta}) > 0$. In particular, $\lambda_{-\log \eta}$ and $\lambda_{-\log c\eta}$ are in two different cusps, and hence there exists $c < r < 1$ such that $\lambda_{-\log r\eta} \in \pi(\mathcal{C}_0)$. Then since $r < 1$,

$$d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \leq d(\lambda_{-\log r\eta}, \lambda_{-\log \eta}) \leq |\log r| \tag{10.6}$$

$$d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) \leq d(\lambda_{-\log r\eta}, \lambda_{-\log c\eta}) \leq \log(r/c) \tag{10.7}$$

Note that since $k(\lambda_{-\log c\eta}) \geq 2$, we have $\delta_\Gamma > 1$. By (10.6) and (10.7), we arrive at

$$(k(\lambda_{-\log \eta}) - \delta_\Gamma) d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \geq (1 - \delta_\Gamma) d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \tag{10.8}$$

$$\geq (\delta_\Gamma - 1)\log c \tag{10.9}$$

$$(k(\lambda_{-\log c\eta}) - \delta_\Gamma) d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) \leq \max\{0, \log(r/c)(k(\lambda_{-\log c\eta}) - \delta_\Gamma)\}$$

We now have two cases. First, assume that $\log(r/c)(k(\lambda_{-\log c\eta}) - \delta_\Gamma) \leq 0$. Then $k(\lambda_{-\log c\eta}) - \delta_\Gamma \leq 0$, so by (10.8), we have that

$$\begin{aligned}
&(k(\lambda_{-\log c\eta}) - \delta_\Gamma) d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log \eta}) - \delta_\Gamma) d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \\
&\leq -(\delta_\Gamma - 1)\log c \\
&= (\delta_\Gamma - 1)|\log c|
\end{aligned}$$

Now, assume that $\log(r/c)(k(\lambda_{-\log c\eta}) - \delta_\Gamma) > 0$, i.e. that $k(\lambda_{-\log c\eta}) - \delta_\Gamma > 0$. Then it follows from (10.7) that

$$\begin{aligned}
&(k(\lambda_{-\log c\eta}) - \delta_\Gamma) d(\pi(\mathcal{C}_0), \lambda_{-\log c\eta}) - (k(\lambda_{-\log \eta}) - \delta_\Gamma) d(\pi(\mathcal{C}_0), \lambda_{-\log \eta}) \\
&\leq (k(\lambda_{-\log c\eta}) - \delta_\Gamma)\log(r/c) - (\delta_\Gamma - 1)\log r.
\end{aligned} \tag{10.10}$$

Now, consider two further cases: $k(\lambda_{-\log c\eta}) - \delta_\Gamma > \delta_\Gamma - 1$ or $k(\lambda_{-\log c\eta}) - \delta_\Gamma \leq \delta_\Gamma - 1$. In the first case, (10.10) is bounded above by

$$(\delta_\Gamma - 1)\log(r/c) - (\delta_\Gamma - 1)\log r = -(\delta_\Gamma - 1)\log c = (\delta_\Gamma - 1)|\log c|.$$

In the second case, note that (10.10) is equal to

$$(k - 2\delta_\Gamma + 1)\log r - (k(\lambda_{-\log c\eta} - \delta_\Gamma)\log c,$$

and our assumption implies that the first term is negative. Thus, an upper bound is

$$-(k(\lambda_{-\log c\eta} - \delta_\Gamma)\log c = (k(\lambda_{-\log c\eta} - \delta_\Gamma)|\log c| \leq (k - \delta_\Gamma)|\log c|,$$

where $k$ is the maximal cusp rank, as before. Note that $k - \delta_\Gamma < \delta_\Gamma$ because $\delta_\Gamma > 2k$ always holds.

Thus, choosing

$$\sigma = \min\{\delta_\Gamma - |k - \delta_\Gamma|, 1\}$$

completes the proof. $\square$

Using Lemma 10.0.3, we obtain the following quantitative Federer-like statement for $\{\mu_x^{\mathrm{PS}}\}_{x^+ \in \Lambda(\Gamma)}$:

**Corollary 10.1.3.** *There exists constants $\sigma_1 = \sigma_1(\Gamma) \geq \delta_\Gamma$, $\sigma_2 = \sigma_2(\Gamma) > 0$ which satisfy the following: let $x \in G$ be such that $x^+ \in \Lambda(\Gamma)$. Then for $c > 0$ and $\eta \ll_\Gamma c^{-1}e^{-\operatorname{height}(x)}$, we have that*

$$\mu_x^{\mathrm{PS}}(B_U(c\eta)) \ll_\Gamma \max\{c^{\sigma_1}, c^{\sigma_2}\}e^{2(\delta_\Gamma + \sigma_1)\operatorname{height}(x)}\mu_x^{\mathrm{PS}}(B_U(\eta)).$$

*Proof.* Fix $g \in G$ which satisfies $x = g\Gamma$ and $\operatorname{height}(x) = d(\pi(\mathcal{C}_0), \pi(g))$. By (3.19), $\operatorname{inj}(x)$ and $\operatorname{height}(x)$ are related, so that for $\eta \ll_\Gamma c^{-1} \operatorname{height}(x)$,

$$\mu_g^{\mathrm{PS}}(B_U(c\eta)) = \mu_x^{\mathrm{PS}}(B_U(c\eta)).$$

For any $0 < \eta \leq 1$ and $u_{\mathbf{t}} \in B_U(\eta)$, we have that

$$\begin{aligned} |\beta_{(u_{\mathbf{t}}g)^+}(o, u_{\mathbf{t}}g(o))| &\leq d(u_{\mathbf{t}}^{-1}(o), g(o)) \\ &\leq d(u_{\mathbf{t}}^{-1}(o), o) + d(o, g(o)) \\ &\leq 2\operatorname{diam}(B_U(1)\pi(\mathcal{C}_0)) + \operatorname{height}(x). \end{aligned}$$

The above gives a bound on the Busemann function for the following when $\eta \leq 1$:

$$\begin{aligned} &e^{-\delta_\Gamma \operatorname{height}(x)} \nu_o(\mathrm{Pr}_{g^-}(B_U(\eta))) \\ &\ll_\Gamma \mu_g^{\mathrm{PS}}(B_U(\eta)) = \int_{\mathbf{t} \in B_U(\eta)} e^{\delta_\Gamma \beta_{(u_{\mathbf{t}}g)^+}(o, u_{\mathbf{t}}g(o))} d\nu_o((u_{\mathbf{t}}g)^+) && (10.11) \\ &\ll_\Gamma e^{\delta_\Gamma \operatorname{height}(x)} \nu_o\left(\mathrm{Pr}_{g^-}(B_U(\eta))\right). && (10.12) \end{aligned}$$

Assume $c \geq 1$. By Lemma 10.0.3 and Lemma 10.1.1, we have that

$$\begin{aligned} \nu_o(B(g^+, \eta)) &= \nu_o(B(g^+, (\tilde{c}e^{\operatorname{height}(x)}\tilde{c}^{-1}e^{-\operatorname{height}(x)}\eta)) \\ &\ll_\Gamma (\tilde{c}e^{\operatorname{height}(x)})^{\sigma_1} \nu_o(B(g^+, \tilde{c}e^{-\operatorname{height}(x)}\eta)). && (10.13) \end{aligned}$$

Let $\tilde{c} > 1$ be as in Corollary 10.0.4. Then as long as

$$\eta \leq \tilde{c}^{-1}c^{-1}e^{-\operatorname{height}(x)},$$

we have the following:

$$\begin{aligned}
\mu_g^{\mathrm{PS}}(B_U(c\eta)) &\ll_\Gamma e^{\delta_\Gamma \operatorname{height}(x)}\nu_o(\mathrm{Pr}_{g^-}(B_U(c\eta))) && \text{by (10.12)}\\
&\ll_\Gamma e^{\delta_\Gamma \operatorname{height}(x)}\nu_o(B(g^+,\tilde{c}e^{\operatorname{height}(x)}c\eta)) && \text{by Corollary 10.0.4}\\
&\ll_\Gamma c^{\sigma}e^{(\delta_\Gamma+\sigma_1)\operatorname{height}(x)}\nu_o(B(g^+,\eta)) && \text{by Lemma 10.1.1}\\
&\ll_\Gamma c^{\sigma_1}e^{(\delta_\Gamma+2\sigma_1)\operatorname{height}(x)}\nu_o(B(g^+,\tilde{c}^{-1}e^{-\operatorname{height}(x)}\eta) && \text{by (10.13)}\\
&\ll_\Gamma c^{\sigma_1}e^{(\delta_\Gamma+2\sigma_1)\operatorname{height}(x)}\nu_o(\mathrm{Pr}_{g^-}(B_U(\eta))) && \text{by Corollary 10.0.4}\\
&\ll_\Gamma c^{\sigma_1}e^{2(\delta_\Gamma+\sigma_1)\operatorname{height}(x)}\mu_g^{\mathrm{PS}}(B_U(\eta)) && \text{by (10.11),}
\end{aligned}$$

which completes the proof in this case.

The case $0 < c < 1$ can be shown in a similar way using Lemma 10.1.2. □

When $x \in \operatorname{supp} m^{\mathrm{BMS}}$, a flowing argument with $\{a_{-s} : s \geq 0\}$ allows us to remove the restriction that $\eta$ must be small in a way that depends on $\operatorname{height}(x)$. More precisely, we obtain:

**Corollary 10.1.4.** *If $\Gamma$ is geometrically finite and Zariski dense, then for any $x \in \operatorname{supp} m^{\mathrm{BMS}}$, the measure $\mu_x^{\mathrm{PS}}$ is doubling, and the constants only depend on $\Gamma$. More precisely, there exist constants $\sigma_1 = \sigma_1(\Gamma) \geq \delta_\Gamma$, $\sigma_2 = \sigma_2(\Gamma) > 0$ such that for every $c > 0$, every $x \in \operatorname{supp} m^{\mathrm{BMS}}$ and every $T > 0$,*

$$\mu_x^{\mathrm{PS}}(B_U(cT)) \ll_\Gamma \max\{c^{\sigma_1}, c^{\sigma_2}\}\mu_x^{\mathrm{PS}}(B_U(T)).$$

*Proof.* On a geometrically finite quotient, there exists a compact set $\Omega_0 \subset X$ such that for every $x \in X$ with $x^- \in \Lambda_r(\Gamma)$, there exists a sequence $s_n \to \infty$ such that $a_{-s_n}x \in \Omega_0$.

Because $\Omega_0$ depends only on $\Gamma$, the height of any point in $\Omega_0$ is bounded by a constant depending only on $\Gamma$. Thus, by Corollary 10.1.3, for all $x \in \Omega_0 \cap \operatorname{supp} m^{\mathrm{BMS}}$ with $x^- \in \Lambda_r(\Gamma)$ and for all $\eta \ll_\Gamma c^{-1}$, we have that

$$\mu_x^{\mathrm{PS}}(B_U(c\eta)) \ll_\Gamma \max\{c^{\sigma_1}, c^{\sigma_2}\}\mu_x^{\mathrm{PS}}(B_U(\eta)). \tag{10.14}$$

Now, fix $x \in \operatorname{supp} m^{\mathrm{BMS}}$ with $x^- \in \Lambda_r(\Gamma)$. Let $T \geq 0$, and let $s > 0$ be sufficiently large so that $e^{-s}T \ll_\Gamma c^{-1}$ and $a_{-s}x \in \Omega_0$. Then

$$\begin{aligned}
\mu_x^{\mathrm{PS}}(B_U(cT)) &= e^{\delta_\Gamma s}\mu_{a_{-s}x}^{\mathrm{PS}}(B_U(ce^{-s}T)) \\
&\ll_\Gamma \max\{c^{\sigma_1}, c^{\sigma_2}\}e^{\delta_\Gamma s}\mu_{a_{-s}x}^{\mathrm{PS}}(B_U(e^{-s}T)) && \text{by (10.14)} \\
&\ll_\Gamma \max\{c^{\sigma_1}, c^{\sigma_2}\}\mu_{a_{-s}x}^{\mathrm{PS}}(B_U(T)),
\end{aligned}$$

so the result holds for $x^- \in \Lambda_r(\Gamma)$.

Since $x \mapsto \mu_x^{\mathrm{PS}}$ is continuous (see Lemma 3.1.1) and the set of $x$ with $x^- \in \Lambda_r(\Gamma)$ is dense in the set of points $y \in X$ which satisfy $y^- \in \Lambda(\Gamma)$, the result then follows for all $x \in \operatorname{supp} m^{\mathrm{BMS}}$. $\square$

## 10.2 Non-planarity of the PS Measure

For a subset $S \subseteq \mathbb{R}^{n-1}$ and $\xi > 0$, let

$$\mathcal{N}_U(S, \xi) = \{u_{\mathbf{t}} \in U \;:\; \exists \mathbf{s} \in S \text{ such that } \|\mathbf{t} - \mathbf{s}\| < \xi\}.$$

In the following, we use the shadow lemma for $\nu_o$ to obtain a stronger version of nonplanarity than that in Definition 10.0.1. From this, we will see that the PS measures when $\Gamma$ is geometrically finite satisfy friendly-like properties. We will obtain stronger estimates for certain hyperplanes that are in the boundary of a ball centered at a BMS point.

**Theorem 10.2.1.** *There exist $\theta = \theta(\Gamma) \geq 1$, $\alpha = \alpha(\Gamma) > 0$ which satisfy the following. For any $w \in \mathbb{H}^n$, $\lambda \in \Lambda(\Gamma)$, $0 < \eta \leq 1$, and $\xi > 0$, we have*

$$\nu_w(\mathcal{N}(L, \xi) \cap B(\lambda, \eta)) \ll_\Gamma e^{2\delta_\Gamma d(o,w)} \frac{\xi^\alpha}{\eta^\theta} \nu_w(B(\lambda, \eta)).$$

*Proof.* First, we show the result for $o$.

According to [DFSU20, Lemma 3.8] there exists $\beta > 0$ such that for any $\eta > 0$, and any affine hyperplane $L \subset \mathbb{R}^n$ we have

$$\nu_o(\mathcal{N}(L, \eta)) \ll_\Gamma \eta^\beta. \tag{10.15}$$

For $\lambda \in \Lambda(\Gamma)$ and for $t \in \mathbb{R}$, let $\lambda_t$ be the unit speed geodesic ray from $o$ to $\lambda$. It follows from the shadow lemma for $\nu_o$ (see [SV95, Theorem 2], also [Sch04, Theorem 3.2]) that for any $\eta > 0$, we have

$$\nu_o\left(B_o\left(\lambda, \eta\right)\right) \gg_\Gamma \eta^{\delta_\Gamma} e^{(k(\lambda_{-\log \eta}) - \delta_\Gamma) d(o, \lambda_{-\log \eta})}$$

where $k(\lambda_{-\log \eta})$ is the rank of the cusp containing $\lambda_{-\log \eta}$ (see §3.2). It follows from the fact that $k(\lambda_{-\log \eta}) \geq 0$ and $d(o, \lambda_{-\log \eta}) \leq -\log \eta$, that

$$\nu_o\left(B_o\left(\lambda, \eta\right)\right) \gg_\Gamma \eta^{2\delta_\Gamma}.$$

Since $\nu_o$ is Federer (by Theorem 10.0.2), using (10.1) we arrive at the same bound for Euclidean balls (with the implied constant changing):

$$\nu_o\left(B\left(\lambda,\eta\right)\right) \gg_\Gamma \eta^{2\delta_\Gamma}. \tag{10.16}$$

Note that by the definition of $\|d_L\|_{\nu_o,B(\lambda,\eta)}$,

$$B(\lambda,\eta)\cap \operatorname{supp}\nu_o \subset \mathcal{N}\left(L, \|d_L\|_{\nu_o,B(\lambda,\eta)}\right).$$

It then follows from (10.15) and (10.16) that

$$\eta^{\delta_\Gamma} \ll_\Gamma \left(\|d_L\|_{\nu_o,B(\lambda,\eta)}\right)^\beta.$$

Hence

$$\|d_L\|_{\nu_o,B(\lambda,\eta)} \gg_\Gamma \eta^{2\delta_\Gamma/\beta}. \tag{10.17}$$

According to Theorem 10.0.2, the PS density is friendly. In particular, it is decaying and nonplanar, so there exists $\alpha > 0$ such that for all $\lambda \in \Lambda(\Gamma)$, $0 < \eta \leq 1$, $\xi > 0$, an affine hyperplane $L \subset \mathbb{R}^n$, and $B = B(\lambda,\eta)$, we have

$$\nu_o(\mathcal{N}(L, \xi\|d_L\|_B)\cap B) \ll_\Gamma \xi^\alpha \nu_o(B). \tag{10.18}$$

The claim now follows for $o$ from (10.17) and (10.18) by taking $\theta = 2\delta_\Gamma/\beta$.

Second, we show the result for a general $w \in \mathbb{H}^n$. Note that

$$e^{-\delta_\Gamma d(o,w)} \ll_\Gamma e^{-\delta_\Gamma \beta_\lambda(w,o)} \ll_\Gamma e^{\delta_\Gamma d(o,w)}.$$

Thus, using this and the fact that $\{\nu_w\}_{w\in\mathbb{H}^n}$ is a conformal density satisfying (3.7), we arrive at

$$\begin{aligned}\nu_w(\mathcal{N}(L,\xi\eta^\theta)\cap B(\lambda,\eta)) &\ll_\Gamma e^{\delta_\Gamma d(o,w)}\nu_o(\mathcal{N}(L,\xi\eta^\theta)\cap B(\lambda,\eta))\\ &\ll_\Gamma e^{\delta_\Gamma d(o,w)}\xi^\alpha\nu_o(B(\lambda,\eta))\\ &\ll_\Gamma e^{2\delta_\Gamma d(o,w)}\xi^\alpha\nu_w(B(\lambda,\eta))\end{aligned}$$

Last, note that by taking $\xi=\eta^{1-\theta}$, we conclude that $\theta\geq 1$. $\square$

**Proposition 10.2.2.** *Let $\Gamma$ be geometrically finite. There exist constants $\alpha=\alpha(\Gamma)>0, \omega=\omega(\Gamma)\geq 0$, and $\theta=\theta(\Gamma)>\alpha$ satisfying the following: for any $x\in G/\Gamma$ with $x^+\in\Lambda(\Gamma)$, and for every $\xi>0$ and $0<\eta\ll_\Gamma e^{-\operatorname{height}(x)}$, we have that for every hyperplane $L$,*

$$\mu_x^{\mathrm{PS}}(\mathcal{N}_U(L,\xi)\cap B_U(\eta))\ll_\Gamma e^{\omega\operatorname{height}(x)}\frac{\xi^\alpha}{\eta^\theta}\mu_x^{\mathrm{PS}}(B_U(\eta)).$$

*Proof.* Let $\alpha=\alpha(\Gamma),\theta=\theta(\Gamma)>0$ satisfy the conclusion of Theorem 10.2.1, and $c'>1$ satisfy the conclusion of Corollary 10.0.4. Fix $g\in G$ which satisfies $x=g\Gamma$ and $\operatorname{height}(x)=d(\pi(\mathcal{C}_0),\pi(g))$.

By the same argument as in the proof of Corollary 10.1.3 to bound the Busemann function when $\eta\leq 1$, we obtain

$$\begin{aligned}&e^{-\delta_\Gamma\operatorname{height}(x)}\nu_o\left(\mathrm{Pr}_{g^-}(\mathcal{N}(L,\xi)x\cap B_U(\eta)x)\right)\\ &\ll_\Gamma \mu_g^{\mathrm{PS}}(\mathcal{N}(L,\xi)\cap B_U(\eta))=\int_{\mathbf{t}\in\mathcal{N}(L,\xi)\cap B_U(\eta)} e^{\delta_\Gamma\beta_{(u_{\mathbf{t}}g)^+}(o,u_{\mathbf{t}}g(o))}d\nu_o((u_{\mathbf{t}}g)^+)\\ &\ll_\Gamma e^{\delta_\Gamma\operatorname{height}(x)}\nu_o\left(\mathrm{Pr}_{g^-}(\mathcal{N}(L,\xi)x\cap B_U(\eta)x)\right)\end{aligned}$$

Thus, for $\eta \ll_\Gamma e^{-\operatorname{height}(x)}$ (so that $ce^{d(o,\pi(x))}\eta \leq 1$ below, and we stay within the injectivity radius at $x$, using (3.19)), we have that

$$
\begin{aligned}
&\mu_x^{\mathrm{PS}}(\mathcal{N}_U(L,\xi) \cap B_U(\eta)) \\
&\ll_\Gamma e^{\delta_\Gamma \operatorname{height}(x)} \nu_o \left(\mathrm{Pr}_{g^-}(\mathcal{N}(L,\xi) \cap B_U(\eta))\right) \\
&\ll_\Gamma e^{\delta_\Gamma \operatorname{height}(x)} \nu_o \left(\mathcal{N}(L', ce^{d(o,\pi(x))}\xi) \cap B\left(g^+, ce^{d(o,\pi(x))}\eta\right)\right) \text{ by Corollary 10.0.5} \\
&\ll_\Gamma e^{\delta_\Gamma \operatorname{height}(x)} \left(\frac{\xi (ce^{d(o,\pi(x))})^{1-\theta}}{\eta^\theta}\right)^\alpha \nu_o\left(B\left(g^+, ce^{d(o,\pi(x))}\eta\right)\right) \text{ by Theorem 10.2.1} \\
&\ll_\Gamma e^{\delta_\Gamma \operatorname{height}(x)} \left(\frac{\xi (e^{d(o,\pi(x))})^{1-\theta}}{\eta^\theta}\right)^\alpha e^{d(o,\pi(x))\sigma} \nu_o(B(g^+,\eta)) \text{ by Lemma 10.1.1} \\
&\ll_\Gamma e^{\delta_\Gamma \operatorname{height}(x)} \left(\frac{\xi (e^{d(o,\pi(x))})^{1-\theta}}{\eta^\theta}\right)^\alpha e^{2d(o,\pi(x))\sigma} \nu_o(B(g^+, c^{-1}e^{-d(o,\pi(x))}\eta)) \text{ by Corollary 10.0.4} \\
&\ll_\Gamma e^{2\delta_\Gamma \operatorname{height}(x) + (\sigma + (1-\theta)\alpha) d(o,\pi(x))} \left(\frac{\xi}{\eta^\theta}\right)^\alpha \mu_x^{\mathrm{PS}}(B_U(\eta)) \\
&\ll_\Gamma e^{(2\delta_\Gamma + \sigma + (1-\theta)\alpha) \operatorname{height}(x)} \left(\frac{\xi}{\eta^\theta}\right)^\alpha \mu_x^{\mathrm{PS}}(B_U(\eta)) \\
&\ll_\Gamma e^{\omega \operatorname{height}(x)} \frac{\xi^\alpha}{\eta^{\theta'}} \mu_x^{\mathrm{PS}}(B_U(\eta)),
\end{aligned}
$$

where

$$\omega = \max\{2\delta_\Gamma + \sigma + (1-\theta)\alpha, 0\}, \quad \theta' = \theta\alpha.$$

□

We now show that much better estimates hold when the hyperplane is on the boundary of the ball. This is because the quantity $\|d_L\|_{\nu_o, B(\lambda,\eta)}$ (for $\lambda \in \Lambda(\Gamma)$, $0 < \eta \leq 1$ and $L$ a hyperplane) can be bounded below by the radius of the ball.

We say that a hyperplane $L$ is *on the boundary* of a closed ball $B$ if

$$\emptyset \neq L \cap B \subseteq \partial(B).$$

Below, we obtain estimates for the PS measure of small neighbourhoods of hyperplanes on the boundary of a ball centered at a BMS point. Though not written here, estimates also hold when the center of the ball is a PS point but not a BMS point, as long as the ball is sufficiently small (bounded by $\eta \ll_\Gamma e^{-\,\mathrm{height}(x)}$).

We caution the reader that the estimates below hold only for hyperplanes on the boundary of such a ball; to obtain such estimates for general hyperplanes, *absolute friendliness* of the PS density is necessary. By [DFSU20, Theorem 1.9], this is satisfied if and only if all cusps of $\mathbb{H}^n/\Gamma$ have maximal rank. This case is discussed in the next section.

**Lemma 10.2.3.** *There exists a constant $\alpha = \alpha(\Gamma) > 0$ satisfying the following: for all $\lambda \in \Lambda(\Gamma)$, $\xi > 0$, $0 < \eta \le 1$, and every hyperplane $L$ that is on the boundary of $B(\lambda, \eta)$, we have that*

$$\nu_o(\mathcal{N}(L,\xi) \cap B(\lambda,\eta)) \ll_\Gamma \left(\frac{\xi}{\eta}\right)^\alpha \nu_o(B(\lambda,\eta)).$$

*Proof.* By [DFSU20, Theorem 1.9], $\nu_o$ is *friendly* when $\Gamma$ is geometrically finite. In particular, this means that there exists $\alpha = \alpha(\Gamma) > 0$ such that for all $\lambda \in \Lambda(\Gamma)$, $\xi > 0$, $0 < \eta \le 1$, and every affine hyperplane $L \subseteq \partial(\mathbb{H}^n)$,

$$\nu_o(\mathcal{N}(L, \xi \left\|d_L\right\|_{\nu_o, B(\lambda,\eta)}) \cap B(\lambda,\eta)) \ll_\Gamma \xi^\alpha \nu_o(B(\lambda,\eta)),$$

where

$$\left\|d_L\right\|_{\nu_o, B(\lambda,\eta)} := \sup\left\{d(\mathbf{y}, L) \;:\; \mathbf{y} \in B(\lambda,\eta) \cap \Lambda(\Gamma)\right\}.$$

Since $\lambda \in \Lambda(\Gamma)$, for any $L$ that is on the boundary of $B(\lambda, \eta)$, we have that

$$\|d_L\|_{\nu_o, B(\lambda,\eta)} \geq \eta/2.$$

Thus, for any $L$ that is on the boundary of $B(\lambda, \eta)$, we have

$$\nu_o(\mathcal{N}(\xi\eta/2) \cap B(\lambda, \eta)) \ll_\Gamma \xi^\alpha \nu_o(B(\lambda, \eta)).$$

Replacing $\xi$ with $2\xi\eta^{-1}$ then implies that for every such $L$,

$$\nu_o(\mathcal{N}(L, \xi) \cap B(\lambda, \eta)) \ll_\Gamma \left(\frac{\xi}{\eta}\right)^\alpha \nu_o(B(\lambda, \eta)),$$

as desired. □

It is of critical importance that the exponents on $\xi$ and $\eta$ match in Lemma 10.2.3. This is the key improvement obtained when the hyperplane is on the boundary. It enables us to flow with $a_{-s}$ for $s > 0$ to obtain estimates for large balls centered at BMS points that do not have a height factor appearing:

**Corollary 10.2.4.** *Let $\alpha = \alpha(\Gamma) > 0$ be as in Lemma 10.2.3. For every $x \in \operatorname{supp} m^{\mathrm{BMS}}$, every $\eta, \xi > 0$, and every hyperplane $L$ in the boundary of $B_U(\eta)x$, we have*

$$\mu_x^{\mathrm{PS}}(\mathcal{N}_U(L, \xi) \cap B_U(\eta)) \ll_\Gamma \left(\frac{\xi}{\eta}\right)^\alpha \mu_x^{\mathrm{PS}}(B_U(\eta)).$$

*Proof.* Since the radial limit points are dense in the all the limit points, using the continuity of the PS-measure, we may assume that $x^- \in \Lambda_r(\Gamma)$.

We will first prove that there exists a constant $c = c(\Gamma) > 0$ so that for all $x \in \operatorname{supp} m^{\mathrm{BMS}} \cap \mathcal{C}_0$ with $x^- \in \Lambda_r(\Gamma)$, $\xi > 0$, $\eta$ satisfying

$$0 < \eta \leq c^{-1},$$

and every hyperplane $L$ in the boundary of $B_U(\eta)x$, the inequality in the statement is satisfied.

By Corollaries 10.0.4 and 10.0.5, there exists a constant $c = c(\Gamma) > 0$ so that for any $0 < \eta \leq c^{-1}$ and $x \in \operatorname{supp} m^{\mathrm{BMS}} \cap \mathcal{C}_0$, writing $x = g\Gamma$, we have

$$
\begin{aligned}
\mu_x^{\mathrm{PS}}(\mathcal{N}_U(L,\xi) \cap B_U(\eta)) &= \mu_g^{\mathrm{PS}}(\mathcal{N}_U(L,\xi) \cap B_U(\eta)) \\
&\ll_\Gamma \nu_o\left(\mathrm{Pr}_{g^-}(\mathcal{N}(L,\xi) \cap B_U(\eta))\right) \\
&\ll_\Gamma \nu_o\left(\mathcal{N}(L',c\xi) \cap B\left(g^+, c\eta\right)\right) && \text{by Corollary 10.0.5} \\
&\ll_\Gamma \left(\frac{\xi}{\eta}\right)^\alpha \nu_o(B(g^+, c\eta)) && \text{by Lemma 10.2.3} \\
&\ll_\Gamma \left(\frac{\xi}{\eta}\right)^\alpha \nu_o(B(g^+, c^{-1}\eta)) && \text{by Lemma 10.1.1/10.1.2} \\
&\ll_\Gamma \left(\frac{\xi}{\eta}\right)^\alpha \mu_x^{\mathrm{PS}}(B_U(\eta)) && \text{by Corollary 10.0.4,}
\end{aligned}
$$

where $L'$ is a hyperplane in the boundary obtained from the projection of $L$.

Now, let $x \in \operatorname{supp} m^{\mathrm{BMS}}$ with $x^- \in \Lambda_r(\Gamma)$ and let $\eta > 0$. Since $a_{-s}x$ has accumulation points in $\mathcal{C}_0$, there exists $s > 0$ so that $e^{-s}\eta < c^{-1}$ and $a_{-s}x \in \mathcal{C}_0$. By the first step of the proof, we then have that

$$
\begin{aligned}
\frac{\mu_x^{\mathrm{PS}}(\mathcal{N}_U(L,\xi) \cap B_U(\eta))}{\mu_x^{\mathrm{PS}}(B_U(\eta))} &= \frac{\mu_{a_{-s}x}^{\mathrm{PS}}(\mathcal{N}_U(L,e^{-s}\xi) \cap B_U(e^{-s}\eta))}{\mu_{a_{-s}x}^{\mathrm{PS}}(B_U(e^{-s}\eta))} \\
&\ll_\Gamma \left(\frac{e^{-s}\xi}{e^{-s}\eta}\right)^\alpha = \left(\frac{\xi}{\eta}\right)^\alpha.
\end{aligned}
$$

$\square$

**Proposition 10.2.5.** *Let $\alpha = \alpha(\Gamma) > 0$ be as in Corollary 10.2.4. Then for all $x \in$*

$\operatorname{supp} m^{\mathrm{BMS}}$, $T > 0$, *and* $0 < \varepsilon \leq 1$, *we have that*

$$\mu_x^{\mathrm{PS}}(B_U((1+2\varepsilon)T)) - \mu_x^{\mathrm{PS}}(B_U(T)) \ll_\Gamma \varepsilon^\alpha \mu_x^{\mathrm{PS}}(B_U(T)).$$

*Proof.* By the geometry of $(B_U((1+2\varepsilon)T) - B_U(T))x$, there exists a constant $m$ depending only on $n$ and hyperplanes $L_1, \ldots, L_m$ in the boundary of $B_U((1+\varepsilon)T)x$ so that

$$(B_U((1+2\varepsilon)T) - B_U(T))x \subseteq \bigcup_{i=1}^{m} \mathcal{N}_U(L_i, \varepsilon T) \cap B_U((1+2\varepsilon)T)x.$$

Then by Corollary 10.2.4, we have that

$$\begin{aligned}
&\mu_x^{\mathrm{PS}}(B_U((1+2\varepsilon)T)) - \mu_x^{\mathrm{PS}}(B_U(T)) \\
&\leq \sum_{i=1}^{m} \mu_x^{\mathrm{PS}}\left(\mathcal{N}_U(L_i, \varepsilon T) \cap B_U((1+2\varepsilon)T)\right) \\
&\ll_\Gamma \left(\frac{\varepsilon T}{(1+2\varepsilon)T}\right)^\alpha \mu_x^{\mathrm{PS}}(B_U((1+2\varepsilon)T)) \\
&\ll_\Gamma \varepsilon^\alpha \mu_x^{\mathrm{PS}}(B_U(T)) && \text{by Corollary 10.1.4}
\end{aligned}$$

□

We can obtain estimates for all $(\varepsilon, s_0)$-Diophantine points for balls that are sufficiently large (in a way that is uniform and linear in $s_0$). In fact, for any compact set $\Omega \subseteq G/\Gamma$, there exists a $T_0 = T_0(\Omega)$ satisfying the statement below for all $x \in \Omega$ with $x^- \in \Lambda(\Gamma)$, see e.g. [MO16, Lemma 3.3]. Thus, the statement below could take many forms and this is not as strong as possible; we simply write it in a way that is useful for our setting.

**Corollary 10.2.6.** *Let $\alpha = \alpha(\Gamma) > 0$ be as in Proposition 10.2.5, let $0 < \varepsilon \leq 1$ and let $s_0 \geq 1$. There exists $T_0 = T_0(\Gamma, s_0) > 0$ so that for every $(\varepsilon, s_0)$-Diophantine point $x \in G/\Gamma$, all $T > 2T_0 + 1$, and all $\xi > 0$,*

$$\mu_x^{\mathrm{PS}}(B_U((1+2\xi)T)) - \mu_x^{\mathrm{PS}}(B_U(T)) \ll_\Gamma \left(\xi + \frac{T_0}{T - T_0}\right)^\alpha \mu_x^{\mathrm{PS}}(B_U(T)). \tag{10.19}$$

*In particular, if $x^- \in \Lambda_r(\Gamma)$, there exists $T_0 = T_0(x) > 0$ so that for all $T \geq 2T_0 + 1$ and all $\xi > 0$,* (10.19) *holds.*

*Proof.* By Lemma 3.2.5, there exists $T_0 = T_0(\Gamma, s_0) > 0$ (in fact, it is linear in $s_0$) so that for every $(\varepsilon, s_0)$-Diophantine point $x$, there exists

$$y \in B_U(T_0)x \cap \operatorname{supp} m^{\mathrm{BMS}}.$$

For $T \geq T_0$, we have

$$B_U(T - T_0)y \subseteq B_U(T)x \subseteq B_U(T + T_0)y.$$

In particular,

$$B_U((1+2\xi)T)x \subseteq B_U((1+2\xi)(T+T_0))y$$

and

$$B_U(T - T_0)y \subseteq B_U(T)x. \tag{10.20}$$

Now assume that $T \geq 2T_0 + 1$ so that we may use Proposition 10.2.5 below:

$$\begin{aligned}
&\mu_x^{\mathrm{PS}}(B_U(1+2\xi)T) - \mu_x^{\mathrm{PS}}(B_U(T)) \\
&\leq \mu_y^{\mathrm{PS}}(B_U(1+2\xi)(T+T_0)) - \mu_y^{\mathrm{PS}}(B_U(T-T_0)) \\
&\leq \mu_y^{\mathrm{PS}}\left(B_U\left((1+2\xi)\left(1+\frac{2T_0}{T-T_0}\right)(T-T_0)\right)\right) - \mu_y^{\mathrm{PS}}(B_U(T-T_0)) \\
&\ll_\Gamma \left(\xi + \frac{T_0}{T-T_0} + \frac{\xi}{T-T_0}\right)^\alpha \mu_y^{\mathrm{PS}}(B_U(T-T_0)) \text{ by Proposition 10.2.5} \\
&\ll_\Gamma \left(\xi + \frac{T_0}{T-T_0} + \frac{\xi}{T-T_0}\right)^\alpha \mu_x^{\mathrm{PS}}(B_U(T)) \text{ by (10.20)}
\end{aligned}$$

Since $T \geq 2T_0 + 1$,

$$\frac{\xi}{T-T_0} \leq \xi,$$

and it can be absorbed into the $\xi$ term, completing the proof. $\square$

Note also that a similar argument can imply a bound for $\mu_x^{\mathrm{PS}}(B_U(T+\xi)) - \mu_x^{\mathrm{PS}}(B_U(T))$, however the presence of the $T_0$ term (which may be bounded by $\ll_\Gamma s_0$) means that this actually yields a (somewhat superficially) worse result in the proofs of the effective equidistribution theorems, requiring $T$ to be larger for the theorems to hold. For this reason, we do not use this result there, and instead use Theorem 5.0.1.

## 10.3 Absolute Friendliness of the PS Measure

When all cusps are of maximal rank, the PS measure is *absolutely friendly*, and stronger results hold. Note that if $\Gamma$ is convex cocompact, then there are no cusps, so this additional assumption is vacuously true.

**Definition 10.3.1.** Let $\mu$ be a measure defined on $\mathbb{R}^m$.

1. $\mu$ is called **absolutely decaying** (respectively, **globally absolutely decaying**) if there exist $\alpha, c_2 > 0$ such that for all $v \in \operatorname{supp} \mu$, all $0 < \xi < \eta \leq 1$ (respectively, $0 < \xi < \eta$), and every affine hyperplane $L \subseteq \mathbb{R}^n$,
$$\mu(\mathcal{N}(L, \xi) \cap B(v, \eta)) \leq c_2 \left(\frac{\xi}{\eta}\right)^{\alpha} \mu(B(v, \eta)).$$
2. $\mu$ is called **absolutely friendly** (respectively, **globally friendly**) if it is Federer (respectively, doubling) and absolutely decaying (respectively, globally absolutely decaying).

◀

It is easy to see that if a measure $\mu$ is globally friendly, then it is also absolutely friendly.

According to [SU40, Theorem 2] if $\Gamma$ is convex cocompact or [DFSU20, Theorem 1.12] if $\Gamma$ is geometrically finite, $\nu_o$ is absolutely friendly if and only if all cusps have maximal rank.

**Theorem 10.3.2.** *Assume that $\Gamma$ is Zariski dense and either convex cocompact or geometrically finite with all cusps having maximal rank. Then the PS-measures $\left\{\mu_x^{\mathrm{PS}}\right\}_{x^- \in \Lambda(\Gamma)}$ are globally friendly, and the constants in Definition 10.3.1 only depend on $\Gamma$ (in particular, they do not depend on $x$).*

This follows by a flowing argument, similar to the results for $x \in \operatorname{supp} m^{\mathrm{BMS}}$ proven before. The key difference is observed by contrasting Definition 10.3.1(1) with Theorem

10.2.1: when the powers of $\xi, \eta$ match, a flowing argument may be used for BMS points. When they do not match, one introduces a power corresponding to how far one flows with $a_{-s}$.

**Corollary 10.3.3.** *Assume that* $\Gamma$ *is Zariski dense and either convex cocompact or geometrically finite with all cusps having maximal rank. There exists* $0 < \alpha = \alpha(\Gamma) < 1$ *such that for any* $x \in \operatorname{supp} m^{\mathrm{BMS}}$*,* $T > 0$*, and* $0 < \xi \leq T$*, we have*

$$\frac{\mu_x^{\mathrm{PS}}(B_U(T+\xi))}{\mu_x^{\mathrm{PS}}(B_U(T))} - 1 \ll_\Gamma \left(\frac{\xi}{T}\right)^\alpha.$$

*Proof.* Let $c_1 = c_1(\Gamma), c_2 = c_2(\Gamma) > 0$ and $\alpha = \alpha(\Gamma) > 0$ satisfy the conclusion of Definition 10.3.1 for $\mu_x^{\mathrm{PS}}$ and $k = 2$.

It follows from the geometry of $B_U(\xi+\eta)x - B_U(\eta)x$ that there exist $L_1, \ldots, L_m$, where $m$ only depends on $n$, such that

$$B_U(\xi+T)x - B_U(T)x \subseteq \bigcup_{i=1}^m \mathcal{N}_U(L_i, 2\xi).$$

Then, by Definition 10.3.1, we have

$$\begin{aligned}\frac{\mu_x^{\mathrm{PS}}(B_U(\xi+T))}{\mu_x^{\mathrm{PS}}(B_U(T))} - 1 &= \frac{\mu_x^{\mathrm{PS}}(B_U(\xi+T) - B_U(T))}{\mu_x^{\mathrm{PS}}(B_U(T))} \\ &\leq mc_2 \left(\frac{\xi}{T}\right)^\alpha \frac{\mu_x^{\mathrm{PS}}\left(B_U\left(\xi+T\right)\right)}{\mu_x^{\mathrm{PS}}(B_U(T))} \\ &\leq mc_1c_2 \left(\frac{\xi}{T}\right)^\alpha.\end{aligned}$$

□

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