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Some quantitative regularity theorems for the Navier-Stokes equations

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ABSTRACT OF THE DISSERTATION

Some quantitative regularity theorems for the Navier-Stokes equations

by

Stanley Palasek
Doctor of Philosophy in Mathematics
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Professor Terence Chi-Shen Tao, Chair

Consider a velocity field u solving the incompressible Navier-Stokes equations on $[0,T]\times\mathbb{R}^d$ ($d\geq 3$) and satisfying $\|u(t)\|_X\leq A$ for all times, where the norm X is critical with respect to the Navier-Stokes scaling. We prove several theorems to the effect that the regularity of the solution can be controlled explicitly in terms of A, building upon Tao's pioneering work on the case $d=3$, $X=L^3(\mathbb{R}^3)$. First we prove a generalization to the critical Lebesgue space in any number of spatial dimensions ($d\geq 4$, $X=L^d(\mathbb{R}^d)$). Then we show a variety of circumstances under which Tao's bounds can be strengthened, including the case in which the solution is nearly axisymmetric. For exactly axisymmetric solutions, we prove regularity in terms of the weak norm $X=L^{3,\infty}(\mathbb{R}^3)$ which implies effective bounds on approximately self-similar behavior.

The dissertation of Stanley Palasek is approved.

Monica Vișan

Christina Kim

Rowan Brett Killip

Terence Chi-Shen Tao, Committee Chair

University of California, Los Angeles

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VITA

2017	A.B. (Mathematics), Princeton University
2019	M.A. (Mathematics), UCLA
2022	UCLA Dissertation Year Fellowship

SELECTED PUBLICATIONS

S. Palasek, Improved quantitative regularity for the Navier-Stokes equations in a scale of critical spaces, 2021, *Archive for Rational Mechanics and Analysis*, 242, 1479-1531.

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CHAPTER 1

Introduction

In this dissertation we consider the incompressible Navier-Stokes equations in $\mathbb{R}^d$, $d \geq 3$,

$$\begin{aligned} \partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{aligned} \tag{1.1}$$

which models the motion of an incompressible viscous fluid with velocity field $u : [0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ and pressure field $p : [0,T) \times \mathbb{R}^d \to \mathbb{R}$. The viscosity ν is a positive dimensional parameter. We perform some standard reductions to transform (1.1) into a more analytically convenient form. First one can normalize the viscosity to $\nu = 1$ via the rescaling

$$u(t,x) \mapsto \nu u(\nu t, x), \quad p(t,x) \mapsto \nu^2 p(\nu t, x).$$

Then, upon projecting to the space of divergence-free vector fields using the Leray projection $\mathbb{P} = 1 - \Delta^{-1} \nabla \operatorname{div}$ which eliminates the pressure term, we arrive at

$$\partial_t u - \Delta u + \mathbb{P} \operatorname{div} u \otimes u = 0. \tag{1.2}$$

Note that $\operatorname{div} u \otimes u = u \cdot \nabla u$ by incompressibility; moreover one no longer needs to separately impose $\operatorname{div} u = 0$ as this condition is preserved by (1.2) (under very mild assumptions).

At times it will still be useful to refer to the pressure since it essentially captures the nonlocal part of the nonlinearity. (1.1) implies that it satisfies the Poisson equation

$$-\Delta p = \operatorname{div} \operatorname{div}(u \otimes u).$$

Assuming, as we often will, that $u(t) \in L^d_x(\mathbb{R}^d)$, this specifies[1] a unique $p \in L^{d/2}_x(\mathbb{R}^d)$ expressible with the formula

$$p = -\Delta^{-1} \operatorname{div} \operatorname{div}(u \otimes u) = \mathcal{F}^{-1}\left(\frac{\xi_i \xi_j}{|\xi|^2} \widehat{u_i u_j}(\xi)\right) \tag{1.3}$$

in which we employ the Einstein summation convention.

To motivate the delicate critical problems that will be our main focus, let us briefly review some well-known classical results in the regularity theory for (1.2). Leray in his seminal paper [35] proved local existence and uniqueness of "strong" solutions of (1.2) with finite energy data u_0 in the spaces $L^p(\mathbb{R}^3)$, $p > 3$. He obtains as well quantitative lower bounds on the norms' divergence: if u is a strong solution blowing up at time $T_* > 0$, meaning the solution cannot be continued smoothly beyond time T_*, then

$$\|u(t)\|_{L^p_x(\mathbb{R}^3)} \geq \frac{c(p)}{(T_* - t)^{\frac{1}{2} - \frac{3}{2p}}} \quad \text{for } t \in [0, T_*) \tag{1.4}$$

for all $p \in (3, \infty]$, where $c(p)$ is a positive constant. In fact, an analogous estimate holds for solutions on $\mathbb{R}^d$, $d \geq 3$ for all $p \in (d, \infty]$.

Let us justify why one would expect this scale of spaces to be capable of detecting blowup as in (1.4). It is straightforward to see that the space of solutions of (1.2) is invariant under the group of transformations

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda u(\lambda^2 t, \lambda x), \quad \lambda > 0, \tag{1.5}$$

which scale the spatial L^p norms according to

$$\|u_\lambda\|_{L^p_x(\mathbb{R}^d)} = \lambda^{1-\frac{d}{p}} \|u\|_{L^p_x(\mathbb{R}^d)}.$$

Taking $\lambda \gg 1$ leads to a transformation which "zooms in" on the small scale behavior of the solution; thus the $L^p_x(\mathbb{R}^d)$ norm is effective at controlling possible concentration at small

[1] Clearly the "normalized pressure" given by (1.3) differs from the original in (1.1) by a harmonic function at every time. However, in the case that u is smooth and finite energy, this difference is just a constant function (in space) for almost all time, as shown in [59, Lemma 4.1]. Thus for such solutions, there is no loss of generality in assuming (1.3).

scales as long as $1-\frac{d}{p}>0$, i.e., when $p>d$. In general, such function spaces X with the homogeneity $\|u_\lambda\|_X=\lambda^a\|u\|_X$, $a>0$, are referred to as "subcritical" with respect to the natural scaling of the PDE.

On the other hand, when $a<0$, the space X is referred to as "supercritical" and has little chance of controlling small scale behavior of the solution. The most relevant example is $L_t^\infty L_x^2([0,T]\times\mathbb{R}^d)$ when $d\geq 3$ which is related to the energy equality

$$\|u(t)\|_{L_x^2(\mathbb{R}^d)}^2+\int_0^t\|\nabla u(t')\|_{L_x^2(\mathbb{R}^d)}^2dt'=\|u(0)\|_{L_x^2(\mathbb{R}^d)}^2 \tag{1.6}$$

for strong solutions of (1.2). (1.6) is the only known coercive conservation law for Navier-Stokes equations; hence all known a priori estimates correspond to spaces at most as strong as $\mathcal{E}:=L_t^\infty L_x^2\cap L_t^2\dot{H}_x^1$. Since the corresponding norm is supercritical (indeed, $\|u_\lambda\|_{\mathcal{E}}=\lambda^{\frac{d}{2}-1}\|u\|_{\mathcal{E}}$), (1.6) appears to be useless by itself for ruling out singular behavior.

The third case, $a=0$, is the subject of the dissertation at hand. This corresponds to the norm respecting the symmetry exactly and thus measuring the solution equally at all scales. For instance, the energy space $\mathcal{E}$ introduced above is critical when $d=2$, and Leray in his classical work had already proved that strong solutions in $\mathbb{R}^2$ exist globally in time. More generally, in $\mathbb{R}^d$, $d\geq 2$, there are a variety of commonly studied homogeneous critical spaces which follow the inclusions

$$\dot{H}^{\frac{d}{2}-1}\subset L^d\subset L^{d,q}\subset\dot{B}_{p,q}^{-1+\frac{d}{p}}\subset BMO^{-1}\subset B_{\infty,\infty}^{-1}$$

where $p,q\in(d,\infty)$. (Of course this is not an exhaustive list, and there are inclusions as well within the $L^{d,q}$ and $\dot{B}_{p,q}^{-1+\frac{d}{p}}$ scales.) Various delicate issues arise in the setting of the weakest spaces in the chain, and in some cases even a satisfying local theory is lacking. For the purpose of this dissertation we restrict our attention primarily to the Lebesgue and Lorentz scales; the endpoint case $L^{d,\infty}=L^{d,w}$ is also of particular interest due to its connection blowup phenomena and will be the subject of Chapter 5.

Let us also mention the critical *spacetime* norms which make up the Prodi-Serrin-Ladyzhenskaya scale,

$$X^{p,q} := L_t^p L_x^q, \quad \text{where} \quad \frac{2}{p} + \frac{d}{q} = 1, \quad d < q \leq \infty.$$

The relationship between p and q enforces criticality. We remark that (1.4) already implies that if u blows up at $t = T_*$, then

$$\|u\|_{X^{p,q}([T_*-\epsilon,T_*]\times\mathbb{R}^d)} = \infty$$

with the $(p,q) = (\infty, d)$ case once again excluded. The stronger result coming from the works of Prodi [49], Serrin [56], and Ladyzhenskaya [31] is that $X^{p,q}$ is enough regularity to ensure weak-strong uniqueness; in other words, if u_1 and u_2 are Leray-Hopf[2] weak solutions of (1.2) with the same initial data and $u_1 \in X^{p,q}$, $q > 3$, then $u_1 = u_2$ and the solutions are smooth.

The endpoint space $L_x^d(\mathbb{R}^d)$ is a glaring gap in the above classical picture. It should be noted that local existence is known for data in L_x^d; however there is no lower bound on the time of existence in terms of $\|u_0\|_{L^d}$ alone. Indeed, dimensional analysis considerations appear to preclude such a lower bound. It is not clear at all that a bound on $\|u\|_{L_t^\infty L_x^d(\mathbb{R}^d)}$ should control the regularity of the solution since the norm does not penalize concentrations at small scales that agree with the PDE's scaling.

One should also compare $L_t^\infty L_x^d = X^{\infty,d}$ to the Prodi-Serrin-Ladyzhenskaya spaces $X^{p,q}$ for $q > d$ in the following manner: if $z_0 \in [0,\infty) \times \mathbb{R}^d$ is a putative blowup point, consider a parabolic cylinder Q with radius r around z_0. If u is bounded in $X^{p,q}$ with $q > 3$, $p < \infty$, then $X^{p,q}(Q)$ can be made arbitrarily small by taking r small and using dominated convergence, from which one can infer regularity in the interior of Q by standard arguments. On the

[2]The Leray-Hopf class is a commonly studied class of weak solutions obeying the additional property that (1.6) holds as an *inequality*. Leray [35] proved the global existence of such weak solutions in $\mathbb{R}^3$ for all reasonable data, yet uniqueness in general remains a major unsolved problem (although there is exciting recent progress [2]). The question of uniqueness of weak solutions to fluid equations is a completely different side to the regularity question and has been intensively studied recently, including by the author and collaborators [10, 11], but will not be in the scope of this dissertation.

other hand, in $X^{\infty,3}$, this argument breaks down due to the L^∞ norm. These considerations suggest that in order to control the regularity of the solution in terms of an endpoint critical norm, or to prove an analogue of (1.4) for L^d_x, nontrivial ideas are needed.

The breakthrough on this problem came in the paper of Escauriaza, Seregin, and Šverák [22] which, as one consequence, implies that for a classical solution u of (1.2) blowing up at time T_*,

$$\limsup_{t\uparrow T_*} \|u(t)\|_{L^3_x(\mathbb{R}^3)} = \infty.$$

The lim sup was upgraded to a pointwise limit as $t \uparrow T_*$ by Seregin [53] (which is still open in the case $d \geq 4$). The most important new tool in [22] was a backward uniqueness theorem for the heat equation on the exterior of a ball, proved in their paper [21]. The idea is as follows: one assumes by contradiction that u blows up at $(t,x) = (0,0)$ while $\|u(t)\|_{L^3_x}$ stays bounded. The scaling symmetry defined in (1.5) allows one to construct a sequence of rescaled solutions u_k which obey the same L^3 bound while zooming in in the singularity. One can justify passing to a subsequence which converges weakly to another solution u_∞; furthermore u_∞ can be shown to be non-trivial near the original singularity, while vanishing far away from 0 at time $t = 0$. The backward uniqueness theorem for the heat equation, applied to the vorticity $\omega = \operatorname{curl} u$, implies that ω vanishes away from $x = 0$ for *all* $t \leq 0$. From there a straightforward unique continuation argument proves that $\omega \equiv 0$ everywhere, which contradicts u_∞ being nontrivial at $(0,0)$.

This argument, while very powerful, suffers from being fundamentally non-quantitative due to the use of compactness. While it does abstractly imply[3] the existence of *some* $F : [0,\infty) \to [0,\infty)$ such that

$$\|u\|_{L^\infty_{t,x}([1/2,1]\times\mathbb{R}^3)} \leq F(\|u\|_{L^\infty_t L^3_x([0,1]\times\mathbb{R}^3)})$$

for all classical solutions u with $\|u\|_{L^\infty_t L^3_x([0,1]\times\mathbb{R}^3)} < \infty$, the growth rate of F is not at all effective.

[3]This observation appears to be due to Hongjie Dong.

Here we briefly survey the subsequent non-quantitative work that made use of the techniques in [22]. Dong and Du have extended these results to dimensions $d \geq 4$ in [19], which will be discussed later in detail. The space L^3_x has been weakened to the non-endpoint Lorentz space $L^{3,q}_x$, $3 < q < \infty$ by Phuc [48] and further to the negative order non-endpoint Besov spaces $B^{-1+3/p}_{p,q}$, $3 < p, q < \infty$ by Gallagher, Koch, and Planchon [25]. It should be noted that this latter work takes a slightly different approach, following the concentration compactness method of Kenig and Merle [27] which had been introduced into the Navier-Stokes setting by Kenig and Koch [26]. The strongest qualitative result for solutions in $\mathbb{R}^3$ to date is due to Albritton and extends Seregin's $\lim_{t\to T_*}$ theorem [53] to the negative order Besov spaces from [25]. Ultimately all of these theorems are non-quantitative and rely on a strategy similar to that of [22].

In 2019, Tao [60] gave the first effective bounds for solutions of (1.2) in $\mathbb{R}^3$ bounded in a critical space. First, he showed that if u is a classical solution with

$$\|u\|_{L^\infty_t L^3_x([0,T]\times\mathbb{R}^3)} \leq A$$

with A large, then one can explicitly control the subcritical norms of u as

$$\|\nabla^j u(t)\|_{L^\infty_x} \leq t^{-\frac{1}{2}(1+j)} \exp\exp\exp(A^{O_j(1)}) \tag{1.7}$$

for all $j \geq 0$ and $t \in [0, T]$. This is a quantitative version of Escauriaza, Seregin, and Šverák's theorem. Taking $j = 0$ and combining this with (1.4), it fills in the missing endpoint of (1.4) (at least along a sequence of times): there exist $t_n \uparrow T_*$ such that

$$\|u(t_n)\|_{L^3_x(\mathbb{R}^3)} \geq \left(\log\log\log\frac{T_*}{T_* - t_n}\right)^c \tag{1.8}$$

where $c > 0$ is an absolute constant.

Tao's idea, which is the starting point of this dissertation, is similar in spirit to Escauriaza, Seregin, and Šverák's argument in the contrapositive (to avoid proof-by-contradiction), replacing qualitative tools including unique continuation theorems with their quantitative analogues. The high-level strategy is as follows: one shows that if the solution concentrates in a high frequency bubble, then the bubble must have been caused by a series of past bubbles at various larger scales. If the solution has sufficient partial regularity, one can use Carleman estimates to show that these previous bubbles lead to slight concentrations of vorticity scattered throughout space, each of which contributes to the L_x^3 norm. The higher the frequency of the initial bubble, the more scales must have had such contributions, which produces the desired conclusion that singular behavior requires a large L_x^3 norm.

The results in [60] leave many questions open that have attracted significant attention since; see [6] for a more detailed survey. First, once again for solutions bounded in $L^3(\mathbb{R}^3)$, Barker and Prange [5] quantify the improvement in [53] toward a pointwise blowup rate of the critical norm. They also prove that for Type I blowups, the norm diverges at least as fast as a *single* logarithm (at the expense of double exponential dependence on the weaker norm that stays bounded). By using tools related to spatial rather than Fourier concentration, they are able to obtain results that are well localized in space. Barker [3] later proved a fully localized blowup theorem using a truncation method.

Another interesting development [4] is that the lower bound (1.8) can be extended to a *slightly supercritical* Orlicz norm on the order of $L^3/\log\log\log\log L$, following an observation of Bulut [9].

To state the results of this dissertation, let us consider the general situation in which the solution is assumed to be bounded uniformly in time as

$$\|u\|_{L^\infty([0,T];X)} \leq A \tag{1.9}$$

with $A \geq 2$, say. The plan is to address three natural questions about solutions obeying this bound:

1. Can analogues of (1.7) and (1.8) be proved in dimensions four and greater? The argument in [60] breaks down in several substantial ways. The fundamental problem is that the energy space is much weaker in high dimensions; thus the partial regularity

which plays an essential role is hard to come by. In Chapter 3 we are able to introduce new quantitative approaches to partial regularity and unique continuation to answer this question in the affirmative.

2. Can the triple exponential and logarithmic bounds in (1.7) and (1.8) be improved? Tao conjectures in [60] that this might be possible by avoiding use of “annuli of regularity” which are problematic due to being distributed sparsely over scales. Unfortunately, while we cannot answer this question in generality, we can answer in the affirmative if $X = L^3(\mathbb{R}^3)$ and u is axisymmetric. In Chapter 4 we prove the following: if either (i) $3 < q < \infty$ or (ii) u is axisymmetric and $2 < q \leq 3$, then with $X = L^q(r^{1-\frac{3}{q}}dx)$, where $r^2 = x_1^2 + x_2^2$, the results (1.7) and (1.8) can be improved by one exponential and logarithm (respectively).

3. Can $X = L^3(\mathbb{R}^3)$ be replaced by the *weak* Lebesgue norm? This question is very relevant for understanding blowup, and is indeed a major open question in the field (even the corresponding qualitative statement). Here we prove a quantitative theorem in the axisymmetric case, which appears inaccessible by the method of [60] (see §5.1), using instead Harnack-type inequalities and favorable parabolic equations obeyed by components of axisymmetric solutions. This will be the subject of Chapter 5 and is joint work with Wojciech Ożański.

CHAPTER 2

Quantitative tools for analyzing critically bounded solutions

2.1 Introduction

This chapter contains results that have appeared in the author's work [46, 47]. The purpose is to prove various estimates and other quantitative properties of solutions of (1.2) that obey a bound of the form (1.9). These will be our main tools toward the regularity theorems in Chapters 3, 4, and 5. There are two main goals that we pursue in the following sections:

(i) We wish to carry out energy estimates on the velocity fields and related quantities, even when the critical space X is poorly suited for this purpose. For instance, if the spatial dimension d is large and we are given only a bound on $\|u\|_{L^d_x}$, then there is not nearly enough spatial decay to control $\int_{\mathbb{R}^d} |u|^2 dx$ at any time; nonetheless, we would still like to make use of energy estimates (for instance, for proving partial regularity). The solution is to employ a Picard-esque decomposition of u into a "flat" part $u^\flat$ which represents a finite number of interactions between different heat flows; and the residual "sharp" part $u^\sharp$. The former has essentially unlimited regularity in high integrability spaces, while the latter has acceptable bounds in spaces with low integrability. Moreover the latter solves a Navier-Stokes-like equation that is conducive for energy estimates. We face similar issues in the case where X is a weighted norm, as in Chapter 4. This is the subject of §2.3.

(ii) We would like to locate regions in spacetime where the solution is regular, such that both the size of the region and regularity of u inside are explicitly controlled in terms of A. Such "partial regularity" results are used in several places in the main arguments but most crucially to produce a region in which Carleman inequalities for the heat equation (Appendix A.1) can be applied (in Chapters 3 and 4). The partial regularity theorems presented in the sequel fall into three categories:

Epochs of regularity When $d = 3$, every time interval contains a subinterval whose length we can bound from below in which the solution is well-controlled. This was proved quantitatively in the $X = L^3_x(\mathbb{R}^3)$ case by Tao [60]. This property is a consequence of energy conservation; in fact, a version for finite-energy solutions was discovered originally by Leray [35].

CKN-type spacetime regularity Caffarelli-Kohn-Nirenberg famously proved [12] that the $d-2$-dimensional (parabolic) Hausdorff measure of the set of spacetime singularities is zero (the epochs of regularity property being a special case). We prove quantitative realizations of this theorem based on the blowup procedure in §2.4. This implies the existence of many thin regions of regularity for even the high-dimensional Navier-Stokes equations which will be essential for our purposes.

"Away-from-the-axis" regularity A simple qualitative consequence of the CKN theorem is that if the axisymmetric Navier-Stokes equations blow up, it may happen only on the axis. In §2.5.3 we produce quantitative versions of this fact in a somewhat more general setting, as well as in some weighted spaces (without symmetry assumptions). The main tools are axisymmetric and weighted Bernstein inequalities (§2.2.2).

We remark that several similar tools have appeared in Tao's work on the $X = L^3_x$ case, although they have been substantially extended for our purposes in this dissertation. For example, the decomposition described in (i) has a predecessor in the decomposition $u^{\text{lin}} + u^{\text{nlin}}$

appearing in [60].[1] Tao also makes use of a three-dimensional "annuli of regularity" proposition which is essentially a manifestation of Caffarelli-Kohn-Nirenberg partial regularity.

2.2 Preliminaries

2.2.1 Notation

The critical spaces X that are of interest often involve some parameter, for instance, the dimension $d \geq 3$ (Chapter 3) or the exponent $q \in (2, \infty)$ (Chapter 4). Since we are not particularly concerned with how the constants depend on these parameters, we use asymptotic notation $x \lesssim y$ or $x = O(y)$ to mean that there is a constant $C = C(X)$ depending on the choice of X such that $|x| \leq Cy$. Indeed, many constants should be expected to deteriorate as $q \downarrow 2$, $q \downarrow 3$, $q \uparrow \infty$, or $d \uparrow \infty$. As in [60], we fix a large constant C_0 that may depend as well on these parameters. With A as in (1.9), we define the hierarchy of large constants $A_j = A^{C_0^j}$.

Let us emphasize that no constants may depend on A or u.

We will occasionally write $x \leq y-$ or $x+ \leq y$ to mean $x < y$. This will make it possible to abbreviate a collection of strict and non-strict inequalities. For example, $x \leq a$, $x \leq b$, $x < c$ can be written as $x \leq \min(a, b, c-)$.

If $I \subset \mathbb{R}$ is a time interval, we use $|I|$ to denote its length. If $\Omega \subset \mathbb{R}^3$, $|\Omega|$ will denote its three-dimensional Lebesgue measure. If $x_0 \in \mathbb{R}^3$ and $R > 0$, we will write $B(x_0, R)$ to denote the closed ball $\{x \in \mathbb{R}^3 : |x - x_0| \leq R\}$. If $x \in \mathbb{R}^3$, then r will denote the radial distance in cylindrical coordinates, that is $r := \sqrt{x_1^2 + x_2^2}$. For a specific point, say $p \in \mathbb{R}^3$, we will write its radial coordinate as $r(p) := \sqrt{p_1^2 + p_2^2}$. For $0 < r_1 < r_2$, we define the cylindrical shell $\mathcal{S}(r_1, r_2) := \{x \in \mathbb{R}^3 : r_1 \leq r \leq r_2\}$ along with the truncated versions $\mathcal{S}(r_1, r_2; M) :=$

[1] In fact, similar decompositions have appeared in other contexts [14, 25, 1]. We thank an anonymous referee for ARMA for bringing these references to our attention.

$\{x \in \mathcal{S}(r_1, r_2) : |x_3| \le M\}$ and $\mathcal{S}(r_1, r_2; M_1, M_2) := \{x \in \mathcal{S}(r_1, r_2) : M_1 \le |x_3| \le M_2\}$.

We say a scalar-valued function is axisymmetric if its derivative in the spatial direction $(-x_2, x_1)^t$ vanishes identically. We say a vector-valued function is axisymmetric if each component is axisymmetric when the function is written in cylindrical coordinates around the x_3-axis.

When studying the nonlinearity of (1.2), we will use the symmetrized tensor product

$$u \odot v := \frac{1}{2}(u \otimes v + v \otimes u)$$

for $u, v \in \mathbb{R}^3$, or in coordinates, $(u \odot v)_{ij} = \frac{1}{2}(u_i v_j + u_j v_i)$. This allows the convenient binomial expansion $(u + v) \otimes (u + v) = u \otimes u + 2u \odot v + v \otimes v$.

For $\Omega \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$, we will use the Lebesgue norms

$$\|f\|_{L^q_x(\Omega)} := \left(\int_\Omega |f(x)|^q dx \right)^{1/q}$$

and

$$\|f\|_{L^p_t L^q_x(I \times \Omega)} := \left(\int_I \|f(t, \cdot)\|^p_{L^q_x(\Omega)} dt \right)^{1/p}$$

with the usual modifications if $p = \infty$ or $q = \infty$.

For a Schwartz function $f : \mathbb{R}^3 \to \mathbb{R}^n$, we define the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) dx$$

and the Littlewood-Paley projection by the formula

$$\widehat{P_{\le N} f}(\xi) := \varphi(\xi/N) \hat{f}(\xi)$$

where $\varphi : \mathbb{R}^3 \to \mathbb{R}$ is a radial bump function supported in $B(0, 1)$ such that $\varphi \equiv 1$ in $B(0, 1/2)$. Then let

$$P_N := P_{\le N} - P_{\le N/2}, \quad P_{>N} := 1 - P_{\le N}, \quad \tilde{P}_N := P_{\le 2N} - P_{\le N/4}.$$

These all commute with other Fourier multipliers such as $\mathbb{P}$, Δ and $e^{t\Delta}$. Estimation of such operators in the weighted spaces $X^p_{\alpha;T}$ is the subject of the next subsection.

When a summation is indexed with a capital letter such as N or M, it should be taken to range over the dyadic integers $2^{\mathbb{Z}}$. Thus we have the shorthand notation

$$\sum_N f(N) := \sum_{N\in 2^{\mathbb{Z}}} f(N), \qquad \sum_{A\le N\le B} f(N) := \sum_{\{N\in 2^{\mathbb{Z}}: A\le N\le B\}} f(N), \quad \text{etc.}$$

2.2.2 Bernstein-type inequalities with axial symmetry and weights

If $\Omega \subset \mathbb{R}^3$ and $\alpha \in \mathbb{R}$, we define the weighted space $X^q_{\alpha;T}(\Omega)$ of smooth vector fields $u : [-T,0]\times\Omega \to \mathbb{R}^3$ such that

$$\|u\|_{X^q_{\alpha;T}(\Omega)} := \|r^\alpha u\|_{L^\infty_t L^q_x([-T,0]\times\Omega)} < \infty.$$

For brevity we will set $X^q_{\alpha;T} := X^q_{\alpha;T}(\mathbb{R}^3)$. The spaces become critical with respect to the Navier-Stokes scaling when $\alpha = \alpha_q$, where

$$\alpha_q := 1 - \frac{3}{q}.$$

The case of interest, in which it is possible to improve the bounds compared to [60], is

$$\text{either } q \in (3,\infty), \text{ or } u \text{ is axisymmetric and } q \in (2,3]. \tag{2.1}$$

See Chapter 4 for details.

We record Hölder's inequality for $X^q_{\alpha;T}$ spaces, which is immediate from the standard version for L^p spaces: assuming $1 \le p,q,r \le \infty$, $\alpha,\beta,\gamma \in \mathbb{R}$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, and $\alpha = \beta+\gamma$,

$$\|r^\alpha uv\|_{X^p_{\alpha;T}} \le \|r^\beta u\|_{X^q_{\beta;T}} \|r^\gamma v\|_{X^r_{\gamma;T}}.$$

The following proposition shows that Bernstein's inequality for Fourier multipliers with compactly supported symbols extends naturally to weighted L^p spaces such as $X^p_{\alpha;T}$. When working with u controlled in an $X^q_{\alpha_q;T}$ space with $q < 3$ one runs into the difficulty that

$\alpha_q < 0$. Proposition 2.1, as well as many of the estimates for other operators we will derive from it, only hold when the weight on the left-hand side has a smaller power than the one on the right (see Remark 2.2), so it is not clear how one would control the components of u with frequency much larger than r^{-1}. Fortunately, in the presence of axial symmetry we can avoid these issues and prove a weighted Bernstein inequality which allows us to exchange some integrability for negative powers of r.

Proposition 2.1. *Let m be a Fourier multiplier supported in $B(0,N)$ with $|\nabla^j m| \leq MN^{-j}$ for $j = 0, 1, \ldots, 100$. If $1 \leq q \leq p \leq \infty$ and either*

1. $\alpha > -\frac{2}{p}$, $\beta < \frac{2}{q'}$, *and* $\alpha \leq \beta$;
2. $p = \infty$, $\alpha = 0$, *and* $0 \leq \beta < \frac{2}{q'}$; *or*
3. $q = 1$, $\beta = 0$, *and* $-\frac{2}{p} < \alpha \leq 0$,

then we have

$$\|r^\alpha T_m u\|_{L^p} \lesssim MN^{\frac{3}{q}-\frac{3}{p}+\beta-\alpha} \|r^\beta u\|_{L^q}. \tag{2.2}$$

If $|u|$ is axisymmetric, then the conditions $\alpha \leq \beta$, $\beta \geq 0$, and $\alpha \leq 0$ can be improved to $\alpha \leq \beta + \frac{1}{q} - \frac{1}{p}$, $\beta \geq -\frac{1}{q}$, or $\alpha \leq 1 - \frac{1}{p}$, respectively.

Proof. In this proof we make use of the standard non-weighted Bernstein inequalities proved, for example, in [60, Lemma 2.1].

When establishing the case of the proposition in which $|u|$ is axisymmetric, let us assume for the moment that the symbol m is likewise axisymmetric.

We begin by rescaling x and m to make $N = M = 1$. Then it clearly suffices to show that the operator $T = r^\alpha T_m r^{-\beta}$ is bounded from $L^q(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$. To do so, we decompose it into spatially localized pieces as

$$T = \sum_{R,S} T_{R,S}, \quad T_{R,S} = r^\alpha \chi_R T_m r^{-\beta} \chi_S$$

where $\chi : \mathbb{R} \to [0,\infty)$ is a smooth function such that the collection $\chi_R(x) = \chi(r/R)$ over $R \in 2^{\mathbb{Z}}$ forms a partition of unity of $\mathbb{R} \setminus \{0\}$. More specifically we may choose χ_R to be supported in $\mathcal{S}(\frac{R}{2}, \frac{3R}{2})$. Then $T_{R,S}$ can be expressed as an integral operator $T_{R,S}f(y) = \int_{\mathbb{R}^3} f(x)K(x,y)dx$ with the kernel

$$K_{R,S}(x,y) = r_y^{\alpha} r_x^{-\beta} \chi_R(y)\chi_S(x)\hat{m}(x-y)$$

satisfying

$$|K_{R,S}(x,y)| \lesssim R^{\alpha} S^{-\beta} \chi_R(y)\chi_S(x)\langle x-y\rangle^{-50}$$

where we let $\langle x\rangle = (1+|x|^2)^{1/2}$. Then, bounding the operator with Hölder's inequality, we have that for R,S such that $\max(R/S, S/R) \geq 100$,

$$\begin{aligned}
\|T_{R,S}\|_{L^q \to L^\infty} &\lesssim \|K_{R,S}\|_{L_y^\infty L_x^{q'}} \\
&\lesssim R^{\alpha} S^{-\beta} \langle \max(R,S)\rangle^{-50} \|\chi_S(x)\langle x_3 - y_3\rangle^{-50}\|_{L_y^\infty L_x^{q'}} \\
&\lesssim R^{\alpha} S^{-\beta+\frac{2}{q'}} \langle \max(R,S)\rangle^{-50}
\end{aligned}$$

and

$$\begin{aligned}
\|T_{R,S}\|_{L^1 \to L^p} &\lesssim \|K_{R,S}\|_{L_x^\infty L_y^{p}} \\
&\lesssim R^{\alpha} S^{-\beta} \langle \max(R,S)\rangle^{-50} \|\chi_R(y)\langle x_3 - y_3\rangle^{-50}\|_{L_x^\infty L_y^{p}} \\
&\lesssim R^{\alpha+\frac{2}{p}} S^{-\beta} \langle \max(R,S)\rangle^{-50}.
\end{aligned}$$

Then by interpolation, if $p \geq q$, it follows that

$$\|T_{R,S}\|_{L^q \to L^p} \lesssim R^{\alpha+\frac{2}{p}} S^{-\beta+\frac{2}{q'}} \langle \max(R,S)\rangle^{-50}.$$

By essentially the same calculation, if $1/100 \leq R/S \leq 100$, then

$$\|T_{R,S}\|_{L^q \to L^p} \lesssim R^{\alpha-\beta+\frac{2}{p}+\frac{2}{q'}}.$$

Unfortunately this estimate is adequate only when $R,S \lesssim 1$, so we separately consider the case where R and S are comparable and $R,S \gg 1$. Fix a $\rho \sim R^{1/10}$ that evenly divides $R/4$.

Let $v = r^{-\beta}\chi_S u$. We need to find a spatial region that we can dilate slightly (as required to use the localized Bernstein inequality, again see [60, Lemma 2.1]) without drastically increasing the L^∞ norm of $T_m v$. Suppose first we do not have such a region, that is

$$\|T_m v\|_{L^\infty(\mathcal{S}(\frac{R}{4},\frac{7R}{4}))} \geq 2\|T_m v\|_{L^\infty(\mathcal{S}(\frac{R}{4}+\rho,\frac{7R}{4}-\rho))} \geq \cdots \geq 2^{\frac{R}{4\rho}}\|T_m v\|_{L^\infty(\mathcal{S}(\frac{R}{2},\frac{3R}{2}))}.$$

Then taking the left- and right-most ends of the inequality, the ordinary Bernstein inequality implies

$$\|T_m v\|_{L^\infty(\mathcal{S}(\frac{R}{2},\frac{3R}{2}))} \leq 2^{-R^{1/2}}\|T_m v\|_{L^\infty} \lesssim R^{-100}\|v\|_{L^q}.$$

It follows that

$$\|T_{R,S}u\|_{L^\infty} \lesssim R^{-50}\|u\|_{L^q}$$

which is an adequate estimate to proceed with the argument. Otherwise, there exists an $R_0 \in [\frac{R}{2}, \frac{3R}{4}]$ such that

$$\|T_m v\|_{L^\infty(\mathcal{S}(R-R_0-\rho,R+R_0+\rho))} \leq \frac{1}{2}\|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))}.$$

Let x_0 be a point in the region $\mathcal{S}(R-R_0, R+R_0)$ such that

$$|T_m v(x_0)| \geq \frac{1}{2}\|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))}.$$

By composing with $P_{\leq 10}$ and applying the local Bernstein inequality from [60], we have the gradient estimate

$$\begin{aligned}\|\nabla T_m v\|_{L^\infty(\mathcal{S}(R-R_0-\frac{\rho}{2},R+R_0+\frac{\rho}{2}))} &\lesssim \|T_m v\|_{L^\infty(\mathcal{S}(R-R_0-\rho,R+R_0+\rho))} + \rho^{-50}\|v\|_{L^q} \\ &\lesssim \|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))} + R^{-5}\|v\|_{L^q}.\end{aligned}$$

Therefore, by the fundamental theorem of calculus,

$$|T_m v(x)| \geq \frac{1}{4}\|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))}$$

for all

$$x \in B\Big(x_0, \frac{1}{O(1)} \frac{\|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))}}{\|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))} + R^{-5}\|v\|_{L^q}}\Big).$$

Importantly, since $\rho \gg 1$ and the radius of this ball is less than 1, it is contained in $\mathcal{S}(R - R_0 - \frac{\rho}{2}, R + R_0 + \frac{\rho}{2})$ where the gradient estimate holds. Without axial symmetry, this implies

$$\|T_m v\|_{L^q} \gtrsim \|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))} \left(\frac{\|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))}}{\|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))} + R^{-5}\|v\|_{L^q}} \right)^{3/q}.$$

No matter which term in the denominator is larger, we conclude (using the ordinary Bernstein inequality for T_m if the first is larger)

$$\|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))} \lesssim \|v\|_{L^q}.$$

Now suppose $|u|$, and consequently $|v|$, is axisymmetric. Let $\tilde{T}_m$ be the operator with kernel $|K(x,y)|$. Then by the triangle inequality, inside the same ball, we have the concentration

$$|\tilde{T}_m(|v|)(x)| \geq \frac{1}{4}\|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))}.$$

Thanks to the assumption that m is axisymmetric, one easily computes that $\tilde{T}_m(|v|)$ is as well. Thus, the bound still holds inside the torus obtained by rotating the ball around the x_3-axis. (See Figure 2.1.) Note that within this torus, $r \gtrsim R$; therefore

$$\begin{aligned}\|\tilde{T}_m(|v|)\|_{L^q} \gtrsim\ & \|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))} R^{1/q} \\ & \times \left(\frac{\|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))}}{\|T_m v\|_{L^\infty(\mathcal{S}(R-R_0,R+R_0))} + R^{-5}\|v\|_{L^q}} \right)^{2/q}.\end{aligned}$$

Once again, no matter which term in the denominator is larger, this implies

$$\|T_m v\|_{L^\infty(\mathcal{S}(\frac{R}{2},\frac{3R}{2}))} \lesssim R^{-\frac{1}{q}}\|v\|_{L^q}.$$

Since $\operatorname{supp}\chi_R \subset \mathcal{S}(\frac{R}{2},\frac{3R}{2})$, it follows that

$$\|T_{R,S}u\|_{L^\infty} \lesssim R^\alpha \|T_m r^{-\beta}\chi_S u\|_{L^\infty(\mathcal{S}(\frac{R}{2},\frac{3R}{2}))} \lesssim R^{\alpha-\beta}\|u\|_{L^q},$$

or

$$\|T_{R,S}u\|_{L^\infty} \lesssim R^{\alpha-\beta-\frac{1}{q}}\|u\|_{L^q}$$

in the presence of axial symmetry. By interpolating with the trivial inequality

$$\|T_{R,S}u\|_{L^1} \lesssim R^{\alpha-\beta}\|u\|_{L^1},$$

we obtain, if $q \leq p$,

$$\|T_{R,S}\|_{L^q\to L^p} \lesssim R^{\alpha-\beta}$$

or

$$\|T_{R,S}\|_{L^q\to L^p} \lesssim R^{\alpha-\beta+\frac{1}{p}-\frac{1}{q}}$$

in the presence of axial symmetry.

Finally, we can sum over $R, S \in 2^{\mathbb{Z}}$ to obtain the desired estimate. Let $\tilde{\chi}_S$ be a dilated version of χ_S such that $\chi_S\tilde{\chi}_S = \chi_S$. Then

$$Tu = \sum_{\max(R/S,S/R)>100} T_{R,S}(\tilde{\chi}_S u) + \sum_{1/100\leq R/S\leq 100} T_{R,S}(\tilde{\chi}_S u)$$

where each $x \in \mathbb{R}^3$ lies in the support of boundedly many terms. This implies that without axial symmetry,

$$\|Tu\|_{L^p}^p \lesssim \sum_R \left(\sum_{\{S\,:\,\max(R/S,S/R)>100\}} R^{\alpha+\frac{2}{p}} S^{-\beta+\frac{2}{q'}} \langle \max(R,S)\rangle^{-100} \|\tilde{\chi}_S u\|_{L^q} \right)^p$$
$$+ \sum_{\substack{1/100\leq R/S\leq 100 \\ \max(R,S)\leq 1}} (R^{\alpha-\beta+\frac{2}{p}+\frac{2}{q'}}\|u\|_{L^q})^p + \sum_{\substack{1/100\leq R/S\leq 100 \\ \max(R,S)>1}} (R^{\alpha-\beta}\|\tilde{\chi}_S u\|_{L^q})^p$$

with the suitable modification if $p = \infty$, in the sense that we are taking an $\ell^p(2^{\mathbb{Z}})$ norm in R. When $p < \infty$, the sums converge as geometric series and are bounded by $\|u\|_{L^q}^p$ as long as $\alpha > -\frac{2}{p}$, $\beta < \frac{2}{q'}$, and $\alpha < \beta$. If $p = \infty$, the expression is similarly bounded as

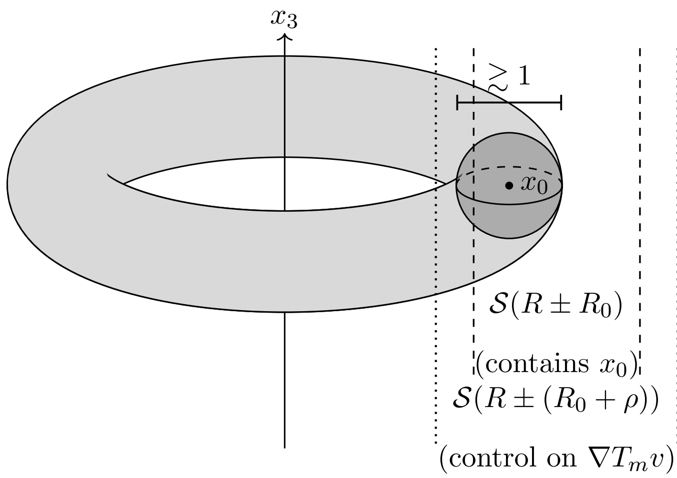

Figure 2.1: We locate a concentration point x_0 inside the cylindrical shell $\mathcal{S}(R - R_0, R + R_0)$ (delimited by the dashed lines) and, using the gradient estimate on $T_m v$ which holds in the region $\mathcal{S}(R - R_0 - \rho, R + R_0 + \rho)$ (delimited by the dotted lines), one can deduce a comparable estimate holds within a ball which (in the most nontrivial case) has radius at least on the order of 1. In the case of axial symmetry, we can infer from the pointwise lower bound in the ball that the same lower bound holds within the solid torus obtained by rotating it around the x_3-axis.

long as $0 \leq \alpha \leq \beta < \frac{2}{q'}$. If $\alpha = \beta$, then summability of the last term follows from the embedding $\ell^q(2^{\mathbb{Z}}) \to \ell^p(2^{\mathbb{Z}})$ (using $p \geq q$), and the fact that $\sum_{R \geq 1} \|\tilde{\chi}_R u\|_{L^q}^q \lesssim \|u\|_{L^q}^q$. A similar argument applies to the first term on the right-hand side in the case $q = 1$, $\beta = 0$.

In the case where $|u|$ has axial symmetry, we carry out an analogous calculation and find the same result except with the last condition relaxed to $\alpha \leq \beta + \frac{1}{q} - \frac{1}{p}$ thanks to the smaller power of R in the last term.

Now we show how to remove the assumption that m is axisymmetric. Note that $P_{\leq 10}$ does have an axisymmetric symbol; moreover $P_{\leq 10} T_m = T_m$. Therefore if $|u|$ is axisymmetric,

$$\|r^\alpha P_{\leq 10} u\|_{L^p} \lesssim \|r^\beta u\|_{L^q}$$

assuming $p \geq q$, $\alpha > -\frac{2}{p}$, $\beta < \frac{2}{q'}$, and $\alpha \leq \beta + \frac{1}{q} - \frac{1}{p}$ (with the appropriate adjustment in the two endpoint cases). Then by the non-axisymmetric version of the theorem,

$$\|r^\alpha T_m u\|_{L^p} \lesssim \|r^\alpha P_{\leq 10} u\|_{L^p}$$

which yields the desired result. Note that we have $\alpha \leq \beta + \frac{1}{q} - \frac{1}{p} < 2 - \frac{1}{q} - \frac{1}{p} \leq \frac{2}{p'}$ as required. □

Remark 2.2. *Later in the chapter, most notably in the proof of Proposition 2.7, we will be applying Proposition 2.1 in an iterative procedure which will lead to some laborious checking of its hypotheses. The reader may find it illuminating to keep in mind some examples which show why each one is necessary. For simplicity we take $N = 1$ and $T_m = P_1$.*

Since $P_1 u$ is approximately constant on balls of radius $O(1)$, when $p < \infty$, in order for $r^{\alpha p}|P_1 u|^p$ to be integrable in such a ball centered on the x_3-axis, we need $\alpha p > -2$, or $\alpha > -\frac{2}{p}$. Of course when $p = \infty$, there is no such integrability issue as long as $\alpha \geq 0$. Next, let $u = \phi(x)/(r^2 + \epsilon^2)$ where u is a bump function supported in $B(0,1)$. By the same *uncertainty principle heuristic, one finds that* $\|r^\alpha P_1 u\|_{L^p}$ *is comparable to* $\log\frac{1}{\epsilon}$*, but*

$$\|r^\beta u\|_{L^q} \sim \begin{cases} \epsilon^{-\frac{2}{q'}+\beta}, & \beta < \frac{2}{q'} \\ \log^{1/q}\frac{1}{\epsilon}, & \beta = \frac{2}{q'} \\ 1, & \beta > \frac{2}{q'} \end{cases}.$$

By taking ϵ *sufficiently small, we find that the proposition can hold only when either* $\beta < \frac{2}{q'}$ *or* $q = 1$ *and* $\beta = 0$*. Let* u *be a bump function supported in* $B(x_0, 1)$ *where* $r(x_0) = R \gg 1$*. Then* (2.2) *asserts* $R^\alpha \lesssim R^\beta$*. By taking* R *sufficiently large, we see* $\alpha \le \beta$*. Similarly, consider a smooth axisymmetric function supported in the annulus* $\{x \in \mathbb{R}^3 : (r-R)^2 + x_3^2 < 1\}$ *where* $R \gg 1$*. Then* (2.2) *becomes* $R^{\alpha+\frac{1}{p}} \lesssim R^{\beta+\frac{1}{q}}$ *which necessitates* $\alpha \le \beta + \frac{1}{q} - \frac{1}{p}$*.*

As in [60], this Bernstein inequality can be localized to a region, at the cost of a global term that can be made small by enlarging the region by a length $\gg N^{-1}$.

Proposition 2.3. *Let* m *be a multiplier with* $\operatorname{supp} m \subset B(0, N)$ *such that*

$$|\nabla^j m| \le M N^{-j}$$

for $j = 0, 1, \ldots, 2K$ *where* $K \ge 100$*. Also let* $\Omega \subset \mathbb{R}^3$ *be open and* $\Omega_{A/N} = \{x \in \mathbb{R}^3 : \operatorname{dist}(x, \Omega) < A/N\}$*. Then*

$$\begin{aligned} \|r^{\alpha_1} T_m u\|_{L^{p_1}(\Omega)} \lesssim_K\ & M N^{\frac{3}{q_1} - \frac{3}{p_1} + \beta_1 - \alpha_1} \|r^{\beta_1} u\|_{L^{q_1}(\Omega_{A/N})} \\ & + A^{-K} M r(\Omega)^{\alpha_1 - \alpha_2} |\Omega|^{\frac{1}{p_1} - \frac{1}{p_2}} N^{\frac{3}{q_2} - \frac{3}{p_2} + \beta_2 - \alpha_2} \|r^{\beta_2} u\|_{L^{q_2}(\mathbb{R}^3)} \end{aligned} \tag{2.3}$$

if $p_i \ge q_i$*,* $p_1 \le p_2$*,* $\alpha_1 \ge \alpha_2$*,* $\alpha_i > -\frac{2}{p_i}$*,* $\beta_i < \frac{2}{q_i'}$*, and* $\alpha_i \le \beta_i$ *for* $i = 1, 2$*. Here* $r(\Omega)$ *denotes* $\sup\{r : x \in \Omega\}$*.*

If $|u|$ *is axisymmetric, the last condition can be weakened to* $\alpha_i \le \beta_i + \frac{1}{q_i} - \frac{1}{p_i}$*. As in Proposition 2.2, the result extends to the* $p_i = \infty$*,* $\alpha_i = 0$ *and* $q_i = 1$*,* $\beta_i = 0$ *endpoints.*

We refer to the second term on the right-hand side of (2.3) as the global term. Regardless of what kind of $X^p_{\alpha;T}$ control is known for u, it is usually possible to make it negligible provided the length scale of Ω is much smaller than N^{-1}.

Proof. Once again we can rescale to achieve $N = M = 1$. Observe by the triangle inequality that it suffices to assume u is supported outside Ω_A, since the part inside can be estimated directly using (2.2). First one uses Hölder's inequality to control the L^{q_1} norm by L^{q_2}, as well as the trivial bound $r^{\alpha_1} \le r^{\alpha_2} r(\Omega)^{\alpha_1-\alpha_2}$. Adopting the notation from the proof of Proposition 2.1, we are concerned with estimating convolutions in the form

$$T_{R,S}(y) = \int_{\mathbb{R}^3} K_{R,S}(x,y)u(x)dx,$$

but with the additional feature that $y \in \Omega$ and $x \notin \Omega_A$, so $|x-y| \ge A$. Therefore, the estimate for the kernel can be improved to

$$|K_{R,S}(x,y)| \lesssim_K R^\alpha S^{-\beta} \chi_R(y)\chi_S(x)\langle x-y\rangle^{-50} A^{-K}$$

and one proceeds as in Proposition 2.1. □

As a special case of Proposition 2.1, with $m = e^{-t|\xi|^2}\psi(\xi/N)$, we get the heat estimate

$$\|r^\alpha e^{t\Delta} P_N \nabla^j u\|_{L^p} \lesssim_j e^{-tN^2/20} N^{j+\frac{3}{q}-\frac{3}{p}+\beta-\alpha} \|r^\beta u\|_{L^q} \tag{2.4}$$

under the same assumptions on the parameters. Then summing over $N \in 2^{\mathbb{Z}}$,

$$\|r^\alpha e^{t\Delta} \nabla^j u\|_{L^p} \lesssim_j t^{-\frac{1}{2}(j+\frac{3}{q}-\frac{3}{p}+\beta-\alpha)} \|r^\beta u\|_{L^q}. \tag{2.5}$$

Let us prove one other local lemma in a similar spirit as Proposition 2.3.

Lemma 2.4. *If $N, K > 0$, $j \ge 0$, $p \le q$, $0 < r_1 < r_2$, $f \in C^\infty(\mathbb{R}^d)$, and $\phi \in C_c^\infty(\mathbb{R}^d)$ with $\phi \equiv 1$ in $B(r_2)$, then*

$$\|P_N \nabla^j f\|_{L^p(B(r_1))} \lesssim_{r_1,r_2,p,q,j,K,\phi} \|P_N(\phi\nabla^j f)\|_{L^p(B(r_1))} + N^{-K}\|f\|_{L^q(\mathbb{R}^d)}.$$

Proof. With $\psi(\xi)$ the Fourier multiplier for P_1, we have

$$P_N \nabla^j f(x) = \int_{\mathbb{R}^d} \check{\psi}(y)(\phi\nabla^j f)(x-y/N)dy + \int_{\mathbb{R}^d\setminus B(cN)} \check{\psi}(y)((1-\phi)\nabla^j f)(x-y/N)dy$$

as long as x is restricted to $B(r_1)$ and c is chosen sufficiently small compared to $r_2 - r_1$. The first term is exactly $P_N(\phi\nabla^j f)(x)$ and the second term is straightforward to estimate using integration by parts, polynomial decay of $\check{\psi}$ and its derivatives, and Hölder's inequality. □

Combining Lemma 2.2 with (1.2) and (1.9), the following bounds on the frequency-localized vector fields are immediate.

Lemma 2.5. *If u solves* (1.2) *on* $[-T,0]$ *and admits the bound* (1.9)*, where either* $X = L^d(\mathbb{R}^d)$ *or* $X = X^q_{\alpha;T}$ *with* (2.1)*, then we have*

$$\|\nabla^j P_N u\|_{L^\infty_{t,x}([-T,0]\times\mathbb{R}^d)} \lesssim_j AN^{1+j}, \quad \|\partial_t P_N u\|_{L^\infty_{t,x}([-T,0]\times\mathbb{R}^d)} \lesssim A^2N^3$$

for all $j \geq 0$, $N > 0$.

Proof. We focus on the $X^p_{\alpha;T}$ case since the proof is slightly less trivial. By (2.2),

$$|\nabla^j P_N u| \lesssim N^{j+3/q+\alpha_q}\|r^{\alpha_q}u\|_{L^q} \lesssim AN^{1+j}.$$

Applying P_N to (1.2) and again using (2.2) and Hölder's inequality,

$$\begin{aligned}|\partial_t P_N u| &\leq \|P_N\mathbb{P}\operatorname{div}(u\otimes u)\|_{L^\infty_x} + \|P_N\Delta u\|_{L^\infty_x}\\ &\lesssim N^{1+\frac{6}{q}+2\alpha_q}\|r^{2\alpha_q}u\otimes u\|_{L^{q/2}_x} + N^{2+\frac{3}{q}+\alpha_q}\|r^{\alpha_q}u\|_{L^q_x}\\ &\lesssim A^2N^3.\end{aligned}$$

Note that the weights satisfy the Bernstein inequality when $q \geq 3$ because then $\alpha_q \geq 0$, and when $2 < q < 3$ because then $\alpha_q > -\frac{1}{q}$. □

2.3 Picard decomposition

2.3.1 The $\mathbb{R}^d$ case

A difficulty of working in L^d_x is that while one would wish to make use of energy methods, the solution does not have enough decay to be in any L^2_x-based spaces. In the cases $d = 3, 4$ one can avoid this problem by some manner of splitting u into one flow solving a linear equation and another that solves a complementary nonlinear equation, see [13, 60]. For example, the method in [60] of considering $u(t) - e^{(t-t_0)\Delta}u(t_0)$, i.e., removing the heat flow part of the evolution, leaves the remaining nonlinear flow in $L_t^\infty L_x^p$ for $p \in [\frac{d}{2}, d]$. Unfortunately when $d \geq 5$, this range excludes the important energy space $L_t^\infty L_x^2$.

In the general case $d \geq 3$ we address this difficulty using the following decomposition of u. We remark that decompositions based on a Picard-type iteration in the same spirit have also appeared in [14, 25, 1]. The idea is essentially to subtract off a Picard iterate starting from an initial condition $u(t_0)$. The critical bound (1.9) implies good subcritical estimates on the iterate thanks to smoothing from the heat propagator, and one can show inductively using Duhamel's formula that the difference lies in lower integrability spaces including $L_t^\infty L_x^2$. Moreover, the difference satisfies a Navier-Stokes-type equation which leads to estimates that will be useful later.

Proposition 2.6. *Suppose u is a classical solution of (1.2) on $[-T, 0]$ with the bound (1.9). Then for every $T_1 \in [0, T/2]$, there exist $u^\flat$ and $u^\sharp$ such that the following hold:*

- *We have the decomposition*

$$u = u^\flat + u^\sharp \text{ on } [-T_1, 0].$$

- *If $d \leq p \leq \infty$ and $j \geq 0$, then*

$$\|\nabla^j u^\flat\|_{L_t^\infty L_x^p([-T_1,0]\times\mathbb{R}^d)} \leq A^{O_j(1)} T_1^{-\frac{1}{2}(1+j-\frac{d}{p})}, \tag{2.6}$$

$$\|P_N u^\flat\|_{L_{t,x}^\infty([-T_1,0]\times\mathbb{R}^d)} \leq A^{O(1)} e^{-T_1 N^2/O(1)} T_1^{-\frac{1}{2}}. \tag{2.7}$$

- *If $1 \leq p \leq d$ and $1 < q < \infty$, then*[2]

$$\|u^\sharp\|_{L_t^\infty L_x^p([-T_1,0]\times\mathbb{R}^d)} \leq A^{O(1)} T_1^{\frac{1}{2}(\frac{d}{p}-1)}, \tag{2.8}$$

$$\|\nabla u\|_{L_{t,x}^2(Q(T_1^{\frac{1}{2}}))} + \|\nabla u^\sharp\|_{L_{t,x}^2([-T_1,0]\times\mathbb{R}^d)} \leq A^{O(1)} T_1^{\frac{d}{4}-\frac{1}{2}}, \tag{2.9}$$

$$\|\nabla u^\sharp\|_{L_t^q L_x^{\frac{d}{2}}([-T_1,0]\times\mathbb{R}^d)} \lesssim_q A^{O(1)} T_1^{\frac{1}{q}} \tag{2.10}$$

where $Q(r) = [-r^2, 0] \times B(r)$.

[2] *We thank an anonymous referee for JMFM for bringing* (2.10) *to our attention which allows a simplification of the original argument.*

- $u^\sharp$ *solves*

$$\partial_t u^\sharp + \mathbb{P}\,\mathrm{div}(u^\sharp \otimes u^\sharp + 2u^\flat \odot u^\sharp) - \Delta u^\sharp = f \tag{2.11}$$

where f *obeys estimates*

$$\|\nabla^j f\|_{L^\infty_t L^p_x([-T_1,0]\times\mathbb{R}^d)} \le A^{O_j(1)} T_1^{-\frac{1}{2}(3+j-\frac{d}{p})} \tag{2.12}$$

for $\frac{d}{2} \le p \le \infty$ *and* $j \ge 0$.

Let us emphasize that different choices of the subinterval $[-T_1, 0]$ lead to entirely different decompositions. In the sequel when we use this proposition, it will be made clear on which interval the decomposition is taken.

Proof. Starting with

$$u^\flat_0 := 0, \quad u^\sharp_0 := u,$$

we inductively define for $n \ge 1$

$$u^\flat_n(t) := e^{(t-\tau_{n-1})\Delta} u_{n-1}(\tau_{n-1}) - \int_{\tau_{n-1}}^t e^{(t-t')\Delta}\mathbb{P}\,\mathrm{div}\, u^\flat_{n-1} \otimes u^\flat_{n-1}(t')dt',$$
$$u^\sharp_n(t) := -\int_{\tau_{n-1}}^t e^{(t-t')\Delta}\mathbb{P}\,\mathrm{div}(u \otimes u - u^\flat_{n-1} \otimes u^\flat_{n-1})(t')dt'$$

where we have chosen a sequence of $O(1)$-many times $-2T_1 < \tau_1 < \tau_2 < \cdots < -T_1$ such that $\tau_i - \tau_{i+1} = T_1/O(1)$. We prove (2.6) on the shrinking time intervals $[\tau_n, 0]$ with $u^\flat$ replaced by $u^\flat_n$ by induction on n. For $n = 0$ it is trivial. Suppose the claim for some $n - 1 \ge 0$. Then, for $t \in [\tau_n, 0]$,

$$\|\nabla^j u^\flat_n(t)\|_{L^p_x(\mathbb{R}^d)} \lesssim (t - \tau_{n-1})^{-\frac{1}{2}(1+j-\frac{d}{p})} A$$
$$+ \int_{\tau_{n-1}}^t (t-t')^{-\frac{1}{2}} \|u^\flat_{n-1}(t')\|_{L^p_x(\mathbb{R}^d)} \|\nabla^j u^\flat_{n-1}(t')\|_{L^\infty_x(\mathbb{R}^d)} dt'$$

which gives the desired bound. Then (2.7) follows similarly by induction using Duhamel's principle and a paraproduct decomposition.

Next, it is convenient to decompose $u_n^\sharp = u_n^{\sharp,1} + u_n^{\sharp,2}$ where

$$u^{\sharp,1}(t) := -2\int_{\tau_{n-1}}^t e^{(t-t')\Delta}\mathbb{P}\operatorname{div} u_{n-1}^\flat \odot u_{n-1}^\sharp(t')dt',$$
$$u^{\sharp,2}(t) := -\int_{\tau_{n-1}}^t e^{(t-t')\Delta}\mathbb{P}\operatorname{div} u_{n-1}^\sharp \otimes u_{n-1}^\sharp(t')dt'.$$

We claim the bound in (2.8) for $u_n^\sharp$, specifically in the range $\max(\frac{d}{n+1},1) \le p < d$ and on the time interval $[\tau_n, 0]$. Thus we will obtain the desired result by taking n large depending on d. Note that the $p = d$ case is immediate from (1.9) and (2.6). As a base case, we consider $n = 1$ for which $u_1^{\sharp,1} = 0$. For $u_1^{\sharp,2}$,

$$\|u_1^{\sharp,2}(t)\|_{L_x^p(\mathbb{R}^d)} \lesssim \int_{\tau_{n-1}}^t (t-t')^{-\frac{1}{2}(3-\frac{d}{p})}\|u(t')\|_{L_x^d(\mathbb{R}^d)}^2 dt'$$

which yields the desired result using (1.9) assuming $\frac{d}{2} \le p < d$. Now assume the desired inequality for some $n-1 \ge 1$. Then

$$\|u^{\sharp,1}(t)\|_{L_x^p(\mathbb{R}^d)} \lesssim \int_{\tau_{n-1}}^t (t-t')^{-\frac{1}{2}(1+\frac{d}{s}+\frac{d}{r}-\frac{d}{p})}\|u_{n-1}^\flat(t')\|_{L_x^s(\mathbb{R}^d)}\|u_{n-1}^\sharp(t')\|_{L_x^r(\mathbb{R}^d)}dt',$$

assuming $\frac{1}{p} \le \frac{1}{s} + \frac{1}{r}$. This is integrable in time, and furthermore we can apply (2.6) and (2.8), by taking $r = \frac{dp}{d-p}$ and $s = d$, and assuming additionally that $\max(\frac{d}{n+1},1) \le p < \frac{d}{2}$. If instead we take $r = \frac{d}{2}$ and $\frac{1}{s} = \max(\frac{1}{p} - \frac{2}{d}, 0)$, we obtain the same result but instead for $\frac{d}{3} \le p < d$. Combining these, we have the full range of p. Next we consider $u^{\sharp,2}$. With $\frac{1}{r} = \frac{1}{2}(\frac{1}{d} + \frac{1}{p}) - \epsilon$,

$$\|u_n^{\sharp,2}(t)\|_{L_x^p(\mathbb{R}^d)} \lesssim \int_{\tau_{n-1}}^t (t-t')^{-1+\epsilon d}\|u_{n-1}^\sharp(t')\|_{L_x^r(\mathbb{R}^d)}^2 dt'$$

implies the desired bound upon taking ϵ sufficiently small depending on p and d. (2.6)-(2.9) therefore hold upon setting $u^\flat, u^\sharp := u_d^\flat, u_d^\sharp$.

One readily computes (2.11) with $f = \mathbb{P}\operatorname{div}(u_{d-1}^\flat \otimes u_{d-1}^\flat - u_d^\flat \otimes u_d^\flat)$. Then (2.12) follows by Hölder's inequality and (2.6). Multiplying (2.11) by $u^\sharp$ and integrating over $\mathbb{R}^d$, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{|u^\sharp|^2}{2}dx = -\int_{\mathbb{R}^d}|\nabla u^\sharp|^2 - \int_{\mathbb{R}^d} u^\sharp\cdot(u^\sharp\cdot\nabla u^\flat) + u^\sharp\cdot f$$

and therefore we can apply (2.8), (2.6), and (2.12) to find

$$\|\nabla u^\sharp\|_{L^2_{t,x}([-T_1,0]\times\mathbb{R}^d)} \lesssim \|u^\sharp\|_{L^\infty_t L^2_x([-T_1,0]\times\mathbb{R}^d)} + \|u^\sharp\|_{L^\infty_t L^2_x([-T_1,0]\times\mathbb{R}^d)}\|\nabla u^\flat\|_{L^\infty_{t,x}([-T_1,0]\times\mathbb{R}^d)} \\ + \|u^\sharp\|_{L^\infty_t L^1_x([-T_1,0]\times\mathbb{R}^d)}\|f\|_{L^\infty_{t,x}([-T_1,0]\times\mathbb{R}^d)}$$

which proves (2.9).

Finally, we note that

$$(\partial_t - \Delta)\nabla u^\sharp = -\nabla \mathbb{P}\operatorname{div}(u\otimes u - u^\flat_{d-1}\otimes u^\flat_{d-1}).$$

Thus by (2.5) and maximal regularity for the heat equation,

$$\|\nabla u^\sharp\|_{L^q_t L^{\frac{d}{2}}_x([-T_1,0]\times\mathbb{R}^d)} \lesssim_q T_1^{-\frac{1}{2}+\frac{1}{q}}\|u^\sharp(-T_1)\|_{L^{\frac{d}{2}}_x(\mathbb{R}^d)} + \|u\otimes u - u^\flat_{d-1}\otimes u^\flat_{d-1}\|_{L^q_t L^{\frac{d}{2}}_x([-T_1,0]\times\mathbb{R}^d)}.$$

We conclude (2.10) by (2.6), (1.9), and Hölder's inequality. □

2.3.2 The axisymmetric case

A similar decomposition is necessary for solutions bounded instead in the $X^q_{\alpha;T}$ scale. (See §2.2.2 for definitions.) As in the $\mathbb{R}^d$ case we begin by defining for $t\in[-T,0]$

$$u^\flat_0 = 0, \quad u^\sharp_0 = u.$$

Let $T_n = (\frac{1}{2}+2^{-n})T$. Then for $n\geq 1$ and $t\in[-T_n,0]$, we iteratively define

$$u^\flat_n(t) = e^{(t+T_n)\Delta}u(-T_n) - \int_{-T_n}^t e^{(t-t')\Delta}\mathbb{P}\operatorname{div} u^\flat_{n-1}\otimes u^\flat_{n-1}(t')dt'$$
$$u^\sharp_n(t) = -\int_{-T_n}^t e^{(t-t')\Delta}\mathbb{P}\operatorname{div}(u\otimes u - u^\flat_{n-1}\otimes u^\flat_{n-1})(t')dt'.$$

By Duhamel's principle applied to the Navier-Stokes on $[-T_n,0]$ we see that for every n, these functions sum to u.

Proposition 2.7. *Assume u is a classical solution of* (1.2) *on $[-T,0]\times\mathbb{R}^3$ satisfying* (1.9) *with $X = X^q_{\alpha;T}$ and* (2.1). *Then we have the following.*

1. *For $n = 1, 2, 3, \ldots$ and $t \in [-\frac{T}{2}, 0]$, u admits the decomposition*

$$u = u_n^\flat + u_n^\sharp. \tag{2.13}$$

2. *If $p \geq q$ and either*

$$q > 3 \quad \textit{and} \quad -\frac{2}{p} < \alpha \leq \alpha_q$$

or

$$u \textit{ is axisymmetric}, \quad 2 < q \leq 3, \quad \textit{and} \quad -\frac{2}{p} < \alpha \leq \alpha_q + \frac{1}{q} - \frac{1}{p},$$

then $u_n^\flat$ satisfies the bound

$$\|\nabla^j u_n^\flat\|_{X^p_{\alpha;T/2}} \lesssim_n T^{(\alpha-\alpha_p-j)/2} A^{O_n(1)}. \tag{2.14}$$

These bounds continue to hold at the $p = \infty$, $\alpha = 0$ endpoint. For $N \geq T^{-1/2}$, there are also the frequency-localized estimates

$$\|P_N u_n^\flat\|_{X^q_{\alpha_q;T/2}} \lesssim_n e^{-TN^2/O_n(1)} A^{O_n(1)} \tag{2.15}$$

and

$$\|P_N u_n^\flat\|_{X^\infty_{0;T/2}} \lesssim_n e^{-TN^2/O_n(1)} N A^{O_n(1)}. \tag{2.16}$$

3. *If $q \in (2, \infty)$ and $p_0 \in (1, 3)$, for any n sufficiently large depending on q and p_0,*

$$\|u_n^\sharp\|_{X^p_{0;T/2}} \lesssim_n T^{-\alpha_p/2} A^{O_n(1)} \tag{2.17}$$

for all $p \in [p_0, 3)$.

Remark 2.8. *A useful observation for Chapter 5 is that Proposition 2.7 still holds when the weighted L^p norm X is replaced with its weak L^p counterpart (see Chapter 5 for definitions), except at the endpoints $p = q$ and $\alpha = \alpha_q$ (resp. $\alpha = \alpha_q + \frac{1}{q} - \frac{1}{p}$) without (resp. with) axial* *symmetry. The reason is that we can easily extend the weighted Bernstein inequality using the Lorentz space version of Young's inequality. Indeed, if Proposition 2.1 gives some bound*

$$\|r^\alpha T_m\|_{L^p} \lesssim \|P_{\leq 10} u\|_{L^{q+\epsilon}}$$

(note that in the notation of Proposition 2.1 we have $T_m P_{\leq 10} = T_m$) then with K the convolution kernel for $P_{\leq 10}$,

$$\|P_{\leq 10} u\|_{L^{q+\epsilon}} \lesssim \|K\|_{L^{1+O(\epsilon)}} \|u\|_{L^{q,\infty}} \lesssim \|u\|_{L^{q,\infty}}.$$

This $\epsilon > 0$ causes the endpoints to fail, and is necessary because of the failure of $\|f * g\|_{L^p} \lesssim \|f\|_{L^1} \|g\|_{L^{p,\infty}}$.

Given Bernstein's inequality, one arrives at estimates for the heat propagator and the proof of Proposition 2.7 can proceed without substantial modification.

Proof. To prove (2.14), we claim slightly more strongly that under the stated conditions on p, q, n, and α,

$$\|\nabla^j u_n^\flat\|_{X^p_{\alpha;T_{n+1}}} \lesssim_j T^{(\alpha-\alpha_p-j)/2} A^{O_n(1)}$$

where $T_n = (\frac{1}{2} + 2^{-n})T$ as above. For $u_1^\flat$, this is immediate from (1.9) and (2.5). Suppose we have the desired inequality for some $u_{n-1}^\flat$, $n-1 \geq 1$. From the triangle inequality,

$$\begin{aligned} \|\nabla^j u_n^\flat\|_{X^p_{\alpha;T_{n+1}}} \lesssim & \|\nabla^j e^{(t+T_n)\Delta} u(-T_n)\|_{X^p_{\alpha;T_{n+1}}} \\ & + \int_{-T_n}^t \|r^\alpha \nabla^j e^{(t-t')\Delta} \mathbb{P} \operatorname{div} u_{n-1}^\flat \otimes u_{n-1}^\flat(t')\|_{L^p_x(\mathbb{R}^3)} dt'. \end{aligned}$$

The first term is estimated in the same way as $u_1^\flat$. For the second term, by Hölder's inequality and (2.5),

$$\begin{aligned} \|\nabla^j e^{(t-t')\Delta} \mathbb{P} \operatorname{div} u_{n-1}^\flat \otimes u_{n-1}^\flat\|_{X^p_{\alpha;T_n}} &\lesssim (t-t')^{-\frac{1}{2}} \|\nabla^j (u_{n-1}^\flat \otimes u_{n-1}^\flat)\|_{X^p_{\alpha;T_n}} \\ &\lesssim (t-t')^{-\frac{1}{2}} \sum_{i_1+i_2=j} \Big(\|\nabla^{i_1} u_{n-1}^\flat\|_{X^{2p}_{\alpha/2;T_n}} \|\nabla^{i_2} u_{n-1}^\flat\|_{X^{2p}_{\alpha/2;T_n}} \Big). \end{aligned}$$

The claimed conditions on p and α are closed under the operation of doubling p and halving α, so we achieve the desired bound on $u_n^\flat$ upon integrating in time.

Now let us address the frequency-localized estimates. We remark that (2.14) can also be proven by estimating $P_N u_n^\flat$ and summing in N, but this is less straightforward than the above method in some endpoint cases.

The $n = 1$ case of (2.15) is immediate from (2.4), and indeed it is true for all $N \geq c_1 T^{-1/2}$ (with a constant depending on c_1). Suppose that for some $n - 1 \geq 1$ we have the following version of (2.15) with a slightly wider time interval,

$$\|P_N u_{n-1}^\flat\|_{X^q_{\alpha_q;T_n}} \lesssim e^{-N^2T/O_n(1)} A^{O_n(1)},$$

for all $N \geq c_1 T^{-1/2}$. Then for $N \geq 1000 c_1$ and $t \in [-T_{n+1}, 0]$, by (2.4) and (1.9),

$$\begin{aligned}\|r^{\alpha_q} P_N u_n^\flat(t)\|_{L^q_x(\mathbb{R}^3)} &\lesssim e^{-N^2T/O_n(1)} A \\ &\quad + \int_{-T_n}^{t} e^{-(t-t')N^2/20} N^2 \|r^{2\alpha_q} \tilde{P}_N(u_{n-1}^\flat \otimes u_{n-1}^\flat)(t')\|_{L^{q/2}_x}.\end{aligned}$$

Integrating in time, taking a paraproduct decomposition of the nonlinearity, and applying Hölder's inequality, the iterative estimate on $P_N u_{n-1}^\flat$, and (2.14), the second term becomes

$$\begin{aligned}&\|\tilde{P}_N(P_{>N/100} u_{n-1}^\flat \otimes u_{n-1}^\flat + P_{\leq N/100} u_{n-1}^\flat \otimes P_{>N/100} u_{n-1}^\flat)\|_{X^{q/2}_{2\alpha_q;T_n}} \\ &\lesssim \sum_{N' > N/100} e^{-(N')^2T/O_n(1)} A^{O_n(1)} \\ &\lesssim e^{-N^2T/O_n(1)} A^{O_n(1)}.\end{aligned}$$

Then (2.15) follows by induction on n. Note that in order to obtain (2.15) for a particular n and all $N \geq T^{-1/2}$, one needs to take c_1 sufficiently small depending on n, since the permissible range for N shrinks by a factor of 1000 at each step (due to the frequency overlap of the Littlewood-Paley projections). Thus the constant in (2.15) depends on n.

From here, (2.16) is immediate. Indeed, by (2.2) and (2.15),

$$\|P_N u_n^\flat\|_{X^\infty_{0;T/2}} = \|\tilde{P}_N P_N u_n^\flat\|_{X^\infty_{0;T/2}} \lesssim N \|P_N u_n^\flat\|_{X^q_{\alpha_q;T/2}} \lesssim_n e^{-N^2T/O_n(1)} N A^{O_n(1)}.$$

Now we turn to estimating $u_n^\sharp$. The desired estimate (2.17) is an immediate consequence of the following more general assertion: if $\max\left(1, \frac{q}{n+1}\right) \le p \le \infty$, and either

$$3 < q < \infty, \quad \alpha_p < \alpha < 2\min\left(\frac{1}{p'}, \alpha_q\right), \quad 1 \le n \le q$$

or

$$u \text{ is axisymmetric}, \quad 2 < q \le 3, \quad \alpha_p < \alpha < \min\left(\frac{2}{p'}, (n+1)\Big(1 - \frac{2}{q}\Big) - \frac{1}{p}\right)$$

then

$$\|u_n^\sharp\|_{X^p_{\alpha;T/2}} \lesssim_n T^{(\alpha-\alpha_p)/2} A^{O_n(1)}. \tag{2.18}$$

It is straightforward to see that by taking $\alpha = 0$ and letting n be large depending on q, these conditions reduce to (4.1) and the hypotheses of (2.17). To prove (2.18), let us decompose $u_n^\sharp = u_n^{\sharp,1} + u_n^{\sharp,2}$ where

$$\begin{aligned} u_n^{\sharp,1}(t) &= -\int_{-T_n}^t e^{(t-t')\Delta}\mathbb{P}\,\mathrm{div}\, u_{n-1}^\sharp \otimes u_{n-1}^\sharp(t')dt', \\ u_n^{\sharp,2}(t) &= -2\int_{-T_n}^t e^{(t-t')\Delta}\mathbb{P}\,\mathrm{div}\, u_{n-1}^\flat \odot u_{n-1}^\sharp(t')dt' \end{aligned}$$

and claim that for $3 < q < \infty$, on the slightly larger interval $[-T_n, 0]$, we have the desired bound for $u_n^{\sharp,1}$ $n \ge 1$, if

$$p \ge \frac{q}{2n}, \quad \alpha_p < \alpha < 2\min\left(\frac{1}{p'}, 2\alpha_q\right)$$

and for $u_n^{\sharp,2}$ if

$$p \ge \frac{q}{n+1}, \quad -\frac{2}{p} < \alpha < 2\min\left(\frac{1}{p'}, \alpha_q\right).$$

The desired result (2.18) will follow by taking the intersection of these two conditions. The base cases where $n = 1$ are immediate from (2.5), Hölder's inequality, and (1.9) (in fact, $u_1^{\sharp,2} \equiv 0$). On the other hand, the induction on n involves fairly complicated relations between the parameters; thus the reader may find it elucidating to refer to Examples 2.9

and 2.10 which provide concrete examples of the iteration in the $q > 3$ and $q \leq 3$ cases respectively.

To induct, we assume (2.18) holds for the $u^{\sharp,1}$ part for some $n-1 \geq 1$. By (2.2) and Hölder's inequality, for $t \in [-T_n, 0]$,

$$\begin{aligned}\|r^\alpha u_n^{\sharp,1}(t)\|_{L_x^p} &\lesssim \int_{-T_n}^t (t-t')^{-\frac{1}{2}(1+\frac{6}{s}-\frac{3}{p}+2\beta-\alpha)} \|r^{2\beta} u_{n-1}^\sharp \otimes u_{n-1}^\sharp(t')\|_{L_x^{s/2}} dt \\ &\lesssim T^{\frac{1}{2}(\alpha-\alpha_p)} A^{O_n(1)}\end{aligned}$$

assuming there exists an $s \in [2, \infty]$ and $\beta \in \mathbb{R}$ such that

$$\alpha \leq 2\beta, \quad \frac{1}{p} \leq \frac{2}{s} \leq 1, \quad \alpha > -\frac{2}{p}, \quad \beta < 1 - \frac{2}{s}$$

which are required for Bernstein's inequality,

$$\frac{6}{s} - \frac{3}{p} + 2\beta - \alpha < 1$$

which is required to integrate in time, and

$$\frac{1}{s} \leq \frac{n}{q}, \quad \alpha_s < \beta < 2\min\left(\alpha_q, \frac{1}{s'}\right)$$

which are needed so that $u_{n-1}^\sharp \in X^s_{\beta;T_{n-1}}$. Letting $\beta = 1 - \frac{2}{s} - \epsilon$ for a positive ϵ which we will take to be as small as needed depending on the other parameters, the conditions on s reduce to

$$\max\left(\frac{1}{2p}, \frac{3}{q} - \frac{1}{2}\right) \leq \frac{1}{s} \leq \min\left(\frac{1}{2}, \frac{2-\alpha}{4}-, \frac{\alpha-\alpha_p}{2}, \frac{n}{q}\right)$$

which one can check is a nonempty interval if in addition to the relations on p, q, n, α in the hypothesis, we assume

$$\alpha \geq \max\left(1 - \frac{2}{p}, \frac{6}{q} - \frac{3}{p}\right). \tag{2.19}$$

Next we let $\beta = \alpha_s + \epsilon$ and the conditions on s become

$$\max\left(\frac{1}{2p}, \frac{2}{q} - \frac{1}{3}+\right) \leq \frac{1}{s} \leq \min\left(\frac{1}{2}, \frac{2-\alpha}{6}, \frac{n}{q}\right)$$

which is nonempty and intersects with $[0,1]$ under the hypotheses, along with the additional assumption

$$\alpha \le 2 - \frac{3}{p}. \tag{2.20}$$

Then it is easy to see that as long as $q \ge 3$, either (2.19) or (2.20) must be true. This completes the estimate of $u_n^{\sharp,1}$. Next we have by (2.5) and Holder, for $t \in [-T_n, 0]$,

$$\|r^\alpha u_n^{\sharp,2}(t)\|_{L^p_x} \lesssim \int_{-T_n}^t (t-t')^{-\frac{1}{2}(1+\frac{3}{s}-\frac{3}{p}+\beta-\alpha)} dt' \|u_{n-1}^\flat\|_{X^{\tilde{q}}_{0;T_n}} \|u_{n-1}^\sharp\|_{X^{\frac{s\tilde{q}}{\tilde{q}-s}}_{\beta;T_n}}$$

which implies the desired bound if there exist $\tilde{q}$, s, and β such that

$$\alpha \le \beta, \quad \frac{1}{p} \le \frac{1}{s}, \quad \alpha > -\frac{2}{p}, \quad \beta < 2 - \frac{2}{s}$$

for Bernstein,

$$\frac{3}{s} - \frac{3}{p} + \beta - \alpha < 1$$

to integrate in time, and

$$\frac{1}{\tilde{q}} \le \frac{1}{q}, \quad 0 \le \frac{1}{s} - \frac{1}{\tilde{q}} \le \min\left(1, \frac{n}{q}\right), \quad 1 - \frac{3}{s} + \frac{3}{\tilde{q}} < \beta < 2\min\left(\alpha_q, 1 - \frac{1}{s} + \frac{1}{\tilde{q}}\right)$$

to make $u_{n-1}^\flat \in X^{\tilde{q}}_{0;T_{n-1}}$ and $u_{n-1}^\sharp \in X^{\frac{s\tilde{q}}{\tilde{q}-s}}_{\beta;T_{n-1}}$. It suffices to use $\frac{1}{\tilde{q}} = \max(\frac{1}{s} - \frac{n}{q}, 0)$. Let us first take $\beta = \alpha$. One can compute that the conditions on s reduce to

$$\alpha > 1 - \frac{3n}{q} \tag{2.21}$$

and

$$\max\left(\frac{1}{p}, \frac{1-\alpha}{3} +\right) \le \frac{1}{s} \le \min\left(1 - \frac{\alpha}{2}-, \frac{1}{3} + \frac{1}{p}-, \frac{n+1}{q}\right)$$

which is a nonempty interval intersecting with $[0,1]$, assuming (2.21) and the original hypotheses on p, q, α, n. Now let us instead take $\beta = 1 - \frac{3n}{q} + \epsilon$. Then the conditions reduce to

$$\alpha \le 1 - \frac{3n}{q}, \quad \max\left(\frac{1}{p}, \frac{n}{q}\right) \le \frac{1}{s} \le \min\left(\frac{1}{2} + \frac{3n}{2q}-, \frac{\alpha}{3} + \frac{1}{p} + \frac{n}{q}-, \frac{n+1}{q}\right)$$

and one can verify that this is a nonempty interval intersecting with $[0, 1]$ if (2.21) fails. Thus there always exists a suitable s.

Next consider the case with $2 < q \leq 3$ and u axisymmetric. In the same manner as above, we actually prove the bound for the nth iterate on the slightly larger time interval $[-T_n, 0]$. As a base for the induction we have

$$\|u_1^\sharp\|_{X^p_{\alpha;T_1}} \lesssim \int_{-T_1}^t (t-t')^{-\frac{1}{2}(2+\alpha_p-\alpha)} \|r^{2\alpha_q} u \otimes u(t')\|_{L_x^{q/2}} dt' \lesssim T^{\frac{1}{2}(\alpha-\alpha_p)} A^2$$

assuming $p \geq \frac{q}{2}$, $-\frac{2}{p} < \alpha \leq 2 - \frac{4}{q} - \frac{1}{p}$, and $\frac{3}{p} + \alpha > 1$, all of which follow from the assumptions.

Next suppose we have the desired estimate for some $u_{n-1}^\sharp$, $n - 1 \geq 1$. Proceeding as in the $q > 3$ case, the result follows for $u^{\sharp,1}$ if there exists an $s \in [2, \infty]$ and $\beta \in \mathbb{R}$ such that

$$\alpha \leq 2\beta + \frac{2}{s} - \frac{1}{p}, \quad \frac{1}{p} \leq \frac{2}{s}, \quad \alpha > -\frac{2}{p}, \quad \beta < 1 - \frac{2}{s}$$

for the axisymmetric Bernstein inequality,

$$\frac{6}{s} - \frac{3}{p} + 2\beta - \alpha < 1$$

to integrate in time, and

$$\frac{1}{s} \leq \frac{n}{q}, \quad \beta < \frac{2}{s'}, \quad \alpha_s < \beta < n\Big(1 - \frac{2}{q}\Big) - \frac{1}{s}$$

so that we have the same bound for $u_{n-1}^\sharp$. First we let $\beta = \alpha_s + \epsilon$ for a sufficiently small (depending on α, q, n, etc.) $\epsilon > 0$. With the given conditions on α, p, n, these constraints reduce to

$$\max\left(\frac{1}{p}, 1 - n\Big(1 - \frac{2}{q}\Big)+, 0+\right) \leq \frac{2}{s} \leq \min\left(1 - \frac{1}{2}\Big(\alpha + \frac{1}{p}\Big), \frac{2n}{q}, 1\right)$$

which one can verify is a nonempty interval if we assume additionally that

$$\alpha \leq 2 - \frac{3}{p}. \tag{2.22}$$

Next we instead take $\beta = \frac{1}{2}\Big(\alpha + \frac{1}{p}\Big) - \frac{1}{s}$, and the conditions reduce to

$$\max\left(\frac{1}{2p}, \frac{1}{4}\Big(2 - \alpha - \frac{1}{p}\Big)+\right) \leq \frac{1}{s} \leq \min\left(1 - \frac{1}{2}\Big(\frac{1}{p} + \alpha\Big)-, \frac{1}{4}\Big(1 + \frac{2}{p}\Big)-, \frac{n}{q}, \frac{1}{2}\right)$$

which is a nonempty interval assuming $\alpha > -\frac{1}{p}$. Clearly if this fails, then instead we can conclude by (2.22).

Next, we have

$$\|r^\alpha u_n^{\sharp,2}(t)\|_{L_x^p} \lesssim \int_{-T_n}^{t} (t-t')^{-\frac{1}{2}(2+\frac{3}{s_1}+\frac{2}{s_2}-\frac{3}{p}-\frac{2}{q}+\beta-\alpha)} \|r^\beta u_{n-1}^\sharp(t')\|_{L_x^{s_1}} \times \|r^{1-\frac{2}{q}-\frac{1}{s_2}} u_{n-1}^\flat(t')\|_{L_x^{s_2}} dt'$$

which can be estimated if there exist s_1, s_2, and β such that

$$\frac{1}{p} \le \frac{1}{s_1} + \frac{1}{s_2}, \quad -\frac{2}{p} < \alpha \le 1 + \beta + \frac{1}{s_1} - \frac{1}{p} - \frac{2}{q}, \quad \beta < 1 + \frac{2}{q} - \frac{2}{s_1} - \frac{1}{s_2}$$

for (2.2),

$$\beta - \alpha + \frac{3}{s_1} + \frac{2}{s_2} < \frac{3}{p} + \frac{2}{q}$$

for integrability in time,

$$\frac{1}{s_1} \le \frac{n}{q}, \quad \beta < \frac{2}{s_1'}, \quad \alpha_{s_1} < \beta \le n\Big(1 - \frac{2}{q}\Big) - \frac{1}{s_1}$$

to control $u_{n-1}^\sharp$, and

$$\frac{1}{s_2} \le \frac{1}{q}, \quad -\frac{1}{s_2} < 1 - \frac{2}{q}$$

to control $u_{n-1}^\flat$. First we take $\beta = \alpha_{s_1} + \epsilon$, and the conditions reduce to the existence of s_1 and s_2 such that

$$\frac{1}{p} - \frac{1}{s_1} \le \frac{1}{s_2} \le \min\left(\frac{1}{2}\Big(\alpha - 1 + \frac{3}{p} + \frac{2}{q}\Big)-, \frac{1}{q}\right)$$

which one computes is nonempty and intersects $[0,1]$ assuming s_1 satisfies

$$\frac{1}{2}\max\left(\frac{1}{p}+, \frac{1}{p'} - \alpha+, \frac{2}{p}, 2 - (n+1)\Big(1 - \frac{2}{q}\Big)\right) \le \frac{1}{s_1} + \frac{1}{q} \le \min\left(1 - \frac{1}{2}\Big(\alpha + \frac{1}{p}\Big), \frac{n+1}{q}\right).$$

Such an s_1 exists in $[0,1]$ assuming

$$\alpha \le \min\left(2 - \frac{3}{q}, 2 - \frac{2}{q} - \frac{1}{p}\right) \tag{2.23}$$

in addition to the given assumptions.

Next we instead take $\beta = \alpha - 1 + \frac{2}{q} - \frac{1}{s_1} + \frac{1}{p}$. The conditions on $\frac{1}{s_2}$ reduce to

$$\max\left(0, \frac{1}{p} - \frac{1}{s_1}\right) \leq \frac{1}{s_2} \leq \min\left(2 - \alpha - \frac{1}{p} - \frac{1}{s_1}-, \frac{1}{2} + \frac{1}{p} - \frac{1}{s_1}, \frac{1}{q}\right)$$

which can be satisfied, along with the other constraints, as long as

$$\max\left(0, \frac{1}{p} - \frac{1}{q}, 1 - \frac{1}{q} - \frac{\alpha}{2} - \frac{1}{2p} +\right) \leq \frac{1}{s_1} \leq \min\left(1, \frac{1}{2} + \frac{1}{p}-, 2 - \alpha - \frac{1}{p}-, \frac{n}{q}\right).$$

There exists such an s_1 if, along with the given assumptions, we have

$$\alpha > 2\max\left(0, 1 - \frac{n}{q}\right) - \frac{2}{q} - \frac{1}{p}.$$

One computes that if this constraint fails, then instead we have (2.23). □

For the reader's convenience, we provide two examples of the iteration for estimating $u_n^\sharp$ in Proposition 2.7. In these special cases, it becomes routine to verify the many conditions at each step such as the hypotheses of Proposition 2.1. Moreover, one can more easily see how the iteration successively makes progress from (1.9) toward the claimed estimates.

Let us assume the more straightforward bounds (2.14) and the $n = 1$ case of (2.18) which follows directly from (1.9). In fact, in this case the upper bound required on α can be weakened slightly to $\alpha \leq 2\alpha_q$ since there is no $u_1^{\sharp,2}$ contribution.

Example 2.9 ($q > 3$ case)**.** *Let u be as in Proposition 2.7. By rescaling, we may assume $T = 1$. With $q = 8$, let us prove $\|u_3^\sharp\|_{X^2_{0;1/2}} \lesssim A^{O(1)}$. We will make use of the estimates*

$$\|u_1^\sharp\|_{X^4_{1/4+\epsilon;3/4}} \lesssim A^2, \quad \|u_n^\flat\|_{X^p_{0;T_{n+1}}} \lesssim A^{2^{n-1}}$$

for $p \geq 8$. We can fix, say, $\epsilon = 1/100$. Putting the first bound into Duhamel's principle using (2.5) *and Hölder's inequality, for $t \in [-3/4, 0]$,*

$$\|r^{4\epsilon} u_2^{\sharp,1}(t)\|_{L^3_x} \lesssim \int_{-3/4}^{t} (t - t')^{-1+\epsilon} \|r^{\frac{1}{2}+2\epsilon} u_1^\sharp \otimes u_1^\sharp(t')\|_{L^2_x} dt' \lesssim A^4.$$

Similarly for $u_2^{\sharp,2}$,

$$\|r^{4\epsilon}u_2^{\sharp,2}(t)\|_{L^3_x} \lesssim \int_{-3/4}^{t}(t-t')^{-\frac{5}{8}+\frac{3}{2}\epsilon}\|r^{\frac{1}{4}+\epsilon}u_1^\sharp \odot u_1^\flat(t')\|_{L^3_x}dt' \lesssim A^3.$$

so in total we have

$$\|u_2^\sharp\|_{X^3_{4\epsilon;3/4}} \lesssim A^4.$$

Again applying this with Duhamel's formula and Hölder's inequality, for $t \in [-5/8, 0]$,

$$\|u_3^{\sharp,1}(t)\|_{L^2_x} \lesssim \int_{-5/8}^{t}(t-t')^{-\frac{3}{4}-4\epsilon}\|r^{8\epsilon}u_2^\sharp \otimes u_2^\sharp(t')\|_{L^{3/2}_x}dt' \lesssim A^8.$$

Next we have

$$\|u_2^{\sharp,1}(t)\|_{L^{8/3}_x} \lesssim \int_{-3/4}^{t}(t-t')^{-\frac{15}{16}-\epsilon}\|r^{\frac{1}{2}+2\epsilon}u_1^\sharp \otimes u_1^\sharp(t')\|_{L^2_x}dt' \lesssim A^4$$

and

$$\begin{aligned}\|u_2^{\sharp,2}(t)\|_{L^{8/3}_x} &\lesssim \int_{-3/4}^{t}(t-t')^{-\frac{5}{8}-\frac{\epsilon}{2}}\|r^{\frac{1}{4}+\epsilon}u_1^\sharp \odot u_1^\flat(t')\|_{L^{8/3}_x}dt' \\ &\lesssim \|r^{\frac{1}{4}+\epsilon}u_1^\sharp\|_{X^4_{1/4+\epsilon;3/4}}\|u_1^\flat\|_{X^8_{0;3/4}} \lesssim A^3\end{aligned}$$

so

$$\|u_2^\sharp\|_{X^{8/3}_{0;3/4}} \lesssim A^4.$$

Finally,

$$\|u_3^{\sharp,2}(t)\|_{L^2_x} \lesssim \int_{-5/8}^{t}(t-t')^{-\frac{1}{2}}\|u_2^\sharp \odot u_2^\flat(t')\|_{L^2_x}dt' \lesssim \|u_2^\sharp\|_{X^{8/3}_{0;5/8}}\|u_2^\flat\|_{X^8_{0;5/8}} \lesssim A^6.$$

In conclusion,

$$\|u_3^\sharp\|_{X^2_{0;1/2}} \lesssim A^8.$$

The argument can be schematized as follows:

$$\begin{array}{ccccccc} & & & & X^3_{4\epsilon;3/4} & & \\ & & & \nearrow & & \searrow & \\ X^8_{5/8;1} & \Longrightarrow & X^4_{1/4+\epsilon;3/4} & & & & X^2_{0;1/2} \\ & & & \searrow & & \nearrow & \\ & & & & X^{8/3}_{0;3/4} & & \\ u_0^\sharp = u & & u_1^\sharp & & u_2^\sharp & & u_3^\sharp \end{array}$$

The arrows indicate that we used that $u^\sharp_{n-1}$ is in one space to prove that $u^\sharp_n$ is in the next space, and each column corresponds to a particular n. The main point of the iteration when $q > 3$ is to prove estimates in lower integrability spaces. One can see that the iteration makes progress by using the quadratic nonlinearity to reduce the exponent at each step. The bottleneck in doing so is the $u^{\sharp,2}$ contribution because of the limited range of p for which (2.14) *holds.*

Example 2.10 (2 < $q \leq 3$ axisymmetric case)**.** *Now we let $q = 5/2$ and set out to prove $\|u_3^\sharp\|_{X^{3/2}_{0;1/2}} \lesssim A^{O(1)}$. We will assume the estimates*

$$\|u_1^\sharp\|_{X^3_{\epsilon;3/4}} \lesssim A^2, \quad \|u_1^\flat\|_{X^\infty_{1/5;3/4}} \lesssim A, \quad \|u_2^\flat\|_{X^{5/2}_{-1/5;5/8}} A^2$$

again for some fixed small $\epsilon > 0$. For $t \in [-3/4, 0]$,

$$\|r^{4\epsilon} u_2^{\sharp,1}(t)\|_{L^3_x} \lesssim \int_{-3/4}^t (t-t')^{-1+\epsilon} \|r^{2\epsilon} u_1^\sharp \otimes u_1^\sharp(t')\|_{L^{3/2}_x} dt' \lesssim A^4$$

and

$$\begin{aligned} \|r^{4\epsilon} u^{\sharp,2}(t)\|_{L^3_x} &\lesssim \int_{-3/4}^t (t-t')^{-\frac{1}{10}+\frac{3}{2}\epsilon} \|r^{\frac{1}{5}+\epsilon} u_1^\sharp \odot u_1^\flat(t')\|_{L^3_x} dt' \\ &\lesssim \|r^\epsilon u_1^\sharp\|_{X^3_{\epsilon;3/4}} \|u_1^\flat\|_{X^\infty_{1/5;3/4}} \lesssim A^3 \end{aligned}$$

which imply

$$\|u_2^\sharp\|_{X^3_{4\epsilon;3/4}} \lesssim A^4.$$

Similarly,

$$\|r^{\frac{1}{5}+4\epsilon} u_2^{\sharp,1}(t)\|_{L^{15/4}_x} \lesssim \int_{-3/4}^t (t-t')^{-1+\epsilon} \|r^{2\epsilon} u_1^\sharp \otimes u_1^\sharp(t')\|_{L^{3/2}_x} dt' \lesssim A^4$$

and

$$\|r^{\frac{1}{5}+4\epsilon}u_2^{\sharp,2}(t)\|_{L_x^{15/4}} \lesssim \int_{-3/4}^{t}(t-t')^{-\frac{3}{5}+\frac{3}{2}\epsilon}\|r^{\frac{1}{5}+\epsilon}u_1^{\sharp}\odot u_1^{\flat}(t')\|_{L_x^3}dt'$$
$$\lesssim \|u_1^{\sharp}\|_{X^3_{\epsilon;3/4}}\|u_1^{\flat}\|_{X^{\infty}_{1/5;3/4}} \lesssim A^3$$

which imply

$$\|u_2^{\sharp}\|_{X^{15/4}_{1/5+4\epsilon;3/4}} \lesssim A^4.$$

Finally, for $t \in [-5/8, 0]$,

$$\|u_3^{\sharp,1}(t)\|_{L_x^{3/2}} \lesssim \int_{-5/8}^{t}(t-t')^{-\frac{1}{2}-\epsilon}\|r^{2\epsilon}u_2^{\sharp}\otimes u_2^{\sharp}(t')\|_{L_x^{3/2}}dt' \lesssim A^8$$

and

$$\|u_3^{\sharp,2}(t)\|_{L_x^{3/2}} \lesssim \int_{-5/8}^{t}(t-t')^{-\frac{1}{2}-2\epsilon}\|r^{4\epsilon}u_2^{\sharp}\odot u_2^{\flat}\|_{L_x^{3/2}} \lesssim \|u_2^{\sharp}\|_{X^{15/4}_{1/5+4\epsilon;5/8}}\|u_2^{\flat}\|_{X^{5/2}_{-1/5;5/8}}$$
$$\lesssim A^6$$

which imply

$$\|u_3^{\sharp}\|_{X^{3/2}_{0;1/2}} \lesssim A^8.$$

Below is the strategy of the iteration.

$$\begin{array}{ccccccc}
 & & & X^3_{4\epsilon;3/4} & & & \\
 & & \nearrow & & \searrow & & \\
X^{5/2}_{-1/2;1} & \Longrightarrow & X^3_{\epsilon;3/4} & & & & X^{3/2}_{0;1/2} \\
 & & \searrow & & \nearrow & & \\
 & & & X^{15/4}_{1/5+4\epsilon;3/4} & & & \\
u_0^{\sharp}=u & & u_1^{\sharp} & u_2^{\sharp} & & & u_3^{\sharp}
\end{array}$$

Here in the $q<3$ case, the main issue is that we require an estimate with $\alpha=0$, but we are only given (1.9) which has $\alpha_q<0$. The iteration above exploits the fact that (2.2) allows one to increase the power α at the cost of increasing the integrability exponent. We can pay this cost thanks to the exponent halving coming from the quadratic nonlinearity and Hölder's inequality.

2.4 Blowup procedure

We will make use of three slightly different consequences of the local energy equality for (1.2). The second is an extension of Lemma 2.2 in [19], now with the dependence on A made explicit. Define the local scale-invariant quantities

$$C(R, z_0) := R^{-\frac{d}{2}-\frac{1}{d+3}+1}\|u\|_{L^{2(d+3)/(d+1)}_{t,x}(Q(z_0,R))}$$

and

$$D(R, z_0) := R^{-\frac{d}{2}-\frac{1}{d+3}+1}\|p\|^{1/2}_{L^{(d+3)/(d+1)}_{t,x}(Q(z_0,R))}.$$

Here we use the standard notation for a parabolic cylinder,

$$Q(z, r) := [t - r^2, t] \times B(x, r)$$

where $r > 0$ and $z = (t, x)$ is a point in spacetime. For brevity, if $Q = Q(z_0, R)$, we write $C(Q)$ in place of $C(R, z_0)$. These quantities appear in [19], although here we have defined them slightly differently so they are 1-homogeneous in u.

Lemma 2.11. *Let u be a smooth solution of* (1.2) *satisfying* (1.9) *with $X = L^d(\mathbb{R}^d)$ on $[-T, 0]$, $r > 0$, and $I \subset [-T, 0]$. Then we have*

$$\sup_I \int_{B(r)} |u|^2 dx + \int_I \int_{B(r)} |\nabla u|^2 dxdt \leq A^{O(1)} r^{d-4}|I|, \tag{2.24}$$

$$\|u\|_{L^\infty_t L^2_x(Q(z_0,r/2))} + \|\nabla u\|_{L^2_{t,x}(Q(z_0,r/2))} \lesssim r^{\frac{d}{2}-1}(C(r, z_0) + D(r, z_0))A^{1/2}, \tag{2.25}$$

and, for $t_0 \leq t \leq t_0 + 10r^2$,

$$\begin{aligned}\int_{Q(z_0,r/2)} \frac{|u(t)|^2}{2}dx - \int_{Q(z_0,r/2)} \frac{|u(t_0)|^2}{2}dx \\ \lesssim \|\nabla u\|_{L^2_{t,x}(Q(z_0,r))}A^2 r^{\frac{d}{2}-1} + D(r, z_0)Ar^{d-2}.\end{aligned} \tag{2.26}$$

Proof. All three estimates are elementary applications of the local energy equality

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{|u|^2}{2}\psi dx+\int_{\mathbb{R}^d}|\nabla u|^2\psi dx=\int_{\mathbb{R}^d}\frac{|u|^2}{2}(\partial_t\psi+\Delta\psi+u\cdot\nabla\psi)+pu\cdot\nabla\psi dx,$$

along with Hölder's inequality, (1.9), integration by parts, and the Calderón-Zygmund estimate for the pressure. □

The following proposition is closely related to Proposition 3.1 in [19]. Their method of proof is by contradiction and uses a compactness argument to find suitable values η and ϵ. Thus such an approach does not give any information on how they depend on A or u more broadly.[3]

Proposition 2.12. *Let u be a smooth solution of* (1.2) *satisfying* (1.9) *with* $X = L^d(\mathbb{R}^d)$. *Then for any $\epsilon \leq A^{-d^3}$, if $z_0 \in [-T/2, 0] \times \mathbb{R}^d$, $\rho \leq T/4$, and*

$$C(\rho, z_0) + D(\rho, z_0) \leq \epsilon,$$

then

$$C(r, z_1) + D(r, z_1) \leq \epsilon A^{O(1)}$$

for any $z_1 \in Q(z_0, \rho/2)$ and $r \in (0, \rho/2)$.

As in [19], Proposition 2.12 is obtained by iteratively applying Lemma 2.13 below. The point is that given a lower bound $C(r, z_1)+D(r, z_1) > \epsilon$ in a small cylinder, the lemma implies the same lower bound in a cylinder dilated by a factor of A. This step can be iterated until it yields a cylinder Q' that is comparable in length to $Q(z_0, \rho)$, the ratio depending on A. Since Q' can be smaller than $Q(z_0, \rho)$, the scaling factors in the definition of C and D lead to the loss of $A^{O(1)}$.

[3]Unfortunately the proof in [19] also appears to be incorrect, as pointed out in a subsequent paper by Dong and Wang [20]. The author is grateful to an anonymous referee for JMFM for bringing this to our attention. Let us emphasize that the results presented here do not rely at all on the lemma in [19] containing the error.

Lemma 2.13. *Let ϵ, u, ρ, and z_0 be as in Proposition 2.12. Then, with $\eta = A^{-1}$,*

$$C(\rho, z_0) + D(\rho, z_0) \leq \epsilon, \tag{2.27}$$

implies

$$C(\eta\rho, z_0) + D(\eta\rho, z_0) \leq \epsilon. \tag{2.28}$$

Proof. We translate and rescale so that $z_0 = 0$ and $\rho = 1$. By (2.25),

$$\|u\|_{L_t^\infty L_x^2(Q(\frac{1}{2}))} + \|\nabla u\|_{L_{t,x}^2(Q(\frac{1}{2}))} \lesssim \epsilon A^{\frac{1}{2}}. \tag{2.29}$$

Fix the large frequency scale $N = \epsilon^{-\frac{1}{d}}$. By interpolation and Lemma 2.3,

$$\begin{aligned}\|P_{>N}u\|_{L_{t,x}^{2+\frac{4}{d+1}}(Q(\eta))} &\leq \sum_{M>N} \|P_M u\|^{\frac{d+1}{d+3}}_{L_t^2 L_x^{2\frac{d+3}{d+1}}(Q(\eta))} \|P_M u\|^{\frac{2}{d+3}}_{L_t^\infty L_x^{2\frac{d+3}{d+1}}(Q(\eta))} \\ &\lesssim \sum_{M>N} (M^{-\frac{3}{d+3}} \|P_M \nabla u\|_{L_{t,x}^2(Q(\frac{1}{4}))} + M^{-100d^3})^{\frac{d+1}{d+3}} \\ &\qquad \times (M^{\frac{d}{d+3}} \|P_M u\|_{L_t^\infty L_x^2(Q(\frac{1}{4}))} + M^{-100d^3})^{\frac{2}{d+3}}.\end{aligned}$$

Fix a spatial cutoff $\varphi \in C_c^\infty(B(1/2))$ with $\varphi \equiv 1$ in $B(1/3)$. By Lemma 2.4, Plancherel, and (2.29),

$$\begin{aligned}\sum_{M>N} \|P_M u\|^2_{L_t^\infty L_x^2(Q(\frac{1}{4}))} &\lesssim \sum_{M>N} \left(\|P_M(\varphi u)\|^2_{L_t^\infty L_x^2(Q(\frac{1}{4}))} + M^{-100d^3} A^2 \right) \\ &\lesssim (\epsilon^2 + N^{-100d^3}) A^2\end{aligned}$$

and by the same reasoning

$$\sum_{M>N} \|P_M \nabla u\|^2_{L_{t,x}^2(Q(\frac{1}{4}))} \lesssim (\epsilon^2 + N^{-100d^3}) A^2.$$

Therefore, by Hölder's inequality, the main term is

$$\begin{aligned}&\sum_{M>N} M^{-\frac{1}{d+3}} \|P_M \nabla u\|^{\frac{d+1}{d+3}}_{L_{t,x}^2(Q(\frac{1}{4}))} \|P_M u\|^{\frac{2}{d+3}}_{L_t^\infty L_x^2(Q(\frac{1}{4}))} \\ &\lesssim N^{-\frac{1}{d+3}} \left(\sum_{M>N} \|P_M \nabla u\|^2_{L_{t,x}(Q(\frac{1}{4}))} \right)^{\frac{1}{2}\frac{d+1}{d+3}} \left(\sum_{M>N} \|P_M u\|^2_{L_t^\infty L_x^2(Q(\frac{1}{4}))} \right)^{\frac{1}{d+3}} \\ &\lesssim N^{-\frac{1}{d+3}} \epsilon + N^{-50d^2} A.\end{aligned}$$

The remaining terms all involve the small global Bernstein error and can be estimated similarly to find

$$\|P_{>N}u\|_{L_{t,x}^{2+\frac{4}{d+1}}(Q(\eta))} \lesssim N^{-\frac{1}{d+3}}\epsilon + N^{-50d^2}A \lesssim C_0^{-1}\eta^{\frac{2+d}{2+\frac{4}{d+1}}-1}\epsilon.$$

To study the low frequencies, let $\phi \in C_c^\infty(Q(1/2))$ be a spacetime cutoff function satisfying $\phi \equiv 1$ in $Q(1/3)$. Using Duhamel's formula we decompose u into local and global parts,

$$u^l(t) := e^{(t+1)\Delta}((\phi u)(-1)) - \int_{-1}^t e^{(t-t')\Delta}\operatorname{div}(\phi(u\otimes u + p\,\mathrm{Id}))(t')dt', \quad u^g := u - u^l.$$

By Hölder, Lemma 2.2, (2.5), (2.29), fractional integration, and (2.27),

$$\begin{aligned}
\|P_{\le N}u^l\|_{L_{t,x}^{2+\frac{4}{d+1}}(Q(\eta))} &\lesssim \eta^{\frac{d+2}{2+\frac{4}{d+1}}}\|e^{(t+1)\Delta}((\phi u)(-1))\|_{L^\infty_{t,x}([-\eta^2,0]\times\mathbb{R}^d)} \\
&\quad + N^{\frac{d}{2}+\frac{1}{d+3}-2}\left\|\int_{-1}^t e^{(t-t')\Delta}\operatorname{div}(\phi(u\otimes u + p\,\mathrm{Id}))(t')dt'\right\|_{L_t^{2+\frac{4}{d+1}}L_x^{\frac{d(d+3)}{d^2-5}}([-\eta^2,0]\times\mathbb{R}^d)} \\
&\lesssim \eta^{\frac{d+2}{2+\frac{4}{d+1}}}\epsilon A^{\frac{1}{2}} + N^{\frac{d}{2}+\frac{1}{d+3}-1}\epsilon^2 \lesssim C_0^{-1}\eta^{\frac{2+d}{2+\frac{4}{d+1}}-1}\epsilon.
\end{aligned}$$

Next observe that $P_{\le N}u^g$ solves the heat equation in $Q(1/3)$ so by Hölder's inequality and well-known parabolic theory,

$$\|P_{\le N}u^g\|_{L_{t,x}^{2+\frac{4}{d+1}}(Q(\eta))} \lesssim \eta^{\frac{d+2}{2+\frac{4}{d+1}}}\|P_{\le N}u^g\|_{L^\infty_{t,x}(Q(\eta))} \lesssim \eta^{\frac{d+2}{2+\frac{4}{d+1}}}\|P_{\le N}u^g\|_{L_{t,x}^{2+\frac{4}{d+1}}(Q(\frac{1}{4}))}.$$

Clearly $P_{\le N}u^g = u - P_{>N}u - P_{\le N}u^l$. The first piece can be estimated using (2.27), while the other two we have already addressed. (Note that the estimates are unaffected by changing the domain to $Q(\frac{1}{4})$ except for the heat propagator part of $P_{\le N}u^l$; however even the worse bound $\epsilon A^{\frac{1}{2}}$ without the improvement from using Hölder on $Q(\eta)$ suffices.) In total,

$$C(\eta) \lesssim C_0^{-1}\epsilon.$$

Next we consider the pressure. From the decomposition

$$P_{>N}p = \Delta^{-1}\operatorname{div}\operatorname{div}P_{>N}\Big(2P_{\le N/5}u \odot P_{>N/5}u + (P_{>N/5}u)^{\otimes 2}\Big) =: \Pi_1 + \Pi_2,$$

we have

$$\|\Pi_1\|_{L^{1+\frac{2}{d+1}}_{t,x}(Q(\eta))} \lesssim \|P_{\leq N/5}u\|_{L^{2+\frac{4}{d+1}}_{t,x}(Q(2\eta))} \|P_{>N/5}u\|_{L^{2+\frac{4}{d+1}}_{t,x}(Q(2\eta))} + (\eta N)^{-50d^2}A^2$$

by Lemma 2.3 and (1.9). By the same calculations by which we estimated C above and the large choice of N, this implies

$$\|\Pi_1\|_{L^{1+\frac{2}{d+1}}_{t,x}(Q(\eta))} \lesssim C_0^{-2}\eta^{\frac{d+2}{1+\frac{2}{d+1}}-2}\epsilon^2.$$

For the other term we have

$$\|\Pi_2\|_{L^{1+\frac{2}{d+1}}_{t,x}(Q(\eta))} \lesssim \|P_{>N/5}u\|^2_{L^{2+\frac{4}{d+1}}_{t,x}(Q(\frac{1}{5}))} + N^{-50d^2}A^2 \lesssim C_0^{-2}\eta^{\frac{d+2}{1+\frac{2}{d+1}}-2}\epsilon^2$$

again by Lemma 2.3 and the calculations above.

Next we turn to the low frequencies. With $\varphi \in C_c^\infty(Q(\frac{1}{5}))$ a new spacetime cutoff satisfying $\varphi \equiv 1$ in $Q(\frac{1}{6})$, define

$$p^l := -\mathcal{N}\operatorname{div}\operatorname{div}(\varphi u \otimes u), \quad p^g := p - p^l$$

where $\mathcal{N}$ is the Newton potential. To estimate the local contribution we employ the paraproduct decomposition

$$P_{\leq N}p^l = -\mathcal{N}\operatorname{div}\operatorname{div} P_{\leq N}\Bigg(\varphi(P_{\leq 5N}u)^{\otimes 2} + \sum_{\substack{N'\sim N'' \\ \max(N',N'')>5N}} \varphi P_{N'}u \otimes P_{N''}u\Bigg) =: \Pi_3 + \Pi_4.$$

The calculations above imply that $P_{\leq N}u$ can be decomposed as $v + w$ where

$$\|v\|_{L^q_{t,x}(Q(\frac{1}{5}))} \leq C_0^{-1}\eta^{-\frac{9}{10}}\epsilon, \quad \|w\|_{L^{2+\frac{4}{d+1}}_{t,x}(Q(\frac{1}{5}))} \leq C_0^{-1}\eta^{\frac{d+2}{2+\frac{4}{d+1}}-1}\epsilon$$

for any $q \geq 1$. (For example, let w be the nonlinear part of $P_{\leq N}u^l$ and v the rest.) Thus, using the Calderón-Zygmund estimate for $\mathcal{N}$,

$$\begin{aligned}\|\Pi_3\|_{L^{1+\frac{2}{d+1}}_{t,x}(Q(\eta))} &\lesssim \eta^{\frac{d+2}{1+\frac{2}{d+1}}-\frac{1}{10}}\|\varphi v \otimes v\|_{L^{2q}_{t,x}([-\eta^2,0]\times\mathbb{R}^d)} + \eta^{\frac{d+2}{2+\frac{4}{d+1}}}\|\varphi v \odot w\|_{L^{2+\frac{4}{d+1}}_{t,x}([-\eta^2,0]\times\mathbb{R}^d)} \\ &\quad + \|\varphi w \otimes w\|_{L^{1+\frac{2}{d+1}}_{t,x}([-\eta^2,0]\times\mathbb{R}^d)} \\ &\lesssim C_0^{-2}\eta^{\frac{d+2}{1+\frac{2}{d+1}}-2}\epsilon^2\end{aligned}$$

where q is taken large but finite to avoid the unboundedness of $\mathcal{N}\,\mathrm{div}\,\mathrm{div}$ at the endpoint. By the calculations for $P_{>N}u$,

$$\|\Pi_4\|_{L_{t,x}^{1+\frac{2}{d+1}}(Q(\eta))} \lesssim \sum_{N'\gtrsim N} (N')^{-\frac{2}{d+3}}\epsilon^2 + (N')^{-100d^2}A^2 \leq C_0^{-2}\eta^{\frac{d+2}{1+\frac{2}{d+1}}-2}\epsilon^2.$$

Finally, observe that $P_{\leq N}p^g$ is harmonic in $Q(\frac{1}{6})$. Therefore

$$\|P_{\leq N}p^g\|_{L_{t,x}^{1+\frac{2}{d+1}}(Q(\eta))} \lesssim \eta^{\frac{d}{1+\frac{2}{d+1}}}\|P_{\leq N}p^g\|_{L_t^{1+\frac{2}{d+1}}L_x^\infty(Q(\eta))} \lesssim \eta^{\frac{d}{1+\frac{2}{d+1}}}\|P_{\leq N}p^g\|_{L_{t,x}^{1+\frac{2}{d+1}}(Q(\frac{1}{6}))}.$$

Then the decomposition $P_{\leq N}p^g = p - P_{>N}p - P_{\leq N}p^l$ along with the above estimates and (2.27) implies the desired bound. This completes the estimate of $D(\eta)$. □

2.5 Partial regularity

2.5.1 Epochs of regularity

An essential step in some of our arguments, for instance the proof of Proposition 4.6, consists of using a Carleman inequality (Proposition A.3) to show that concentrations of the solution near $x = 0$ imply additional concentration in regions far from the x_3-axis. Since the Carleman estimate demands some pointwise regularity of the solution, it is important that it be applied within an "epoch of regularity" which we construct in three dimensions in Proposition 2.15 for solutions bounded in the spaces $X^q_{\alpha;T}$ with (2.1). This is an extension of Proposition 3.1(iii) in [60] which we state first (for solutions in a slightly weaker space than the original, but without introducing any new difficulties for the reasons mentioned in Remark 2.8).

Proposition 2.14. *Let $u : [t_0 - T, t_0] \times \mathbb{R}^3 \to \mathbb{R}^3$ be a classical solution of* (1.2) *satisfying* (1.9) *with $X = L^{3,\infty}(\mathbb{R}^3)$. Then for any interval I in $[t_0 - T/2, t_0]$, there is a subinterval $I' \subset I$ with $|I'| \gtrsim A^{-O(1)}|I|$ such that*

$$\|\nabla^j u\|_{L^\infty_{t,x}(I\times\mathbb{R}^3)} \lesssim A^{O(1)}|I|^{-(j+1)/2}.$$

This only differs from the corresponding statement in [60] in the L^3 norm being weak, but it is easy to see that the proof holds without substantial modification.

On the other hand, additional effort is needed to handle the axial cases.

Proposition 2.15. *Proposition 2.14 holds as well with* $X = X^q_{\alpha;T}$ *and* (2.1).

Proof. By shifting time and rescaling, we may assume that $I = [0,1]$ and $[-1,1] \subset [t_0 - T, t_0]$.

For n sufficiently large, (2.17) implies that

$$\|u_n^\sharp\|_{L_t^\infty L_x^p([-1/2,1]\times\mathbb{R}^3)} \lesssim_p A^{O(1)}$$

for all $p \in [\min(q', \frac{q}{2}), 3)$. By differentiating the definition of $u_n^\sharp$ in time, we see that it satisfies

$$\partial_t u_n^\sharp + \mathbb{P}\operatorname{div}(u\otimes u - u_{n-1}^\flat \otimes u_{n-1}^\flat) - \Delta u_n^\sharp = 0.$$

Thus, defining

$$E_0(t) := \frac{1}{2}\int_{\mathbb{R}^3} |u_n^\sharp(x,t)|^2 dx, \quad E_1(t) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n^\sharp(x,t)|^2 dx,$$

we have the equality

$$\frac{d}{dt}E_0(t) + \int_{\mathbb{R}^3} u_n^\sharp \cdot \operatorname{div}(u\otimes u - u_{n-1}^\flat \otimes u_{n-1}^\flat - u_n^\sharp \otimes u_n^\sharp)dx + 2E_1(t) = 0,$$

using the fact that $\operatorname{div} u_n^\sharp = 0$ and therefore $\int u_n^\sharp \cdot \operatorname{div} u_n^\sharp \otimes u_n^\sharp = 0$. Then integrating in time, using (2.17) with $p = 2$, expanding the product $u\otimes u - u_n^\sharp \otimes u_n^\sharp = 2u_n^\flat \odot u_n^\sharp + u_n^\flat \otimes u_n^\flat$, integrating by parts, and applying Young's inequality,

$$\begin{aligned}\int_{-\frac{1}{2}}^1 E_1(t)dt &= A^{O(1)} + \int_{-\frac{1}{2}}^1 \int_{\mathbb{R}^3} \Big(\nabla u_n^\sharp : u_n^\flat \odot u_n^\sharp - \frac{1}{2} u_n^\sharp \cdot \operatorname{div}(u_n^\flat \otimes u_n^\flat - u_{n-1}^\flat \otimes u_{n-1}^\flat\Big) dxdt \\ &\le A^{O(1)} + \frac{1}{2}\int_{-\frac{1}{2}}^1 E_1(t)dt + 2\int_{-\frac{1}{2}}^1\int_{\mathbb{R}^3} \Big(8|u_n^\flat \odot u_n^\sharp|^2 \\ &\quad + \frac{1}{2}|u_n^\sharp||\operatorname{div}(u_n^\flat \otimes u_n^\flat - u_{n-1}^\flat \otimes u_{n-1}^\flat)|\Big) dxdt.\end{aligned}$$

Therefore, by Holder's inequality,

$$\begin{aligned}\int_{-\frac{1}{2}}^{1} E_1(t)dt &\lesssim A^{O(1)} + \|u_n^\flat\|^2_{L^\infty_{t,x}([-\frac{1}{2},1]\times\mathbb{R}^3)}\|u_n^\sharp\|^2_{L^\infty_t L^2_x([-\frac{1}{2},1]\times\mathbb{R}^3)} \\ &\quad + \left(\|u_{n-1}^\flat\|_{L^\infty_t L^{2p}_x([-\frac{1}{2},1]\times\mathbb{R}^3)} + \|u_n^\flat\|_{L^\infty_t L^{2p}_x([-\frac{1}{2},1]\times\mathbb{R}^3)}\right)^2 \|u_n^\sharp\|_{L^\infty_t L^{p'}_x([-\frac{1}{2},1]\times\mathbb{R}^3)} \\ &\lesssim A^{O(1)}.\end{aligned} \tag{2.30}$$

The above inequality is consistent with the hypotheses of (2.14) and (2.17) if we take $p = \max(q, \frac{q}{q-2})$. Plancherel's theorem then implies

$$\sum_N N^2 \|P_N u_n^\sharp\|^2_{L^2_t L^2_x([-\frac{1}{2},1]\times\mathbb{R}^3)} \lesssim A^{O(1)} \tag{2.31}$$

and then by Sobolev embedding,

$$\|u_n^\sharp\|_{L^2_t L^6_x([-\frac{1}{2},1]\times\mathbb{R}^3)} \lesssim A^{O(1)}. \tag{2.32}$$

Next, using the equation satisfied by $u_n^\sharp$, integration by parts and the identity $u_{n-1}^\sharp = u_n^\sharp + u_n^\flat - u_{n-1}^\flat$, we have

$$\begin{aligned}\frac{d}{dt}E_1 &= -\int_{\mathbb{R}^3} |\nabla^2 u_n^\sharp|^2 + \int_{\mathbb{R}^3} \Delta u_n^\sharp \cdot \operatorname{div}(u_{n-1}^\sharp \otimes u_{n-1}^\sharp + 2u_{n-1}^\flat \odot u_{n-1}^\sharp) \\ &= -\int_{\mathbb{R}^3} |\nabla^2 u_n^\sharp|^2 + \int_{\mathbb{R}^3} \Delta u_n^\sharp \cdot \operatorname{div}(u_n^\sharp \otimes u_n^\sharp + 2u_n^\flat \odot u_n^\sharp) \\ &\quad + \int_{\mathbb{R}^3} u_n^\sharp \cdot \Delta \operatorname{div}(u_{n-1}^\flat + u_n^\flat) \odot (u_n^\flat - u_{n-1}^\flat).\end{aligned}$$

Note that for a vector field u, $|\nabla^2 u|^2$ denotes the quantity $\partial_{ij}u_k\partial_{ij}u_k$. By Hölder's inequality, Sobolev embedding, interpolation, and Gagliardo-Nirenberg,

$$\begin{aligned}\|\operatorname{div} u_n^\sharp \otimes u_n^\sharp\|_{L^2_x(\mathbb{R}^3)} &\lesssim \|u_n^\sharp\|_{L^6_x(\mathbb{R}^3)}\|\nabla u_n^\sharp\|_{L^3_x(\mathbb{R}^3)} \lesssim \|\nabla u_n^\sharp\|^{3/2}_{L^2_x(\mathbb{R}^3)}\|\nabla u_n^\sharp\|^{1/2}_{L^6_x(\mathbb{R}^3)} \\ &\lesssim E_1(t)^{3/4}\|\nabla^2 u_n^\sharp\|^{1/2}_{L^2_x(\mathbb{R}^3)}.\end{aligned}$$

By Hölder's inequality, (2.14), and (2.17),

$$\begin{aligned}\|\operatorname{div} u_n^\flat \odot u_n^\sharp\|_{L^2_x(\mathbb{R}^3)} &\lesssim \|\nabla u_n^\flat\|_{L^\infty_x(\mathbb{R}^3)}\|u_n^\sharp\|_{L^2_x(\mathbb{R}^3)} + \|u_n^\flat\|_{L^\infty_x(\mathbb{R}^3)}\|\nabla u_n^\sharp\|_{L^2_x(\mathbb{R}^3)} \\ &\lesssim A^{O(1)}(1 + E_1(t)^{1/2})\end{aligned}$$

and

$$\|\Delta\operatorname{div}(u_{n-1}^\flat + u_n^\flat)\odot(u_n^\flat - u_{n-1}^\flat)\|_{L_x^p(\mathbb{R}^3)} \lesssim A^{O(1)}$$

where again $p = \max(q, \frac{q}{q-2})$. Combining these estimates for the nonlinearity with Young's inequality, Hölder's inequality, and (2.17),

$$\begin{aligned}\frac{d}{dt}E_1 &= -\frac{1}{2}\int_{\mathbb{R}^3}|\nabla^2 u_n^\sharp|^2 + O(E_1(t)^{3/2}\|\nabla^2 u_n^\sharp\|_{L_x^2(\mathbb{R}^3)} + A^{O(1)}(1+E_1(t)))\\ &\leq -\frac{1}{4}\int_{\mathbb{R}^3}|\nabla^2 u_n^\sharp|^2 + O(E_1(t)^3 + A^{O(1)}(1+E_1(t)).\end{aligned} \tag{2.33}$$

From (2.30), there exists a time $t_1 \in [0, \frac{1}{2}]$ such that $E_1(t_1) \lesssim A^{O(1)}$. A continuity argument then implies that there is an absolute constant C such that within the interval $I_0 = [t_1, t_1 + A^{-C}]$, we have $E_1(t) \leq A^C$. More generally we define the truncated intervals $I_j = [t_1 + \frac{j}{10}A^{-C}, t_1 + A^{-C}]$. Along with (2.33), this implies

$$\int_{I_0}\int_{\mathbb{R}^3}|\nabla^2 u_n^\sharp|^2 dxdt \lesssim A^{O(1)}. \tag{2.34}$$

Using this along with the bound on E_1 within the Gagliardo-Nirenberg inequality

$$\|u_n^\sharp\|_{L_x^\infty(\mathbb{R}^3)} \lesssim \|\nabla u_n^\sharp\|_{L_x^2(\mathbb{R}^3)}^{1/2}\|\nabla^2 u_n^\sharp\|_{L_x^2(\mathbb{R}^3)}^{1/2},$$

then applying Hölder's inequality in time and (2.14), yield

$$\|u\|_{L_t^4 L_x^\infty(I_0\times\mathbb{R}^3)} \leq \|u_n^\flat\|_{L_t^4 L_x^\infty(I_0\times\mathbb{R}^3)} + \|u_n^\sharp\|_{L_t^4 L_x^\infty(I_0\times\mathbb{R}^3)} \lesssim A^{O(1)}.$$

Duhamel's principle on I_0, (2.5), and Young's inequality give

$$\begin{aligned}\|u\|_{L_t^8 L_x^\infty(I_1\times\mathbb{R}^3)} &\lesssim \|e^{(t-t_1)\Delta}u(t_1)\|_{L_t^8 L_x^\infty(I_1\times\mathbb{R}^3)} + \left\|\int_{t_1}^t (t-t')^{-\frac{1}{2}}\|u(t')\|_{L_x^\infty(\mathbb{R}^3)}^2 dt'\right\|_{L_t^8(I_1)}\\ &\lesssim A^{O(1)},\end{aligned}$$

where we truncate the time interval to I_1 so the heat propagator in the linear term stays away from the initial time. Bootstrapping and truncating the interval one more time in the same manner, we arrive at

$$\|u\|_{L_{t,x}^\infty(I_2\times\mathbb{R}^3)} \lesssim A^{O(1)}. \tag{2.35}$$

We also have, by (2.34) and Sobolev embedding,

$$\|\nabla u_n^\sharp\|_{L_t^2 L_x^6(I_0\times\mathbb{R}^3)} \lesssim \|\nabla^2 u_n^\sharp\|_{L_t^2 L_x^2(I_0\times\mathbb{R}^3)} \lesssim A^{O(1)} \tag{2.36}$$

which we apply to the Duhamel formula

$$\nabla u(t) = e^{(t-t_1-\frac{1}{5}A^{-C})\Delta}\nabla u(t_1+\frac{1}{5}A^{-C}) - \int_{t_1+\frac{1}{5}A^{-C}}^{t} e^{-(t-t')\Delta}\nabla\mathbb{P}\operatorname{div} u\otimes u(t')dt'$$

for $t\in I_2$. Using (2.5), (2.35), and (2.14) and assuming $t\in I_3$,

$$\begin{aligned}\|\nabla u(t)\|_{L_x^\infty} &\lesssim A^{O(1)} + \int_{t_1+\frac{1}{5}A^{-C}}^{t} (t-t')^{-3/4}\|u\cdot\nabla u_n^\sharp(t')\|_{L_x^6} \\ &\quad + (t-t')^{-1/2}\|u\cdot\nabla u_n^\flat(t')\|_{L_x^\infty}dt' \\ &\lesssim A^{O(1)}\left(1+\int_{t_1}^{t}(t-t')^{-3/4}\|\nabla u_n^\sharp(t')\|_{L_x^6}dt'\right)\end{aligned}$$

and therefore

$$\|\nabla u\|_{L_t^4 L_x^\infty(I_3\times\mathbb{R}^3)} \lesssim A^{O(1)}$$

by fractional integration and (2.36). Finally, for $t\in I_4$, by this and (2.35),

$$\|\nabla u(t)\|_{L_x^\infty(\mathbb{R}^3)} \lesssim A^{O(1)} + \int_{t_1+\frac{3}{10}A^{-C}}^{t}(t-t')^{-1/2}\|u\|_{L_{t,x}^\infty(I_3\times\mathbb{R}^3)}\|\nabla u(t')\|_{L_x^\infty(\mathbb{R}^3)}dt'$$

and so

$$\|\nabla u\|_{L_{t,x}^\infty(I_4\times\mathbb{R}^3)} \lesssim A^{O(1)} \tag{2.37}$$

again by Young's inequality.

The estimates (2.35) and (2.37) imply regularity of the coefficients of the vorticity equation in $I_4\times\mathbb{R}^3$ and therefore the estimates for ω and $\nabla\omega$ follow by (4.17) and parabolic regularity. □

2.5.2 Annuli and slices of regularity

The first application of Proposition 2.12 is that the smallness of C and D implies good pointwise bounds on the solution. We state Proposition 2.16 as a more quantitative variant of Theorem 4.1 in [19].

Proposition 2.16. *Let u, z_0, and ρ be as in Proposition 2.12 and suppose that for every $z_1 \in Q(z_0, \rho_1/2)$, $\rho \in (0, \rho_1/2)$ we have*

$$C(\rho, z_1) + D(\rho, z_1) \leq \epsilon \leq A_1^{-1}.$$

Then, for $j = 0, 1, 2$,

$$\|\nabla^j u\|_{L^\infty_{t,x}(Q(z_0,\rho_1/4))} \leq A^{O(1)} \epsilon^{1/O(1)} \rho_1^{-1-j}.$$

Proof. Let us normalize $\rho_1 = 1$ and $z_0 = 0$. By the argument in the proof of Theorem 4.1 in [19] using the bound on p coming from (1.9), one finds

$$\|u\|_{L^\infty_{t,x}(Q(1/3))} \leq A^{O(1)} \epsilon^{1/O(1)}. \tag{2.38}$$

We may bootstrap the estimates for higher derivatives using Duhamel's formula. Let us fix a decreasing sequence of $O(1)$-many lengths $\frac{1}{3} > r_1 > r_2 > r_3 > \cdots > \frac{1}{4}$ satisfying $r_n - r_{n+1} = 1/O(1)$. For a frequency $N \gg 1$ to be specified, (2.38), (2.6), Lemma 2.3, and Duhamel's formula for (1.2) starting from $t = -1/3$ imply

$$\|P_N u\|_{L^\infty_{t,x}(Q(r_1))} \lesssim e^{-N^2/O(1)} NA + N^{-1} A^{O(1)} \epsilon^{1/O(1)} + N^{-50} A^2.$$

Clearly with N large enough, the first term (from the linear propagator) is negligible compared to the third (the global contribution to Bernstein). Therefore, again by Duhamel's formula, (2.38), and a paraproduct decomposition of $P_N(u \otimes u)$,

$$\begin{aligned}\|P_N u\|_{L^\infty_{t,x}(Q(r_2))} &\lesssim e^{-N^2/O(1)} NA + N^{-1} \|P_{\lesssim N} u \odot P_{\sim N} u\|_{L^\infty_{tx}(Q(r_1))} \\ &\quad + N^{-1} \sum_{N' \gtrsim N} \|P_{N'} u\|^2_{L^\infty_{tx}(Q(r_1))} + N^{-49} A^2 \\ &\lesssim N^{-1}(A^{O(1)} \epsilon^{1/O(1)} + N^{-49} A)(N^{-1} A^{O(1)} \epsilon^{1/O(1)} + N^{-50} A^2) \\ &\quad + N^{-1} \sum_{N' \gtrsim N} ((N')^{-1} A^{O(1)} \epsilon^{1/O(1)} + (N')^{-50} A^2) + N^{-50} N A^2 \\ &\lesssim N^{-2} A^{O(1)} \epsilon^{1/O(1)} + N^{-49} A^{O(1)}.\end{aligned}$$

Thus, once again by Duhamel's formula and (2.5), for any $N_0 > 0$,

$$\begin{aligned}\|\nabla u\|_{L^\infty_{t,x}(Q(r_3))} &\lesssim N_0 \|u\|_{L^\infty_{t,x}(Q(1/3))} + N_0^{-48} A + \sum_{N > N_0} (N \|P_N u\|_{L^\infty_{t,x}(Q(r_2))} + N^{-48} A) \\ &\leq N_0 A^{O(1)} \epsilon^{1/O(1)} + N_0^{-48} A^{O(1)}.\end{aligned}$$

By taking N_0 to be a suitable power of ϵ^{-1}, we arrive at

$$\|\nabla u\|_{L^\infty_{t,x}(Q(r_3))} \leq A^{O(1)} \epsilon^{1/O(1)}.$$

Proceeding in the same way, one can obtain the higher order estimates as well. □

Taking Propositions 2.12 and 2.16 together, we obtain the useful fact that if $C(Q) + D(Q) \leq A_1^{-1}$, then we have good pointwise bounds for u in $Q/2$. (Clearly we may also replace $Q/2$ with, say, $9Q/10$ by trivially modifying the proofs.) As an application, we prove the first partial regularity result. As discussed in more depth in §3.3, by letting the region of regularity expand in space (as opposed to taking, say, $Q_0 \times \mathbb{R}^{d-k}$ for some small $Q_0 \subset \mathbb{R}^k$), we obtain better estimates upon iterating unique continuation. We remark that we do not claim this to be the optimal result; indeed one should expect that regular regions exist that are unconstrained in up to three of the $d+2$ parabolic dimensions, (cf. epochs of regularity when $d = 3$ which are unbounded in all three spatial dimensions). In this case, the region is unbounded in only one spatial dimension, i.e., radially toward θ.

Proposition 2.17 (Slices of regularity). *Assume u is smooth and satisfies* (1.2) *and* (1.9) *with $X = L^d(\mathbb{R}^d)$ on $[-T, 0]$, $z_0 \in [-T/2, 0] \times \mathbb{R}^d$, and $R^2 \leq T/4$. Then there exist a direction $\theta \in S^{d-1}$ and a time interval $I \subset [t_0 - R^2, t_0]$ with $|I| = A_2^{-2} R^2$ such that within the slice*

$$S = I \times \{x \in \mathbb{R}^d : \mathrm{dist}(x, x_0 + \mathbb{R}_+\theta) \leq 10A_2^{-1}|(x - x_0) \cdot \theta|,\ |x - x_0| \geq 20R\} \subset [-T, 0] \times \mathbb{R}^d,$$

for $j = 0, 1, 2$, we have

$$\|\nabla^j u\|_{L^\infty_{t,x}(S)} \leq A_1^{-1} \left(\frac{R}{A_2}\right)^{-1-j}.$$

Proof. We normalize $R = 1$ and $z_0 = 0$, then apply Proposition 2.6 on the interval $[-2, 0]$. Let $\mathcal{S}_0$ be the collection of all spacetime regions of the form

$$I \times \{x \in \mathbb{R}^d : \mathrm{dist}(x, x_0 + \mathbb{R}_+\theta) \leq 20A_2^{-1}|(x - x_0) \cdot \theta|,\ |x - x_0| \geq 10\}$$

ranging over all $\theta \in S^{d-1}$ and $I = [-10A_2^{-2}k, -10A_2^{-2}(k-1)]$ where $k \in [1, A_2^2/10] \cap \mathbb{N}$. Clearly we may find a disjoint subcollection $\mathcal{S}_1$ containing $\gtrsim A_2^{d+1}$ such slices. We seek to find one where we can apply Propositions 2.12 and 2.16. To find a region where D is small, observe that by the Calderón-Zygmund estimate for $\mathrm{div}\,\mathrm{div}\,/\Delta$, Hölder's inequality, Sobolev embedding, and (2.9),

$$\|\Delta^{-1}\,\mathrm{div}\,\mathrm{div}\, u^\sharp \otimes u^\sharp\|_{L^1_t L^{\frac{d}{d-2}}_x([-1,0]\times\mathbb{R}^d)} \lesssim \|u^\sharp\|^2_{L^2_t L^{\frac{2d}{d-2}}_x([-1,0]\times\mathbb{R}^d)} \leq A^{O(1)}.$$

By interpolation with the $L^\infty_t L^{\frac{d}{2}}_x$ bound from (2.8),

$$\|\Delta^{-1}\,\mathrm{div}\,\mathrm{div}\, u^\sharp \otimes u^\sharp\|_{L^2_{t,x}([-1,0]\times\mathbb{R}^d)} \leq A^{O(1)}.$$

As a result, of the $\gtrsim A_2^{d+1}$ slices in $\mathcal{S}_1$, at least 99% must have

$$\|\Delta^{-1}\,\mathrm{div}\,\mathrm{div}\, u^\sharp \otimes u^\sharp\|_{L^2_{t,x}(S)} \leq A_1 A_2^{-\frac{d+1}{2}}.$$

Using (2.6) and Hölder's inequality, it is easy to see that the same can be said for $u^\flat \odot u^\sharp$ and $u^\flat \otimes u^\flat$. Let $\mathcal{S}_2 \subset \mathcal{S}_1$ be the collection of all such slices. Combining these estimates and applying Hölder's inequality, we have

$$D(Q) \leq A_1 A_2^{-\frac{3}{4}} \tag{2.39}$$

for every parabolic cylinder $Q \subset S$ of length $\sim A_2^{-1}$ and every $S \in \mathcal{S}_2$. By the same argument along with (2.9), most of the $S \in \mathcal{S}_2$ satisfy

$$\|\nabla u^\sharp\|_{L^2_{t,x}(S)} \leq A_1 A_2^{-\frac{d+1}{2}}, \tag{2.40}$$

so in fact the family $\mathcal{S}_3$ of slices satisfying both (2.39) and (2.40) has $\#(\mathcal{S}_3) \geq C_0^{-1} A_2^{d+1}$. Each of these slices occupies one of $\sim A_2^2$ time intervals, so by the pigeonhole principle, there is an interval $I = [t_0, t_0 + A_2^{-2}]$ which contains at least $C_0^{-2} A_2^{d-1}$ slices in $\mathcal{S}_3$. By (1.9), there must be one of these slices S_0 such that

$$\|u(t_0)\|_{L^d_x(S_{0,x})} \lesssim A_2^{-1+\frac{1}{d}} A$$

where $S_{0,x} \subset \mathbb{R}^d$ is the projection of S_0 to the spatial components. Then by Hölder's inequality, for every ball of length A_2^{-1} inside $S_{0,x}$,

$$\|u(t_0)\|_{L^2_x(B)} \lesssim A_2^{-\frac{d}{2}+\frac{1}{d}} A.$$

By (2.26), (2.39), and (2.40), for any $Q \subset S_0$ of length $A_2/2$,

$$\|u\|_{L^\infty_t L^2_x(Q/2)} \leq A_2^{-\frac{d}{2}+\frac{5}{8}} A_1.$$

Note that the bound (2.40) on $u^\sharp$ can be restricted to any such $Q \subset S_0$ and extended to the full solution u using (2.6) and Hölder's inequality. We conclude from the above and the local Gagliardo-Nirenberg inequality (see e.g., Lemma 2.1 in [19]) that

$$C(Q) \lesssim A_2^{\frac{d}{2}-1} \left(\|\nabla u\|_{L^2_{t,x}(Q)}^{\frac{d}{d+3}} \|u\|_{L^\infty_t L^2_x(Q)}^{\frac{3}{d+3}} + \|u\|_{L^\infty_t L^2_x(Q)} \right) \leq A_1 A_2^{-\frac{3}{8}}$$

for any $Q \subset S_0$ of length $A_2/2$. This along with (2.39) leads to the claimed bounds by Propositions 2.12 and 2.16. □

The next proposition should be compared to Proposition 3.1(vi) in [60]. In the case $d \geq 4$ it will be necessary locate even wider annuli where the solution enjoys good subcritical bounds, at the expense of needing to search a larger range of length scales. Note that in

[60] a key ingredient of the proof is the bounded total speed property which is unavailable in high dimensions. For this reason we proceed in the manner of Barker and Prange who use an ϵ-regularity criterion to find quantitative annuli of regularity; see [5, Section 6].

Proposition 2.18 (Annuli of regularity)**.** *Let u be a smooth solution of* (1.2) *satisfying* (1.9) *on* $[-10, 0]$. *For any $R_0 \geq 2$, there exists a scale $R \in [R_0, R_0^{\exp(A_4)}]$ such that for $j = 0, 1, 2$,*

$$\|\nabla^j u\|_{L^\infty_{t,x}([-1,0]\times\{R\leq|x|\leq R^{2A_4}\})} \leq A_4^{-1/O(1)}.$$

Proof. Since, by (1.9),

$$\int_{[-10,0]\times\{R_0\leq|x|\leq R_0^{\exp A_4}\}} (|u|^d + |p|^{d/2}) dxdt \leq A^{O(1)},$$

the pigeonhole principle implies that there exists R in the desired range such that

$$\int_{[-10,0]\times\{R/10\leq|x|\leq 10R^{2A_4}\}} (|u|^d + |p|^{d/2}) dxdt \leq A_4^{-\frac{1}{2}}$$

and therefore, by Hölder's inequality, for every parabolic cylinder $Q \subset [-10, 0] \times \{R/10 \leq |x| \leq 10R^{2A_4}\}$,

$$\|u\|_{L^{2+\frac{4}{d+1}}_{t,x}(Q)} + \|p\|_{L^{1+\frac{2}{d+1}}_{t,x}(Q)} \lesssim A_4^{-\frac{1}{2}}.$$

This implies that the region $[-1, 0] \times \{R \leq |x| \leq R^{2A_4}\}$ can be covered by a collection of cylinders $Q_j/2$ such that $Q_j \subset [-10, 0] \times \{R/10 \leq |x| \leq 10R^{2A_4}\}$ and $C(Q_j) + D(Q_j) \lesssim A_4^{-\frac{1}{2}}$. Successively applying Propositions 2.12 and 2.16 in all the Q_j yields the desired bounds. □

2.5.3 Regularity away from the axis

The following appeared in [41] refining a result in [46]. One can heuristically justify that these are likely the sharp pointwise bounds for critically bounded axisymmetric solutions, perhaps up to the ϵ power loss far from the axis.

Proposition 2.19 (Pointwise bounds away from the axis)**.** *Let u solve (1.2) on $[0,1]$ satisfying (1.9) with $X = L^{3,\infty}$. Then for every $\epsilon \in (0, 4/15)$, we have*

$$|\nabla^j u| \leq \left(r^{-1-j} + r^{-\frac{1}{3}+\epsilon}\right) A^{O_{\epsilon,j}(1)}$$

for each $t \in [1/2, 1]$. We also have

$$\|u\|_{L^p(\{r\geq 1\})} \leq A^{O_p(1)}$$

for each such t, and $p \in (3, \infty]$.

Proof. We first pick any $\alpha \in (1/3 - \epsilon/2, 1/3)$ and $c = c(j) > 0$ sufficiently small so that

$$(1-\alpha+j)c < \epsilon/2 \quad \text{and} \quad c < \alpha/(1-\alpha). \tag{2.41}$$

We also pick $n = n(j) \in \mathbb{N}$ sufficiently large so that

$$n \geq (2+j)\left(1+\frac{1}{c}\right) \tag{2.42}$$

We set $t_k \coloneqq 1/2 - (1/2)^k$ and we define a sequence of regions $\{x \in \mathbb{R}^d : r \geq R/2\} = \Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n = \{x \in \mathbb{R}^d : r \geq R\}$ such that $\operatorname{dist}(\Omega_i, \Omega_{i+1}) \geq R/2n$.

Given such a sequence of times we now consider the corresponding Picard iterates $u_k^\flat$, $u_k^\sharp$, for $k \in \{0, 1, \dots, n\}$.

`Step 1.` We show that

$$\|P_N u_k^\flat(t)\|_{L^\infty(\{r\geq R/2\})}, \|P_N u_k^\sharp(t)\|_{L^\infty(\{r\geq R/2\})} \lesssim R^{-\alpha} N^{1-\alpha} A^{O_k(1)} \tag{2.43}$$

for all $\alpha \in [0, \frac{1}{3})$, $R > 0$ and $t \in [t_k, 1]$, $k \geq 0$.

In fact, we first observe that Lemma 2.2 gives that

$$\|r^\alpha P_N u(t)\|_\infty \lesssim N^{1-\alpha} \|u(t)\|_{L^{3,\infty}} \lesssim N^{1-\alpha} A^{O(1)}. \tag{2.44}$$

Thus, since the first inequality above is valid for any axisymmetric function, it remains to note that the second inequality is also valid for each $u_k^\flat$, $u_k^\sharp$, on $[t_k,1]$, $k\geq 0$. Indeed, the case $k=0$ follows trivially, while the inductive step follows by applying Young's inequality (5.21) for weak L^p spaces, and Hölder's inequality (5.20) for Lorentz spaces

$$\begin{aligned}\|u_k^\flat(t)\|_{L^{3,\infty}} &\lesssim \|\Psi(t-t_k)\|_1\|u(t_k)\|_{L^{3,\infty}} + \int_{t_k}^t \|\nabla\Psi(t-t')\|_1 \|(u_{k-1}^\flat\otimes u_{k-1}^\flat)(t')\|_{L^{3/2,\infty}}\mathrm{d}t' \\ &\leq C_k A + C_k \|u_{k-1}^\flat\|^2_{L^\infty([t_{k-1},1];L^{3,\infty})}\int_{t_k}^t (t-t')^{-\frac12}\mathrm{d}t' \leq A^{O_k(1)}\end{aligned}$$

for $t\in[t_k,1]$, as required, where we also used the heat kernel bounds (5.18).

`Step 2.` We show that the inequality from Step 1 can be improved for $u_k^\sharp$ for large k, namely

$$\|P_N u_k^\sharp\|_{L^\infty([\frac12,1]\times\{r\geq R\})} \leq N A^{O_k(1)}((RN)^{-(k-1)\alpha} + N^{-(k-1)}) \tag{2.45}$$

for every $k\geq 1$ and $N\in 2^{\mathbb{N}}\cap[100^k\max(1,R^{-1}),\infty)$.

We will show that,

$$X_{k,N} \leq N^{-\frac45}A^{O_k(1)}((RN)^{-(k-1)\alpha} + N^{-(k-1)}), \tag{2.46}$$

for $k\geq 1$ and $N\geq 100^k\max(1,R^{-1})$, using induction with respect to k, where

$$X_{k,N} \coloneqq \|P_N u_k^\sharp\|_{L^\infty([t_{k+1},1];L^{5/3}(\Omega_k))}.$$

Then (2.45) follows by the local Bernstein inequality (2.3).

As for the base case $k=1$ we note that (2.5) gives that

$$\begin{aligned}\|P_N u_1^\sharp(t)\|_{5/3} &\lesssim \int_{t_1}^t \|P_N \mathrm{e}^{(t-t')\Delta}\mathbb{P}\,\mathrm{div}(u\otimes u)(t')\|_{5/3}\mathrm{d}t' \\ &\lesssim \int_{t_1}^t \mathrm{e}^{-(t-t')N^2/O(1)} N^{\frac65}\|(u\otimes u)(t')\|_{L^{\frac32,\infty}}\mathrm{d}t' \\ &\lesssim N^{\frac65}\|\mathrm{e}^{-tN^2/O(1)}\|_{L^1(t_1,1)}\|u\|^2_{L^{3,\infty}}\end{aligned}$$

for $t \in [t_1, 1]$. Thus

$$X_{1,N} \leq \|P_N u_1^\sharp\|_{L^\infty([t_2,1];L^{5/3})} \leq N^{-\frac{4}{5}} A^{O(1)}, \tag{2.47}$$

due to Hölder's inequality for Lorentz spaces (5.20).

As for the inductive step, we use the Duhamel formula for $u_k^\sharp$ and the local Bernstein inequality (2.3) to obtain

$$\begin{aligned}
\|P_N u_k^\sharp(t)\|_{L^{5/3}(\Omega_k)} &\lesssim \int_{t_k}^t \|P_N \mathrm{e}^{(t-t')\Delta}\mathbb{P}\operatorname{div}(u\otimes u - u_{k-1}^\flat \otimes u_{k-1}^\flat)\|_{L^{5/3}(\Omega_k)}\mathrm{d}t' \\
&\leq \int_{t_k}^t N\mathrm{e}^{-(t-t')N^2/O(1)}\mathrm{d}t' \left(\|P_N(u\otimes u - u_{k-1}^\flat \otimes u_{k-1}^\flat)\|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))}\right. \\
&\qquad \left. + (NR)^{-(k-1)\alpha}\|P_N(u\otimes u - u_{k-1}^\flat \otimes u_{k-1}^\flat)\|_{L^\infty([t_k,1];L^{5/3})}\right) \\
&\lesssim N^{-1}\left(\|P_N(u\otimes u - u_{k-1}^\flat \otimes u_{k-1}^\flat)\|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} + N^{\frac{1}{5}}(NR)^{-(k-1)\alpha}A^{O(1)}\right),
\end{aligned}$$

where we used the weak L^3 bound (5.4) and Lemma 2.2 for the $u\otimes u$ term and (2.6) for the $u_{k-1}^\flat \otimes u_{k-1}^\flat$ term. Thus we can use the paraproduct decomposition in the first term on the right-hand side to obtain

$$X_{k,N} \lesssim N^{-1}\|Y_1 + \cdots + Y_5\|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} + N^{-\frac{4}{5}}(NR)^{-(k-1)\alpha}A^{O_k(1)}, \tag{2.48}$$

where

$$\begin{aligned}
Y_1 &:= 2\sum_{N'\sim N} P_{N'}u_{k-1}^\sharp \odot P_{\leq N/100}u_{k-1}^\sharp, \\
Y_2 &:= \sum_{N_1\sim N_2 \gtrsim N} P_{N_1}u_{k-1}^\sharp \otimes P_{N_2}u_{k-1}^\sharp, \\
Y_3 &:= \sum_{N_1\sim N_2 \gtrsim N} P_{N_1}u_{k-1}^\flat \otimes P_{N_2}u_{k-1}^\sharp, \\
Y_4 &:= 2\sum_{N'\sim N} P_{N'}u_{k-1}^\flat \odot P_{\leq N/100}u_{k-1}^\sharp, \\
Y_5 &:= 2\sum_{N'\sim N} P_{\leq N/100}u_{k-1}^\flat \odot P_{N'}u_{k-1}^\sharp.
\end{aligned}$$

Using (2.43),

$$\|Y_1\|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} \lesssim \sum_{N'\sim N} X_{k-1,N'} \sum_{N'\lesssim N} R^{-\alpha}(N')^{1-\alpha}A^{O_k(1)}$$
$$\lesssim R^{-\alpha}N^{1-\alpha}A^{O_k(1)} \sum_{N'\sim N} X_{k-1,N'}$$

and

$$\|Y_2\|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} \lesssim R^{-\alpha}A^{O_k(1)} \sum_{N'\gtrsim N} (N')^{1-\alpha}X_{k-1,N'}.$$

Moreover, the frequency-localized bounds (2.5) for $u^\flat_{k-1}$ give that

$$\|Y_3\|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} \lesssim A^{O_k(1)} \sum_{N'\gtrsim N} \mathrm{e}^{-(N')^2/O_k(1)}N'X_{k-1,N'},$$

and (2.8), as well as boundedness of $P_{\le N/100}$ on $L^{5/3}$ give that

$$\|Y_4\|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} \lesssim A^{O_k(1)} \sum_{N'\sim N} \mathrm{e}^{-(N')^2/O_k(1)}N' \lesssim \mathrm{e}^{-N^2/O_k(1)}A^{O_k(1)}.$$

Finally, using boundedness of $P_{\le N/100}$ on L^∞ and (2.6) we obtain

$$\|Y_5\|_{L^\infty([t_k,1];L^{5/3}(\Omega_{k-1}))} \lesssim A^{O_k(1)} \sum_{N'\sim N} X_{k-1,N'}.$$

Combining these estimates into (2.48), we have shown

$$X_{k,N} \le A^{O_k(1)} \Bigg(((RN)^{-\alpha}+N^{-1}) \sum_{N'\sim N} X_{k-1,N'} + N^{-1}R^{-\alpha} \sum_{N'\gtrsim N} (N')^{1-\alpha}X_{k-1,N'}$$
$$+N^{-1} \sum_{N'\gtrsim N} \mathrm{e}^{-(N')^2/O_k(1)}N'X_{k-1,N'} + N^{-\frac{4}{5}}(NR)^{-(k-1)\alpha} + N^{-1}\mathrm{e}^{-N^2/O_k(1)} \Bigg). \tag{2.49}$$

Since the upper bounds on $X_{k-1,N'}$ provided by the inductive assumption (2.46) are comparable for all $N'\sim N$, up to constants depending only on k, we thus obtain that

$$\sum_{N'\sim N} X_{k-1,N'} \le A^{O_k(1)}N^{-\frac{4}{5}}\left((RN)^{-\alpha(k-2)} + N^{-k-2}\right),$$

$$R^{-\alpha}\sum_{N'\gtrsim N}(N')^{1-\alpha}X_{k-1,N'} \le A^{O_k(1)}R^{-\alpha}\sum_{N'\gtrsim N}(N')^{1-\alpha-\frac{4}{5}}\left((RN')^{-\alpha(k-2)} + (N')^{-(k-2)}\right)$$
$$\le A^{O_k(1)}N^{\frac{1}{5}}\left((RN)^{-\alpha(k-1)} + N^{-(k-1)}\right),$$

where, in the last line we used the fact that $(k-1)(1-\alpha)-4/5<0$ for any $k\geq 2$. A similar estimate for $\sum_{N'\gtrsim N} \mathrm{e}^{-(N')^2/O_k(1)}N'X_{k-1,N'}$ now allows us to deduce from (2.49) that

$$X_{k,N}\leq N^{-\frac{4}{5}}A^{O_k(1)}((RN)^{-(k-1)\alpha}+N^{-(k-1)}),$$

as required.

`Step 3.` We prove the claim.

We first consider the case $R\geq 100^{n/c}$, and we note that, by (2.43)

$$\|P_{N\leq R^c}\nabla^j u_n^\sharp\|_{L^\infty_{t,x}([\frac{1}{2},1]\times\{r\geq R\})}\leq \sum_{N\leq R^c} A^{O_n(1)}N^{1-\alpha+j}R^{-\alpha}\leq A^{O_n(1)}R^{-\alpha+(1-\alpha+j)c}\leq A^{O_n(1)}R^{-\frac{1}{3}+\varepsilon},$$

where we used the choice of $\alpha>1/3-\epsilon/2$ and the first property of our choice (2.41) of c in the last inequality. On the other hand for $N>R^c$ we can use (2.45) with $k=n$ to obtain arbitrarily fast decay in N. Comparing the terms on the right-hand side of (2.45) we see that $N^{-(n-2)}$ dominates $(RN)^{-(n-2)\alpha}$ if and only if $N\leq R^{\alpha/(1-\alpha)}$, which allows us to apply the decomposition

$$\begin{aligned}\|P_{N>R^c}\nabla^j u_n^\sharp\|_{L^\infty_{t,x}([\frac{1}{2},1]\times\{r\geq R\})}&\leq \sum_{R^c<N\leq R^{\alpha/(1-\alpha)}} A^{O_n(1)}N^{-n+2+j}\\&\quad+\sum_{N>R^{\alpha/(1-\alpha)}} A^{O_n(1)}N^{1+j}(RN)^{-(n-1)\alpha}\\&\leq A^{O_n(1)}R^{c(-n+2+j)}\\&\leq A^{O_n(1)}R^{-1-j}.\end{aligned}$$

where we used the second property of our choice (2.41) of c in the second inequality, and the choice (2.42) of n in the last inequality.

We now suppose that $R\leq 100^{n/c}$. The low frequencies can be estimated directly from the weak L^3 bound (5.4),

$$\|P_{\leq 100^{2n/c}R^{-1}}\nabla^j u\|_{L^\infty_{t,x}([\frac{1}{2},1]\times\{r\geq R\})} \lesssim_{n,c} A^{O(1)}R^{-1-j}.$$

On the other hand, for $N > 100^{2n/c}R^{-1}$ we have in particular $N > R^{\alpha/(1-\alpha)}$, which shows that the dominant term on the right-hand side of (2.45) is $(RN)^{-(n-2)\alpha}$, and so

$$\|P_{>100^{2n/c}R^{-1}}\nabla^j u_n^\sharp(t)\|_{L^\infty(\{r\geq R\})} \leq \sum_{N>100^{2n/c}R^{-1}} N^{1+j}A^{O_n(1)}(RN)^{-(n-1)\alpha} \leq A^{O_n(1)}R^{-1-j}$$

for every $t \in [1/2, 1]$, as desired. As for the estimate for $u^\flat$ we use (2.6) to obtain

$$\|\nabla^j u_n^\flat\|_{L^\infty(\{r\geq R\})} \leq R^{-1/3+\epsilon}\|r^{1/3-\epsilon}\nabla^j u_n^\flat\|_\infty \lesssim_\epsilon R^{-1/3+\epsilon}A^{O_{\epsilon,j}(1)},$$

as needed.

The estimate for $\|u\|_{L^p(\{r\geq 1\})}$ follows by an L^p analogue of Step 1, as well as applying the $X_{k,N}$ estimates (2.46) in the L^p variant of Step 3. □

We prove an axial version of Proposition 2.19. We do not attempt to be as precise because the bound is sure to deteriorate near the thresholds of (2.1), and it will not be necessary for the application in Chapter 4.

Proposition 2.20. *Suppose u is a classical solution of* (1.2) *obeying* (1.9) *with $X = X^q_{\alpha;T}$ and* (2.1)*. If $T' \in [0, T/2]$ and $R \geq (T')^{1/2}$, then in the region*

$$\Omega = \{(t,x) \in [t_0 - T', t_0] \times \mathbb{R}^3 : r \geq R\},$$

we have

$$\|\nabla^j u\|_{L^\infty_{t,x}(\Omega)} \lesssim (T')^{-\frac{j+1}{2}}\left(\frac{R^2}{T'}\right)^{-1/O_j(1)} A^{O(1)}.$$

for $j \geq 0$.

Proof. The strategy of the proof is similar to that of Proposition 2.19. Let us shift time and rescale to achieve $t_0 = 0$ and $T' = 1$. First we note that by (2.2), (2.14), and (1.9), if $q > 3$

and f is either $u_n^\flat$ or $u_n^\sharp$, we have

$$R^{1-\frac{3}{q}}\|P_N f\|_{L^\infty_{t,x}([-1/2,0]\times\{r\geq R/10\})} \lesssim \|r^{1-\frac{3}{q}}P_N f\|_{L^\infty_{t,x}([-1/2,0]\times\mathbb{R}^3)} \lesssim N^{\frac{3}{q}}A^{O_n(1)}$$

and if instead $q \leq 3$ and u is axisymmetric, then

$$R^{1-\frac{2}{q}}\|P_N f\|_{L^\infty_{t,x}([-1/2,0]\times\{r\geq R/10\})} \lesssim \|r^{1-\frac{2}{q}}P_N f\|_{L^\infty_{t,x}([-1/2,0]\times\mathbb{R}^3)} \lesssim N^{\frac{2}{q}}A^{O_n(1)}.$$

Let us therefore define $\gamma_q = \frac{3}{q}$ in the former case and $\gamma_q = \frac{2}{q}$ in the latter so that we always have

$$\|P_N u_n^\flat\|_{L^\infty_{t,x}([-1/2,0]\times\{r\geq R/10\})},\ \|P_N u_n^\sharp\|_{L^\infty_{t,x}([-1/2,0]\times\{r\geq R/10\})} \lesssim N^{\gamma_q}R^{-1+\gamma_q}A^{O_n(1)}. \tag{2.50}$$

Importantly, in either case, $\gamma_q < 1$. The point is that while staying uniformly away from the x_3-axis, this is a subcritical estimate and we can iteratively improve it with Duhamel's principle. Let us begin with a straightforward application of (2.4), Hölder's inequality, and (1.9) to obtain, for $t \in [-1/2, 0]$,

$$\begin{aligned}
\|r^{2\alpha_q}P_N u_1^\sharp(t)\|_{L_x^{q/2}(\mathbb{R}^3)} &\leq \int_{-1/2}^t \|r^{2\alpha_q}P_N e^{(t-t')\Delta}\mathbb{P}\operatorname{div}(u\otimes u)(t')\|_{L_x^{3/2}(\mathbb{R}^3)}dt' \\
&\lesssim \int_{-1/2}^t e^{-(t-t')N^2/20}N\|r^{2\alpha_q}u\otimes u\|_{L_x^{q/2}(\mathbb{R}^3)}dt' \\
&\lesssim N^{-1}A^2.
\end{aligned}$$

Next, we have

$$P_N u_n^\sharp(t) = \int_{-1/2}^t P_N e^{(t-t')\Delta}\operatorname{div}\tilde{P}_N(u_{n-1}^\sharp\otimes u_{n-1}^\sharp + 2u_{n-1}^\sharp\odot u_{n-1}^\flat)dt'.$$

By (2.3), for $t\in[-1/2,0]$, we have

$$\begin{aligned}
\|r^{2\alpha_q}P_N u_n^\sharp(t)\|_{L_x^{q/2}(r\geq(\frac{1}{2}-2^{-n})R)} &\lesssim_n N^{-1}\|r^{2\alpha_q}(Y_1+Y_2+Y_3+Y_4+Y_5)\|_{L_x^{q/2}(r\geq(\frac{1}{2}-2^{-(n-1)})R)} \\
&\quad + (NR)^{-50}N^{-1}\|r^{2\alpha_q}(u_{n-1}^\sharp\otimes u_{n-1}^\sharp + 2u_{n-1}^\sharp\odot u_{n-1}^\flat)\|_{L_x^{q/2}(\mathbb{R}^3)}
\end{aligned}$$

where we decompose $P_N(u^\sharp_{n-1} \otimes u^\sharp_{n-1} + 2u^\sharp_{n-1} \odot u^\flat_{n-1})$ with the paraproducts

$$\begin{aligned}
Y_1 &= 2 \sum_{N' \sim N} P_{N'} u^\sharp_{n-1} \odot P_{\leq N/100} u^\sharp_{n-1} \\
Y_2 &= \sum_{N_1 \sim N_2 \gtrsim N} P_{N_1} u^\sharp_{n-1} \otimes P_{N_2} u^\sharp_{n-1} \\
Y_3 &= \sum_{N_1 \sim N_2 \gtrsim N} P_{N_1} u^\flat_{n-1} \otimes P_{N_2} u^\sharp_{n-1} \\
Y_4 &= 2 \sum_{N' \sim N} P_{N'} u^\flat_{n-1} \odot P_{\leq N/100} u^\sharp_{n-1} \\
Y_5 &= 2 \sum_{N' \sim N} P_{\leq N/100} u^\flat_{n-1} \odot P_{N'} u^\sharp_{n-1}
\end{aligned}$$

By Hölder's inequality, (2.14), and (1.9), the global Bernstein term is bounded by $(NR)^{-50} N^{-1} A^{O_n(1)}$. Let $\Omega_n = [-1/2, 0] \times \{r \geq (\frac{1}{2} - 2^{-n})R\}$. Assuming $N \gtrsim_{n,c} R^c$, where $c > 0$ is a small constant depending on q, by Hölder's inequality, (2.2), (2.14), (1.9), (2.50), (2.15), and (2.16),

$$\begin{aligned}
\|r^{2\alpha_q} Y_1\|_{L^\infty_t L^{q/2}_x(\Omega_{n-1})} &\lesssim \sum_{N' \sim N} \|r^{2\alpha_q} P_{N'} u^\sharp_{n-1}\|_{L^\infty_t L^{q/2}_x(\Omega_{n-1})} \\
&\quad \times \sum_{N' \lesssim n} A^{O_n(1)} N' \max(N'R, 1)^{-1+\gamma_q} \\
&\lesssim A^{O_n(1)} N^{\gamma_q} R^{-1+\gamma_q} \sum_{N' \sim N} \|r^{2\alpha_q} P_{N'} u^\sharp_{n-1}\|_{L^\infty_t L^{q/2}_x(\Omega_{n-1})},
\end{aligned}$$

$$\|r^{2\alpha_q} Y_2\|_{L^\infty_t L^{q/2}_x(\Omega_{n-1})} \lesssim \sum_{N_1 \sim N_2 \gtrsim N} A^{O_n(1)} N_1^{\gamma_q} R^{-1+\gamma_q} \|r^{2\alpha_q} P_{N_2} u^\sharp_{n-1}\|_{L^\infty_t L^{q/2}_x(\Omega_{n-1})},$$

$$\|r^{2\alpha_q} Y_3\|_{L^\infty_t L^{q/2}_x(\Omega_{n-1})} \lesssim \sum_{N_1 \sim N_2 \gtrsim N} e^{-N_1^2/O_n(1)} N_1 A^{O_n(1)} \|r^{2\alpha_q} P_{N_2} u^\sharp_{n-1}\|_{L^\infty_t L^{q/2}_x(\Omega_{n-1})},$$

$$\begin{aligned}
\|r^{2\alpha_q} Y_4\|_{L^\infty_t L^{q/2}_x(\Omega_{n-1})} &\lesssim \sum_{N' \sim N} \|P_{N'} u^\flat_{n-1}\|_{X^q_{\alpha_q;1}} \|P_{\lesssim N} u^\sharp_{n-1}\|_{X^q_{\alpha_q;1}} \\
&\lesssim e^{-N^2/O_n(1)} A^{O_n(1)},
\end{aligned}$$

and

$$\|r^{2\alpha_q}Y_5\|_{L_t^\infty L_x^{q/2}(\Omega_{n-1})} \lesssim \sum_{N'\lesssim N} e^{-(N')^2/O_n(1)} N' A^{O_n(1)} \sum_{N'\sim N} \|r^{2\alpha_q}P_{N'}u_{n-1}^\sharp\|_{L_t^\infty L_x^{q/2}(\Omega_{n-1})}$$
$$\lesssim A^{O_n(1)} \sum_{N'\sim N} \|r^{2\alpha_q}P_{N'}u_{n-1}^\sharp\|_{L_t^\infty L_x^{q/2}(\Omega_{n-1})}.$$

In total,

$$\begin{aligned}\|r^{2\alpha_q}P_N u_n^\sharp\|_{L_t^\infty L_x^{q/2}(\Omega_n)} &\lesssim A^{O_n(1)}((NR)^{1-\gamma_q}+N)^{-1}\sum_{N'\sim N}\|r^{2\alpha_q}P_N u_{n-1}^\sharp\|_{L_t^\infty L_x^{q/2}(\Omega_{n-1})}\\ &\quad + R^{-1+\gamma_q}N^{-1}A^{O_n(1)}\sum_{N_1\gtrsim N} N_1^{\gamma_q}\|r^{2\alpha_q}P_{N_1}u_{n-1}^\sharp\|_{L_t^\infty L_x^{q/2}(\Omega_{n-1})}\\ &\quad + N^{-1}A^{O_n(1)}e^{-N^2/O_n(1)}.\end{aligned}$$

Iteratively applying this, we find

$$\|r^{2\alpha_q}P_N u_n^\sharp\|_{X^{q/2}_{2\alpha_q;1}(r\geq R/2)} \lesssim_n A^{O_n(1)}N^{-1}\min((NR)^{1-\gamma_q},N)^{-n+1}, \tag{2.51}$$

noting that the assumption $N \gtrsim_{n,c} R^c$ implies

$$e^{-N^2/O_n(1)} \lesssim ((NR)^{1-\gamma_q},N)^{-n+1}.$$

In order to make use of (2.51), we take the Littlewood-Paley decomposition of $\nabla^j u_n^\sharp$ and apply (2.2) and (2.3) to find

$$\begin{aligned}\|\nabla^j u_n^\sharp\|_{L^\infty_{t,x}(\Omega)} &\lesssim \|P_{\leq R^{-1}}\nabla^j u_n^\sharp\|_{X^\infty_{0;1}} + \sum_{R-1<N\lesssim_{n,c}R^c} N^j\|P_N u_n^\sharp\|_{L^\infty_{t,x}(\Omega)}\\ &\quad + \sum_{N\gtrsim_{n,c}R^c}\Big(N^{2+j}\|P_N u_n^\sharp\|_{X^{q/2}_{2\alpha_q;1}(r\geq R/2)} + (NR)^{-50}N^{1+j}\|P_N u_n^\sharp\|_{X^q_{\alpha_q;1}}\Big).\end{aligned}$$

Thanks to (2.2), (2.14), and (1.9), the first term is bounded by $R^{-1-j}A^{O_n(1)}$. The global Bernstein term is estimated the same way, and summing the geometric series, we obtain the bound $A^{O_n(1)}R^{-40}$. For the intermediate frequency term, we apply (4.13) and sum the geometric series to find

$$\sum_{R^{-1}<N\lesssim_{n,c}R^c} N^j\|P_N u_n^\sharp\|_{L^\infty_{t,x}(\Omega)} \lesssim_{n,c} A^{O_n(1)}R^{-1+\gamma_q+(\gamma_q+j)c}.$$

For the high frequency term, we have to split the sum once again depending on how the minimum is attained in (2.51).

$$\begin{aligned}\sum_{N \gtrsim_{n,c} R^c} N^{2+j} \|P_N u_n^\sharp\|_{X^{q/2}_{2\alpha_q;1}(r \geq R/2)} &\lesssim \sum_{R^c \lesssim_{n,c} N \lesssim R^{\frac{1}{\gamma_q}-1}} A^{O_n(1)} N^{j-n+2} \\ &\quad + \sum_{N \gtrsim R^{\frac{1}{\gamma_q}-1}} A^{O_n(1)} N^{1+j} (NR)^{-(1-\gamma_q)(n-1)} \\ &\lesssim_{n,c} A^{O_n(1)} R^{\min(\frac{1}{\gamma_q}-1, c)(j-n+2)}\end{aligned}$$

where $j = 0, 1, 2, 3$, assuming $n > 10/(1-\gamma_q)$ in order to make the series summable. By taking n and c^{-1} sufficiently large depending on q, all the powers on R can be made uniformly negative, that is to say

$$\|\nabla^j u_n^\sharp\|_{L^\infty_{t,x}(\Omega)} \lesssim A^{O_n(1)} R^{-1/O_n(1)}.$$

Moreover, by essentially the same argument we used for (2.50), we have

$$\|\nabla^j u_n^\flat\|_{L^\infty_{t,x}(\Omega)} \lesssim A^{O_n(1)} R^{-(1-\gamma_q)}.$$

Since $u = u_n^\flat + u_n^\sharp$ and $\gamma_q < 1$, this proves the estimates for $\nabla^j u$. $\square$

CHAPTER 3

Regularity in dimensions four and higher

3.1 Introduction

The contents of this chapter appeared in the author's paper [47]. We prove a quantitative blowup rate analogous to (1.7) for $d \geq 4$, answering a question of Tao, see Remark 1.6 in [60]. As in [60], we assume for convenience that u is a classical solution, meaning it is smooth with derivatives in $L_t^\infty L_x^2([0,T] \times \mathbb{R}^d)$. Since our results depend quantitatively on only $\|u\|_{L_t^\infty L_x^d}$, they can in principle be extended, for instance to the Leray-Hopf class as in [22].

Theorem 3.1. *Suppose u is a classical solution of* (1.2) *that blows up at $t = T_*$ and $d \geq 4$. Then*

$$\limsup_{t \uparrow T_*} \frac{\|u(t)\|_{L_x^d(\mathbb{R}^d)}}{(\log\log\log\log \frac{1}{T_*-t})^c} = \infty$$

for a constant $c = c(d) > 0$ depending only on the dimension.

This is a straightforward consequence of our other main theorem which asserts that a solution satisfying the critical bound (1.9) with

$$X := L_x^d(\mathbb{R}^d)$$

is regular; in particular we can quantify its subcritical norms in terms of A. Let us take A to be at least 2.

Theorem 3.2. *If u is a classical solution of* (1.2) *on* $[0,T]$ *satisfying* (1.9) *with $d \geq 4$, then*

$$\|\nabla^j u(t)\|_{L_x^\infty(\mathbb{R}^d)} \leq \exp\exp\exp\exp(A^C) t^{-\frac{1+j}{2}}$$

for $t \in (0, T]$, where $C = C(j, d)$ depends only on $j \geq 0$ and the dimension.

Remark 3.3. *Using ideas from [46], particularly Proposition 8, it is possible to improve the bounds in Theorems 3.1 and 3.2 if some mild symmetry assumptions are made on u. For example, suppose u is axisymmetric*[1] *about the $x_3, x_4, \ldots, x_d$-plane. When $d = 4$, one* log *and one* exp *can be removed from Theorems 3.1 and 3.2 respectively. When $d \geq 5$, we may remove two* logs *and two* exps. *In the latter case, in the proof of Proposition 3.7, we find the desired concentration at length scale $\ell = A^{-O(1)}$ using the slightly improved energy bound* (2.10)*, while when $d = 4$, we resort to pigeonholing the energy over $A^{O(1)}$-many length scales which yields an ℓ as small as* $\exp(-A^{O(1)})$. *An argument similar to Proposition 8 in [46] allows one to avoid losing additional exponentials when locating annuli of regularity as in Proposition 2.18.*

Let us summarize why the approach in [60] breaks down in greater than three dimensions. The first set of difficulties arises when one would use the "bounded total speed" property, i.e., control on $\|u\|_{L^1_t L^\infty_x}$, see Proposition 3.1(ii) in [60]. One expects (for example, based on the heuristics following Proposition 9.1 in [59]) that this property fails when $d \geq 4$. In other words, one cannot expect any kind of "speed limit" for elements convected by u. Instead, we derive a procedure to propagate concentrations of the velocity and pressure from fine to coarse scales, encapsulated in Proposition 2.12, which is a quantitative version of Lemma 3.2 in [19]. From this we can extract several important results including an ϵ-regularity criterion (Proposition 2.16) and the backward-propagation lemma (Proposition 3.4).

The second and more significant challenge in high dimensions is due to the lack of quantitative epochs of regularity as in Proposition 3.1(iii) in [60]. In the qualitative analysis, it suffices to use epochs of regularity for which one has absolutely no lower bound on the length, nor any explicit upper bound on $|u|$, $|\nabla u|$, etc. (For example, see the use of Proposi-

[1] *By this we mean the following: when regarded in the coordinate system which consists of polar coordinates (r, θ) in the x_1, x_2-plane and Cartesian coordinates in the rest, we have $u(x) = R_\theta(u(R_{-\theta}x))$ where R_θ denotes counterclockwise rotation by θ in the x_1, x_2-plane.*

tion 2.4 in the proof of Proposition 5.3 in [19].) This becomes a problem when one needs to propagate concentrations of vorticity through space and into a distant annulus of regularity, as the width of the time interval on which one has regularity determines the lower bound one can extract from unique continuation for the heat equation. We will remedy this by substituting spacetime partial regularity in place of epochs of regularity. This creates some new difficulties; first that when one propagates a high frequency concentration of the solution backward in time, a priori there is no guarantee that the resulting concentration has any of its $L^2_{t,x}$ mass inside the regular region. There is a particular fractal arrangement of concentrations in spacetime which is consistent with this obstruction; indeed the objective of Proposition 3.7 is to locate a scale where we may rule it out.

The second difficulty faced when propagating the vorticity using only spacetime partial regularity is the following: the usual Carleman inequality for unique continuation has as its domain a large ball in space (compared to the length of the time interval); however we wish to propagate the vorticity for a great distance through a thin spacetime slice. We are able to accomplish this without the bounds suffering too badly (losing only one additional exponential compared to the $d = 3$ case) by repeatedly applying the Carleman inequality in a series of moving and expanding balls lying in an expanding slice of spacetime. We show that the iteration of unique continuation accelerates exponentially away from the initial vorticity concentration. The positive feedback loop this creates is essential for arriving at the claimed bounds, as unique continuation through a uniformly thin slice would lead to an unbounded number of logarithms and exponentials in Theorems 3.1 and 3.2.

3.2 High dimensional back propagation

Next we prove a high-dimensional analogue of Proposition 3.1(v) in [60]. The proof given there is obtained by iterating a lemma for very short back-propagation, with the bounded total speed property (Proposition 3.1(ii) in [60]) preventing the sequence of concentrations from traveling too far through space. Although the bounded total speed is unlikely to hold when $d \geq 4$, Proposition 2.12 is a suitable replacement.

Proposition 3.4. *Suppose u is smooth and satisfies* (1.2) *and* (1.9) *with $X = L^d(\mathbb{R}^d)$ on $[-T, 0]$ where $T \geq 100$. If $N_0 \geq 10A_1$ and*

$$|P_{N_0} u(0)| \geq A_1^{-1} N_0,$$

then there exist $z_1 \in [-1, -A_2^{-1}] \times B(A_2)$ and $N_1 \in [A_2^{-1}, A_2]$ such that

$$|P_{N_1} u(z_1)| \geq A_2^{-1}.$$

Proof. Using Lemma 2.5 to deduce that there must be a parabolic cylinder about $z = 0$ where we still have the lower bound on $|P_{N_0} u|$, we have

$$A_1^{-1} N_0 r^{\frac{(d+1)(d+2)}{2(d+3)}} \leq \|P_{N_0} u\|_{L^{2+\frac{4}{d+1}}_{t,x}(Q(r))} \lesssim \|u\|_{L^{2+\frac{4}{d+1}}_{t,x}(Q(A_1^3 r))} + A_1^{-50} r^{\frac{(d+1)(d+2)}{2(d+3)} - 1} A$$

with $r = A_1^{-2} N_0^{-1}$, using Lemma 2.3. Rearranging, this implies

$$C(A_1 N_0^{-1}, 0) \geq A_1^{-3d}. \tag{3.1}$$

Because $N_0 \geq 10A_1$, we can apply Proposition 2.12 in the contrapositive to find

$$C(1, 0) + D(1, 0) \geq A_1^{-4d}.$$

Suppose first that $C(1, 0) \geq \frac{1}{2} A_1^{-4d}$. Using some large parameter M to be specified, we split u into three pieces to estimate $C(0, 1)$: low frequencies

$$\|P_{<M^{-1}} u\|_{L^{2+\frac{4}{d+1}}_{t,x}(Q(1))} \lesssim \|P_{<M^{-1}} u\|_{L^\infty_{t,x}([-1,0]\times\mathbb{R}^d)} \lesssim \frac{A}{M},$$

intermediate frequencies

$$\|P_{M^{-1} \leq \cdot \leq M} u\|_{L^{2+\frac{4}{d+1}}_{t,x}(Q(1))} \lesssim \log(M) \max_{M^{-1} \leq N \leq M} \|P_N u\|_{L^{2+\frac{4}{d+1}}_{t,x}(Q(1))},$$

and high frequencies

$$\|P_{>M}u\|_{L_{t,x}^{2+\frac{4}{d+1}}(Q(1))} \lesssim \sum_{N>M}\Big(\|P_N u^\flat\|_{L_{t,x}^\infty([-1,0]\times\mathbb{R}^d))} + N^{\frac{d}{d+3}}\|P_N u^\sharp\|_{L_{t,x}^2([-1,0]\times\mathbb{R}^d)}^{1-\frac{2}{d+3}}\|P_N u^\sharp\|_{L_t^\infty L_x^2([-1,0]\times\mathbb{R}^d)}^{\frac{2}{d+3}}\Big).$$

Here we have used the decomposition from Proposition 2.6 on, say, $[-2,0]\times\mathbb{R}^d$ followed by Lemma 2.3 and Hölder's inequality in space and interpolation in time. For the first term, by (2.7),

$$\sum_{N>M}\|P_N u^\flat\|_{L_{t,x}^\infty([-1,0]\times\mathbb{R}^d)} \lesssim M^{-50}A^{O(1)}.$$

For the second, by Hölder's inequality, Plancherel, (2.8), and (2.9),

$$\begin{aligned}
&\sum_{N>M} N^{-\frac{1}{d+3}}(N\|P_N u^\sharp\|_{L_{t,x}^2([-1,0]\times\mathbb{R}^d)})^{1-\frac{2}{d+3}}\|P_N u^\sharp\|_{L_t^\infty L_x^2([-1,0]\times\mathbb{R}^d)}^{\frac{2}{d+3}} \\
&\lesssim M^{-\frac{1}{d+3}}\left(\sum_N N^2\|P_N u^\sharp\|_{L_{t,x}^2([-1,0]\times\mathbb{R}^d)}^2\right)^{\frac{1}{2}-\frac{1}{d+3}}\left(\sum_N \|P_N u^\sharp\|_{L_t^\infty L_x^2([-1,0]\times\mathbb{R}^d)}^2\right)^{\frac{1}{d+3}} \\
&\le M^{-\frac{1}{d+3}}A^{O(1)}.
\end{aligned}$$

Combining the above estimates, we conclude

$$\frac{1}{2}A_1^{-4d} \le C(1,0) \lesssim \frac{A}{M} + \log(M)\max_{M^{-1}\le N\le M}\|P_N u\|_{L_{t,x}^\infty(Q(1))} + M^{-\frac{1}{d+3}}A^{O(1)}.$$

With $M = A_1^{O(1)}$, we obtain $z_1 \in Q(1)$ and $N_1 \in [A_2^{-1}, A_2]$ such that

$$|P_{N_1}u(z_1)| \ge A_1^{-O(1)}.$$

Suppose instead that $D(1,0) \ge \frac{1}{2}A_1^{-4d}$. By Hölder's inequality, Lemma 2.3, and (1.9), also using the fact that $p = -\Delta^{-1}\operatorname{div}\operatorname{div}(u\otimes u)$, we have

$$\|P_{<10M^{-1}}p\|_{L_{t,x}^{1+\frac{2}{d+1}}(Q(1))} \lesssim \|P_{<10M^{-1}}p\|_{L_{t,x}^\infty(\mathbb{R}^d)} \lesssim M^{-2}A^2.$$

To handle the intermediate and high frequencies, we use the paraproduct decomposition

$$\begin{aligned}P_{\geq 10M^{-1}}(u \otimes u) &= P_{\geq 10M^{-1}}\big(2(P_{<M^{-1}}u) \odot (P_{M^{-1}\leq \cdot \leq M}u) + 2u \odot (P_{>M}u) + (P_{M^{-1}\leq \cdot \leq M}u)^{\otimes 2}\big)\\ &= \Pi_1 + \Pi_2 + \Pi_3.\end{aligned}$$

For the first term, by Hölder's inequality, Lemma 2.2, and (1.9),

$$\begin{aligned}\|\Delta^{-1}\operatorname{div}\operatorname{div}\Pi_1\|_{L_{t,x}^{1+\frac{2}{d+1}}(Q(1))} &\lesssim \|P_{<M^{-1}}u\|_{L^\infty_{t,x}([-1,0]\times\mathbb{R}^d)}\|P_{M^{-1}\leq \cdot \leq M}u\|_{L^\infty_t L^d_x([-1,0]\times\mathbb{R}^d)}\\ &\lesssim A^2 M^{-1}.\end{aligned}$$

Next, by Proposition 2.6, Hölder's inequality, (1.9), (2.7), and estimating $P_{>M}u^\sharp$ using Plancherel and (2.9) as above, we have

$$\begin{aligned}\|\Delta^{-1}\operatorname{div}\operatorname{div}\Pi_2\|_{L_{t,x}^{1+\frac{2}{d+1}}(Q(1))} &\lesssim A\Big(\|P_{>M}u^\flat\|_{L^\infty_{t,x}([-1,0]\times\mathbb{R}^d)} + \|P_{>M}u^\sharp\|_{L_{t,x}^{2+\frac{4}{d+1}}([-1,0]\times\mathbb{R}^d)}\Big)\\ &\lesssim M^{-\frac{1}{d+3}}A^{O(1)}.\end{aligned}$$

Finally, by Hölder's inequality, Lemma 2.3, and (1.9),

$$\|\Delta^{-1}\operatorname{div}\operatorname{div}\Pi_3\|_{L_{t,x}^{1+\frac{2}{d+1}}(Q(1))} \lesssim \|P_{M^{-1}\leq \cdot \leq M}u\|^2_{L_{t,x}^{2+\frac{4}{d+1}}([-1,0]\times B(M^2))} + M^{-50}A^2.$$

In total,

$$\frac{1}{2}A_1^{-4d} \leq D(0,1) \lesssim M^{-\frac{1}{2}}A + M^{-\frac{1}{2(d+3)}}A^{O(1)} + \log(M)\max_{N\in[M^{-1},M]}\|P_N u\|_{L^\infty_{t,x}(Q[-1,0]\times B(M^2))}.$$

Once again with $M = A_1^{O(1)}$, we obtain N_1 and z_1 with the claimed properties. Finally we address the possibility that this t_1 falls in $[-A_2^{-1}, 0]$ instead of the desired interval. By the fundamental theorem of calculus and Lemma 2.5,

$$|P_{N_1}u(t_1 - A_2^{-1}, x_1)| \geq A_3^{-O(1)} - O(N_1^3 A^2 A_2^{-1})$$

which implies we can redefine t_1 to be in $[-1, -A_2^{-1}]$ while maintaining the lower bound on $|P_{N_1}u|$. □

3.3 Iterated unique continuation

Clearly the Carleman inequality Proposition A.3 is incompatible with the geometry of Proposition 2.17 since $B(r)$ would have to be contained in the thin slice in order to guarantee (A.2), while simultaneously we need $r^2 \gg t_0$ in order for the first error term to be suppressed. Instead we iteratively apply the Carleman inequality outward in space, starting near the vertex of the slice. The point is that as the iteration proceeds, the center for the Carleman inequality moves further in the θ direction, so r can be taken to be larger, which makes the Carleman inequality stronger. Thus combining Propositions 2.17 and 3.5 leads to a feedback loop which leads to substantially better estimates; specifically, only $\sim \log(R_2/R_1)$ iterations of Proposition A.3 (by way of Lemma 3.6) are needed[2] to propagate a concentration from length scale R_1 to R_2.

Proposition 3.5 (Iterated unique continuation Carleman inequality). *Suppose $T_1 > 0$, $0 < \eta \leq C_0^{-1}$, and u is smooth on S with*

$$\|\nabla^j u\|_{L^\infty_{t,x}(S)} \leq (\eta T_1)^{-1-\frac{j}{2}}, \quad |Lu| \leq \frac{|u|}{C_0 \eta T_1} + \frac{|\nabla u|}{(C_0 \eta T_1)^{\frac{1}{2}}} \quad \forall (t,x) \in S \tag{3.2}$$

for $j = 0, 1$, where, for some direction $\theta \in S^{d-1}$,

$$S = [-\eta T_1, 0] \times \{x \in \mathbb{R}^d : |x| > 10 T_1^{\frac{1}{2}}, \operatorname{dist}(x, \mathbb{R}_+ \theta) \leq \eta |x \cdot \theta|\}.$$

Moreover, assume that for every $t \in [-\eta T_1, 0]$, we have

$$\int_{B(R_0\theta, \eta^5 R_0)} |u(t)|^2 dx \geq \epsilon T_1^{\frac{d}{2}-2}$$

[2] If instead one were to iterate the Carleman inequality through a region of the form $Q_0 \times \mathbb{R}^{d-k}$ for some small $Q_0 \subset \mathbb{R}^k$, one would need a number of iterations on the order of R_2/R_1. This would lead to an extra exponential in the vorticity lower bound, which would in turn require us to ensure a much smaller error when the backward uniqueness Carleman inequality is applied in the proof of Proposition 3.8. It would be necessary then to find a much larger annulus of regularity in Proposition 2.18 which would result (rather unsatisfyingly) in tower exponential bounds in Theorem 3.2.

where $20T_1^{\frac{1}{2}} \leq R_0 \leq \eta^{-2}T_1^{\frac{1}{2}}$ *and* $\epsilon \leq \eta^8$*. Then for every* $t \in [-\eta T_1/2, 0]$ *and* $R \geq 2R_0$*, we have*

$$\int_{B(R\theta,\eta^5 R)} |u(t)|^2 dx \geq \epsilon^{(R/R_0)^{\eta^{-4}}} T_1^{\frac{d}{2}-2}.$$

Given the following lemma, Proposition 3.5 will follow by iteration.

Lemma 3.6. *Assume* u*,* T_1*, and* η *are as in Proposition 3.5 and that there is some* $R \geq 20T_1^{\frac{1}{2}}$ *and* $a \in (\frac{1}{2}, 1)$ *such that for every* $t \in [-aT_1, 0]$*,*

$$\int_{B(R\theta,\eta^5 R)} |u(t)|^2 dx \geq \epsilon_0 T_1^{\frac{d}{2}-2}$$

where

$$\epsilon_0 \leq \min(\eta^8, (R^2/T_1)^{-50d\eta}, e^{-2000d\eta^4 R^2/T_1}).$$

Then for every $t \in [-aT_1 + 2\eta^5 R^2 \log^{-1}\frac{1}{\epsilon_0}, 0]$*,*

$$\int_{B(R'\theta,\eta^5 R')} |u(t)|^2 dx \geq \epsilon_0^{\eta^{-2}} T_1^{\frac{d}{2}-2}$$

where $R' := (1+\eta^3)R$*.*

Proof of Proposition 3.5. Let us normalize $T_1 = 1$. One iterates Lemma 3.6 on the time intervals $[-a_k, 0]$ for $k = 0, 1, \ldots, n$, where $n = \lceil \log_{1+\eta^3}(R/R_0) \rceil$. Specifically, the kth application of the lemma is centered at the point $R_k\theta \in \mathbb{R}^d$ and uses the lower bound ϵ_k, where

$$\epsilon_k = \epsilon^{\eta^{-2k}}, \quad R_k = R_0(1+\eta^3)^k, \quad a_k = \eta - 2\sum_{i=0}^{k} \eta^5 R_i^2 \log^{-1}\frac{1}{\epsilon_i}.$$

One computes that

$$a_k = \eta - 2\eta^5 R_0^2 \log^{-1}\frac{1}{\epsilon}\sum_{i=0}^{k}(\eta+\eta^4)^{2i} \geq \eta - 4\eta^5 R_0^2 \log^{-1}\frac{1}{\epsilon}.$$

Recall that $R_0 \le \eta^{-2}$ and $\epsilon \le \eta^8$. Thus, with η sufficiently small, $a_k \ge \frac{\eta}{2}$ so the claimed bound holds on $[-\frac{\eta}{2}, 0]$. The final lower bound resulting from the iteration is given by

$$\epsilon_n = \epsilon^{\eta^{-2\lceil \log_{1+\eta^3}(R/R_0)\rceil}} \ge \epsilon^{\eta^{-2}(R/R_0)^{\frac{\log \eta^{-2}}{\log(1+\eta^3)}}}.$$

With η sufficiently small, we have $\frac{\log \eta^{-2}}{\log(1+\eta^3)} \le \eta^{-\frac{7}{2}}$ and $\eta^{-2} \le (R/R_0)^{\eta^{-\frac{1}{2}}}$, using that $R \ge 2R_0$. Thus $\epsilon_n \ge \epsilon^{(R/R_0)^{\eta^{-4}}}$ as claimed. □

Proof of Lemma 3.6. Again, we rescale so that $T_1 = 1$. Fix any $t' \in [-a + 2\eta^5 R^2 \log^{-1} \frac{1}{\epsilon_0}, 0]$. We apply Proposition A.3 to the function

$$(t, x) \mapsto u(t' - t, x + R'\theta)$$

on the time interval $[0, T_c]$ with the parameters

$$T_c = \min(\eta/2, \eta^5 R^2), \quad r = \eta^2 R, \quad t_0 = \eta^5 R^2 \log^{-1} \frac{1}{\epsilon_0}, \quad t_1 = \eta^{15} R^2 \log^{-1} \frac{1}{\epsilon_0}.$$

Clearly (3.2) implies (A.2) is satisfied. Consider the three terms in the Carleman inequality which takes the form $Z \le X + Y$. For the left-hand side, since $B(R'\theta, \eta^2 R/2) \supset B(R\theta, \eta^5 R)$,

$$Z \gtrsim t_0 T_c^{-1} \int_{B(R'\theta, \eta^2 R/2)} |u|^2 e^{-|x-R'\theta|^2/4t_0} dx \gtrsim \max(\eta^4 R^2, 1) \log^{-1}\left(\frac{1}{\epsilon_0}\right) \epsilon_0^{1+\eta^5/4} \ge \epsilon_0^2,$$

using that η and ϵ_0 are small, $R \ge 20$, and $\eta \ge \epsilon_0^{\frac{1}{8}}$. Next, by (3.2),

$$X \le \epsilon_0^{\eta^{-1/1000}} \eta^{2+2d} R^d$$

which is negligible compared to Z due to the constraint $\eta \le C_0^{-1}$. For the remaining term in the Carleman inequality,

$$Y \le \epsilon_0^{-\eta^{-2}} \int_{|x-R'\theta| \le \eta^2 R} |u(t', x)|^2 e^{-|x-R'\theta|^2/4t_1} dx.$$

By (3.2), the contribution to this term from the region where $|x - R'\theta| > \eta^5 R'$ is negligible compared to Z:

$$\epsilon_0^{-\eta^{-2}} \int_{\eta^6 R' < |x-R'\theta| \le \eta^2 R} |u(t', x)|^2 e^{-|x-R'\theta|^2/4t_1} dx \lesssim \eta^{2d-2} R^d \epsilon_0^{\eta^{-3/4} - \eta^{-2}} \ll \epsilon_0^2,$$

using that $\epsilon_0 \le R^{-100d\eta}$. Thus Z is bounded by the contribution to Y from $B(R'\theta, \eta^6 R')$ which proves the lemma. □

3.4 Main propositions

Next we proceed to the main propositions which will lead to the theorems claimed in this chapter. The philosophy is similar to [60] but will face additional obstacles in higher dimensions without quantitative epochs of regularity. As a result, given a spacetime point where u has a high frequency concentration, it is far from clear that the vorticity lower bound implied by Proposition 3.4 intersects at all with a spacetime region where the solution is regular, let alone an entire epoch $I \times \mathbb{R}^d$ as in the three-dimensional case. From a qualitative perspective, since ω is locally $L^2_{t,x}$ and the measure of the spacetime set where $|\nabla^j u| \lesssim \ell^{-1-j}$ shrinks to zero as $\ell \to 0$, there must be some small ℓ and cylinders $Q' \subset\subset Q$ of length $\sim \ell$ such that $|\nabla^j u| \lesssim \ell^{-1-j}$ holds in Q while $\int_{Q'} |\omega|^2 dxdt$ is bounded from below (see Figure 3.1). The problem is that in order to prove a quantitative theorem, we need an effective lower bound on this ℓ.

As one sees in the proof of Proposition 3.7, the worst-case scenario is that at each small scale ℓ, there are $\sim \ell^{-d+2}$ parabolic cylinders of length $\sim \ell$ where $\int |\omega|^2 dxdt \gtrsim \ell^{d-2}$, and in the complement the solution obeys $|\nabla^j u| \lesssim \ell^{-1-j}$. At each scale, this fractal configuration is consistent with the energy inequality. We rule out this scenario in dimensions $d \geq 5$ by applying the improved energy bound (2.10) at a sufficiently small scale $\ell = A^{-C}$. In $d = 4$ we cannot quite use this improvement and are forced to take ℓ as small as $\exp(-A^C)$. Here the idea is that each scale ℓ contributes roughly a fixed amount to the energy. A significant fraction of the contribution comes from the frequencies around ℓ^{-1}, so by summing over many scales we can contradict this scenario.

Note that the exponential smallness of ℓ when $d = 4$ does not affect the final estimates because it contributes in parallel with exponentials appearing at other points in the argument.[3]

[3]It is conceivable that the $d \geq 5$ case can be handled using the same energy pigeonholing approach, although it is less straightforward because of the spatial overlaps of the concentrations caused by the fact that Q' is a factor δ smaller than Q. As a result ℓ would depend exponentially on δ^{-1} which would cause

Proposition 3.7 (Backward propagation into a regular region)**.** *Suppose u is a classical solution of* (1.2) *on* $[t_0 - T, t_0]$ *satisfying* (1.9) *with* $X = L^d(\mathbb{R}^d)$*, and that there are* $x_0 \in \mathbb{R}^d$ *and* $N_0 > 0$ *such that at the point* $z_0 = (t_0, x_0)$*,*

$$|P_{N_0} u(z_0)| \geq A_1^{-1} N_0.$$

Then for any $T_1 \in [A_1^2 N_0^{-2}, T/100]$*, there exist* $\ell > 0$ *and* $Q = Q(z_0', \ell/2) \subset [-T_1, -A_2^{-1} T_1] \times B(A_3 T_1^{\frac{1}{2}})$ *such that*

$$\|\nabla^j u\|_{L^\infty_{t,x}(Q)} \leq A_2^{-1} \ell^{-j-1} \tag{3.3}$$

for $j = 0, 1, 2$ *and*

$$\|\omega\|_{L^2_{t,x}(Q')} \geq A_3^{-O(1)} (\delta \ell)^{\frac{d}{2}+1} T_1^{-1} \tag{3.4}$$

where $Q' = Q(z_0' - (\ell^2/8, 0), \delta\ell)$*. We may take* $\delta = A_4^{-1}$ *and* ℓ *such that*

$$\begin{aligned} &\ell \in [\exp(-A_4), A_4^{-1}], \quad && d = 4, \\ &\ell = A_4^{-2d-1}, && d \geq 5. \end{aligned} \tag{3.5}$$

Proof. Without loss of generality we may let $z_0 = 0$ and $T_1 = 1$. Let us begin with the case $d \geq 5$. By Proposition 3.4, there exists a point $z_1 \in [-1, -A_2^{-1}] \times B(A_2)$ and a frequency $N_1 \in [A_2^{-1}, A_2]$ such that

$$|P_{N_1} u(z_1)| \geq A_2^{-1}.$$

Combining this with Lemma 2.5, we find that the lower bound persists in a parabolic cylinder:

$$|P_{N_1} u(z)| \gtrsim A_2^{-1}, \quad \forall z \in Q(z_1, A_2^{-4}). \tag{3.6}$$

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problems in the proof of Proposition 3.8, as the smallness of δ is necessary to create favorable geometry for the Carleman estimates. It is preferable for other reasons to have ℓ depend polynomially on A; for example see Remark 3.3.

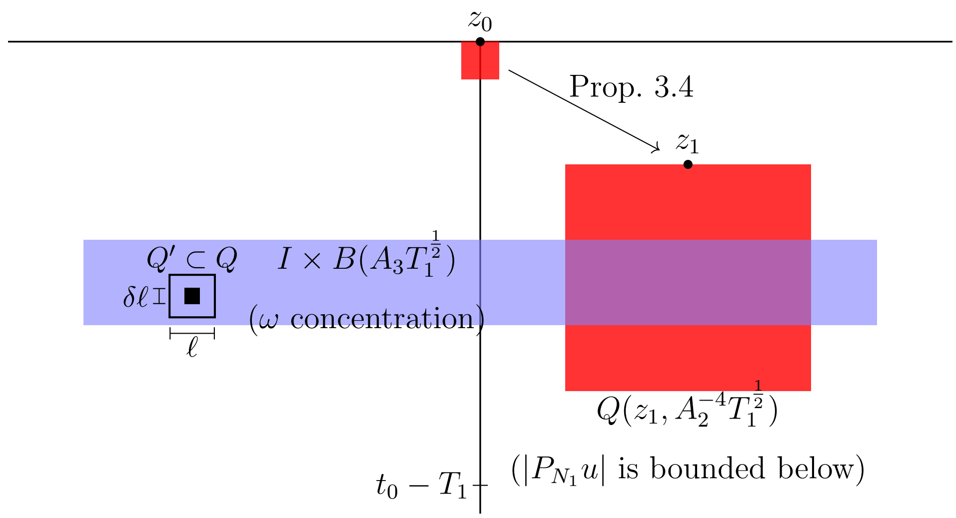

Figure 3.1: We schematize some key steps in the proof of Proposition 3.7. The high frequency concentration at z_0 is propagated backward in time to z_1. The concentration of $P_{N_1} u$ persists in a parabolic cylinder (red) which we convert into a lower bound on $\|\omega\|_{L^2_{t,x}}$ (blue). The objective is to locate a small cylinder Q such that u obeys subcritical bounds in the interior and the vorticity concentrates on a smaller subcylinder Q'.

We apply Proposition 2.6 on $[t_1 - 2A_2^{-4}, t_1]$ to obtain a decomposition $u = u^\flat + u^\sharp$. Let I be the contraction of the time interval $[t_1 - A_2^{-4}, t_1]$ by a factor of $\frac{1}{5}$ about its center. By (2.10), Hölder's inequality, and (2.6),

$$\|\nabla u\|_{L^{\frac{d}{2}}_{t,x}(5I \times B(2A_3))} \leq A_3^{O(1)}. \tag{3.7}$$

Defining the vorticity $\omega := dv$ where d is the exterior derivative on $\mathbb{R}^d$ and v is the covelocity field of u, we apply the codifferential δ to obtain $-\Delta v = \delta\omega$. Thus we have a version of the Biot-Savart law,

$$v = -\Delta^{-1}\delta\omega.$$

It follows from (3.6) and Lemma 2.3 that for all $t \in I$,

$$\begin{aligned} A_2^{-O(1)} &\lesssim \|P_{N_1}\Delta^{-1}\delta\omega(t)\|_{L^2_x(B(A_3/2))} \\ &\lesssim N_1^{-1}\|\omega(t)\|_{L^2_x(B(A_3))} + (A_3 N_1)^{-50d} N_1^{-1}(\|\nabla u^\sharp\|_{L^2_x(\mathbb{R}^d)} + A_3^{\frac{d}{2}}\|\nabla u^\flat\|_{L^\infty_x(\mathbb{R}^d)}). \end{aligned}$$

Taking the $L^2_t(I)$ norm, bounding the $u^\sharp$ global error term with (2.9), and the $u^\flat$ term with (2.6), we obtain

$$\|\omega\|_{L^2_{t,x}(I \times B(A_3))} \geq A_2^{-O(1)}. \tag{3.8}$$

Consider the collection of parabolic cylinders

$$\mathcal{C}_0 := \{Q(z, \ell) : z \in ((\delta\ell)^2\mathbb{Z} \times (\delta\ell\mathbb{Z})^d) \cap (2I \times B(2A_3))\}$$

of which there are $\sim A_2^{-5}A_3^d(\delta\ell)^{-d-2}$. (Once again $2I$ denotes dilation of the interval about its center.) We seek to understand in which cylinders u is regular. By the L^p-boundedness of $\operatorname{div}\operatorname{div}/\Delta$, Hölder's inequality, Sobolev embedding, and (2.9),

$$\|\Delta^{-1}\operatorname{div}\operatorname{div} u^\sharp \otimes u^\sharp\|_{L^1_t L_x^{\frac{d}{d-2}}([t_1 - A_2^{-4}, t_1] \times \mathbb{R}^d)} \lesssim \|u^\sharp\|^2_{L^2_t L_x^{\frac{2d}{d-2}}([t_1 - A_2^{-4}, t_1] \times \mathbb{R}^d)} \leq A_2^{O(1)}.$$

By interpolation with the $L^\infty_t L^{\frac{d}{2}}_x$ bound coming from (2.8),

$$\|\Delta^{-1}\operatorname{div}\operatorname{div} u^\sharp \otimes u^\sharp\|_{L^2_{t,x}([t_1 - A_2^{-4}, t_1] \times \mathbb{R}^d)} \leq A_2^{O(1)}.$$

Using this, (1.9), and (2.9),

$$\sum_{Q\in\mathcal{C}_0}\left(\|\nabla u^\sharp\|^2_{L^2_{t,x}(Q)}+\|\Delta^{-1}\operatorname{div}\operatorname{div}u^\sharp\otimes u^\sharp\|^2_{L^2_{t,x}(Q)}+\|u\|^d_{L^d_{t,x}}\right)\le\delta^{-d-2}A_2^{O(1)},\tag{3.9}$$

since the sets in $\mathcal{C}_0$ can overlap up to $O(\delta^{-d-2})$ times. Define

$$\mathcal{C}_1:=\left\{Q\in\mathcal{C}_0:\max\left(\|\nabla u^\sharp\|_{L^2_{t,x}(Q)},\|\Delta^{-1}\operatorname{div}\operatorname{div}u^\sharp\otimes u^\sharp\|_{L^2_{t,x}(Q)},\|u\|^{d/2}_{L^d_{t,x}(Q)}\right)>A_3^{-1}\ell^{\frac{d}{2}-1}\right\}.$$

From (3.9), we clearly have

$$\#(\mathcal{C}_1)\le\delta^{-d-2}\ell^{2-d}A_3^2A_2^{O(1)}\le\frac{1}{100}\#(\mathcal{C}_0).$$

Consider an arbitrary $Q_0=I_0\times B_0\in\mathcal{C}_0\setminus\mathcal{C}_1$. Additionally using (2.6) and (2.8), we have

$$\begin{aligned}\|p\|_{L^2_{t,x}(Q_0)}&\le A_3^{-1}\ell^{\frac{d}{2}-1}+\|\Delta^{-1}\operatorname{div}\operatorname{div}(2u^\sharp\odot u^\flat+u^\flat\otimes u^\flat)\|_{L^2_{t,x}(Q_0)}\\&\lesssim A_3^{-1}\ell^{\frac{d}{2}-1}+\ell^{\frac{d}{2}}A_2^{O(1)}.\end{aligned}$$

Then by Hölder's inequality

$$D(Q_0)\lesssim A_3^{-1}+\ell A_2^{O(1)}\lesssim A_3^{-1}.\tag{3.10}$$

Next we address $C(Q_0)$. Let $I_{1/10}$ be the first $\frac{1}{10}$ of the interval I_0. Using again that $Q_0\in\mathcal{C}_0\setminus\mathcal{C}_1$,

$$\int_{I_{1/10}}\|u\|^d_{L^d_x(B)}dt\le A_3^{-2}\ell^{d-2}$$

and so by the pigeonhole principle and Hölder's inequality, there exists a $\tau_0\in I_{1/10}$ such that

$$\|u(\tau_0)\|_{L^2_x(B_0)}\lesssim\ell^{\frac{d}{2}-1}\|u(\tau_0)\|_{L^d_x(B_0)}\lesssim A_3^{-\frac{2}{d}}\ell^{\frac{d}{2}-\frac{4}{d}}.$$

With this we can apply (2.26), (3.10), and the fact that $Q_0\notin\mathcal{C}_1$ (along with Hölder's inequality and (2.6) for the $u^\flat$ part) to obtain

$$\|u\|_{L^\infty_tL^2_x(3Q_0/4)}\lesssim A_3^{-\frac{2}{d}}\ell^{\frac{d}{2}-\frac{4}{d}}+A_3^{-\frac{1}{2}}\ell^{\frac{d}{2}-1}A+A_2^{O(1)}\ell^{\frac{d}{2}}.$$

A bound for $\|\nabla u\|_{L^2_{t,x}(3Q_0/4)}$ similarly follows from the definition of $\mathcal{C}_1$ and (2.6). Then by Gagliardo-Nirenberg interpolation,

$$C(3Q_0/4) \lesssim A_3^{-\frac{2}{d}} A + \ell A_2^{O(1)}.$$

With this and (3.10), we arrive at (3.3) in $Q_0/2$ by Propositions 2.12 and 5.1.

For every $Q = Q(z, \ell) \in \mathcal{C}_0$, let $Q' := Q(z - (\ell^2/8, 0), \delta\ell)$. Since $\{Q' : Q \in \mathcal{C}_0\}$ covers $I \times B(R)$, (3.8) implies

$$\sum_{Q \in \mathcal{C}_0} \|\omega\|^2_{L^2_{t,x}(Q')} \geq 2A_3^{-1}.$$

There are two cases. First, suppose

$$\sum_{Q \in \mathcal{C}_0 \setminus \mathcal{C}_1} \|\omega\|^2_{L^2_{t,x}(Q')} \geq A_3^{-1}.$$

By the pigeonhole principle, since the family $\mathcal{C}_0 \setminus \mathcal{C}_1$ has cardinality $A_3^{O(1)}(\delta\ell)^{-d-2}$, there is a $Q \in \mathcal{C}_0 \setminus \mathcal{C}_1$ such that

$$\|\omega\|_{L^2_{t,x}(Q')} \geq A_3^{-O(1)}(\delta\ell)^{\frac{d}{2}+1}. \tag{3.11}$$

This pair Q, Q' satisfies the conclusion of the proposition. In the other case,

$$\sum_{Q \in \mathcal{C}_1} \|\omega\|^2_{L^2_{t,x}(Q')} \geq A_3^{-1}. \tag{3.12}$$

If so, we seek to derive a contradiction with (3.7). We compare the lower bound (3.12) with

$$\|\omega\|^2_{L^2_{t,x}(Q')} \leq A^{O(1)}(\delta\ell)^{d-2}$$

from (2.24), and the fact that $\mathcal{C}_1$ contains at most $A_3^3\delta^{-d-2}\ell^{-d+2}$ cylinders. Indeed, defining the family of disjoint cylinders

$$\mathcal{C}_2 := \{Q' : Q \in \mathcal{C}_1,\ \|\omega\|^2_{L^2_{t,x}(Q')} > A_3^{-5}\delta^{d+2}\ell^{d-2}\},$$

we have, using that the contracted cylinders $\{Q'\}_{Q\in\mathcal{C}_0}$ are disjoint,

$$A_3^{-1} \leq \sum_{Q\in\mathcal{C}_1} \|\omega\|^2_{L^2_{t,x}(Q')} \leq \#(\mathcal{C}_2)A^{O(1)}(\delta\ell)^{d-2} + \#(\mathcal{C}_1 \setminus \mathcal{C}_2)A_3^{-5}\delta^{d+2}\ell^{d-2}.$$

It follows that

$$\#(\mathcal{C}_2) \geq A_3^{-2}(\delta\ell)^{-d+2}. \tag{3.13}$$

For all $Q' \in \mathcal{C}_2$ and $p \geq 2$, by Hölder's inequality,

$$\|\omega\|_{L^{\frac{d}{2}}_{t,x}(Q')} \gtrsim A_3^{-\frac{5}{2}}\ell^{\frac{4}{d}}\delta^{2+\frac{4}{d}}.$$

Summing over $\mathcal{C}_2$,

$$\|\omega\|_{L^{\frac{d}{2}}_{t,x}(2I\times B(2A_3))} \geq A_3^{-O(1)}\ell^{\frac{8}{d}-2}\delta^{\frac{8}{d}}. \tag{3.14}$$

With ℓ sufficiently small as in (3.5), this is in contradiction with (3.7).

Next consider the case $d = 4$. We define

$$\mathcal{C}_3 := \left\{Q \in \mathcal{C}_0 : \|\nabla^j u\|_{L^\infty_{t,x}(Q/2)} \leq A_2^{-1}\ell^{-j-1} \text{ for } j = 0,1,2\right\}.$$

There are two cases: first, suppose $\bigcup_{Q\in\mathcal{C}_0\setminus\mathcal{C}_3} 5Q$ projected to the time axis does not cover I. Then there exists an interval $I' \subset I$ of length ℓ^2 such that

$$\|\nabla^j u\|_{L^\infty_{t,x}(I'\times B(A_3))} \leq A_2^{-1}\ell^{-j-1}$$

for $j = 0, 1, 2$. The existence of a large slab of regularity makes this case relatively straightfoward so we argue briefly. One appeals once again to (3.6) and repeats the calculations leading to (3.8); however now when we take the L^2_t norm of the Bernstein inequality it is only over I' which yields the lower bound $\|\omega\|_{L^2_{t,x}(I'\times B(A_3))} \geq A_2^{-O(1)}\ell$. Analogous to the definition of $\mathcal{C}_0$, we partition a slight dilation of $I' \times B(A_3)$ into overlapping parabolic cylinders of length ℓ offset by length $\delta\ell$. Using the regularity assumed within I' and applying the pigeonhole principle to the vorticity lower bound, it is clear that there exist Q and Q' obeying (3.3) and (3.4).

Otherwise, suppose $\bigcup_{Q \in \mathcal{C}_0 \setminus \mathcal{C}_3} 5Q$ when projected to the time axis does cover I. Then we may take a $\mathcal{C}_4 \subset \mathcal{C}_0 \setminus \mathcal{C}_3$ such that the projections of $\{5Q\}_{Q \in \mathcal{C}_4}$ form a subcover which is minimal in the sense that no more than two intersect at once. It follows that

$$\#(\mathcal{C}_4) \geq A_2^{-O(1)} \ell^{-2}.$$

Due to our definition of $\mathcal{C}_3$, for every $Q \in \mathcal{C}_4$, applying Propositions 2.12 and 5.1 in the converse yields

$$C(Q) + D(Q) > A_2^{-O(1)}.$$

By the argument from the proof of Proposition 3.4, there exist $N \in [A_3^{-1} \ell^{-1}, A_3 \ell^{-1}]$ and $z \in A_3 Q$ such that

$$|P_N u(z)| \geq A_3^{-1} \ell^{-1}.$$

It follows by Lemmas 2.5 and 2.3, as well as Hölder, (2.6), and (2.9) to estimate the global Bernstein error, that

$$\|P_N \nabla u\|_{L^2_{t,x}(A_3^2 Q)} \geq A_3^{-O(1)} \ell.$$

Using Hölder's inequality and (2.7), one computes that the contribution from $u^\flat$ is negligible thanks to the smallness of ℓ. (Note that we continue to refer to the decomposition obtained by applying Proposition 2.6 on $[t_1 - 2A_2^{-4}, t_1]$.) By the properties of $\mathcal{C}_4$, particularly the at most $A_3^{O(1)}$-fold boundedness of the overlap, we obtain

$$\sum_{N \in [A_3^{-1} \ell^{-1}, A_3 \ell^{-1}]} \|P_N \nabla u^\sharp\|^2_{L^2_{t,x}(2I \times \mathbb{R}^d)} \geq A_3^{-O(1)}. \tag{3.15}$$

On the other hand, by Plancherel and (2.9),

$$\sum_N \|P_N \nabla u^\sharp\|^2_{L^2_{t,x}(I \times \mathbb{R}^d)} \leq A_2^{O(1)}.$$

If (3.15) holds for all $\ell \in [\exp(-A_4), A_4^{-1}]$, we reach a contradiction by summing over a geometric sequence of scales in this range. Thus the proposition is satisfied by fixing ℓ to be any scale for which (3.15) fails. □

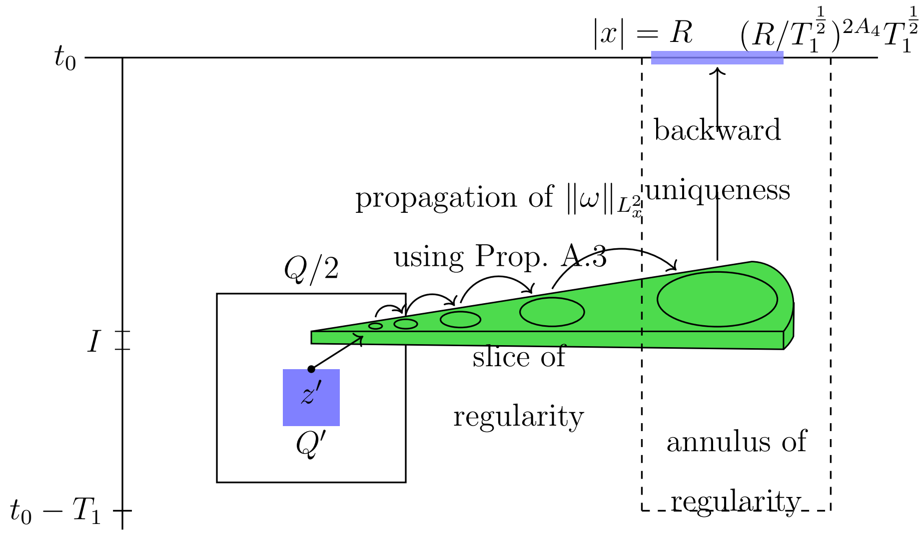

Figure 3.2: In the proof of Proposition 3.8 we begin with a vorticity concentration in a parabolic cylinder Q', which in turned is contained in a $Q/2$ where u possesses subcritical bounds. We use Proposition A.3 to propagate the vorticity lower bound into a slice of regularity obtained from Proposition 2.17. Then we iteratively apply Proposition A.3 to locate a vorticity concentration in a distant annulus where u is regular. In this annulus we may apply a backward uniqueness Carleman inequality to conclude the existence of a $\|u\|_{L^d_x}$ concentration at the final time.

Having obtained a suitable vorticity concentration within a cylinder where the solution is regular, we need only to propagate this lower bound back to time t_0 using a series of Carleman inequalities. For every scale T_1 between N_0^{-2} and T, this scheme leads to a triple-exponentially small amount of L^3_x mass at t_0. Summing over $\log(TN_0^2)$-many geometrically separated scales and comparing the result to (1.9), we will conclude the following.

Proposition 3.8 (Propagation forward to the final time)**.** *Suppose u, z_0, and N_0 are as in Proposition 3.7. Then*

$$TN_0^2 \leq \exp\exp\exp\exp(A_6).$$

Proof. Let us once again fix an arbitrary $T_1 \in [A_1^2 N_0^{-2}, T/100]$. For now, we normalize $z_0 = 0$

and $T_1 = 1$. We continue to use the notation of Proposition 3.7 and its proof; in particular let us take Q and Q' satisfying the the conclusion. Let $z' := (t', x')$ be the center of Q'. We apply Proposition 2.17 centered at $z' + (100(\delta\ell)^2, 0)$ (i.e., shifted forward in time) at length scale $R = \delta\ell$. This yields a slice of regularity which, by rotating, we may assume has $\theta = e_1$. Specifically, there is an $I \subset [t' + 99(\delta\ell)^2, t' + 100(\delta\ell)^2]$ of length $(\delta\ell)^2 A_2^{-2}$ such that within

$$S := I \times \{x \in \mathbb{R}^d : \operatorname{dist}(x, x' + \mathbb{R}_+ e_1) \leq 10A_2^{-1}|x_1 - x_1'|,\ |x - x'| > 20\delta\ell\},$$

we have for $j = 0, 1, 2$

$$\|\nabla^j u\|_{L^\infty_{t,x}(S)} \leq A_1^{-1}(\delta\ell/A_2)^{-1-j}. \tag{3.16}$$

Let $t'' \in I$ be arbitrary. In order to propagate vorticity concentration into this cone, we apply Proposition A.3 to the function

$$(t, x) \mapsto \omega(t'' - t, x + x' + 50\delta\ell e_1)$$

on the interval $[0, C_0(\delta\ell)^2]$ with $t_0 = 75(\delta\ell)^2$, $t_1 = (A_3^{-2}\delta^3\ell)^2$, and $r = \ell/2$. The differential inequality (A.2) for ω is clear from the coordinate form of the vorticity equation

$$\partial_t\omega_{ij} - \Delta\omega_{ij} + u \cdot \nabla\omega_{ij} + (\partial_i u_k)\omega_{kj} - (\partial_j u_k)\omega_{ik} = 0,$$

combined with the estimates in (3.3). Considering each the terms in the Carleman inequality which takes the form $Z \leq X + Y$, by (3.4) the left-hand side obeys

$$Z \geq A_3^{-O(1)}(\delta\ell)^d$$

while for the first term on the right-hand side,

$$X \leq e^{-1/O(\delta^2)}\ell^{d-4}.$$

The latter is negligible compared to the former given (3.5); thus the Carleman inequality becomes

$$\int_{B(x'+50\delta\ell e_1, \ell/2)} |\omega(t'')|^2 e^{-|x-x'-50\delta\ell e_1|^2/4t_1} dx \geq \ell^d \exp(-\delta^{-3}).$$

Finally we narrow the domain of integration using the fact that the contribution from outside $B(x' + 50\delta\ell e_1, A_3^{-1}\delta\ell)$ is negligible compared to the left-hand side which follows from (3.3) and (3.5). This yields

$$\int_{B(x'+50\delta\ell e_1, A_3^{-1}\delta\ell)} |\omega(t'')|^2 dx \geq \exp(-A_4^5) \tag{3.17}$$

for every $t'' \in I$.

Next we apply Proposition 2.18 to find an $R \in [A_4, \exp\exp(A_4)]$ such that

$$\|\nabla^j u\|_{L^\infty_{t,x}([-1,0]\times\{|x|\in[R,R^{2A_4}]\})} \leq A_4^{-1/O(1)} \tag{3.18}$$

for $j = 0, 1, 2$. Then define $x_* = x' + 100Re_1$ and let $\tau = \sup I$. We apply Proposition 3.5 to the function

$$(t, x) \mapsto \omega(t + \tau, x + x')$$

on the interval $[0, 4(\delta\ell)^2]$ with $R_0 = 50\delta\ell$, $\eta = A_2^{-3}$, and $\epsilon = e^{-A_4^6}$ to find

$$\int_{B(x_*, A_3^{-1}R)} |\omega(t)|^2 dx \geq e^{-R^{A_4}} \tag{3.19}$$

for every $t \in [\tau - e^{-3A_4}, \tau]$. Note that the initial lower bound follows from (3.17) and that we have (3.2) thanks to (3.16) and the vorticity equation.

Next we propagate this concentration forward in time using a Carleman inequality for backward uniqueness, see Proposition 4.2 in [60] (the extension of which to higher dimensions was proved in [46], Proposition 9). In particular, by applying it to the function $(t, x) \mapsto \omega(-t, x)$ on the interval $[0, 1]$ with $r_- = 5R$ and $r_+ = R^{2A_4}/10$, we have $Z \leq X + Y$ where

$$\begin{aligned}
Z &\gtrsim e^{-(5R^2)^{A_4}}, \\
X &\lesssim e^{-R^{5A_4}} \int_{-1}^{0} \int_{5R\leq|x|\leq R^{2A_4}/10} e^{2|x|^2/C_0}(|\omega|^2 + |\nabla\omega|^2)\,dx\,dt, \\
Y &\leq e^{R^{10A_4}} \int_{5R\leq|x|\leq r_+} |\omega(0, x)|^2 dx.
\end{aligned}$$

(Note that this and all subsequent applications of Carleman inequalities are valid because (A.2) is implied by (3.18) and the vorticity equation.) Thus there are two cases:

$$\int_{5R\leq|x|\leq R^{2A_4}/10}|\omega(0,x)|^2dx\geq e^{-R^{20A_4}}\tag{3.20}$$

and

$$\int_{-1}^{0}\int_{5R\leq|x|\leq R^{2A_4}/10}e^{2|x|^2/C_0}(|\omega|^2+|\nabla\omega|^2)dxdt\geq e^{R^{A_4}}.\tag{3.21}$$

First assuming (3.21), we essentially follow the proof of Theorem 5.1 in [60]. By the pigeonhole principle, there exists an $R'\in[5R,R^{2A_4}/10]$ such that

$$\int_{-1}^{0}\int_{R'\leq|x|\leq 2R'}(|\omega|^2+|\nabla\omega|^2)dxdt\geq e^{-4(R')^2/C_0}.$$

By (3.18), the contribution to the left-hand side from the time interval $[-e^{-(R')^2},0]$ is negligible compared to the right so essentially the same lower bound holds with the integral evaluated on $[-1,-e^{-(R')^2}]$. We apply the pigeonhole principle, now in time, to find a $T_0\in[e^{-(R')^2},1]$ in this time interval such that

$$\int_{-2T_0}^{-T_0}\int_{R'\leq|x|\leq 2R'}(|\omega|^2+|\nabla\omega|^2)dxdt\geq e^{-(R')^2}.$$

Having obtained length and time scales where the vorticity concentrates, we cover the annulus $\{R'\leq|x|\leq 2R'\}$ by $O(R'/T_0^{\frac{1}{2}})^d$ balls of radius $T_0^{\frac{1}{2}}$. The pigeonhole principle then provides an $x_0\in\{R'\leq|x|\leq 2R'\}$ such that

$$\int_{Q((-T_0,x_0),T_0^{1/2})}(|\omega|^2+|\nabla\omega|^2)dxdt\geq e^{-O(R')^2}.$$

Finally we may apply Proposition A.3 on $[0,1000dT_0]$ to the function

$$(t,x)\mapsto\omega(-t,x+x_0)$$

with $t_0=T_0$, $t_1=C_0^{-3}T_0$, and $r=C_0R'T_0^{\frac{1}{2}}$. The Carleman inequality becomes

$$e^{-O(R')^2}\leq e^{-C_0(R')^2}T_0^{\frac{d}{2}}+e^{O(C_0^2(R')^2)}\int_{B(x_0,C_0R'T_0^{\frac{1}{2}})}|\omega(0,x)|^2e^{-C_0^3|x-x_0|^2/4T_0}dx.$$

With a sufficiently large choice of C_0, the first term on the right-hand side is negligible compared the the left. Moreover, the contribution to the second term on the right from outside $B(x_0, R'/2)$ is also negligible by (3.18). Thus

$$\int_{B(x_0,R'/2)} |\omega(0,x)|^2 dx \geq e^{-C_0^3 (R')^2}.$$

In both cases (3.20) and (3.21), we can thus conclude

$$\int_{2R \leq |x| \leq R^{2A_4}/4} |\omega(0,x)|^2 dx \geq \exp(-\exp\exp(2A_4)).$$

Now let us fix an $x_* \in \{2R \leq |x| \leq R^{2A_4}/4\}$ where

$$|\omega(0,x_*)| \geq \exp(-\exp\exp(3A_4)).$$

By repeating the simple mollification argument from [60] to convert the concentration of vorticity into the critical space, we obtain

$$\int_{A_4 T_1^{\frac{1}{2}} \leq |x| \leq \exp\exp(3A_4) T_1^{\frac{1}{2}}} |u(0,x)|^d dx \geq \exp(-\exp\exp A_5).$$

At this point we undo the original rescaling so that T_1 is explicit. This estimate can be summed over geometrically separated scales $T_1 \in [A_1^2 N_0^{-2}, T/100]$ to conclude

$$\int_{\mathbb{R}^d} |u(0,x)|^d dx \geq \exp(-\exp\exp A_5) \log(T N_0^2)$$

which implies the result when compared to the upper bound (1.9). □

3.5 Proof of Theorems 3.1 and 3.2

As in [60], Theorem 3.1 is obtained easily from Theorem 3.2 combined with, say, Leray's blowup criterion.

Proof of Theorem 3.2. We increase A so that $A \geq C_0$ and rescale so that $t = 1$. By Propositions 3.7 and 3.8 in the converse, we have that

$$\|P_N u\|_{L^\infty_{t,x}([\frac{1}{2},1]\times\mathbb{R}^d)} \leq A_1^{-1} N \tag{3.22}$$

for all

$$N \geq N_* := 2 \exp \exp \exp \exp(A_6).$$

Starting with the decomposition $u = u^\flat + u^\sharp$ on $[0,1]$ and differentiating to reach $\omega = \omega^\flat + \omega^\sharp$, we define the enstrophy-type quantities

$$E_n(t) := \int_{\mathbb{R}^d} \frac{|\nabla^n \omega^\sharp(t)|^2}{2} dx$$

and compute

$$\begin{aligned} E_0'(t) = &- \int_{\mathbb{R}^d} |\nabla \omega^\sharp|^2 dx - \int_{\mathbb{R}^d} \omega^\sharp \cdot \langle \nabla u^\sharp, \omega^\sharp \rangle dx \\ &- \int_{\mathbb{R}^d} \omega^\sharp \cdot (\langle \nabla u^\sharp, \omega^\flat \rangle + \langle \nabla u^\flat, \omega^\sharp \rangle + u^\sharp \cdot \nabla \omega^\flat - f) dx \\ = &-X_1 + X_2 + X_3. \end{aligned}$$

Here we have defined $\langle \nabla u, \omega \rangle_{ij} := (\partial_i u_k)\omega_{kj} - (\partial_j u_k)\omega_{ik}$ for a vector field u and 2-form ω so that we may represent the Lie derivative as $\mathcal{L}_u \omega = \langle \nabla u, \omega \rangle + u \cdot \nabla \omega$.

Clearly $X_1 \geq 0$. By Littlewood-Paley decomposition and Plancherel we have

$$\begin{aligned} X_2(t) &= - \sum_{N_1, N_2, N_3} \int_{\mathbb{R}^d} P_{N_1} \omega^\sharp \cdot \langle \nabla P_{N_2} u^\sharp, P_{N_3} \omega^\sharp \rangle dx \\ &\lesssim \sum_{N_1 \sim N_2 \gtrsim N_3} \|P_{N_1} \omega^\sharp\|_{L^2_x(\mathbb{R}^d)} \|P_{N_2} \omega^\sharp\|_{L^2_x(\mathbb{R}^d)} \|P_{N_3} \omega^\sharp\|_{L^\infty_x(\mathbb{R}^d)}. \end{aligned}$$

Applying Lemma 2.5 and (3.22) for N_3 smaller or larger than N_* respectively, we arrive at

$$\begin{aligned} X_2(t) &\lesssim \sum_{N_1} \|P_{N_1} \omega^\sharp(t)\|^2_{L^2_x(\mathbb{R}^d)} (A^{O(1)} N_*^2 + A_1^{-1} N_1^2) \\ &\lesssim \|\nabla u^\sharp(t)\|^2_{L^2_x(\mathbb{R}^d)} A^{O(1)} N_*^2 + A_1^{-1} X_1. \end{aligned}$$

By Hölder's inequality, (2.6), (2.8), and (2.12), we have for $t \in [\frac{1}{2}, 1]$

$$X_3(t) \leq (\|\nabla u^\sharp(t)\|^2_{L^2_x} + 1) A^{O(1)}.$$

Integrating in time using (2.9) and Gronwall's inequality, we find that for any $\frac{1}{2} \leq t_1 \leq t_2 \leq 1$,

$$E_0(t_2) - E_0(t_1) \leq N_*^2 A^{O(1)}.$$

At the same time, by (2.9), there exists a $t_0 \in [1/2, 3/4]$ such that $E_0(t_0) \leq A^{O(1)}$. Thus

$$\sup_{t \in [\frac{3}{4}, 1]} E_0 + 2 \int_{\frac{3}{4}}^{t} E_1(t) dt \leq N_*^2 A^{O(1)}. \tag{3.23}$$

Next we compute using (2.11)

$$E_n'(t) = -Y_1 + Y_2 + Y_3 + Y_4 + Y_5$$

where

$$\begin{aligned}
Y_1(t) &= \int_{\mathbb{R}^d} |\nabla^{n+1} \omega^\sharp|^2, \\
Y_2(t) &= -\sum_{k=0}^{n} \binom{n}{k} \int_{\mathbb{R}^d} \nabla^n \omega^\sharp \cdot \langle \nabla \nabla^{n-k} u^\sharp, \nabla^k \omega^\sharp \rangle dx, \\
Y_3(t) &= -\sum_{k=1}^{n} \binom{n}{k} \int_{\mathbb{R}^d} \nabla^n \omega^\sharp \cdot (\nabla^k u^\sharp \cdot \nabla \nabla^{n-k} \omega^\sharp) dx, \\
Y_4(t) &= -\sum_{k=1}^{n} \binom{n}{k} \int_{\mathbb{R}^d} \nabla^n \omega^\sharp \cdot (\nabla^k u^\flat \cdot \nabla \nabla^{n-k} \omega^\sharp) dx \\
Y_5(t) &= -\int_{\mathbb{R}^d} \nabla^n \omega^\sharp \cdot \nabla^n (\langle \nabla u^\sharp, \omega^\flat \rangle + \langle \nabla u^\flat, \omega^\sharp \rangle - u^\sharp \cdot \nabla \omega^\flat - \operatorname{curl} f) dx.
\end{aligned}$$

We then take the Littlewood-Paley decompositions and estimate

$$\begin{aligned}
Y_2(t) &= -\sum_{k=0}^{n} \binom{n}{k} \sum_{N_1, N_2, N_3} \int_{\mathbb{R}^d} \nabla^n P_{N_1} \omega^\sharp \cdot \langle \nabla \nabla^{n-k} P_{N_3} u^\sharp, \nabla^k P_{N_2} \omega^\sharp \rangle dx \\
&\leq I + II
\end{aligned}$$

where we decompose based on whether the top order derivatives that fall on the high frequency factors. Specifically, by Hölder, Lemma 2.2, (1.9), and (3.22),

$$
\begin{aligned}
I &\lesssim_n \sum_{k=0}^{n} \sum_{N_1\sim N_2\gtrsim N_3} \|\nabla^n P_{N_1}\omega^\sharp\|_{L^2_x(\mathbb{R}^d)}\|\nabla^k P_{N_2}\omega^\sharp\|_{L^2_x(\mathbb{R}^d)}\|\nabla^{n-k+1}P_{N_3}u^\sharp\|_{L^\infty_x(\mathbb{R}^d)} \\
&\lesssim \sum_{k=0}^{n} \sum_{N_1\sim N_2} \|\nabla^n P_{N_1}\omega^\sharp\|_{L^2_x(\mathbb{R}^d)}\|\nabla^k P_{N_2}\omega^\sharp\|_{L^2_x(\mathbb{R}^d)}(A^{O(1)}N_*^{n-k+2} + A_1^{-1}N_1^{n-k+2}) \\
&\lesssim \sum_{k=0}^{n} A^{O(1)}N_*^{n-k+2}E_k(t)^{\frac12}E_n(t)^{\frac12} + A_1^{-1}Y_1(t),
\end{aligned}
$$

and

$$
\begin{aligned}
II &\lesssim_n \sum_{k=1}^{n-1} \sum_{N_1\sim N_2\gtrsim N_3} \|\nabla^k P_{N_1}\omega^\sharp\|_{L^2_x(\mathbb{R}^d)}\|\nabla^{n-k} P_{N_2}\omega^\sharp\|_{L^2_x(\mathbb{R}^d)}\|\nabla^{n}P_{N_3}\omega^\sharp\|_{L^\infty_x(\mathbb{R}^d)} \\
&\lesssim \sum_{k=1}^{n-1} \sum_{N_1\sim N_2} \|\nabla^k P_{N_1}\omega^\sharp\|_{L^2_x(\mathbb{R}^d)}\|\nabla^{n-k} P_{N_2}\omega^\sharp\|_{L^2_x(\mathbb{R}^d)}(A^{O(1)}N_*^{n+2} + A_1^{-1}N_1^{n+2}) \\
&\lesssim \sum_{k=1}^{n-1} A^{O(1)}N_*^{n+2}E_k(t)^{\frac12}E_{n-k}(t)^{\frac12} + A_1^{-1}Y_1(t).
\end{aligned}
$$

Next, $Y_3(t)$ contains essentially the same terms and admits the same bounds (note crucially the exclusion of $k = 0$ by incompressiblity). By Cauchy-Schwarz and (2.6),

$$Y_4(t) \lesssim_n \sum_{k=1}^{n} A^{O(1)}E_n(t)^{\frac12}E_{n-k+1}(t)^{\frac12}.$$

Finally, by (2.6), (2.8), (2.12), and integration by parts,

$$Y_5(t) \lesssim_n A^{O(1)}E_n(t)^{\frac12}\Big(1 + \sum_{k=0}^{n} E_k(t)^{\frac12}\Big) + A^{O(1)}.$$

In total, combining some terms with Young's inequality,

$$E_n'(t) \le A^{O_n(1)}N_*^2E_n(t) + N_*^{2n+2}\sum_{k=0}^{n-1} E_k(t) + A^{O_n(1)}.$$

Inductively applying Gronwall's inequality (at each step using the pigeonhole principle to find an initial time), starting with (3.23) as a base case, implies

$$\sup_{t\in[t_n,1]}\int_{\mathbb{R}^d} |\nabla^n\omega^\sharp(t)|^2dx + \int_{t_n}^1\int_{\mathbb{R}^d} |\nabla^{n+1}\omega^\sharp(t)|^2dxdt \le N_*^{O_n(1)}$$

for an increasing sequence $t_n \in [\frac12, 1]$. The claimed $L^\infty_{t,x}$ estimates are immediate by (2.6) and Sobolev embedding, taking n sufficiently large depending on d. □

CHAPTER 4

Regularity for approximately axisymmetric solutions

4.1 Introduction

This chapter contains results that appeared in the author's paper [46]. We will be concerned with solutions of (1.2) satisfying the critical bound

$$\|u\|_X := \|r^{1-\frac{3}{q}} u\|_{L^q_x(\mathbb{R}^3)} \le A$$

where u and q fall into one of two cases:

$$\text{either } q \in (3, \infty), \text{ or } u \text{ is axisymmetric and } q \in (2, 3]. \tag{4.1}$$

These conditions have already been mentioned in §2.2.2 but we repeat them for convenience. In fact the assumption of axial symmetry (when $q \in (2, 3]$) can be weakened to $|u|$ being *comparable* to an axisymmetric function. In other words, it suffices that there exist $f : \mathbb{R} \times \mathbb{R}^3 \to [0, \infty)$ and $C > 0$ such that f is axisymmetric and $C^{-1} f \le |u| \le C f$. Indeed, we will only invoke the axial symmetry assumption by way of Propositions 2.1 and 2.3. This is in contrast with the bulk of the literature on the axisymmetric Navier-Stokes equations in which one takes advantage of the special structures coming from this symmetry; Chapter 5 is an example, for instance by making use of the very favorable PDE (5.11) solved by ru_θ.

Without loss of generality, let us take $A \ge 2$. Then we have the following theorems, which mirror those in [60] but offer improvements of the quantitative bounds.

Theorem 4.1. *If $u : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$ is a classical solution of* (1.2) *satisfying* (1.9) *with*

$X = X^q_{\alpha;T}$ and (4.1), then it satisfies the bounds

$$|\nabla_x^j u(t,x)| \leq \exp\exp(A^{O_j(1)})t^{-\frac{j+1}{2}}$$

for $t \in (0,T]$, $x \in \mathbb{R}^3$, and $j \geq 0$.

Theorem 4.2. *Let $u : [0,T_*) \times \mathbb{R}^3 \to \mathbb{R}^3$ be a classical solution to (1.2) which blows up at time $t = T_*$. If u and q satisfy (4.1), then*

$$\limsup_{t \uparrow T_*} \frac{\|r^{1-\frac{3}{q}} u(t)\|_{L^q_x(\mathbb{R}^3)}}{(\log\log \frac{1}{T_*-t})^c} = +\infty$$

for a constant $c > 0$ depending only on q.

Let us emphasize that in the case $q \in (3,\infty)$, we do not assume any symmetry on u. We also wish to stress that here r is not the distance to the origin, but the distance to the x_3-axis. When $q > 3$, it should be possible to extend these arguments even to the case where r is replaced with, say, $|x_1|$. However we choose instead to work in the axial setting in order to make the results comparable to other regularity theorems in the literature.

One noteworthy special case of Theorems 4.1 and 4.2 is when $q = 3$, which is the endpoint of the famous Prodi-Serrin-Ladyzhenskaya scale. By assuming additionally that u is axisymmetric, we obtain the same result as [60] but with one fewer exp or log in the estimates. Also notable is that when q gets large, we approach the the well-known criterion from [28] and [15] cited above, but without needing to assume any kind of symmetry on u. Unfortunately, it seems unlikely that this result can be extended all the way to $q = \infty$ using these techniques. Not only do many of the estimates in this chapter degenerate as $q \to \infty$, but L^∞-based critical spaces seem to be out of reach of these quantitative methods since the argument relies on locating concentrations in many different spacial regions which then contribute additively to the critical norm. (See Proposition 4.6.)

On the other hand, it seems likely that the $q = 2$ case is achievable, although we expect Proposition 2.19 to fail at this endpoint and pigeonholing would again be necessary to apply the Carleman estimates. Thus one may have to settle for triple exponential and logarithmic bounds. We can justify this as follows. Although all the conditions defined by (1.9) and (4.1) are critical with respect to the Navier-Stokes scaling, we claim that when $q = 3$ or $q = 2$ with u axisymmetric, the criticality is homogeneous in the sense that the norms measure all concentrations of the solution identically everywhere in space; on the other hand, if, say, $r^{1-\frac{3}{q}}u \in L_t^\infty L_x^q$ where $q > 3$ or u is axisymmetric and $q > 2$, the norm becomes subcritical far from the x_3-axis and supercritical near it. (The opposite would be true if $q < 3$ or $q < 2$ respectively.) This explains why we can handle these cases without gaining a third exp or log; indeed, by working sufficiently far from the axis, we can guarantee that the velocity and its derivatives are suitably small compared to the scale of the spacetime region. One can see this phenomenon concretely by considering a concentration of $P_N u$ at an $x_0 \in \mathbb{R}^3$ which lies a distance r_0 from the axis. (Refer to Section 2.2.1 for the definition of Littlewood-Paley projections.) Using the same heuristic for (1.2) from [59, p. 67], u behaves essentially as a solution to the heat equation unless the advection term in (1.2) dominates the viscosity, which happens when $|P_N u(x_0)| \gg N$. By the uncertainty principle, such a concentration must occupy a length scale of at least N^{-1}. In the case that $N \gg r_0^{-1}$, the ball $B(x_0, N^{-1})$ does not intersect the x_3-axis and therefore, roughly speaking, it contributes at least $(r_0 N)^{1-\frac{3}{q}}$ to the critial norm $\|r^{1-\frac{3}{q}}u\|_{L_t^\infty L_x^q}$. Thus by assuming (1.9) with $q > 3$, we expect to be able to rule out nonlinear effects with amplitude much larger than r_0^{-1}. If we assume axial symmetry and $N \gg r_0^{-1}$, then this concentration exists not just in $B(x_0, N^{-1})$ but also in the torus obtained by rotating this ball around the x_3-axis; thus the contribution to $\|r^{1-\frac{3}{q}}u\|_{L_t^\infty L_x^q}$ can be strengthened to $(r_0 N)^{1-\frac{2}{q}}$, and we only need $q > 2$ to reach the same conclusion. These heuristics are formalized in the proofs of Propositions 2.1 and 2.19.

In order to work in these weighted spaces, we employ the decomposition $u = u^\flat + u^\sharp$ detailed in §2.3. The overall strategy is analogous to [60]. An essential new element is the observation that far from the x_3-axis, the solution is regular enough to use a Carleman inequality to propagate concentration forward in time (Proposition 2.19). We then prove a backward uniqueness-type Carleman inequality with geometry suited for use with Proposition 2.19. It becomes necessary to work in cylindrical regions where r and z are localized, rather than annular regions as in [60], and in the appendix we prove a backward uniqueness Carleman inequality suited to such a region.

4.2 Axisymmetric back propagation

The "bounded total speed property" (see [59]) is useful for iterating the back propagation—although, as we saw in Chapter 3, there are alternatives. Proposition 4.3 is an extension of the version that appears in [60].

Proposition 4.3. *Let u solve* (1.2) *on $[t_0 - T, t_0] \times \mathbb{R}^3$ with* (1.9)*, $X = X^q_{\alpha;T}$. For any time interval $I \subset [t_0 - T/2, t_0]$, we have*

$$\|u\|_{L^1_t L^\infty_x(I\times\mathbb{R}^3)} \lesssim A^{O(1)}|I|^{1/2}. \tag{4.2}$$

Proof. By symmetries we may assume without loss of generality that $I = [0, 1]$. Once again we let n be sufficiently large so that

$$\|u^\sharp_n\|_{L^\infty_t L^p_x([-1/2,1]\times\mathbb{R}^3)} \lesssim_p A^{O(1)}$$

for all $p \in [q', 3)$.

From the equation for $u^\sharp$ we have

$$P_N u^\sharp_n(t) = -\int_{-1/2}^t P_N e^{(t-t')\Delta}\mathbb{P}\operatorname{div}\tilde{P}_N(u\otimes u - u^\flat_{n-1}\otimes u^\flat_{n-1})(t')dt'$$

and so

$$\begin{aligned}
&\|P_N u^\sharp_n\|_{L^1_t L^\infty_x([-1/2,1]\times\mathbb{R}^3)} \\
&\quad\lesssim \left\| \int_{-1/2}^t N e^{-N^2(t-t')/20}\|\tilde{P}_N(u\otimes u - u^\flat_{n-1}\otimes u^\flat_{n-1})(t')\|_{L^\infty_x(\mathbb{R}^3)}dt' \right\|_{L^1_t([-1/2,1])} \\
&\quad\lesssim N^{-1}\|\tilde{P}_N(u\otimes u - u^\flat_{n-1}\otimes u^\flat_{n-1})\|_{L^1_t L^\infty_x([-1/2,1]\times\mathbb{R}^3)}.
\end{aligned}$$

We split $u \otimes u = u_n^\sharp \otimes u_n^\sharp + 2u_n^\flat \odot u_n^\sharp + u_n^\flat \otimes u_n^\flat$ and estimate, by (2.2) and (2.14),

$$\|\tilde{P}_N(u_n^\flat \otimes u_n^\flat)\|_{L_t^1 L_x^\infty([-1/2,1]\times\mathbb{R}^3)} \lesssim \|u_n^\flat\|^2_{L^\infty_{t,x}([-1/2,1]\times\mathbb{R}^3)} \lesssim A^{O(1)}$$

and similarly for $u_{n-1}^\flat \otimes u_{n-1}^\flat$. By (2.2), Hölder's inequality in time, and (2.32),

$$\begin{aligned}\|\tilde{P}_N(u_n^\flat \odot u_n^\sharp)\|_{L_t^1 L_x^\infty([-1/2,1]\times\mathbb{R}^3)} &\lesssim N^{1/2}\|\tilde{P}_N(u_n^\flat \odot u_n^\sharp)\|_{L_t^1 L_x^6([-1/2,1]\times\mathbb{R}^3)} \\ &\lesssim N^{1/2}\|u_n^\flat\|_{L^\infty_{t,x}([-1/2,1]\times\mathbb{R}^3)}\|u_n^\sharp\|_{L_t^2 L_x^6([-1/2,1]\times\mathbb{R}^3)} \\ &\lesssim A^{O(1)}N^{1/2}.\end{aligned}$$

Finally, we decompose $u_n^\sharp = P_{\leq N}u_n^\sharp + P_{>N}u_n^\sharp$ and estimate the three terms that appear when $u_n^\sharp \otimes u_n^\sharp$ is expanded. By (2.2) and Hölder's inequality,

$$\begin{aligned}\|\tilde{P}_N(P_{\leq N}u_n^\sharp \otimes P_{\leq N}u_n^\sharp)\|_{L_t^1 L_x^\infty([-1/2,1]\times\mathbb{R}^3)} &\lesssim \|P_{\leq N}u_n^\sharp\|^2_{L_t^2 L_x^\infty([-1/2,1]\times\mathbb{R}^3)}, \\ \|\tilde{P}_N(P_{\leq N}u_n^\sharp \odot P_{>N}u_n^\sharp)\|_{L_t^1 L_x^\infty([-1/2,1]\times\mathbb{R}^3)} &\lesssim N^{3/2}\|P_{\leq N}u_n^\sharp \odot P_{>N}u_n^\sharp\|_{L_t^1 L_x^2([-1/2,1]\times\mathbb{R}^3)} \\ &\lesssim N^{3/2}\|P_{\leq N}u_n^\sharp\|_{L_t^2 L_x^\infty([-1/2,1]\times\mathbb{R}^3)} \\ &\quad \times \|P_{>N}u_n^\sharp\|_{L_t^2 L_x^2([-1/2,1]\times\mathbb{R}^3)}, \\ \|\tilde{P}_N(P_{>N}u_n^\sharp \otimes P_{>N}u_n^\sharp)\|_{L_t^1 L_x^\infty([-1/2,1]\times\mathbb{R}^3)} &\lesssim N^3\|P_{>N}u_n^\sharp \otimes P_{>N}u_n^\sharp\|_{L_t^1 L_x^1([-1/2,1]\times\mathbb{R}^3)} \\ &\lesssim N^3\|P_{>N}u_n^\sharp\|^2_{L_t^2 L_x^2([-1/2,1]\times\mathbb{R}^3)}.\end{aligned}$$

In total, by Young's inequality,

$$\begin{aligned}\|\tilde{P}_N(u_n^\sharp \otimes u_n^\sharp)\|_{L_t^1 L_x^\infty([-1/2,1]\times\mathbb{R}^3)} &\lesssim \|P_{\leq N}u_n^\sharp\|^2_{L_t^2 L_x^\infty([-1/2,1]\times\mathbb{R}^3)} \\ &\quad + N^3\|P_{>N}u_n^\sharp\|^2_{L_t^2 L_x^2([-1/2,1]\times\mathbb{R}^3)}.\end{aligned}$$

Inserting this into the estimate for $u_n^\sharp$,

$$\begin{aligned}\|P_N u_n^\sharp\|_{L_t^1 L_x^\infty([-1/2,1]\times\mathbb{R}^3)} &\lesssim N^{-1}\|P_{\leq N}u_n^\sharp\|^2_{L_t^2 L_x^\infty([-1/2,1]\times\mathbb{R}^3)} \\ &\quad + N^2\|P_{>N}u_n^\sharp\|^2_{L_t^2 L_x^2([-1/2,1]\times\mathbb{R}^3)} + A^{O(1)}(N^{-1}+N^{-1/2}).\end{aligned} \tag{4.3}$$

By (2.2) and Cauchy-Schwarz,

$$\begin{aligned}\|P_{\leq N} u_n^{\sharp}\|_{L_t^2 L_x^{\infty}([-1/2,1]\times\mathbb{R}^3)}^2 &\lesssim \left(\sum_{N'\leq N} (N')^{3/2} \|P_{N'} u_n^{\sharp}\|_{L_t^2 L_x^2([-1/2,1]\times\mathbb{R}^3)}\right)^2 \\ &\lesssim N^{1/2} \sum_{N'\leq N} (N')^{5/2} \|P_{N'} u_n^{\sharp}\|_{L_t^2 L_x^2([-1/2,1]\times\mathbb{R}^3)}^2\end{aligned}$$

and by Plancherel's theorem,

$$\|P_{>N} u_n^{\sharp}\|_{L_t^2 L_x^2([-1/2,1]\times\mathbb{R}^3)}^2 \lesssim \sum_{N'>N} \|P_{N'} u_n^{\sharp}\|_{L_t^2 L_x^2([-1/2,1]\times\mathbb{R}^3)}^2.$$

Plugging these into (4.3), we obtain an estimate for the high frequency component,

$$\begin{aligned}\|P_{\geq 1} u_n^{\sharp}\|_{L_t^1 L_x^{\infty}([-1/2,1]\times\mathbb{R}^3)} &\lesssim \sum_{N\geq 1} \Bigg(N^{-1/2} \sum_{N'\leq N} (N')^{5/2} \|P_{N'} u_n^{\sharp}\|_{L_t^2 L_x^2([-1/2,1]\times\mathbb{R}^3)}^2 \\ &\qquad + N^2 \sum_{N'>N} \|P_{N'} u_n^{\sharp}\|_{L_t^2 L_x^2([-1/2,1]\times\mathbb{R}^3)}^2 \Bigg) + A^{O(1)} \\ &\lesssim \sum_{N} N^2 \|P_N u_n^{\sharp}\|_{L_t^2 L_x^2([-1/2,1]\times\mathbb{R}^3)}^2 + A^{O(1)} \\ &\lesssim A^{O(1)}.\end{aligned}$$

by (2.31). For the remaining parts of u, by Hölder's inequality in time, (2.2), (2.14), and (2.8),

$$\|u_n^{\flat}\|_{L_t^1 L_x^{\infty}([-1/2,1]\times\mathbb{R}^3)} \lesssim \|u_n^{\flat}\|_{L_{t,x}^{\infty}([-1/2,1]\times\mathbb{R}^3)} \lesssim A^{O(1)}$$

and

$$\|P_{<1} u_n^{\sharp}\|_{L_t^1 L_x^{\infty}([-1/2,1]\times\mathbb{R}^3)} \lesssim \|u_n^{\sharp}\|_{L_t^{\infty} L_x^2([-1/2,1]\times\mathbb{R}^3)} \lesssim A^{O(1)}$$

which completes the proof. □

Now we can prove the back propagation proposition from [60] with the more general critical control on u.

Proposition 4.4. *Let u be as in Proposition 4.3. Suppose there exist $(t_1, x_1) \in [t_0 - \frac{T}{2}, t_0] \times \mathbb{R}^3$ and $N_1 \geq A_3 T^{-\frac{1}{2}}$ such that*

$$|P_{N_1} u(t_1, x_1)| \geq A_1^{-1} N_1. \tag{4.4}$$

Then there exists $(t_2, x_2) \in [t_0 - T, t_1] \times \mathbb{R}^3$ and $N_2 \in [A_2^{-1} N_1, A_2 N_1]$ such that

$$A_3^{-1} N_1^{-2} \leq t_1 - t_2 \leq A_3 N_1^{-2},$$

$$|x_2 - x_1| \leq A_4 N_1^{-1},$$

and

$$|P_{N_2} u(t_2, x_2)| \geq A_1^{-1} N_2.$$

Proof. First consider $q > 3$. We scale and translate so that $N_1 = 1$ and $t_1 = 0$. Then $[-2A_3, 0] \subset [t_0 - T, t_0]$. Then by assumption we have

$$|P_1 u(0, x_1)| \geq A_1^{-1}. \tag{4.5}$$

Assume for contradiction that the claim fails, which would imply

$$\|P_N u\|_{L^\infty_{t,x}([-A_3, -A_3^{-1}] \times B(x_1, A_4))} \leq A_1^{-1} N$$

for $N \in [A_2^{-1}, A_2]$. From the pointwise bound on $\partial_t P_N u$ and the fundamental theorem of calculus, the time interval can be enlarged up to $t = 0$,

$$\|P_N u\|_{X^\infty_{0;A_3}(B(x_1, A_4))} \lesssim A_1^{-1} N + A_3^{-1} A^2 N^3 \lesssim A_1^{-1} N. \tag{4.6}$$

For $t \in [-A_3, 0]$, Duhamel's formula, Hölder's inequality for the linear term, and (2.2) give us

$$\begin{aligned}
\|r^\alpha P_N u(t)\|_{L^{q/2}_x(B(x_1, A_4))} &\leq A_4^{3/q} \|r^\alpha e^{(t+2A_3)\Delta} P_N u(-2A_3)\|_{L^q_x(B(x_1, A_4))} \\
&\quad + \int_{-2A_3}^{t} \|r^\alpha e^{(t-t')\Delta} P_N \operatorname{div} u \otimes u(t')\|_{L^{q/2}_x(\mathbb{R}^3)} dt' \\
&\lesssim A_4^{3/q} e^{-N^2 A_3/20} N^{\alpha_q - \alpha} A + \int_{-2A_3}^{t} e^{-(t-t')N^2/20} N^{1+2\alpha_q - \alpha} A^2 dt'
\end{aligned}$$

assuming $-\frac{2}{q} < \alpha \le \alpha_q$. Therefore, for $N \ge A_2^{-1}$,

$$\|P_N u\|_{X^{q/2}_{\alpha; A_3}(B(x_1, A_4))} \lesssim A^2 N^{1-\frac{6}{q}-\alpha}. \tag{4.7}$$

Starting from this base case, we claim inductively that

$$\|P_N u\|_{X^{q/n}_{\alpha; T_n}(B_n)} \lesssim N^{1-\frac{3n}{q}-\alpha} A^{O_n(1)} \tag{4.8}$$

for all $N \ge A_2^{-\frac{1}{2}-\frac{1}{n}}$, where $T_n = (\frac{1}{2}+\frac{1}{n})A_3$ and $B_n = B(x_1, (\frac{1}{2}+\frac{1}{n})A_4)$, if $2 \le n \le \min(q, \frac{q+5}{2}-)$ and $-\frac{2}{q} < \alpha \le \min(\alpha_q, 2-\frac{2n}{q}-)$. Suppose (4.8) holds for some $n-1 \ge 2$. For $t \in [-T_n, 0]$,

$$\begin{aligned}\|r^\alpha P_N u(t)\|_{L^{q/n}_x(B_n)} \le{}& \|r^\alpha e^{(t+T_{n-1})\Delta} P_N u(-T_{n-1})\|_{L^{q/n}_x(B_n)} \\ &+ \int_{-T_{n-1}}^{t} \|r^\alpha e^{(t-t')\Delta} P_N \operatorname{div} \tilde{P}_N(u\otimes u)(t')\|_{L^{q/n}_x(B_n)} dt'.\end{aligned}$$

The linear term can be handled exactly as in the previous case, using Hölder and (2.2). Then by (2.3) and a paraproduct decomposition of the nonlinearity,

$$\begin{aligned}\|r^\alpha e^{(t-t')\Delta} P_N \operatorname{div}(u\otimes u)\|_{L^{q/n}_x(B_n)} \lesssim_n{}& e^{-(t-t')N^2/20} N^{1-\alpha} \\ &\times \Big(N^{\alpha_q+\beta} \|r^{\alpha_q+\beta}(P_{>N/100}u\otimes u + P_{\le N/100}u \otimes P_{>N/100}u)\|_{L^{q/n}_x(B_{n-1})} \\ &+ (NA_4)^{-50n} A_4^{\frac{3}{q}(n-2)} N^{2\alpha_q} \|r^{2\alpha_q} u\otimes u\|_{L^{q/2}_x(\mathbb{R}^3)}\Big),\end{aligned}$$

assuming additionally that $\alpha - \alpha_q \le \beta < 1 - \frac{2n-3}{q}$. This implies

$$\begin{aligned}\|P_N u\|_{X^{q/n}_{\alpha;T_n}(B_n)} \lesssim_n{}& (NA_4)^{-10} + N^{-\frac{3}{q}+\beta-\alpha}\Big(\|(P_{>N/100}u)\otimes u\|_{X^{q/n}_{\alpha_q+\beta; T_{n-1}}(B_{n-1})} \\ &+ \|(P_{\le N/100}u)\otimes(P_{>N/100}u)\|_{X^{q/n}_{\alpha_q+\beta; T_{n-1}}(B_{n-1})}\Big) \\ &+ (NA_4)^{-50n} A_4^{\frac{3}{q}(n-2)} N^{1-\frac{6}{q}-\alpha} A^2.\end{aligned}$$

For the first nonlinear term, for $N \ge A_2^{-\frac{1}{2}-\frac{1}{n}}$,

$$\|P_{>N/100}u\otimes u\|_{X^{q/n}_{\alpha_q+\beta; T_{n-1}}(B_{n-1})} \lesssim A^{O_n(1)} N^{1-\frac{3(n-1)}{q}-\beta}$$

using Holder's inequality, (4.8), (1.9), and summing the geometric series, assuming additionally that $\max(-\frac{2}{q}, 1-\frac{3(n-1)}{q}) < \beta \leq \min(\alpha_q, 2-\frac{2(n-1)}{q}-)$. The "low-high" term is analogous. Thus (4.8) is proved, assuming that there exists a β such that all the stated conditions on α, β, and n are satisfied. One easily checks that this is follows from the hypotheses given above.

It is straightforward to see that by taking the largest permissible n satisfying the constraints of (4.8), we can always make $q/n \in (1, 2]$. Therefore we can apply (4.8) with $\alpha = 0$ along with (2.3) and (1.9) to find

$$\|P_N u\|_{X^2_{0;A_3/2}(B(x_1,A_4/2))} \lesssim N^{\frac{3n}{q}-\frac{3}{2}} \|P_N u\|_{X^{q/n}_{0;A_3/2}(B_n)} + (A_4 N)^{-50} A_4^{\frac{3}{2}-\frac{3}{q}} N^{\alpha_q} A.$$

This implies, using (4.8), Hölder's inequality, and (1.9),

$$\|P_N u\|_{X^2_{0;A_3/2}(B(x_1,A_4/2))} \lesssim N^{-\frac{1}{2}} A^{O(1)} \tag{4.9}$$

for $N \geq A_2^{-\frac{1}{2}}$.

We bootstrap this estimate one more time to bring in factors of A_1^{-1}. The linear and global terms are estimated the same way as before so we neglect them and focus on the remaining parts. By the same calculation above, for $N \geq A_2^{-1/3}$,

$$\|P_N u\|_{X^2_{0;A_3/2}(B(x_1,A_4/4))} \lesssim N^{-1} \|\tilde{P}_N (u \otimes u)\|_{X^2_{0;A_3/4}(B(x_1,A_4/3))}.$$

Using the paraproduct decomposition and (2.3),

$$\begin{aligned} \|\tilde{P}_N (u \otimes u)\|_{X^2_{0;A_3/2}(B(x_1,A_4/3))} \lesssim & \sum_{N_1 \lesssim N_2 \sim N} \|P_{N_1} u \odot P_{N_2} u\|_{X^2_{0;A_3/2}(B(x_1,A_4/2))} \\ & + N^{3/4} \sum_{N_1 \sim N_2 \gtrsim N} \|P_{N_1} u \otimes P_{N_2} u\|_{L^\infty_t L^{4/3}_x (B(x_1,A_4/2))}. \end{aligned}$$

By Hölder's inequality, (4.9), Lemma 2.5, and (4.6),

$$\begin{aligned} \sum_{N_1 \lesssim N_2 \sim N} \|P_{N_1} u \odot P_{N_2} u\|_{X^2_{0;A_3/2}(B(x_1,A_4/2))} & \lesssim \sum_{N_1 \lesssim N} A_1^{-1} A^{O(1)} N^{-1/2} \|P_{N_1} u\|_{X^\infty_{0;A_3/2}} \\ & \lesssim A^{O(1)} (A_2^{-1} + A_1^{-1} N^{1/2}) \end{aligned}$$

and, using Young's inequality, interpolation, (4.9), Lemma 2.5, and (4.6),

$$\begin{aligned}\sum_{N_1\sim N_2\gtrsim N}\|P_{N_1}u\otimes P_{N_2}u\|_{X^{4/3}_{0;A_3/2}(B(x_1,A_4/2))} &\lesssim \sum_{N\lesssim N_1\le A_2} A^{O(1)}N_1^{-1/4}A_1^{-1/2}\\ &\quad+\sum_{N_1\ge A_2}A^{O(1)}N_1^{-1/4}\\ &\lesssim A^{O(1)}(A_1^{-1/2}N^{-1/4}+A_2^{-1/4}).\end{aligned}$$

Thus we conclude

$$\|P_N u\|_{X^2_{0;A_3/2}(B(x_1,A_4/4))}\lesssim A^{O(1)}((A_1N)^{-1/2}+(A_2N)^{-1/4}). \tag{4.10}$$

To reach the contradiction, Duhamel's formula and (4.5) give us

$$\begin{aligned}A_1^{-1}&\le |P_1u(0,x_1)|\\ &\le |e^{A_3\Delta/4}P_1u(-A_3/4)|(x_1)+\int_{-A_3/4}^0 |e^{(t-t')\Delta}P_1\operatorname{div}\tilde P_1(u\otimes u)(t',x_1)|dt'.\end{aligned}$$

For the first term, by (2.4) and (1.9),

$$|e^{A_3\Delta/4}P_1u(-A_3/4)|(x_1)\lesssim e^{-A_3/80}A$$

which is negligible compared to A_1^{-1}. Therefore

$$\int_{-A_3/4}^0 |e^{(t-t')\Delta}P_1\operatorname{div}\tilde P_1(u\otimes u)(t',x_1)|dt'\gtrsim A_1^{-1}. \tag{4.11}$$

By (2.3), we have

$$|e^{(t-t')\Delta}P_1\operatorname{div}\tilde P_1(u\otimes u)(t',x_1)|\lesssim e^{-(t-t')/20}(\|\tilde P_1(u\otimes u)(t')\|_{L^\infty_x(B(x_1,A_4/8))}+A_4^{-50})$$

which admits the paraproduct decomposition

$$\begin{aligned}\|\tilde P_1(u\otimes u)\|_{X^\infty_{0;A_3/4}(B(x_1,A_4/8))}&\lesssim\sum_{N_1\lesssim N_2\sim 1}\|P_{N_1}u\odot P_{N_2}u\|_{X^\infty_{0;A_3/4}(B(x_1,A_4/4))}\\ &\quad+\sum_{1\lesssim N_1\sim N_2\le A_2}\|P_{N_1}u\otimes P_{N_2}u\|_{X^{4/3}_{0;A_3/4}(B(x_1,A_4/4))}\\ &\quad+\sum_{N_1\sim N_2\ge A_2}\|P_{N_1}u\otimes P_{N_2}u\|_{X^1_{0;A_3/4}(B(x_1,A_4/4))}.\end{aligned}$$

We estimate each piece using Hölder's inequality, (4.6), Lemma 2.5, (4.9), interpolation, and (4.10):

$$\sum_{N_1 \lesssim N_2 \sim 1} \|P_{N_1} u \odot P_{N_2} u\|_{X^{\infty}_{0;A_3/4}(B(x_1,A_4/4))} \lesssim \sum_{N_1 \leq A_2^{-1}, N_2 \sim 1} A^2 N_1 + \sum_{A_2^{-1} \leq N_1 \lesssim N_2 \sim 1} A_1^{-2} N_1$$
$$\lesssim A^2 A_2^{-1} + A_1^{-2},$$

$$\sum_{1 \lesssim N_1 \sim N_2 \leq A_2} \|P_{N_1} u \otimes P_{N_2} u\|_{X^{4/3}_{0;A_3/4}(B(x_1,A_4/4))}$$
$$\lesssim \sum_{1 \lesssim N_1 \leq A_2} A^{O(1)}((A_1 N_1)^{-3/4} + (A_2 N_1)^{-3/8})(A_1^{-1} N_1)^{1/2}$$
$$\lesssim A^{O(1)}(A_1^{-5/4} + A_1^{-1/2} A_2^{-1/4}),$$

and

$$\sum_{N_1 \sim N_2 \geq A_2} \|P_{N_1} u \otimes P_{N_2} u\|_{X^{1}_{0;A_3/4}(B(x_1,A_4/4))} \lesssim \sum_{N_1 \gtrsim A_2} A^{O(1)} N_1^{-1} \lesssim A^{O(1)} A_2^{-1}.$$

Comparing this upper bound with the lower bound (4.11), we reach the contradiction

$$A_1^{-1} \lesssim A^{O(1)}(A_2^{-1} + A_2^{-2} + A_1^{-5/4} + A_1^{-1/2} A_2^{-1/4} + A_2^{-1}).$$

Now we consider the $2 < q < 3$ case. By (4.4), (2.2), and (1.9),

$$A_1^{-1} \leq |P_1 u(0, x_1)| \leq r(x_1)^{-1+\frac{2}{q}} \|r^{1-\frac{2}{q}} P_1 u(0)\|_{L^\infty_x} \lesssim r(x_1)^{-1+\frac{2}{q}} A.$$

Therefore we may assume $r(x_1) \lesssim A_2$ which will be useful at certain points to bound the heat propagator term when the power on the weight is larger than what we can handle using (2.2).

First, we show by induction on $n \geq 1$ that

$$\|P_N u\|_{X^q_{\alpha;T_n}(B_n)} \lesssim N^{\alpha_q - \alpha} A^{O(1)} \tag{4.12}$$

if $N \geq A_2^{-1-\frac{1}{n}}$, $-\frac{2}{q} < \alpha < 2 - \frac{3}{q}$ and $\alpha \leq n(1-\frac{2}{q}) - \frac{1}{q}$, where now $T_n = (\frac{1}{2} + \frac{1}{2n})A_3$ and $B_n = B(x_1, (\frac{1}{2} + \frac{1}{2n})A_4)$. For $n = 2$ and $t \in [-3A_3/4, 0]$, first consider the linear term. Using the fact that $r(B_1) \leq A_4$, (2.4), and (1.9),

$$\begin{aligned}\|P_N e^{(t+2A_3)\Delta} u(-2A_3)\|_{X^q_{\alpha;A_3}(B_1)} &\lesssim A_4^{\alpha-\gamma} \|P_N e^{(t+2A_3)\Delta} u(-2A_3)\|_{X^q_{\gamma;A_3}} \\ &\lesssim A_4^{\alpha-\gamma} N^{\alpha_q - \gamma} e^{-A_3 N^2/20} A \\ &\lesssim (AN)^{-10}\end{aligned}$$

upon taking, say, $\gamma = \min(\alpha_q, \alpha)$. Therefore, by Hölder's inequality, (2.3), and (1.9),

$$\begin{aligned}\|r^\alpha P_N u(t)\|_{L^q_x(B_1)} &\lesssim (AN)^{-10} + \int_{-2A_3}^t \|r^\alpha P_N e^{(t-t')\Delta} \mathbb{P} \operatorname{div}(u \otimes u)(t')\|_{L^q_x(\mathbb{R}^3)} dt' \\ &\lesssim (AN)^{-10} + \int_{-2A_3}^t e^{-(t-t')N^2/20} N^{2+\alpha_q-\alpha} \|r^{2\alpha_q} u \otimes u\|_{L^{q/2}_x(\mathbb{R}^3)} dt' \\ &\lesssim N^{\alpha_q - \alpha} A^2\end{aligned}$$

if $-\frac{2}{q} < \alpha \leq 2 - \frac{5}{q}$. Note that the lower bound $N \geq A_2^{-3/2}$ is essential to make the contribution of the linear term negligible. This completes the proof of the base case.

Next suppose we have the desired bound for some $n - 1 \geq 2$. The linear term can be treated as in the previous case so we do not repeat the argument. Then, again by Hölder's inequality and (2.3), for $t \in [-T_n, 0]$,

$$\begin{aligned}\|r^\alpha P_N u(t)\|_{L^q_x(B_n)} &\lesssim \|P_N e^{(t+T_{n-1})\Delta} u(-T_{n-1})\|_{X^q_{\alpha;T_n}(B_n)} \\ &\quad + \int_{-T_{n-1}}^t \|r^\alpha P_N e^{(t-t')\Delta} \mathbb{P} \operatorname{div}(u \otimes u)(t')\|_{L^q_x(B_n)} dt' \\ &\lesssim (A_4 N)^{-10} + (A_4 N)^{-50} A_3 A_4^{\frac{3}{q}+\alpha-\gamma} N^{3-\frac{3}{q}-\gamma} \|u \otimes u\|_{X^{q/2}_{2\alpha_q;T_{n-1}}} \\ &\quad + N^{2+\beta-\alpha} \int_{-T_{n-1}}^t e^{-(t-t')N^2/20} \Big(\|r^{\beta+\alpha_q} P_{>N/100} u \otimes u\|_{L^{q/2}_x(B_{n-1})} \\ &\quad + \|r^{\beta+\alpha_q} P_{\leq N/100} u \otimes P_{>N/100} u)\|_{L^{q/2}_x(B_{n-1})} \Big) dt' \\ &\lesssim (A_4 N)^{-10} + N^{\beta-\alpha} \|P_{>N/100} u\|_{X^q_{\beta;T_{n-1}(B_{n-1})}} \|u\|_{X^q_{\alpha_q;T_{n-1}}} \\ &\lesssim N^{1-\frac{3}{q}-\alpha} A^{O(1)}\end{aligned}$$

assuming $\gamma \leq \alpha$ and $-\frac{2}{q} < \gamma \leq 2 - \frac{5}{q}$ for the global Bernstein term, $-\frac{2}{q} < \alpha \leq \beta + \alpha_q + \frac{1}{q}$ and $\beta < 1 - \frac{1}{q}$ for the local Bernstein term, and $-\frac{2}{q} < \beta < 2 - \frac{3}{q}$ and $\beta \leq (n-1)(1-\frac{2}{q}) - \frac{1}{q})$ for the inductive bound on $P_N u$. One computes that such a β and γ exist under the stated conditions on α and q.

For fixed q, since $q > 2$, the upper bound $n(1-\frac{2}{q}) - \frac{1}{q}$ becomes arbitrarily large by taking n large so eventually the only constraint on α becomes $-\frac{2}{q} < \alpha < 2 - \frac{3}{q}$. Therefore we have

$$\|P_N u\|_{X^q_{\alpha;T/2}(B(x_1,1/2))} \lesssim N^{\alpha_q - \alpha} A^{O(1)} \tag{4.13}$$

for all such α if $N \geq A_2^{-1}$. Now as in the $q > 3$ case, we bootstrap this estimate one more time with (4.6) to bring in powers of A_1^{-1}. In the usual Duhamel formula for $P_N u$ on $[-T_{n-1}, 0]$, we neglect the linear term and the global Bernstein term since they can be dealt with as above. Then by (2.3) and a paraproduct decomposition, for $t \in [-T_n, 0]$,

$$\begin{aligned}\|P_N u(t)\|_{L^q_x(B_n)} \lesssim N^{-1} &\sum_{N' \sim N} \|P_{N'} u \odot P_{\lesssim N} u\|_{X^q_{0;T_{n-1}}(B_{n-1})} \\ &+ N^{1-\frac{2}{q}} \sum_{N_1 \sim N_2 \gtrsim N} \|P_{N_1} u \otimes P_{N_2} u\|_{X^{3q/4}_{2-3/q;T_{n-1}}(B_{n-1})}.\end{aligned}$$

For the first term, by Hölder's inequality, (4.13) with $\alpha = 0$, Lemma 2.5, and (4.6), we have for $A_2^{-1/2} \leq N \leq A_2^{1/2}$

$$\begin{aligned}\sum_{N' \sim N} \|P_{N'} u \odot P_{\lesssim N} u\|_{X^q_{0;T_{n-1}}(B_{n-1})} &\lesssim N^{\alpha_q} A^{O(1)} \left(\sum_{N_1 \leq A_2^{-1}} A N_1 + \sum_{A_2^{-1} \leq N_1 \lesssim N} A_1^{-1} N_1 \right) \\ &\lesssim N^{1+\alpha_q} A^{O(1)} A_1^{-1}.\end{aligned}$$

For the second term, by Young's inequality, the trivial interpolation inequality

$$\|r^\beta f^2\|_{L^{3q/4}} \leq \|r^{3\beta/4} f\|_{L^q}^{4/3} \|f\|_{L^\infty}^{2/3},$$

(4.13) with $\alpha = \frac{3}{2} - \frac{9}{4q}$, (4.6), and Lemma 2.5,

$$\begin{aligned}\sum_{N_1 \sim N_2 \gtrsim N} \|P_{N_1} u \otimes P_{N_2} u\|_{X^{3q/4}_{2-\frac{3}{q}; T_{n-1}}(B_{n-1})} &\lesssim \sum_{N \lesssim N_1 \leq A_2} (N_1^{\alpha_q - (\frac{3}{2} - \frac{9}{4q})} A^{O(1)})^{4/3} (A_1^{-1} N_1)^{2/3} \\ &\quad + \sum_{N_1 \geq A_2} (N_1^{\alpha_q - (\frac{3}{2} - \frac{9}{4q})} A^{O(1)})^{4/3} (A N_1)^{2/3} \\ &\lesssim A^{O(1)} (A_1^{-2/3} N^{-1/q} + A_2^{-1/q}).\end{aligned}$$

Therefore, if $A_2^{-1/2} \leq N \leq A_2^{1/2}$,

$$\|P_N u\|_{X^q_{0; A_3/4}(B(x_1, A_4/4))} \lesssim A^{O(1)} A_1^{-2/3} N^{\alpha_q}. \tag{4.14}$$

Now returning to (4.11) and applying a paraproduct decomposition, Hölder's inequality, (4.14), Lemma 2.5, (4.6), and (4.13), we have

$$\begin{aligned}A_1^{-1} &\lesssim \int_{-A_3/4}^{0} |e^{(t-t')\Delta} P_1 \operatorname{div}(u \otimes u)(t', x_1)| dt' \\ &\lesssim \sum_{N' \sim 1} \|P_{N'} u \odot P_{\lesssim 1} u\|_{X^q_{0; A_3/4}(B(x_1, A_4/4))} \\ &\quad + \sum_{N_1 \sim N_2 \gtrsim 1} \|P_{N_1} u \otimes P_{N_1} u\|_{X^{q/2}_{0; A_3/4}(B(x_1, A_4/4))} \\ &\lesssim A^{O(1)} A_1^{-2/3} \left(\sum_{N_1 \leq A_2^{-1}} A N_1 + \sum_{A_2^{-1} \leq N_1 \lesssim 1} A_1^{-1} N_1 \right) \\ &\quad + \sum_{1 \lesssim N_1 \leq A_2^{1/2}} A^{O(1)} A_1^{-4/3} N_1^{2\alpha_q} + \sum_{N_1 \geq A_2^{1/2}} N_1^{2\alpha_q} A^{O(1)} \\ &\lesssim A^{O(1)} (A_1^{-2/3} A_2^{-1} + A_1^{-5/3} + A_1^{-4/3} + A_2^{\alpha_q})\end{aligned}$$

which is the desired contradiction, recalling that $\alpha_q < 0$ in this case. □

This proposition can be iterated exactly as in [60] to obtain the back propagation result we will need in the main argument.

Proposition 4.5 ([60, Proposition 3.1(v)]). *Let $x_0 \in \mathbb{R}^3$ and $N_0 > 0$ be such that*

$$|P_{N_0} u(t_0, x_0)| \geq A_1^{-1} N_0.$$

Then for every $A_4 N_0^{-2} \leq T_1 \leq A_4^{-1} T$*, there exists*

$$(t_1, x_1) \in [t_0 - T_1, t_0 - A_3^{-1} T_1] \times \mathbb{R}^3$$

and

$$N_1 = A_3^{O(1)} T_1^{-\frac{1}{2}}$$

such that

$$x_1 = x_0 + O(A_4^{O(1)} T_1^{\frac{1}{2}})$$

and

$$|P_{N_1} u(t_1, x_1)| \geq A_1^{-1} N_1.$$

We do not repeat the proof from [60] because it would proceed in the exact same manner now that we have all the building blocks: the back propagation proposition (Proposition 4.4), the pointwise bounds for the frequency-localized vector fields (Lemma 2.5), and the bounded total speed property (Proposition 4.3).

4.3 Main blowup proposition

Theorems 4.1 and 4.2 will follow without much difficulty from the following proposition.

Proposition 4.6. *Let* u *be as in Proposition 2.15, with* $A \geq C_0$*. Suppose that there exist* $x_0 \in \mathbb{R}^3$ *and* $N_0 > 0$ *such that*

$$|P_{N_0} u(t_0, x_0)| \geq A_1^{-1} N_0.$$

Then

$$T N_0^2 \leq \exp(\exp(A_6^{O(1)})).$$

Proof. By translating the solution, we may assume $t_0 = (x_0)_3 = 0$. Note that we can shift to make the third component of x_0 vanish but not the first two as the norm $X^q_{\alpha;T}$ is not shift-invariant in those directions.

As shown in [60], by propagating the concentration of $|P_{N_0}u|$ backward in time using Proposition 4.5, converting it into a lower bound on the vorticity, and applying Proposition A.3 within an epoch of regularity provided by Proposition 2.15, one can deduce the following: for every $T_1 \in [A_4 N_0^{-2}, A_4^{-1}T]$, and every $x_* \in \mathbb{R}^3$ with $|x_* - x_0| \geq A_4 T_1^{1/2}$, we have the concentration

$$\int_{B(x_*,|x_*|/2)} |\omega(t,x)|^2 dx \gtrsim \exp(-O(A_5^3 |x_*|^2/T_1))T_1^{-1/2}$$

for all $t \in I$ where $I \subset [-T_1, -A_3^{-O(1)}T_1]$ is a time interval with $|I| = A_3^{-O(1)}T_1$. (We do not repeat these arguments because they hold in our setting without modification.)

In order to make use of this lower bound, we need some control on the location of x_0. As in the proof of Proposition 2.20, letting $\gamma_q = \frac{2}{q}$ in the case where $q \in (2,3]$ with u axisymmetric and $\gamma_q = \frac{3}{q}$ in the case where $3 < q < \infty$, we compute using (2.2) and (1.9)

$$\begin{aligned} A_1^{-1}N_0 \leq |P_{N_0}u(t_0,x_0)| &\leq r(x_0)^{-1+\gamma_q} \|r^{1-\gamma_q} P_{N_0}u(t_0)\|_{L^\infty_x(\mathbb{R}^3)} \\ &\lesssim r(x_0)^{-1+\gamma_q} N_0^{\gamma_q} A \end{aligned}$$

and therefore

$$|x_0| \lesssim A_1^{O(1)} N_0^{-1} \leq A_3^{-1} T_1^{1/2}.$$

Thus we deduce the lower bound

$$\int_{-T_1}^{-A_4^{-1}T_1} \int_{\mathcal{S}(R,10R;10R)} |\omega(t,x)|^2 dx dt \gtrsim \exp(-O(A_5^3 R^2/T_1))T_1^{1/2} \tag{4.15}$$

for any $R \geq A_4^2 T_1^{1/2}$, since the domain of this integral necessarily contains a ball $B(x_*, |x_*|/2)$ such that $2A_4 T_1^{1/2} \leq |x_*| \lesssim R$. In order to propagate this concentration forward in time, we need some regularity on u and ω. For any $T_2 \in [A_4^2 N_0^{-2}, A_4^{-1} T]$, by Proposition 2.20, we have

$$|\nabla^j u(x,t)| \leq T_2^{-\frac{1+j}{2}} A_5^{-1/O(1)}, \quad |\nabla^j \omega(x,t)| \leq T_2^{-\frac{2+j}{2}} A_5^{-1/O(1)} \tag{4.16}$$

for $j = 0, 1$ and all $(t,x) \in [-T_2, 0] \times \{r \geq A_5 T_2^{1/2}\}$. This allows us to apply Proposition A.2 on $[0, T_2/C_0]$ with $r_- = A_5^2 T_2^{1/2}$, $r_+ = A_6 T_2^{1/2}$, and u replaced by the function

$$(t,x) \mapsto \omega(-t,x).$$

The vorticity equation

$$\partial_t \omega - \Delta \omega = \omega \cdot \nabla u - u \cdot \nabla \omega \tag{4.17}$$

along with the coefficient bounds for the right-hand side coming from (4.16) imply (A.2). Letting

$$X = \int_{-T_2/C_0}^{0} \int_{\mathcal{S}(A_5^2 T_2^{1/2}, A_6 T_2^{1/2}; A_6 T_2^{1/2})} e^{2|x|^2/T_2} (T_2^{-1} |\omega|^2 + |\nabla \omega|^2) dx dt,$$

$$Y = \int_{\mathcal{S}(A_5^2 T_2^{1/2}, A_6 T_2^{1/2}; A_6 T_2^{1/2})} |\omega(0,x)|^2 dx,$$

and

$$Z = T_2^{-1} \int_{-T_2/4C_0}^{0} \int_{\mathcal{S}(10 A_5^2 T_2^{1/2}, A_6 T_2^{1/2}/2; A_6 T_2^{1/2}/2)} |\omega(x,t)|^2 dx dt,$$

the Carleman estimate gives

$$Z \lesssim e^{-A_5^2 A_6/4} X + e^{2A_6^2} Y.$$

From (4.15), we have

$$Z \gtrsim T_2^{-1/2} e^{-O(A_5^5)}.$$

Thus either

$$X \gtrsim T_2^{-1/2} e^{A_6} \tag{4.18}$$

or

$$Y \gtrsim T_2^{-1/2} e^{-3A_6^2}. \tag{4.19}$$

First let us assume that the concentration comes from (4.18) which is the harder case. Then

$$\int_{-T_2/C_0}^{0} \int_{\mathcal{S}(A_5^2 T_2^{1/2}, A_6 T_2^{1/2}; A_6 T_2^{1/2})} e^{2|x|^2/T_2} (T_2^{-1}|\omega|^2 + |\nabla\omega|^2) dxdt \gtrsim T_2^{-1/2} e^{A_6}.$$

By (4.16), the integrand is bounded by $T_2^{-3} e^{4A_6^2}$ and the region of integration in the (t, x_1, x_2) variables has volume $O(A_6^2 T_2^2)$. Therefore the range of x_3 in the integral can be narrowed without changing the inequality to

$$\int_{-T_2/C_0}^{0} \int_{\mathcal{S}(A_5^2 T_2^{1/2}, A_6 T_2^{1/2}; e^{-A_6^3} T_2^{1/2}, A_6 T_2^{1/2})} e^{2|x|^2/T_2} (T_2^{-1}|\omega|^2 + |\nabla\omega|^2) dxdt$$
$$\gtrsim T_2^{-1/2} e^{A_6}.$$

The region $\mathcal{S}(A_5^2 T_2^{1/2}, A_6 T_2^{1/2}; e^{-A_6^3} T_2^{1/2}, A_6 T_2^{1/2})$ can be covered by $O(A_6^4)$ sets of the form $\mathcal{S}(\rho, 2\rho; z, 2z)$, so by the pigeonhole principle there exist $\rho \in [A_5^2 T_2^{1/2}, A_6 T_2^{1/2}]$ and $|z| \in [e^{-A_6^3} T_2^{1/2}, A_6 T_2^{1/2}]$ such that

$$\int_{-T_2/C_0}^{0} \int_{\mathcal{S}(\rho, 2\rho; z, 2z)} e^{2|x|^2/T_2} (T_2^{-1}|\omega|^2 + |\nabla\omega|^2) dxdt \gtrsim T_2^{-1/2} e^{A_6/2}.$$

Therefore,

$$\int_{-T_2/C_0}^{0} \int_{\mathcal{S}(\rho, 2\rho; z, 2z)} (T_2^{-1}|\omega|^2 + |\nabla\omega|^2) dxdt \gtrsim T_2^{-1/2} \exp(-O(\rho^2 + z^2)/T_2).$$

By an analogous argument using (4.16), the upper time limit in the integral can be shortened to $-\frac{T_2^{5/2}}{\rho^2 z} e^{-O(\rho^2+z^2)/T_2}$ which, by Young's inequality, is less than $-T_2 e^{-O(\rho^2+z^2)/T_2}$. Thus

$$\int_{-T_2/C_0}^{-e^{-O(\rho^2+z^2)/T_2} T_2} \int_{\mathcal{S}(\rho, 2\rho; z, 2z)} (T_2^{-1}|\omega|^2 + |\nabla\omega|^2) dxdt \gtrsim T_2^{-1/2} \exp(-O(\rho^2 + z^2)/T_2).$$

The interval $[-T_2/C_0, -e^{-O(\rho^2+z^2)/T_2} T_2]$ can be covered by $O((\rho^2 + z^2)/T_2)$ intervals of the form $[-2t_0, -t_0]$ so by the pigeonhole principle, there exists a $t_0 \in [e^{-O(\rho^2+z^2)/T_2} T_2, T_2/C_0]$ such that

$$\int_{-2t_0}^{-t_0} \int_{\mathcal{S}(\rho, 2\rho; z, 2z)} (T_2^{-1}|\omega|^2 + |\nabla\omega|^2) dxdt \gtrsim T_2^{-1/2} \exp(-O(\rho^2 + z^2)/T_2).$$

Moreover, since $t_0 \geq e^{-O(\rho^2+z^2)/T_2} T_2$, the spatial domain of integration can be covered by $e^{O(\rho^2+z^2)/T_2} T_2^{-3/2} \rho^2 z$, which again is smaller than $e^{O(\rho^2+z^2)/T_2}$, balls of radius $t_0^{1/2}$. Therefore there exists an x_* in the region $\mathcal{S}(\rho, 2\rho; z, 2z)$ such that

$$\int_{-2t_0}^{-t_0} \int_{B(x_*, t_0^{1/2})} (T_2^{-1}|\omega|^2 + |\nabla\omega|^2) dx dt \gtrsim T_2^{-1/2} \exp(-O(|x_*|^2)/T_2). \tag{4.20}$$

From here we apply Proposition A.3 to the function

$$(t, x) \mapsto \omega(-t, x_* + x)$$

on the interval $[0, 1000t_0]$ with $\rho_{\text{carleman}} = C_0^{1/4}(t_0/T_2)^{1/2}|x_*|$ and $t_1 = t_0$. Note that $r \leq |x_*|/C_0^{1/4}$ and $\rho_{\text{carleman}} \geq A_5^2 T_2^{1/2}$ imply that $B(x_*, r)$ is contained in the region of regularity guaranteed by (4.16). Therefore

$$Z' \lesssim e^{-C_0^{1/2}|x_*|^2/500T_2} X' + t_0^{3/2} e^{O(C_0^{1/2}|x_*|^2/T_2)} Y' \tag{4.21}$$

where

$$X' = \int_{-1000t_0}^{0} \int_{B(x_*, C_0^{1/4}(t_0/T_2)^{1/2}|x_*|)} (t_0^{-1}|\omega|^2 + |\nabla\omega|^2) dx dt,$$

$$Y' = \int_{B(x_*, C_0^{1/4}(t_0/T_2)^{1/2}|x_*|)} |\omega(0, x)|^2 t_0^{-3/2} e^{-|x-x_*|^2/4t_0} dx,$$

and, since $t_0^{1/2} \leq r/2$,

$$Z' = \int_{-2t_0}^{-t_0} \int_{B(x_*, t_0^{1/2})} (t_0^{-1}|\omega|^2 + |\nabla\omega|^2) dx dt.$$

By (4.20), we have

$$Z' \gtrsim T_2^{-1/2} \exp(-O(|x_*|^2)/T_2).$$

Using (4.16),

$$e^{-C_0^{1/2}|x_*|^2/500T_2} X' \lesssim e^{-C_0^{1/2}|x_*|^2/500T_2} C_0^{3/4} t_0^{3/2} T_2^{-7/2} |x_*|^3 \lesssim e^{-C_0^{1/2}|x_*|^2/1000T_2} T_2^{-1/2}.$$

Therefore within (4.21), the X' term is negligible compared to the Z' term and we are left with

$$\int_{B(x_*, C_0^{1/4}(t_0/T_2)^{1/2}|x_*|)} |\omega(0,x)|^2 e^{-|x-x_*|^2/4t_0} dx \gtrsim \exp(-O(C_0^{1/2}|x_*|^2/T_2))T_2^{-1/2}.$$

It follows that

$$\int_{B(x_*, C_0^{-1/4}|x_*|)} |\omega(0,x)|^2 dx \gtrsim \exp(-O(A_6^3))T_2^{-1/2}$$

for some x_* in $\mathcal{S}(A_5^2 T_2^{1/2}, 2A_6 T_2^{1/2}; e^{-A_6^3} T_2^{1/2}, 2A_6 T_2^{1/2})$. In conclusion,

$$\int_{\mathcal{S}(A_5 T_2^{1/2}, A_6^2 T_2^{1/2}; A_6^2 T_2^{1/2})} |\omega(0,x)|^2 dx \gtrsim \exp(-O(A_6^3))T_2^{-1/2} \tag{4.22}$$

for all $T_2 \in [A_4^2 N_0^{-2}, A_4^{-1} T]$. If instead of (4.18) we had (4.19), then (4.22) is immediate.

Next we convert (4.22) back into a lower bound on the velocity. By the pigeonhole principle, there exists an x_* in $\mathcal{S}(A_5 T_2^{1/2}, A_6^2 T_2^{1/2}; A_6^2 T_2^{1/2})$ where

$$|\omega(0,x_*)| \gtrsim \exp(-O(A_6^3))T_2^{-1}.$$

The gradient estimate in (4.16) implies that this concentration persists up to a distance of at least $\exp(-O(A_6^3))T_2^{1/2}$ from x_*, and therefore

$$\left|\int_{\mathbb{R}^3} \omega(0, x_* - \rho y)\phi(y) dy\right| \gtrsim \exp(-O(A_6^3))T_2^{-1}$$

for a bump function ϕ supported in $B(0,1)$, for some $\rho = \exp(-O(A_6^3))T_2^{1/2}$. Then writing $\omega = \operatorname{curl} u$ and integrating by parts,

$$\left|\int_{\mathbb{R}^3} u(0, x_* - \rho y) \operatorname{curl} \phi dy\right| \geq \exp(-O(A_6^3))T_2^{-1/2}.$$

Then by Hölder's inequality,

$$\int_{B(0,1)} |u(0, x_* - \rho y)|^q dy \gtrsim \exp(-O(A_6^3))T_2^{-q/2}.$$

Within $B(x_*, \rho)$, since $\rho \leq \frac{1}{100} r(x_*)$, r is comparable to $r(x_*) \in [A_5 T_2^{1/2}, A_6^2 T_2^{1/2}]$. Therefore

$$\int_{B(x_*,\rho)} r^{q-3} |u(0,x)|^q dx \gtrsim \exp(-O(A_6^3)). \tag{4.23}$$

Since such an x_* appears within every set $\mathcal{S}(A_5^2 T_2^{1/2}, 2A_6 T_2^{1/2})$, and T_2 can take any value in $[A_4^2 N_0^{-2}, A_4^{-1} T]$, there are at least $\log(TN_0^2)/\log A_6$ disjoint concentrations of the form (4.23). Therefore

$$\frac{\log(TN_0^2)}{\log A_6} \exp(-O(A_6^3)) \lesssim \int_{\mathbb{R}^3} r^{q-3} |u(0,x)|^q dx \leq A^q$$

by (4.23) and (1.9), and the desired conclusion follows. □

4.4 Proof of main theorems

Proof of Theorem 4.1. Once again, we can roughly follow [60], but we must be a bit more careful due to our slightly worse control of $u_n^\flat$. By increasing A, we can make $A \geq C_0$. By rescaling, it suffices to prove the theorem with $t = 1$. Proposition 4.6 implies that

$$\|P_N u\|_{L^\infty_{t,x}([1/2,1] \times \mathbb{R}^3)} \leq A_1^{-1} N \tag{4.24}$$

whenever $N \geq N_*$, where

$$N_* = \exp(\exp(A_7)).$$

We apply the decomposition $u = u_n^\flat + u_n^\sharp$ on $[0,1]$ so that on $[1/2, 1]$, we have all the estimates from Proposition 2.7. Taking the curl, we analogously have $\omega = \omega_n^\flat + \omega_n^\sharp$ and define

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^3} |\omega_n^\sharp(t,x)|^2 dx$$

where we fix an n sufficiently large so that (2.17) gives bounds on $u_n^\sharp$ for $p \in [\min(q', \frac{q}{2}), 3)$.

With (4.17) and integration by parts, we compute

$$\frac{d}{dt} E(t) = -Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8$$

where

$$\begin{aligned}
Y_1(t) &= \int_{\mathbb{R}^3} |\nabla \omega_n^\sharp|^2 dx, \\
Y_2(t) &= -\int_{\mathbb{R}^3} \omega_n^\sharp \cdot (u_n^\flat \cdot \nabla \omega_n^\flat) dx, \\
Y_3(t) &= -\int_{\mathbb{R}^3} \omega_n^\sharp \cdot (u_n^\sharp \cdot \nabla \omega_n^\flat) dx, \\
Y_4(t) &= \int_{\mathbb{R}^3} \omega_n^\sharp \cdot (\omega_n^\sharp \cdot \nabla u_n^\sharp) dx, \\
Y_5(t) &= \int_{\mathbb{R}^3} \omega_n^\sharp \cdot (\omega_n^\sharp \cdot \nabla u_n^\flat) dx, \\
Y_6(t) &= \int_{\mathbb{R}^3} \omega_n^\sharp \cdot (\omega_n^\flat \cdot \nabla u_n^\sharp) dx, \\
Y_7(t) &= \int_{\mathbb{R}^3} \omega_n^\sharp \cdot (\omega_n^\flat \cdot \nabla u_n^\flat) dx, \\
Y_8(t) &= -\int_{\mathbb{R}^3} \omega_n^\sharp \cdot \mathrm{curl}(u_n^\flat \cdot \nabla u_n^\flat) dx.
\end{aligned}$$

By Hölder's inequality, (2.17), and (2.14), we have for $t \in [1/2, 1]$

$$\begin{aligned}
|Y_2(t)| &\lesssim \|\omega_n^\sharp\|_{L_t^\infty L_x^{p'}([1/2,1]\times\mathbb{R}^3)} \|u_n^\flat\|_{L_t^\infty L_x^{2p}([1/2,1]\times\mathbb{R}^3)} \|\nabla \omega_n^\flat\|_{L_t^\infty L_x^{2p}([1/2,1]\times\mathbb{R}^3)} \\
&\lesssim A^{O(1)},
\end{aligned}$$

taking $p = \max(q, \frac{q}{q-2})$. The same argument applies for Y_7 and Y_8. For Y_3, by Hölder's inequality and (2.14), we have

$$|Y_3(t)| \lesssim E(t)^{1/2} \|u_n^\sharp\|_{L_t^\infty L_x^2([1/2,1]\times\mathbb{R}^3)} \|\nabla \omega_n^\flat\|_{L_{t,x}^\infty([1/2,1]\times\mathbb{R}^3)} \lesssim E^{1/2} A^{O(1)}.$$

For Y_5, Hölder's inequality and (2.14) easily give

$$|Y_5(t)| \lesssim E(t) \|\nabla u_n^\flat\|_{L_{t,x}^\infty([1/2,1]\times\mathbb{R}^3)} \lesssim E(t) A^{O(1)}.$$

The same is true for Y_6, since Plancherel's theorem and incompressibility imply

$$\|\nabla u_n^\sharp\|_{L_x^2(\mathbb{R}^3)} \lesssim \|\omega_n^\sharp\|_{L_x^2(\mathbb{R}^3)}.$$

From here one proceeds to estimate Y_4 and conclude exactly as in [60], making use of (2.30) and (4.24). (In that paper the analogous term is called Y_3.) $\square$

As in [60], Theorem 4.2 follows immediately from Theorem 4.1 combined with essentially any classical blowup criterion.

CHAPTER 5

Type I regularity theorem for axisymmetric solutions

5.1 Introduction

This chapter reflects joint work with Wojciech Ożański [41] which is to appear. We aim to understand the regularity properties of solutions bounded in the weak space $L^{3,\infty}$ which is connected to Type I blowup and other self-similar-type behavior.

5.1.1 Tao's stacking argument and Type I blowup

In order to illustrate the main difficulty in the endpoint space $L^{3,\infty}$, let us recall the main strategy of Tao [60] to show that, if u concentrates at a particular time, then there exists a widely separated sequence of length scales $(R_k)_{k=1}^K$ and $\alpha = \alpha(A) > 0$ such that $\|u\|_{L^3(\{|x|\sim R_k\})} \geq \alpha$ for all k, which implies that

$$\|u\|_3^3 = \int_{\mathbb{R}^3} |u|^3 \geq \sum_k \int_{|x|\sim R_k} |u|^3 \geq \alpha^3 K. \tag{5.1}$$

The more singularly u concentrates at the origin, the larger one can take K; thus the L^3 norm controls the regularity of u. More precisely, if $\|u\|_3 \leq A$ and u concentrates at a large frequency N at time T, then one can take $\alpha = \exp(-\exp(A^{O(1)}))$ and $K \sim \log(NT^{\frac{1}{2}})$, which leads to the conclusion $N \leq T^{-\frac{1}{2}} \exp\exp\exp(A^{O(1)})$.

Let us contrast this L^3 situation with that of general Lorentz spaces with interpolation exponent $q \geq 3$. In that case, $\|u\|_{L^{3,q}(\{|x|\sim R_k\})} \geq \alpha$ implies (as a simple application of Tonelli's theorem)

$$\|u\|_{L^{3,q}(\mathbb{R}^3)} \gtrsim \left\| \|u\|_{L^{3,q}(\{|x|\sim R_k\})} \right\|_{\ell^q_k} \geq \alpha K^{\frac{1}{q}},$$

and so one should expect the bounds from the stacking argument used in the Lorentz space $L^{3,q}$ extension [23] to degenerate as $q \to \infty$. Indeed, if $|u(x)| = |x|^{-1}$, we have $\|u\|_{L^{3,\infty}(\{|x|\sim R\})} = 1/O(1)$ for every $R > 0$, yet $\|u\|_{L^{3,\infty}(\mathbb{R}^3)} \sim 1$ which shows that the first inequality in (5.1) fails for the $L^{3,\infty}$ norm. For this reason, the approach of Tao [60] (and, for related reasons, of Escauriaza-Seregin-Šverák) to the L^3 problem cannot be extended to $L^{3,\infty}$.

This issue is in fact closely related to the study of Type 1 blowups and approximately self-similar solutions to (1.2). Leray famously conjectured the existence of backwards self-similar solutions that blow up in finite time, a possibility later ruled out by Nečas, Růžička, and Šverák [40] for finite-energy solutions and by Tsai [61] for locally-finite energy solutions. The latter reference identifies the following as a very natural ansatz for blowup:

$$u(t,x) = \frac{1}{(T_0 - t)^{\frac{1}{2}}} U\left(\frac{x}{(T_0 - t)^{\frac{1}{2}}}\right), \quad U(y) = a\left(\frac{y}{|y|}\right)\frac{1}{|y|} + o\left(\frac{1}{|y|}\right) \text{ as } |y| \to \infty, \tag{5.2}$$

where $a : S^2 \to \mathbb{R}^3$ is smooth. While Tsai [61] shows that there are no solutions *exactly* of this form, solutions that approximate this profile or attain it in a discretely self-similar way are promising candidates for singularity formation, as demonstrated by the Scheffer constructions [42, 43, 51, 52], for example. Unfortunately, criteria pertaining to L^3 such as those in [22, 60, 46] are not effective at controlling such solutions because $|x|^{-1} \notin L^3(\mathbb{R}^3)$, which shows the relevance of the weak norm $L^{3,\infty}$.

Specializing to the case of axial symmetry, it is known, for instance due to Seregin's result [54], that finite-time blowup cannot be of Type I. Thus, roughly speaking, no axisymmetric solution can approximate the profile (5.2) all the way up to a putative blowup time T_0. However, this regularity is only qualitative (indeed, the proof uses an argument by contradiction based on a "zooming in" procedure), and so explicit bounds on the solution have not been available.

The main purpose of this work is to make this regularity quantitative, in the same sense that Tao [60] made quantitative the of Escauriaza-Seregin-Šverák theorem [22]. This allows us to not only to rule out Type I singularies, but also to control how singular they can possibly become. For example it lets us estimate the length scale down to which a solution can be approximated by a self-similar profile; see Corollary 5.10 for details.

5.1.2 The main regularity theorem

We suppose that a strong solution to (1.2) on the time interval $[0,T]$ is axisymmetric, meaning that

$$\partial_\theta u_r = \partial_\theta u_3 = \partial_\theta u_\theta = 0, \tag{5.3}$$

where u_r, u_θ, u_3 denote (respectively) the radial, angular, and vertical components of u, so that

$$u = u_r e_r + u_\theta e_\theta + u_3 e_3$$

in cylindrical coordinates, where e_r, e_θ, e_3 denote the cylindrical basis vectors.

We assume further that u remains bounded in $L^{3,\infty}$,

$$\|u\|_{L^\infty([0,T];L^{3,\infty}(\mathbb{R}^3))} \leq A \tag{5.4}$$

for some $A \gg 1$. We prove the following.

Theorem 5.1 (Main result)**.** *Suppose u is a classical axisymmetric solution of* (1.2) *on* $[0,T]\times\mathbb{R}^3$ *obeying* (5.4)*. Then*

$$\|\nabla^j u(t)\|_{L^\infty_x(\mathbb{R}^3)} \leq t^{-\frac{1+j}{2}} \exp\exp(A^{O_j(1)})$$

for all $j \geq 0$, $t \in [0,T]$.

Our main ingredients are parabolic methods applied to the swirl $\Theta := ru_\theta$ near the axis, as well as localized energy estimates on

$$\Phi := \frac{\omega_r}{r} \quad \text{and} \quad \Gamma := \frac{\omega_\theta}{r}. \tag{5.5}$$

These quantities will be our avenue to transfer regularity of Θ to regularity of the full solution.

To be more precise, our proof builds on the work of Chen, Fang, and Zhang [16], who showed that the energy norm of Φ, Γ,

$$\|\Phi\|_{L_t^\infty L_x^2} + \|\Gamma\|_{L_t^\infty L_x^2} + \|\nabla\Phi\|_{L_t^2 L_x^2} + \|\nabla\Gamma\|_{L_t^2 L_x^2}, \tag{5.6}$$

controls u via an estimate on $\|u_\theta^2/r\|_{L^2}$ (see [16, Lemma 3.1]). They also observed that one can indeed estimate this energy norm as long as the angular velocity u_θ remains small in any neighbourhood of the axis, namely if

$$\|r^d u_\theta\|_{L_t^\infty([0,T];L^{3/(1-d)}(\{r\leq\alpha\}))} \text{ is sufficiently small for some } \alpha>0 \text{ and } d\in(0,1). \tag{5.7}$$

In fact, this can be observed from the PDEs satisfied by Φ, Γ, namely that

$$\begin{aligned} &\left(\partial_t + u\cdot\nabla - \Delta - \frac{2}{r}\partial_r\right)\Gamma + \frac{2}{r^2}u_\theta\omega_r = 0,\\ &\left(\partial_t + u\cdot\nabla - \Delta - \frac{2}{r}\partial_r\right)\Phi - (\omega_r\partial_r + \omega_3\partial_3)\frac{u_r}{r} = 0, \end{aligned} \tag{5.8}$$

which shows that, in order to control the energy of Γ, Φ one needs to control u_r/r, ω_r, ω_3 and u_θ. However, u_r/r can be controlled by Γ, in the sense that

$$\frac{u_r}{r} = \Delta^{-1}\partial_3\Gamma - 2\frac{\partial_r}{r}\Delta^{-2}\partial_3\Gamma \tag{5.9}$$

(see [16, p. 1929] for details), which is one of the main properties of function Γ. In particular, (5.9) lets us use the Calderón-Zygmund inequality to obtain that

$$\left\| D^2\frac{u_r}{r}\right\|_{L^q} \leq \|\nabla\Gamma\|_{L^q} \tag{5.10}$$

for $q\in(1,\infty)$ (see [16, Lemma 2.3] for details). Moreover we have $\omega_r = r\Phi$, and $\omega_3 = \partial_r(ru_\theta)/r$, which shows that the L^2 estimate of Φ and Γ relies only on control of u_θ. In fact, away from from the axis, one can easily control u_θ, while near the axis the smallness condition (5.7) is required in an absorption argument by the dissipative part of the energy; see [16, (3.11)–(3.14)] for details.

In this work we obtain adequate control on u_θ thanks to the weak-L^3 bound (5.4) combined with the parabolic theory developed by Nazarov and Ural'tseva [38] in the spirit of the Harnack inequality. Namely, noting that the swirl satisfies the autonomous PDE

$$\Big(\partial_t + \Big(u + \frac{2}{r}e_r\Big)\cdot\nabla - \Delta\Big)\Theta = 0 \tag{5.11}$$

everywhere except for the axis, one can deduce (as observed in [38, Section 4]) Hölder continuity of Θ near the axis. A similar observation, but in a case of limited regularity of u was used by Seregin [54] in his proof of no Type I blow-ups for axisymmetric solutions. We quantify this approach (see Proposition A.4 below) to obtain an estimate on the Hölder exponent in terms of the weak-L^3 norm, and hence we obtain sufficient control of the swirl in a very small neighbourhood of the axis. As for the outside of the neighbourhood, we make use of the pointwise estimates proved in §2.5.3. This would enable one to close the energy estimates for the quantities in (5.6) if there exist sufficiently many starting times where the energy norms are finite. Unfortunately, there are no times when we can explicitly control these energies in terms of A due to lack of quantitative decay in the x_3 direction. The standard approach of propagating L^2 control of Φ, Γ from the initial data at $t = 0$ (for instance, as in [16]) would lead to additional exponentials in Theorem 5.1. To avoid this issue and prove efficient bounds, we replace (5.6) with L^2 norms that measure Φ and Γ uniformly-locally in x_3: namely, we consider

$$\|\Phi\|_{L^\infty_t L^2_{3-uloc}} + \|\Gamma\|_{L^\infty_t L^2_{3-uloc}} + \|\nabla\Phi\|_{L^2_t L^2_{3-uloc}} + \|\nabla\Gamma\|_{L^2_t L^2_{3-uloc}}. \tag{5.12}$$

See Proposition 5.7 below for an estimate of such energy norm, as well as (5.17) for the precise definition of the L^2_{3-uloc} space. This issue gives rise to further challenges, such as the $x_3 - uloc$ control of the solution u itself in terms of (5.12), as well as an estimate on u_r. We show that the former difficulty can be resolved by an $x_3 - uloc$ generalization of the L^4 estimate on $u_\theta/r^{1/2}$ introduced by [16, Lemma 3.1], together with a $x_3 - uloc$ bootstrapping via $\|u\|_{L^\infty_t L^6_{3-uloc}}$ and an inductive argument for the norms $\|u\|_{L^\infty_t W^{k-1,6}_{uloc}}$ with respect to $k \geq 1$, where "*uloc*" refers to the uniformly locally integrable spaces (in all variables, not only x_3).

As for the latter difficulty, we derive new $x_3 - uloc$ estimates of u_r in terms of Γ. To be more precise, instead of the global estimate (5.10), we require L^2_{3-uloc} control of u_r/r, which is much more challenging, particularly considering the bilaplacian term in (5.9) above. To this end we develop bilaplacian Poisson-type estimate in L^2_{3-uloc} (see Lemma 5.9), which enables us to show that

$$\left\| \nabla \partial_r \frac{u_r}{r} \right\|_{L^2_{3-uloc}} + \left\| \nabla \partial_3 \frac{u_r}{r} \right\|_{L^2_{3-uloc}} \lesssim \|\Gamma\|_{L^2_{3-uloc}} + \|\nabla \Gamma\|_{L^2_{3-uloc}}, \tag{5.13}$$

see Lemma 5.6. This lets us close the estimate of (5.12), and thus control all subcritical norms of u in terms of $\|u\|_{L^{3,\infty}}$.

5.1.3 Blowup rate and comparison to the literature

We note that Theorem 5.1, together with the well-known blow-up criterion $\|u(t)\|_\infty \geq c/(T_* - t)^{1/2}$ (see [45, Corollary 6.25], for example), where $T_* > 0$ is a putative blow-up time, immediately implies the following lower bound on the blow-up rate of $\|u(t)\|_{L^{3,\infty}}$.

Corollary 5.2 (Blow-up rate of the weak-L^3 norm)**.** *If u is a classical axisymmetric solution of* (1.2) *that blows up at T_*, then*

$$\limsup_{t \to T_*^-} \frac{\|u(t)\|_{L^{3,\infty}(\mathbb{R}^3)}}{(\log \log (T_* - t)^{-1})^c} = +\infty. \tag{5.14}$$

This corollary is also a consequence of a recent theorem of Chen, Tsai, and Zhang [17], who prove[1]

$$\limsup_{t \to T_*^-} \frac{\|b(t)\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)}}{\left(\log \log \frac{100}{T_* - t}\right)^{\frac{1}{48}-}} = +\infty,$$

where $b \coloneqq u_r e_r + u_3 e_3$ denotes the swirl-less part of the velocity field u (see [34, Section 3.3] for the relevant definition of $\dot{B}^{-1}_{\infty,\infty}$). Thus, since $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \supset L^{3,\infty}$, the above blow-up rate

[1] Let us note the existence of a substantial misprint in the published version of [17]: in their Theorem 1.4, as in our Corollary 5.2, the blowup rate is *double*-logarithmic.

implies (5.14). We conjecture that a variant of Theorem 5.1 holds with the weak-L^3 norm replaced by such a critical Besov norm and can be proved using the ideas presented here.

In order to describe the relation of Corollary 5.2 to [17], we note that the argument in [17] proceeds by proving a pointwise estimate of the form

$$|ru_\theta| \leq C\exp(-c|\log r|^\tau), \tag{5.15}$$

where $c, C > 0$, $\tau \in (0,1)$, for axisymmetric solutions obeying the slightly supercritical bound

$$\frac{1}{R^{\frac{1}{2}}}\|u\|_{L^\infty((-R^2,0);L^2(B_R))} \leq K\left(\log\log\frac{100}{R}\right)^\beta \qquad \text{for all } R\in(0,1/4]$$

for some $\beta \in (0,\frac{1}{8})$ and $K > 0$. This is yet another application of Harnack inequality methods to axisymmetric Navier-Stokes equations. Rather than proving Hölder continuity of Θ under a global control of a critical norm as we do in Proposition A.4, [17] obtains (5.15) by an "almost Hölder continuity,"

$$\operatorname*{osc}_{Q_\rho}\Theta \leq \exp\left(-c\left(\left(\log\frac{100}{\rho}\right)^\tau - \left(\log\frac{100}{R}\right)^\tau\right)\right)\operatorname*{osc}_{Q_R}\Theta \tag{5.16}$$

for $0 < \rho < R \leq 1/4$, $\tau \in (0,1)$; see [17, Proposition 1.2]. A similar result in the case of $\tau = 1/4$ has been obtained independently by Seregin [55, Proposition 1.3]. Note that the case of $\tau = 1$ corresponds to Hölder continuity.

Let us emphasize that the main point of our work is not to improve the blowup rate but to give an explicit bound on u and its derivatives in terms of only the critical norm—this is a strictly stronger result in the sense that it pertains to *all* axisymmetric classical solutions, even those not blowing up. A naïve attempt to prove a similar quantitative theorem (e.g., using ideas of estimating axisymmetric vector fields from [32]) would lead to a bound which, compared to Theorem 5.1, would contain more iterated exponentials as well as severe dependence on the time t and subcritical norms of the initial data. Instead, Theorem 5.1 parallels the results in [60] and improves on those in [46] in the sense that the final bound depends only on $\|u\|_{L^\infty_t L^{3,\infty}_x}$ and a dimensional factor in t. This also leads to additional interesting corollaries: for instance, an explicit rate of convergence for $u(t) \to 0$ as $t \to +\infty$, and the non-existence of nontrivial ancient axisymmetric solutions in $L^\infty_t L^{3,\infty}_x$.

A comparison of these results with the work of Chen, Tsai and Zhang [17] raises the following question: Is it possible to efficiently control (in the sense of Theorem 5.1) u and its derivatives in terms of only b measured in some critical norm? In fact, in our proof of Hölder continuity of Θ near the axis (Proposition A.4) one can easily replace (5.4) with boundedness of $\|b(t)\|_{L^{3,\infty}}$ in time, since "u" in (5.11) can be replaced by "b", due to axial symmetry. However, we do require $L^{3,\infty}$ control of all components of u for other quantitative estimates leading to Theorem 5.1. These include the basic estimates, quantitative decay away from the axis (Proposition 2.19), as well as energy estimates on Γ and Φ (Proposition 5.7) and their implementation in the main argument.

A related open problem is to explicitly control u in terms of only u_θ. Despite a great deal of work [16, 29, 33, 39, 55, 62] on the properties of the swirl ru_θ, its role in the regularity problem for axisymmetric solutions remains unclear.

5.2 Preliminaries

Given $f\colon \Omega \to \mathbb{R}$ we let

$$\operatorname*{osc}_{\Omega} f \coloneqq \sup_{\Omega} f - \inf_{\Omega} f$$

denote the oscillation of f over Ω. We also denote by $⨍_\Omega \coloneqq \frac{1}{|\Omega|}\int_\Omega$ the average over Ω.

In this chapter, given $p \in [1,\infty]$, we will make use of the uniformly local L^p norms,

$$\|u\|_{L^p_{uloc}} \coloneqq \sup_{x\in\mathbb{R}^3} \|u\|_{L^p_x(B(x,1)))} \quad \text{ and } \|u\|_{L^p_{t,x-uloc}} \coloneqq \left\| \|u\|_{L^p_{uloc}} \right\|_{L^p_t},$$

as well as the norms that are uniformly local in x_3 only,

$$\|f\|_{L^p_{3-uloc}(\mathbb{R}^3)} \coloneqq \sup_{z\in\mathbb{R}} \|f\|_{L^p_x(\mathbb{R}^2\times[z-1,z+1])}. \tag{5.17}$$

Let us also define the heat kernel $\Psi(x,t) \coloneqq (4\pi t)^{-3/2} \mathrm{e}^{-x^2/4t}$ which satisfies

$$\|\nabla^k \Psi(t)\|_p = C_{k,p} t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{k}{2}}. \tag{5.18}$$

5.2.0.1 Lorentz spaces

We recall the Lorentz spaces, defined by

$$\|f\|_{L^{p,q}} \coloneqq p^{1/q} \|\lambda |\{|f| \geq \lambda\}|^{1/p}\|_{L^q(\mathbb{R}_+, \frac{d\lambda}{\lambda})} \tag{5.19}$$

for $q < \infty$ and the endpoint

$$\|f\|_{L^{p,\infty}} \coloneqq \|\lambda |\{|f| \geq \lambda\}|^{1/p}\|_{L^\infty(\mathbb{R}_+, \frac{d\lambda}{\lambda})}.$$

There is an analogue of the Hölder inequality,

$$\|fg\|_{L^{p,q}} \leq C_{p_1,p_2,q_1,q_2} \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}, \tag{5.20}$$

whenever $1/p = 1/p_1 + 1/p_2$, $1/q = 1/q_1 + 1/q_2$, $p_1, p_2, p \in (0,\infty)$, $q_1, q_2, q \in (0,\infty]$. We refer the reader to [58, Theorem 6.9] for a proof of (5.20). The Hölder inequality can be very useful when estimating some localized integrals in terms of the $L^{p,\infty}$ norm. For example, if $\phi \in C_0^\infty(\Omega)$ is a smooth cutoff function then we have the simple estimate

$$\|\phi\|_{L^{p,1}} = p \int_0^\infty |\{|\phi| \geq \lambda\}|^{1/p} d\lambda \leq p \int_0^{\|\phi\|_\infty} |\{|\phi| \geq \lambda\}|^{1/p} d\lambda \leq p|\Omega|^{1/p} \|\phi\|_\infty,$$

which shows that, for example

$$\int_\Omega fg \leq \|f\|_{L^{3,\infty}} \|g\|_2 |\Omega|^{1/6}.$$

This simple method allows us to use the weak L^3 space to estimate some integrals over a region close to the axis of symmetry.

We also note two Young's inequalities involving weak L^p spaces

$$\|f * g\|_{L^{p,\infty}} \lesssim \|f\|_1 \|g\|_{L^{p,\infty}} \qquad \text{for } p \in (1,\infty), \tag{5.21}$$

$$\|f * g\|_p \lesssim \|f\|_r \|g\|_{L^{q,\infty}} \quad \text{for } p,q,r \in (1,\infty) \text{ with } \frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}, \tag{5.22}$$

see [34, Proposition 2.4(a)] and [50, Theorem A.16] for details (respectively).

5.2.0.2 The Bogovskiĭ operator

We recall that, given $p \in (1,\infty)$, an open ball $B \subset \mathbb{R}^3$, $b \in W^{1,p}(B)$ such that $\mathrm{div}\, b = 0$, and $\phi \in C_0^\infty(B;[0,1])$ such that $\phi = 1$ on $B/2$ there exists $\overline{b} \in W^{1,p}(\mathbb{R}^3)$ such that $\overline{b} = 0$ outside B and inside $B/2$,

$$\mathrm{div}\, \overline{b} = div(\phi b) \quad \text{ and } \quad \|\overline{b}\|_{W^{1,p}} \lesssim \|b\|_{W^{1,p}(B)}, \tag{5.23}$$

due to the Bogovskiĭ lemma (see [7, 8] or [24, Lemma III.3.1], for example). We note that the Bogovskiĭ lemma often assumes that the domain is star-shaped (which is not the case for $B \setminus B/2$), but it can be overcome in this particular setting by applying the partition of unity to ϕ; see [44, Section 2.3] for example. Note as well that although the implicit constant in (5.23) depends on the scale of B (by inhomogeneity), in our applications B will be of unit scale.

5.2.0.3 A Poisson-type tail estimate

Here we are concerned with a Poisson equation of the form $-\Delta f = D^2 g$, and we show that any $W^{k,\infty}(B(0,1))$ norm of ∇f can be bounded by the L^1_{uloc} norm of g, if $g = 0$ on $B(0,2)$.

To be more precise, we let $\psi \in C_c^\infty(B(0,1);[0,1])$ be such that $\psi = 1$ on $B(0,1/2)$. Given $y \in \mathbb{R}^3$ we set

$$\psi_y(x) \coloneqq \psi(x-y). \tag{5.24}$$

and

$$\widetilde{\psi} \coloneqq \sum_{\substack{j\in\mathbb{Z}^3\\ |j|\le 10}} \psi_j .$$

Lemma 5.3. *Suppose that $f = D^2(-\Delta)^{-1}(g(1-\widetilde{\psi}))$ for some $g \in L^2$. Then*

$$\|\psi \nabla f\|_{W^{k,\infty}} \lesssim_k \|g\|_{L^1_{uloc}} \qquad \text{for } k \ge 0.$$

Proof. We note that

$$\partial_i f(x) = \int \frac{(x_i - y_i) g(y)(1-\tilde{\phi}(y))}{|x-y|^5} dy$$

for $x \in \text{supp}\ \phi$, and so

$$\begin{aligned}|\nabla f(x)| &\le \int_{\{|x-y|\ge 5\}} \frac{|g(y)|}{|x-y|^4} dy \\ &\le \sum_{\substack{j\in\mathbb{Z}^3\\ |j|\ge 2}} \int_{x_1+j_1}^{x_1+j_1+1} \int_{x_2+j_2}^{x_2+j_2+1} \int_{x_3+j_3}^{x_3+j_3+1} \frac{|g(y)|}{|x-y|^4} dy_3\, dy_2\, dy_1 \\ &\lesssim \|g\|_{L^1_{uloc}} \sum_{\substack{j\in\mathbb{Z}^3\\ |j|\ge 2}} |j|^{-4} \lesssim \|g\|_{L^1_{uloc}},\end{aligned}$$

as required. An analogous argument applies to higher derivatives of f. □

The above proof demonstrates a simple method of tail estimation which we will later use to obtain a L^2_{3-uloc} estimate of u_r/r in terms of Γ, mentioned in the introduction (recall (5.13)). In fact, to this end, a similar strategy can be applied in the x_3 direction only, and can be extended to the more challenging bi-Laplacian Poisson equation (see Lemma 5.9 below).

5.2.0.4 Cylindrical coordinates

Given $x \in \mathbb{R}^3$ we denote by $x' := (x_1, x_2)$ the horizontal variables, and $r := (x_1^2 + x_2^2)^{1/2}$ denotes the radius in the cylindrical coordinates. We often use the notation

$$\{r < r_0\} := \{x \in \mathbb{R}^3 \colon r < r_0\}$$

for a given $r_0 > 0$.

We recall a version of the Hardy inequality

$$\|r^{-1} f\|_{L^q(\Omega)} \lesssim C(\Omega)\|f\|_{L^q(\Omega)} + \|\nabla f\|_{L^q(\Omega)} \tag{5.25}$$

where Ω is a bounded domain and $q \in (1, 2]$; see [16, Lemma 2.4] for a proof.

We recall the divergence operator in cylindrical coordinates: if $v = v_r e_r + v_\theta e_\theta + v_3 e_3$ then

$$\text{div}\ v = \frac{1}{r}\partial_r(r v_r) + \frac{1}{r}\partial_\theta v_\theta + \partial_3 v_3 \tag{5.26}$$

For an axisymmetric vector field v, one can compute the length of the partial gradient,

$$|\nabla' v|^2 = (\partial_r v_r)^2 + (\partial_r v_\theta)^2 + (\partial_r v_3)^2 + \frac{1}{r^2}(v_r^2 + v_\theta^2), \tag{5.27}$$

which implies the pointwise bounds

$$\frac{|v_r|}{r}, \frac{|v_\theta|}{r} \leq |\nabla' v|.$$

Here ∇' refers to the gradient with respect to the horizontal variables x' only. Moreover,

$$|\partial_{rr} f| \lesssim |D^2 f|. \tag{5.28}$$

Indeed, since

$$\partial_r = \cos\theta\, \partial_1 + \sin\theta\, \partial_2 = \frac{x_1}{|x'|}\partial_1 + \frac{x_2}{|x'|}\partial_2,$$

we can compute that

$$\partial_{rr} = \frac{x_1^2}{|x'|^2}\partial_{11} + 2\frac{x_1 x_2}{|x'|^2}\partial_1\partial_2 + \frac{x_2^2}{|x'|^2}\partial_{22},$$

from which (5.28) follows. One can compute more generally that

$$\partial_r^N = \sum_{n=0}^{N} \binom{N}{n} |x'|^{-N} (x_1\partial_1)^n (x_2\partial_2)^{N-n}$$

from which it follows

$$|D^3_{r,x_3} f| \lesssim |D^3 f| \qquad \text{and} \qquad |D^4_{r,x_3} f| \lesssim |D^4 f| \tag{5.29}$$

for any axisymmetric f, where D^4 refers to all fourth order derivatives with respect to x_1, x_2, x_3.

5.2.0.5 A quantified version of the Hardy inequality

From the classical Hardy inequality

$$\|r^{-\frac{3}{p}+\frac{1}{2}} f\|_p \lesssim_p (\|f\|_2 + \|\nabla f\|_2)$$

for $p \in (2,6)$ (see [16, Lemma 2.6], for example), we prove a variant that is localized in the horizontal variables, uniformly local in x_3, and whose failur near the $p = 2$ endpoint is explicitly controlled. In particular,

Lemma 5.4 (Quantified Hardy inequality)**.** *The is a $C > 0$ such that for all $p \in (2, 6-\epsilon)$,*

$$\|r^{-\frac{3}{p}+\frac{1}{2}} f\|_{L^p_{3-uloc}(\{r\le 1\})} \lesssim_\epsilon (p-2)^{-C} \left(\|f\|_{L^2_{3-uloc}(\{r\le 1\})} + \|\nabla f\|_{L^2_{3-uloc}(\{r\le 1\})} \right).$$

Proof. From the Sobolev embedding

$$\|u\|_{L^{2p/(2-p)}(\mathbb{R}^2)} \lesssim (2-p)^{-O(1)} \|\nabla u\|_{L^p(\mathbb{R}^2)}$$

for $p < 2$, (see, e.g., [57] where the sharp constant is computed), one can prove the two-dimensional Gagliardo-Nirenberg inequality

$$\|f\|_{L^q(B(1))} \lesssim q \left(\|f\|^{\frac{6}{q}}_{L^6(B(1))} \|\nabla f\|^{1-\frac{6}{q}}_{L^2(B(1))} + \|f\|_{L^p(B(1))} \right) \tag{5.30}$$

for $q > 6$. Fix $\epsilon > 0$ to be specified. Then

$$\left\| \frac{f}{r^{\frac{3}{q}-\frac{1}{2}}} \right\|_{L^q_{x'}(r\ge\epsilon)} \le \|r^{-\frac{3}{q}+\frac{1}{2}}\|_{L^{6q/(6-q)}_{x'}(\{r\ge\epsilon\})} \|f\|_{L^6_{x'}(\mathbb{R}^2)} \lesssim \epsilon^{-\frac{1}{q}+\frac{1}{6}} \|f\|_{L^6_{x'}(\mathbb{R}^2)}.$$

Inside, for any $\frac{1}{s} \in (\frac{3}{2p} - \frac{1}{4}, \frac{1}{p})$, by (5.30),

$$\begin{aligned} \left\| \frac{f}{r^{\frac{3}{p}-\frac{1}{2}}} \right\|_{L^p_{x'}(r\le\min(1,\epsilon))} &\le \|r^{-\frac{3}{p}+\frac{1}{2}}\|_{L^s_{x'}(r<\min(1,\epsilon))} \|f\|_{L^{ps/(s-p)}_{x'}(B(1))} \\ &\lesssim \left(\frac{1}{s} - \frac{3}{2p} + \frac{1}{4}\right)^{-\frac{1}{s}} \left(\frac{1}{p} - \frac{1}{s}\right)^{-1} \\ &\quad \times \left(\epsilon^{-\frac{3}{p}+\frac{1}{2}+\frac{2}{s}} \|f\|^{\frac{6}{p}-\frac{6}{s}}_{L^6_{x'}(B(1))} \|\nabla f\|^{1-\frac{6}{p}+\frac{6}{s}}_{L^2_{x'}(B(1))} + \|f\|_{L^p_{x'}(B(1))} \right). \end{aligned}$$

Upon taking $\epsilon = \|f\|_6^3 / \|\nabla f\|_2^3$ and $\frac{1}{s} = \frac{4}{3p} - \frac{1}{6}$,

$$\left\| \frac{f}{r^{\frac{3}{p}-\frac{1}{2}}} \right\|_{L^p_{x'}(B(1))} \lesssim (p-2)^{-O(1)} \left(\|f\|^{\frac{3}{2}-\frac{3}{p}}_{L^6_{x'}(B(1))} \|\nabla f\|^{-\frac{1}{2}+\frac{3}{p}}_{L^2_{x'}(B(1))} + \|f\|_{L^p_{x'}(B(1))} \right).$$

Finally by Hölder's inequality, Sobolev embedding, and Gagliardo-Nirenberg interpolation, we find

$$\left\| \frac{f}{r^{\frac{3}{p}-\frac{1}{2}}} \right\|_{L^p_x(B_{\mathbb{R}^2}(1)\times B_{\mathbb{R}}(z,1))} \lesssim (p-2)^{-O(1)} \|f\|_{H^1_x(B_{\mathbb{R}^2}(1)\times B_{\mathbb{R}}(z,1))},$$

as required. ∎

5.2.0.6 Second derivative estimates

The following second derivative estimate is a consequence of energy conservation and is related to theorems of Constantin [18], Lions [36], and Vasseur [63].

Lemma 5.5 (2nd order derivatives estimates)**.** *If u solves* (1.2) *on* $[0,T]$ *and obeys* (5.4)*, then*

$$\|\nabla^2 u\|_{L^p_{t,x-uloc}([\frac{T}{2},T]\times\mathbb{R}^3)} \lesssim_p A^{O(1)} T^{\frac{5}{2p}-\frac{3}{2}}$$

for $p \in [1, \frac{4}{3})$*, where the local norm is measured at spatial scale* $T^{\frac{1}{2}}$*.*

Proof. We use the approach due to Constantin [18]. First rescale to make $T = 1$. For every $\epsilon \in (0, \frac{1}{2})$, we define the approximation to the function $\langle x\rangle := (1+|x|^2)^{\frac{1}{2}}$,

$$q(x) := \langle x\rangle - \frac{1}{2(1-\epsilon)}\langle x\rangle^{1-\epsilon}$$

which satisfies the properties

$$|\nabla q| \leq 1, \tag{5.31}$$

$$\xi^T \nabla^2 q(x)\xi > \frac{\epsilon}{2}\langle x\rangle^{-(1+\epsilon)}|\xi|^2, \tag{5.32}$$

$$\frac{1-2\epsilon}{2-2\epsilon}\langle x\rangle \leq q(x) \leq \langle x\rangle. \tag{5.33}$$

With τ a time scale to be specified, we define $w := q(\tau\omega)$ which obeys the equation

$$(\partial_t + u\cdot\nabla - \Delta)w = \tau\nabla q(\tau\omega)\cdot(\omega\cdot\nabla u) - \tau^2\,\mathrm{tr}(\nabla\omega^T\nabla^2 q\nabla\omega).$$

Multiplying by a spatial cutoff at length scale R and integrating over $\mathbb{R}^d$,

$$\frac{\mathrm{d}}{dt}\int_{\mathbb{R}^3} w\psi \leq \int_{\mathbb{R}^3}(u\cdot\nabla\psi + \Delta\psi)w + O(\tau|\nabla u|^2)\psi - \frac{\epsilon}{2}\tau^2\langle\tau\omega\rangle^{-(1+\epsilon)}|\nabla\omega|^2\psi.$$

Let $\tilde\psi$ be an enlarged cutoff function so that $R|\nabla\psi| + R^2|\Delta\psi| \leq 10\tilde\psi$. We define the $L^p_{x-uloc,R}$ norm to be the supremum of the L^p norm restricted to balls of radius R. Integrating in time starting from a t_0 to be specified and taking a supremum over the balls,

$$\begin{aligned}
&\|w\psi(t)\|_{L^1_{x-uloc,R}} + \frac{\epsilon}{2}\tau^2 \int_{t_0}^{t}\int_{\mathbb{R}^3} \langle \tau\omega\rangle^{-(1+\epsilon)} |\nabla\omega|^2 \psi dxdt \\
&\qquad \lesssim \|w(t_0)\|_{L^1_{x-uloc,R}} + \int_{t_0}^{t} (R^{-2} + R^{-1}\|u\|_\infty)\|w(t')\|_{L^1_{x-uloc,R}} dt' + \tau\|\nabla u\|^2_{L^2_{t,x-uloc,R}}.
\end{aligned}$$

Grönwall's inequality and (2.9) imply

$$\|w(t)\|_{L^1_{uloc,R}} \lesssim \left(\|w(t_0)\|_{L^1_{uloc,R}} + \tau R A^{O(1)}\right) \exp(R^{-2}|t-t_0| + R^{-1}A^{O(1)}|t-t_0|^{\frac{1}{2}}).$$

Setting $R = A^{C_1}$ and $\tau = A^{-2C_1}$ for a sufficiently large C_1, we find

$$\|\langle \tau\omega(t)\rangle\|_{L^1_{x-uloc,R}} \lesssim \|\langle \tau\omega(t_0)\rangle\|_{L^1_{x-uloc,R}}.$$

By (2.9) and Hölder's inequality, we can find a $t_0 \in [1/4, 1/2]$ where the right-hand side is bounded by $A^{O(1)}$. Therefore

$$\int_{t_0}^{t}\int_{\mathbb{R}^3} \langle \tau\omega\rangle^{-(1+\epsilon)} |\nabla\omega|^2 \psi \, dxdt \le \epsilon^{-1} A^{O(1)}.$$

We use Hölder's inequality with the decomposition

$$|\nabla\omega|^{\frac{4}{3+\epsilon}} = \left(|\nabla\omega|^{\frac{4}{3+\epsilon}} \langle \tau\omega\rangle^{-2\frac{1+\epsilon}{3+\epsilon}}\right) \langle \tau\omega\rangle^{2\frac{1+\epsilon}{3+\epsilon}}$$

to conclude

$$\|\nabla\omega\|_{L^{4/(3+\epsilon)}_{t,x-uloc}([t_0,t]\times\mathbb{R}^3)} \le \epsilon^{-O(1)} A^{O(1)}.$$

To convert this into a bound on $\nabla^2 u$, fix a unit ball $B \subset \mathbb{R}^3$ and a cutoff function $\varphi \in C_c^\infty(3B)$ with $\varphi \equiv 1$ in $2B$. We decompose $\nabla^2 u = a + b$ where $a = \nabla^2 \Delta^{-1} \operatorname{curl}(\varphi\omega)$. Note that $b = \nabla f$ where $f = \nabla\Delta^{-1}\operatorname{curl}((1-\varphi)\omega)$ is harmonic in $2B$ so for any $p \in [1, \frac{4}{3})$,

$$\|a\|_{L^p_{t,x}([t_0,t]\times B)} \lesssim \|\nabla\omega\|_{L^p_{t,x}([t_0,t]\times 3B)} + \|\nabla\varphi\|_{L^\infty}\|\omega\|_{L^2_{t,x-uloc}([t_0,t]\times\mathbb{R}^3)} \le \epsilon^{-O(1)}A^{O(1)}$$

and

$$\begin{aligned}
\|b\|_{L^p_{t,x}([t_0,t]\times B)} &\lesssim \|\nabla\Delta^{-1}\operatorname{curl}((1-\varphi)\omega)\|_{L^2_{t,x}([t_0,t]\times 2B)} \\
&\lesssim \|\omega^\sharp\|_{L^2_{t,x}([t_0,t]\times\mathbb{R}^3)} + \|\omega^\flat\|_{L^\infty_{t,x}([t_0,t]\times\mathbb{R}^3)} \le A^{O(1)}
\end{aligned}$$

where we have used (2.9), Hölder's inequality, and (2.6). □

5.3 A Poisson-type estimate on u_r/r

Here we discuss how derivatives of u_r/r can be controlled by Γ using the representation (5.9),

$$\frac{u_r}{r} = \Delta^{-1}\partial_3\Gamma - 2\frac{\partial_r}{r}\Delta^{-2}\partial_3\Gamma, \tag{5.34}$$

see [16, p. 1929], which will be an essential part of our 3-uloc energy estimates for Φ and Γ (see Proposition 5.7 below).

Lemma 5.6 (The L^2_{3-uloc} estimate on u_r/r)**.**

$$\left\|\nabla\partial_r\frac{u_r}{r}\right\|_{L^2_{3-uloc}} + \left\|\nabla\partial_3\frac{u_r}{r}\right\|_{L^2_{3-uloc}} \lesssim \|\Gamma\|_{L^2_{3-uloc}} + \|\nabla\Gamma\|_{L^2_{3-uloc}}$$

We defer the proof to §5.5.0.1.

A version of the above estimate without the localization in x_3 has appeared in [16, Lemma 2.3]. As mentioned in the introduction, the localization makes the estimate much more challenging, particularly due to the bilaplacian term in (5.34).

5.4 Energy estimates for ω/r

In this section, we assume the weak L^3 bound (5.4) on time interval $[0,1]$ and prove an energy bound for $\Phi^2+\Gamma^2$ at time 1.

We first note that u_θ satisfies

$$\left(\partial_t + u\cdot\nabla - \Delta + \frac{1}{r^2}\right)u_\theta + \frac{u_r}{r}u_\theta = 0, \tag{5.35}$$

from which one computes that the swirl $\Theta = ru_\theta$ obeys

$$\left(\partial_t + \left(u + \frac{2}{r}e_r\right)\cdot\nabla - \Delta\right)\Theta = 0 \tag{5.36}$$

in $(\mathbb{R}^3\setminus\{r=0\})\times(0,T)$. It then follows that, at each time, $(r,x_3)\mapsto u_\theta(r,x_3,t)$ is a continuous function on $\overline{\mathbb{R}}_+\times\mathbb{R}$ with $u_\theta(0,x_3,t)=0$ for all x_3 (see [37, Lemma 1] for details). In particular

$$\Theta(0,0,x_3) = 0 \quad \text{for all } x_3 \in \mathbb{R}. \tag{5.37}$$

Moreover, since ω is a smooth vector field with bounded derivatives (on which we have no effective bounds yet, of course) (5.27) implies Φ and Γ are locally bounded near the x_3-axis.

Proposition 5.7 (An L^2_{3-uloc} energy estimate for Φ and Γ)**.** *Let u be a classical solution of* (1.2) *satisfying the weak L^3 bound* (5.4) *on* $[0,1]$*. Then*

$$\|\Phi(1)\|_{L^2_{3-uloc}(\mathbb{R}^3)} + \|\Gamma(1)\|_{L^2_{3-uloc}(\mathbb{R}^3)} \le \exp\exp A^{O(1)}. \tag{5.38}$$

We note that we will only use (in (5.43) below) the bound on Γ.

Proof. We fix a cutoff function $\phi \in C_c^\infty((-1,1);[0,1])$ such that $\phi \equiv 1$ in $[-1/2,1/2]$, and we define the translate

$$\phi_z(y) \coloneqq \phi(y-z).$$

Clearly, we have the pointwise inequality

$$\phi_z', \phi_z'' \lesssim \sum_{i=-2}^{2} \phi_{z+i}.$$

We will consider the energies

$$\begin{aligned} E(t) &\coloneqq \sup_{z\in\mathbb{R}} E_z(t), \quad E_z(t) \coloneqq \frac{1}{2}\int_{\mathbb{R}^3} (\Phi(t,x)^2 + \Gamma(t,x)^2)\phi_z(x_3)dx, \\ F(t) &\coloneqq \sup_{z\in\mathbb{R}} F_z(t), \quad F_z(t) \coloneqq \int_{t_0}^t \int_{\mathbb{R}^3} (\nabla\Phi(s,x)^2 + \nabla\Gamma(s,x)^2)\phi_z(x_3)dx\, ds \end{aligned}$$

for $t \in [t_0,1]$, where $t_0 \in [0,1]$ will be chosen in Step 3 below. Given $z \in \mathbb{R}$, we multiply the equations (5.8) by $\phi_z\Gamma$ and $\phi_z\Phi$, respectively, and integrate to obtain, at a given time t,

$$\begin{aligned} E_z' &\le \int_{\mathbb{R}^3} \Big(-(|\nabla\Phi|^2 + |\nabla\Gamma|^2)\phi_z + \frac{1}{2}(\Phi^2+\Gamma^2)(u_z\phi_z' + \phi_z'') \\ &\qquad + (\omega_r\partial_r + \omega_3\partial_3)\frac{u_r}{r}\Phi\phi_z - 2r^{-1}u_\theta\Phi\Gamma\phi_z\Big)dx \\ &=: -F_z'(t) + I_1 + I_2 + I_3. \end{aligned} \tag{5.39}$$

The second term on the right hand side can be bounded directly,

$$I_1 \lesssim (1 + \|u_z\|_{L^\infty_x(\mathbb{R}^3)})E(t). \tag{5.40}$$

The remaining terms I_2, I_3 are more challenging—we estimate them and choose t_0 as follows.

Step 1. We use the Hölder continuity proved in Proposition A.4 to show that $|\Theta| \le r^\gamma A^{O(1)}$ whenever $r \le \frac{1}{2}$, where $\gamma = \exp(-A^{O(1)})$.

To this end we note that, due to incompressibility, $\operatorname{div}(u + \frac{2}{r}e_r) = 4\pi\delta_{\{x'=0\}}$, which enables us to apply Proposition A.4 to the equation for the swirl Θ (recall (5.11)).

Moreover, in the notation of Proposition A.4, for every $R < \frac{1}{2}$, $t_0 \in [\frac{1}{2}, 1]$ and $x_0 \in (0,0) \times \mathbb{R}$ (i.e., on the x_3-axis),

$$R^{-\frac{4}{5}} \|u + \frac{e_r}{r}\|_{L^\infty_t L^{\frac{5}{3}}_x(Q((t_0,x_0),R))} \lesssim R^{-\frac{1}{2}} \|u\|_{L^\infty_t L^2_{uloc}([t_0 - R^2, t_0] \times \mathbb{R}^3)} + 1 \le A^{O(1)}$$

by Hölder's inequality and (2.9) applied on the timescale R^2. (In particular note that each scale R leads to a different decomposition $u = u_n^\flat + u_n^\sharp$, but they all obey the same bounds up to being suitably rescaled.) Thus, for every $r \in (0, 1/2)$, $\operatorname{osc}_{B(x_0,r)} \Theta(t_0) \lesssim r^\gamma \operatorname{osc}_{Q(1/2)} \Theta$ for $r \in (0, 1/2)$, which implies the claim.

Step 2. We show that

$$\int_{t_0}^t |I_2 + I_3| \lesssim \frac{1}{2} F(t) + r_0^{-10} + \int_{t_0}^t GE$$

for each $t_0 \in [t/2, t]$, where

$$r_0 \coloneqq \mathrm{e}^{-\gamma^{-2}}, \tag{5.41}$$

$\gamma = \exp(-A^{O(1)})$ is given by Step 1, and

$$G \coloneqq r_0^{-3} + \|u\|_\infty + \|D^2 u\|_{L^{5/4}_{uloc}} + \|\nabla u\|_{L^2_{uloc}}$$

at each $t' \in [t_0, t]$.

To this end, we proceed similarly to [16]. Using integration by parts, we compute

$$
\begin{aligned}
I_2 &= 2\pi \int_{\mathbb{R}} \int_0^\infty (-\partial_3 u_\theta \partial_r \frac{u_r}{r}\Phi + \frac{\partial_r(ru_\theta)}{r}\partial_3 \frac{u_r}{r}\Phi)\phi_z(x_3) r\, dr\, dx_3 \\
&= \int_{\mathbb{R}^3} u_\theta (\partial_r \frac{u_r}{r}\partial_3 \Phi \phi_z - \partial_3 \frac{u_r}{r}\partial_r \Phi \phi_z + \partial_r \frac{u_r}{r}\Phi \phi_z') \\
&=: I_{2,1} + I_{2,2} + I_{2,3}.
\end{aligned}
$$

Let us further decompose $I_{2,i} = I_{2,i,in} + I_{2,i,out}$ $(i = 1,2,3)$ by writing

$$
\int = \int_{\{r<r_0\}} + \int_{\{r \geq r_0\}},
$$

and yet further

$$
I_{2,1,in} = I_{2,1,in,1} + I_{2,1,in,2}
$$

where

$$
I_{2,1,in,1} \coloneqq \int_{\{r<r_0\}} u_\theta \left(\fint_\Omega \partial_r \frac{u_r}{r} \right) \partial_3 \Phi \phi_z
$$

and $\Omega \coloneqq \{x' \colon r < 1\} \times \operatorname{supp} \phi_z$. We compute using Hölder's inequality and Sobolev embedding

$$
\begin{aligned}
\left| \int_\Omega \partial_r \frac{u_r}{r} \right| &\leq \|r^{-1}\partial_r u_r\|_{L^1(\Omega)} + \|r^{-2}u_r\|_{L^1(\Omega)} \\
&\lesssim \|r^{-1}\|_{L^{15/8}(\Omega)} \|\nabla u\|_{L^{15/7}(\Omega)} \lesssim \|\nabla^2 u\|_{L^{5/4}(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \lesssim G.
\end{aligned}
$$

Thus, integrating by parts, and applying Hölder's inequality in Lorentz spaces (5.20), and Young's inequality, we obtain

$$
\begin{aligned}
|I_{2,1,in,1}| &\leq G \int_{B(r_0)\times\mathbb{R}} \left(r\Phi^2 \phi_z + |u_\theta \Phi \phi_z'| \right) dx \\
&\lesssim G(r_0 E + \|u_\theta\|_{L_x^{3,\infty}(\mathbb{R}^3)} \|\Phi\|_{L_x^2(\Omega)} |\Omega|^{\frac{1}{6}}) \\
&\lesssim G(E + A^{O(1)}).
\end{aligned}
$$

As for $I_{2,1,in,2}$ we note that $p=2(1-\gamma)/(1-2\gamma)$ is such that $p-2=2\gamma/(1-2\gamma)\geq\gamma$ and so we can use the quantitative Hardy inequality (Lemma 5.4) to obtain for $t\in[\frac{1}{2},1]$ that

$$\begin{aligned}|I_{2,1,in,2}| &\lesssim \|r^{\frac{3}{p}-\frac{1}{2}}u_\theta\|_{L^{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}}(\{r\leq r_0\}\cap \operatorname{supp}\phi_z)} \left\| r^{-\frac{3}{p}+\frac{1}{2}}(\partial_r \frac{u_r}{r} - \fint_\Omega \partial_r \frac{u_r}{r})\phi_z^{\frac{1}{2}} \right\|_{L^p(\{r\leq 1\})} \|\partial_3\Phi\phi_z^{\frac{1}{2}}\|_2\\
&\lesssim \gamma^{-O(1)} r_0^{\gamma/3} \left\|\nabla\partial_r \frac{u_r}{r}\right\|_{L^2_{3-uloc}} \|\nabla\Phi\phi_z^{\frac{1}{2}}\|_2\\
&\leq \mathrm{e}^{-\gamma^{-1}/4}(\|\nabla\Gamma\|_{L^2_{3-uloc}(\mathbb{R}^3)} + \|\Gamma\|_{L^2_{3-uloc}(\mathbb{R}^3)})\|\nabla\Phi\phi_z^{\frac{1}{2}}\|_2,\end{aligned}$$

where we have also applied Poincaré's inequality and our choice (5.41) of r_0. Thus

$$\int_{t_0}^t I_{2,1,in,2} \leq \frac{1}{20}F(t) + \int_{t_0}^t E.$$

An analogous argument with "∂_r" and "∂_3" interchanged yields the same bound for $I_{2,2,in,2}$. As for $I_{2,2,in,1}$, we integrate by parts and apply (5.20) and Young's inequality to obtain

$$\begin{aligned}|I_{2,2,in,1}| &\leq \left|\fint_\Omega \partial_3 \frac{u_r}{r}\phi_z\right| \int_{\{r\leq r_0\}\cap\operatorname{supp}\phi_z} |u_\theta\partial_r\Phi|\\
&\lesssim \left|\fint_\Omega \frac{u_r}{r}\phi_z'\right| \|u_\theta\|_{L^{3,\infty}} \|\nabla\Phi\|_{L^2_x(\operatorname{supp}\phi_z)} r_0^{\frac{1}{3}}\\
&\lesssim \sum_{i=-2}^{2} \|\nabla u\|_{L^1(\Omega)} A (F_{z+i}')^{\frac{1}{2}} r_0^{\frac{1}{3}}\\
&\lesssim GAr_0^{1/3}\left(\sum_{i=-2}^{2} F_{z+i}'\right)^{\frac{1}{2}},\end{aligned}$$

which, thanks to the smallness of $r_0=\exp(-\exp(A^{O(1)}))$ (recall (5.41)), gives

$$\int_{t_0}^t |I_{2,2,in,1}| \leq \frac{1}{20}F(t) + (t-t_0).$$

We similarly decompose $I_{2,3,in} = I_{2,3,in,1} + I_{2,3,in,2}$ to find

$$\begin{aligned}|I_{2,3,in,1}| = \left|\fint_\Omega \partial_r \frac{u_r}{r}\right| \left|\int_{\{r\leq r_0\}} u_\theta \Phi\phi_z'\right| &\lesssim (\|\nabla u\|_{L^2(\Omega)} + \|\nabla^2 u\|_{L^{5/4}(\Omega)}) A E^{\frac{1}{2}} r_0^{\frac{1}{3}}\\
&\lesssim G(E+1)\end{aligned}$$

where we have used Lemma 5.4 and change of variables, the pointwise estimate $|u_r/r| \le |\nabla u|$, and Hölder's inequality to bound

$$\begin{aligned}\left|\fint_\Omega \partial_r \frac{u_r}{r}\right| &\lesssim \int_{z-10}^{z+10}\int_0^1 \left(|\partial_r u_r| + \frac{|u_r|}{r}\right) dr\, dz\\ &\lesssim \|r^{-1}\partial_r u_r\|_{L^1(\Omega)} + \|r^{-1}\nabla u\|_{L^1(\Omega)}\\ &\lesssim \|r^{-1}\nabla u\|_{L^{5/4}(\Omega)}\\ &\lesssim \|\nabla u\|_{L^2(\Omega)} + \|\nabla^2 u\|_{L^{5/4}(\Omega)},\end{aligned}$$

where we used (5.27) in the third line, and the Hardy inequality (5.25) in the last line. Next

$$\begin{aligned}|I_{2,3,in,2}| &= \left|\int_{\{r\le r_0\}} u_\theta \left(dd_r\frac{u_r}{r} - \fint_\Omega \partial_r\frac{u_r}{r}\right)\Phi\phi_z'\right|\\ &\lesssim \|ru_\theta\|_{L^3(\{r\le r_0\})}\left\|r^{-\frac{1}{2}}\left(dd_r\frac{u_r}{r} - \fint_\Omega \partial_r\frac{u_r}{r}\right)\right\|_{L^3(\mathbb{R}^2\times \operatorname{supp}\phi_z)}\|r^{-\frac{1}{2}}\Phi\|_{L^3(\mathbb{R}^2\times\operatorname{supp}\phi_z)}\\ &\le A^{O(1)} r_0^{\frac{2}{3}}\left\|\nabla\partial_r\frac{u_r}{r}\right\|_{L^2_{3-uloc}}\|\nabla\Phi\|_{L^2_{3-uloc}},\end{aligned}$$

where we have used the Hardy inequality (Lemma 5.4). Thus Lemma 5.6 and Young's inequality imply that

$$\int_{t_0}^t |I_{2,3,in,2}| \le \frac{1}{20}F(t) + \int_{t_0}^t E.$$

Next let us consider the contributions to I_2 from outside $B(r_0)$. Using Hölder's inequality, we obtain that

$$\begin{aligned}|I_{2,1,out}| &= \left|\int_{\{r>r_0\}} u_\theta \partial_r\frac{u_r}{r}\partial_3\Phi\phi_z\, dx\right|\\ &\le \|u_\theta\|_{L^6_{3-uloc}(\{r>r_0\})}\|r^{-1}\partial_r u_r - r^{-2}u_r\|_{L^3_{3-uloc}(\{r>r_0\})}\|\nabla\Phi\|_{L^2_{3-uloc}(\mathbb{R}^3)}.\end{aligned}$$

Hence, since Proposition 2.19 shows that $|u| \le A^{O(1)}(r^{-1}+r^{-1/4})$ and $|\partial_r u_r| \le A^{O(1)}(r^{-2}+r^{1/4})$, we see that the first two norms on the right hand side are finite and bounded by, say, r_0^{-10}. Thus, an application of Young's inequality gives that

$$\int_{t_0}^t |I_{2,1,out}| \le \frac{1}{20}F(t) + r_0^{-10}(t-t_0).$$

The remaining outer parts of I_2, i.e. $I_{2,2,out}$ and $I_{2,3,out}$ can be estimated in a similar way, with the latter bounded by, say, $E + r_0^{-10}$.

Finally let us consider I_3. Taking p such that, for example, $\frac{1}{p} = \frac{1}{2} - \frac{\gamma}{4}$, we have $p - 2 = 2\gamma/(2-\gamma) \geq \gamma$, and so our quantified Hardy's inequality (Lemma 5.4) shows that

$$\begin{aligned}|I_{3,in}| &\leq \left\| r^{-2+\frac{6}{p}} u_\theta \right\|_{L^{\left(1-\frac{2}{p}\right)^{-1}}(\{r\leq r_0\})} \|r^{-\frac{3}{p}+\frac{1}{2}}\Phi\|_{L^p_{3-uloc}} \|r^{-\frac{3}{p}+\frac{1}{2}}\Gamma\|_{L^p_{3-uloc}} \\ &\lesssim \gamma^{-O(1)} r_0^{\gamma/2} \left(\|\Phi\|_{L^2_{3-uloc}} + \|\nabla\Phi\|_{L^2_{3-uloc}} \right) \left(\|\Gamma\|_{L^2_{3-uloc}} + \|\nabla\Gamma\|_{L^2_{3-uloc}} \right),\end{aligned}$$

which gives that $\int_{t_0}^t |I_{3,in}| \leq \frac{1}{20} F(t) + \int_{t_0}^t E$. On the other hand, for $r \geq r_0$ we have the simple bound

$$|I_{3,out}| \leq 2\|r^{-1}u_\theta\|_{L^\infty_x(\{r\geq r_0\})} \|\Phi\|_{L^2_{3-uloc}} \|\Gamma\|_{L^2_{3-uloc}} \leq r_0^{-5/4} E,$$

as required.

Step 3. Given $\tau > 0$ we use the choice of time of regularity (Proposition 2.14) to find $t_0 \in [1-\tau, 1]$ such that $E(t_0) \lesssim A^{O(1)}\tau^{-3}$.

Indeed, Proposition 2.14 lets us choose $t_0 \in [1-\tau, 1]$ such that

$$\|\nabla^2 u(t_0)\|_\infty \leq A^{O(1)} \tau^{-\frac{3}{2}}.$$

It follows from the axial symmetry and (5.27) that $|\Phi| + |\Gamma| \leq |\nabla\omega|$, and so

$$\|\Phi(t_0)\phi_z^{1/2}\|_{L^2(\{r\leq 1\})} + \|\Gamma(t_0)\phi_z^{1/2}\|_{L^2(\{r\leq 1\})} \lesssim \|\nabla\omega(t_0)\|_{L^\infty(B(1)\times\mathbb{R})} \leq A^{O(1)}\tau^{-\frac{3}{2}} \tag{5.42}$$

for every $z \in \mathbb{R}$. Using the decomposition $\omega = \omega_1^\sharp + \omega_1^\flat$ on the interval $[0,1]$, by (2.9), (2.6), and Hölder's inequality,

$$\begin{aligned}\|\Phi(t_0)\phi_z^{1/2}\|_{L^2(\{r>1\})} + \|\Gamma(t_0)\phi_z^{1/2}\|_{L^2(\{r>1\})} &\lesssim \|\omega_1^\sharp\|_{L^2(\mathbb{R}^3)} + \|r^{-1}\omega_1^\flat\|_{L^2(\{r>1\}\cap \text{supp }\phi_z)} \\ &\lesssim \|\nabla u_1^\sharp\|_{L^2(\mathbb{R}^3)} + \|r^{-1}\|_{L^4_{x'}(B(1)^c)} \|\omega_1^\flat\|_{L^4(\mathbb{R}^3)} \\ &\leq A^{O(1)}.\end{aligned}$$

This and (5.42) proves the claim of this step.

Step 4. We prove the desired estimate on Φ and Γ.

Integration in time of the energy inequality (5.39) from initial time t_0 chosen in Step 3 above, taking $\sup_{z\in\mathbb{R}}$, and applying the estimate (5.40) for I_1 and Step 2 for I_2, I_3 we find that

$$E(t)+\frac{1}{2}F(t)\le \underbrace{E(t_0)}_{\le A^{O(1)}\tau^{-3}} +r_0^{-10}+\int_{t_0}^{t} O(r_0^{-3}+\|u\|_\infty+\|\nabla^2 u\|_{L^{5/4}_{uloc}}+\|\nabla u\|_{L^2_{uloc}})E(t')dt'$$

for $t\in[t_0,1]$. Thus, by Grönwall's inequality,

$$E(1)\le (A^{O(1)}\tau^{-3}+r_0^{-10})\exp\left(O\left(r_0^{-3}(t-t_0)+A^{O(1)}(t-t_0)^{\frac{1}{5}}\right)\right).$$

Setting $\tau\coloneqq r_0^4$, we see that the last exponential function is $O(1)$, and the prefactor gives the required estimate (5.38). □

5.5 Main type I theorem

In this section we prove Theorem 5.1. Namely, given the $L^{3,\infty}$ bound (5.4) on time interval $[0,1]$, we show that $|\nabla^j u|\le \exp\exp A^{O_j(1)}$ at time 1.

`Step 1.` We show that $\|b\|_{L^p_{3-uloc}(\mathbb{R}^3)}\le C_p\exp\exp A^{O(1)}$ for each $p\in[3,\infty)$, $t\in[1/2,1]$, where $b\coloneqq u_r e_r+u_z e_z$ denotes the swirl-free part of the velocity field.

To this end we apply Proposition 5.7 to find

$$\|\Gamma\|_{L^\infty_t L^2_{3-uloc}([\frac{1}{2},1]\times\mathbb{R}^3)}\le \exp\exp A^{O(1)}. \tag{5.43}$$

On the other hand Proposition 2.19 shows that

$$\|r^2\omega\|_{L^\infty_x(\{r\le 10\})} \le A^{O(1)}.$$

Interpolating between this inequality and (5.43) we obtain

$$\|\omega_\theta\|_{L^p_{3-uloc}(\{r\le 10\})} = \|\Gamma^{\frac{2}{3}}(r^2\omega_\theta)^{\frac{1}{3}}\|_{L^p_{3-uloc}(\{r\le 10\})} \lesssim \|\Gamma\|^{\frac{2}{3}}_{L^2_{3-uloc}} \|r^2\omega_\theta\|^{\frac{1}{3}}_{L^\infty_x(\{r\le 10\})} \le \exp\exp A^{O(1)}$$

for all $p \le 3$.

Recalling that

$$\operatorname{curl} b = \omega_\theta e_\theta, \quad \operatorname{div} b = 0$$

almost everywhere, we localize b to obtain an L^p estimate near the axis. Namely, for any unit ball $B \subset \{r \le 10\}$, let $\phi \in C^\infty_c(B)$ such that $\phi \equiv 1$ on $B/2$. Observe that for all $p \in [1,3)$ we can use Hölder's inequality for Lorentz spaces (5.20) to obtain

$$\|\operatorname{div}(\phi b)\|_{L^p(\mathbb{R}^3)} = \|b\cdot\nabla\phi\|_p \lesssim \|b\|_{L^{3,\infty}}\|\nabla\phi\|_{L^{3p/(3-p),1}} \lesssim A.$$

Applying the Bogovskiĭ operator (5.23) to $\operatorname{div}(\phi b)$ on the domain $B\setminus(B/2)$, we find $\tilde b \in W^{1,p}$ such that $\operatorname{div}\tilde b = 0$, $\|b - \tilde b\|_{W^{1,p}(B)} \le A^{O(1)}$, $\tilde b \equiv b$ in $B/2$, and $\tilde b \equiv 0$ outside B. Then for any $p \in (1,3)$,

$$\begin{aligned}\|b\|_{L^{3p/(3-p)}(B/2)} &\le \|\tilde b\|_{3p/(3-p)} \lesssim \|\nabla \tilde b\|_p \lesssim \|\operatorname{curl}\tilde b\|_{L^p(B)}\\ &\le \|\omega_\theta\|_{L^p(B)} + \|b-\tilde b\|_{W^{1,p}(B)}\\ &\le \exp\exp A^{O(1)},\end{aligned}$$

which is our desired localized estimate. Here we have used the boundedness of the operator $\operatorname{curl} f \mapsto \nabla f$ in L^p, $p \in (1,\infty)$, (which is a consequence of the identity $\operatorname{curl}\operatorname{curl} f = \nabla(\operatorname{div} f) - \Delta f$, which in turn implies that $\nabla f = \nabla(-\Delta)^{-1}\operatorname{curl}\operatorname{curl} f$ for divergence-free f). Combining this with the pointwise estimates away from the axis (Proposition 2.19) gives the claim of this step.

`Step 2.` We show that there exists $C_0>1$ such that

$$\Big\|\frac{u_\theta(t)}{r^{\frac12}}\Big\|^4_{L^4_{3-uloc}} \le \Big\|\frac{u_\theta(t_0)}{r^{\frac12}}\Big\|^4_{L^4_{3-uloc}} + 1 + \exp\exp A^{C_0}\int_{t_0}^t \Big\|\frac{u_\theta}{r^{\frac12}}\Big\|_{L^4_{3-uloc}}$$

for each $t_0\in[1/2,1]$ and $t\in[t_0,1]$.

To this end we provide a localization of the estimate of $u_\theta/r^{1/2}$ in the spirit of [16, Lemma 3.1]. Indeed, one can calculate from the equation (5.35) for u_θ that for a smooth cutoff $\psi=\psi(x_3)$,

$$\begin{aligned}\frac14\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{u_\theta^4}{r^2}\psi + \frac34\int_{\mathbb{R}^3}\Big|\nabla\frac{u_\theta^2}{r}\Big|^2\psi + \frac34\int_{\mathbb{R}^3}\frac{u_\theta^4}{r^4}\psi_z\\ = -\frac32\int_{\mathbb{R}^3}\frac{1}{r^3}u_r u_\theta^4\psi + \frac18\int_{\mathbb{R}^3}\frac{1}{r^2}u_\theta^2(2u_\theta^2u_z-\partial_z(u_\theta^2))\psi' =: I_1+I_2+I_3.\end{aligned}$$

As before, we choose $\psi\in C_c^\infty((-2,2))$ with $\psi\equiv1$ in $[-1,1]$ and define the translates $\psi_z(x):=\psi(x-z)$ for all $z\in\mathbb{R}$. Consider the energies

$$E_z(t):=\frac14\int_{\mathbb{R}^3}\frac{u_\theta^4}{r^2}\psi_z,\qquad F_z(t):=\frac34\int_{t_0}^t\int_{\mathbb{R}^3}\Big|\nabla\frac{u_\theta^2}{r}\Big|^2\psi_z,$$
$$E(t):=\sup_{z\in\mathbb{R}}E_z(t),\qquad F(t):=\sup_{z\in\mathbb{R}}F_z(t).$$

By Step 1 and Sobolev embedding,

$$\begin{aligned}|I_1| &\lesssim \|u_r\|_{L^6_{3-uloc}}\Big\|r^{-\frac12}\frac{u_\theta^2}{r}\Big\|^2_{L^{12/5}(\Omega)}\\ &\le \exp\exp A^{O(1)}\left(\Big\|\frac{u_\theta^2}{r}\Big\|^{\frac12}_{L^2(\Omega)}\Big\|\nabla\frac{u_\theta^2}{r}\Big\|^{\frac32}_{L^2(\Omega)} + \Big\|\frac{u_\theta^2}{r}\Big\|^{\frac12}_{L^2(\Omega)}\right),\end{aligned}$$

where $\Omega:=\mathbb{R}^2\times\operatorname{supp}\psi$. It follows that

$$\int_{t_0}^t|I_1|\le\frac{1}{20}F(t)+\exp\exp A^{O(1)}\int_{t_0}^t E+(t-t_0).$$

Similarly,

$$\begin{aligned}|I_2| &\lesssim \|u_z\|_{L^6_{3-uloc}}\Big\|\frac{u_\theta^2}{r}\Big\|_{L^2_{3-uloc}}\Big\|\frac{u_\theta^2}{r}\Big\|_{L^3(\Omega)}\\ &\le \exp\exp A^{O(1)}E^{\frac12}\left(\Big\|\frac{u_\theta^2}{r}\Big\|^{\frac12}_{L^2(\Omega)}\Big\|\nabla\frac{u_\theta^2}{r}\Big\|^{\frac12}_{L^2(\Omega)} + \Big\|\frac{u_\theta^2}{r}\Big\|_{L^2(\Omega)}\right)\end{aligned}$$

which yields the same bound as I_1. Finally,

$$|I_3| = \frac{1}{8}\left|\int_{\mathbb{R}^3} \frac{u_\theta^2}{r}\partial_3\frac{u_\theta^2}{r}\psi'\right| \lesssim \left\|\frac{u_\theta^2}{r}\right\|_{L^2_{3-uloc}}\left\|\nabla\frac{u_\theta^2}{r}\right\|_{L^2(\Omega)}$$

so we have

$$\int_{t_0}^{t} |I_3| \leq \frac{1}{20}F(t) + \int_{t_0}^{t} O(E).$$

Summing and taking the supremum over $z \in \mathbb{R}$ gives the claim of this step.

`Step 3.` We deduce that

$$\|u\|_{L_t^\infty L^6_{3-uloc}([t_0,1]\times\mathbb{R}^3)} \leq \exp\exp A^{O(1)}, \tag{5.44}$$

where

$$t_0 := 1 - \exp(-\exp A^{O(1)}).$$

Indeed, Lemma 2.14 and Proposition 2.19 give a $t_0 \in [1-\exp(-\exp A^{C_0}), 1]$ such that $\|r^{-\frac{1}{2}}u_\theta(t_0)\|_{L^4_x(\mathbb{R}^3)} \leq \exp\exp A^{2C_0}$. Therefore, applying Grönwall's inequality to the claim of the previous step,

$$\left\|\frac{u_\theta}{r^{\frac{1}{2}}}\right\|_{L_t^\infty L^4_{3-uloc}([t_1,1]\times\mathbb{R}^3)} \leq \exp\exp A^{O(1)}.$$

Combining this with Proposition 2.19 and Hölder's inequality,

$$\begin{aligned}\|u_\theta\|_{L_t^\infty L^6_{3-uloc}([t_1,1]\times\mathbb{R}^3)} &\leq \|ru_\theta\|^{\frac{1}{3}}_{L^\infty_x(\{r\leq 1\})}\|r^{-\frac{1}{2}}u_\theta\|^{\frac{2}{3}}_{L_t^\infty L^4_{3-uloc}([t_1,1]\times\mathbb{R}^3)} + \|u\|_{L_t^\infty L^6_x([t_1,1]\times\{r>1\})}\\ &\leq \exp\exp A^{O(1)},\end{aligned}$$

which, together with Step 1, implies (5.44).

We note that Step 3 already provides a subcritical local regularity condition of the type of Ladyzhenskaya-Prodi-Serrin, which guarantees local boundedness of all spatial derivatives of u, and can be proved by employing the vorticity equation for example (see [50, Theorem 13.7]). In the last step below we use a robust tail estimate of the pressure function (recall Lemma 5.3) to provide a simpler justification of pointwise bounds by $\exp\exp A^{O(1)}$.

`Step 4.` We prove that, if $\|u\|_{L^\infty([1-t_1,1];W^{k-1,6}_{uloc})} \lesssim \exp\exp A^{O(1)}$ for some $k \geq 1$ and $t_1 = \exp(-\exp A^{O(1)})$, then the same is true for k (with some other t_1 of the same order).

Let $I = [a,b] \subset [t_1, 1]$, and let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(t) = 0$ for $t < a + (b-a)/8$ and $\chi(t) = 1$ for $t > (a+b)/2$. We set $\phi \in C_c^\infty(B(0,2);[0,1])$ such that $\phi = 1$ on $B(0,1/2)$ and $\sum_{j\in\mathbb{Z}^3} \phi_j = 1$, where $\phi_j \coloneqq \phi(\cdot - j)$ for each $j \in \mathbb{R}^3$.

Letting $v \coloneqq \chi\phi\nabla^k u$ we see that $v(t_1) = 0$, and

$$
\begin{aligned}
v_t - \Delta v &= \underbrace{-\chi'\phi\nabla^k u - 2\chi\nabla\phi\cdot\nabla(\nabla^k u) - \chi\Delta\phi(\nabla^k u)}_{=:f_1} - \chi\phi\,\mathrm{div}(1+T)\nabla^k(u\otimes u) \\
&= f_1 - \phi\,\mathrm{div}(1+T)((\chi\nabla^k u\otimes u + u\otimes\chi\nabla^k u)\tilde{\phi}) \\
&\qquad - \chi\phi\,\mathrm{div}(1+T)\sum_{\substack{|\alpha|+|\beta|+|\gamma|=k\\ |\alpha|,|\beta|<k}} C_{\alpha,\beta,\gamma}(D^\alpha u\otimes D^\beta u D^\gamma\tilde{\phi}) - \chi\phi\,\mathrm{div}\,T\nabla^k(u\otimes u(1-\tilde{\phi})) \\
&=: f_1 + f_2 + f_3 + f_4.
\end{aligned}
$$

We can now estimate $\|v(t)\|_6$, by extracting the same norm on the right-hand side and ensuring that the length of the interval is sufficiently small, so that the norm can be absorbed.

Namely,

$$
\begin{aligned}
\|v(t)\|_6 &= \left\| \int_a^t \mathrm{e}^{(t-t')\Delta} f_1(t')\mathrm{d}t' + \int_a^t \mathrm{e}^{(t-t')\Delta} f_2(t') dt' + \int_a^t \mathrm{e}^{(t-t')\Delta} f_3(t')\mathrm{d}t' + \int_a^t \mathrm{e}^{(t-t')\Delta} f_4(t')\mathrm{d}t' \right\|_6 \\
&\leq \left(\|\chi \nabla^k u \tilde{\phi}\|_{L^\infty([a,t];L^6)} + \|\chi' \nabla^{k-1} u \tilde{\phi}\|_{L^\infty([a,1];L^6)} \right) \int_a^t \|\Psi(t-t')\|_{W^{1,1}} dt' \\
&\qquad + \|\chi \nabla^k u \tilde{\phi}^{1/2}\|_{L^\infty([a,t];L^6)} \|u \tilde{\phi}^{1/2}\|_{L^\infty([a,t];L^6)} \int_a^t \|\Psi(t-t')\|_{W^{1,6/5}} dt' \\
&\qquad + \|u\|^2_{L^\infty([a,1];W^{k-1,6}_{uloc})} \int_a^t \|\Psi(t-t')\|_{W^{1,6/5}} dt' \\
&\qquad + \|\operatorname{div} T(u\otimes u(1-\tilde{\phi}))\|_{L^\infty([a,1];W^{k,6}(B(0,2)))} \int_a^t \|\Psi(t-t')\|_1 dt' \\
&\leq \|\chi \nabla^k u\|_{L^\infty([a,t];L^6_{uloc})} \left((b-a)^{1/2} + \exp\exp A^{O(1)} (b-a)^{1/4} \right) + \exp\exp A^{O(1)}
\end{aligned}
$$

for each $t \in (a,b)$, where we used Young's inequality, heat estimates (5.18) and the Calderón-Zygmund inequality. By replacing ϕ (in the definition of v) by ϕ_z for any $z \in \mathbb{R}^3$, we obtain the same bound, and so

$$
\|\chi \nabla^k u\|_{L^\infty([a,b];L^6_{uloc})} \leq \|\chi \nabla^k u\|_{L^\infty([a,b];L^6_{uloc})} (b-a)^{1/4} \exp\exp A^{O(1)} + \exp\exp A^{O(1)}.
$$

Thus, for any b,a such that $t_1 \leq a < b \leq 1$ and $(b-a)^{1/4} \leq \exp\exp A^{O(1)}/2$ we can absorb the first term on the right-hand side by the left-hand side to obtain

$$
\|\nabla^k u\|_{L^\infty([(a+b)/2,b];L^6_{uloc})} \leq \exp\exp A^{O(1)}.
$$

Since the upper bound is independent of the location of $[a,b] \subset [t_1,1]$, we obtain the claim.

5.5.0.1 Proof of Lemma 5.6

Here we prove Lemma 5.6, namely that

$$
\left\| \nabla \partial_r \frac{u_r}{r} \right\|_{L^2_{3-uloc}} + \left\| \nabla \partial_3 \frac{u_r}{r} \right\|_{L^2_{3-uloc}} \lesssim \|\Gamma\|_{L^2_{3-uloc}} + \|\nabla \Gamma\|_{L^2_{3-uloc}}. \tag{5.45}
$$

To this end we recall (5.34) that

$$
\frac{u_r}{r} = \Delta^{-1}\partial_3 \Gamma - 2\frac{\partial_r}{r}\Delta^{-2}\partial_3\Gamma.
$$

Since

$$\frac{\partial_r}{r} = \Delta' - \partial_{rr},$$

we have that

$$\frac{u_r}{r} = -\Delta^{-1}\partial_3\Gamma + 2(\partial_{rr} - \Delta')\Delta^{-2}\partial_3\Gamma. \tag{5.46}$$

Thus, since $|\nabla\partial_3 \frac{u_r}{r}| = |(\partial_r\partial_3 \frac{u_r}{r}, \partial_3\partial_3 \frac{u_r}{r})|$ (and similarly for $|\nabla\partial_r \frac{u_r}{r}|$), we can use (5.28) and (5.29) to observe that

$$\begin{aligned}\left|\nabla\partial_3 \frac{u_r}{r}\right| + \left|\nabla\partial_r \frac{u_r}{r}\right| &\lesssim |D^2_{r,x_3}\Delta^{-1}\partial_3\Gamma| + |D^2_{r,x_3}(\partial_{rr} - \Delta')\Delta^{-2}\partial_3\Gamma| \\ &\lesssim |\nabla\Gamma| + |D^2\Delta^{-1}\nabla'\Gamma| + |D^4\Delta^{-2}\nabla'\Gamma|,\end{aligned}$$

where we used $\partial_{33} = \Delta - \Delta'$ in the last line. In particular, each of the terms on the right-hand side involves at least one derivative in the horizontal variables. Thus, in order to estimate the left-hand side of (5.45) it suffices to find suitable bounds on the last two terms, which we achieve in Lemmas 5.8–5.9 below. Their claims give us (5.45), as required.

Lemma 5.8. *Let $f = \Delta^{-1}\nabla'\Gamma$. Then*

$$\|D^2 f\|_{L^2_{3-uloc}} \leq \|\Gamma\|_{L^2_{3-uloc}} + \|\nabla\Gamma\|_{L^2_{3-uloc}}$$

Proof. Let $I(x)$ denote the kernel matrix of $D^2(-\Delta)^{-1}$. We have that

$$|\nabla^j I(x)| \leq \frac{C}{|x|^{3+j}} \qquad \text{for } j = 0, 1,$$

and

$$\begin{aligned}D^2 f(x) &= \text{p.v.} \int_{\mathbb{R}^3} I(x-y)\nabla'\Gamma(y)dy \\ &= \text{p.v.} \int_{\mathbb{R}^3} \nabla'\Gamma(y)\tilde{\phi}(y_3)I(x-y)dy + \text{p.v.} \int_{\mathbb{R}^3} \Gamma(y)(1-\tilde{\phi}(y_3))\nabla' I(x-y)dy \\ &=: f_1(x) + f_2(x).\end{aligned}$$

The Calderón-Zygmund inequality gives that

$$\|f_1\|_{L^2_{3-uloc}} \leq \|\Gamma\|_{L^2_{3-uloc}} + \|\nabla\Gamma\|_{L^2_{3-uloc}}.$$

Moreover, noting that $\int_{\mathbb{R}^2} \frac{dx_1\, dx_2}{(a^2+x_1^2+x_2^2)^2} = Ca^{-2}$, we can use Young's inequality for convolutions to obtain

$$\begin{aligned}
\|f_2(\cdot, x_3)\|_{L^2} &\leq \int_{\mathbb{R}} \frac{\|\Gamma(\cdot, y_3)\|_{L^2}(1-\tilde{\phi}(y_3))}{|x_3-y_3|^2} dy_3 \\
&\leq \sum_{j\geq 1} \int_{\{|x_3-y_3|\in(j,j+1)} \frac{\|\Gamma(\cdot, y_3)\|_{L^2}(1-\tilde{\phi}(y_3))}{|x_3-y_3|^2} dy_3 \\
&\leq \sum_{j\geq 1} j^{-2} \int_{\{|x_3-y_3|\in(j,j+1)} \|\Gamma(\cdot, y_3)\|_{L^2} dy_3 \\
&\leq \|\Gamma\|_{L^2_{3-uloc}}.
\end{aligned}$$

integration in x_3 over supp ϕ finishes the proof. □

For the double Laplacian term one needs to work harder:

Lemma 5.9. *Let $f = D^4\Delta^{-2}\nabla'\Gamma$. Then*

$$\|f\|_{L^2_{3-uloc}} \leq \|\Gamma\|_{L^2_{3-uloc}} + \|\nabla\Gamma\|_{L^2_{3-uloc}}.$$

Proof. We have that

$$f(x) = \text{p.v.} \int_{\mathbb{R}^3} \text{p.v.} \int_{\mathbb{R}^3} \partial_3\Gamma(z) I(x-y) I(y-z) dz\, dy$$

Recalling that $\tilde{\phi} = \sum_{|j|\leq 10} \phi_j$, and $\tilde{\tilde{\phi}} = \sum_{|j|\leq 20} \phi_j$ we use the partition of unity,

$$\begin{aligned}
1 &= \tilde{\tilde{\phi}}(z_3) + (1-\tilde{\tilde{\phi}}(z_3))\tilde{\phi}(y_3) + \sum_{\substack{|j|>10 \\ |k|>20}} \phi_j(y_3)\phi_k(z_3) \\
&= \tilde{\tilde{\phi}}(z_3) + (1-\tilde{\tilde{\phi}}(z_3))\tilde{\phi}(y_3) \\
&\quad + \sum_{|j|>10} \phi_j(y_3) \left(\sum_{\substack{|k|>20 \\ |k-j|\leq 10}} \phi_k(z_3) + \sum_{\substack{|k|>20 \\ |k-j|>10 \\ k\leq j/2}} \phi_k(z_3) + \sum_{\substack{|k|>20 \\ |k-j|>10 \\ j/2<k\leq 2j}} \phi_k(z_3) + \sum_{\substack{|k|>20 \\ |k-j|>10 \\ k>2j}} \phi_k(z_3) \right),
\end{aligned}$$

to decompose f accordingly,

$$
\begin{aligned}
f(x) &= \text{p.v.} \int_{\mathbb{R}^3} \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z) \tilde{\tilde{\phi}}(z_3) I(x-y) I(y-z) dy\, dz \\
&\quad + \text{p.v.} \int_{\mathbb{R}^3} I(x-y) \tilde{\phi}(y_3) \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z)(1-\tilde{\tilde{\phi}}(z_3)) I(y-z) dz\, dy \\
&\quad + \text{p.v.} \int_{\mathbb{R}^3} I(x-y) \sum_{|j|>10} \phi_j(y_3) \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z) \sum_{\substack{|k|>20 \\ |k-j|\le 10}} \phi_k(z_3) I(y-z) dz\, dy \\
&\quad + \text{p.v.} \int_{\mathbb{R}^3} I(x-y) \sum_{|j|>10} \phi_j(y_3) \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z) \sum_{\substack{|k|>20 \\ |k-j|>10 \\ k \le j/2}} \phi_k(z_3) I(y-z) dz\, dy \\
&\quad + \text{p.v.} \int_{\mathbb{R}^3} I(x-y) \sum_{|j|>10} \phi_j(y_3) \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z) \sum_{\substack{|k|>20 \\ |k-j|>10 \\ j/2<k\le 2j}} \phi_k(z_3) I(y-z) dz\, dy \\
&\quad + \text{p.v.} \int_{\mathbb{R}^3} I(x-y) \sum_{|j|>10} \phi_j(y_3) \text{p.v.} \int_{\mathbb{R}^3} \nabla' \Gamma(z) \sum_{\substack{|k|>20 \\ |k-j|>10 \\ k>2j}} \phi_k(z_3) I(y-z) dz\, dy \\
&=: f_1(x) + f_2(x) + f_3(x) + f_4(x) + f_5(x) + f_6(x).
\end{aligned}
$$

Clearly f_1 involves localization of $\nabla' \Gamma$ in z_3, and so we can use the Calderón-Zygmund inequality twice to obtain

$$
\|f_1\|_{L^2} \lesssim \|\nabla \Gamma\|_{L^2_{3-uloc}}.
$$

As for f_2 we integrate by parts in the z-integral (note that this does not conflict with the principal value, as the singularity has been cut off, and the far field has sufficient decay) and apply the Calderón-Zygmund estimate in x to obtain

$$
\begin{aligned}
\|f_2\|_{L^2} &\lesssim \left\| \tilde{\phi}(y_3) \int_{\mathbb{R}^3} \frac{|\Gamma(z)|(1-\tilde{\tilde{\phi}}(z_3))}{|y-z|^4} dz \right\|_{L^2} \lesssim \sup_{y_3 \in \text{supp } \tilde{\phi}} \left\| \int_{\mathbb{R}^3} \frac{|\Gamma(z)|(1-\tilde{\tilde{\phi}}(z_3))}{|y-z|^4} dz \right\|_{L^2_{y'}} \\
&\lesssim \sup_{y_3 \in \text{supp } \tilde{\phi}} \int_{\mathbb{R}} \frac{\|\Gamma(\cdot, z_3)\|_{L^2}(1-\tilde{\tilde{\phi}}(z_3))}{|y_3-z_3|^2} dz_3 \\
&\lesssim \sup_{y_3 \in \text{supp } \tilde{\phi}} \sum_{j\ge 1} j^{-2} \int_{|z_3-y_3| \in (j, j+1)} \|\Gamma(\cdot, z_3)\|_{L^2_{z'}} dz_3 \lesssim \|\Gamma\|_{L^2_{3-uloc}}
\end{aligned}
$$

where we used Young's inequality in the second line (as in the lemma above).

As for f_3, we integrate by parts in z and then in y to obtain

$$|f_3(x)| \lesssim \sum_{|j|>10} \int_{\mathbb{R}^3} \frac{\phi_j(y_3)}{|x-y|^4} \left| \text{p.v.} \int_{\mathbb{R}^3} \Gamma(z) \sum_{\substack{|k|>20\\ |k-j|\le 10}} \phi_k(z_3) I(y-z) dz \right| dy.$$

We note that the integration by parts is justified as

$$f_3 = D^2(-\Delta)^{-1} \left((1 - \sum_{|j|\le 10} \phi_j(y_3)) D^2(-\Delta)^{-1} \left(\nabla' \Gamma (1 - \sum_{k\in I} \phi_k(z_3) \right) \right),$$

where $I \coloneqq \{-20,\dots,20\} \cup \{j-10,\dots,j+10\}$ is a finite index set. Thus, the operation of integration by parts above is equivalent to moving ∇' outside of the outer brackets, which in turn holds since the sums do not depend on x' and ∇' commutes with other differential symbols .

Thus, using Young's inequality in x'

$$\begin{aligned}
\|f_3(\cdot,x_3)\|_{L^2_{x'}} &\lesssim \sum_{|j|>10} \int_{\mathbb{R}} \frac{\phi_j(y_3)}{|x_3-y_3|^2} \left\| \text{p.v.} \int_{\mathbb{R}^3} \Gamma(z) \sum_{\substack{|k|>6\\ |k-j|\le 2}} \phi_k(z_3) I(y-z) dz \right\|_{L^2_{y'}} dy_3 \\
&\lesssim \sum_{|j|>2} j^{-2} \left\| \text{p.v.} \int_{\mathbb{R}^3} \Gamma(z) \sum_{\substack{|k|>20\\ |k-j|\le 10}} \phi_k(z_3) I(y-z) dz \right\|_{L^2_y} \\
&\lesssim \sum_{|j|>10} j^{-2} \left\| \Gamma(z) \sum_{\substack{|k|>20\\ |k-j|\le 10}} \phi_k(z_3) \right\|_{L^2} \lesssim \|\Gamma\|_{L^2_{3-uloc}}
\end{aligned}$$

for each $x_3 \in \text{supp } \phi$, where we applied the Cauchy-Schwarz inequality (in y_3) in the second line.

As for f_4 we note that

$$|y_3 - z_3| \ge |y_3| - |z_3| \ge (j-1) - (k+1) \ge \frac{j}{2} - 2 \ge (j+2)/4 \ge (|y_3|+1)/4 \ge |y_3 - x_3|/4$$

Thus, we can integrate by parts in z to obtain

$$|f_4(x)| \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \cap \{|y_3 - z_3| \geq |x_3 - y_3|/4\}} \frac{|\Gamma(z)|(1-\tilde{\phi}(y_3))(1-\tilde{\phi}(y_3 - z_3))}{|x-y|^3 |y-z|^4} dz\, dy$$

Thus, applying Young's inequality in x' and then in y' we obtain

$$\begin{aligned}\|f_4(\cdot, x_3)\|_{L^2} \leq \int_{\mathbb{R}} \left\| \int_{\mathbb{R}^3 \cap \{|y_3 - z_3| \geq |x_3 - y_3|/4\}} \frac{\Gamma(z)(1-\tilde{\phi}(y_3))(1-\tilde{\phi}(y_3 - z_3))}{|y-z|^4} dz \right\|_{L^2_{y'}} \\ \cdot \underbrace{\int_{\mathbb{R}^2} \frac{\mathrm{d}x_1\, dx_2}{\left(|x_3 - y_3|^2 + x_1^2 + x_2^2\right)^{3/2}}}_{=C|x_3 - y_3|^{-1}} \mathrm{d}y_3 \\ \lesssim \int_{\mathbb{R}} \int_{\mathbb{R} \cap \{|y_3 - z_3| \geq |x_3 - y_3|/4\}} \frac{\|\Gamma(\cdot, z_3)\|_{L^2}(1-\tilde{\phi}(y_3))(1-\tilde{\phi}(y_3 - z_3))}{|x_3 - y_3|\, |y_3 - z_3|^2} dz_3\, dy_3.\end{aligned} \tag{5.47}$$

Hence

$$\begin{aligned}\|f_4(\cdot, x_3)\|_{L^2} &\leq \int_{\mathbb{R}} \frac{1-\tilde{\phi}(y_3)}{|x_3 - y_3|^{3/2}} \left(\sum_{j \geq 1} \int_{\{|y_3 - z_3| \in (j, j+1)\}} \frac{\|\Gamma(\cdot, z_3)\|_{L^2}}{|y_3 - z_3|^{3/2}} dz_3 \right) dy_3 \\ &\lesssim \|\Gamma\|_{L^2_{3-uloc}} \int_{\mathbb{R}} \frac{1-\tilde{\phi}(y_3)}{|x_3 - y_3|^{3/2}} dy_3 \lesssim \|\Gamma\|_{L^2_{3-uloc}}.\end{aligned}$$

As for f_5 we have

$$\frac{1}{4} \leq \frac{|x_3 - y_3|}{|x_3 - z_3|} \leq 4,$$

since

$$|x_3 - y_3| \leq |y_3| + |x_3| \leq j + 2 \leq 2j - 8 \leq 4k - 8 \leq 4(|z_3| - |x_3|) \leq 4|x_3 - z_3|$$

and

$$|x_3 - z_3| \leq |z_3| + |x_3| \leq k + 2 \leq 2j + 2 \leq 4(j-2) \leq 4(|y_3| - |x_3|) \leq 4|x_3 - y_3|.$$

In particular, the triangle inequality gives that

$$|y_3 - z_3| \leq 5|x_3 - z_3|.$$

Thus we can integrate by parts twice (in z and then in y, so that the derivative falls on $I(x-y)$), and then use Young's inequality twice (as in (5.47) above) and Tonelli's Theorem to obtain

$$
\begin{aligned}
\|f_5(\cdot,x_3)\|_{L^2} &\le \int_{\mathbb{R}} \int_{\{|x_3-y_3|/4\le|x_3-z_3|\le 4|x_3-y_3|\}} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}(1-\tilde{\phi}(y_3-z_3))(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-y_3|^2|y_3-z_3|} dz_3\, dy_3 \\
&\le \int_{\mathbb{R}} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-z_3|^2} \int_{\{|y_3-z_3|\le 5|x_3-x_3|\}} \frac{1-\tilde{\phi}(y_3-z_3)}{|y_3-z_3|} dy_3\, dz_3 \\
&\lesssim \int_{\mathbb{R}} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-z_3|^2} \log(5|x_3-z_3|)\mathrm{d}z_3 \\
&\lesssim \sum_{j\ge 1} \int_{|z_3-x_3|\in(j,j+1)} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}}{|x_3-z_3|^2} \log(5|x_3-z_3|) dz_3 \\
&\lesssim \sum_{j\ge 1} j^{-2}\log(5j)\|\Gamma\|_{L^2_{3-uloc}} \lesssim \|\Gamma\|_{L^2_{3-uloc}}
\end{aligned}
$$

Finally, for f_6 we observe that

$$
\frac{1}{4} \le \frac{|x_3-z_3|}{|y_3-z_3|} \le 4,
$$

since

$$
|y_3-z_3| \ge |z_3|-|y_3| \ge k-j-2 > \frac{k-8}{2} \ge \frac{k+2}{4} \ge \frac{|x_3|+|z_3|}{4} \ge \frac{|x_3-z_3|}{4}
$$

and

$$
|y_3-z_3| \le |y_3|+|z_3| \le j+k+2 \le \frac{3k+4}{2} \le 4(k-2) \le 4(|z_3|-|x_3|) \le 4|x_3-z_3|.
$$

In particular, the triangle inequality gives that

$$
|x_3-y_3| \le 5|x_3-z_3|.
$$

Thus, similarly to the case of f_5 (although without integrating by parts in y), we apply

Young's inequality twice and Tonelli's Theorem to obtain

$$
\begin{aligned}
\|f_6(\cdot,x_3)\|_{L^2} &\le \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}(1-\tilde{\phi}(y_3))(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-y_3||y_3-z_3|^2} dz_3\, dy_3 \\
&\le \int_{\mathbb{R}} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-z_3|^2} \int_{\{\frac{1}{4}|z_3-x_3|\le|y_3-z_3|\le 4|z_3-x_3|\}} \frac{1-\tilde{\phi}(y_3)}{|x_3-y_3|} dy_3\, dz_3 \\
&\le \int_{\mathbb{R}} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-z_3|^2} \int_{\{1\le|x_3-y_3|\le 5|x_3-z_3|\}} \frac{1}{|x_3-y_3|} dy_3\, dz_3 \\
&\lesssim \int_{\mathbb{R}} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}(1-\tilde{\tilde{\phi}}(z_3))}{|x_3-z_3|^2} \log(5|x_3-z_3|) dz_3 \\
&\lesssim \sum_{j\ge 1} \int_{|z_3-x_3|\in(j,j+1)} \frac{\|\Gamma(\cdot,z_3)\|_{L^2}\log(5|x_3-z_3|)}{|x_3-z_3|^2} dz_3 \\
&\lesssim \sum_{j\ge 1} \log(5j) j^{-2} \|\Gamma\|_{L^2_{3-uloc}} \lesssim \|\Gamma\|_{L^2_{3-uloc}}
\end{aligned}
$$

for $x_3 \in \operatorname{supp}\phi$. Integration of the squares of the above estimates for f_3, f_4, f_5, f_6 gives the claim. $\square$

5.6 Lower bounds on the self-similar length scale

An application of the quantitative estimate in Theorem 5.1 above is an estimate on the length scale up to which an axisymmetric solution to the NSE (1.2) can be approximated by a self-similar profile as in (5.2).

In order to make this precise, we will say that a vector field $b \in L^\infty(\mathbb{R}^3;\mathbb{R}^3)$ is *nearly-spherical* if there exists $\delta \in (0, 1/2)$ such that for every $R > 0$, there exists $x_0 \in \mathbb{R}^3$ with $|x_0| = R$ such that

$$
|b(x_0)| \ge \frac{\|b\|_\infty}{2} \quad \text{and} \quad |b(x)-b(x_0)| \le \frac{\|b\|_\infty}{4} \quad \text{for all } x \in B(x_0, \delta|x_0|). \tag{5.48}
$$

Clearly any spherical profile $b(x) = a(x/|x|)$ is nearly-spherical for every $a \in C(\partial B(0,1))$ (in which case the choice of δ for (5.48) to hold can be made by a simple continuity argument). Let $\psi \in C_c^\infty(\mathbb{R}^3;[0,1])$ be such that $\int \psi = 1$, and let $\psi_l(x) := l^{-3}\psi(x/l)$ denote a mollifier of a given length scale $l > 0$. We also set $\widetilde{\psi}_l := \psi_l * \psi_l$.

We note that, letting $R \coloneqq 2l/\delta$, we can find $x_0 \in \mathbb{R}^3$ with $|x_0| = 2l/\delta$ and satisfying (5.48). In particular

$$\left|\left(\widetilde{\psi}_l * \frac{b(\cdot)}{|\cdot|}\right)(x_0)\right| = \left|\int_{B(x_0,2l)} \widetilde{\psi}_l(x_0 - y)\frac{b(y)}{|y|}\mathrm{d}y\right| \gtrsim \frac{|b(x_0)| - \|b\|_\infty/4}{(1+\delta)|x_0|} \geq \frac{\delta\|b\|_\infty}{16l},$$

which shows that

$$\left\|\widetilde{\psi}_l * \frac{b(\cdot)}{|\cdot|}\right\|_\infty \geq \frac{\delta\|b\|_\infty}{16l} \tag{5.49}$$

for every length scale $l > 0$. This simple fact lets us deduce from Theorem 5.1 that, if an axisymmetric solution approximates a self-similar profile $b(t,x)/|x|$ up to length scale $l(t)$, where b is nearly-spherical uniformly on $[0,t]$, then $l(t)$ cannot be smaller than a particular quantitative threshold.

Corollary 5.10. *If u is a strong axisymmetric solution u of (1.2) on $[0,T]$,*

$$\left\|u(t) - \psi_{l(t)} * \frac{b(t,x)}{|x|}\right\|_{L^{3,\infty}} \leq \sigma\|b(t)\|_\infty \tag{5.50}$$

for $t \in [0,T]$, and $\sigma < c\delta$, where $c > 0$ is a sufficiently small constant and $b(T)$ is nearly-spherical with constant δ, then

$$l(T) \gtrsim \delta T^{\frac{1}{2}}\|b(T)\|_\infty \exp\left(-\exp\left(\|b\|^{O(1)}_{L^\infty_{t,x}([0,T]\times\mathbb{R}^3)}\right)\right).$$

Proof. We note that, at time T, we can use (5.49) to obtain that

$$\begin{aligned}
\|u\|_\infty &\gtrsim \|\psi_l * u\|_\infty \\
&\geq \left\|\widetilde{\psi}_l * \frac{b(\cdot)}{|\cdot|}\right\|_\infty - \left\|\psi_l * \left(u - \psi_l * \frac{b(\cdot)}{|\cdot|}\right)\right\|_\infty \\
&\geq \frac{\delta\|b\|_\infty}{16l} - Cl^{-1}\left\|u - \psi_l * \frac{b(\cdot)}{|\cdot|}\right\|_{L^{3,\infty}} \\
&\geq \left(\frac{\delta}{16} - C\,\sigma\right)\frac{\|b\|_\infty}{l}.
\end{aligned}$$

Thus $\|u(T)\|_\infty \geq \delta\|b(T)\|_\infty/32l$ if $\sigma \in (0, \delta/32C)$. Since also

$$\|u(t)\|_{L^{3,\infty}} \leq \left\|\widetilde{\psi}_{l(t)} * \frac{b(t,\cdot)}{|\cdot|}\right\|_{L^{3,\infty}} + \left\|u(t) - \psi_{l(t)} * \frac{b(t,\cdot)}{|\cdot|}\right\|_{L^{3,\infty}} \leq C\|b(t,\cdot)\|_\infty,$$

for *all* $t \in [0,T]$, Theorem 5.1 implies that

$$\frac{\delta \|b(T)\|_\infty}{32\, l(T)} \leq \|u(T)\|_\infty \lesssim T^{-1/2} \exp\exp\left(\|b\|^{O(1)}_{L^\infty([0,T]\times\mathbb{R}^3)}\right),$$

from which the claim follows. □

APPENDIX A

Quantitative parabolic theory

A.1 Carleman inequalities for the heat equation

We quote from [60, Lemma 4.1] the general Carleman inequality for the backward heat operator $L = \partial_t + \Delta$ from which Carleman inequalities for specific domains and weight functions can be derived. Note that it is conventional to work with the backward heat operator even though we intend to apply these estimates to the forward heat equation.

Lemma A.1. *Let $[t_1, t_2]$ be a time interval and $u : C_c^\infty([t_1, t_2] \times \mathbb{R}^d \to \mathbb{R}^m)$ solve the backwards heat equation*

$$Lu = f.$$

Fix a smooth weight function $g : [t_1, t_2] \times \mathbb{R}^d \to \mathbb{R}$ and define

$$F = \partial_t g - \Delta g - |\nabla g|^2.$$

Then we have

$$\begin{aligned} &\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left(\frac{1}{2}(LF)|u|^2 + 2D^2 g(\nabla u, \nabla u) \right) e^g dxdt \\ &\quad \le \frac{1}{2} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |Lu|^2 e^g dxdt + \int_{\mathbb{R}^d} \left(|\nabla u|^2 + \frac{1}{2} F|u|^2 \right) e^g dx \Big|_{t=t_1}^{t=t_2}. \end{aligned} \tag{A.1}$$

Our first application of this lemma is to a Carleman estimate resembling the one used to prove backward uniqueness for the heat operator in [22] and the quantitative analog appearing in [60]. Unfortunately that estimate relies on the differential inequality (A.2) holding in an annular region (or, in the qualitative case, the complement of a ball), which cannot possibly be contained in the cylindrical regions of regularity provided by Proposition 2.19. Thus we prove a variant that is suited to this geometry.

In this section only, since we give the result in the general setting $\mathbb{R}^{d_1+d_2}$ (where the last d_2 coordinates correspond to the "axis"), we extend the definition of r and define $|z|$ to be

$$r := \sqrt{x_1^2 + \cdots + x_{d_1}^2}, \quad |z| := \sqrt{x_{d_1+1}^2 + \cdots x_{d_1+d_2}^2}.$$

The regions $\mathcal{S}(r_-, r_+)$, etc. are defined in the same way as before but in terms of the generalized r and $|z|$ coordinates, where naturally $|z|$ replaces $|x_3|$.

Proposition A.2 generalizes a quantitative Carleman inequality from [60] which corresponds to the case $d_1 = 3$, $d_2 = 0$. In the case at hand, we will be using the case $d_1 = 2$, $d_2 = 1$.

Proposition A.2 (Backward uniqueness Carleman estimate)**.** *Let $d_1 \geq 1$, $d_2 \geq 0$, $T > 0$, $0 < r_- < r_+$, and $\mathcal{C}$ denote the spacetime region*

$$\mathcal{C} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^{d_1+d_2} : t \in [0, T],\ r_- \leq r \leq r_+,\ |z| \leq r_+\}.$$

Let $u : \mathcal{C} \to \mathbb{R}$ be a smooth function obeying the differential inequality

$$|Lu| \leq \frac{1}{C_0 T}|u| + \frac{1}{(C_0 T)^{1/2}}|\nabla u| \tag{A.2}$$

on $\mathcal{C}$. Assume the inequality

$$r_-^2 \geq 4 C_0 T.$$

Then one has

$$\int_0^{T/4} \int_{\mathcal{S}(10r_-, \frac{r_+}{2}; \frac{r_+}{2})} (T^{-1}|u|^2 + |\nabla u|^2) dx dt \lesssim C_0 e^{-\frac{r_- r_+}{4 C_0 T}} (X + e^{\frac{2r_+^2}{C_0 T}} Y)$$

where

$$X = \int \int_{\mathcal{C}} e^{2|x|^2/C_0 T} (T^{-1}|u(t,x)|^2 + |\nabla u(t,x)|^2) dx dt$$

and

$$Y = \int_{\mathcal{S}(r_-,r_+;r_+)} |u(0,x)|^2 dx.$$

Proof. We may assume $r_+ \geq 20r_-$. The pigeonhole principle implies the existence of a $T_0 \in [T/2, T]$ such that

$$\int_{\mathcal{S}(r_-,r_+;r_+)} e^{2|x|^2/C_0T}(T^{-1}|u(T_0,x)|^2 + |\nabla u(T_0,x)|^2)dx \lesssim T^{-1}X. \tag{A.3}$$

With the weight

$$g(x,t) = \frac{r_+(T_0-t)}{2C_0T^2} r + \frac{1}{C_0T}|x|^2,$$

we apply the general Carleman inequality to ψu, where ψ is a smooth spatial cutoff supported in $\mathcal{S}(r_-,r_+;r_+)$ that equals 1 in $\mathcal{S}(2r_-,r_+/2;r_+/2)$ and obeys $|\nabla^j\psi(x)| \lesssim r_-^{-j}$ for $j=0,1,2$. Since the function r is convex, we have

$$D^2 g \geq \frac{2}{C_0T} Id$$

as quadratic forms. With $F = \partial_t g - \Delta g - |\nabla g|^2$, we compute

$$\begin{aligned} F = &-\frac{r_+}{2C_0T^2}r - \frac{r_+(T_0-t)}{2C_0T^2}\frac{d_1-1}{r} - \frac{2(d_1+d_2)}{C_0T} - \frac{r_+^2(T_0-t)^2}{4C_0^2T^4} - \frac{4}{C_0^2T^2}|x|^2 - \frac{2r_+(T_0-t)}{C_0^2T^3}r \\ &\leq 0. \end{aligned}$$

It follows that

$$\begin{aligned} LF = \frac{r_+^2(T_0-t)}{2C_0^2T^4} + \frac{2r_+}{C_0^2T^3}r - \frac{r_+(T_0-t)}{2C_0T^2}\frac{(d_1-1)(3-d_1)}{r^3} \\ -\frac{8(d_1+d_2)}{C_0^2T^2} - \frac{2r_+(T_0-t)}{C_0^2T^3}\frac{d_1-1}{r}. \end{aligned}$$

By using the bounds $2(C_0T)^{1/2} \leq r_- \leq r \leq r_+$, one finds that

$$\frac{r_+(T_0-t)}{2C_0T^2}\frac{(d_1-1)(3-d_1)}{r^3} + \frac{8(d_1+d_2)}{C_0^2T^2} + \frac{2r_+(T_0-t)}{C_0^2T^3}\frac{d_1-1}{r} \leq \frac{3(d_1+d_2)r_+}{C_0^3T^3}r.$$

Therefore, letting $C_0 \geq 3(d_1+d_2)$,

$$LF \geq \frac{r_+ r}{C_0^2 T^3} \geq \frac{4}{C_0 T^2}.$$

Putting this information into the general inequality (A.1), we have

$$\begin{aligned}
&\int_0^{T_0} \int_{\mathcal{S}(2r_-,\frac{r_+}{2};\frac{r_+}{2})} \left(\frac{2}{C_0 T^2}|u|^2 + \frac{4}{C_0 T}|\nabla u|^2 \right) e^g dxdt \\
&\quad \leq \frac{1}{2}\int_0^{T_0} \int_{\mathbb{R}^{d_1+d_2}} |L(\psi u)|^2 e^g dxdt + \int_{\mathbb{R}^{d_1+d_2}} |\nabla(\psi u)(T_0,x)|^2 e^{g(T_0,x)} dx \\
&\quad\quad + \frac{1}{2}\int_{\mathbb{R}^{d_1+d_2}} |F(0,x)||\psi u(0,x)|^2 e^{g(0,x)} dx.
\end{aligned}$$

In the region $\mathcal{S}(2r_-,\frac{r_+}{2};\frac{r_+}{2})$, ψ is identically 1 so thanks to the pointwise bound on Lu, this part of the integral in the first term on the right-hand side can be absorbed into the left-hand side. Moreover, throughout all of $\mathcal{C}$, using the bounds on $\nabla^j\psi$ and r_-,

$$|L(\psi u)|^2 = |\psi L u + 2\nabla\psi\cdot\nabla u + (\Delta\psi)u|^2 \lesssim (C_0T)^{-2}|u|^2 + (C_0T)^{-1}|\nabla u|^2.$$

Similarly,

$$|\nabla(\psi u)|^2 = |\psi\nabla u + \nabla\psi u|^2 \lesssim |\nabla u|^2 + (C_0T)^{-1}|u|^2.$$

By limiting the time interval for the integral on the left-hand side to $[0,T/4]$ and the r interval to $[10r_-, r_+/2]$, we find that on this region of integration

$$g(x,t) \geq \frac{5r_- r_+}{4C_0 T}.$$

Therefore

$$\begin{aligned}
&e^{\frac{5r_-r_+}{4C_0T}} \int_0^{T/4} \int_{\mathcal{S}(10r_-,\frac{r_+}{2};\frac{r_+}{2})} \left(\frac{1}{C_0T^2}|u|^2 + \frac{1}{C_0T}|\nabla u|^2 \right) dxdt \\
&\quad \lesssim \int_0^{T_0} \int_{\mathcal{S}(r_-,2r_-;\frac{r_+}{2})\cup\mathcal{S}(\frac{r_+}{2},r_+;\frac{r_+}{2})\cup\mathcal{S}(r_-,r_+;\frac{r_+}{2},r_+)} \left(\frac{1}{(C_0T)^2}|u|^2 + \frac{1}{C_0T}|\nabla u|^2 \right) e^g dxdt \\
&\quad\quad + \int_{\mathcal{S}(r_-,r_+;r_+)} \left(|\nabla u(T_0,x)|^2 + \frac{1}{C_0T}|u(T_0,x)|^2 \right) e^{g(T_0,x)} \\
&\quad\quad + \int_{\mathcal{S}(r_-,r_+;r_+)} |F(0,x)||u(0,x)|^2 e^{g(0,x)} dx.
\end{aligned}$$

Consider the first term on the right-hand side. Within the region of integration, we have

$$g(x,t) - \frac{2|x|^2}{C_0 T} - \frac{5r_- r_+}{4C_0 T} = \frac{r_+(T_0 - t)}{2C_0 T^2} r - \frac{1}{C_0 T}|x|^2 - \frac{5r_- r_+}{4C_0 T} \leq -\frac{r_+ r_-}{4C_0 T}.$$

Indeed, in $\mathcal{S}(r_-, 2r_-) \cup \mathcal{S}(\frac{r_+}{2}, r_+)$, this is maximized at $r = 2r_-$, $|z| = 0$ where the given upper bound holds. In $\{|z| \in [r_+/2, r_+]\}$, the quantity is clearly largest when $|z| = r_+/2$, so we have the upper bound

$$\frac{r_+}{2C_0 T} r - \frac{1}{C_0 T}\left(r^2 + \frac{r_+^2}{4}\right) - \frac{5r_- r_+}{4C_0 T}$$

which is largest when $r = r_+/4$, yielding an upper bound of $-\frac{3r_+^2}{16C_0 T} \leq -\frac{r_+ r_-}{4C_0 T}$.

In conclusion, after dividing both sides of the inequality by $e^{\frac{5r_- r_+}{4C_0 T}}$, the first term on the right-hand side has a weight bounded by $e^{\frac{2|x|^2}{C_0 T} - \frac{r_+ r_-}{4C_0 T}}$ so the whole term can be absorbed into $e^{-\frac{r_+ r_-}{4C_0 T}} X/T$. Similarly, $e^{g(T_0, x)} = e^{\frac{|x|^2}{C_0 T}}$ so by the definition of T_0, the second term on the right has the same upper bound. Thus we have

$$\int_0^{T/4} \int_{\mathcal{S}(10r_-, \frac{r_+}{2}; r_+/2)} \left(\frac{1}{C_0 T^2}|u|^2 + \frac{1}{C_0 T}|\nabla u|^2\right) dxdt$$
$$\lesssim e^{-\frac{r_+ r_-}{4C_0 T}} \left(T^{-1} X + \int_{\mathcal{S}(r_-, r_+; r_+)} |F(0,x)||u(0,x)|^2 e^{g(0,x)} dx\right).$$

To conclude, we easily have

$$|F(0,x)| \leq \frac{r_+^2}{C_0 T^2}$$

and

$$e^{g(0)} \leq e^{\frac{3r_+^2}{2C_0 T}}$$

when x is restricted to $\mathcal{S}(r_-, r_+; r_+)$. Therefore

$$|F(0,x)| e^{g(0)} \leq e^{\frac{2r_+^2}{C_0 T}} T^{-1}$$

which completes the proof. □

The next Carleman inequality we quote directly from [60].

Proposition A.3 (Unique continuation Carleman inequality)**.** *Define the cylindrical space-time region*

$$\mathcal{C} = \{(t,x) \in \mathbb{R} \times \mathbb{R}^3 : t \in [0,T],\ |x| \leq \rho\}.$$

Let $u : \mathcal{C} \to \mathbb{R}^3$ *be a smooth function obeying* (A.2) *on* $\mathcal{C}$. *Assume*

$$\rho^2 \geq 4000T.$$

Then for any

$$0 < t_1 \leq t_0 \leq \frac{T}{1000},$$

one has

$$\int_{t_0}^{2t_0} \int_{|x|\leq\frac{\rho}{2}} (T^{-1}|u|^2 + |\nabla u|^2) e^{-|x|^2/4t} dxdt \lesssim e^{-\frac{\rho^2}{500t_0}} X + t_0^{3/2} (et_0/t_1)^{O(\rho^2/t_0)} Y$$

where

$$X = \int_0^T \int_{|x|\leq\rho} (T^{-1}|u|^2 + |\nabla u|^2) dxdt$$

and

$$Y = \int_{|x|\leq\rho} |u(0,x)|^2 t_1^{-3/2} e^{-|x|^2/4t_1} dx.$$

A.2 Harnack-type inequalities

Here we consider the parabolic equation

$$\mathcal{M}V := \partial_t V - \Delta V + b \cdot \nabla V = 0 \tag{A.4}$$

in a space-time cylinder

$$Q_R(x_0,t_0) := B(x_0,R) \times (t_0 - R^2, t_0).$$

We assume that at each point of Q_R

$$\text{either div } b = 0 \quad \text{or} \quad V = 0. \tag{A.5}$$

We also assume that

$$\mathcal{N}(R) := 2 + \sup_{R' \le R} (R')^{-\alpha} \|b\|_{L_t^\ell L_x^q(Q_{R'})} < \infty \tag{A.6}$$

where $\alpha := \frac{3}{q} + \frac{2}{\ell} - 1 \in [0, 1)$. In such setting [38, Corollary 3.6] observed that V must be Hölder continuous in the interior of Q_R, and in the proposition below we state a version of their result in which we quantify the dependence of the Hölder exponent in terms of $\mathcal{N}$.

Proposition A.4. *If V is a Lipschitz solution of* (A.4) *then*

$$\operatorname*{osc}_{B(r)} V(0) \lesssim \left(\frac{r}{R}\right)^{\gamma} \operatorname*{osc}_{Q(R)} V$$

for all $r \le R$, where $\gamma = \exp(-\mathcal{N}^{O(1)})$.

We note that the swirl Θ satisfies (A.4) with $b := u + 2e_r/r$ (recall (5.36) above). Moreover div $b = 0$ everywhere except for the axis, since div $u = 0$, $\mathrm{div}(e_r/r) = 0$ (recall (5.26)) there. Moreover, $V = 0$ on the axis (recall (5.37)), and so the assumption (A.5) holds. Thus Proposition A.4 shows that Θ is Hölder continuous in a neighbourhood of the axis. We explore this in more detail in the proof of Theorem 5.1, where we quantify $\mathcal{N}$ in terms of the weak-L^3 bound A (see Step 1 in subsection 5.4).

Here we prove Proposition A.4. Namely, we consider parabolic cylinders

$$Q_R^{\lambda,\theta}(t_0, x_0) := [t_0 - \theta R^2, t_0] \times B(x_0, \lambda R), \quad Q_R^{\lambda,\theta} := Q_R^{\lambda,\theta}(0,0), \quad Q_R := Q_R^{1,1}$$

and we consider Lipschitz solutions V of $\mathcal{M}V = 0$ on $Q_R^{\lambda,\theta}$, namely we suppose that

$$\int_{\mathbb{R}} \int (\partial_t V \phi + \nabla V \cdot \nabla \phi + b \cdot \nabla V \phi) = 0 \tag{A.7}$$

for all $\phi \in C_c^\infty(Q_R^{\lambda,\theta})$, where the (distributional) supports of $\mathrm{div}\, b$ and V are disjoint. Moreover we assume that (A.6) holds, namely

$$\mathcal{N}(R) := 2 + \sup_{R' \le 2R} (R')^{-\alpha} \|b\|_{L_t^\ell L_x^q(Q_{R'})} < \infty$$

where $\alpha := \frac{n}{q} + \frac{2}{\ell} - 1 \in [0,1)$. We also say that V is a *subsolution* (or *supersolution*) of $\mathcal{M}V = 0$, i.e. $\mathcal{M}V \le 0$ (or $\mathcal{M}V \ge 0$), if (A.7) holds with "=" replaced by "≤" (or "≥") for all nonnegative test functions.

We will show that

$$\operatorname*{osc}_{B(r)} V(0) \lesssim \left(\frac{r}{R}\right)^{\gamma} \operatorname*{osc}_{Q(R)} V \tag{A.8}$$

for all $r \le R$, where $\gamma = \exp(-\mathcal{N}^{O(1)})$.

To this end we first prove the Harnack inequality for Lipschitz subsolutions of $\mathcal{M}V = 0$.

Lemma A.5 (based on Lemma 3.1 in [38])**.** *Let V be a Lipschitz solution of $\mathcal{M}V \le 0$ in $Q_R^{\lambda,\theta}$ where $\lambda \in (1,2]$ and $\theta \in (0,1]$. Then*

$$\sup_{Q_R^{1,\theta/2}} V_+ \le (\mathcal{N}/\theta)^C \left(\fint_{Q_R^{\lambda,\theta}} V_+^2 \right)^{\frac{1}{2}}.$$

Proof. We first note that for any r, a such that

$$\frac{3}{r} + \frac{2}{a} \in \left[\frac{3}{2}, \frac{5}{2}\right]$$

we have the interpolation inequality

$$\|\zeta U\|_{L_t^a L_x^r (Q_R^{\lambda,\theta})} \lesssim_{\lambda,\theta} R^{\frac{3}{r} + \frac{2}{a} - \frac{3}{2}} \|\zeta U\|_{\mathcal{V}(Q_R^{\lambda,\theta})} \tag{A.9}$$

by [30, (3.4) in Chapter II], where $\mathcal{V}$ is the energy space $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$.

Since V is a subsolution, we have, for a non-negative test function η,

$$\int_{Q_R^{\lambda,\theta}} (\partial_t V \eta + \nabla V \cdot \nabla \eta + b \cdot \nabla V \eta) \le 0.$$

We let $\eta = \varphi'(V)\xi$ where ξ is a cutoff function vanishing on a neighborhood of the boundary of $Q_R^{\lambda,\theta}$, and φ is a convex function vanishing on $\mathbb{R}_-$. Taking $U := \varphi(V)$ then gives

$$\int_{Q_R^{\lambda,\theta} \cap \{V > 0\}} \left(\partial_t U \xi + \nabla U \cdot \nabla \xi + \frac{\varphi''(V)}{\varphi'(V)^2} |\nabla U|^2 \xi + b \cdot \nabla U \xi \right) \le 0.$$

We now take

$$\varphi(\tau) \coloneqq \tau_+^p \quad (p>1) \qquad \text{and} \qquad \xi \coloneqq \chi_{\{t<\bar t\}} U\zeta^2,$$

where ζ is a smooth cutoff function in $Q_R^{\lambda,\theta}$ and $\bar t \in (-\theta R^2, 0)$,

$$\int_{B_{\lambda R}} (\zeta U)^2(\bar t)dx + \int_{Q_R^{\lambda,\theta}\cap\{t<\bar t\}} (2-p^{-1})|\nabla U|^2\zeta^2 + U\nabla U\cdot\nabla(\zeta^2) + \frac{1}{2} b\cdot\nabla(U^2)\zeta^2 - \partial_t(\zeta^2)U^2 \le 0. \tag{A.10}$$

Using integration by parts and recalling the assumption div $b \ge 0$, we can apply Hölder's inequality to obtain

$$\begin{aligned}\int_{Q_R^{\lambda,\theta}\cap\{t<\bar t\}} b\cdot\nabla(U^2)\zeta^2 &\ge -\int_{Q_R^{\lambda,\theta}\cap-\{t<\bar t\}} b\cdot\nabla(\zeta^2)U^2 \\ &\ge -\|b\|_{L_t^\ell L_x^q(Q_R^{\lambda,\theta})} \||U|^{\frac{1}{s}}\zeta^{\frac{1}{s}-1}\nabla\zeta\|_{L^{2s}_{t,x}(Q)} \|(\zeta|U|)^{2-\frac{1}{s}}\|_{L_t^{(1-\frac{1}{2s}-\frac{1}{\ell})^{-1}} L_x^{(1-\frac{1}{2s}-\frac{1}{q})^{-1}}(Q)} \\ &= -\|b\|_{L_t^\ell L_x^q(Q_R^{\lambda,\theta})} \|U\zeta^{1-s}|\nabla\zeta|^s\|^{\frac{1}{s}}_{L^2_{t,x}(Q_R^{\lambda,\theta})} \|\zeta U\|^{2-\frac{1}{s}}_{L_t^a L_x^r(Q_R^{\lambda,\theta})}\end{aligned}$$

where $s>2$ and r and a are defined by

$$\frac{1}{2s} + \frac{1}{q} + \frac{1}{r}\Big(2-\frac{1}{s}\Big) = 1, \quad \frac{1}{2s} + \frac{1}{\ell} + \frac{1}{a}\Big(2-\frac{1}{s}\Big) = 1.$$

Applying Young's inequality to separate the last term, and utilizing the interpolation inequality (A.9) (which is valid since

$$\frac{3}{r} + \frac{2}{a} = \frac{3}{2} + 1 - 2\left(1 + 2/\left(\frac{3}{q}+\frac{2}{\ell}\right)\right)^{-1} \in (3/2, 11/6),$$

as needed) we obtain, after plugging into the local energy inequality (A.10),

$$\begin{aligned}\sup_{t\in[-\theta R^2,0]} \int_{B_{\lambda R}} (\zeta U)^2 dx + \int_{Q_R^{\lambda,\theta}\cap\{t<\bar t\}} (2-p^{-1})|\nabla U|^2\zeta^2 + U\nabla U\cdot\nabla(\zeta^2) - \partial_t(\zeta^2)U^2 \\ -O\left(R^2\|b\|^{2s}_{L_t^\ell L_x^q(Q_R^{\lambda,\theta})} \|U\zeta^{1-s}|\nabla\zeta|^s\|^2_{L^2_{t,x}(Q_R^{\lambda,\theta})}\right) - \frac{1}{10}\|\zeta U\|^2_{\mathcal{V}(Q_R^{\lambda,\theta})} \le 0.\end{aligned}$$

Absorbing ∇U from the term on the third term on the left-hand side by the second term we obtain

$$\|\zeta U\|^2_{\mathcal{V}(Q_R^{\lambda,\theta})} \lesssim \int_{Q_R^{\lambda,\theta}} \left(|\nabla\zeta|^2 + \zeta|\partial_t\zeta| + R^2\|b\|^{2s}_{L_t^\ell L_x^q(Q_R^{\lambda,\theta})}\zeta^{2-2s}|\nabla\zeta|^{2s}\right) U^2.$$

We now set

$$\lambda_m := 1 + 2^{-m}(\lambda - 1) \qquad \text{and} \qquad \theta_m := \frac{1}{2}\theta(1 + 4^{-m}),$$

and we substitute ζ with ζ_m such that

$$\zeta_m \equiv 1 \text{ in } Q_R^{\lambda_{m+1},\theta_{m+1}}, \quad \zeta_m \equiv 0 \text{ outside } Q_R^{\lambda_m,\theta_m}, \quad |\partial_t \zeta_m| \leq \frac{4^m C}{\theta R^2}, \quad \frac{|\nabla \zeta_m|}{\zeta_m^{1-\frac{1}{s}}} \leq \frac{2^m C}{R},$$

where C may depend on λ. Then the energy estimate and (A.9), taken with $r = l = 10/3$, yield

$$\|\zeta_m U\|_{L^{10/3}_{t,x}(Q_R^{\lambda,\theta})} \lesssim \|\zeta_m U\|_{\mathcal{V}(Q_R^{\lambda,\theta})} \leq C R^{-1}(\theta^{-\frac{1}{2}} + 2^m + \mathcal{N}^s) 2^{ms} \|U\|_{L^2_{t,x}(Q_R^{\lambda,\theta})}.$$

Recalling the definition of U and replacing p with $p_m := (5/3)^m$, Hölder's inequality implies

$$\begin{aligned}\left(\fint_{Q_R^{\lambda_{m+1},\theta_{m+1}}} u_+^{2p_{m+1}} \right)^{\frac{1}{2p_{m+1}}} &\leq \left(C \fint_{Q_R^{\lambda_m,\theta_m}} (\zeta_m U)^{10/3} \right)^{\frac{1}{rp_m}} \\ &\leq \left(C \theta_m^{-1} \mathcal{N}^{2s} 4^{m(s+1)} \fint_{Q_R^{\lambda_m,\theta_m}} u_+^{2p_m} \right)^{\frac{1}{2p_m}}.\end{aligned}$$

Iterating, we have

$$\left(\fint_{Q_R^{\lambda_m,\theta_m}} u_+^{2p_m} \right)^{\frac{1}{2p_m}} \leq \prod_{k=0}^{m-1} \left(\frac{C}{\theta} 4^{k(s+1)} \mathcal{N}^s \right)^{\frac{1}{2p_k}} \left(\fint_{Q_R^{\lambda,\theta}} u_+^2 \right)^{\frac{1}{2}}$$

and we conclude by taking $m \to \infty$. □

In the next three lemmas we focus on nonnegative solutions to $\mathcal{M}V \leq 0$ and we find lower bounds on the mass distribution of such solutions. We first show that if $V \geq k$ in Q_R, except for a small (quantified) "portion of Q_R", then in fact $V \geq k/2$ everywhere in a smaller cylinder.

Lemma A.6 (based on part 2 of Corollary 3.1 in [38])**.** *If V is a non-negative solution of $\mathcal{M}V \geq 0$ in $Q_R^{\lambda,\theta}$ and*

$$|\{V < k\} \cap Q_R^{\lambda,\theta}| \leq (\mathcal{N}/\theta)^{-5C} |Q_R^{\lambda,\theta}|,$$

then

$$V \geq \frac{k}{2} \quad in\ Q_R^{1,\theta/2}.$$

Proof. We apply Lemma A.5 to $k - V$ to find

$$\sup_{Q_1^{1,\theta/2}} (k-V)_+ \leq (\mathcal{N}/\theta)^C \left(\fint_{Q_R^{\lambda,\theta}} (k-V)_+^2 \right)^{\frac{1}{2}} \leq \mathcal{N}^{-1} k$$

which implies the result. □

We now show that, if the cylinder $Q_R^{1,\theta}$ is flat enough, then a lower bound on the bottom lid of $Q_R^{1,\theta}$ (i.e. at $t = -\theta R^2$) implies a similar lower bound at every t.

Lemma A.7 (based on Lemma 3.2 in [38])**.** *Suppose V is non-negative with $\mathcal{M}V \geq 0$ in a neighbuorhood of Q_R^{1,θ_0} and*

$$|\{V(-\theta_0 R^2) \geq k\} \cap B_R| \geq \delta_0 |B_R|$$

for some $\delta_0 > 0$ and $\theta_0 \leq C^{-1}\delta_0^6 \mathcal{N}^{-1}$. Then

$$|\{V(\bar{t}) \geq \frac{1}{3}\delta_0 k\} \cap B_R| \geq \frac{1}{3}\delta_0 |B_R|$$

for all $\bar{t} \in [-\theta_0 R^2, 0]$.

Proof. By the calculations in [38], with ζ a smooth cutoff function supported in B_R,

$$\int_{B_R} (V(\bar{t}) - k)_-^2 \zeta^2 + \int_{Q_R^{1,\theta_0}} \chi_{\{t<\bar{t}\}} |\nabla (V-k)_-|^2 \zeta^2 \leq \int_{B_R} (V(-\theta_0 R^2) - K)_-^2 \zeta^2 \tag{A.11}$$

$$+ \int_{Q_R^{1,\theta_0}} \chi_{\{t<\bar{t}\}} (V-k)_-^2 \big(O(|\nabla\zeta|^2) + b\cdot\nabla(\zeta^2) + (\operatorname{div} b)\zeta^2 \big). \tag{A.12}$$

We choose ζ such that $\zeta \equiv 1$ in $B_{(1-\sigma)R}$ and $|\nabla\zeta| \leq \frac{2}{\sigma R}$ where $\sigma < 1$ is to be specified. Note that due to (A.5),

$$\begin{aligned}\int_{Q_R^{1,\theta_0}} \chi_{\{t<\bar{t}\}} (V-k)_-^2 (\operatorname{div} b)\zeta^2 &\leq k^2 \int_{Q_R^{1,\theta_0}} \chi_{\{t<\bar{t}\}} (\operatorname{div} b)\zeta^2 \\ &= -k^2 \int_{Q_R^{1,\theta_0}} \chi_{\{t<\bar{t}\}} b\cdot\nabla(\zeta^2).\end{aligned}$$

Then the right-hand side of (A.11) is bounded by

$$k^2\Big((1-\delta_0)|B_R| + O(\theta_0\sigma^{-2}|B_R|) + \frac{4}{\sigma R}\|b\|_{L_t^\ell L_x^q(Q_R)}\|1\|_{L_t^{\ell'}L_x^{q'}(Q_R^{1,\theta_0})}\Big).$$

From here one can proceed with the argument exactly as in [38] to arrive at

$$\left|\left\{V(\bar{t}) < \frac{1}{3}\delta_0 k\right\} \cap B_R\right| \leq \left(1-\frac{1}{3}\delta_0\right)^{-2}(1-\delta_0 + O(\sigma + \sigma^{-2}\theta_0 + \sigma^{-1}\theta_0^{2/\ell'}\mathcal{N})).$$

Setting $\sigma = C^{-1/5}\delta_0^2$ and θ_0 as above proves the claimed bound. □

We now show that for any given "portion of $Q_R^{1,\theta}$" (in the sense of a set with the measure arbitrarily close to $|Q^{1,\theta}|$) V is greater or equal a constant multiple of some lower bound, if, for each t, the lower bound occurs at least on some "portion of B_R". Although this enables us to obtain a lower bound on almost the entire cylinder, we lose an exponential in the process.

Lemma A.8 (based on Lemma 3.3 in [38])**.** *Let $V \geq 0$ be a solution of $\mathcal{M}V \geq 0$ in $Q_R^{\lambda,\theta}$ satisfying*

$$|\{V(t) \geq k_0\} \cap B_R| \geq \delta_1|B_R| \text{ for all } t \in [-\theta R^2, 0]$$

for some $k_0 > 0$, $\delta_1 > 0$. Then for any $\mu > 0$ and $s > C(\mathcal{N} + \theta^{-1})/(\delta_1\mu)^2$,

$$|\{V < 2^{-s}k_0\} \cap Q_R^{1,\theta}| \leq \mu|Q_R^{1,\theta}|.$$

Proof. With $k_m = 2^{-m}k_0$, we define

$$\mathcal{E}_m(t) \coloneqq \{x \in B_R \colon k_{m+1} \leq V(x,t) < k_m\}; \quad \mathcal{E}_m \coloneqq \{(t,x) \in Q_R^{1,\theta} \colon x \in \mathcal{E}_m(t)\}.$$

Integrating the inequality $\mathcal{M}V \geq 0$ against the test function $\eta = (V-k_m)_-\xi(x)^2$ where ξ is a smooth cutoff vanishing in a neighborhood of $\partial B_{\lambda R}$ and satisfying $\xi \equiv 1$ in B_R,

$$\begin{aligned}\int_{Q_R^{\lambda,\theta}\cap\{V<k_m\}} |\nabla V|^2\xi^2 &\leq \int_{Q_R^{\lambda,\theta}} |\nabla(V-k_m)_-|^2\xi^2 \lesssim \int_{B_{\lambda R}\cap\{V<k_m\}} (V-k_m)_-^2\xi^2\Big|_{t=-\theta R^2} \\ &\quad + \int_{-\theta R^2}^0\int_{B_{\lambda R}\cap\{V<k_m\}} (V-k_m)_-^2|\nabla\xi|^2 + 2(V-k_m)_-^2\xi b\cdot\nabla\xi \\ &\lesssim k_m^2 R^n(1+\theta\mathcal{N}) \qquad \text{(A.13)}\end{aligned}$$

by Hölder's inequality and the trivial bound $0 \leq (V - k_m)_- \leq k_m$. From De Giorgi's inequality [30, (5.6) in Chapter II],

$$(k_m - k_{m+1})|\{V(t) < k_{m+1}\} \cap B_R| \lesssim \frac{R}{\delta_1} \int_{\mathcal{E}_m(t)} |\nabla V(t)|$$

for all $t \in [-\theta R^2, 0]$. Integrating in time, squaring, and applying Cauchy-Schwarz gives

$$k_{m+1}^2 \left|\{V < k_{m+1}\} \cap Q_R^{1,\theta}\right|^2 \lesssim \frac{R^2}{\delta_1^2} \int_{\mathcal{E}_m} |\nabla V|^2 dxdt |\mathcal{E}_m|.$$

Combined with (A.13), this gives

$$\left|\{V < k_{m+1}\} \cap Q_R^{1,\theta}\right|^2 \lesssim \delta_1^{-2} R^{n+2}(1 + \theta\mathcal{N})|\mathcal{E}_m|.$$

We conclude

$$\begin{aligned} s\left|\{V < k_s\} \cap Q_R^{1,\theta}\right|^2 &\leq \sum_{m=0}^{s-1} \left|\{V < k_{m+1}\} \cap Q_R^{1,\theta}\right|^2 \\ &\lesssim \delta_1^{-2} R^{n+2}(1 + \theta\mathcal{N}) \sum_{m=0}^{s-1} |\mathcal{E}_m| \\ &\lesssim \delta_1^{-2}(\theta^{-1} + \mathcal{N})|Q_R^{1,\theta}|^2. \end{aligned}$$

□

We can now combine Lemmas A.6–A.8 to obtain a pointwise lower bound for V in the interior of a cylinder, with an exponential dependence on $\mathcal{N}$.

Lemma A.9 (based on part 1 of Corollary 3.2 in [38]). *If V is a non-negative solution of $\mathcal{M}V \geq 0$ in $Q_R^{2,1}$ and*

$$|\{V(-\Theta R^2) \geq k\} \cap B_R| \geq \delta |B_R|$$

for some $k > 0$ and $\Theta \leq C^{-1}\delta^6\mathcal{N}^{-1}$, then

$$V \geq \exp(-\delta^{-2}(\mathcal{N}/\Theta)^{20C})k \quad \text{in } Q_R^{1,\Theta/2}.$$

Proof. This is a straightforward application of Lemmas A.7, A.8, and A.6 in sequence, with the latter two applied with $R \to \frac{3}{2}R$ to compensate for the shrinking domain in Lemma A.6. □

By considering $V - \inf V$ and $\sup V - V$ the above lemma now allows us to estimate oscillations of solutions to $\mathcal{M}V = 0$ with no sign restrictions.

Lemma A.10 (based on Lemma 3.5 of [38]). *If V solves $\mathcal{M}V = 0$ in $Q_R^{2,1}$ then*

$$\operatorname*{osc}_{Q^{(1)}} V \le (1 - \exp(-\mathcal{N}^{50C})) \operatorname*{osc}_{Q^{(2)}} V$$

where $Q^{(1)} = Q_R^{1,\Theta/2}$, $Q^{(2)} = Q_R^{2,1}$, and $\Theta = C^{-2}\mathcal{N}^{-1}$.

Proof. Consider the positive supersolutions $V_1 = V - \inf_{Q^{(2)}} V$ and $V_2 = \sup_{Q^{(2)}} V - V$. With $k = \operatorname{osc}_{Q^{(2)}} V$, clearly we must have $|\{V_i(-\Theta R^2) \ge k\} \cap B_{2R}| \ge |B_{2R}|/2$ for either $i = 1$ or $i = 2$. Fix this i, so V_i obeys the hypotheses of Lemma A.9. Let us assume for concreteness that $i = 1$; the other case is analogous. Then by the lemma,

$$\inf_{Q^{(2)}} V + \exp(-\mathcal{N}^{50C}) \operatorname*{osc}_{Q^{(2)}} V \le V \le \sup_{Q^{(2)}} V$$

for all $(t, x) \in Q^{(1)}$, which immediately implies the result. □

Finally, iterating Lemma A.10 we obtain the required Hölder continuity (A.8), i.e. we can prove Proposition A.4.

Proof of Proposition A.4. Iterating Lemma A.10, we have

$$\operatorname*{osc}_{Q^{2,1}_{(\Theta/2)^{k/2}R/2}} V \le (1 - \exp(-\mathcal{N}^{50C}))^k \operatorname*{osc}_{Q^{2,1}_{R/2}} V.$$

We conclude upon taking $k = \lfloor \log \frac{R}{r} (\log \frac{2}{\Theta})^{-1} \rfloor$. □

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