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Capacity of ultra-wideband power-constrained ad hoc networks

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sitions are removed. The general algorithm to construct permutation DPMs from the multilevel construction, is

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Input: (x_1, x_2, ..., x_M)
Output: (y_1, y_2, ..., y_M)
begin
  (y_1, y_2, ..., y_M) ← (1, 2, ..., M)
  if x_1 = 1 then
    for i from 1 to ⌊(M+2)/4⌋
      swap(y_{4i-3}, y_{4i-2})
  if x_2 = 1 then
    for i from 1 to ⌊M/4⌋
      swap(y_{4i-1}, y_{4i})
  for i from 1 to L-1
    for j from 1 to 2^i
      if j ≤ M - 2^i then
        if x_{j+2^i} = 1 then
          for k from 1 to ⌊(M - j + 2^i)/2^{i+1}⌋
            p = j + 2^{i+1}(k-1)
            swap(y_p, y_{p+2^i})
end.
```

This algorithm then produces mappings with the same $|\mathbf{E}|$ values as those listed in [8] for the multilevel mappings. In almost all cases these values are larger than those for previous mappings.

VI. Conclusion

We have shown how the use of graphs can give new insight into the analysis and construction of permutation DPMs. The graphs can visually aid one in determining the positions of symbols at certain stages in a mapping, as well as showing why some mappings cannot be distance-preserving. These graphs can also be used in the decoding of permutation codes obtained from mapping algorithms [11].

Although the multilevel construction of [8] was flexible and could produce numerous different mappings for the same M value, some empirical work or computer searches were still necessary to obtain the mappings. Using the trellis representation of the graphs, we were able to construct a general algorithm for this construction, even though only a subset of all the possible mappings are obtained for a certain M.

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Capacity of Ultra-Wideband Power-Constrained Ad Hoc Networks

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***Abstract*—In this correspondence, we show that the uniform throughput capacity of an ultra-wideband (UWB) power-constrained ad hoc network is given by $\Theta(P_0(\sqrt{n/\log n})^{\alpha-1})$, where P_0 is the per-node power constraint, α is the fading exponent of radio signal and n is the number of nodes randomly distributed inside a disk of unit area. This is a stronger result than the upper bound $O(P_0(\sqrt{n\log n})^{\alpha-1})$ and the lower bound $O(P_0\sqrt{n^{\alpha-1}/(\log n)^{\alpha+1}})$ previously shown by Negi and Rajeswaran. Our proof is simple given the prior work by Gupta and Kumar.**

***Index Terms*—Capacity, power-constrained ad hoc networks, ultra-wideband (UWB) ad hoc networks.**

I. Introduction

Recently, there has been much interest in analyzing the capacity of ad hoc networks since the seminal paper [1]. For an ad hoc wireless network where n nodes are independently and uniformly distributed inside a disk of unit area, the throughput obtainable by each node in the network is shown in [1] to be $\Theta(W/\sqrt{n\log n})$ where W is the bandwidth available for the network. When n approaches infinity, the throughput approaches zero. Several research efforts have been made to investigate the ad hoc network capacity under different settings [2]–[9]. In [2] and [3], it is shown that node mobility improves the capacity bound, and two-hop routes are sufficient to achieve the per-node throughput order $\Theta(1)$ even if the node mobility is further constrained to a one-dimensional pattern. The issue of packet delay caused by mobility is further addressed in [5] and [6]. In [7], capacity improvement by infrastructure support is investigated, where a hybrid wireless network is formed by placing a sparse network (in a hexagonal pattern) of m base stations in an ad hoc network of total n mobile nodes. Their results show that if m grows faster than $\sqrt{n}$, the throughput capacity increases linearly with the number of base stations, i.e., the capacity order becomes $\Theta(m)$. In [8], directional antennas with transmitter beamwidth α and receiver beamwidth β are considered, and the capacity is shown to be improved by a factor of $4\pi^2/\alpha\beta$. In [9], the extreme case where the transmitter can generate arbitrarily narrow beams is considered. It is shown that by choosing a very small beam width α, the capacity can only be improved by an order of $O(\log^2(n))$.

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All the above research activities are based on the assumption that the whole network is bandwidth-constrained. In contrast to this bandwidth-limited scenario, there exists another scenario where the available (radio frequency) bandwidth may be sufficiently large to be modeled as unlimited but the transmission power of each node is limited. This scenario arises from many possible ad hoc wireless sensor networks where ultra-wideband (UWB) techniques are suitable for short range transmissions. In [10], it is shown that when the bandwidth W is infinite and each node has a power constraint of P_0, the per-node throughput capacity is upper bounded by $O(P_0(\sqrt{n \log n})^{\alpha-1})$ and lower bounded by $O(P_0\sqrt{n^{\alpha-1}/(\log n)^{\alpha+1}})$, where α is the (large scale) path loss exponent. In this paper, we show a much stronger result, that is, the exact order of the per-node throughput capacity of a power-constrained ad hoc network is $\Theta(P_0(\sqrt{n/\log n})^{\alpha-1})$. This result closes the gap between the upper bound and the lower bound given in [10].

II. Background

In this section, we first quantify the assumptions of bandwidth-constrained networks versus power-constrained networks, then define the performance metric of uniform throughput capacity, and finally introduce the Voronoi tessellation and the Chernoff bounds, both of which are important for our derivation of network capacity.

A. Bandwidth-Constrained Versus Power-Constrained

In the bandwidth-constrained scenario, all nodes share a limited bandwidth W. Let X_i denote both the node and its position in a unit area disk. A power P_i is used by node X_i to transmit data to a randomly chosen node Y_i. Let g_i denote the attenuation due to path loss. Then the SINR (signal to interference and noise ratio) at the receiver Y_i is

$$\mathrm{SINR} = \frac{P_i g_i}{W N_0 + \sum_{k \in I} P_k g_k} \tag{1}$$

where N_0 represents the (single-sided) power density of noise, WN_0 is the power of noise falling within the bandwidth W, I is the set of nodes that cause interference to node Y_i. The transmission from X_i to Y_i is assumed to be successful if SINR is above a threshold β, i.e.

$$\mathrm{SINR} \geq \beta. \tag{2}$$

This is called *physical model* in [1]. Furthermore, in [1], each node is assumed to use a common power P for transmission. When n is sufficiently large, the term WN_0 can be discarded because of the limited bandwidth assumption, and hence all transmissions are only limited by the interference generated by neighboring concurrent transmissions.

In the power-constrained scenario, we assume that the maximal power consumed by each node for transmission is P_0 and furthermore the bandwidth W is very large, i.e., $W \gg n_I P_0/N_0$ where n_I is the maximum number of nodes within an interference range of any receiving node. In this case, SINR is only affected by ambient noise, i.e.

$$\mathrm{SINR} = \frac{P_i g_i}{W N_0}. \tag{3}$$

Therefore, the Shannon capacity of a link $X_i \to Y_i$ is now given by

$$r_i = \lim_{W \to \infty} W \log\left(1 + \frac{P_i g_i}{W N_0}\right) = \frac{P_i g_i}{N_0}. \tag{4}$$

Our capacity analysis will be based on the power-constrained scenario. Further assumptions of the network are as follows. There are n stationary nodes that are independently and uniformly distributed on a disk of unit area. At anytime, each node can play the role of either source or sink, and each node can serve as a relay for multiple source-destination pairs. Each node can transmit a packet to its next-hop node and receive a new packet from its previous-hop node at the same time. This is possible because of the unlimited bandwidth assumption. At anytime, there can be maximum $n/2$ pairs of source and destination, and all these pairs are randomly chosen. The path loss model for each link $X_i \to Y_i$ is

$$g_i = \left(\frac{d_0}{|X_i - Y_j|}\right)^{\alpha}, \qquad \text{for } |X_i - Y_j| > d_0 \tag{5}$$

where α is the path loss exponent, $|X_i - Y_j|$ is the distance between X_i and Y_j, and d_0 is a very small number that corresponds to the dimension of each node after the network is scaled to a unit area disk. However, d_0 is independent of the number n of nodes in the network.

B. Uniform Throughput Capacity

If each source node is able to transmit data to its chosen destination node at a rate of $r(n)$ bits per second, the data rate $r(n)$ is called a uniform throughput of the network. Note that the length (number of hops) of a route affects the delay of a packet from source to destination but not the data rate. The maximum feasible uniform throughput in the network is called the uniform throughput capacity of the network. A uniform throughput capacity $r(n)$ is said to be of the order $\Theta(f(n))$ if there exits constants $0 < c_2 < c_1 < \infty$ such that

$$\lim_{n \to \infty} \Pr[r(n) = c_1 f(n) \text{ is feasible}] < 1 \tag{6}$$

$$\lim_{n \to \infty} \Pr[r(n) = c_2 f(n) \text{ is feasible}] = 1 \tag{7}$$

where "feasible" means that the data can be transmitted at the specified rate with virtually no error (via optimal encoding and decoding).

C. Routing Based on Voronoi Tessellation

A Voronoi tessellation is a partition of space based on the nearest neighboring criterion [12]. It was used in deriving lower bounds on uniform throughput capacity in [1] and [10]. Let $\{a_1, a_2, \ldots, a_p\}$ be a set of reference points on the disk. A Voronoi cell $V(a_i)$ is the set of all points which are closer to a_i than to any other points $a_j, j \neq i$. The reference point a_i is also called the generator of the Voronoi cell $V(a_i)$. It is also known that for any $\varepsilon > 0$, a Voronoi tessellation can be constructed such that each Voronoi cell contains a disk of radius ε and is itself contained in a disk of radius 2ε [1].

The routing scheme based on a Voronoi tessellation is shown in Fig. 1. X_i and Y_i are two randomly chosen points or nodes to form a pair of source and destination. X_i and Y_i are independently and uniformly distributed on the disk. The straight line segment L_i connecting X_i and Y_i are also independently and uniformly distributed. The route from X_i and Y_i consists of the cells intersected by L_i in the tessellation. Packets from X_i are relayed from the source cell, through the cells intersected by L_i, to the destination cell and finally to Y_i. Each cell may contain multiple nodes.

D. Chernoff Bounds

Let $A_1, A_1, \ldots, A_n$ be independent indicator (binary) random variables with $\Pr[A_i = 1] = p_i$ and $\Pr[A_i = 0] = 1 - p_i$, where $0 < p_i < 1$. Define the sum of the binary random variables: $A = \sum_{i=1}^{n} A_i$ where $\mu = E[A] = \sum_{i=1}^{n} p_i$. As shown in the Appendix, for any $0 < \delta < 1$, the following two inequalities hold:

$$\Pr[A > (1+\delta)\mu] \leq e^{-\mu\delta^2/3} \tag{8}$$

$$\Pr[A < (1-\delta)\mu] \leq e^{-\mu\delta^2/2}. \tag{9}$$

An implication of the above bounds is that as $n \to \infty$, the probability for A to be larger than its mean by a fixed (however small) fraction of

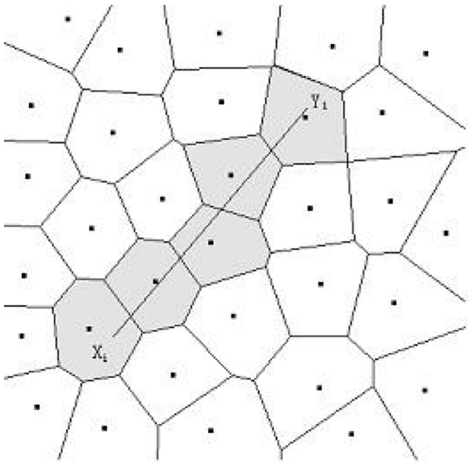

Fig. 1. Packets from X_i are relayed from the source cell, along the cells intersected by L_i in order, to the destination cell and finally to Y_i.

the mean approaches zero exponentially, and likewise the probability for A to be smaller than its mean by a fixed (however small) fraction of the mean approaches zero exponentially.

III. Bounds on Uniform Throughput Capacity

A. An Upper Bound on Uniform Throughput Capacity

We denote a route R_i from a source node $X_{i,1}$ to a destination node X_{i,K_i+1} by $R_i = [X_{i,1}, X_{i,2}, \ldots, X_{i,K_i}, X_{i,K_i+1}]$ where there are K_i links or hops. Let $r_i(n)$ be the throughput achieved by this route. Then, the power consumed by the kth link $X_{i,k} \to X_{i,k+1}$ is

$$P_{i,k} \geq \frac{r_i(n)N_0}{g_{i,k}} \tag{10}$$

where $g_{i,k}$ denotes the path power attenuation. The total power consumed by route R_i is

$$\begin{aligned} P_i(n) &= \sum_{k=1}^{K_i} P_{i,k} \geq r_i(n)N_0 \sum_{k=1}^{K_i} \frac{1}{g_{i,k}} \\ &= \frac{r_i(n)N_0}{d_0^\alpha} \sum_{k=1}^{K_i} d_{i,k}^\alpha \end{aligned} \tag{11}$$

where $d_{i,k} = |X_{i,k} - X_{i,k+1}|$. There are $n/2$ concurrent routes in the network. According to the definition of uniform throughput capacity, we have $r(n) \leq r_i(n)$ for all $1 \leq i \leq n/2$. Taking the sum of (11) leads to

$$r(n) \leq \frac{d_0^\alpha}{N_0} \frac{\sum_{i=1}^{n/2} P_i(n)}{\sum_{i=1}^{n/2} \sum_{k=1}^{K_i} d_{i,k}^\alpha} \tag{12}$$

where the numerator is the sum of power used by all routes in the network, which is upper bounded by total allowable transmission power nP_0, i.e.

$$\sum_{i=1}^{n/2} P_i(n) \leq nP_0. \tag{13}$$

Here, P_0 is the maximum power consumed by each node in the network.

We now denote

$$H = \sum_{i=1}^{n/2} \sum_{k=1}^{K_i} 1 \tag{14}$$

which is the total number of hops in the network. Since d^α is a convex function of d as long as $\alpha \geq 1$ (although for radio signals we have $\alpha \geq 2$), we can write

$$\frac{1}{H} \sum_{i=1}^{n/2} \sum_{k=1}^{K_i} d_{i,k}^\alpha \geq \left(\frac{1}{H} \sum_{i=1}^{n/2} \sum_{k=1}^{Ki} d_{i,k} \right)^\alpha \geq d^\alpha \tag{15}$$

where d is the minimum distance among all hops in the network and the equalities in (15) hold if and only if $d_{i,k} = d$, for all $1 \leq k \leq K_i$ and $1 \leq i \leq n/2$. This together with (12) implies that the optimal routing occurs if and only if all hops in the network are using the same transmission range d.

We further observe that

$$\sum_{k=1}^{K_i} d_{i,k} \geq D_i(n) \tag{16}$$

where $D_i(n) = |X_{i,1} - X_{i,K_i+1}|$ denotes the (direct) distance of the ith pair of source and destination. Combining (15) and (16) yields

$$\begin{aligned} \sum_{i=1}^{n/2} \sum_{k=1}^{K_i} d_{i,k}^\alpha &\geq H \left(\frac{1}{H} \sum_{i=1}^{n/2} \sum_{k=1}^{Ki} d_{i,k} \right)^{\alpha-1} \left(\frac{1}{H} \sum_{i=1}^{n/2} \sum_{k=1}^{Ki} d_{i,k} \right) \\ &\geq d^{\alpha-1} \sum_{i=1}^{n/2} D_i(n) \end{aligned} \tag{17}$$

Combining (17) with (13) and (12) gives

$$r(n) \leq \frac{P_0}{N_0} \frac{2d_0^\alpha}{d^{\alpha-1}\overline{D}} \tag{18}$$

where $\overline{D} = \frac{2}{n} \sum_{i=1}^{n/2} D_i(n)$.

We have shown that to achieve the maximum uniform throughput, all hops must have the same distance d. But to ensure the connectivity of the network with high probability (i.e., with probability approaching one as n approaching infinity), d can not be arbitrarily small. Indeed, it is shown in [13] that if all hops in the network have the same distance, to guarantee that all nodes in the network are connected with high probability, it is sufficient and necessary that

$$d \geq \sqrt{(\log n + c(n))/n\pi} \tag{19}$$

where $c(n)$ can grow much more slowly than $\log n$ but must become infinity when n becomes infinity. For example, we can choose $c(n) = \log\log n$.

By using (19) in (18), we have

$$r(n) \leq \frac{P_0}{N_0} \frac{2\pi^{\frac{\alpha-1}{2}} d_0^\alpha}{\overline{D}} \left(\sqrt{\frac{n}{\log n}} \right)^{\alpha-1} \tag{20}$$

which defines an upper bound of the uniform throughput capacity of the network where the total network power is upper bounded by (13) and the network connectivity is guaranteed with high probability.

We now explain that the value of $\overline{D}$ becomes a constant as n becomes large. The problem of $D_i(n)$ is known as disk line picking [11], i.e., $D_i(n)$ is the distance between two randomly picked two points in a disk of unit area. The mean and variance of $D_i(n)$ are known to be

$$\begin{aligned} \mu = E(D_i(n)) &= \frac{128}{45\pi^{3/2}} \simeq 0.5 \\ \sigma^2 &= \mathrm{Var}(D_i(n)) \\ &= \frac{1}{\pi} \left[1 - \left(\frac{128}{45\pi} \right)^2 \right] \simeq 0.06 \end{aligned}$$

According to the central limit theorem, $\overline{D} = \frac{2}{n}\sum_{i=1}^{n/2} D_i(n)$ is asymptotically Gaussian with mean μ and variance $\sigma_D^2 = \frac{\sigma^2}{n/2} \to 0$.

From (20), we have that for large n,

$$r(n) \le c_1 P_0 \left(\sqrt{\frac{n}{\log n}}\right)^{\alpha-1} \tag{21}$$

where c_1 is a constant. In other words, we have the following theorem.

Theorem 1: There exists $c_1 < \infty$ such that

$$\lim_{n\to\infty} \Pr[r(n) = c_1 P_0(\sqrt{n/\log n})^{\alpha-1} \text{ is feasible}] < 1 \tag{22}$$

where P_0 is the maximum power per node. (The constant c_1 here should be larger than the constant c_1 in (21).)

This theorem has established an upper bound on the uniform throughput. In the next section, we establish that a uniform throughput of the same order $O(P_0(\sqrt{n/\log n})^{\alpha-1})$ can be achieved with the per-node power P_0.

B. A Constructive Lower Bound on Uniform Throughput Capacity

The meaning of a lower bound here is an achievable uniform throughput in the network with high probability. Recall that the maximum uniform throughput of the network is achieved when all hops are of equal distance, and the minimum of such distance must be in the order of $\sqrt{\log n/n}$ to ensure network connectivity. (We have neglected the term $c(n)$ in $\sqrt{(\log n + c(n))/n}$ as $c(n)$ becomes negligible as n increases.) We now construct a Voronoi tessellation of a unit area disk as in [1], i.e., each Voronoi cell contains a disk of radius $\rho(n) = c\sqrt{\log n/n}$ and is itself contained in a disk of radius $2\rho(n)$ where c is any constant. A Voronoi cell of the above defined dimension contains at least one node (and hence the network connectivity) with high probability. Furthermore, every node in such a cell is within a maximum possible distance of $8\rho(n)$ from any node in its own cell or in its adjacent cell. Adjacent cells are defined as cells sharing common edges or vertices.

The route for a source–destination pair is a sequence of cells intersected by the line segment that connects the source node and the destination node. There are $n/2$ concurrent working routes in the network at any time. Each cell in the tessellation can be intersected by multiple line segments. Each intersection to a cell adds traffic load to that cell. The total traffic in a cell will be carried by all the nodes in the cell. We will establish a lower bound on the uniform throughput capacity of the network by relating the answers to the following two questions:

Q1: Given a uniform throughput in the network, what is the maximum traffic that each cell has to serve with high probability?
Q2: What is the capability that each cell has with high probability?

The following two lemmas will answer the above two questions respectively and hence lead to a feasible lower bound on the uniform throughput capacity. Lemma 1 is also proved in [1]. However, we give a simpler proof based on the Chernoff bounds (8) and (9), instead of the Vapnik–Chervonenkis theorem as used in [1]. The Chernoff bound is also effective to prove Lemma 2. The fact that all nodes in a cell can work simultaneously was not noticed in [10].

Lemma 1: Let V_k denote a Voronoi cell in the tessellation, and $T(V_k(n))$ denote the total traffic that cell V_k has to carry. If all transmission pairs transmit with a uniform rate of $r(n)$, then there exists $\delta(n) \to 0$ such that

$$\Pr(\sup T(V_k(n)) \le c_3 r(n)\sqrt{n\log n}) \ge 1 - \delta(n). \tag{23}$$

Proof: Let $L_i \cap V_k \ne \phi$ denote that a segment line L_i intersects a cell V_k. It is shown in [1] that

$$\Pr(L_i \cap V_k \ne \phi) \le c_4\sqrt{\frac{\log n}{n}}. \tag{24}$$

There are totally $n/2$ lines in the network. Since each pair of nodes communicate at a rate of $r(n)$ bits per second, each line L_i carries $r(n)$ bits per second. From the fact that $\{L_i\}_{i=1}^{n/2}$ are independent with each other, we have the mean value of traffic that a cell V_k has to carry

$$E[T(V_k(n))] \le \frac{nr(n)}{2}c_4\sqrt{\frac{\log n}{n}} = c_5 r(n)\sqrt{n\log n}. \tag{25}$$

Define the random indicators

$$A_i = \begin{cases} 1, & L_i \cap V_k \ne \phi \\ 0, & L_i \cap V_k = \phi \end{cases}. \tag{26}$$

Then, $T(V_k(n)) = r(n)\sum_{i=1}^{n/2} A_i$. Using the Chernoff bound (8), we have

$$\begin{aligned}&\Pr(T(V_k(n)) \\ &\quad \le (1+\delta)c_5 r(n)\sqrt{n\log n}) \\ &\quad \ge 1 - e^{-c_5\sqrt{n\log n}\delta^2/3}, \quad \text{for all } 0<\delta<1.\end{aligned}$$

Let $\delta(n) = e^{-c_5\sqrt{n\log n}\delta^2/3}$ and $c_3 = (1+\delta)c_5$. We have $\lim_{n\to\infty}\delta(n) = 0$. Lemma 1 is proved.

Lemma 2: Let $C(V_k(n))$ denote the traffic that a Voronoi cell V_k is able to carry. Then, there exists $\delta'(n) \to 0$ such that

$$\Pr(C(V_k(n)) \ge c_6\frac{P_0}{N_0}\frac{n^{\alpha/2}}{(\log n)^{\alpha/2-1}}) \ge 1 - \delta'(n) \tag{27}$$

where P_0 is the transmission power of each node.

Proof: In our routing scheme, each packet is relayed from a Voronoi cell to a neighboring cell. Any pair of nodes in two neighboring cell are within the distance $8\rho(n)$. The Shannon capacity between any two neighboring nodes is

$$C \ge \frac{P_0}{N_0}\left(\frac{d_0}{8\rho(n)}\right)^{\alpha} = \frac{P_0(\pi d_0/8)^{\alpha}}{N_0}\left(\sqrt{\frac{n}{\log n}}\right)^{\alpha}. \tag{28}$$

Note that in a power-constrained network, interference between transmission pairs are negligible in comparison to the ambient noise. All nodes in a cell can work simultaneously, which is different from the setting used in [10]. In [10], only one node in each cell is allowed to work at any time.

Let $|V_k(n)|$ denote the number of nodes in cell V_k. Since all nodes are independently and uniformly distributed, the expected number of nodes in each cell is lower bounded by a lower bound on the area of each cell times n, i.e., there is constant c_7 such that the expected number of nodes in cell V_k is

$$E(|V_k(n)|) \ge c_7\rho^2(n)n = c_8\log n \tag{29}$$

Note that

$$E(C(V_k(n))) \ge C\inf E(|V_k(n)|). \tag{30}$$

Define the random indicators

$$A_i = \begin{cases} 1, & X_i \in V_k \\ 0, & X_i \bar{\in} V_k. \end{cases} \tag{31}$$

Then, $|V_k(n)| = \sum_{i=1}^{n} A_i$. Therefore, for all $0<\delta<1$, the Chernoff bound (9) gives

$$\Pr(\inf |V_k(n)| \ge c_9\log n) \ge 1 - \delta'(n) \tag{32}$$

where $c_9 = (1-\delta)c_8$ and $\delta'(n) = e^{-c_8\log n\delta^2/2} \to 0$. Combining (28), (30) and (32), we get (27). Lemma 2 is proved.

Lemmas 1 and 2 state that any cell V_k in the Voronoi tessellation has to carry a maximum traffic no larger than $c_3 r(n)\sqrt{n\log n}$ with high probability, and at the same time the cell V_k is able to carry a traffic more than $c_6 \frac{P_0}{N_0}\frac{n^{\alpha/2}}{(\log n)^{\alpha/2-1}}$ with high probability. Therefore, there exists such $r(n)$ that $c_3 r(n)\sqrt{n\log n} \ge c_6 \frac{P_0}{N_0}\frac{n^{\alpha/2}}{(\log n)^{\alpha/2-1}}$, i.e.

$$r(n) \ge \frac{c_6}{c_3}\frac{P_0}{N_0}\left(\sqrt{\frac{n}{\log n}}\right)^{\alpha-1}. \tag{33}$$

As n becomes large, the above achievable lower bound on the uniform throughput $r(n)$ holds with high probability. Therefore, we have the following theorem.

Theorem 2: There exists $c_2 > 0$ such that

$$\lim_{n\to\infty} \Pr[r(n) = c_2 P_0 \left(\sqrt{n/\log n}\right)^{\alpha-1} \text{ is feasible}] = 1 \tag{34}$$

where P_0 is the transmission power of each node.

Combining Theorems 1 and 2, we can draw the conclusion that the per-node uniform throughput capacity of the power-constrained network is in the order of $\Theta(P_0(\sqrt{n/\log n})^{\alpha-1})$.

IV. Conclusion

We have shown that for a power-constrained ad hoc network where the nodes are randomly distributed in a unit area disk, the uniform (per-node) throughput capacity is $\Theta(P_0(\sqrt{n/\log n})^{\alpha-1})$ where P_0 is the maximum transmission power consumed by each node.

Appendix I
Proof of Chernoff Bounds

We are not aware of any prior source of the exact forms of the Chernoff bounds (8) and (9). Hence, we include a proof here. For any $t > 0$

$$\Pr(A > (1+\delta)\mu) = \Pr(e^{tA} > e^{t(1+\delta)\mu}). \tag{35}$$

For $\delta > 0$, we have the Markov inequality:

$$\Pr(e^{tA} > e^{t(1+\delta)\mu}) < \frac{E[e^{tA}]}{e^{t(1+\delta)\mu}} \tag{36}$$

where $E[e^{tA}] = \prod_{i=1}^n E[e^{tA_i}] = \prod_{i=1}^n (p_i e^t + 1 - p_i) = \prod_{i=1}^n (1 + p_i(e^t - 1)) < \prod_{i=1}^n e^{p_i(e^t-1)} = e^{(e^t-1)\sum_{i=1}^n p_i} = e^{(e^t-1)\mu}$. Combining (35) and (36), we have

$$\Pr(A > (1+\delta)\mu) < e^{\mu(e^t - 1 - t(1+\delta))} \tag{37}$$

Minimizing the right side of (37) with respect to $t > 0$ gives

$$\Pr(A > (1+\delta)\mu) < e^{\mu(\delta - (1+\delta)\log(1+\delta))} \tag{38}$$

which corresponds to (37) with $t = \log(1+\delta) > 0$. By the Taylor's series expansion of $\log(1+\delta)$, we have that

$$\begin{aligned}
&\delta - (1+\delta)\log(1+\delta)\\
&\quad = -\frac{\delta^2}{2} + \frac{\delta^3}{6} - (\frac{1}{3} - \frac{1}{4})\delta^4 + (\frac{1}{4} - \frac{1}{5})\delta^5 - \cdots\\
&\quad < \left(-\frac{1}{2} + \frac{1}{6}\right)\delta^2 - \left[\left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{1}{4} - \frac{1}{5}\right)\right]\delta^4 - \cdots
\end{aligned}$$

where the inequality holds under the condition $0 < \delta < 1$, and all the terms with orders higher than two are negative. Therefore

$$\delta - (1+\delta)\log(1+\delta) < -\delta^2/3. \tag{39}$$

Hence, we have

$$\Pr[A > (1+\delta)\mu] \le e^{-\mu\delta^2/3} \tag{40}$$

which is (8).

Similarly, for any $t > 0$ and $0 < \delta < 1$

$$\Pr(A < (1-\delta)\mu) = \Pr(e^{-tA} > e^{-t(1-\delta)\mu}). \tag{41}$$

By the Markov inequality, we have

$$\Pr(e^{-tA} > e^{-t(1-\delta)\mu}) < \frac{E[e^{-tA}]}{e^{-t(1-\delta)\mu}} \tag{42}$$

where we can show that $E[e^{-tA}] < e^{(e^{-t}-1)\mu}$. Combining (41) and (42) gives

$$\Pr(A < (1-\delta)\mu) < e^{\mu(e^{-t} - 1 + t(1-\delta))}. \tag{43}$$

Minimizing the right side of (43) with respect to $t > 0$ yields

$$\Pr(A < (1-\delta)\mu) < e^{-\mu(\delta + (1-\delta)\log(1-\delta))} \tag{44}$$

which corresponds to (43) with $t = -\log(1-\delta) > 0$. By the Taylor's series expansion of $\log(1-\delta)$, we have

$$\delta + (1-\delta)\log(1-\delta) = \frac{\delta^2}{2} + \left(\frac{1}{2} - \frac{1}{3}\right)\delta^3 + \left(\frac{1}{3} - \frac{1}{4}\right)\delta^4 + \cdots.$$

Since the terms with orders higher than two are positive, we have

$$\delta + (1-\delta)\log(1-\delta) > \delta^2/2. \tag{45}$$

Therefore, we have

$$\Pr[A < (1-\delta)\mu] \le e^{-\mu\delta^2/2}. \tag{46}$$

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