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For Reference

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THEORY OF TRANSITION RADIATION FROM A SMOOTHLY VARYING BOUNDARY*

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November 20, 1975

ABSTRACT

Some new results on transition radiation are presented. In particular, equations are developed for the situation where the boundaries of the medium are not sharp. These equations are then used to show that the sharp boundary approximation is valid for relativistic particles since the distance over which the radiation develops is always large compared to the thickness of the boundary.

1. INTRODUCTION

The transition radiation is of interest to high energy physicists since it depends on the energy (γ) of the charged particle rather than its velocity as in the Cerenkov radiation. Furthermore, over an interesting range of energies (γ 's in the tens of thousands range) the radiation is approximately proportional to γ . It thus offers a useful method for particle detection which is being widely studied.¹

The prediction of transition radiation was first made by Ginzburg and Frank² in 1946 and since that time the theory has been extensively developed using phenomenological models for the dielectric

constant and the permeability of a medium which change discontinuously at a boundary. Garibian has written an excellent review of the subject which contains an exhaustive list of references to other work.³ Such a discontinuity is an idealization of the effect of atomic phenomena in which changes at the boundary occur over distances of the order of atomic radii, and since the radiation considered is typically in the X-ray region, one may question the validity of calculations based on such models. In this paper we investigate the radiation obtained from a model in which the media have smoothly varying dielectric constants, and for simplicity, we assume that the radiating particle approaches the boundary at normal incidence. We then show that if the "formation zone" is large (as contrasted with the wavelength of the radiation) compared to the size of the boundary region the results of the discontinuous models are valid. Since this condition is satisfied for particles in which $\gamma \gg 1$, the usual results hold.

One might question whether the medium can be treated as continuous, or whether one must deal with a collection of electron-atom interactions. If the formation zone is large, as is true even when the important wavelengths become small, it seems reasonable that such a collection of atoms may be described in terms of space-time average properties such as the dielectric constant and the permeability. This belief is reinforced by the result from our model that the formation zone provides the appropriate scale of length for evaluating the effects of diffuseness of a medium boundary. In our model we assume that the change in the dielectric constant takes place in a

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distance t which is small compared to the length of the formation zone. The permeability μ will be taken to be unity.

2. CALCULATION FOR A SMOOTHLY VARYING DIELECTRIC CONSTANT

For the model calculation we use a Fourier time decomposition.

Then we set

$$\vec{D}(\vec{r}, \omega) = \epsilon(z, \omega) \vec{E}(\vec{r}, \omega). \quad (1)$$

The Maxwell equations, specialized to the case of frequency ω , may then be combined to give differential equations for $\vec{E}$ and $\vec{B}$. For $\vec{E}$ we find:

$$\nabla^2 \vec{E} + \frac{\omega^2}{c^2} \epsilon \vec{E} + \nabla \left(\frac{\vec{E} \cdot \nabla \epsilon}{\epsilon} \right) = \nabla \left(\frac{4\pi \rho}{\epsilon} \right) - 4\pi \frac{1}{c^2} \frac{\partial \rho}{\partial t} \vec{v}. \quad (2)$$

The charge density is initially $e\delta(\vec{r} - \vec{v}t)$. On Fourier decomposition it is given by

$$\rho(\vec{r}, \omega) = \frac{e}{2\pi} \frac{\delta(x)\delta(y)}{v} e^{ik_z z}. \quad (3)$$

where $k_z = \omega/v$. If we now write

$$\vec{E}(\vec{r}, \omega) = \int \vec{E}(\vec{k}, \omega, z) e^{i\vec{k} \cdot \vec{r}_\perp} d^2 k, \quad (4)$$

we obtain the following equations for $\vec{E}_\perp(\vec{k}, \omega, z)$ and $E_z(\vec{k}, \omega, z)$:

$$\frac{d^2 \vec{E}_\perp}{dz^2} + \left(\epsilon \frac{\omega^2}{c^2} - \kappa^2 \right) \vec{E}_\perp + \frac{1}{\epsilon} \frac{d\epsilon}{dz} \left(\frac{\epsilon'}{\epsilon} \vec{E}_\perp \right) = 4\pi i \rho_0 \frac{\vec{k}}{\epsilon} e^{ik_z z}, \quad (5)$$

and

$$\frac{d^2 E_z}{dz^2} + \left(\epsilon \frac{\omega^2}{c^2} - \kappa^2 \right) E_z + \frac{d}{dz} \left(\frac{\epsilon'}{\epsilon} E_z \right) = 4\pi \rho_0 \left[\frac{d}{dz} \left(\frac{e^{ik_z z}}{\epsilon} \right) - ik_z \beta^2 e^{ik_z z} \right]. \quad (6)$$

In a similar manner, one may obtain the equation for $\vec{B}$. If the direction of observation is taken to lie in the x, z plane, the only nonvanishing component of the magnetic field is then B_y :

$$\frac{d^2 B_y}{dz^2} - \frac{\epsilon'}{\epsilon} \frac{dB_y}{dz} + \left(\epsilon \frac{\omega^2}{c^2} - \kappa^2 \right) B_y = 4\pi \rho_0 i \beta \kappa_x e^{ik_z z}. \quad (7)$$

In the above equations,

$$\epsilon' = \frac{d\epsilon}{dz}, \quad \rho_0 = \frac{e}{(2\pi)^3 v}, \quad \text{and} \quad \beta = \frac{v}{c}. \quad (8)$$

In principle, a discussion of the emitted radiation may be based either on Eq. (6) or on Eq. (7). However, in what follows we wish to demonstrate the adequacy of the sharp boundary approximation when ϵ is small, as is true in cases of interest, and when the formation zone is large compared to t . In making this approximation we must be sure that the radiation field is divergenceless, i.e.

$$\vec{k} \cdot \vec{E}_{r\perp}^{(2)} + \frac{dE_{rz}^{(2)}}{dz} = 0. \quad (9)$$

The equation for $E_{rz}^{(2)}$ in Ref. 4 shows that $E_{rz}^{(2)}$ is proportional to κ^2 so that when $\kappa = 0$, $E_{rz}^{(2)}$ vanishes. Our approximation must also lead to this result since it follows generally that there is no radiation in the forward direction. We see that if we work with Eq. (6), care must be taken to satisfy Eq. (9). On the other hand, if we choose to work with Eq. (7) for B_y , the divergence condition for the radiation field

$$\vec{k} \cdot \vec{B}_{r\perp} = 0,$$

is automatically satisfied. When $\kappa = 0$, the right-hand side of

$$\underline{x}^{(\ell+1)} = \underline{x}^{(0)} + P_\ell(K) \underline{z}^{(0)}.$$

Then if

$$Z = [\underline{z}^{(0)}, K\underline{z}^{(0)}, \dots, K^\ell \underline{z}^{(0)}], \quad (2.6)$$

$$\underline{x}^{(\ell+1)} = \underline{x}^{(0)} + Z \underline{\beta}^{(\ell)}. \quad (2.7)$$

Consider the weighted error function:

$$E(\underline{x}^{(\ell+1)}) = \frac{1}{2} (\underline{x} - \underline{x}^{(\ell+1)})^T (M-N) (\underline{x} - \underline{x}^{(\ell+1)}). \quad (2.8)$$

Assuming that $(M-N)$ is nonsingular, we obtain, using

$$\underline{z}^{(0)} = K(\underline{x} - \underline{x}^{(0)}),$$

the relations

$$\begin{aligned} E(\underline{x}^{(\ell+1)}) &= \frac{1}{2} \underline{z}^{(0)T} (I - KP_\ell(K))^T M(M-N)^{-1} M(I - KP_\ell(K)) \underline{z}^{(0)} \\ &= \frac{1}{2} \underline{e}^{(0)T} (I - KP_\ell(K))^T (M-N) (I - KP_\ell(K)) \underline{e}^{(0)}, \end{aligned} \quad (2.9)$$

where

$$\underline{e}^{(0)} = \underline{x} - \underline{x}^{(0)}.$$

Equivalently, we can use (2.7) and re-write (2.8) as

$$E(\underline{x}^{(\ell+1)}) = \frac{1}{2} (K^{-1} \underline{z}^{(0)} - Z \underline{\beta}^{(\ell)})^T (M-N) (K^{-1} \underline{z}^{(0)} - Z \underline{\beta}^{(\ell)}). \quad (2.10)$$

The quantity $E(\underline{x}^{(\ell+1)})$ is minimized when we choose $\underline{\beta}^{(\ell)}$ so that

$$G \underline{\beta}^{(\ell)} = \underline{h},$$

where

$$G = Z^T (M-N) Z, \quad \underline{h} = Z^T M \underline{z}^{(0)}.$$

the application of principal interest for us and for which the generalized CG method seems particularly effective, convergence to desired accuracy often occurs within a number of iterations small compared with n .

2. OPTIMALITY PROPERTIES

From (1.6), we obtain

$$\tilde{z}^{(k+1)} = \tilde{z}^{(k-1)} - \omega_{k+1} (\alpha_k (I - M^{-1}N) \tilde{z}^{(k)} + \tilde{z}^{(k-1)} - \tilde{z}^{(k)}). \quad (2.1)$$

Define

$$K = I - M^{-1}N. \quad (2.2)$$

We have $\tilde{z}^{(1)} = (I - \alpha_0 K) \tilde{z}^{(0)}$, and there follows by induction that

$$\tilde{z}^{(\ell+1)} = [I - K P_\ell(K)] \tilde{z}^{(0)} \quad (2.3)$$

where

$$P_\ell(K) = \sum_{j=0}^{\ell} \beta_j^{(\ell)} K^j. \quad (2.4)$$

We denote

$$p_\ell(\lambda) = \sum_{j=0}^{\ell} \beta_j^{(\ell)} \lambda^j \quad (2.5)$$

and from (2.1) we have for $k = 2, 3, \dots, \ell$

$$p_k(\lambda) = \omega_{k+1} (1 - \alpha_k \lambda) p_{k-1}(\lambda) - (\omega_{k+1} - 1) p_{k-2}(\lambda) + \alpha_k \omega_{k+1},$$

and

$$p_0(\lambda) = \alpha_0, \quad p_1(\lambda) = \omega_2 (\alpha_0 + \alpha_1 - \alpha_0 \alpha_1 \lambda).$$

The coefficients $\{\beta_j^{(\ell)}\}_{j=0}^{\ell}$ can be generated directly.

From (2.3) and the relation $\tilde{z}^{(\ell+1)} = \tilde{z}^{(0)} + K(\tilde{x}^{(\ell+1)} - \tilde{x}^{(0)})$,

there follows

(3) Compute

$$a_k = \frac{\tilde{z}^{(k)T} M \tilde{z}^{(k)}}{\tilde{p}^{(k)T} (M-N) \tilde{p}^{(k)}},$$

$$\tilde{x}^{(k+1)} = \tilde{x}^{(k)} + a_k \tilde{p}^{(k)}.$$

In the computation of the numerators of a_k and b_k one need not recompute $M \tilde{z}^{(k)}$, since it can be saved from step (1). Also, instead of computing the right hand side of step (1) explicitly at each iteration, it is often advantageous to compute it recursively from

$$[\tilde{c} - (M-N)\tilde{x}^{(k+1)}] = [\tilde{c} - (M-N)\tilde{x}^{(k)}] - a_k (M-N)\tilde{p}^{(k)}, \quad (1.12)$$

which equation is obtained from step (3). The quantity $(M-N)\tilde{p}^{(k)}$ appearing in (1.12) may be saved from the computation of a_k . Similar remarks hold for the algorithm in its first form as well. There is evidence that the use of (1.12) is no less accurate than use of the explicit computation (see [18], [3] for particular examples).

The calculated vectors $\{\tilde{z}^{(k)}\}_{k=0}^n$ will not generally be M-orthogonal in practice because of rounding errors. One might consider forcing the newly calculated vectors to be M-orthogonal by a procedure such as Gram-Schmidt. However, this would require the storage of all the previously obtained vectors.

Our basic approach is to permit the gradual loss of orthogonality and with it the finite termination property of CG. We consider primarily the iterative aspects of the algorithm. In fact, for solving large sparse systems arising from the discretization of elliptic partial differential equations,

$$\omega_{k+1} = \left(1 - \frac{\alpha_k}{\alpha_{k-1}} \frac{\tilde{z}^{(k)T} M \tilde{z}^{(k)}}{\tilde{z}^{(k-1)T} M \tilde{z}^{(k-1)}} \cdot \frac{1}{\omega_k} \right)^{-1}, \quad (k \geq 1),$$

$$\omega_1 = 1.$$

(3) Compute

$$\tilde{x}^{(k+1)} = \tilde{x}^{(k-1)} + \omega_{k+1} (\alpha_k \tilde{z}^{(k)} + \tilde{x}^{(k)} - \tilde{x}^{(k-1)}).$$

Note that the algorithm can be viewed as an acceleration of the underlying first order iteration ($\omega_{k+1} \equiv 1$), $\tilde{x}^{(k+1)} = \tilde{x}^{(k)} + \alpha_k \tilde{z}^{(k)}$. As with other higher order methods, the storage requirements of the algorithm are greater than those of the underlying first order iteration being accelerated.

The algorithm presented above is given primarily for expository purposes. For actual computation, the following equivalent form can be more efficient in terms of storage [3].

Algorithm (alternative form)

Let $\tilde{x}^{(0)}$ be a given vector and arbitrarily define $\tilde{p}^{(-1)}$. For $k = 0, 1, \dots$

(1) Solve $M \tilde{z}^{(k)} = \tilde{c} - (M-N) \tilde{x}^{(k)}$.

(2) Compute

$$b_k = \frac{\tilde{z}^{(k)T} M \tilde{z}^{(k)}}{\tilde{z}^{(k-1)T} M \tilde{z}^{(k-1)}}, \quad k \geq 1,$$

$$b_0 = 0,$$

$$\tilde{p}^{(k)} = \tilde{z}^{(k)} + b_k \tilde{p}^{(k-1)}.$$

$$\omega_{k+1} = \left(1 - \frac{\alpha_k}{\alpha_{k-1}} \frac{\tilde{z}^{(k)T} M \tilde{z}^{(k)}}{\tilde{z}^{(k-1)T} M \tilde{z}^{(k-1)}} \frac{1}{\omega_k} \right)^{-1}$$

From (1.6), for $j < k-1$

$$\tilde{z}^{(j)T} M \tilde{z}^{(k+1)} = \alpha_k \omega_{k+1} \tilde{z}^{(j)T} N \tilde{z}^{(k)}.$$

But

$$M \tilde{z}^{(j+1)} = M \tilde{z}^{(j-1)} - \omega_{j+1} (\alpha_j (M-N) \tilde{z}^{(j)} + M(\tilde{z}^{(j-1)} - \tilde{z}^{(j)})),$$

so that

$$\tilde{z}^{(k)T} N \tilde{z}^{(j)} = 0.$$

Thus, since $N = N^T$,

$$\tilde{z}^{(j)T} M \tilde{z}^{(k+1)} = 0 \quad \text{for } j < k-1.$$

Hence by induction we obtain (1.7) and (1.8).

The generalized CG method is summarized as follows.

Algorithm

Let $\tilde{x}^{(0)}$ be a given vector and arbitrarily define $\tilde{x}^{(-1)}$. For $k = 0, 1, \dots$

(1) Solve $M \tilde{z}^{(k)} = \tilde{c} - (M-N) \tilde{x}^{(k)}.$

(2) Compute

$$\alpha_k = \frac{\tilde{z}^{(k)T} M \tilde{z}^{(k)}}{\tilde{z}^{(k)T} (M-N) \tilde{z}^{(k)}},$$

We derive the above result by induction. Assume

$$\tilde{z}^{(p)T} M\tilde{z}^{(q)} = 0 \quad (1.9)$$

for $p \neq q$ and $p, q = 0, 1, \dots, k$.

Then if

$$\alpha_k = \tilde{z}^{(k)T} M\tilde{z}^{(k)} / \tilde{z}^{(k)T} (M-N)\tilde{z}^{(k)}, \quad (1.10)$$

there holds

$$\tilde{z}^{(k)T} M\tilde{z}^{(k+1)} = 0,$$

and if

$$\omega_{k+1} = \left(1 - \alpha_k \frac{\tilde{z}^{(k-1)T} N\tilde{z}^{(k)}}{\tilde{z}^{(k-1)T} M\tilde{z}^{(k-1)}} \right)^{-1} \quad (1.11)$$

then

$$\tilde{z}^{(k-1)T} M\tilde{z}^{(k+1)} = 0.$$

We can simplify the above expression for ω_{k+1} as follows.

From (1.6) we obtain

$$M\tilde{z}^{(k)} = M\tilde{z}^{(k-2)} - \omega_k (\alpha_{k-1} (M-N)\tilde{z}^{(k-1)} + M(\tilde{z}^{(k-2)} - \tilde{z}^{(k-1)})),$$

and then from (1.9)

$$\tilde{z}^{(k)T} N\tilde{z}^{(k-1)} = \tilde{z}^{(k)T} M\tilde{z}^{(k)} / (\omega_k \alpha_{k-1}).$$

Since

$$\tilde{z}^{(k-1)T} N\tilde{z}^{(k)} = \tilde{z}^{(k)T} N\tilde{z}^{(k-1)},$$

it follows

For the Richardson or Chebyshev methods, the optimal parameters (ω_{k+1}, α_k) are given as simple, easy-to-compute functions of the smallest and largest eigenvalues of the iteration matrix $M^{-1}N$ [11]; thus good estimates of these eigenvalues are required for the methods to be efficient. The methods do not take into account the values of any of the interior eigenvalues of the iteration matrix.

The CG method, on the other hand, needs no a priori information on the extremal eigenvalues and does take into account the interior ones, but at a cost of increased computational requirements for evaluating ω_{k+1} and α_k . In § 3, we describe a technique to provide directly from the CG method good estimates for the extreme eigenvalues of the iteration matrix.

From equations (1.4) and (1.5), we obtain the relation

$$M\tilde{z}^{(k+1)} = M\tilde{z}^{(k-1)} - \omega_{k+1}(\alpha_k(M-N)\tilde{z}^{(k)} + M(\tilde{z}^{(k-1)} - \tilde{z}^{(k)})). \quad (1.6)$$

For the generalized CG method the parameters $\{\alpha_k, \omega_{k+1}\}$ are computed so that

$$\tilde{z}^{(p)T} M\tilde{z}^{(q)} = 0 \quad (1.7)$$

for $p \neq q$ and $p, q = 0, 1, \dots, n-1$.

Since M is $n \times n$ positive-definite, (1.7) implies that for some $k \leq n$

$$\tilde{z}^{(k)} = 0$$

and hence

$$\tilde{x}^{(k)} = \tilde{x}. \quad (1.8)$$

That is, the iteration converges in no more than n steps.

examples. Based on our investigations, we feel that the generalized CG method has the potential for widespread application in the numerical solution of boundary value problems for elliptic partial differential equations. Additional experience should further indicate how best to take full advantage of the method's inherent possibilities.

1. DERIVATION OF THE METHOD

Consider the system of equations

$$Ax = b, \quad (1.1)$$

where A is an $n \times n$, symmetric, positive-definite matrix and b is a given vector. It is frequently desirable to rewrite (1.1) as

$$Mx = Nx + c, \quad (1.2)$$

where M is positive-definite and symmetric and N is symmetric. In § 4 we describe several decompositions of the form (1.2). We are interested in those situations for which it is a much simpler computational task to solve the system

$$Mz = d \quad (1.3)$$

than it is to solve (1.1).

We consider an iteration of the form

$$x^{(k+1)} = x^{(k-1)} + \omega_{k+1} (\alpha_k z^{(k)} + x^{(k)} - x^{(k-1)}), \quad (1.4)$$

where

$$Mz^{(k)} = c - (M-N)x^{(k)}. \quad (1.5)$$

Many iterative methods can be described by (1.4); e.g. the Chebyshev semi-iterative method and the Richardson second order method (cf. [11]). The generalized CG method is also of this form.

Eq. (7) vanishes. For this methodological reason we choose to work with the magnetic field.

We now turn to a discussion of the solution of Eq. (7).

The direction of observation will be chosen in the x, z plane so we will write $\kappa = \kappa_x$ in the following. It will be assumed that ϵ is a continuous function of z which changes from unity in region 1 to an asymptotic value ϵ_∞ in region 2 ($z > 0$) as $z \rightarrow \infty$. If we define λ by

$$\lambda^2 = \epsilon \frac{\omega^2}{c^2} - \kappa^2, \quad (10)$$

then for $z < 0$, $\lambda = k_{1z} = [(\omega/c)^2 - \kappa^2]^{\frac{1}{2}}$, and as $z \rightarrow +\infty$, $\lambda \rightarrow k_{2z} = [\epsilon_\infty (\omega/c)^2 - \kappa^2]^{\frac{1}{2}}$. For all z , λ is positive.

$$\frac{d^2 B_y}{dz^2} - \frac{\epsilon'}{\epsilon} \frac{dB_y}{dz} + \lambda^2 B_y = 4\pi\rho_0 i \kappa \beta e^{ik_z z}. \quad (11)$$

To solve this inhomogeneous equation we first construct a Green's function, G , for which

$$\frac{d^2 G}{dz^2} - \frac{\epsilon'}{\epsilon} \frac{dG}{dz} + \lambda^2 G = \delta(z - z'). \quad (12)$$

In this equation δ represents the Dirac delta function, and G may be constructed in terms of the two solutions X_1 and X_2 of the homogeneous part of Eq. (12). These satisfy

$$\frac{d^2 X}{dz^2} - \frac{\epsilon'}{\epsilon} \frac{dX}{dz} + \lambda^2 X = 0. \quad (13)$$

Because of the asymptotic behavior of X in regions 1 and 2 we write:

$$X = a(z) \exp\left(1 \int_0^z \lambda(z') dz'\right) + b(z) \exp\left(-1 \int_0^z \lambda(z') dz'\right). \quad (14)$$

Furthermore a and b are expected to be approximately constant in the case of interest. In calculating dX/dz ($\equiv X'$) we write

$$X' = \lambda \left[a \exp\left(1 \int_0^z \lambda(z') dz'\right) - b \exp\left(-1 \int_0^z \lambda(z') dz'\right) \right], \quad (15)$$

and thus impose the condition that

$$a' \exp\left(1 \int_0^z \lambda(z') dz'\right) + b' \exp\left(-1 \int_0^z \lambda(z') dz'\right) = 0. \quad (16)$$

We then find the equations for a and b :

$$\begin{aligned} a' + \frac{1}{2} \left(\frac{\lambda'}{\lambda} - \frac{\epsilon'}{\epsilon} \right) \left[a - b \exp\left(-21 \int_0^z \lambda(z') dz'\right) \right] &= 0 \\ -b' + \frac{1}{2} \left(\frac{\lambda'}{\lambda} - \frac{\epsilon'}{\epsilon} \right) \left[a \exp\left(21 \int_0^z \lambda(z') dz'\right) - b \right] &= 0 \end{aligned} \quad (17)$$

where primes denote differentiation with respect to z . The substitutions

$$a = A \left(\frac{\epsilon}{\lambda} \right)^{\frac{1}{2}}, \quad b = B \left(\frac{\epsilon}{\lambda} \right)^{\frac{1}{2}} \quad (18)$$

then lead to⁵

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$$A' = \frac{1}{2} \left(\frac{\lambda'}{\lambda} - \frac{\epsilon'}{\epsilon} \right) B \exp \left(-2i \int_0^z \lambda(z') dz' \right) \quad (19)$$

and

$$B' = \frac{1}{2} \left(\frac{\lambda'}{\lambda} - \frac{\epsilon'}{\epsilon} \right) A \exp \left(2i \int_0^z \lambda(z') dz' \right) \quad (20)$$

Our object is to construct two linearly independent solutions of Eq. (13) from which we can obtain the Green's function. These may be defined by choosing solutions χ_1 and χ_2 of the form given by Eq. (14) with

$$A_1(-\infty) = 1; \quad B_1(-\infty) = 0, \quad (21)$$

and

$$A_2(-\infty) = 0; \quad B_2(-\infty) = 1.$$

We note here that the Wronskian, W , of these two solutions

$$W = (\chi_1 \chi_2' - \chi_2 \chi_1') = -2i\epsilon(A_1 B_2 - A_2 B_1) \quad (22)$$

has the value

$$W = -2i\epsilon(z). \quad (23)$$

We choose to construct a Green's function which then gives a particular solution of Eq. (11) for B_y in the form

$$B_y(z) = \int_0^z \mathcal{G}(z, z') R(z') dz' \quad (24)$$

where $R(z')$ denotes the right-hand side of Eq. (11). The Green's function may be written as ⁶

$$\mathcal{G}(z, z') = F(z') \left[\chi_1(z) \chi_2'(z') - \chi_2(z) \chi_1'(z') \right], \text{ for } z > z' \quad (25)$$

$$\mathcal{G}(z, z') = 0, \text{ for } z \leq z'.$$

The function F is determined by the condition:

$$\left. \frac{d\mathcal{G}(z, z')}{dz} \right|_{z=z'} = 1. \quad (26)$$

One finds that for our choice of χ 's, $F = -W^{-1}$, where W is the Wronskian. Thus $\mathcal{G}$ is given by:

$$\mathcal{G}(z, z') = \frac{1}{2i\epsilon(z)} \left[\chi_1(z) \chi_2'(z') - \chi_2(z) \chi_1'(z') \right]. \quad (27)$$

The general solution of Eq. (11) may now be written as

$$B_y(z) = \left[C_1 \chi_1(z) + C_2 \chi_2(z) \right] + \int_0^z \mathcal{G}(z, z') R(z') dz'. \quad (28)$$

In this expression C_1 and C_2 are constants which must be chosen to fit the boundary conditions.

In what follows it will be convenient to introduce the functions, χ_{1A} and χ_{2A} which represent the asymptotic behavior of χ_1 and χ_2 for large positive values of z :

$$\begin{aligned} \chi_{1A} &= \left(\frac{\epsilon}{k_{2z}} \right)^{\frac{1}{2}} \left[A_1(\infty) e^{i(\sigma+k_{2z}z)} + B_1(\infty) e^{-i(\sigma+k_{2z}z)} \right] \\ \chi_{2A} &= \left(\frac{\epsilon}{k_{2z}} \right)^{\frac{1}{2}} \left[A_2(\infty) e^{i(\sigma+k_{2z}z)} + B_2(\infty) e^{-i(\sigma+k_{2z}z)} \right] \end{aligned} \quad (29)$$

where

$$\sigma = \int_0^t [\lambda(z') - k_{2z}] dz' . \quad (30)$$

It is also useful to introduce the function $\mathcal{A}_A(z, z')$, defined by

$$\mathcal{A}_A(z, z') = \frac{1}{2i\epsilon} \left[\chi_1(z) \chi_{2A}(z') - \chi_2(z) \chi_{1A}(z') \right] . \quad (31)$$

When z becomes very large ($z \rightarrow \infty$), $\mathcal{A}_A$ has the value

$$\mathcal{A}_A(z, z') = \frac{1}{2ik_{2z}} \left[e^{ik_{2z}(z-z')} - e^{-ik_{2z}(z-z')} \right], \quad z > z' . \quad (32)$$

We now turn to the determination of the constant C_1 in Eq. (28). Consider the case when $z \rightarrow -\infty$. Since ϵ is constant for $z < 0$, and then $\lambda = +k_{1z}$, one easily finds that

$$B_y(z) = \left[\frac{C_1}{(k_{1z})^{\frac{1}{2}}} e^{+ik_{1z}z} + \frac{C_2}{(k_{1z})^{\frac{1}{2}}} e^{-ik_{1z}z} \right] + \frac{1}{2ik_{1z}} \int_0^z \left[e^{+ik_{1z}(z-z')} - e^{-ik_{1z}(z-z')} \right] R(z') dz' . \quad (33)$$

We now define

$$R(z') = r e^{ik_z z'} \quad (34)$$

where

$$r = 4\pi p_0 i k \beta . \quad (35)$$

Equation (33) may be integrated to yield

$$B_y(z) = \left[\frac{C_1}{(k_{1z})^{\frac{1}{2}}} + \frac{r}{2k_{1z}} \left(\frac{1}{k_z - k_{1z}} \right) \right] e^{+ik_{1z}z} + \left[\frac{C_2}{(k_{1z})^{\frac{1}{2}}} - \frac{r}{2k_{1z}} \left(\frac{1}{k_z + k_{1z}} \right) \right] e^{-ik_{1z}z} - \frac{r e^{ik_z z}}{(k_z^2 - k_{1z}^2)} . \quad (36)$$

The terms in $B_y(z)$, in the order in which they appear, represent incoming radiation from $z = -\infty$, outgoing radiation and the field of the particle. Since there is no incoming radiation

$$C_1 + \frac{r}{2(k_{1z})^{\frac{1}{2}}} \cdot \frac{1}{k_z - k_{1z}} = 0 . \quad (37)$$

This determines the constant C_1 .

To determine the constant C_2 it is necessary to consider the field as $z \rightarrow \infty$. Since we are interested here in the asymptotic fields it is convenient to write

$$B_y(z) = \left[C_1 \chi_1(z) + C_2 \chi_2(z) \right] + \int_0^z \mathcal{A}_A(z, a') r e^{ik_z z'} dz' + \int_0^z \left[\mathcal{A}(z, z') - \mathcal{A}_A(z, z') \right] r e^{ik_z z'} dz' . \quad (38)$$

Between the two integrals, the major term in the asymptotic field is given by the first integral involving $\mathcal{A}_A$, since the difference $[\mathcal{A}(z, z') - \mathcal{A}_A(z, z')]$ vanishes when z and z' are in the region

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$z, z' > t$. This will be shown in detail in what follows, but first let us consider the contributions from $\oint_A$ and the free-field terms to $B_y(z)$:

$$B_y^A(z) = \left[C_1 X_1(z) + C_2 X_2(z) \right] + \int_0^z \oint_A(z, z') r e^{ik_z z'} dz'. \quad (39)$$

When z is large we have

$$B_y^A(z) = \left(\frac{\epsilon}{k_{2z}} \right)^{\frac{1}{2}} \left\{ C_1 \left[A_1(\infty) e^{i(\sigma+k_{2z}z)} + B_1(\infty) e^{-i(\sigma+k_{2z}z)} \right] + C_2 \left[A_2(\infty) e^{i(\sigma+k_{2z}z)} + B_2(\infty) e^{-i(\sigma+k_{2z}z)} \right] \right\} + \frac{1}{2ik_{2z}} \int_0^z \left[e^{ik_{2z}(z-z')} - e^{-ik_{2z}(z-z')} \right] r e^{ik_z z'} dz'. \quad (40)$$

In this equation we have written ϵ for the value of the dielectric constant in the second medium. The integral for σ (Eq. 30) is of the order t/d_F , where d_F is the "formation zone" length⁷ and so by assumption is exceedingly small and may be neglected. We thus have

$$B_y^A(z) \cong \left(\frac{\epsilon}{k_{2z}} \right)^{\frac{1}{2}} \left\{ C_1 \left[A_1(\infty) e^{ik_{2z}z} + B_1(\infty) e^{-ik_{2z}z} \right] + C_2 \left[A_2(\infty) e^{ik_{2z}z} + B_2(\infty) e^{-ik_{2z}z} \right] \right\} - \frac{r e^{ik_z z}}{(k_z^2 - k_{2z}^2)} + \frac{r e^{ik_{2z}z}}{2k_{2z}(k_z - k_{2z})} - \frac{r e^{-ik_{2z}z}}{2k_{2z}(k_z + k_{2z})}, \quad (41)$$

a result which may be compared with Eq. (36). As there, the

various terms represent incoming and outgoing radiation and the field of the charge.

The condition that there be no incoming radiation from $+\infty$ is

$$\left(\frac{\epsilon}{k_{2z}} \right)^{\frac{1}{2}} \left[C_1 B_1(\infty) + C_2 B_2(\infty) \right] - \frac{r}{2k_{2z}(k_z + k_{2z})} = 0. \quad (42)$$

The field of the charge is given by $B_{cy}(z)$:

$$B_{cy}(z) = - \frac{r e^{ik_z z}}{(k_z^2 - k_{2z}^2)}$$

and the radiation field, $B_{ry}^{(2)}(z)$, is

$$B_{ry}^{(2)}(z) \cong \left(\frac{\epsilon}{k_{2z}} \right)^{\frac{1}{2}} \left[C_1 A_1(\infty) e^{ik_{2z}z} + C_2 A_2(\infty) e^{ik_{2z}z} \right] + \frac{r e^{ik_{2z}z}}{2k_{2z}(k_z - k_{2z})}. \quad (43)$$

Since a detailed comparison of these results would be both tedious and unrewarding we now note that two simplifications may be made. First, for the case of interest the change in dielectric constant between the two media is extremely small and second, $(k_z + k_{2z})$ is very large compared to $k_z - k_{2z}$. Accordingly the comparison which we shall make between these two results will be to first order in $\delta\epsilon$, the change in dielectric constants, and terms involving $(k_z + k_{2z})^{-1}$ will be neglected.

Let us now consider the approximate solution of Eqs. (19)

and (20) for A_1 and B_1 . The region over which λ' is different

from zero is t . It is easily seen that the quantity $(\lambda'/\lambda - \epsilon'/\epsilon)$ is a monotonic function of z if $\epsilon(z)$ is monotonic.⁸ Therefore, since

$$\left| \frac{dA}{dz} \right| = \frac{1}{2} \left| \frac{\lambda'}{\lambda} - \frac{\epsilon'}{\epsilon} \right| \cdot |B|,$$

we see that

$$\frac{dA}{dz} < -\frac{1}{2} \left(\frac{\lambda'}{\lambda} - \frac{\epsilon'}{\epsilon} \right) \cdot B,$$

assuming that $B > 0$, and that λ/ϵ is monotonic decreasing.

The latter is true if the angle of emitted radiation θ is less than $\sim 45^\circ$, while for $\theta \gtrsim 45^\circ$ the sign of the

inequality must be reversed. A similar relation holds for B . Both A and B will be bounded if the inequalities are replaced by equalities, in which case the equations can be integrated to obtain:

$$\begin{aligned} A(z) &\leq \left[\left(\frac{\lambda_0}{\lambda} \epsilon \right)^{\frac{1}{2}} + \left(\frac{\lambda}{\lambda_0} \epsilon \right)^{\frac{1}{2}} \right] \frac{A_0}{2}, \\ \text{and} \\ B(z) &\leq \left[\left(\frac{\lambda_0}{\lambda} \epsilon \right)^{\frac{1}{2}} - \left(\frac{\lambda}{\lambda_0} \epsilon \right)^{\frac{1}{2}} \right] \frac{A_0}{2}. \end{aligned} \quad (44)$$

In this case we have assumed that $B_0 = 0$. A similar result holds if $A_0 = 0$. Since B reaches its final value $B(\infty)$ at t we have

$$B_1(\infty) \gtrsim \frac{1}{2} \left[\left(-\frac{\epsilon k_{1z}}{k_{2z}} \right)^{\frac{1}{2}} - \left(-\frac{k_{2z}}{\epsilon k_{1z}} \right)^{\frac{1}{2}} \right]. \quad (45)$$

The quantity $[k_{2z}/(\epsilon k_{1z})]^2$ may be written as

$$\left(\frac{k_{2z}}{\epsilon k_{1z}} \right)^2 = 1 - \frac{(\epsilon - 1)}{\epsilon^2 k_{1z}^2} \left[\epsilon \frac{\omega^2}{c^2} - (\epsilon + 1) k^2 \right] \quad (46)$$

so that B_1 is proportional to $(\epsilon - 1) = \delta\epsilon$, and

$$B_1(\infty) \gtrsim \frac{|(\epsilon - 1)| \omega^2}{4\epsilon^2 k_{1z}^2 c^2}. \quad (47)$$

Since B_1 is proportional to $\delta\epsilon$ it follows that the change in A_1 , δA_1 , is proportional to $(\delta\epsilon)^2$ and so may be neglected. A similar result holds for A_2 and B_2 with the roles of A and B interchanged. Thus, to first order in $\delta\epsilon$ we may take $A_1(\infty) = B_2(\infty) = 1$ and replace $B_1(\infty) = A_2(\infty)$ by the value given in Eq. (47).

With the approximations discussed above we find from Eq. (42):

$$C_2 = C_1 B_1(\infty). \quad (48)$$

Thus C_2 is of order $\delta\epsilon$. If we now turn to the expression for $B_y(z)$, the term involving $C_2 A_2$ is proportional to $(\delta\epsilon)^2$ and may be neglected. Thus

$$B_{ry}^{(2)}(z) \approx (\epsilon/k_{2z})^{\frac{1}{2}} C_1 e^{ik_{2z}z} + \frac{r}{2k_{2z}} \frac{e^{ik_{2z}z}}{(k_z - k_{2z})}. \quad (49)$$

Upon combining this with Eq. (37) one finds

$$B_{ry}^{(2)}(z) \approx \left[\frac{1}{k_{2z}(k_z - k_{2z})} - \frac{\epsilon^{\frac{1}{2}}}{(-k_{1z} k_{2z})^{\frac{1}{2}} (k_z - k_{1z})} \right] \frac{r}{2} e^{ik_{2z}z}. \quad (50)$$

It is interesting but not surprising to note that the "backward" radiation given in Eq. (36) is very small compared to the "forward" radiation since it contains no small denominator like those in Eq. (50). This completes the determination of $B_{ry}^{(2)}$.

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We may now find $E_{rz}^{(2)}(z)$ using the Maxwell equation for radiation, thus

$$E_{rz}^{(2)} = -\frac{\kappa c}{\epsilon \omega} B_{rz}^{(2)}.$$

The final result is

$$E_{rz}^{(2)}(z) = \frac{1}{4\pi^2} \frac{e \kappa^2}{k_z v_z} \left[\frac{-1/\epsilon}{k_{2z}(k_z - k_{2z})} + \frac{1}{(-\epsilon k_{1z} k_{2z})^{\frac{1}{2}} (k_z - k_{1z})} \right]. \quad (51)$$

This is now to be compared with the well-known result,³ for this case,

$$E_{rz}^{(2)}(z) = \frac{1}{2\pi^2} \frac{e \kappa^2}{v_z k_z} \left[\frac{1}{(k_{2z} - \epsilon k_{1z})} \right] \left[\frac{(-1/\epsilon)(k_z - \epsilon k_{1z})}{(k_z^2 - k_{2z}^2)} + \frac{1}{k_z - k_{1z}} \right]. \quad (52)$$

To make the comparison, we note that k_z , k_{1z} and k_{2z} are approximately equal. Furthermore, $\delta\epsilon \approx -\omega_p^2/\omega^2$ where ω_p is the plasma frequency for the atomic electrons, and, as has been shown by Garibian,³ the important frequency region for the radiated power lies below $\gamma\omega_p$, so that in this region $|\delta\epsilon| > \gamma^{-2}$, and as $|\delta\epsilon| \rightarrow \gamma^{-2}$ the radiation falls to an insignificant value.

We now consider the terms $(k_z - k_{1z})$ and $(k_z - k_{2z})$ in either of the above expressions. We have, since $\kappa_0 = (\epsilon^{\frac{1}{2}} \omega/c) \sin \theta$,

$$(k_z - k_{1z})^{-1} \approx \frac{2\gamma^2 c}{\omega(1 + \gamma^2 \theta^2)}$$

and

$$(k_z - k_{2z})^{-1} \approx \frac{2c}{\omega(\gamma^{-2} + \delta\epsilon + \theta^2)}.$$

Thus the terms involving $(k_z - k_{1z})^{-1}$ are much larger than those involving $(k_z - k_{2z})^{-1}$ and we conclude that Eqs. (51) and (52) agree to a high degree of precision. This is the major result of this note.

Our remaining task is to justify the neglect of the terms in Eq. (37) which involve $\left[\mathcal{J}(z, z') - \mathcal{J}_A(z, z') \right]$. These are given by

$$B_y^{\Delta}(z') = \int_0^z \left[\mathcal{J}(z, z') - \mathcal{J}_A(z, z') \right] r e^{ik_z z'} dz'. \quad (53)$$

Consider a typical term in this expression:

$$I = -\frac{\chi_2(z)}{2i} \int_0^z \left[\frac{\chi_1(z')}{\epsilon(z')} - \frac{\chi_{1A}(z')}{\epsilon} \right] r e^{ik_z z'} dz'. \quad (54)$$

If we now use Eqs. (14) and (29), I may be written as

$$I = \frac{\chi_2(z)}{2i} \int_0^t \left\{ \left[\epsilon(z') \lambda(z') \right]^{\frac{1}{2}} \left[A_1(z') \exp \left(i \int_0^{z'} \lambda(z'') dz'' \right) + B_1(z') \exp \left(-i \int_0^{z'} \lambda(z'') dz'' \right) \right] - (\epsilon k_{2z})^{-\frac{1}{2}} \left[A_1(\omega) e^{i(\sigma + k_{2z} z')} + B_1(\omega) e^{-i(\sigma + k_{2z} z')} \right] \right\} r e^{ik_z z'}. \quad (55)$$

It is to be noted that the integral extends only to $z = t$ since the integrand vanishes beyond that point. Consider first the term I_1

arising from the A's in Eq. (55):

$$I_1 = \frac{\chi_2(z)}{2i} \int_0^t \left\{ \left[\epsilon(z') \lambda(z') \right]^{-\frac{1}{2}} A_1(z') \exp \left[i \left(\int_0^{z'} \lambda(z'') dz'' - \sigma - k_{2z} z' \right) \right] \right. \\ \left. - (\epsilon k_{2z})^{-\frac{1}{2}} A_1(\infty) \right\} r \exp \left[i(k_z z' + k_{2z} z' + \sigma) \right] dz' . \quad (56)$$

We see that the terms in the exponential multiplying $A_1(z')$ are negligible if the formation zone is large so that

$$I_1 \approx \frac{\chi_2(z)}{2i} \int_0^t \left\{ \left[\epsilon(z') \lambda(z') \right]^{-\frac{1}{2}} A_1(z') - (\epsilon k_{2z})^{-\frac{1}{2}} A_1(\infty) \right\} \\ \times r e^{i(k_z + k_{2z})z'} dz' . \quad (57)$$

As k_z becomes small compared to t , this term will become very small because of the cancelling oscillations associated with the exponential. Even if the cancellations are ignored, however, in this expression one can demonstrate that $\epsilon(z) \lambda(z)$ is monotonic in ϵ , and thus the integral is less than $\approx \left\{ (-k_{1z})^{-\frac{1}{2}} - (\epsilon k_{2z})^{-\frac{1}{2}} \right\} rt$. Compared to the corresponding terms in B_y^A , this term is of order $t\delta\epsilon$ whereas the other is of order $d_F \delta\epsilon$. Hence the integral is small compared to those terms which have been discussed above and so may be neglected. Next consider the terms, I_2 , arising from the B's in Eq. (55).

$$I_2 = \frac{\chi_2(z)}{2i} \int_0^t \left\{ \left[\epsilon(z') \lambda(z') \right]^{-\frac{1}{2}} B_1(z') \exp \left[-i \left(\int_0^{z'} \lambda(z'') dz'' + k_{2z} z' - \sigma \right) \right] \right. \\ \left. - (\epsilon k_{2z})^{-\frac{1}{2}} B_1(\infty) \right\} r \exp \left[i(k_z z' - k_{2z} z' - \sigma) \right] dz' . \quad (58)$$

Using the same approximations which led to Eq. (56) we have

$$I_2 \approx \frac{\chi_2(z)}{2i} \int_0^t \left\{ \left[\epsilon(z') \lambda(z') \right]^{-1} B_1(z') \right. \\ \left. - (\epsilon k_{2z})^{\frac{1}{2}} B_1(\infty) \right\} r e^{i(k_z - k_{2z})z'} dz' . \quad (59)$$

The exponential may be set equal to one since it depends on the formation zone d_F . Further, since $\delta\epsilon \ll 1$, and B_1 is of order $\delta\epsilon$, the remaining integral is of order $t\delta\epsilon$ and hence can be neglected as well. Corresponding arguments can be applied to the other parts of B_y^A to show that they too are negligible in our approximation. We thus conclude that when the formation zone is large compared to the region over which the dielectric constant varies, the simple result, obtained by assuming a sharp dielectric boundary, is valid.

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FOOTNOTES AND REFERENCES

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 4. G. M. Garibian, JETP 6, 1079 (1958), Eqs. (11) and (12).
 5. The well-known W.K.B. approximation follows from these transformations by noting that if $\lambda'/\lambda \ll 1$, a rapidly oscillating exponential makes A, B approximately constant.
 6. Cf., E. L. Ince, in Ordinary Differential Equations, Chap. XI (Dover Publications, Inc., New York, 1943).
 7. Garibian defines the "formation zone" length in terms of the radiation from a slab of thickness l . Radiation from the two surfaces has a phase difference which arises from the difference between the phase variation of the radiation and of the charged particle in passing through the slab. For $\gamma \gg 1$, and radiation produced at angle θ , one easily finds

$$d_F^{-1} \simeq k_{1z}(\gamma^{-2} + \theta^2 + 1 - \epsilon)/2\pi.$$

The expression for σ involves λ which changes monotonically

7. Cont.

from k_{1z} to k_{2z} and can be bounded as $\sigma \lesssim -(\omega/c)(\epsilon - 1)t/2$, which is of the order of t/d_F for $\theta = 0$, in the limit $\gamma \rightarrow \infty$.

8. Explicitly, $\lambda'/\lambda - \epsilon'/\epsilon = (\epsilon'/\epsilon)(2 \sin^2 \theta - \epsilon)/(2\epsilon - \sin^2 \theta)$.

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